GEOMETRY OF ABSTRACT NULL HYPERSURFACES AND MATCHING OF SPACETIMES

Author:
Miguel Manzano

Supervisor:
Prof. Marc Mars

A Thesis submitted for the
Degree of Doctor of Philosophy
at Universidad de Salamanca

Departamento de Física Fundamental
Facultad de Ciencias
Instituto Universitario de Física Fundamental y Matemáticas

Universidad de Salamanca
Salamanca, España

October 2023

GEOMETRY OF ABSTRACT NULL HYPERSURFACES AND MATCHING OF SPACETIMES, © October 2023

Author:
Miguel Manzano

Supervisor:
Prof. Marc Mars

Institute:
Universidad de Salamanca
Instituto Universitario de Fśica Fundamental y Matemáticas (IUFFYM)

CONTENTS

Abstract vii

Declaration of supervisor ix

List of publications xi

Acknowledgements xiii

List of Figures xvii

Abbreviations xix

1 introduction 1
1.1 Context and motivation 1
1.2 Aim of this thesis 13
1.3 Contents 13

2 preliminaries 19
2.1 Notation and conventions 19
2.2 Formalism of hypersurface data 20
2.2.1 Metric hypersurface data 20
2.2.1.1 Gauge structure 25
2.2.2 Hypersurface data 28
2.2.2.1 Hypersurface connection $\overline{\nabla}$ 30
2.2.2.2 Curvature of the metric hypersurface connection $\overset{\circ}{\nabla}$ 31
2.2.3 Matter-hypersurface data and constraint equations 32
2.3 Geometry of submanifolds 34
2.4 Geometry of null hypersurfaces 39
2.4.1 Quotient space in a null hypersurface 43
2.4.2 Raychaudhuri equation on a null hypersurface 45
2.4.3 Totally geodesic null hypersurfaces 47
2.5 Non-expanding, weakly isolated and isolated horizons 49
2.6 Killing horizons 55
2.6.1 Multiple Killing horizons 58
2.6.2 Constancy of the surface gravity 63

2.7 Matching of spacetimes ... 63
2.7.1 Thin shells: formalism of hypersurface data ... 65
2.7.2 Cut-and-paste method ... 68

3 new results on the formalism of hypersurface data ... 73
3.1 General hypersurface data ... 74
3.1.1 The Lie derivative of the connection $\overset{\circ}{\nabla}$ along n ... 78
3.1.2 Curvature of the metric hypersurface connection $\overset{\circ}{\nabla}$ ... 82
3.2 Null hypersurface data ... 85
3.2.1 Gauge-fixing results ... 88
3.2.2 Curvature of the metric hypersurface connection $\overset{\circ}{\nabla}$: null case 92
3.2.3 Transverse submanifolds ... 94
3.2.4 Null metric hypersurface data admitting a cross-section ... 100
3.3 Gauge-invariant vector along the degenerate direction ... 103

4 the constraint tensor ... 107
4.1 Definition and first properties ... 108
4.2 Constraint tensor: null case ... 113
4.2.1 Constraint tensor along the null direction n ... 114
4.2.2 Constraint tensor on a transverse submanifold S ... 115
4.3 Gauge invariant quantities on a transverse submanifold S ... 117

5 embedded hypersurface data and ambient vector fields ... 123
5.1 Lie derivative of $\mathbf{Y}$ ... 124
5.2 Vector field along the degenerate direction ... 128
5.3 The tensor $\Sigma_\eta \stackrel{\text{def}}{=} \pounds_\eta \nabla$ ... 131
5.4 Abstract Killing horizons of order zero and one ... 138
5.4.1 Some aspects of abstract Killing horizons of order zero ... 144
5.4.2 Some aspects of abstract Killing horizons of order one ... 145
5.4.3 Connection with non-expanding and isolated horizons ... 146

6 generalized master equation ... 149
6.1 Covariant master equation on a general null hypersurface ... 150
6.2 Deformation tensor K^η proportional to the metric ... 155
6.3 Abstract Killing horizons of order zero and one ... 161
6.4 Generalized master equation on a transverse submanifold S ... 163
6.5 Vacuum degenerate Killing Horizons of order one ... 167

7 matching from a spacetime viewpoint ... 171
7.1 Prior considerations and setup ... 172

7.2 Matching of spacetimes with boundaries of any causality ... 177
7.3 Matching of spacetimes across a null hypersurface ... 181
7.3.1 The step function ... 185
7.3.1.1 Second fundamental forms and multiple matchings 188
7.3.2 Energy-momentum tensor of the shell ... 189
7.3.2.1 Gauge behaviour of the energy-momentum tensor . 199
7.3.3 Matching of two Minkowski regions across a null hyperplane 201
7.3.3.1 C^0-form of the metric on the resulting spacetime ... 210

8 matching across abstract killing horizons of order zero . 217
8.1 Case of $\eta^{\pm}$ degenerate ... 221
8.2 Case of $\eta^{\pm}$ non-degenerate ... 222
8.3 Case of η^{-} degenerate, η^{+} non-degenerate ... 225
8.4 Killing Horizons with bifurcation surfaces ... 227
8.4.1 Surface layer equations ... 233
8.5 Spherical, plane or hyperbolic symmetric spacetimes ... 235
8.5.1 Schwarzschild spacetime ... 238
8.5.2 Schwarzschild-de Sitter spacetime ... 240

9 matching from an abstract viewpoint ... 243
9.1 Abstract matching without topological assumptions ... 244
9.1.1 Null boundaries ... 248
9.1.1.1 Pressure of the shell ... 256
9.1.2 Multiple matchings ... 258
9.2 Null boundaries with product topology $S \times \mathbb{R}$ ... 262
9.3 Cut-and-paste construction: (anti-)de Sitter spacetime ... 267

10 conclusions ... 273
10.1 Formalism of hypersurface data ... 273
10.2 Matching of spacetimes ... 278

a general identities for curvature tensors ... 285

b a generalized gauss identity ... 287

c gauge behaviour of $\boldsymbol{\mathsf{S}}_{\parallel}$ ... 291

d coordinates near a null hypersurface ... 295
d.1 Coordinates near a null hypersurface ... 295
d.1.1 Generalized Rácz-Wald coordinates ... 298
d.1.2 Gaussian null coordinates ... 299

d.2 Coordinates near a Killing horizon 299

bibliography 301

ABSTRACT

This thesis consists of two distinct parts. Chapters 3, 4, 5, 6 are devoted to study the geometry of null hypersurfaces by means of the so-called formalism of hypersurface data. In Chapters 7, 8, 9 we address the problem of matching two completely general spacetimes across a null hypersurface.

The formalism of hypersurface data allows one to study hypersurfaces of any causal character without considering them as embedded on any ambient space. In Chapter 3, we present some new notions and results that are to be used elsewhere in the thesis. Chapter 4 is devoted to the so-called constraint tensor R. This tensor can be defined at the abstract level so that, when the hypersurface happens to be embedded on an ambient space, it codifies a certain combination of components of the ambient Riemann tensor. At null points, R coincides with the pull-back of the ambient Ricci tensor to the hypersurface. Chapters 5 and 6 concentrate on the implications at the abstract level due to the existence of a vector field in a neighbourhood of a hypersurface. Some core results in this context are the new abstract notions of Killing horizons of order zero and one and the so-called *generalized master equation*. From the latter we can recover, as particular cases, the well-known *near horizon equation* of isolated horizons (see e.g. [1]) as well as the so-called *master equation* of multiple Killing horizons (see e.g. [2]).

The problem of matching two general spacetimes with null boundaries is firstly addressed in Chapter 7. In this chapter we assume that the boundaries have product topology $S \times \mathbb{R}$ (S being a spacelike cross-section). We prove that all the information about the matching can be encoded in a scalar function H and a diffeomorphism Ψ between the sets of null generators of both sides. We find explicit expressions for the matter-energy content of any null thin shell. In Chapter 8 we apply the matching formalism to the case when the boundaries are abstract Killing horizons of order zero. Finally, in Chapter 9 we provide a fully abstract formulation of the matching problem for boundaries of any causality and *any topology*. The null case is analyzed in detail, proving that all information about the matching is codified by a diffeomorphism (whose components are precisely $\{H, \Psi\}$) and obtaining explicit expressions for the gravitational/matter-energy content of the shell. Our results are connected with those from the so-called cut-and-paste matching procedure (see e.g. [3], [4], [5], [6], [7]).

DECLARATION OF SUPERVISOR

Dr. D. Marc Mars Lloret, Catedrático de Física Teórica en el Departamento de Física Fundamental de la Universidad de Salamanca,

CERTIFICA:

Que el trabajo de investigación que se recoge en la siguiente memoria titulada *Geometry of abstract null hypersurfaces and matching of spacetimes*, presentada por D. Miguel Manzano Rodríguez para optar al Título de Doctor por la Universidad de Salamanca con la Mención de Doctorado Internacional, ha sido realizada en su totalidad bajo su dirección y autoriza su presentación.

En Salamanca, a 2 de octubre de 2023.

Firmado por MARS LLORET MARC - ***8571** el día 02/10/2023 con un certificado emitido por AC FNMT Usuarios

Dr. D. Marc Mars Lloret,
Catedrático de Física Teórica,
Departamento de Física Fundamental,
Universidad de Salamanca.

