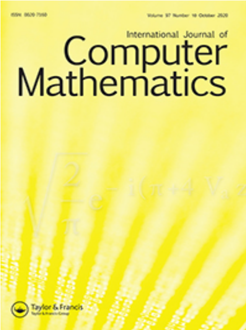


International Journal of Computer Mathematics

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/gcom20

A perturbation-based approach for solving fractional-order Volterra–Fredholm integro differential equations and its convergence analysis

Pratibhamoy Das , Subrata Rana & Higinio Ramos

To cite this article: Pratibhamoy Das , Subrata Rana & Higinio Ramos (2020) A perturbation-based approach for solving fractional-order Volterra–Fredholm integro differential equations and its convergence analysis, International Journal of Computer Mathematics, 97:10, 1994-2014, DOI: 10.1080/00207160.2019.1673892

To link to this article: https://doi.org/10.1080/00207160.2019.1673892

Accepted author version posted online: 27 Sep 2019.
Published online: 09 Oct 2019.

INTERNATIONAL JOURNAL OF COMPUTER MATHEMATICS
2020, VOL. 97, NO. 10, 1994–2014
https://doi.org/10.1080/00207160.2019.1673892

Taylor & Francis
Taylor & Francis Group

ARTICLE

A perturbation-based approach for solving fractional-order Volterra–Fredholm integro differential equations and its convergence analysis

Pratibhamoy Das[a], Subrata Rana[a] and Higinio Ramos[b]

[a]Department of Mathematics, Indian Institute of Technology, Patna, India; [b]Department of Applied Mathematics, University of Salamanca, Salamanca, Spain

ABSTRACT

The present work considers the approximation of solutions of a type of fractional-order Volterra–Fredholm integro-differential equations, where the fractional derivative is introduced in Caputo sense. In addition, we also present several applications of the fractional-order differential equations and integral equations. Here, we provide a sufficient condition for existence and uniqueness of the solution and also obtain an a priori bound of the solution of the present problem. Then, we discuss about the higher-order model equation which can be written as a system of equations whose orders are less than or equal to one. Next, we present an approximation of the solution of this problem by means of a perturbation approach based on homotopy analysis. Also, we discuss the convergence analysis of the method. It is observed through different examples that the adopted strategy is a very effective one for good approximation of the solution, even for higher-order problems. It is shown that the approximate solutions converge to the exact solution, even for higher-order fractional differential equations. In addition, we show that the present method is highly effective compared to the existed method and produces less error.

ARTICLE HISTORY
Received 4 June 2019
Revised 5 August 2019
Accepted 23 September 2019

KEYWORDS
Fractional integro differential equation; Caputo fractional derivative; Volterra–Fredholm integral equation; approximation theory; convergence analysis; perturbation approach; experimental evidence

2010 MATHEMATICS SUBJECT CLASSIFICATIONS
26A33; 1A58; 34K05

1. Introduction

Fractional calculus has a major role in molecular theory on which the viscoelastic property of polymer solids with no cross linking can be addressed. By using the fractional calculus, it can be shown that the results of these molecular theories are equivalent to constitutive relationships written in terms of fractional differential models [4]. In addition, the fractional calculus is also frequent in speech signal modelling. This process is based on integer-order module which is constructed with the celebrated Linear Predictive Coding (LPC). It is observed that by considering several integrals of fractional orders as basis functions, the speech signal can be modelled accurately [3]. Fractional coupled Burgers equations can model the sedimentation of two kinds of particles in fluid suspensions or colloids [22]. In addition, integral equations and fractional differential equations appear in several other places as in the modelling of particle motion in Physics and Mechanics. To illustrate this use, we present below two models which lead to these type of problems.

Let us consider the equation of motion of a flying vehicle whose initial velocity along horizontal and vertical axis is denoted as (u_0, v_0) during its take off, and the flying vehicle is moving upwards with constant horizontal velocity till it reaches to its desired altitude h. Here, we assume that the flying

CONTACT Higinio Ramos higra@usal.es

vehicle is moving free of friction forces. In this scenario, if we consider $T(y)$ as the time fixed by the air traffic control to reach at an altitude y, then the equation of motion involving the length of the path $\sigma(y^*)$ at the height y^* can be calculated by considering the Energy Conservation Law. If the mass of the flying vehicle is m (assuming that the mass lost due to fuel consumption is neglected) and the gravitational constant is g, then the conservation of energy of the flying vehicle with tangential velocity $v = \mathrm{d}\sigma/\mathrm{d}t$ at time t leads to

$$\frac{1}{2}mv^2 + mgy^* = \frac{1}{2}m(u_0^2 + v_0^2) \Rightarrow v^2 = (u_0^2 + v_0^2 - 2gy^*)$$
$$\Rightarrow \frac{\mathrm{d}\sigma}{\mathrm{d}t} = (u_0^2 + v_0^2 - 2gy^*)^{1/2} \Rightarrow \frac{\mathrm{d}\sigma}{(u_0^2 + v_0^2 - 2gy^*)^{1/2}} = \mathrm{d}t.$$

At $t = 0$ it is $y^* = 0$, and when $y^* = y$, we consider the time as some known function of y, i.e. $t = T(y)$. Then, by integrating it, we get

$$T(y) = \int_0^y \sigma'(y^*)\frac{\mathrm{d}y^*}{(u_0^2 + v_0^2 - 2gy^*)^{1/2}}. \tag{1}$$

Since $u_0 > 0$, we can conclude that $K(y, y^*) = (u_0^2 + v_0^2 - 2gy^*)^{-1/2} > 0$ for all $0 \le y^* \le y \le h$, where h is the maximum height at which the pilot wanted to fly. During the motion, let us assume that the vertical component of velocity v_0, only depends on y. At the maximum permissible altitude, the vertical component will be zero and flying vehicle will move based on horizontal velocity. Again, at any height $y \in [0, h]$, the conservation of energy provides

$$\tfrac{1}{2}mu_0^2 + mgy = \tfrac{1}{2}m(u_0^2 + v_0^2) \Rightarrow v_0^2 = 2gy. \tag{2}$$

Hence, by substituting (2) into (1), we obtain

$$T(y) = \int_0^y \frac{\sigma'(y^*)\,\mathrm{d}y^*}{(u_0^2 + 2g(y - y^*))^{1/2}} \quad \text{where } \sigma(0) = 0. \tag{3}$$

Using Leibniz's rule, we get from (3) that

$$\begin{aligned} &\sigma'(y) = g(y) + \frac{c}{2}\int_0^y \frac{\sigma'(y^*)\,\mathrm{d}y^*}{(c^2 + (y - y^*))^{3/2}}, \\ &\sigma(0) = 0, \end{aligned} \tag{4}$$

where $g(y) = u_0 T'(y)$ and $c = u_0/\sqrt{2g}$.

Here, our aim is to find out the length of the path travelled by the flying vehicle to reach at its desired altitude. Note that at the desired altitude, the vertical velocity v_0 is zero. Therefore, flying vehicle will be governed by horizontal velocity. Once the flying vehicle has reached to its desired altitude of travelling, it will follow its own direction with an average speed. Hence the required travel distance from the departure place to destination place on the sky can be calculated by an algebraic equation.

Let us see one example where fractional equations are obtained by modelling the motion of a particle. To show this, let us consider a particle which is constrained to fall along a path on vertical plane with constant vertical acceleration, say g. Given that the time taken to fall is prescribed by a function of height y, say $T_1(y)$, our aim is to find the length of the path σ at any time t through which the particle moves. Now we consider the motion of a particle from height y with zero initial velocity. Here $\sigma(y)$ denotes the length of the trajectory from height y to origin. Hence, the velocity at any point

$\bar{y}$ is $v = d\sigma/dt$, if the particle moves from y to $\bar{y}$ at time dt. At the height $\bar{y}$, we have by the Energy Conservation Law that

$$\frac{1}{2}m\left(\frac{d\sigma}{dt}\right)^2 + mg\bar{y} = mgy \Rightarrow \frac{d\sigma}{dt} = -\sqrt{2g(y-\bar{y})} \Rightarrow \frac{d\sigma}{\sqrt{(y-\bar{y})}} = -\sqrt{2g}\,dt.$$

Here, a negative sign is taken in the above equation because the distance decreases as time increases. At $t = 0$ it is $\bar{y} = y$, and for $t = T_1(y)$ it is $\bar{y} = 0$. Then, by integrating, we get

$$\sqrt{2g}T_1(y) = \int_0^y \sigma'(\bar{y})\frac{d\bar{y}}{\sqrt{(y-\bar{y})}} \Rightarrow D^{1/2}\sigma(y) = \Gamma(1/2)\sqrt{2g}T_1(y), \tag{5}$$

where $\sigma(y)$ is the desired unknown. This problem is also popularly known as Abel's mechanical problem [15].

In the present work, we consider a fractional-order Volterra–Fredholm type integro-differential equation [13,24] and provide an approximation of its solution. We consider the approximation method and its convergence analysis by a semi-analytical approach [1,17,18]. This approach can be considered as an alternative approach based on adaptive techniques, provided in [6–12]. It is observed that the present approach can be extended to a higher-order fractional integro-differential equation, by reducing it to an equivalent system of fractional differential equations of order less than or equal to one. Some numerical examples are provided to show the effectiveness of the present approximation method for integro differential equations in fractional case.

