Journal of Computational and Applied Mathematics 354 (2019) 348–359

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Solving initial and boundary value problems of fractional ordinary differential equations by using collocation and fractional powers

M.F. Simões Patrício [a], Higinio Ramos [b,c,*], Miguel Patrício [d,e]

[a] CMUC, Departamento de Matematica, University of Coimbra, 3001-454, Coimbra, Portugal
[b] Scientific Computing Group, Universidad de Salamanca, Spain
[c] Escuela Politécnica Superior, Campus Viriato, 49022, Zamora, Spain
[d] Laboratory of Biostatistics and Medical Informatics and IBILI, Faculty of Medicine, University of Coimbra, Portugal
[e] CMUC, University of Coimbra, Coimbra, Portugal

A R T I C L E   I N F O

Article history:
Received 17 April 2018
Received in revised form 21 July 2018

MSC:
34A08
26A33

Keywords:
Caputo fractional derivative
Fractional differential equations
Gamma function
Collocation method
Initial and boundary value problems

A B S T R A C T

This paper presents a novel approach for solving initial and boundary-values problems on ordinary fractional differential equations. The strategy considered is based on the use of a suitable truncated series expressed in terms of fractional powers of the independent variable, and a collocation approach. With this strategy accurate numerical approximations can be obtained. The error criterion is based on the concepts of residue and relative error. Two algorithms fitted to the nature (linear or nonlinear) of the considered fractional differential equation are proposed. Finally, some problems illustrating the efficiency of the proposed method are presented.

1. Introduction

Fractional calculus and in particular the concept of fractional derivatives are by no means new. Applying this theory new fractional-order models have been proposed. The latter are more adequate than the previously used integer-order models, because fractional-order derivatives enable the description of the memory and hereditary properties of different substances (see [1,2]). Fractional derivatives have been successfully applied to problems in system biology [3], physics [4], chemistry and biochemistry [5], hydrology [6], and so on. Different approximations and numerical methods have been proposed to solve fractional differential equations: spectral method [7], backward differentiation formulas [8], Pseudo-Spectral Method [9], Adomian Decomposition Method [10], spline collocation methods [11,12], fractional splines [13], and Chebyshev collocation methods [14]. Some strategies for the numerical calculation of fractional derivatives and integrals have been recently studied in [15] and [16].

This paper is structured as follows. In Section 2 we review the main definitions and analyze an adequate series that will provide the basis for the method to be developed in the next section. In Section 3 the ordinary fractional differential equation (FDE) is presented. Using the truncated series, either a linear system (SL) or a non-linear system (SnL) is obtained to provide the approximate solution according to the characteristics of the FDE. In Section 4 we address how to obtain the solution of

* Corresponding author at: Scientific Computing Group, Universidad de Salamanca, Spain.
E-mail addresses: mfernandapatricio@gmail.com (M.F. Simões Patrício), higra@usal.es (H. Ramos), miguelpatricio@gmail.com (M. Patrício).

https://doi.org/10.1016/j.cam.2018.07.034
0377-0427/

the system SL or SnL using appropriate numerical methods. An error criterion for the approximation based on the concepts of residue and relative error is established. Furthermore, two algorithms have been presented, adapted to the type of the system, SL or SnL. In Section 5 some numerical results are given to illustrate the validity and applicability of the presented algorithms. Finally, some concluding remarks are given.

2. Some preliminaries

In the study of fractional differential equations the definition of fractional derivative is fundamental. In the literature different concepts have been proposed (see. e.g. [1]). The two most commonly used definitions are the Riemann–Liouville operator and the Caputo operator. In this paper we will use the last one.

Definition 2.1 (*see [17]*)**.** The Caputo fractional derivative operator of order $\alpha > 0$, D^α, is defined in the following form:

$$D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}}\, dt \tag{1}$$

where $m-1 < \alpha < m$, $m \in \mathbb{N}$, $x > 0$, and Γ represents the Gamma function given by

$$\Gamma(k) = \int_0^\infty t^{k-1} e^{-t}\, dt\,, \qquad k \in \mathbb{R} \setminus \mathbb{Z}_0^-\,. \tag{2}$$

We note that, from Caputo's derivative we have that

$$D^\alpha x^\tau = \begin{cases} 0, \ \text{for } \tau \in \mathbb{N} \cup \{0\} \text{ and } \tau < \lceil \alpha \rceil \\ \dfrac{\Gamma(\tau+1)}{\Gamma(\tau+1-\alpha)}\, x^{\tau-\alpha}, \ \text{ for } \tau \in \mathbb{N} \cup \{0\} \\ \quad \text{and } \tau > \lceil \alpha \rceil \text{ or } \tau \notin \mathbb{N} \text{ and } \tau > \lfloor \alpha \rfloor \end{cases} \tag{3}$$

where the ceiling function $\lceil \alpha \rceil$ denotes the smallest integer greater than or equal to α, and the floor function $\lfloor \alpha \rfloor$ denotes the largest integer greater than or equal to α. In case $\alpha \in \mathbb{N}$ the Caputo derivative coincides with the usual derivative.

Now, in order to motivate the procedure that will be developed later on, let us consider that a function $y(x)$ can be represented in the form

$$y(x) = \sum_{k=0}^{\infty} \frac{x^{\nu k}}{\Gamma(\nu k + 1)} \tag{4}$$

where $\nu > 0$ is a rational number (this could be considered as a variation of the Taylor series).