LIST OF PUBLICATIONS

1. M. Manzano and M. Mars, "Null shells: General matching across null boundaries and connection with cut-and-paste formalism", *Classical and Quantum Gravity*, 2021. DOI: 10.1088/1361-6382/abfd91.
2. M. Manzano and M. Mars, "General matching across Killing horizons of zero order", *Physical Review D*, vol. 106, no. 4, p. 044 019, 2022. DOI: 10.1103/PhysRevD.106.044019.
3. M. Manzano and M. Mars, "The Constraint Tensor: General Definition and Properties", 2023. ArXiv:2309.14813 [gr-qc].
4. M. Manzano and M. Mars, "Abstract Formulation of the Spacetime Matching Problem and Null Thin Shells", 2023. ArXiv:2309.14874 [gr-qc].
5. M. Manzano and M. Mars, "Embedded Hypersurface Data and Ambient Vector Fields: Abstract Killing Horizons of order zero/one", in preparation.
6. M. Manzano and M. Mars, "Master Equation for General Null Hypersurfaces", in preparation.

ACKNOWLEDGEMENTS

Me gustaría dedicar las primeras palabras de agradecimiento a mi supervisor Marc Mars Lloret. En primer lugar, porque su excelente labor docente durante mi último curso de grado hizo surgir en mí la curiosidad por el apasionante mundo de la Gravitación y de la Relatividad General y, en lo que se refiere a los últimos cuatro años, por su enorme dedicación, por su disponibilidad constante, por su curiosidad contagiosa, por haberme enseñado la importancia del entendimiento profundo y del trabajo minucioso. En definitiva, porque su inestimable ayuda ha hecho posible el proyecto de tesis que a continuación se expone.

Quiero dar las gracias también a todo el personal del Departamento, así como a los miembros del grupo de investigación GRACUS. En particular, quiero acordarme de Marina de la Torre, con quien ha sido un placer trabajar, y de Jose, Florencia y las 64 casillas que nos han separado en más de una ocasión.

Asimismo, al personal investigador de la Fakultät für Mathematik de la Universität Wien, que me ha acogido durante mis estancias, les estoy profundamente agradecido. En especial quiero dar las gracias a Roland Steinbauer y Argam Ohanyan.

Una tesis doctoral es un proceso constante de lucha contra uno mismo en el que se asientan las bases de una futura carrera académica. Aunque ésta culmina con un doctor, junto a él se encuentran las personas que lo sostienen, a quienes quiero agradecer la paciencia, el apoyo y el cariño en las siguientes líneas.

A mis padres, siempre con un libro abierto en las manos, a quienes debo todo lo que soy. A la Yaya, por encarnar el amor más puro que jamás he conocido. A Rosalía, siempre valiente, siempre luchando, siempre riendo, siempre madre. A María y a Daniela, luz y vida para todos. A Javier, artista genial. A Chanel y a Coco, porque el amor no es solo humano.

No me olvido de Espi, Jaime, Rodri y Sánchez, con quienes siempre me entiendo, aunque no nos entendamos siempre.

Tampoco de Maite, quien a partir de piezas sueltas de un puzle ha construido la imagen de un paisaje de paz.

Estaré también eternamente agradecido a todos los compañeros de la USAL: Alejandro, Duvier, Marina, Mario, Miguel, Néstor, Riccardo, David y David, y sobre todo, a Paz, Carlos y Gabri.

Finalmente, quiero dar las gracias a mis hermanos Cristian y Lorena, en quienes siempre he encontrado un hogar; a Sara: inigualable, brillante, única, siempre conmigo; y a Ceci, siempre en mi corazón y hacia quien siento una gratitud mayor de la que puede expresarse con palabras.

Esta tesis doctoral ha sido posible gracias a la financiación de los proyectos de investigación PGC2018-096038-B-I00 (Ministerio de Ciencia, Innovación y Universidades and FEDER) y SA096P20 (JCyL), así como al contrato predoctoral FPU17/03791 (Ministerio de Ciencia, Innovación y Universidades).

"*To have been loved so deeply, even though the person who loved us is gone, will give us some protection forever.*"

J. K. Rowling in *Harry Potter and the Sorcerer's Stone*

LIST OF FIGURES

Figure 1.1 Fragment of the first page of the paper "Die Feldgleichungen der Gravitation" by Albert Einstein, published in 1915 in the scientific journal *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften*. 2

Figure 1.2 Null hypersurface $\mathcal{N}$ with a rigging vector field ζ. 6

Figure 7.1 Matching of two spacetimes $(\mathcal{M}^{\pm}, g^{\pm})$ with boundaries $\mathcal{N}^{\pm}$ of any causality, where $\mathcal{N}$ is an abstract hypersurface embedded in $(\mathcal{M}^{\pm}, g^{\pm})$ with embeddings $\phi^{\pm}$, $\zeta^{\pm}$ are matching riggings and Φ is the corresponding matching map. 180

Figure 7.2 Left part: (a)-(d) depict the possible scenarios in which a null past rigging ζ^{-} along $\mathcal{N}^{-}$ points inwards (resp. outwards) and a null rigging ζ^{+} along $\mathcal{N}^{+}$ pointing outwards (resp. inwards) happens to be future. Right part: (a)-(d) are the corresponding matchings after having changed the time-orientation of $(\mathcal{M}^{+}, g^{+})$ 182

Figure 7.3 Matching of two regions of the spacetime of Minkowski: plot of the pressure p, step function H and energy density ρ given by (7.135)-(7.137) along the null generator $\{y^A = 0\}$ for the particular values $a = 1$, $b = 1$, $\beta(y^A) = 1$ and $H(y^A) = \frac{1}{2(n-1)}\delta_{IJ}y^I y^J$ 209

Figure 8.1 Possible matchings of two spacetimes $(\mathcal{M}^{\pm}, g^{\pm})$ across their respective boundaries $\mathcal{N}^{\pm}$ in the case when these are non-degenerate embedded AKH$_0$s without a fixed points set. Here boundaries directly in front of each other are to be identified and the dot represents the point at which the fixed points set would be located if the horizons extended further. 225

Figure 8.2 For $r_0 = 1$ and $\beta(\theta)$ given by (8.72), the up-left, up-right and bottom plots show $\beta(\theta)$, $j^{\theta}(\theta)$ and the energy density $\rho(\lambda, \theta)$ for $\lambda = 1$, $\lambda = 0$ and $\lambda = -1$ respectively. The figure corresponds to the case when $\mathcal{N}^{-}$ lies in the future of $\mathcal{M}^{-}$ so that the past rigging ζ^{-} points inwards, hence $\epsilon = -1$. . . 240

ABBREVIATIONS

AKH$_0$	**A**bstract **K**illing **H**orizon of order **zero**.
KH$_0$	**K**illing **H**orizon of order **zero**.
AKH$_1$	**A**bstract **K**illing **H**orizon of order **one**.
KH$_{1/2}$	**K**illing **H**orizon of order **½**.

1

INTRODUCTION

1.1 context and motivation

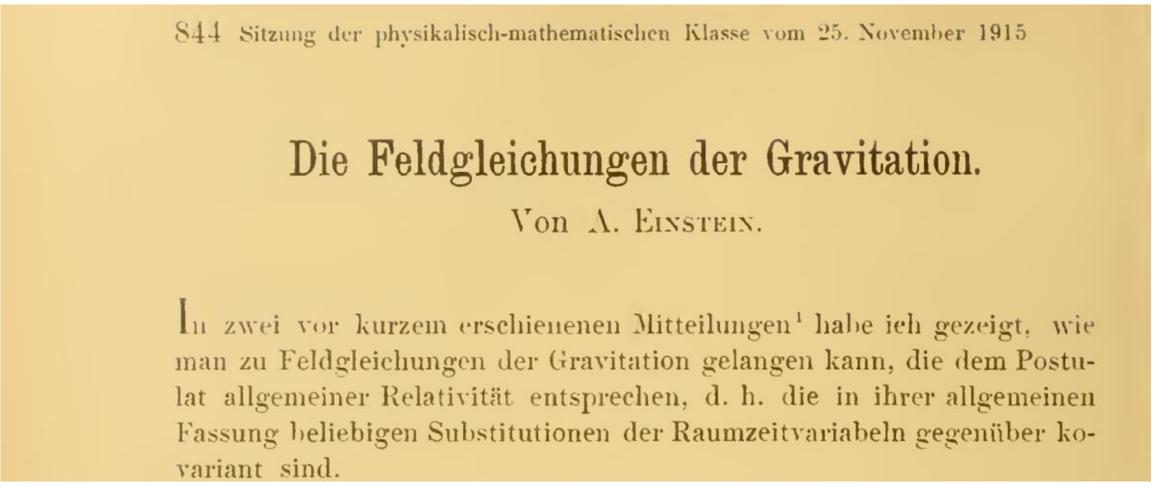
On November 25th 1915, the scientific journal *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften* (*Proceedings of the Royal Prussian Academy of Sciences*) published a paper [8] entitled "*Die Feldgleichungen der Gravitation*"[1] (see Figure 1.1). In this work, Albert Einstein provides the first geometric formulation of the (latter known as Einstein) equations of the gravitational field. The original version of these equations, derived in a 4-dimensional framework, reads as

$$\mathbf{Ric}_g = \chi\left(\boldsymbol{T} - \frac{1}{2}g\mathrm{T}\right), \tag{1.1}$$

where g is the metric, $\mathbf{Ric}_g$ is the Ricci tensor of g, $\boldsymbol{T}$ is the energy-momentum tensor and T its g-trace. With the publication of these equations, Newton's theory of gravitation faded into the background in favour of a new geometric theory of gravity, space and time: the theory of *General Relativity*.

The theory of General Relativity, presented fully in [9] for the first time, has proven to be the most accurate fundamental theory to describe gravitational effects at large scales. From its early predictions (the precession of the perihelion of Mercury [10], the deflection of light rays [11], the gravitational redshift [12], [13] and the emission of gravitational waves [14], [15]) to the more recent ones (e.g. the existence of black holes [16], [17], the expansion of the Universe [18] and the Big Bang [19], [17]), General Relativity seems to anticipate with extreme accuracy many of the natural phenomena supported afterwards by empirical observations. Already from its birth, General Relativity has proven to be unswerving and fully consistent with the experimental observations, no matter the increasing level of precision of the observational results. The robustness of General Relativity makes it the most accepted theory of gravity nowadays.