Throughout the paper, the set of all natural numbers will be denoted by $\mathbb{N}$. The closure of the domain Ω is denoted as $\overline{\Omega}$. In addition, the set of all continuous functions on Ω will be $C(\Omega)$ while $C^n(\Omega) = \{f(x)|f^{(j)}(x), j = 1, \ldots, n$ exist and are continuous for $x \in \Omega\}$. Where necessary, the supremum norm of a function $f(x)$ will be considered, which is denoted by $\|f(x)\|_\infty = \|f(x)\| = \sup_{x\in\Omega}|f(x)|$ in a domain Ω. Finally, $C([0,T],\mathbb{R})$ denotes the set of all continuous functions from $[0,T]$ to $\mathbb{R}$.

Now we introduce some preliminaries on the topics of fractional integrals and fractional derivatives, and some properties, which will be used later for further analysis. A more complete discussion of these topics can be found in [19,20].

Definition: The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $f(x)$ is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1}f(\tau)\,d\tau, \quad x > 0,\ \alpha \in \mathbb{R}^+, \tag{6}$$

where $\mathbb{R}^+$ is the set of all positive real numbers.

Definition: The Liouville–Caputo fractional derivative of a function f is defined as

$$D^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\tau)^{n-\alpha-1}f^{(n)}(\tau)\,d\tau & \text{if } n-1 < \alpha < n, \\ [8pt]\frac{d^n f(x)}{dx^n}, & \alpha = n,\ n \in \mathbb{N}, \end{cases} \tag{7}$$

where $\alpha \in \mathbb{R}^+$ is called the order of the derivative.

Some Properties of the Fractional Integrals and Fractional Derivatives:

(i) If $0 < \alpha < 1$, then we have

$$D^\alpha J^\alpha f(x) = f(x) \quad \text{and} \quad J^\alpha D^\alpha f(x) = f(x) - f(0+).$$

(ii) For $\delta \in \mathbb{R}$ and $n \in \mathbb{N}$ we have

$$D^{\alpha} x^{\delta} = \begin{cases} 0 & \text{if } n-1 < \alpha < n,\ \delta \le n-1, \\ \dfrac{\Gamma(\delta+1)}{\Gamma(\delta-\alpha+1)} x^{\delta-\alpha} & \text{if } n-1 < \alpha < n,\ \delta > n-1. \end{cases}$$

In case $\alpha \in \mathbb{N}$, the Liouville–Caputo derivative coincides with the usual derivative.

(iii) Liouville–Caputo derivative satisfies the linearity property for the differentiation, i.e.

$$D^{\alpha}(\delta_1 f(x) + \delta_2 g(x)) = \delta_1 D^{\alpha} f(x) + \delta_2 D^{\alpha} g(x),$$

where δ_1 and δ_2 are constants.

2. Volterra–Fredholm integro differential model

Consider the following fractional-order Volterra–Fredholm integro-differential equation:

$$\begin{aligned} D^{\alpha} u(x) &= f(x) + \int_0^x k_1(x,t) F_1(u(t),t)\,\mathrm{d}t + \int_0^1 k_2(x,t) F_2(u(t),t)\,\mathrm{d}t, \quad x \in \Omega, \\ u(0) &= u_0, \end{aligned} \tag{8}$$

where $0 < \alpha \le 1$ and $\Omega = (0,1]$. Here, we assume that $f(x)$ is continuous on $\overline{\Omega} = [0,1]$.

In addition, we will make some assumptions which will provide most of the sufficient conditions for the existence and uniqueness of the model problem. Those assumptions are given below:

(I) $F_1(u(x),x), F_2(u(x),x)$ are functions that satisfy the Lipschitz condition with respect to $u(x)$, with Lipschitz constants $L_1(>0)$ and $L_2(>0)$ respectively, and $F_1(0,x) = F_2(0,x) = 0$, for all $x \in \overline{\Omega}$.

(II) The kernels $k_1(x,t)$ and $k_2(x,t)$ are continuous on $\overline{\Omega} \times \overline{\Omega}$, and therefore bounded by $M_1(>0)$ and $M_2(>0)$ in $\overline{\Omega} \times \overline{\Omega}$.

Our goal is to find under these assumptions a sufficiently smooth solution $u(x)$ of the problem in (8), defined on $\overline{\Omega}$.

3. Analytical properties

Theorem 3.1: *If the condition*

$$\frac{M_1 L_1 + (\alpha+1) M_2 L_2}{\Gamma(\alpha+2)} (= \gamma) < 1$$

is satisfied, then the above Volterra–Fredholm initial value problem in (8) *under the assumptions in* (I)-(II), *has a unique solution* $u(x)$, *for all* $x \in \overline{\Omega}$.

Proof: The idea of the proof is given in [16] (Theorem 7). This idea is required to obtain the error analysis of the present approach. Hence, we provide the sketch in few lines.

We apply J^{α} on both sides of (8) to obtain $u(x) = \Lambda u(x)$ where

$$\Lambda u(x) = u_0 + J^{\alpha}(f(x)) + J^{\alpha}\left[\int_0^x k_1(x,t) F_1(u(t),t)\,\mathrm{d}t + \int_0^1 k_2(x,t) F_2(u(t),t)\,\mathrm{d}t\right]. \tag{9}$$

Considering that $u_1(x), u_2(x) \in C[0,1]$, we have

$$\begin{aligned} |\Lambda u_1(x) - \Lambda u_2(x)| &\le \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \left[\int_0^t |k_1(t,s)|(|F_1(u_1(s),s) - F_1(u_2(s),s)|)\,\mathrm{d}s \right. \\ &\quad \left. + \int_0^1 |k_2(t,s)|(|F_2(u_1(s),s) - F_2(u_2(s),s)|)\,\mathrm{d}s \right] \mathrm{d}t \\ &\le \frac{M_1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \left[\int_0^t L_1 |u_1(s) - u_2(s)|\,\mathrm{d}s \right] \mathrm{d}t \\ &\quad + \frac{M_2}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \left[\int_0^1 L_2 |u_1(s) - u_2(s)|\,\mathrm{d}s \right] \mathrm{d}t \\ &\le \frac{(M_1L_1 + (\alpha+1)M_2L_2)}{\Gamma(\alpha+2)} \|u_1 - u_2\|, \end{aligned}$$

which, according to the hypothesis, states that Λ is a contraction mapping since $(C[0,1], \|\cdot\|)$ is a Banach space. Hence, by the Banach's Fixed-Point Theorem, we can conclude that the initial value problem in (8) has a unique solution in $C[0,1]$ assuming that

$$\gamma = \frac{M_1L_1 + (\alpha+1)M_2L_2}{\Gamma(\alpha+2)} < 1. \tag{10}$$

This completes the proof. ∎

Now we readily get the following result which provides the a priori bound of the solution.

3.1. A priori bound of the solution

Under the hypotheses in the above theorem, we can obtain a priori bound of the solution $u(x)$ of (8). Let $B_r = \{w \in C([0,1],\mathbb{R}) : \|w\| \le r\}$ be the set of bounded continuous functions in $[0,1]$, with bound $r > 0$. We can estimate a value of r as follows. Note that, $h(x) = u_0 + J^\alpha(f(x))$ is continuous on $\overline{\Omega}$, since $f(x)$ is continuous on $\overline{\Omega}$. Therefore, there exists a real number $M_0 > 0$ such that $|h(x)| \le M_0$. Assuming that the unique solution $u(x)$ is in B_r, we apply J^α on both sides of (8) and get

$$\begin{aligned} |u(x)| &\le |h(x)| + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \left[\int_0^t |k_1(t,s)||F_1(u(s),s)|\,\mathrm{d}s + \int_0^1 |k_2(t,s)||F_2(u(s),s)|\,\mathrm{d}s \right] \mathrm{d}t \\ &\le M_0 + \frac{M_1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \left[\int_0^t L_1|u(s)|\,\mathrm{d}s \right] \mathrm{d}t + \frac{M_2}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \left[\int_0^1 L_2|u(s)|\,\mathrm{d}s \right] \mathrm{d}t \\ &\le M_0 + \frac{(M_1L_1 + (\alpha+1)M_2L_2)r}{\Gamma(\alpha+2)}. \end{aligned}$$

Hence,

$$\|u\| \le M_0 + \frac{(M_1L_1 + (\alpha+1)M_2L_2)r}{\Gamma(\alpha+2)}.$$

Now, we consider a real number $r > 0$ such that

$$r \ge \frac{M_0\Gamma(\alpha+2)}{\Gamma(\alpha+2) - (M_1L_1 + (\alpha+1)M_2L_2)} = \frac{M_0}{1-\gamma}.$$

This provides an a priori bound for $u(x)$.