A similar formulation will allow us to build the numerical solution of the FDE. In practice, we will take $\nu = 1/2$, and only the first $m+1$ terms of the series will be considered, that is, we will consider the approximation given by

$$y_m(x) = \sum_{k=0}^{m} \frac{x^{k/2}}{\Gamma(k/2+1)}\,. \tag{5}$$

The nature of the function in (4) is established in the following proposition.

Proposition 2.1. *Let be $\nu = 1/2$. The series given by $y(x) = \sum_{k=0}^{\infty} \frac{x^{\nu k}}{\Gamma(\nu k+1)}$ is convergent for $x \geq 0$, and has sum $S(x) = e^x \left(1 + Erf(\sqrt{x})\right)$, where Erf is the Gauss error function.*

Proof. Let us decompose the series into two other series, in the form

$$y(x) = \sum_{k=0}^{\infty} \frac{x^{\nu k}}{\Gamma(\nu k + 1)} = \sum_{k=0}^{\infty} a_i x^i + \sum_{k=0}^{\infty} b_i x^{(2i-1)/2} \tag{6}$$

where

$$a_i = \frac{1}{i!}\,, \qquad b_i = \frac{2^i}{\sqrt{\pi}\prod_{j=1}^{i}(2j-1)}\,. \tag{7}$$

We can conclude easily that the two series in the right of (6) converge for $x \geq 0$. Consequently the series $y(x)$ converges. Even more, representing by T_1 and T_2 the sum of each of the above series we have

$$T_1 = \lim_{k\to\infty} \sum_{i=0}^{k} a_i x^i = e^x\,, \tag{8}$$

Table 1
Bounds for $A(m)$ for different values of m and ν.

m	$\nu = 1/4$	$\nu = 1/2$	$\nu = 3/4$	$\nu = 5/4$
5	3.4E−0	3.3E−1	2.7E−2	7.7E−6
10	7.9E−1	5.8E−3	1.8E−5	2.4E−11
20	1.5E−2	1.2E−7	1.1E−13	1.2E−27
30	1.1E−4	2.6E−13	2.0E−23	1.3E−46

$$T_2 = \lim_{k\to\infty} \sum_{i=0}^{k} b_i x^{(2i-1)/2} = e^x \mathrm{Erf}(\sqrt{x}), \tag{9}$$

and the conclusion follows. □

The following proposition allows us to obtain an estimate for the error when we replace $y(x)$ by $y_m(x)$ given in (4) and (5), respectively.

Proposition 2.2. *Let be $\nu = 1/2$ and $y(x)$, $y_m(x)$ given by*

$$y(x) = \sum_{k=0}^{\infty} \frac{x^{\nu k}}{\Gamma(\nu k + 1)}, \qquad y_m(x) = \sum_{k=0}^{m} \frac{x^{\nu k}}{\Gamma(\nu k + 1)}$$

with $m \in \mathbb{N}$. Then, for each $x \in [0, 1]$ we have the following bounds

$$|y(x) - y_m(x)| = A(m)$$

where $A(m)$ is a decreasing function of m that verifies

$$A(m) \le \begin{cases} 3.3\mathrm{E}{-1} & \text{for } m \ge 5 \\ 5.8\mathrm{E}{-3} & \text{for } m \ge 10 \\ 1.2\mathrm{E}{-7} & \text{for } m \ge 20 \\ 2.6\mathrm{E}{-13} & \text{for } m \ge 30\,. \end{cases} \tag{10}$$

Proof. From above, for each $x \in [0, 1]$ we can put

$$A(m) = |y(x) - y_m(x)| = \sum_{k=m+1}^{\infty} \frac{x^{\nu k}}{\Gamma(\nu k + 1)} \tag{11}$$

$$\le \sum_{k=m+1}^{\infty} \frac{1}{\Gamma(\nu k + 1)} \tag{12}$$

and being Γ a positive function, $\Gamma(k) > 0$ for $k > 0$, the conclusion follows. □

To obtain the bounds in the above proposition we have used the system Mathematica. For other values of ν different than 1/2 it is easy to obtain similar bounds. In Table 1 different bounds are presented for different values of ν and m. We note that as ν increases the bounds decrease, in other words this means that to obtain a similar accuracy on the approximation of $y(x)$ by $y_m(x)$, the higher is ν the fewer terms in the series are needed.

An approximation for the derivative of $y(x)$ is obtained in the next proposition.

Proposition 2.3. *Let be $\nu = 1/2$ and $y_m(x)$ given by (5). The Caputo fractional derivative of order $\alpha > 0$ may be written in the form*

$$D^\alpha y_m(x) = \sum_{i=1}^{m_1} a_i \frac{\Gamma(i+1)}{\Gamma(i+1-\alpha)} x^{i-\alpha} + \sum_{i=1}^{m_2} b_i \frac{\Gamma(i+1/2)}{\Gamma(i+1/2-\alpha)} x^{i-1/2-\alpha}$$

with the a_i and b_i the same as in (7), and $m = m_1 + m_2$ where $m_1 + 1$ is the number of integer powers of x, and m_2 is the number of rational powers of x.