[1] *The field equations of gravitation*



844 Sitzung der physikalisch-mathematischen Klasse vom 25. November 1915

Die Feldgleichungen der Gravitation.

Von A. Einstein.

In zwei vor kurzem erschienenen Mitteilungen[1] habe ich gezeigt, wie man zu Feldgleichungen der Gravitation gelangen kann, die dem Postulat allgemeiner Relativität entsprechen, d. h. die in ihrer allgemeinen Fassung beliebigen Substitutionen der Raumzeitvariabeln gegenüber kovariant sind.

Figure 1.1: Fragment of the first page of the paper "Die Feldgleichungen der Gravitation" by Albert Einstein, published in 1915 in the scientific journal *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften*.

The first remarkable achievement of General Relativity is the already mentioned prediction of the precession of Mercury's orbit around the Sun. The problem of the discrepancy between the observational and the (Newtonian) theoretical values for the precession of Mercury was addressed by Einstein in 1905, shortly after publishing the theory of Special Relativity [20]. After a burdensome research, he was able to apply General Relativity to study the orbit of Mercury [10], obtaining the exact value of precession supported by the observations. This event translated into Einstein's revolutionary theory achieving support from the vast majority of the scientific community. The results by Arthur S. Eddington and Frank W. Dyson [11], which constituted an experimental proof of the effect of deflection of light rays because of gravity, also contributed significantly to the quick acceptance of the theory. In the words of the mathematician David Hilbert in 1920 [21],

> "*Die Aufstellung der allgemeinen Relativitätstheorie ist m.E. eine der größten Leistungen in der Geschichte der Wissenschaften. Den von Pythagoras begonnenen, von Newton ausgestalteten, Bau hat Einstein zum Abschluß gebracht.*"
> (*"The establishment of the General Theory of Relativity is, in my opinion, one of the greatest achievements in the scientific history. What was begun by Pythagoras and designed by Newton has been completed by Einstein."*)

Not all predictions of the theory were demonstrated observationally within Einstein's lifetime. For instance, the emission of gravitational waves (already considered by Einstein [14], [15], see also the later works [22], [23], [24]) took much longer to be endorsed by empirical experience. It was not until 2015 (coincidentally a hundred years after the publication of [8]) that the Laser Interferometer Gravitational-Wave Observatory (LIGO) detected gravitational waves from the merger of two black holes. The corresponding publication [25] was released in 2016.

Beyond Einstein's imagination, the most astonishing prediction of General Relativity is the existence of black holes. Although the first black hole solution was found by Karl Schwarzschild [26] soon after the publication of the field equations (1.1) in [8], the discussion on whether black holes were of actual physical nature or rather a pathological aspect of the theory kept the scientific community divided for a long time. It was in 1965, ten years after the demise of Einstein, that the work [16] by Roger Penrose provided the first significant improvement in this regard. In [16] (and later in [17] in collaboration with Stephen W. Hawking), Penrose proved that the absence of spherical symmetry still allowed for the formation of singularities[2], contrary to what was believed at the time. Almost contemporaneously, some experimental results suggested the existence of black holes (see e.g. [29]). Over the years many observational facts on the real existence of black holes have accumulated. This effort has culminated with the striking first direct observation of the black hole shadow by the Event Horizon Telescope [30], [31], [32], [33], [34], [35] (see also [36]). All these observational facts have turned black holes into widely accepted physical objects within the scientific community.

So far the theory of General Relativity has not been refuted by any of its tests, but this does not mean that it is a complete theory of gravity. There are various reasons supporting this last claim, among which we stress that General Relativity predicts the existence of singularities and, perhaps more important, that the three main fundamental theories in current physics, namely *Quantum Mechanics* [37] (a theoretical framework to understand dynamics at atomic and subatomic scales), the *Standard Model of Particle Physics* [38] (which describes all matter that can be observed directly as well as its non-gravitational interactions) and General Relativity, cannot be matched. Thus far these three theories continue providing (sometimes astonishingly) accurate predictions, despite they seem to be incompatible.

Of course, there has been various attempts to unify these three theories and construct one single theory of quantum gravity. The most important ones are *Loop Quantum Gravity* [39] (which provides a quantum description of gravity, space and time) and *String Theory* [40] (where point-like particles are substituted by one-dimensional objects called strings which interact with each other). Other al-

[2]The existence of naked singularities contradicts the Cosmic Censorship Conjecture [27], [28]. Thus, if General Relativity allows for the formation of singularities and (assuming that the Cosmic Censorship Conjecture is true) they must be "clothed" by an event horizon, then General Relativity predicts the existence of black holes.

ternative theories are the so-called *Modified Theories of Gravity* [41], which propose different extensions of General Relativity that generically lead to different field equations, and with which scientists have tried to endow many cosmological phenomena with an appropriate explanation. It seems however clear that, within its range of applicability (i.e. far enough from quantum scales), General Relativity is a suitable theory.

Depending on the approach and on the sort of problems that are addressed, General Relativity is divided in several branches. We find *Numerical General Relativity* [42], based on numerical methods and programming codes; *Relativistic Astrophysics* [43] and *Cosmology* [44], concentrated on providing theoretical and computational models as well as on experimental aspects of the theory; and *Mathematical General Relativity* [45], which addresses fundamental questions of gravitational physics within a rigorous mathematical framework. It is precisely the area of Mathematical General Relativity in which this thesis is framed.

Despite its long lifetime, Mathematical General Relativity is far from being completely unravelled. To mention some of its most important open problems, we stress the (strong and weak versions of the) Cosmic Censorship (see e.g. [27], [28], [46], [47]) and the final state conjecture [48] (and related problems such as uniqueness of black holes, or stability of Kerr-Newman black holes). This makes the field of Mathematical General Relativity a highly active research ground.

The scientific discipline in which Mathematical General Relativity relies on is Geometry, whose fundamental objects are manifolds and tensors. In particular, one of the milestones of Geometry is the study of hypersurfaces (i.e. codimension one submanifolds embedded in an ambient space). Depending on the causal character of the hypersurface, they are called null, timelike, spacelike or mixed.

For spacelike hypersurfaces, the first and second fundamental forms (which we shall denote by γ and $\mathbf{K}$) codify the intrinsic and the extrinsic geometry respectively. In this case, there is no need to introduce additional tensor fields. This also happens with timelike hypersurfaces or, more precisely, with any hypersurface embedded in a semi-Riemannian manifold of any signature, provided that the first fundamental form is everywhere non-degenerate on the hypersurface. When the ambient manifold is strictly Riemannian the property that γ and $\mathbf{K}$ encode all the geometric information holds for any embedded hypersurface. However, for any other ambient signature (in particular in the Lorentzian case), there exist many types of hypersurfaces which does not fulfil this condition.

Null hypersurfaces constitute the main object of study in this thesis. They are precisely defined to be such that the first fundamental form γ is everywhere degenerate. Therefore, the first fundamental form does not encode the whole intrinsic geometry of the hypersurface. As we shall see later in the thesis, in order to characterize the intrinsic part of the hypersurface one needs, in addition, a scalar function and a one-form field along the hypersurface.

The prime example of null hypersurface is the light-cone (either future or past) at any point within a spacetime, which defines a smooth null hypersurface after removing the origin of the cone as well as caustics that may form. There are however countless scenarios where null hypersurfaces are involved. For instance, they play a core role in causality, in the context of emission of gravitational waves, in the analysis of the geometry of the null infinity, in the characteristic problem, in the study of any sort of horizon such as Cauchy horizons, event horizons of black holes, non-expanding or (weakly) isolated horizons, (multiple) Killing horizons, cosmological horizons... Understanding the geometry of null hypersurfaces is therefore key for the comprehension of the physical and mathematical aspects of General Relativity. In fact null hypersurfaces are essential in General Relativity because they describe (at least locally) the trajectories of light rays that are emitted perpendicularly to a spacelike surface of codimension two.

Given a point p of a null hypersurface $\overline{N}$, we call it null whenever the first fundamental form γ at p is degenerate, otherwise p is referred to as non-null. It turns out that, besides hypersurfaces that are fully null, there also exists cases of physically relevant hypesurfaces containing both non-null and null points. As an example, we can mention the ergosphere of the Kerr spacetime, which is a spacelike hypersurface off the axis of symmetry and null where the ergosphere cuts the axis. In this thesis, we shall mainly focus on null hypersurfaces, but some of the results will hold for hypesurfaces of any causality.

In most of the literature, the geometry of hypersurfaces is studied by considering them as embedded in an ambient manifold. However, in many circumstances this approach is definitely not the most convenient. An illustrative example of this is the standard Cauchy problem [49], [50], [51], [52], or its null version, i.e. the characteristic initial value problem [53], [54] [55], [56], [57]. In these two situations, one needs to prescribe some data on a spacelike or null hypersurface, and then study the existence and uniqueness of the would-be spacetime where the data is to be embedded. Throughout this thesis we shall see many more examples of scenarios where the study of hypersurfaces in a detached way from the space where they may be embedded turns out to be a great advantage. Indeed, most of the results in this thesis will be based on a formalism that allows one to do precisely this: the so-called *hypersurface data formalism*.