Note 1: Let us consider the following fractional-order Volterra–Fredholm integro-differential equation (see [2,19])

$$\begin{aligned} D^{\alpha}u(x) &= f(x) + \int_0^x k_1(x,t)F_1(u(t),t)\,\mathrm{d}t + \int_0^1 k_2(x,t)F_2(u(t),t)\,\mathrm{d}t, \quad x \in \Omega, \\ u^{(i)}(0) &= \bar{u}_i, \quad i = 0,1,\ldots,n-1, \end{aligned} \tag{11}$$

where $n-1 < \alpha \le n, n \in \mathbb{N}$ and $\Omega = (0,1]$. Here, $f(x)$ is a continuous function on $\overline{\Omega} = [0,1]$. With the assumptions of (I) and (II) and the condition at Theorem 3.1, the equation (11) has a unique continuous solution on $\overline{\Omega}$ which can be found out in a similar way, as was described in the proof of Theorem 3.1.

Now, we can reduce this higher-order fractional integro-differential equation into an equivalent system of fractional equations of order less than or equal to one. Therefore, obtaining the solution of this equivalent system is similar to obtain the solution of (11) (for more details see [5]). Equation (11) can be written in the following form:

$$\begin{aligned} D^{\alpha_1}u_1(x) &= u_2(x), \\ D^{\alpha_2}u_2(x) &= u_3(x), \\ D^{\alpha_3}u_3(x) &= u_4(x), \\ \cdots \quad & \cdots \quad \cdots \\ D^{\alpha_m}u_m(x) &= f(x) + \int_0^x k_1(x,t)F_1(u_1(t),t)\,\mathrm{d}t + \int_0^1 k_2(x,t)F_2(u_1(t),t)\,\mathrm{d}t, \quad x \in \Omega, \end{aligned} \tag{12}$$

where $0 < \alpha_j \le 1$, $j = 1,2,\ldots,m$, and $\sum_{j=1}^m \alpha_j = \alpha$ for some $m \in \mathbb{N}$. In addition, we choose the $\{\alpha_j\}_{j=1,\ldots,m}$ in such a way that there exists some $j \ge 2$ such that $\sum_{l=1}^{j-1} \alpha_l = k$, where $k \in \{1,2,\ldots,n-1\}$ and the value of k can not be more than $j-1$ for any $j \ge 2$. Hence, the initial conditions $u_j(0)$ for $j = 1,2,\ldots,m$, are given by

$$u_j(0) = \begin{cases} \bar{u}_0 & \text{if } j = 1, \\ \bar{u}_k & \text{if } \displaystyle\sum_{l=1}^{j-1} \alpha_l = k,\ k \in \{1,2,\ldots,n-1\} \text{ will be obtained in ascending order}, \\ 0 & \text{else}. \end{cases} \tag{13}$$

Equation (11) is equivalent to the system of equations (12) with initial conditions (13) in the following sense:

(i) If $u \in C^n(\overline{\Omega})$ is a solution of (11), then $U = \{u_1,u_2,\ldots,u_m\}^{\mathrm{T}}$ with

$$u_j(x) = \begin{cases} u(x) & \text{if } j = 1, \\ D^{\alpha_1+\alpha_2+\cdots+\alpha_{j-1}}u(x) & \text{if } j \ge 2, \end{cases}$$

is a solution of (12) with initial conditions (13).

(ii) If $U = \{u_1,u_2,\ldots,u_m\}^{\mathrm{T}}$ is a solution of (12) with initial conditions (13), then $u(x) = u_1(x)$ is a solution of (11).

4. Approximation of the solution and its convergence analysis

Now we consider the approximation of the solution of (8) based on the homotopy perturbation method. Let us define the operator

$$\mathrm{A}(v) \equiv D^{\alpha} v(x) - \int_0^x k_1(x,t) F_1(v(t),t)\,\mathrm{d}t - \int_0^1 k_2(x,t) F_2(v(t),t)\,\mathrm{d}t = f(x), \tag{14}$$

where $x \in \Omega$, and $v = v(x)$ is a sufficiently smooth function in $\overline{\Omega}$. Let us decompose the operator A as $\mathrm{A} = \mathrm{A}_1 + \mathrm{A}_2$, where A_1 is a differential operator and A_2 is an integral operator of the original problem. By means of a homotopy, we define a function $H(v,p),\ p \in [0,1]$, such that

$$H(v,0) \equiv \mathrm{A}_1(v) - f(x), \quad H(v,1) \equiv \mathrm{A}(v) - f(x).$$

Now, we construct a homotopy by using the convex combination of $\mathrm{A}_1(v)$ and $\mathrm{A}(v)$ as

$$H(v,p) \equiv (1-p)(\mathrm{A}_1(v) - f(x)) + p(\mathrm{A}(v) - f(x)) = 0. \tag{15}$$

Here p defines the perturbation parameter which satisfies $0 \le p \le 1$. This embedding parameter p can be increased monotonically from zero to one in such a way that the solution of the simpler problem $\mathrm{A}_1(u) - f(x) = 0$ is continuously deformed to the solution of the original problem $\mathrm{A}(u) - f(x) = 0$. Now considering p as a small parameter lying in $0 \le p \le 1$, the homotopy-based perturbation method constructs the solution of (15) as a series in p, which is defined as follows:

$$v = v_0 + p v_1 + p^2 v_2 + p^3 v_3 + \cdots \tag{16}$$

where as $p \to 1$, it can be considered as the solution of $\mathrm{A}(v) - f(x) = 0$, that is,

$$u(x) = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \cdots. \tag{17}$$

From (16) and (15), we obtain a relation by equating the coefficients of same powers of p, which gives the approximate solution. The above components can be obtained as follows:

$$\begin{aligned} v_0(x) &= h(x) = u_0 + J^{\alpha}(f(x)), \\ v_n(x) &= J^{\alpha}\left[\int_0^x k_1(x,t) F_1(v_{n-1}(t),t)\,\mathrm{d}t + \int_0^1 k_2(x,t) F_2(v_{n-1}(t),t)\,\mathrm{d}t\right], \end{aligned} \tag{18}$$

where J^{α} are defined in (6). In our problem, $F_1(u)$ or $F_2(u)$ can be considered as a non-linear term which is taken as $N(u)$. To obtain the numerical solution, we use He's polynomial [14] denoted by H_m which can be obtained from the non-linear term $N(u)$ and is defined by

$$H_m(u_0, u_1, \ldots, u_m) = \frac{1}{m!} \frac{\partial^m}{\partial p^m} N\left(\sum_{k=0}^{m} p^k u_k\right)_{p=0}, \quad m = 0, 1, 2, \ldots \tag{19}$$

These polynomials are used to make the solution procedure simple, easy and more straightforward. In addition, for a two variable nonlinear terms, i.e. $N(u) = u(x)u(y)$, He's polynomials denoted by B_m and is obtained by (19) as

$$B_m(u_0, u_1, \ldots, u_m) = \sum_{k=0}^{m} u_k(x) u_{m-k}(y).$$

4.1. Convergence analysis

Here, we discuss about the convergence of the above approach analysing the series in (17).

Theorem 4.1: *Let us assume that* (I) *and* (II) *hold true and the function* $f(x)$ *is continuous on* $\overline{\Omega}$. *In addition, assume that* $0 < \gamma < 1$ *as in* (10). *Then, the series in* (16) *is uniformly convergent on* $\overline{\Omega}$. *In particular, an approximate solution of* (8) *can be obtained from* (17).

Proof: Note that v_0 is continuous on $\overline{\Omega}$. Therefore, there exist a real number $M_0 > 0$, such that $|v_0| \leq M_0$ for all $x \in \overline{\Omega}$. Having in mind the stated assumptions and (18) we will show that the ith term of the series (16) satisfies the following relations:

$$|v_i(x)| \leq M_0 \gamma^i. \tag{20}$$

We prove it by mathematical induction. For $i = 1$,

$$\begin{aligned}
|v_1(x)| &= \left| J^\alpha \left[\int_0^x k_1(x,t) F_1(v_0(t),t)\,\mathrm{d}t + \int_0^1 k_2(x,t) F_2(v_0(t),t)\,\mathrm{d}t \right] \right| \\
&\leq J^\alpha \left[\int_0^x |k_1(x,t)||F_1(v_0(t),t)|\,\mathrm{d}t + \int_0^1 |k_2(x,t)||F_2(v_0(t),t)|\,\mathrm{d}t \right] \\
&\leq \frac{M_1 L_1}{\Gamma(\alpha+2)} |v_0(x)| x^{\alpha+1} + \frac{M_2 L_2}{\Gamma(\alpha+1)} |v_0(x)| x^{\alpha} \\
&\leq \gamma |v_0(x)| \leq M_0 \gamma,
\end{aligned}$$

where γ was defined in (10). Let us assume that for $i = k-1$, (20) is true, i.e. $|v_{k-1}(x)| \leq M_0 \gamma^{k-1}$. Now, we show that the result holds for $i = k$. We have

$$\begin{aligned}
|v_k(x)| &= \left| J^\alpha \left[\int_0^x k_1(x,t) F_1(v_{k-1}(t),t)\,\mathrm{d}t + \int_0^1 k_2(x,t) F_2(v_{k-1}(t),t)\,\mathrm{d}t \right] \right| \\
&\leq \gamma |v_{k-1}(x)| \leq M_0 \gamma^k.
\end{aligned}$$

Hence, by mathematical induction, we get the desired result.