Proof. As $y_m(x)$ may be written in the form

$$y_m(x) = \sum_{i=0}^{m_1} a_i x^i + \sum_{i=1}^{m_2} b_i x^{(2i-1)/2} \tag{13}$$

where m_1+1 is the number of integer powers of x and m_2 is the number of rational powers of x, with the a_i and b_i as in (7), using (3) the Caputo fractional derivative of order $\alpha>0$ may be written as

$$D^{\alpha}y_m(x)=\sum_{i=1}^{m_1}a_i\,\frac{\Gamma(i+1)}{\Gamma(i+1-\alpha)}\,x^{i-\alpha}+\sum_{i=1}^{m_2}b_i\,\frac{\Gamma(i+1/2)}{\Gamma(i+1/2-\alpha)}\,x^{i-1/2-\alpha}\,.$$

In particular, for $\alpha=1/2$ the above formula results in

$$D^{1/2}y_m(x)=\sum_{i=1}^{m_1}a_i\,\frac{\Gamma(i+1)}{\Gamma(i+1/2)}\,x^{i-1/2}+\sum_{i=1}^{m_2}b_i\,\frac{\Gamma(i+1/2)}{\Gamma(i)}\,x^{i-1}\,. \quad \square$$

3. The discretization of the fractional differential equation

In this paper we are looking for an approximate solution of the fractional ordinary differential equation (FDE) in the form

$$D^{\alpha}u(x)=F(x,u(x)) \tag{14}$$

with initial conditions and/or boundary conditions for $x\in[0,1]$, where $\alpha>0$ refers to the fractional order of the spatial derivative and F is a given function.

In Section 2 we had considered a series generated by $\left\{x^{\nu k}\right\}_{k=0,1,2,\ldots}$ with $\nu>0$ a rational number. Similarly, let us now consider a formal series with unknown coefficients. Assuming that ν is an irreducible fraction, that is, $\nu=p/q$ with p and q with no common divisors except 1, we have that $m_1=c$, and $m_2=m_1(q-1)+r$, where c and r are respectively the quotient and the remainder of the euclidean division, $m=cq+r$. Taking in mind (13) we can approximate the solution of the problem in (14) by

$$u_m(x)=\sum_{i=0}^{m_1}a_ix^i+\sum_{i=1}^{m_2}b_ix^{\nu\sigma_i}\,, \tag{15}$$

where a_i and b_i are unknown coefficients and $\{\sigma_i\}_{i=1,\ldots,m_2}$ is the subset of the first m natural numbers $\mathbb{N}_m=\{1,\ldots,m\}$, which are not divisible by m.

The fractional derivative of order $\alpha>0$ is

$$D^{\alpha}u_m(x)=\sum_{i=1}^{m_1}a_i\,\frac{\Gamma(i+1)}{\Gamma(i+1-\alpha)}\,x^{i-\alpha}+\sum_{i=1}^{m_2}b_i\,\frac{\Gamma(\nu\sigma_i+1)}{\Gamma(\nu\sigma_i+1-\alpha)}\,x^{\nu\sigma_i-\alpha}\,. \tag{16}$$

Consequently, by using (15) and (16) in the FDE in (14) it results in the equation

$$\sum_{i=1}^{m_1}a_i\,\frac{\Gamma(i+1)}{\Gamma(i+1-\alpha)}\,x^{i-\alpha}+\sum_{i=1}^{m_2}b_i\,\frac{\Gamma(\nu\sigma_i+1)}{\Gamma(\nu\sigma_i+1-\alpha)}\,x^{\nu\sigma_i-\alpha}=F\left(x,\sum_{i=0}^{m_1}a_ix^i+\sum_{i=1}^{m_2}b_ix^{\nu\sigma_i}\right)\,. \tag{17}$$

In particular, for the simple value $\nu=1/2$ we have that the differential equation in (14) may be discretized by the following equation

$$\sum_{i=1}^{m_1}a_i\frac{\Gamma(i+1)}{\Gamma(i+1-\alpha)}\,x^{i-\alpha}+\sum_{i=1}^{m_2}b_i\,\frac{\Gamma(i+1/2)}{\Gamma(i+1/2-\alpha)}\,x^{i-1/2-\alpha}=F\left(x,\sum_{i=0}^{m_1}a_ix^i+\sum_{i=1}^{m_2}b_ix^{(2i-1)/2}\right)\,. \tag{18}$$

We have to take into account the initial and boundary conditions of the problem. So, if the number of these conditions is m_c let be $m_s = m + 1 - m_c$. Using collocation techniques at m_s points $\{x_p\}_{p=1,\ldots,m_s}$ in [0, 1] together with the initial and boundary conditions we obtain a linear system (SL) or a non linear system (SnL) depending on whether F is a linear or a nonlinear function. This system allows us to calculate the $m + 1$ unknowns a_i and b_i. In the first case, (SL), we apply a direct method to obtain the solution and in the second some iterative method. In both cases, we will establish in the next section their algorithms with some details in order to facilitate the calculations.

4. Implementation: algorithms

According to the previous section the approximate solution of the fractional ordinary differential equation in (14) will be obtained using the equation in (17). Taking in (17) m_s intermediate values $x = x_p$, $p = 1, \ldots, m_s$ and considering the initial and boundary conditions we are led to a system with the $m + 1$ unknowns a_i and b_i, with $m + 1 = m_s + m_c$, where m_s corresponds to the number of collocation equations and m_c to the number of initial and boundary conditions. To obtain the solution we apply a direct method or any iterative method (as the fixed point method or Newton-type ones) depending on whether F is a linear or a nonlinear function respectively. From Proposition 2.2 we note that the higher m is in (5) the smaller is the upper bound of the truncation error. Of course, in the latter case the computational effort will be greater. In this paper we control the error of approximation measuring the residue $\mathcal{R}(u_m)$ and the relative error $r(u_m)$ of the numerical approximation $u_m(x)$. These are defined, respectively, by

$$\mathcal{R}(u_m) = \max_{x\in[0,1]} |D^\alpha u_m(x) - F(x, u_m(x))|$$

and

$$r(u_m) = \max_{x\in[0,1]} \left| \frac{u_m(x) - u_{m-1}(x)}{u_m(x)} \right| .$$

The solution $u_m(x)$ will be rejected if $\mathcal{R}(u_m)$ or $r(u_m)$ are greater than a predefined tolerance. In this case the calculations will be done again after incrementing m.