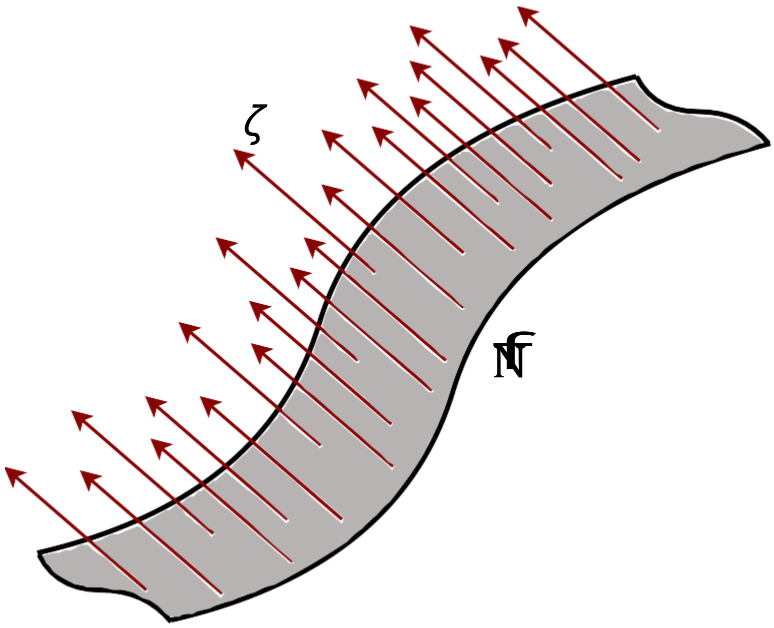


Figure 1.2: Null hypersurface $\mathcal{N}$ with a rigging vector field ζ.

The formalism of hypersurface data [58] [59] allows one to study hypersurfaces of any causal character from a completely abstract viewpoint, namely without them being embedded. We shall see some interesting applications of this formalism throughout the thesis. However, just to mention another recent achievement of the formalism, we stress the works [60], [61] on the characteristic problem in General Relativity.

The main difficulty that one must solve in order to codify the geometry of a hypersurface abstractly is that at null points the directions normal to the hypersurface are tangent as well. Thus, there does not exists a proper notion of tangent part and normal part of a vector field. The other important obstacle is that the first fundamental form γ at a null point is degenerate, so γ does not define a metric on the hypersurface. This means, in particular, that one cannot construct a canonical covariant derivative induced from the ambient geometry.

The seminal idea to deal with hypersurfaces containing null points was originally presented by Jan A. Schouten [62], and it consists of introducing an additional structure based on a vector field ζ which is everywhere transverse along the hypersurface. The vector field ζ is known as *rigging*, and it was used later e.g. to study the problem of matching two spacetimes across boundaries of any causality (see [63], [64], [65]). It is worth mentioning that the rigging vector field is highly non-unique. This lack of uniqueness translates, in the language of the formalism of hypersurface data, into an inherent gauge freedom.

The formalism of hypersurface data is based on the two notions of *metric hypersurface data* and *hypersurface data*. The first one is given by an abstract manifold $\mathcal{N}$, a symmetric 2-covariant tensor field γ, a one-form $\boldsymbol{\ell}$ and a scalar function $\ell^{(2)}$. The tuple $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ codifies the intrinsic geometry of the hypesurface so that when it is embedded on an ambient space, γ coincides with the first fundamental form, $\boldsymbol{\ell}$ is the pull-back of a rigging one-form (i.e. a one-form metrically related to a rigging) and $\ell^{(2)}$ is the norm of such rigging. In this way, a metric hypersurface data set codifies the full ambient metric at any point of the hypersurface. A hypersurface data is equipped, in addition, with another symmetric 2-covariant tensor $\mathbf{Y}$, which encodes the extrinsic geometry of the hypersurface. In the embedded picture, the tensor $\mathbf{Y}$ is related to transverse first derivatives of the metric at points on the hypersurface. These two data sets are endowed with an inherent gauge freedom, justified at the embedded level by the fact that rigging vector fields are highly non-unique, as we mentioned before.

It turns out that one can define two natural torsion-free covariant derivatives within the formalism of hypersurface data. The first one, denoted by $\overset{\circ}{\nabla}$, is constructed from the metric part of the data. Thus, it only depends on the intrinsic geometry. One can show that $\overset{\circ}{\nabla}$ coincides with the Levi-Civita covariant derivative whenever the hypersurface is everywhere non-null (and one makes a suitable choice of gauge, or of rigging at the embedded level). The second torsion-free connection is denoted by $\overline{\nabla}$ and is built from a hypersurface data set. It therefore depends not only on the intrinsic but also on the extrinsic geometric properties of the hypersurface. In particular, the connection $\overline{\nabla}$ coincides with the covariant derivative projected from the ambient space along the rigging whenever the hypersurface happens to be embedded (see [64] for details on the so-called *rigged connection*). We emphasize, however, that these two connections can be constructed at a purely abstract level, i.e. without requiring the existence of an ambient space.

The first part of this thesis will be devoted to develop the formalism of hypersurface data. Our main interest is to understand the implications at the abstract level due to the existence of a privileged vector field along the hypersurface. This scenario is addressed both at the abstract and at the embedded levels, where we let the vector field extend off the hypersurface in any manner. Our aim is to codify as much information as possible in terms of the deformation tensor associated to such special vector field.

The idea of studying the geometry of hypersurfaces equipped with an additional vector field arises when addressing the problem of matching two spacetimes across Killing horizons. As we shall see later, in these circumstances the matching freedom is huge, which makes the situation specially interesting. It is natural to ask whether the two spacetimes can be matched so that the Killing vectors are identified in the process of matching, as this would give rise to a resulting spacetime with a global symmetry. In this context (and in many others), the understanding of the geometry of hypersurfaces admitting a privileged vector becomes essential.

The framework in this thesis is however much more general. We shall mostly focus on abstract null hypersurfaces and on the case when the vector field is null and tangent. The reason why our work is of interest is that we keep as much generality as possible, e.g. by not making any global topological assumptions, or by refraining ourselves from restricting the set of zeroes of the vector field. As a consequence, our results can be particularized to a variety of situations, for instance to non-expanding, (weakly) isolated and Killing horizons.

The second part of this thesis, addressed in Chapters 7, 8 and 9, is devoted to the problem of matching two spacetimes across a hypersurface, which we briefly introduce next.

The question of under which conditions two general spacetimes can be matched across a hypersurface and give rise to a new spacetime is a fundamental problem in any metric theory of gravity. The properties of the resulting spacetime (in particular on the matching hypersurface) are certainly worth analyzing.

One prime example of this appears when studying the gravitational field generated by a self-gravitating object, e.g. a neutron star. In this context, the matter content in the interior region of the star is non-zero, hence the gravitational field must verify the Einstein equations (or the field equations of any other theory of gravity) with a non-vanishing source term. On the other hand, in the exterior region there is no matter and therefore the gravitational field must be a solution of the vacuum field equations. This argument does not change even if one considers magnetic fields, any flux of matter or the existence of an interstellar medium, as the matter content differs from the inner and the outer part in any case. The Einstein equations are not the same in the inside and in the outside so the solutions are necessarily distinct. However, the spacetime is not separated in two regions, which makes it essential to match the exterior and the interior solutions in one only solution.

In many physically interesting situations, the transition zone between the exterior and the interior regions is thin enough (compared with the dimensions of the problem) for one to address the problem by considering that a hypersurface (e.g. the surface of a star) separates the outer and the inner regions. Then, the core problem is to identify the conditions that must satisfy the two spacetime regions for the matching to be possible.

Another framework where a matching theory is required occurs in any physical situation where a substantial amount of matter-energy is located in a region of the spacetime which is thin enough with respect to the dimensions of the problem. Sometimes, the matter-content can be modelled as concentrated on a hypersurface (this is analogous to considering a surface distribution of charge in a theory of electromagnetism). This thin layer of matter-energy possesses its own gravity and therefore affects the spacetime geometry, and the key problem lies now upon finding the specific relationship between the matter-energy content of the layer and the properties of the spacetime containing it. Here one also finds two distinct spacetime regions (one at each side of the layer) that must be matched according to the corresponding theory.

Over the last hundred years, many authors have contributed to the problem of matching in General Relativity. The standard approach consists of considering two (a priori different) spacetimes and then constructing the matched spacetime by fulfilling two tasks. First, one must construct a differentiable manifold from the two initial spacetimes. For this purpose, one must cut each original spacetime so that one obtains two differentiable manifolds with boundary, and then provide an identification between the boundary points (which in particular requires that both boundaries are diffeomorphic). This process results in a new differentiable manifold without boundary, and allows one to construct a C^1 atlas. However, it is well-known that any differentiable boundary admits a C^∞ subatlas (see e.g. [66]), and hence the resulting spacetime can be treated (with full generality) as a smooth manifold.

The second task is to endow the resulting spacetime with a Lorentzian metric. Since the metric must exist everywhere on the manifold (in particular on the matching hypersurface), one needs to identify not only the points of the boundaries but also their tangent spaces. The directions tangent to the boundaries are automatically identified as a consequence of the mapping between the points. Therefore, it suffices to provide an identification between two transverse directions, one on each spacetime. The full construction requires, in addition, that one of these transverse directions points from the boundary inwards while the other points outwards. Of course, in general any two transverse directions cannot be identified in such a way that the metric of the resulting spacetime is well-defined and continuous. The two spacetimes to be matched must verify certain conditions which we briefly describe next.