Therefore, for all $x \in \overline{\Omega}$,

$$\sum_{i=0}^{\infty} |v_i(x)| \leq \sum_{i=0}^{\infty} M_0 \gamma^i. \tag{21}$$

For $0 < \gamma < 1$, $\sum_{i=0}^{\infty} M_0 \gamma^i$ is a convergent geometric series. Therefore, by Weierstrass M-test, we conclude that $\sum_{i=0}^{\infty} v_i(x)$ converges uniformly on $\overline{\Omega}$. Note that for all $p \in [0,1]$ and for all $x \in \overline{\Omega}$, it is

$$\sum_{i=0}^{\infty} p^i v_i(x) \leq \sum_{i=0}^{\infty} |v_i(x)|.$$

Hence, the series $\sum_{i=0}^{\infty} |v_i(x)|$ converges uniformly on $\overline{\Omega}$. Therefore, by Weierstrass M-test, we obtain the uniform convergence of (16) on $\overline{\Omega}$. Hence, from (17) we have $u(x) = \lim_{p \to 1} v(x,p)$, which after truncation provides an approximate solution of the problem in (8). ∎

4.2. Error analysis

Let $u(x) = \sum_{i=0}^{\infty} v_i(x)$ be the solution of (8). We approximate the exact solution by taking a finite number of terms, say N terms, of the series (17). Note that the upper bound of the absolute error, based on this partial sum $\sum_{i=0}^{N-1} v_i(x)$ will be given by $M_0\gamma^N/(1-\gamma)$ (see (21) of Theorem 4.1), where γ is noted in Theorem 3.1 and M_0 is defined in Theorem 4.1.

In practice, we will approximate the series solution by its partial sum. Hence, the error will be bounded by $M_0\gamma^N/(1-\gamma)$ for Nth partial sum.

The following remark suggests that we can estimate the lower bound of N for a given tolerance of error.

Remark 4.1: Let ϵ be a tolerable error based on the partial sum up to N terms. Therefore, we can provide a bound of N, for which the ϵ tolerable error is admissible. This is obtained by $N \geq \lfloor (\ln(\epsilon(1-\gamma)/M_0))/\ln(\gamma) \rfloor + 1$, where $\lfloor x \rfloor$ defines the floor function, which gives the greatest integer less than or equal to x, γ is mentioned in Theorem 3.1 and M_0 is defined in Theorem 4.1.

5. Numerical experiments

Here, we provide the key steps for HPM-based approximate solution by describing the corresponding algorithm. For an efficient solution, it will be beneficial to use this algorithm till the number of terms in the series (17) becomes N, as depicted in Remark 4.1. In addition, we provide few test examples of linear and non-linear Volterra–Fredholm integro-differential equations to demonstrate the procedure of obtaining an approximate solution and show its effectiveness by using the Mathematica system. For simplicity, in the test problems we will only consider kernels given by polynomial functions. However, it would be possible to compute the approximate solutions for any kernel satisfying the assumption (II), by approximating this kernel by polynomial functions.

5.1. Algorithm

Step 1. Fix ϵ as an user desired accuracy and, find $n = \lfloor (\ln(\epsilon(1-\gamma)/M_0))/\ln(\gamma) \rfloor + 1$ based on Remark 4.1.
Step 2. Obtain v_i from (17) by using (18) for $i = 0, 1, \ldots, n-1$.
Step 3. Consider $\Phi_0(x) = 0$. Now, compute the approximate solution by $\Phi_i(x) = \Phi_{i-1}(x) + v_{i-1}(x)$ for $i = 1, \ldots, n$. Then, $\Phi_n(x)$ is the approximate solution sought.

Example 5.1:

$$D^{3/4}u(x) = f(x) + \frac{1}{20}\int_0^x (3x+4t)u(t)\,\mathrm{d}t + \frac{1}{20}\int_0^1 (3x-t)u^2(t)\,\mathrm{d}t, \qquad u(0) = 0, \tag{22}$$

with $x \in (0, 1]$ and

$$f(x) = \frac{47}{450} + \frac{2x^{1/4}}{\Gamma(5/4)} + \frac{3\sqrt{\pi}x^{3/4}}{4\Gamma(7/4)} - \frac{229x}{260} - \frac{41x^{7/2}}{350} - \frac{17x^3}{60}.$$

For this choice of $f(x)$, the exact solution of (22) is $u(x) = x^{3/2} + 2x$. Let us define $\mathrm{A}_1 \equiv D^{3/4}$, a linear fractional differential operator and $H_0(u_0) = u_0^2$ as the He's polynomial which is obtained by (19). Now, substituting (16) into (15) and by equating the powers of p and the initial condition, we have the following outcome:

$$\mathrm{A}_1(v_0(x)) - f(x) = 0$$
$$\Rightarrow v_0(x) = 2x + \frac{94x^{3/4}}{675\Gamma(3/4)} - \frac{229x^{7/4}}{735\Gamma(3/4)} + x^{3/2} - \frac{17x^{15/4}}{10\Gamma(19/4)} - \frac{123\sqrt{\pi}x^{17/4}}{160\Gamma(21/4)}.$$

$$A_1(v_1(x)) = \frac{1}{20}\int_0^x (3x+4t)v_0(t)\,\mathrm{d}t - \frac{1}{20}\int_0^1 (3x-t)H_0(v_0(t))\,\mathrm{d}t$$
$$\Rightarrow v_1(x) = \frac{17x^{15/4}}{10\Gamma(19/4)} + \frac{123\sqrt{\pi}x^{17/4}}{160\Gamma(21/4)} - \frac{441x^{3/4}}{125\Gamma(21/4)} + \frac{1603x^{7/4}}{200\Gamma(21/4)} - \frac{579\sqrt{\pi}}{125\Gamma(6)}x^{7/2}$$
$$- \frac{147\sqrt{\pi}}{250\Gamma(6)}x^{9/2} - \frac{837\sqrt{\pi}}{1250\Gamma(7)}x^{13/2} - \frac{9\pi^{3/2}}{125\Gamma(7)}x^7.$$

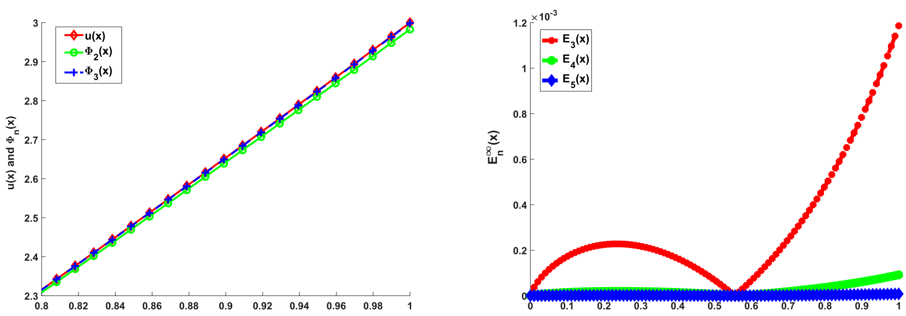
Similarly, we obtain the next three terms $v_2(x), v_3(x), v_4(x)$ of (17). The approximate solution $\Phi_n(x)$ is defined by $\Phi_n(x) = \sum_{m=0}^{n-1} v_m(x)$. Now, we produce the absolute error $E_n^\infty(x)$ to show the effectiveness of our present method. These point-wise errors are defined as follows [6–12,23]:

$$E_n^\infty(x) = |u(x) - \Phi_n(x)| = \left| u(x) - \sum_{m=0}^{n-1} v_m(x) \right|.$$

For Example 5.1, we took $n=4$ and $n=5$ respectively, to obtain the errors. These are given in Table 1. In addition, we have also provided a comparison of figures between exact solution and the approximate solutions and the absolute pointwise computational errors for different values of n at Figure 1 for Example 5.1. Note that Figure 1 is plotted in a magnified scale to show the exact and approximate solutions variation. This clearly shows that the approximate solution converges as the number of terms in the series solution increases.

Table 1. Absolute point-wise errors of Example 5.1.

x	$E_4^\infty(x)$	$E_5^\infty(x)$
0.1	1.46828E−05	1.27630E−06
0.2	1.91732E−05	1.66545E−06
0.3	1.87294E−05	1.62459E−06
0.4	1.44619E−05	1.25061E−06
0.5	6.83671E−06	5.85146E−07
0.6	4.06397E−06	3.62415E−07
0.7	1.85019E−05	1.61092E−06
0.8	3.71150E−05	3.20943E−06
0.9	6.10025E−05	5.24325E−06
1	9.18714E−05	7.84443E−06



(a) (b)

Figure 1. Solution and errors plot for Example 5.1 with different values of n. (a) Comparison of exact and approximate solutions. (b) Absolute point-wise errors.

Table 2. L^2 norm errors of Example 5.1.

x	$n=2$	$n=3$	$n=4$	$n=5$
1	5.99306E−02	4.17825E−03	3.26599E−04	2.80768E−05

Here, we produce the error with respect to L^2 norm over $\overline{\Omega}$ as

$$E_n^2 = \left(\int_0^1 (u(x) - \Phi_n(x))^2 \, \mathrm{d}x \right)^{1/2},$$

at Table 2 to show the effectiveness of our present approach.