Remark. Note that m can be incremented in different ways. In this paper if $m_1 < m_2$ we begin by incrementing m_1, and in other case we increment m_2. If necessary we continue alternately by increasing m_1 and m_2 until we reach the predefined tolerance.

But the values of m_1 and m_2 can be taken independently, as we have done in Example 3, where we have taken $m_1 = 3$ and $m_2 = 0$.

From the above we can establish the following algorithms, to solve the problem in (14). The following algorithms are linked to the linear and nonlinear cases respectively.

Algorithm 1: FOR LINEAR PROBLEMS

Data: α, *tol*
Result: approximation u_m to the FDE

1 Let $m = 3$;
2 Solve (17) to obtain a_i, b_i
3 From (13) obtain $u_m(x)$
4 **if** $r(u_m) < tol$ *and* $\mathcal{R}(u_m) < tol$ **then**
5 | go to 9;
6 **else**
7 | $m = m + 1$ go to 2;
8 **end**
9 Accept $u(x) \approx u_m(x)$
10 End

Before establishing the structure of the second algorithm to obtain an approximation of the FDE let us rewrite in the case $\nu = 1/2$ the equation in (18) in the form

$$\begin{aligned} &\sum_{i=1}^{m_1} a_i^{(s)} \frac{\Gamma(i+1)}{\Gamma(i+1-\alpha)} x^{i-\alpha} \\ &+ \sum_{i=1}^{m_2} b_i^{(s)} \frac{\Gamma(i+1/2)}{\Gamma(i+1/2-\alpha)} x^{i-1/2-\alpha} \\ &= F\left(x, \sum_{i=0}^{m_1} a_i^{(s-1)} x^i + \sum_{i=1}^{m_2} b_i^{(s-1)} x^{(2i-1)/2}\right), \end{aligned} \tag{19}$$

in order to use an iterative fixed point algorithm for solving the resulting system of equations in the nonlinear case, where $s = 1, 2, \ldots$ denotes the current iteration. According to the above, the approximation u_m has also to be rewritten as

$$u_m^{(s)}(x) = \sum_{i=0}^{m_1} a_i^{(s)} x^i + \sum_{i=1}^{m_2} b_i^{(s)} x^{(2i-1)/2} \,. \tag{20}$$

Algorithm 2: FOR NON-LINEAR PROBLEMS

Data: α, tol, tol_1, $iter$, $a_i^{(0)}$, $b_i^{(0)}$
Result: approximation u_m to the FDE
1 Let $m = 3, s = 1$;
2 Solve (19) to obtain $a_i^{(s)}, b_i^{(s)}$.
3 **if** $s > iter$ **then**
4   Write: Failed after *iter* iterations. Go to 18;
5 **else**
6   From (20) obtain $u_m^{(s)}(x)$;
7   **if** $\|u_m^{(s)}(x) - u_m^{(s-1)}(x)\| > tol_1$ **then**
8         $s = s + 1$. Go to 2;
9   **else**
10   **if** $r(u_m) < tol$ *and* $\mathcal{R}(u_m) < tol$ **then**
11   Go to 17;
12   **else**
13             $m = m + 1, s = 1$. Go to 2;
14   **end**
15   **end**
16 **end**
17 Accept $u(x) \approx u_m(x)$
18 End

Remark. Note the following details regarding the algorithms above:

- The algorithms begin considering $m = 3$ but any other values can be used. Usually $m_1 = 2$ and $m_2 = 1$ is a good starting choice.
- In the error criterion the same tolerances are considered for the relative error and for the residue. However, two different tolerances can be used.
- If convergence in the fixed point algorithm in (19) fails then the initial approximation can be changed. The number of iterations may be also changed, or even the collocation points may be modified.
- If the approximation is taken as in (15) where $m_1 + 1$ values are used for the a_i and m_2 values are used for the b_i the method will be referred to as method (m_1, m_2).

5. Numerical results

In this section we present some numerical examples in order to illustrate the performance of the proposed method, considering $\nu = 1/2$, that is, the equation in (18). Let us now apply the preceding algorithms for solving some fractional ordinary differential equations.

5.1. Linear problems

5.1.1. Example 1

Let us consider the fractional differential problem appeared in [18]

$$D^\alpha u(x) = u(x) + 1 \,, \quad u(0) = 0 \,, \quad x \in [0, 1] \tag{21}$$

with $\alpha = 1/2$, whose exact solution is given by

$$y(x) = x^\alpha E_{\alpha,\alpha+1}(x^\alpha)$$

where $E_{\alpha,\beta}(z)$ is a generalized Mittag-Leffler function defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \,.$$

Table 2
Results for numerical approximations of problem (21).