Let us consider two $(n+1)$-dimensional spacetimes $(\mathrm{M}^{\pm}, g^{\pm})$ with respective boundaries ₦ $^{\pm}$. As already mentioned, both boundaries must be diffeomorphic for the matching to be possible. The standard way of imposing this is by requiring the existence of a diffeomorphism $\Phi : \mathfrak{N}^- \longrightarrow \mathfrak{N}^+$. One then defines the resulting manifold $\mathcal{M}$ as the union of $\mathcal{M}^+$ and $\mathcal{M}^-$ where any two points $p^\pm \in \mathfrak{N}^\pm$ are identified if and only if $\Phi(p^-) = p^+$. The manifold $\mathcal{M}$ must also admit an everywhere continuous metric g which coincides with the metrics $g^\pm$ on the spacetime regions that have been matched. The necessary and sufficient conditions for such metric g to exist where firstly studied by Christopher J. S. Clarke and Tevian Dray [67] for boundaries with any constant causal character. They obtained that the two first fundamental forms must coincide. Later on Marc Mars and José M. M. Senovilla [64] proved that the reasoning by Clarke and Dray extends to boundaries with arbitrary causal character. These conditions, while necessary in all cases, are not quite sufficient when there are null points. This was noticed by Mars, Senovilla and Raül Vera in [65], where it was shown that one must add a condition on the relative orientation of the riggings.

Combining the results from [67], [64], and [65], one concludes that the necessary and sufficient conditions for the matching of two completely general spacetimes are (*i*) that the first fundamental forms from both boundaries coincide, (*ii*) that there exists a pair of riggings $\zeta^\pm$ along the boundaries $\mathfrak{N}^\pm$ with the same norm and such that their metrically-related one forms $g^\pm(\zeta^\pm, \cdot)$ are the same and (*iii*) that $\zeta^\pm$ are such that one points inwards and the other outwards, as we pointed out before.

When the boundaries do not contain null points, the coincidence of the first fundamental forms of each boundary automatically guarantees the matching. This is because one can always select the riggings to be unit and normal to the boundaries and fix their orientations so that one points inwards and the other outwards. This works differently when there exist null points on the boundaries. Then, the existence of the riggings verifying (*i*)-(*iii*) does not follow from the equality of the first fundamental forms [65]. This makes it necessary to include the third requirement (*iii*). One can indeed fix one orientation but then the other is automatically fixed, and it can well happen that (*iii*) does not hold.

Once we have identified the conditions that allow for the existence of a spacetime $(\mathcal{M}, g)$ resulting from the matching and with a continuous metric g, the next step is to analyze the physical properties of the matched spacetime. In particular, it is interesting to study whether $\mathcal{M}$ contains matter or energy located on the matching hypersurface. This problem was addressed in the spacelike case by George Darmois [68], Stephen O'Brien and John L. Synge [69] and André Lichnerowicz [70]. These three works rely on distinct approaches, but the relation between them is well-known [71], [72]. The Darmois matching conditions are coordinate-independent and require the first and second fundamental forms of the boundaries to coincide. In these circumstances, there exists a subatlas of $\mathcal{M}$ in which the resulting metric g is of C^1 type [73], [71]. The Riemann tensor can be discontinuous at the matching hypersurface (this is because second transverse derivatives of the metric do not need to coincide at the boundaries) but it must be regular everywhere. Hence, in this case one concludes that there is no concentration of matter or energy on the matching hypersurface. On the other hand, if (still in the spacelike case) we relax the condition on the second fundamental forms and allow for a jump on them, then the Riemann tensor of g can only be defined everywhere on $\mathcal{M}$ by using distribution theory (among the many contributions to the field of theory of distributions, we mention [70], [74], [75], [64], [76], [77], [78], [79], [80], [81]). Applying distributional calculus to compute the Riemann tensor of g yields a Dirac delta distribution with support on the matching hypersurface. Physically, this corresponds to a concentration of energy-momentum (and also of gravitational field) on a thin layer, i.e. along the matching hypersurface. The singular part of the Einstein tensor of g is given by the discontinuity of the second fundamental form $\mathbf{K}$ through the tensor $\mathcal{T}$, namely

$$\boldsymbol{\mathcal{T}} \stackrel{\mathrm{def}}{=} -([\mathbf{K}] - [\mathrm{tr}_\gamma \mathbf{K}]\gamma^\sharp), \tag{1.2}$$

where we have defined $[\mathcal{Q}] \stackrel{\mathrm{def}}{=} \mathcal{Q}^+ - \mathcal{Q}^-$ for any tensor pair $\{\mathcal{Q}^+, \mathcal{Q}^-\}$ along $\mathcal{N}$ and $\gamma^\sharp$ is the contravariant metric of $\mathcal{N}$ (i.e. the inverse of the non-degenerate first fundamental form γ). The tensor $\mathcal{T}$ must satisfy certain equations that were originally obtained by Kornel Lanczos [82], [83] and later by Werner Israel [73] using a more covariant method. These equations are currently known as Israel equations, thin shell equations or surface distribution equations, and they arise from the distributional equation $\nabla_\mu \underline{G}^{\mu\nu} = 0$ for the distribution $\underline{G}^{\mu\nu}$ associated to the Einstein tensor of $(\mathcal{M}, g)$. The Israel equations read

$$(\mathrm{K}^+_{ab} + \mathrm{K}^-_{ab})\mathcal{T}^{ab} = 2[(\mathbf{Ein}_g)(\nu,\nu)], \qquad \nabla^{(\gamma)}_b \mathcal{T}^{\ b}_{a} = [\mathrm{J}_a], \tag{1.3}$$

where $\mathbf{Ein}_{g^\pm}^{\pm}$ are the Einstein tensors on each side, $\nu^\pm$ are the unique normals satisfying $g^\pm(\zeta^\pm, \nu^\pm)|_{\mathcal{N}^\pm} = 1$ for a pair of riggings $\zeta^\pm$ satisfying the matching conditions, $\nabla^{(\gamma)}$ is the Levi-Civita connection of γ and $\mathbf{J}^\pm$ are the pull-back to the boundaries of the components $\mathbf{Ein}_{g^\pm}^{\pm}(\nu^\pm, \cdot)$.

A similar argument can be followed to derive the Israel equations for the timelike case. The null case, on the other hand, is intrinsically different. However, Claude Barrabés and Israel [63] derived the shell equations by means of a limiting procedure in which the null hypersurface is approximated by a series of spacelike hypersurfaces. Later on [64] the distribution formalism was exploited to study the matching across hypersurfaces with arbitrary causal character. One of the core results in [64] is the explicit expression for the singular part of the Einstein tensor of the matched spacetime (M, g). For both the Riemann and the Einstein tensors of (M, g), the singular part is given by a tensor field multiplied by a Dirac delta distribution with support on the matching hypersurface (which we denote by $\overline{N}$). The second (distributional) Bianchi identity as well as its contracted version $\nabla_\mu \underline{G}^{\mu\nu} = 0$ can be used to derive the Israel equations for thin shells of arbitrary causal character by means of the distributional formalism. This program has been carried our recently by Senovilla [84]. Prior to that, the Israel equations for the case of arbitrary causal character had been obtained in [58] as one of the first applications of the formalism of hypersurface data. Later on we will have more to comment on this result.

We conclude the first part of the introduction by discussing another procedure to construct spacetimes containing null thin shells, namely the so-called *cut-and-paste method*. This different approach was introduced by Penrose [3], [85], [86], [87] in the sixties.

The cut-and-paste method describes the shell by means of a metric with a Dirac delta distribution with support on the shell. In these coordinates, the metric is therefore very singular, and standard tensor distributional calculus is not sufficient to study its geometry. However, by a suitable change of coordinates the metric becomes continuous and the method can be reinterpreted as follows. Given a spacetime (M, g) containing an embedded null hypersurface $\overline{N}$, the cut-and-paste procedure uses lightlike coordinates adapted to $\overline{N}$. Then, $\overline{N}$ is removed by a cut, which leaves two separated manifolds $(M^{\pm}, g^{\pm})$ corresponding to both sides of $\overline{N}$. Finally, those regions are reattached by identifying their boundaries so that there exists a jump on the coordinates when crossing the matching null hypersurface. This jump is responsible for the appearance of the Dirac delta term in the metric, which is interpreted as a concentration of matter and energy located on the matching hypersurface. By means of this useful geometrical approach, Penrose was able to study certain classes of impulsive plane-fronted and spherically-fronted waves propagating in a Minkowski background. Later works of Jirˇí Podolskỳ et al. e.g. in [88], [89], [90], [91], [92], [4], [5], [6], [7] (and references therein) apply the cut-and-paste procedure to generate spacetimes whose metric again contains a Dirac delta function with support on the null hypersurface. The most general construction so far describes pp-waves with additional gyratonic terms [5]. The cut-and-paste method is, by construction, strongly linked to the use of appropriate coordinate systems adapted both to the spacetime and to the null hypersurface where the cut is performed.

1.2 aim of this thesis

The purpose of this thesis is two-fold. As already mentioned, we are firstly interested in the geometry of null hypersurfaces (see Chapters 3, 4, 5, 6). In this context the formalism of hypersurface data becomes a powerful mathematical framework. Our second aim (and actually the starting point of the thesis) is the study of the problem of matching two completely general spacetimes across a null hypersurface, which we address in Chapters 7, 8, 9.

Concerning the part of the thesis where we expand the formalism of hypersurface data, the motivations described above have lead us to study how to characterize curvature information at the abstract level (see Chapter 4). Also, they have allowed us to understand how the data is affected by the existence of a privileged vector field (Chapter 5). In particular, this has permitted that we construct abstract notions of Killing horizons of order zero and one which do not require of any ambient space and which generalize the concepts of non-expanding, (weakly) isolated and Killing horizons. Finally, we have been able to derive an equation, called *generalized master equation*, that governs the geometry of null hypersurfaces with an extra null tangent vector field (Chapter 6). The analysis of this equation reveals properties about the surface gravity of such vector and about homothetic Killing horizons and Killing horizons of order zero and one. Moreover, it allows us to recover, as particular cases, the well-known *near horizon equation* of isolated horizons as well as the so-called *master equation* of multiple Killing horizons.