Example 5.2: We consider the following Volterra–Fredholm fractional integro-differential equation:

$$\begin{aligned} D^{4/5}u(x) &= f(x) - \frac{2}{13}\int_0^x (x-4t)u(t)\,\mathrm{d}t - \frac{1}{12}\int_0^1 (2+3x-t)u(t)\,\mathrm{d}t, \\ u(0) &= 1, \end{aligned} \tag{23}$$

with $x \in (0, 1]$ and

$$f(x) = \frac{125x^{6/5}}{33\Gamma(1/5)}(11+6x) + \frac{251}{720} + \frac{19x}{24} - \frac{2}{13}\left(x^2 + \frac{10}{3}x^4 + \frac{11}{10}x^5\right).$$

Here, we construct the homotopy as follows:

$$\begin{aligned} H(v,p) \equiv D^{4/5}v(x) &- \frac{125x^{6/5}}{33\Gamma(1/5)}(11+6x) + p\left(\frac{2}{13}\left(x^2 + \frac{10}{3}x^4 + \frac{11}{10}x^5\right) - \frac{251}{720}\right. \\ &\left. - \frac{19x}{24} + \frac{2}{13}\int_0^x (x-4t)u(t)\,\mathrm{d}t + \frac{1}{12}\int_0^1 (2+3x-t)u(t)\,\mathrm{d}t\right) = 0. \end{aligned}$$

Substituting (16) into the above equation and equating the coefficients of identical powers of p, we obtain

$$D^{4/5}v_0(x) - \frac{125x^{6/5}}{33\Gamma(1/5)}(11+6x) = 0.$$

Hence, by using initial condition, we have $v_0(x) = 1 + 2x^3 + 5x^2$. Again we have

$$\begin{aligned} D^{4/5}v_1(x) &+ \frac{2}{13}\left(x^2 + \frac{10}{3}x^4 + \frac{11}{10}x^5\right) - \frac{251}{720} - \frac{19x}{24} + \frac{2}{13}\int_0^x (x-4t)v_0(t)\,\mathrm{d}t \\ &+ \frac{1}{12}\int_0^1 (2+3x-t)v_0(t)\,\mathrm{d}t = 0. \end{aligned}$$

By solving above equation, we get $v_1 = 0$. Continuing the same process, we get $v_2 = v_3 = v_4 = \cdots = 0$. Hence

$$u(x) = v_0(x) = 2x^3 + 5x^2 + 1,$$

which turns out to be the exact solution of Example 5.2.

Example 5.3: Consider

$$D^{1/2}u(x) = f(x) + \frac{1}{16}\int_0^x (3x-2t+1)u(t)\,\mathrm{d}t + \frac{1}{24}\int_0^1 (2x+3t)u^2(t)\,\mathrm{d}t, \quad 0 < x \le 1, \tag{24}$$
$$u(0) = 0,$$

where

$$f(x) = \frac{4x^{3/2}}{5\Gamma(5/2)} - \frac{1}{600}\left(\frac{389}{275} + \frac{559}{165}x\right) + \frac{\Gamma(7/2)x^2}{10} - \frac{x^3}{5040}(42 + 18\sqrt{x} + 63x + 26x^{3/2}).$$

The exact solution of Example 5.3 is unknown. The approximate solution of Example 5.3 is $\Phi_n(x)$, defined by $\Phi_n(x) = \sum_{m=0}^{n-1} v_m(x)$. Using (18) and the initial condition, we obtain the following results:

$$v_0(x) = J^{1/2}(f(x))$$
$$\Rightarrow v_0(x) = -\frac{x^{1/2}}{50\sqrt{\pi}}\left(\frac{389}{1638} + \frac{559}{1485}x\right) + \frac{2}{5}x^2 + \frac{1}{5}x^{5/2} - \frac{x^{7/2}}{60\Gamma(9/2)}(3+4x) - \frac{3\sqrt{\pi}x^4}{256\Gamma(6)}(10+13x),$$
$$v_1(x) = J^{1/2}\left[\frac{1}{16}\int_0^x (3x-2t+1)v_0(t)\,\mathrm{d}t + \frac{1}{24}\int_0^1 (2x+3t)H_0(v_0(t))\,\mathrm{d}t\right]$$
$$\Rightarrow v_1(x) = \frac{x^{1/2}}{25\sqrt{\pi}}\left(\frac{501}{500} - \frac{4}{21}x^3 + \frac{16}{65}x^4\right) + \frac{1494}{3125}\frac{x^{3/2}}{\Gamma(6)} - \frac{389}{130}\frac{x^2}{\Gamma(9)} + \frac{51}{2500}\frac{x^3}{\Gamma(6)} + \left(\frac{45}{64}\sqrt{\pi} - \frac{559}{960}\right)\frac{x^4}{\Gamma(7)} + \frac{x^5}{160}\left(\frac{13}{64}\sqrt{\pi} - \frac{1}{215}\right) - \frac{x^{11/2}}{22\Gamma(8)} - \frac{27x^6}{640\Gamma(7)} - \frac{29x^{13/2}}{286\Gamma(8)} - \frac{51x^7}{40\Gamma(9)} - \frac{3x^{15/2}}{55\Gamma(8)},$$

where $J^{1/2}$ is Riemann–Liouville fractional integral of order 1/2 defined by (6) and $H_0(v_0) = v_0^2$ is the He's polynomial, which is obtained by (19). In a similar way, we can obtain the next terms of the series (17) by (18). Now, we produce the absolute residue error $E_n^{\infty}(x)$ to show the effectiveness of our present method. It is defined as follows:

$$E_n^{\infty}(x) = |\mathrm{A}(\Phi_n(x)) - f(x)| = \left|\mathrm{A}\left(\sum_{m=0}^{n-1} v_m(x)\right) - f(x)\right|,$$

where A is defined as in (14) and the errors are shown in Table 3. For Example 5.3, we have plotted the exact solution and the approximate solutions at Figure 2 to show the convergence behaviour as n increases. It can be clearly seen from Figure 2 that the absolute pointwise errors are converging to zero as the number of terms n in the approximate solution increases. Note that the scale of the errors are of 10^{-3} order.

In addition, we also produce the residual error with respect to L^2 norm over $\overline{\Omega}$ as

$$E_n^2 = \left(\int_0^1 (\mathrm{A}(\Phi_n(x)) - f(x))^2\,\mathrm{d}x\right)^{1/2},$$

to show the effectiveness of the present method. Table 4 clearly shows that the approximate solution converges and the errors are negligible for larger values of n.

Table 3. Absolute residue errors of Example 5.3.

x	$E_3^{\infty}(x)$	$E_4^{\infty}(x)$
0.1	6.52738E−05	7.20244E−06
0.2	7.26372E−05	7.99446E−06
0.3	8.16522E−05	8.95061E−06
0.4	9.26773E−05	1.01055E−05
0.5	1.06161E−04	1.15023E−05
0.6	1.22657E−04	1.31936E−05
0.7	1.42849E−04	1.52439E−05
0.8	1.67604E−04	1.77324E−05
0.9	1.98031E−04	2.07572E−05
1	2.35579E−04	2.44403E−05

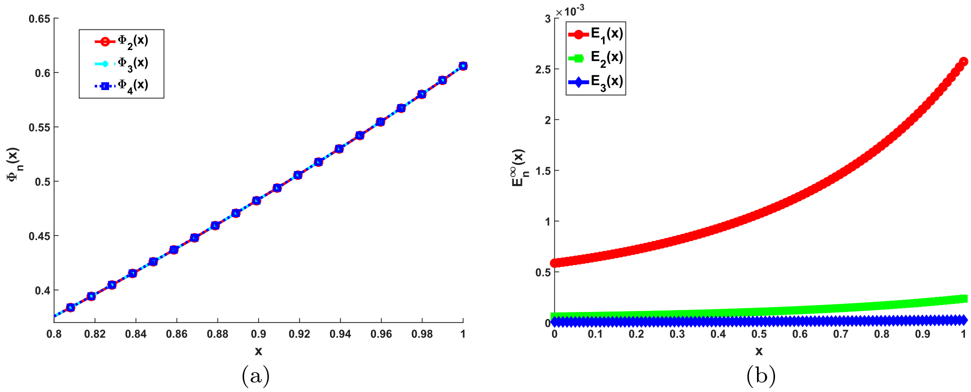


Figure 2. Computed solution and error plots for Example 5.3 for different values of n. (a) Comparison of approximate solutions. (b) Absolute point-wise errors.