Method	$\mathcal{R}(u_m)$	$r(u_m)$	*AbsErr*
(2,3)	1.0E−2	2.0E−2	1.2E−2
(3,3)	1.7E−3	7.3E−3	2.3E−3
(3,4)	2.0E−4	8.0E−4	7.3E−4
(12,12)	4.5E−14	2.7E−13	2.3E−15

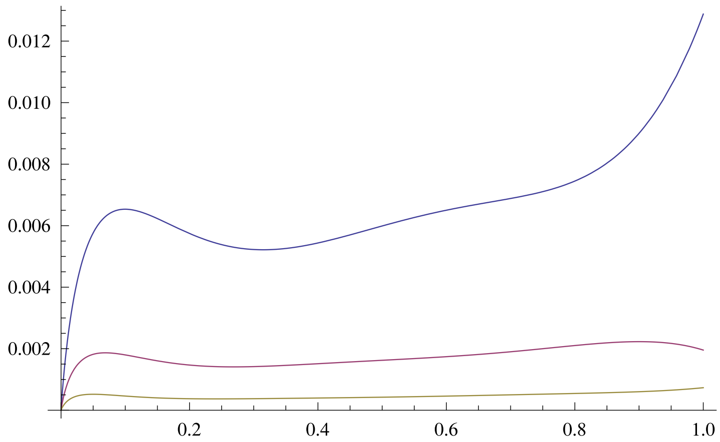

Fig. 1. Absolute error module of approximations for problem (21) using methods (2, 3), (3, 3) and (3, 4) (from top to bottom).

Note that the solution may be rewritten in the form

$$y(x) = x^\alpha E_{\alpha,\alpha+1}(x^\alpha)x^\alpha = x^\alpha \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(\alpha k + \alpha + 1)}$$
$$= \sum_{k=0}^{\infty} \frac{x^{\alpha k+\alpha}}{\Gamma(\alpha k + \alpha + 1)} = \sum_{k=1}^{\infty} \frac{x^{\alpha k}}{\Gamma(\alpha k + 1)}$$

which, in view of the result in Proposition 2.1, may be written as

$$y(x) = e^x \left(1 + \mathrm{Erf}(\sqrt{x})\right) - 1.$$

Using the proposed method, we have considered different values of m until reaching the tolerance $tol = 1.0\mathrm{E}{-3}$. The intermediate points used for collocation were the equally spaced points $x_p = \{j/m\}_{j=0,\ldots,m-1}$.

Taking initially $m = 5$ ($m_1 = 2$, $m_2 = 3$), that is, the method (2, 3), the error criterion failed, and the same occurs for the method (3, 3). The method (3, 4) allows us an approximation satisfying the pre-established requirements. The numerical results concerning to residue, relative error, and the maximum absolute error of the approximations

$$AbsErr = \max_{x\in[0,1]} |u(x) - u_m(x)|$$

are shown in Table 2. Note that the first two methods provide acceptable approximations although they failed the error criterion.

In Fig. 1 we plot the absolute error module of the approximations obtained with the three methods considered. We can get high accuracy just taking bigger values of m. For example, taking $m = 24$ ($m_1 = 12$, $m_2 = 12$) we obtain the values included in the last row of Table 2, highlighting the great performance of the method. Fig. 2 shows the absolute error module of this approximation.

5.1.2. Example 2

Consider the linear fractional initial-value problem appeared in [7]

$$\begin{aligned} &D^\alpha u(x) + u(x) = x^2 + \frac{2}{\Gamma(3-\alpha)} x^{2-\alpha}, \\ &u(0) = 0, \quad x \in [0, 1] \end{aligned} \tag{22}$$

whose exact solution is $u(x) = x^2$. Using the proposed method we have obtained approximate solutions for this problem, considering $tol = 1.0\mathrm{E}{-4}$ and collocation points equally spaced for the parameters $\alpha = 1/10$ and $\alpha = 1/2$.

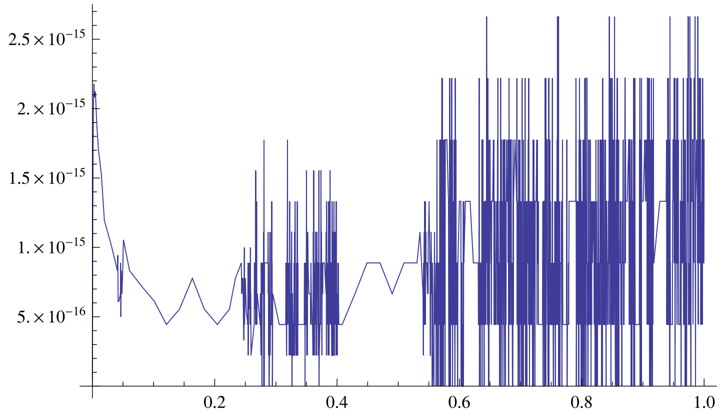

Fig. 2. Absolute error module of approximation for problem (21) using method (12, 12).

Table 3
Results for numerical approximations of problem (22).

α	Method	$\mathcal{R}(u_m)$	*AbsErr*
1/10	(1,2)	9.5E−2	4.1E−2
	(2,2)	0.0E+0	0.0E+0
1/2	(1,2)	1.5E−1	4.5E−2
	(2,2)	0.0E+0	0.0E+0

We considered initially the method with $m = 3\,(m_1 = 1, m_2 = 2)$, but this method did not achieve the desired numerical solution. However, the exact solution arises in the two cases with the method $(2, 2)$ or any other with $m > 3$, either for $\alpha = 1/10$ or $\alpha = 1/2$. In Table 3 the numerical results obtained with the proposed method are shown for the different cases considered. We note that the relative error has not been included as the denominator becomes zero for some values of $x \in [0, 1]$. The solutions provided in [7] for the two values of α involve a larger number of calculations than the present method, and still only approximate solutions were obtained.