The problem of matching two spacetimes across a null hypersurface constitutes the second part of the thesis. In a spacetime context and by requiring a simple topology of the boundaries, we have been able to encode the whole matching information in a function and a diffeomorphism between the set of null generators of both matching hypersurfaces. We have also derived explicit expressions for the matter-energy content of the shell. Finally, we have exploited the formalism of hypersurface data to address the problem of matching in a completely abstract context and without requiring topological restrictions upon the boundaries. This approach, as we will see, has many advantages that will be discussed later.

1.3 contents

This thesis is divided in three parts. In the first one, corresponding to Chapter 2, we discuss the mathematical definitions, tools and results from the literature that are required later throughout the thesis. We start by establishing our notational conventions in Section 2.1. In Section 2.2 we introduce the formalism of hypersurface data as it is presented in [58], [59], including the definitions of (metric) hypersurface data, the construction of the covariant derivatives $\overset{\circ}{\nabla}$, $\overline{\nabla}$ already mentioned and several useful results within the formalism. Then, in Sections 2.3 and 2.4, we revisit some key aspects of the geometry of submanifolds, in particular concerning the geometry of embedded null hypersurfaces. In Sections 2.5 and 2.6, we review the definitions and geometric properties of several types of null hypersurfaces that play an essential role later in the thesis, namely non-expanding, weakly isolated, isolated and (multiple) Killing horizons. Finally, in Section 2.7 we include some prior considerations concerning the matching of two given spacetimes across a hypersurface.

The rest of the thesis presents the new results that have been obtained in this work. The second part is devoted to the development of the formalism of hypersurface data. This is done in Chapters 3, 4, 5, 6, whose contents we describe next.

The structure of Chapter 3 is as follows. In Section 3.1, we start by providing several new results within he framework of the formalism of hypersurface data. In particular, in Section 3.1.1 we introduce the tensor field "Lie derivative of a connection" along a privileged vector field Z (denoted by Σ_Z). We define Σ_Z in a completely general context, derive various identities involving it and then focus our analysis on the tensor field "Lie derivative of $\overset{\circ}{\nabla}$" along a vector field n which can be defined from any metric hypersurface data set. In Section 3.2 we study null hypersurface data, i.e. data describing (abstractly) an everywhere null hypersurface. Therein we include gauge-fixing results, new identities involving $\overset{\circ}{\nabla}$ as well as its associated curvature and Ricci tensors and a detailed discussion on geometric aspects of non-degenerate smooth submanifolds within the abstract null hypersurface. Finally, in Section 3.3 we analyze the case when a null hypersurface data set is equipped with an extra null gauge-invariant vector field.

Chapter 4 is devoted to the so-called *constraint tensor* R. The constraint tensor is defined for any abstract hypersurface and, when the data happens to be embedded in a semi-Riemannian manifold, it captures a certain combination of components of the Riemann curvature tensor of the ambient space. In Section 4.1, we motivate its abstract definition and derive some of its properties. In Section 4.2, we particularize our analysis to the null case, finding the contractions of R with a null generator and providing its pull-back to any non-degenerate submanifold within the abstract hypersurface. In particular, we compute its relation with the Ricci tensor of the induced metric on such Riemannian submanifold. The chapter concludes with Section 4.3, where we introduce several quantities that are either invariant under gauge transformations or have a simple gauge behaviour. The results in this chapter are of use in other parts of the thesis.

Chapter 5 constitutes one of the core parts of this thesis. Its main concern is to study the case when a hypersurface admits a privileged vector field. The chapter is divided in four sections. In Section 5.1, we consider completely general hypersurface data embedded in a semi-Riemannian manifold equipped with a special vector field y. Initially, we allow y to be completely arbitrary, in particular not necessarily tangent to the hypersurface. In this context we derive explicit expressions for the Lie bracket of y with any extension of a rigging vector field. Then, we focus on the case when y is tangent and obtain the Lie derivative of the data tensor $\mathbf{Y}$ along y (recall that $\mathbf{Y}$ encodes the extrinsic geometric information of the hypersurface, as we mentioned above). All results in Section 5.1 involve the deformation tensor of y. In Section 5.2 we focus on the case when the hypersurface is null and y is null and tangent to the hypersurface. In this context, we derive several identities to be used later in the thesis. Section 5.3 is devoted to the tensor "Lie derivative along y of the Levi-Civita connection", namely Σ_y. We compute the explicit form of Σ_y in terms of the data plus an additional tensor field $\daleth^y$ which happens to play a crucial role in the analysis of the so-called abstract Killing horizons of order zero and one. These are new abstract notions of horizons which we motivate and present in Section 5.4, where we also compare them with other definitions of horizons at the embedded level, including Killing horizons, non-expanding horizons and isolated horizons.

With Chapter 6 we conclude our development of the formalism of hypersurface data. This chapter focuses on the derivation and consequences of the so-called *generalized master equation* (see (6.61)). The generalized master equation is an identity that holds for any null hypersurface admitting a privileged null, tangent vector field y. It involves the proportionality function between y and a null generator of the hypersurface, the constraint tensor R, the tensor $\daleth^y$ mentioned above and various tensor fields of the data. The generalized master equation, together with its contractions with a null generator, are included in Section 6.1. In Section 6.2 we particularize the analysis to the case when the deformation tensor of y is proportional to the metric. In this context, we obtain several interesting results concerning the fixed points set of y, the regularity of the Ricci tensor of the ambient space and the constancy of the surface gravity κ of y. In Section 6.3, we particularize the previous results for abstract Killing horizons of order zero and one. This allows us to identify some consequences of κ not being constant. In Section 6.4 we provide another key result of this thesis, namely the restriction of the generalized master equation to any non-degenerate submanifold within a null hypersurface. As particular case, we recover the so-called master equation of multiple Killing horizons (see e.g. [2], [93], [94]) as well as the so-called near horizon equation of isolated horizons (see e.g. [95], [96], [97], [1], [98], [99]). Finally, in Section 6.5 we apply the prior results to the case of a vacuum degenerate Killing horizon.

The third part of this thesis, corresponding to Chapters 7, 8 and 9, is devoted to the problem of matching two completely general spacetimes across a null hypersurface. The summary of the contents of these chapters is as follows.

In Chapter 7 we address the matching problem from a spacetime viewpoint, namely without considering the boundaries of the spacetimes to be matched in a detached way. Throughout the chapter, we assume that the boundaries can be foliated by a family of spacelike cross-sections. In Section 7.1 we provide some preliminary results and identities for its later use. Section 7.2 offers a brief discussion on the problem of matching in the general case, i.e. when the boundaries of the spacetimes have any causality. Section 7.3 focuses on the null case and constitutes the main part of the chapter. We start by writing the junction conditions in terms of a basis of vector fields. This allows us to determine the necessary and sufficient conditions for the matching to be possible. In Section 7.3.1, we prove that the whole matching information can be codified in the so-called *step function* and a diffeomorphism between the set of null generators of each boundary. We also analyze a scenario in which an infinite number of matchings are feasible, namely when the boundaries are totally geodesic. We obtain explicit expressions for the matter-energy content of the most general null shell resulting from the matching (Section 7.3.2). We conclude the chapter by applying the results to the case of the matching of two regions of the spacetime of Minkowski across a null hyperplane (see Section 7.3.3), connecting our results with those from the cut-and-paste constructions in the literature.

In Chapter 8, we study a particular case of the above, namely when the boundaries of the spacetimes to be matched are embedded abstract Killing horizons of order zero (this notion has been introduced in Chapter 5). The idea is to analyze the situation in which the matching procedure identifies the zero order "Killing" vector fields. In Sections 8.1, 8.2 and 8.3, we address the matching problem for three different scenarios: both boundaries being non-degenerate, both being degenerate and one being degenerate and the other being non-degenerate. In Section 8.4, we particularize the results for the case of Killing horizons with bifurcation surfaces. We conclude with Section 8.5, where we examine the case of the spacetimes to be matched being spherical, plane or hyperbolic symmetric.

Chapter 9 concludes our discussion on the matching problem. This chapter constitutes another core part of the thesis for several reasons. First, because we study the problem of matching from a purely abstract viewpoint (i.e. without requiring the matching hypersurfaces to be embedded) and secondly because the results are completely general (in the sense that we do not enforce any topological restrictions, nor any other condition whatsoever on the null hypersurfaces and the spacetimes). In Section 9.1, we first provide a theorem for boundaries of any causality that establishes the abstract construction of the matching. We then focus on the null case (see Section 9.1.1), obtaining the necessary and sufficient conditions that allow for the matching and obtaining explicit expressions for the gravitational/matter-energy content of the resulting null shell. We also analyze the multiple matchings scenario (cf. Section 9.1.2). In Section 9.2, we recover the results from Chapter 7 whenever the boundaries have product topology. Finally, we conclude with an example of matching across a totally geodesic null hypersurface in the spacetimes of (anti-)de Sitter, see Section 9.3.

The last chapter of the thesis, namely Chapter 10, is devoted to collect the conclusions of our work as well as some future prospects.

Finally, this thesis includes four appendices. In Appendix A, we prove various general identities concerning the curvature tensor of a torsion-free connection. Appendix B is devoted to the derivation of a generalized form of the Gauss identity. In Appendix C, we offer a consistency check on the gauge behaviour of a tensor field introduced in Chapter 4. The thesis concludes with Appendix D, where we present a new geometric construction of coordinates near any null hypersurface. The essential point of such construction is that it allows one to recover the so-called Gaussian null coordinates (see e.g. [100]) and Rácz-Wald coordinates [101] in a neighbourhood of a null hypersurface and a bifurcation surface respectively.

2

PRELIMINARIES

As we have mentioned in the Introduction, this chapter includes all mathematical tools and results that have already been obtained in the literature. The contents of this chapter will be of use later elsewhere in the thesis.