Table 4. L^2 norm-based residue errors of Example 5.3.

x	$n = 2$	$n = 3$	$n = 4$	$n = 5$
1	1.34966E−02	1.29633E−03	1.37827E−04	1.56569E−05

Example 5.4: Consider

$$D^{1/2}u(x) = 1 + 3x^2 + x^3 + \frac{1}{8}\int_0^x (2 + x + 2t)u(t)\,dt, \quad 0 < x \leq 1, \qquad u(0) = 0. \tag{25}$$

The exact solution of Example 5.4 is unknown. The approximate solution of Example 5.4 is $\Phi_n(x)$, defined by $\Phi_n(x) = \sum_{m=0}^{n-1} v_m(x)$. Using (18) and the initial condition, we obtain the following results:

$$v_0(x) = J^{1/2}(1 + 3x^2 + x^3)$$
$$\Rightarrow v_0(x) = \frac{2x^{1/2}}{\sqrt{\pi}} + \frac{16x^{5/2}}{5\sqrt{\pi}} + \frac{32x^{7/2}}{35\sqrt{\pi}},$$
$$v_1(x) = J^{1/2}\left[\frac{1}{8}\int_0^x (2 + x + 2t)v_0(t)\,dt\right]$$

Table 5. Comparison of absolute residue errors of Example 5.4 by HPM and FDTM.

x	$E^{\infty}_{4,\mathrm{HPM}}(x)$	$E^{\infty}_{5,\mathrm{HPM}}(x)$
0.1	7.58951E−12	3.31162E−15
0.2	6.62831E−10	8.74676E−13
0.3	1.00974E−08	2.60318E−11
0.4	7.46321E−08	3.13490E−10
0.5	3.69425E−07	2.28496E−09
x	$E^{\infty}_{12,\mathrm{FDTM}}(x)$	$E^{\infty}_{13,\mathrm{FDTM}}(x)$
0.1	8.47897E−09	7.94806E−10
0.2	5.85645E−07	9.43516E−08
0.3	7.15798E−06	1.57034E−06
0.4	4.29831E−05	1.16529E−05
0.5	1.74682E−04	5.54825E−05

Table 6. Comparison of absolute residue errors of Example 5.4 by HPM and FDTM.

x	$E^{\infty}_{4,\mathrm{HPM}}(x)$	$E^{\infty}_{5,\mathrm{HPM}}(x)$
0.6	1.41513E−06	1.20747E−08
0.7	4.53412E−06	5.09772E−08
0.8	1.27329E−05	1.82284E−07
0.9	3.23076E−05	5.73457E−07
1	7.56300E−05	1.62930E−06
x	$E^{\infty}_{12,\mathrm{FDTM}}(x)$	$E^{\infty}_{13,\mathrm{FDTM}}(x)$
0.6	5.54176E−04	1.99409E−04
0.7	1.48118E−03	5.90098E−04
0.8	3.49073E−03	1.51439E−03
0.9	7.47050E−03	3.48553E−03
1	1.48131E−02	7.36028E−03

$$\Rightarrow v_1(x) = \frac{x^2}{8} + \frac{11}{96}x^3 + \frac{x^4}{16} + \frac{27}{320}x^5 + \frac{29x^6}{1920},$$

where $J^{1/2}$ is Riemann–Liouville fractional integral of order 1/2 defined by (6). Similarly, we obtain the next three terms $v_2(x), v_3(x)$ and $v_4(x)$ by the relation

$$v_{m+1}(x) = J^{1/2}\left[\frac{1}{8}\int_0^x (2+x+2t)v_m(t)\,\mathrm{d}t\right] \quad \text{for } m = 1, 2, 3.$$

Now, we produce the absolute residue error $E^{\infty}_n(x)$ to show the effectiveness of our present method. It is defined as follows:

$$E^{\infty}_n(x) = |\mathrm{A}(\Phi_n(x)) - (1+3x^2+x^3)| = \left|\mathrm{A}\left(\sum_{m=0}^{n-1} v_m(x)\right) - (1+3x^2+x^3)\right|,$$

where A is defined as in (14) and the errors are shown in Tables 5 and 6. In addition, we also produce the absolute residual error of Example 5.4 by using fractional differential transform method (FDTM), more details of the method can be found in [2,21].

From Tables 5 and 6, we observe that the existed FDTM produces higher errors than HPM, if we take only four terms in the series. Hence, it can be concluded that the number of terms required by FDTM to produce the same errors like in HPM, should be more than four. Therefore, we conclude that our method gives more accurate result corresponding to a few number of terms.

Table 7. Comparison of absolute residue errors of Example 5.5 by HPM and FDTM.

x	$E^{\infty}_{5,\mathrm{HPM}}(x)$	$E^{\infty}_{6,\mathrm{HPM}}(x)$
0.1	4.36193E−06	3.42731E−07
0.2	2.53136E−06	2.38621E−07
0.3	6.28764E−07	1.59598E−07
0.4	1.56847E−06	8.95925E−08
0.5	4.44733E−06	3.94877E−10
x	$E^{\infty}_{11,\mathrm{FDTM}}(x)$	$E^{\infty}_{13,\mathrm{FDTM}}(x)$
0.1	8.71073E−05	3.22812E−05
0.2	5.06581E−05	1.85150E−05
0.3	2.26396E−05	5.92564E−06
0.4	3.56597E−05	1.51665E−06
0.5	1.82236E−04	3.20027E−05

Example 5.5: Consider

$$D^{1/2}u(x) = 2 + \frac{1}{3}x^2 + \frac{1}{6}\int_0^x (4x+t)u(t)\,\mathrm{d}t + \frac{1}{8}\int_0^1 (3x-1)u(t)\,\mathrm{d}t, \quad 0 < x \le 1, \qquad (26)$$
$$u(0) = 0.$$

The exact solution of Example 5.5 is unknown. The approximate solution of Example 5.5 is $\Phi_n(x)$, defined by $\Phi_n(x) = \sum_{m=0}^{n-1} v_m(x)$. Using (18) and the initial condition, we obtain the following results:

$$v_0(x) = J^{1/2}\left(2 + \frac{1}{3}x^2\right)$$
$$\Rightarrow v_0(x) = \frac{4x^{1/2}}{\sqrt{\pi}} + \frac{16x^{5/2}}{45\sqrt{\pi}},$$
$$v_1(x) = \left[\frac{1}{6}\int_0^x (4x+t)u(t)\,\mathrm{d}t + \frac{1}{8}\int_0^1 (3x-1)u(t)\,\mathrm{d}t\right]$$
$$\Rightarrow v_1(x) = -\frac{218x^{1/2}}{315\pi} + \frac{436x^{3/2}}{315\pi} + \frac{23x^3}{36} + \frac{43}{2160}x^5,$$

where $J^{1/2}$ is Riemann–Liouville fractional integral of order 1/2 defined by (6). Similarly, we obtain the next terms $v_2(x)$, $v_3(x)$, $v_4(x)$ and $v_5(x)$ by the relation

$$v_{m+1}(x) = J^{1/2}\left[\frac{1}{6}\int_0^x (4x+t)v_m(t)\,\mathrm{d}t + \frac{1}{8}\int_0^1 (3x-1)v_m(t)\,\mathrm{d}t\right] \quad \text{for } m = 1,2,3,4.$$

Now, we produce the absolute residue error $E^{\infty}_n(x)$ to show the effectiveness of our present method. It is defined as follows:

$$E^{\infty}_n(x) = \left|\mathrm{A}(\Phi_n(x)) - \left(2 + \frac{1}{3}x^2\right)\right| = \left|\mathrm{A}\left(\sum_{m=0}^{n-1} v_m(x)\right) - \left(2 + \frac{1}{3}x^2\right)\right|,$$

where A is defined as in (14) and the errors are shown in Tables 7 and 8. In addition, we also produce the absolute residual error of Example 5.5 by using fractional differential transform method (FDTM) and compare both error point wisely.

From Tables 7 and 8, we conclude that our method gives more accurate result corresponding to a few number of terms.

Table 8. Comparison of absolute residue errors of Example 5.5 by HPM and FDTM.

x	$E^{\infty}_{5,\mathrm{HPM}}(x)$	$E^{\infty}_{6,\mathrm{HPM}}(x)$
0.6	8.59472E−06	1.59331E−07
0.7	1.48095E−05	4.69966E−07
0.8	2.40013E−05	1.06997E−06
0.9	3.67491E−05	2.17705E−06
1	5.20183E−05	4.10160E−06
x	$E^{\infty}_{11,\mathrm{FDTM}}(x)$	$E^{\infty}_{13,\mathrm{FDTM}}(x)$
0.6	6.73690E−04	1.73159E−04
0.7	1.92839E−03	6.02523E−04
0.8	4.69933E−03	1.68446E−03
0.9	1.02458E−02	4.09778E−03
1	2.05615E−02	9.01710E−03

Example 5.6: Consider the following fractional integro-differential equation:

$$D^{3/2}u(x) = f(x) + \frac{1}{5}\int_0^x (3x - 2t + 1)u(t)\,\mathrm{d}t + \frac{1}{4}\int_0^1 (5x + 2t - 3)u(t)\,\mathrm{d}t, \quad 0 < x \le 1,$$
$$u^i(0) = 0 \quad \text{for } i = 0, 1, \tag{27}$$

where

$$f(x) = \frac{281}{560} + \frac{3\sqrt{\pi}}{2} + \frac{6\sqrt{x}}{\sqrt{\pi}} - \frac{13x}{8} - \frac{x^{5/2}}{5}\left(\frac{4}{5} + \frac{\sqrt{x}}{2} + \frac{44x}{35} + \frac{3x^{3/2}}{4}\right).$$

Therefore, the fractional system of equations corresponding to Equation (27) can be written as

$$D^1 u_1(x) = u_2(x), \quad 0 < x \le 1,$$
$$D^{1/2}u_2(x) = f(x) + \frac{1}{5}\int_0^x (3x - 2t + 1)u_1(t)\,\mathrm{d}t + \frac{1}{4}\int_0^1 (5x + 2t - 3)u_1(t)\,\mathrm{d}t, \tag{28}$$
$$u_1(0) = 0, \quad u_2(0) = 0.$$