5.2. Nonlinear problems

5.2.1. Example 3

The proposed method may also be applied for solving a multi-order FDE, for which one has to consider for each differential operator the corresponding approximation in (16). Let us illustrate the procedure considering the nonlinear fractional ordinary differential equation that appeared in [7]

$$D^{\alpha}u(x) + D^{\beta}u(x) + D^{\gamma}u(x) + u(x)^3 = f(x)\,, \quad x \in [0, 1] \tag{23}$$

where $\alpha = 11/5$, $\beta = 3/4$, $\gamma = 5/4$, and

$$f(x) = \frac{2x^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{2x^{3-\beta}}{\Gamma(4-\beta)} + \frac{2x^{3-\gamma}}{\Gamma(4-\gamma)} + \frac{x^9}{27}.$$

The following three types of boundary conditions will be considered

1. First type

$$u(0) = 0\,, \quad u'(3/5) = 9/25\,, \quad u(1) = 1/3$$

2. Second type

$$u''(0) = 0\,, \quad u'(3/5) = 9/25\,, \quad u(1) = 1/3$$

3. Third type

$$u(0) = 0\,, \quad u'(7/10) = 49/100\,, \quad u''(1) = 2$$

In all cases, the exact solution is $y(x) = \frac{x^3}{3}$.

Table 4
Numerical results for the approximations of problem (23), first case.

s	$AbsErrV$	$\mathcal{R}(u_m)$	$r(u_m)$	$AbsErr$
1	1.2E−1	1.0E−2	3.7E−1	4.4E−4
2	4.3E−5	3.0E−5	1.2E−3	8.6E−7
3	8.6E−7	5.1E−8	2.5E−6	1.6E−9

First type:

Let us consider the first type conditions. We begin with $m = 3$ taking ($m_1 = 3, m_2 = 0$). From the first initial condition we easily may conclude that $a_0 = 0$ whereas a_1, a_2, a_3 are unknowns. These are calculated considering the system in (19) after including the boundary conditions. As we have still two conditions we need only one equation to form a system of three equations. This last equation is obtained through (19) after evaluating in an intermediate point. The resulting system is

$$\begin{cases} a_1^{(s)} + a_2^{(s)} + a_3^{(s)} = \dfrac{1}{3} \\ a_1^{(s)} + 2x_0 a_2^{(s)} + 3x_0^2 a_3^{(s)} = \dfrac{9}{25} \\ a_1^{(s)} \dfrac{\Gamma(2)}{\Gamma(2-\beta)} x_1^{1-\beta} + a_2^{(s)} \left(\dfrac{\Gamma(3)}{\Gamma(3-\beta)} x_1^{2-\beta} + \dfrac{\Gamma(3)}{\Gamma(3-\gamma)} x_1^{2-\gamma} \right) \\ +a_3^{(s)} \left(\dfrac{\Gamma(4)}{\Gamma(4-\beta)} x_1^{3-\beta} + \dfrac{\Gamma(4)}{\Gamma(4-\gamma)} x_1^{3-\gamma} + \dfrac{\Gamma(4)}{\Gamma(4-\alpha)} x_1^{3-\alpha} \right) \\ = f(x_1) - \left(\displaystyle\sum_{i=1}^{3} a_i^{(s-1)} x_1^i \right)^3 \end{cases}$$

where $x_0 = 3/5$, and the intermediate point x_1 can be selected arbitrarily. We choose $x_1 = x_0$.

In order to apply the fixed point iteration for solving the above system we rewrite it in the form

$$A\,V^{(s)} = B + C^{(s-1)}, \quad s = 1, 2, \ldots$$

with

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2x_0 & 3x_0^2 \\ \dfrac{\Gamma(2)x_1^{1-\beta}}{\Gamma(2-\beta)} & \displaystyle\sum_{j=\beta,\gamma} \dfrac{\Gamma(3)x_1^{2-j}}{\Gamma(3-j)} & \displaystyle\sum_{j=\beta,\gamma,\alpha} \dfrac{\Gamma(4)x_1^{3-j}}{\Gamma(4-j)} \end{pmatrix}$$

$$V^{(s)} = \left(a_1^{(s)}, a_2^{(s)}, a_3^{(s)}\right)^T, \qquad B = \left(\frac{1}{3}, \frac{9}{25}, f(x_1)\right)^T$$

$$C^{(s-1)} = \left(0, 0, -\left(\sum_{i=1}^{3} a_i^{(s-1)} x_1^i\right)^3\right)^T$$

As for the starting value, for $s = 0$, we take $u^{(0)}(x) = \frac{1}{3}x$, that is, $a_1^{(0)} = 1/3$, $a_2^{(0)} = a_3^{(0)} = 0$. We also consider $iter = 3$ and $tol = tol_1 = 5.0\text{E}{-6}$.

For the first two iterations, $s = 1$ and $s = 2$ we do not obtain that $\|u_m^{(s)}(x) - u_m^{(s-1)}(x)\| > tol_1$, as can be seen in the numerical results presented in Table 4.