2.1 notation and conventions

All manifolds are smooth, connected and, unless otherwise indicated, without boundary. Given a manifold $\mathcal{M}$ and a point $p \in \mathcal{M}$, the tangent and cotangent spaces at p are denoted by $T_p\mathcal{M}$, $T^\star_p\mathcal{M}$ respectively. As usual, $T\mathcal{M}$ refers to the corresponding tangent bundle and $\Gamma(T\mathcal{M})$ to its sections. Given a differentiable map ϕ between manifolds, we use the standard notation of $\phi^\star$ and $\phi_\star$ for its pullback and push-forward respectively. We also let $\mathcal{F}(\mathcal{M}) \underset{\text{def}}{=} C^\infty(\mathcal{M}, \mathbb{R})$ and $\mathcal{F}^\star(\mathcal{M}) \subset \mathcal{F}(\mathcal{M})$ its subset of no-where zero functions. We use the symbols $\pounds$, d to denote Lie derivative and exterior derivative respectively. Both tensorial and abstract index notation will be henceforth employed at our convenience. To help distinguishing objects, we will often use boldface to define covariant tensors in index-free notation. When indices are used, we shall use standard font, not boldface, for them. We will use Greek, lower case Latin and capital Latin indices for $(n+1)$-dimensional, n-dimensional and $(n-1)$-dimensional manifolds as follows

$$\alpha, \beta, ... = 0, 1, 2, ..., n; \qquad a, b, ... = 1, 2, ..., n; \qquad A, B, ... = 2, ..., n, \tag{2.1}$$

where $n \geq 1$ (or $n \geq 2$ whenever indices A, B, ... are involved) will always be assumed. Parenthesis (resp. brackets) will denote symmetrization (resp. antisymmetrization) of indices, e.g. for a given two-covariant tensor field A we write

$$A_{[ab]} \stackrel{\text{def}}{=} \frac{1}{2}(A_{ab} - A_{ba}), \qquad A_{(ab)} \stackrel{\text{def}}{=} \frac{1}{2}(A_{ab} + A_{ba}). \tag{2.2}$$

Given two tensors A and B, we let $A \otimes_s B \equiv \frac{1}{2}(A \otimes B + B \otimes A)$ and denote the trace of A with respect to B as $\mathrm{tr}_B A$. For any semi-Riemannian manifold $(\mathcal{M}, g)$, we use the symbol $\sharp$ for the corresponding contravariant metric, i.e. $g^\sharp$. We let ∇ denote the Levi-Civita covariant derivative of g and $g(X, Y)$ (also $\langle X, Y\rangle_g$) will be the scalar product of two vector fields $X, Y \in \Gamma(T\mathcal{M})$. For any connection D on a manifold $\mathcal{M}$, our notation and convention for its associated curvature operator are

$$R^D(X, W)Z \stackrel{\text{def}}{=} \left(D_X D_W - D_W D_X - D_{[X,W]} \right) Z, \qquad \forall X, W, Z \in \Gamma(T\mathcal{M}), \tag{2.3}$$

except if $D = \nabla$, in which case we simply write R for the curvature operator. Our signature convention for Lorentzian manifolds $(\mathcal{M}, g)$ is $(-, +, ..., +)$.

2.2 formalism of hypersurface data

In this section we introduce the main formalism exploited throughout this thesis, called *formalism of hypersurface data*. Originally presented in [58], [59] (with precursor [64]), it has proven useful in the study of first order perturbations of a general hypersurface [102], in the study of the characteristic problem in General Relativity [60], [61] and in the context of matching spacetimes across null boundaries [103], [104], which constitutes a core part of this thesis (see Chapters 7, 8 and 9). The key advantages of this formalism are firstly that it allows one to describe hypersurfaces of arbitrary causal character at a purely abstract level, namely without making any reference to an ambient space where they may be embedded; and secondly that it can be adapted to many different situations of interest by means of an inherent gauge freedom. We discuss all the details below.

2.2.1 *Metric hypersurface data*

The formalism of hypersurface data relies upon the two core notions of *metric hypersurface data* and *hypersurface data*, which conceptually involve different levels of geometric information. Specifically, a metric hypersurface data set encodes all information concerning the intrinsic geometry of an abstract hypersurface $\mathcal{N}$ while a hypersurface data set codifies, in addition, the extrinsic geometry of $\mathcal{N}$.

It is convenient to start with the intrinsic part. The underlying idea behind the notion of metric hypersurface data is that, whenever $\mathcal{N}$ is embedded in a semi-

Riemannian manifold, we are able to recover all the information about the ambient geometry along the hypersurface.

Definition 2.2.1. *(Metric hypersurface data) Let* $\mathcal{N}$ *be an* n*-dimensional manifold endowed with a* 2*-covariant symmetric tensor* γ*, a covector* $\boldsymbol{\ell}$ *and a scalar function* $\ell^{(2)}$*. The four-tuple* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ *defines metric hypersurface data provided that the symmetric* 2*-covariant tensor* $\boldsymbol{\mathcal{A}}|_p$ *on* $T_p\mathcal{N} \times \mathbb{R}$ *given by*

$$\begin{aligned} \boldsymbol{\mathcal{A}}|_p\left((W,a),(Z,b)\right) &\stackrel{\text{def}}{=} \gamma|_p(W,Z) + a\,\boldsymbol{\ell}|_p(Z) + b\,\boldsymbol{\ell}|_p(W) + ab\,\ell^{(2)}|_p, \\ & W,Z \in T_p\mathcal{N}, \quad a,b \in \mathbb{R} \end{aligned} \tag{2.4}$$

is non-degenerate at every $p \in \mathcal{N}$*.*

Since $\boldsymbol{\mathcal{A}}|_p$ is non-degenerate there exists a unique inverse contravariant tensor $\mathcal{A}|_p$ on $T_p^\star\mathcal{N} \times \mathbb{R}$. Splitting its action as

$$\begin{aligned} \mathcal{A}|_p\left((\boldsymbol{\alpha},a),(\boldsymbol{\beta},b)\right) &\stackrel{\text{def}}{=} P|_p(\boldsymbol{\alpha},\boldsymbol{\beta}) + a\,n|_p(\boldsymbol{\beta}) + b\,n|_p(\boldsymbol{\alpha}) + abn^{(2)}|_p, \\ & \boldsymbol{\alpha},\boldsymbol{\beta} \in T_p^\star\mathcal{N}, \quad a,b \in \mathbb{R} \end{aligned} \tag{2.5}$$

defines a symmetric 2-contravariant tensor P, a vector n and a scalar $n^{(2)}$ in $\mathcal{N}$. By definition, they are smooth fields satisfying [58]:

$$\gamma_{ab}n^b + n^{(2)}\ell_a = 0, \tag{2.6}$$

$$\ell_a n^a + n^{(2)}\ell^{(2)} = 1, \tag{2.7}$$

$$P^{ab}\ell_b + \ell^{(2)}n^a = 0, \tag{2.8}$$

$$P^{ab}\gamma_{bc} + n^a\ell_c = \delta^a_c. \tag{2.9}$$

Observe that we have imposed no conditions on the signature of the tensor $\boldsymbol{\mathcal{A}}$. This will be relevant later when we introduce the concept of embedded (metric) hypersurface data. It is also worth stressing that there are no restrictions upon γ besides being a symmetric 2-covariant tensor field. In particular, γ can be degenerate. In any case, for arbitrary metric hypersurface data (irrespective of the signature of $\boldsymbol{\mathcal{A}}$), one can prove that the radical of γ at a point $p \in \mathcal{N}$ (i.e. the set $\operatorname{Rad}\gamma|_p \stackrel{\text{def}}{=} \{X \in T_p\mathcal{N} \mid \gamma(X,\cdot) = 0\}$ of vectors anhihilated by γ) is [59] either zero- or one-dimensional. Moreover, the latter case occurs if and only if $n^{(2)}|_p = 0$, which together with (2.6) means that $\operatorname{Rad}\gamma|_p = \langle n|_p \rangle$. Thus, $n|_p$ is non-zero (by (2.7)) and defines the degenerate direction of $\gamma|_p$. This property suggests introducing the following definitions of null and non-null points.

Definition 2.2.2. *(Null and non-null point) Let* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ *be metric hypersurface data. A point* p *is called null if* $\dim(\mathrm{Rad}\gamma)|_p = 1$ *and non-null otherwise.*

Thus, at a non-null (resp. null) point $p \in \mathcal{N}$, it holds $n^{(2)}|_p \neq 0$ (resp. $n^{(2)}|_p = 0$).

Given metric hypersurface data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$, it is useful to define the tensors

$$\mathbf{F} \stackrel{\mathrm{def}}{=} \frac{1}{2} d\boldsymbol{\ell}, \tag{2.10}$$

$$\boldsymbol{s} \stackrel{\mathrm{def}}{=} \mathbf{F}(n, \cdot), \tag{2.11}$$

$$\mathbf{U} \stackrel{\mathrm{def}}{=} \frac{1}{2} \pounds_n \gamma + \boldsymbol{\ell} \otimes_s dn^{(2)}. \tag{2.12}$$

Observe that $\mathbf{U}$ is symmetric and $\mathbf{F}$ is a 2-form. These tensor fields satisfy the following identities [59]:

$$\pounds_n \boldsymbol{\ell} = 2\boldsymbol{s} - d(n^{(2)} \ell^{(2)}), \tag{2.13}$$

$$\mathbf{U}(n, \cdot) = -n^{(2)} \boldsymbol{s} + \frac{1}{2} dn^{(2)} + \frac{1}{2} (n^{(2)})^2 d \ell^{(2)}. \tag{2.14}$$

It is also worth mentioning that one can construct a volume form on the abstract manifold $\mathcal{N}$ provided it is oriented. The corresponding definition is as follows.