The exact solution of Example 5.6 is unknown. Therefore, the approximate solution of Example 5.6 is obtained by the method described in previous section. The approximate solution $\Phi_{i,n}(x)$ corresponding to $u_i(x)$ is defined by $\Phi_{i,n}(x) = \sum_{m=0}^{n-1} v_{i,m}(x)$, for $i = 1,2$. Using (18) and the initial condition, we obtain the following results:

$$v_{2,0}(x) = J^{1/2}f(x)$$
$$\Rightarrow v_{2,0}(x) = 3\sqrt{x} + 3x + \sqrt{\frac{x}{\pi}}\left(\frac{281}{280} - \frac{13x}{6} - \frac{16x^2}{175} - \frac{64x^4}{525}\right) - \frac{\sqrt{\pi}}{160}x^3(8 + 11x),$$
$$v_{1,0}(x) = \int_0^x v_{2,0}(x)$$
$$\Rightarrow v_{1,0}(x) = 2x^{3/2} + \frac{x^{3/2}}{15\sqrt{\pi}}\left(\frac{281}{28} - 13x - \frac{32x^3}{105} - \frac{64x^4}{385}\right) + \frac{3}{2}x^2 - \frac{\sqrt{\pi}}{800}x^4(10 + 11x),$$

where $J^{1/2}$ is Riemann–Liouville fractional integral of order 1/2 defined by (6). For $i = 0, 1, 2, \ldots$, we obtain the next terms of the series by the following relations:

$$v_{2,i+1}(x) = J^{1/2}\left[\frac{1}{5}\int_0^x (3x - 2t + 1)v_{1,i}(t)\,\mathrm{d}t + \frac{1}{4}\int_0^1 (5x + 2t - 3)v_{1,i}(t)\,\mathrm{d}t\right],$$

Table 9. Absolute residual errors of $\Phi_{2,n}(x)$ of Example 5.6.

x	$E^{\infty}_{2,4}(x)$	$E^{\infty}_{2,5}(x)$
0.1	5.41809E−06	2.22558E−07
0.2	4.28063E−06	1.35351E−07
0.3	2.71330E−06	3.91375E−08
0.4	6.70087E−07	6.86347E−08
0.5	1.79090E−06	1.89128E−07
0.6	4.48111E−06	3.21546E−07
0.7	7.05283E−06	4.62672E−07
0.8	8.97039E−06	6.06736E−07
0.9	9.48332E−06	7.45816E−07
1	7.60301E−06	8.71119E−07

$$v_{1,i+1}(x) = \int_0^x v_{2,i+1}(t)\,\mathrm{d}t.$$

Now, we produce the absolute residual error $E^{\infty}_{i,n}(x)$, $i = 1, 2$, corresponding to the approximate solution $\Phi_{i,n}(x)$ to show the effectiveness of our present method. These errors are defined as follows:

$$E^{\infty}_{1,n}(x) = |D^1(\Phi_{1,n}(x)) - \Phi_{2,n}(x)| = \left|D^1\left(\sum_{m=0}^{n-1} v_{1,m}(x)\right) - \sum_{m=0}^{n-1} v_{2,m}(x)\right|$$

and

$$E^{\infty}_{2,n}(x) = |\mathrm{A}(\Phi_{2,n}(x)) - f(x)| = \left|\mathrm{A}\left(\sum_{m=0}^{n-1} v_{2,m}(x)\right) - f(x)\right|,$$

where $\mathrm{A}(\Phi_{2,n})(x)$ is defined by

$$\mathrm{A}(\Phi_{2,n})(x) \equiv D^{1/2}(\Phi_{2,n}(x)) - \frac{1}{5}\int_0^x (3x - 2t + 1)\Phi_{1,n}(t)\,\mathrm{d}t - \frac{1}{4}\int_0^1 (5x + 2t - 3)\Phi_{1,n}(t)\,\mathrm{d}t.$$

For any values of n, the residual errors of $\Phi_{1,n}(x)$ are zero and the residual errors of $\Phi_{2,n}(x)$ are shown in Table 9.

Table 9 clearly shows that the approximate solution converges as n increases and the errors become negligible.

Example 5.7: Consider the following equation:

$$D^{3/2}u(x) = x^2 + 3x^3 + \frac{1}{5}\int_0^x (xt + 1)u(t)\,\mathrm{d}t + \frac{1}{6}\int_0^1 (2 + x + t^2)u(t)\,\mathrm{d}t, \quad 0 < x \le 1,$$
$$u(0) = 0, \quad u'(0) = 2. \tag{29}$$

The fractional equation (29) can be written as the following equivalent form:

$$\begin{aligned} &D^{1/2}u_1(x) = u_2(x), \quad 0 < x \le 1,\\ &D^{1/2}u_2(x) = u_3(x),\\ &D^{1/2}u_3(x) = x^2 + 3x^3 + \frac{1}{5}\int_0^x (xt + 1)u_1(t)\,\mathrm{d}t + \frac{1}{6}\int_0^1 (2 + x + t^2)u_1(t)\,\mathrm{d}t,\\ &u_1(0) = 0, \quad u_2(0) = 0, \quad u_3(0) = 2. \end{aligned} \tag{30}$$

The exact solution of Example 5.7 is unknown. The approximate solution $\Phi_{i,n}(x)$ corresponding to $u_i(x)$ is defined by $\Phi_{i,n}(x) = \sum_{m=0}^{n-1} v_{i,m}(x)$, for $i = 1,2,3$. Using (18) and the initial condition, we obtain the following results:

$$v_{3,0}(x) = 2 + J^{1/2} f(x) \;\Rightarrow\; v_{3,0}(x) = 2 + \frac{16x^{5/2}}{15\sqrt{\pi}} + \frac{96x^{7/2}}{35\sqrt{\pi}},$$
$$v_{2,0}(x) = J^{1/2} v_{3,0}(x) \;\Rightarrow\; v_{2,0}(x) = \frac{4x^{1/2}}{\sqrt{\pi}} + \frac{x^3}{3} + \frac{3x^4}{4},$$
$$v_{1,0}(x) = J^{1/2} v_{2,0}(x) \;\Rightarrow\; v_{1,0}(x) = 2x + \frac{32x^{7/2}}{105\sqrt{\pi}} + \frac{64x^{9/2}}{105\sqrt{\pi}},$$

where $J^{1/2}$ is Riemann–Liouville fractional integral of order 1/2, which is defined by (6). For $i = 0, 1, 2, \ldots$, we obtain the consequent terms of the series by the following recurrence relations:

$$v_{3,i+1}(x) = J^{1/2}\left[\frac{1}{5}\int_0^x (xt+1)v_{1,i}(t)\,\mathrm{d}t + \frac{1}{6}\int_0^1 (2+x+t^2)v_{1,i}(t)\,\mathrm{d}t\right],$$
$$v_{2,i+1}(x) = J^{1/2} v_{3,i+1}(t)\,\mathrm{d}t,$$
$$v_{1,i+1}(x) = J^{1/2} v_{2,i+1}(t)\,\mathrm{d}t.$$

Now, we produce the absolute residual error $E^{\infty}_{i,n}(x)$ corresponding to the approximate solution $\Phi_{i,n}(x)$ to show the effectiveness of our present method. These errors are defined as follows:

$$E^{\infty}_{1,n}(x) = |D^{1/2}(\Phi_{1,n}(x)) - \Phi_{2,n}(x)| = \left|D^{1/2}\left(\sum_{m=0}^{n-1} v_{1,m}(x)\right) - \sum_{m=0}^{n-1} v_{2,m}(x)\right|,$$
$$E^{\infty}_{2,n}(x) = |D^{1/2}(\Phi_{2,n}(x)) - \Phi_{3,n}(x)| = \left|D^{1/2}\left(\sum_{m=0}^{n-1} v_{2,m}(x)\right) - \sum_{m=0}^{n-1} v_{3,m}(x)\right|$$

and

$$E^{\infty}_{3,n}(x) = |\mathrm{A}(\Phi_{3,n}(x)) - x^2 - 3x^3| = \left|\mathrm{A}\left(\sum_{m=0}^{n-1} v_{3,m}(x)\right) - x^2 - 3x^3\right|,$$

where $\mathrm{A}(\Phi_{3,n})(x)$ is defined as

$$\mathrm{A}(\Phi_{3,n})(x) \equiv D^{1/2}(\Phi_{3,n}(x)) - \frac{1}{5}\int_0^x (xt+1)\Phi_{1,n}(t)\,\mathrm{d}t - \frac{1}{6}\int_0^1 (2+x+t^2)\Phi_{1,n}(t)\,\mathrm{d}t.$$

For any value of n, the residual errors of $\Phi_{1,n}(x)$ and $\Phi_{2,n}(x)$ are zero and the residual errors of $\Phi_{3,n}(x)$ are shown in Table 10.

Table 10 clearly shows that the approximate solution converges as n increases and the errors become negligible.

Now, we consider a fractional-order nonlinear Volterra population growth model of a closed system where neither emigration nor immigration occurs for an individual [25,26], to show the effectiveness of our present approach.