However, with $s = 3$ we have $|u_m^{(3)}(x) - u_m^{(2)}(x)| = 8.6\text{E}{-7} < tol_1$. The relative errors and the residues are also shown in Table 4. It can be seen that they satisfy the error criterion concerning the predefined tolerances, so $u_3(x)$ is the desired approximation. In Table 4 we have also included the maximum absolute value of the error in the obtained approximation on each iteration $s = 1, 2, 3$

$$AbsErrV = \max_{s=0,1,2\ldots} \|V^{(s)} - V^{(s-1)}\|.$$

In order to make an acceptable representation of the errors we show in Fig. 3 the logarithm of the absolute error for the successive approximations $u_m^{(s)}(x)$, $s = 0, 1, 2, 3$. As expected, the errors are smaller the greater s is.

Second type:

Now, consider again the problem in (23) with the second type conditions. The Algorithm 2 is applied as before with $m = 3$ and ($m1 = 3, m2 = 0$), $iter = 4$, $tol = tol_1 = 5.0\text{E}{-6}$. Using the boundary condition at $x = 0$ we have that $a_2^{(0)} = 0$.

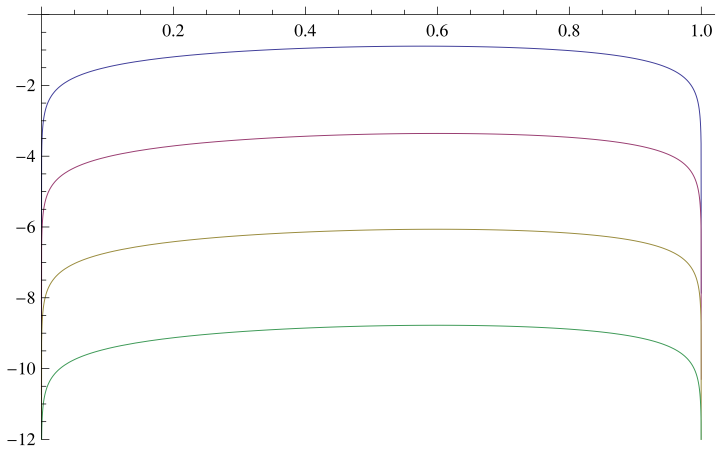

Fig. 3. Logarithmic absolute error module of successive approximations for problem (23), in the first case, for $u_3^{(s)}$, $s = 0, 1, 2, 3$ (from top to bottom).

Table 5
Numerical results for the approximations of problem (23), second case.

s	*AbsErrV*	$\mathcal{R}(u_m)$	$r(u_m)$	*AbsErr*
1	1.3E−1	1.1E−2	3.7E−1	4.8E−4
2	4.8E−4	9.8E−6	1.2E−3	4.8E−7
3	4.9E−7	9.7E−9	2.5E−6	4.8E−10

For the initial approximation in solving the system we put $a_1^{(0)} = 1/3$, $a_2^{(0)} = a_3^{(0)} = 0$. Taking as in the previous case the intermediate point $x_1 = 3/5$ and $x_0 = x_1$ we obtain the system

$$\begin{cases} a_0^{(s)} + a_1^{(s)} + a_3^{(s)} = \dfrac{1}{3} \\ a_1^{(s)} + 3x_0^2 a_3^{(s)} = \dfrac{9}{25} \\ a_1^{(s)} \dfrac{\Gamma(2)}{\Gamma(2-\beta)} x_1^{1-\beta} \\ +a_3^{(s)} \left(\dfrac{\Gamma(4)}{\Gamma(4-\beta)} x_1^{3-\beta} + \dfrac{\Gamma(4)}{\Gamma(4-\gamma)} x_1^{3-\gamma} + \dfrac{\Gamma(4)}{\Gamma(4-\alpha)} x_1^{3-\alpha} \right) \\ = f(x_1) - \left(\displaystyle\sum_{i=0}^{3} a_i^{(s-1)} x_1^i \right)^3 \end{cases}$$

and proceeding as before we obtain the results in Table 5.

Third type:

Finally, consider the problem in (23) with the third type conditions. We use the values $tol_1 = 1.5\text{E}{-4}$, $tol = 5.0\text{E}{-5}$, $x_0 = x_1 = 7/10$ maintaining the other necessary values equal to the previous cases. Now from the first initial condition we get that $a_0^{(s)} = 0$, and the system is

$$\begin{cases} 2a_2^{(s)} + 6a_3^{(s)} = 2 \\ a_1^{(s)} + 2x_0 a_2^{(s)} + 3x_0^2 a_3^{(s)} = \dfrac{49}{100} \\ a_1^{(s)} \dfrac{\Gamma(2)}{\Gamma(2-\beta)} x_1^{1-\beta} + a_2^{(s)} \left(\dfrac{\Gamma(3)}{\Gamma(3-\beta)} x_1^{2-\beta} + \dfrac{\Gamma(3)}{\Gamma(3-\gamma)} x_1^{2-\gamma} \right) \\ +a_3^{(s)} \left(\dfrac{\Gamma(4)}{\Gamma(4-\beta)} x_1^{3-\beta} + \dfrac{\Gamma(4)}{\Gamma(4-\gamma)} x_1^{3-\gamma} + \dfrac{\Gamma(4)}{\Gamma(4-\alpha)} x_1^{3-\alpha} \right) \\ = f(x_1) - \left(\displaystyle\sum_{i=0}^{3} a_i^{(s-1)} x_1^i \right)^3 \end{cases}$$

Table 6
Numerical results for the approximations of problem (23), in the third case, taking $x_1 = 7/10$.

s	*AbsErrV*	$\mathcal{R}(u_m)$	$r(u_m)$	*AbsErr*
1	1.0E−1	1.6E−2	3.9E−1	3.8E−3
2	3.7E−3	5.9E−4	1.1E−2	1.4E−4
3	1.3E−4	2.0E−5	4.1E−4	4.8E−6

Table 7
Numerical results for the approximations of problem (23) in the third case, taking $x_1 = 1/2$.

s	*AbsErrV*	$\mathcal{R}(u_m)$	$r(u_m)$	*AbsErr*
1	1.2E−1	8.6E−3	3.9E−1	2.0E−3
2	2.0E−3	4.3E−5	6.1E−3	1.0E−5
3	1.0E−5	1.9E−7	3.0E−5	4.7E−8

Table 8
Absolute errors for problem (23) with the method $(3, 0)$ and that by Bhrawy [7].