Example 5.8: Consider

$$\begin{aligned} kD^{3/4}u(x) &= u(x) - u^2(x) - u(x)\int_0^x u(t)\,\mathrm{d}t, \quad 0 < x \le 1, \\ u(0) &= \frac{1}{10}, \end{aligned} \tag{31}$$

where $k = c/ab$ with $a > 0, b > 0$ and $c > 0$ denote the birth rate coefficient, crowding coefficient and toxicity coefficient respectively. Here, $u(x)$ denotes the population of an identical species at time x

Table 10. Absolute residual errors of $\Phi_{3,n}(x)$ of Example 5.7.

x	$E_{3,6}^{\infty}(x)$	$E_{3,7}^{\infty}(x)$
0.1	3.69345E−05	5.53340E−06
0.2	3.85381E−05	5.77365E−06
0.3	4.04256E−05	6.05642E−06
0.4	4.27189E−05	6.39998E−06
0.5	4.55740E−05	6.82773E−06
0.6	4.91947E−05	7.37017E−06
0.7	5.38463E−05	8.06705E−06
0.8	5.98713E−05	8.96970E−06
0.9	6.77097E−05	1.01440E−05
1	7.79245E−05	1.16744E−05

and $u(0)$ defines the initial population. The integral terms of the above equation, represents the total amount of toxins, initiated from initial time. Since the toxicity coefficient is present in the closed system, it leads to extinct the population for a long run.

The exact solution of Example 5.8 is unknown. The approximate solution of Example 5.8 is denoted by $\Phi_n(x) = \sum_{m=0}^{n-1} v_m(x)$. Using (18) and the initial condition, we obtain the following results, by taking $k = 1$,

$$v_0(x) = u(0) = \frac{1}{10},$$

$$v_1(x) = J^{3/4}\left[v_0(x) - H_0(v_0) - \int_0^x B_0(v_0)\,\mathrm{d}t\right]$$

$$\Rightarrow v_1(x) = \frac{3x^{3/4}}{25\Gamma(3/4)} - \frac{4x^{7/4}}{525\Gamma(3/4)},$$

where $J^{3/4}$ is Riemann–Liouville fractional integral of order 3/4 defined by (6) and $H_0(v_0) = v_0^2$ and $B_0(v_0) = v_0(x)v_0(t)$ are the He's polynomial, which is obtained by (19). In a similar way, we can obtain the next terms of the series (17) by (18). Now, we produce the absolute residue error $E_n^{\infty}(x)$ to show the effectiveness of our present method. It is defined as follows $E_n^{\infty}(x) = |\mathrm{A}(\Phi_n(x))| = |\mathrm{A}(\sum_{m=0}^{n-1} v_m(x))|$, where A is defined as

$$\mathrm{A}(\Phi_n)(x) \equiv D^{3/4}(\Phi_n(x)) - \Phi_n(x) + \Phi_n(x)\Phi_n(x) + \Phi_n(x)\int_0^x \Phi_n(t)\,\mathrm{d}t,$$

and the errors are shown in Table 11. For Example 5.8, these errors are converging to zero as the number of terms n in the approximate solution, increases.

Table 11. Absolute residue errors of Example 5.8.

x	$E_5^{\infty}(x)$	$E_6^{\infty}(x)$
0.1	5.87799E−06	6.44373E−07
0.2	5.09843E−05	8.37354E−06
0.3	1.82931E−04	3.65374E−05
0.4	4.54223E−04	1.01183E−04
0.5	9.18056E−04	2.16876E−04
0.6	1.62433E−03	3.92499E−04
0.7	2.61589E−03	6.27109E−04
0.8	3.92497E−03	9.06023E−04
0.9	5.57007E−03	1.19733E−03
1	7.55312E−03	1.44905E−03

6. Conclusions

The present research provides real-life examples which lead to fractional integro-differential equations. In addition, we also provide an a priori bound of the solution. We develop a semi-analytical approach by considering a homotopy perturbation technique, which can be used to approximate the solutions of the problems, considered here. Several numerical examples are included demonstrating the uniformly effective behaviour of the considered approach. Comparison with the existed fractional differential transform method shows that our method provides more accurate results with the comparatively less number of terms in the approximation.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

The author, Pratibhamoy Das, wants to thank the Science and Engineering Research Board (SERB), Govt. of India, for providing the research support for the present work under the Early Career Research Award with the grant ECR/2017/001973.

References

[1] M.A. AL-Jawary, M.M. Azeez and G.H. Radhi, *Analytical and numerical solutions for the nonlinear Burgers and advection–diffusion equations by using a semi-analytical iterative method*, Comput. Math. Appl. 76 (2018), pp. 155–171.
[2] A. Arikoglu and I. Ozkol, *Solution of fractional integro-differential equations by using fractional differential transform method*, Chaos Soliton Fract. 40 (2009), pp. 521–529.
[3] K. Assaleh and W.M. Ahmad, *Modeling of speech signals using fractional calculus*, 9th International Symposium on Signal Processing and Its Applications, 2007.
[4] R.L. Bagley and P.J. Torvik, *A theoretical basis for the application of fractional calculus to viscoelasticity*, J. Rheol. 27 (1983), pp. 201–210.
[5] D. Baleanu, K. Diethelm, E. Scalas and J.J. Trujillo, *Fractional Calculus: Models and Numerical Methods*, World Scientific, New Jersey, 2016.
[6] M. Chandru, P. Das and H. Ramos, *Numerical treatment of two-parameter singularly perturbed parabolic convection diffusion problems with non-smooth data*, Math. Methods Appl. Sci. 41 (2018), pp. 5359–5387.
[7] M. Chandru, T. Prabha, P. Das and V. Shanthi, *A numerical method for solving boundary and interior layers dominated parabolic problems with discontinuous convection coefficient and source terms*, Differ. Equ. Dyn. Syst. 27(1) (2019), pp. 91–112.
[8] P. Das, *Comparison of a priori and a posteriori meshes for singularly perturbed nonlinear parameterized problems*, J. Comput. Appl. Math. 290 (2015), pp. 16–25.
[9] P. Das, *A higher order difference method for singularly perturbed parabolic partial differential equations*, J. Differ. Equ. Appl. 24(3) (2018), pp. 452–477.
[10] P. Das, *An a posteriori based convergence analysis for a nonlinear singularly perturbed system of delay differential equations on an adaptive mesh*, Numer. Algorithms 81 (2019), pp. 465–487.
[11] P. Das and V. Mehrmann, *Numerical solution of singularly perturbed convection–diffusion–reaction problems with two small parameters*, BIT 56 (2016), pp. 51–76.
[12] P. Das and J. Vigo-Aguiar, *Parameter uniform optimal order numerical approximation of a class of singularly perturbed system of reaction diffusion problems involving a small perturbation parameter*, J. Comput. Appl. Math. 354 (2019), pp. 533–544.
[13] C. Dong, Z. Chen and W. Jiang, *A modified homotopy perturbation method for solving the non-linear mixed Volterra–Fredholm integral equation*, J. Comput. Appl. Math. 239 (2013), pp. 359–366.
[14] A. Ghorbani, *Beyond adomian polynomials: He polynomials*, Chaos Soliton Fract. 39 (2009), pp. 1486–1492.
[15] R. Gorenflo and S. Vessella, *Abel Integral Equations: Analysis and Applications*, Springer-Verlag, Berlin, 1991.
[16] A.A. Hamoud, K.P. Ghadle, M.Sh. Bani Issa and G. Giniswamy, *Existence and uniqueness theorems for fractional Volterra–Fredholm integro-differential equations*, Int. J. Appl. Math. 31 (2018), pp. 333–348.
[17] S. Kheybari, M.T. Darvishi and A.M. Wazwaz, *A semi-analytical approach to solve integro-differential equations*, J. Comput. Appl. Math. 317 (2017), pp. 17–30.
[18] S.A. Khuri and A. Sayfy, *Generalizing the variational iteration method for BVPs: Proper setting of the correction functional*, Appl. Math. Lett. 68 (2017), pp. 68–75.

[19] A.A. Kilbas, H.M. Srivastav and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier B.V., Amsterdam, 2006.

[20] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley-Interscience, New York, 1993.

[21] D. Nazari and S. Shahmorad, *Application of the fractional differential transform method to fractional-order integro-differential equations with nonlocal boundary conditions*, J. Comput. Appl. Math. 234 (2010), pp. 883–891.

[22] A. Prakash, M. Kumar and K.K. Sharma, *Numerical method for solving fractional coupled Burgers equations*, Appl. Math. Comput. 260 (2015), pp. 314–220.

[23] H. Ramos and J. Vigo-Aguiar, *A new algorithm appropriate for solving singular and singularly perturbed autonomous initial-value problems*, Int. J. Comput. Math. 85 (2008), pp. 603–611.

[24] P.K. Sahu and S. Saha Ray, *A novel Legendre wavelet Petrov Galerkin method for fractional Volterra integro-differential equations*, Comput. Math. Appl. 290 (2016), pp. 1–5.

[25] F.M. Scudo, *Vito Volterra and theoretical ecology*, Theor. Popul. Biol. 2 (1971), pp. 1–23.

[26] R.D. Small, *Population growth in a closed system*, SIAM Rev. 25 (1983), pp. 93–95.