Type	Method $(3, 0)$	Method in [7]
First	1.6E−9	8.0E−8
Second	4.8E−10	1.2E−7
Third	4.7E−8	6.3E−5

Proceeding similarly as before we obtain the results that are shown in Table 6. Changing only the collocation point, taking $x_1 = 1/2$, the results obtained are shown in Table 7. We note that with $x_1 = 7/10$ and 3 iterations the algorithm failed to obtain the required accuracy (see Table 6). To obtain the required solution we can increase m or *iter*. Here we choose to change the collocation point, taking $x_1 = 1/2$ resulting that the integration is possible under the same requirements without changing the number of iterations (see Table 7).

This nonlinear problem has been solved by Bhrawy in [4]. In the present paper with a lower computational effort we got smaller absolute errors than those found in [4] as can be seen in Table 8. Here, the second column displays the absolute errors found with method $(3, 0)$ and $iter = 3$ while the third column reports the results obtained by Bhrawy: 20 points were used to solve the problem with first or second type boundary conditions and 24 points when the third type was considered.

6. Conclusion

In this paper we present a method to provide an approximate solution for initial/boundary value problems of linear and nonlinear fractional ordinary differential equations.

The method is obtained using a truncated series being the fractional derivatives described in the Caputo sense. An important feature of this method is that an analytical solution can be obtained, as has been shown in the first example. We can get numerical solutions with high precision increasing the value of the parameter m. A suitable choice of the collocation points can reduce the computational effort, as has been shown in the last example. A proper selection of the intermediate points is a problem to be studied.

Acknowledgments

The authors are grateful to the referees for several constructive comments and remarks.

References

[1] I. Podlubny, Fractional Differential Equations, in: Mathematics in Science and Engineering, vol. 198, Academic Press Inc., San Diego, CA, 1999.
[2] R. Almeida, N.R.O. Bastos, M.T.T. Monteiro, Modeling some real phenomena by fractional differential equations, Math. Methods Appl. Sci. 39 (2016) 4846–4855.
[3] S.B. Yuste, K. Lindenberg, Subdiffusion-limited $A + A$ reactions, Phys. Rev. Lett. 87 (11) (2001) 1–4.
[4] E. Barkai, R. Metzler, J. Klafter, From continuous time random walks to the fractional Fokker–Planck equation, Phys. Rev. E 61 (1) (2000) 132–138.
[5] S.B. Yuste, L. Acedo, K. Lindenberg, Reaction front in an $A + B \to C$ reaction-subdiffusion process, Phys. Rev. E 69 (3) (2004) 1–10.
[6] F. Liu, V. Anh, I. Turner, Numerical solution of the space fractional Fokker–Planck equation, J. Comput. Appl. Math. 166 (1) (2004) 209–219.
[7] A.H. Bhrawy, M.M. Al-Shomran, A shifted Legendre spectral method for fractional-order multi-point boundary value problems, Adv. Difference Equ. 4 (2012) 1–19.
[8] T.A. Biala, S.N. Jator, Block backward differentiation formulas for fractional differential equations, Int. J. Eng. Math. (2015) 650425, 1–14.
[9] Y. Yang, Multi-order fractional differential equation using legendre pseudo-spectral method, Appl. Math. 4 (2013) 113–118.
[10] V. Gejji, H. Jafari, Solving a multi-order fractional differential equation using adomian decomposition, Appl. Math. Comput. 189 (1) (2007) 541–548.

[11] A. Pedas, E. Tamme, Numerical solution of nonlinear fractional differential equations by spline collocation methods, J. Comput. Appl. Math. 255 (2014) 216–230.
[12] M.N. Sherif, I. Abouelfarag, T.S. Amer, Numerical Solution of fractional delay differential equations using spline functions, Int. J. Pure Appl. Math. 90 (2014) 73–83.
[13] F.K. Hamasalh, P.O. Muhammad, Numerical solution of fractional differential equations by using fractional spline model, J. Inform. Comput. Sci. 10 (2) (2015) 98–105.
[14] H. Azizi, G.B. Loghmani, A numerical method for space fractional diffusion equations using a semi-discrete scheme and Chebyshev collocation method, J. Math. Comput. Sci. 8 (2014) 226–235.
[15] S. Pooseh, R. Almeida, Delfim F.M. Torres, Numerical approximations of fractional derivatives with applications, Asian J. Control 15 (2013) 698–712.
[16] D.W. Brzeziński, Accuracy problems of numerical calculation of fractional order derivatives and integrals applying the Riemann–Liouville/caputo formulas, Appl. Math. Nonlinear Sci. 1 (2016) 23–44.
[17] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, first ed., Minsk, Belarus, 2006.
[18] S. Kazem, Exact solution of some linear fractional differential equations by laplace transform, Int. J. Nonlinear Sci. 16 (1) (2013) 3–11.