Definition 2.2.3. *(Volume form) Consider metric hypersurface data* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ *and assume* $\mathcal{N}$ *to be oriented. Then the following local expression in any chart* $\{x^a\}$ *defines a volume form on* $\mathcal{N}$*:*

$$W^{(\boldsymbol{\ell})}_{\mathrm{vol}} = e \sqrt{|\det \boldsymbol{\mathcal{A}}|}\, E, \tag{2.15}$$

where $e = +1(-1)$ *if the chart is positively (negatively) oriented,* E *is the Levi-Civita totally antisymmetric symbol and* $|\cdot|$ *denotes the absolute value.*

In general, $\mathcal{N}$ is *not* a semi-Riemannian manifold because it is not endowed with a metric tensor. This means that generically (in particular when there exist null points in $\mathcal{N}$) we cannot define a Levi-Civita covariant derivative on $\mathcal{N}$. As we shall see next, in spite of this fact it turns out that there exists a canonical notion of covariant derivative on $\mathcal{N}$. This fundamental property makes the whole formalism relevant both mathematically and physically, as the possibility of doing calculus with tensor fields is key in any differential geometric theory.

This canonical covariant derivative, denoted by $\overset{\circ}{\nabla}$, can be defined from its action on some tensor fields constructed from the metric hypersurface data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ [59, Prop. 4.3].

Theorem 2.2.4. *For any given metric hypersurface data* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$, *the conditions*

$$(\overset{\circ}{\nabla}_X \gamma)(Z,W) = -\mathbf{U}(X,Z)\boldsymbol{\ell}(W) - \mathbf{U}(X,W)\boldsymbol{\ell}(Z), \tag{2.16}$$

$$(\overset{\circ}{\nabla}_X \boldsymbol{\ell})(Z) + (\overset{\circ}{\nabla}_Z \boldsymbol{\ell})(X) = -2\,\ell^{(2)}\mathbf{U}(X,Z), \qquad \forall X,Z,W \in \Gamma(T\mathcal{N}) \tag{2.17}$$

define a unique torsion-free connection $\overset{\circ}{\nabla}$ *on* $\mathcal{N}$.

We call the connection $\overset{\circ}{\nabla}$ *metric hypersurface connection*, as it is constructed solely from the metric hypersurface data. The following identities [59] for $\overset{\circ}{\nabla}$-derivatives of the tensor fields γ, $\boldsymbol{\ell}$, n and P will be used later:

$$\overset{\circ}{\nabla}_a \gamma_{bc} = -\ell_b \mathrm{U}_{ac} - \ell_c \mathrm{U}_{ab}, \tag{2.18}$$

$$\overset{\circ}{\nabla}_a \ell_b = \mathrm{F}_{ab} - \ell^{(2)}\mathrm{U}_{ab}, \tag{2.19}$$

$$\overset{\circ}{\nabla}_b n^c = n^c\left(s_b - n^{(2)}(d\ell^{(2)})_b\right) + P^{ac}\left(\mathrm{U}_{ba} - n^{(2)}\mathrm{F}_{ba}\right), \tag{2.20}$$

$$\overset{\circ}{\nabla}_a P^{bc} = -\left(n^b P^{cf} + n^c P^{bf}\right)\mathrm{F}_{af} - n^b n^c \overset{\circ}{\nabla}_a \ell^{(2)}. \tag{2.21}$$

An interesting particular case occurs when $\boldsymbol{\ell} = 0$. Then, $\boldsymbol{A}$ being non-degenerate requires γ to be non-degenerate and $\ell^{(2)} \neq 0$ everywhere on $\mathcal{N}$. This means that $(\mathcal{N}, \gamma)$ is a semi-Riemannian manifold to which one can associate a Levi-Civita connection $\nabla^{(\gamma)}$. Moreover, (2.8) entails that $n = 0$, and hence $\mathbf{U}$ and $\mathbf{F}$ all vanish identically. This transforms (2.16)-(2.17) into $(\overset{\circ}{\nabla}_X \gamma)(Z,W) = 0$, from where we conclude that $\overset{\circ}{\nabla} = \nabla^{(\gamma)}$. Thus, the formalism recovers the usual definition of metric connection whenever $\mathcal{N}$ has no null points and $\boldsymbol{\ell} = 0$. We emphasize, however, that the definition of $\overset{\circ}{\nabla}$ works in general and that one can treat not only the case when $\mathcal{N}$ includes null points (or when all of them are null) but also when $\mathcal{N}$ consists of non-null points only but one wants to use $\boldsymbol{\ell} \neq 0$.

As already mentioned, the purpose of the formalism of hypersurface data is to study general hypersurfaces independently of any ambient manifold where they may be embedded. However, at some point it will become necessary to connect the abstract formalism with the geometry of embedded hypersurfaces. The relationship between them, which relies upon the notion of embedded metric hypersurface data, also allows one to understand how the abstract data is affected by the freedom in the choice of an everywhere transversal vector field along the hypersurface.

Definition 2.2.5. *(Rigging, embedded metric hypersurface data) A metric hypersurface data* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ *is said to be embedded in a semi-Riemannian manifold* $(\mathcal{M}, g)$ *of* *dimension $n+1$ provided there exists an embedding $\phi : \mathcal{N} \longrightarrow \mathcal{M}$ and a rigging vector field ζ (i.e. a vector field along $\phi(\mathcal{N})$, everywhere transversal to it) satisfying*

$$\phi^\star(g) = \gamma, \qquad \phi^\star(g(\zeta,\cdot)) = \boldsymbol{\ell}, \qquad \phi^\star(g(\zeta,\zeta)) = \ell^{(2)}. \tag{2.22}$$

Notation 2.2.6. *In the following, whenever it does not lead to misunderstandings we shall identify scalar functions on $\mathcal{N}$ and on $\phi(\mathcal{N})$ as well as vector fields on $\mathcal{N}$ with their corresponding images through $\phi_\star$.*

In view of (2.22), in the embedded case the tensors $\{\gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ coincide respectively with the first fundamental form of the hypersurface, the tangent part of the rigging covector $g(\zeta,\cdot)$ and the norm of the rigging. Observe that any metric hypersurface data with $\boldsymbol{\ell} = 0$ is related, when one embeds it, to the case when the rigging is normal to $\phi(\mathcal{N})$ (which of course requires that $\phi(\mathcal{N})$ has no null points, as the normal would not be everywhere transverse otherwise).

It is also worth stressing that from any metric hypersurface data set one can reconstruct the full metric g along $\phi(\mathcal{N})$ (this is the reason behind the terminology "metric hypersurface data"), as it holds that

$$\boldsymbol{A}|_p\left((W,a),(Z,b)\right) = g|_{\phi(p)}(\phi_* W + a\zeta, \phi_* Z + b\zeta). \tag{2.23}$$

Thus, $\boldsymbol{A}$ completely encodes the metric g at points on $\phi(\mathcal{N})$. This justifies referring to $\boldsymbol{A}$ as *ambient metric* and provides its geometrical interpretation.

In order to relate the quantities $\{n, n^{(2)}\}$ with geometric objects in the ambient space, we now consider the following setup.

Setup 2.2.7. *We let $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ be metric hypersurface data embedded in a semi-Riemannian manifold $(\mathcal{M}, g)$ with embedding ϕ and rigging vector ζ. We select any local basis $\{\hat{e}_a\}$ of $\Gamma(T\mathcal{N})$ and define $e_a \stackrel{\text{def}}{=} \phi_\star(\hat{e}_a)$. By transversality of the rigging, $\{\zeta, e_a\}$ constitutes a (local) basis of $\Gamma(T\mathcal{M})|_{\phi(\mathcal{N})}$. The hypersurface $\phi(\mathcal{N})$ admits a unique normal covector $\boldsymbol{\nu}$ satisfying $\boldsymbol{\nu}(\zeta) = 1$. By construction, this covector belongs to the dual basis of $\{\zeta, e_a\}$, which we denote by $\{\boldsymbol{\nu}, \boldsymbol{\theta}^a\}$. We define the vector fields $\nu \stackrel{\text{def}}{=} g^\sharp(\boldsymbol{\nu},\cdot)$, $\theta^a \stackrel{\text{def}}{=} g^\sharp(\boldsymbol{\theta}^a,\cdot)$.*

In these circumstances, $\{\nu, \theta^a\}$ satisfy, by definition of dual basis,

$$g(\nu,\zeta) = 1, \qquad g(\nu, e_a) = 0, \qquad g(\theta^a,\zeta) = 0, \qquad g(\theta^a, e_b) = \delta^a_b, \tag{2.24}$$

and can be decomposed in the basis $\{\zeta, e_a\}$ as

$$\nu = n^{(2)}\zeta + n^a e_a, \tag{2.25}$$

$$\theta^a = P^{ab} e_b + n^a \zeta. \tag{2.26}$$

This can be proven by taking (2.25)-(2.26) as an ansatz and checking (2.24):

$$g(\nu, e_a) = n^{(2)} \boldsymbol{\ell}_a + \gamma_{ab} n^b \overset{(2.6)}{=} 0, \qquad g(\theta^a, \zeta) = P^{ab} \boldsymbol{\ell}_b + n^a \ell^{(2)} \overset{(2.8)}{=} 0,$$

$$g(\nu, \zeta) = n^{(2)} \ell^{(2)} + n^a \boldsymbol{\ell}_a \overset{(2.7)}{=} 1, \qquad g(\theta^a, e_b) = P^{ac}\gamma_{bc} + n^a \boldsymbol{\ell}_b \overset{(2.9)}{=} \delta^a_b.$$

Since in the embedded case the components of $\boldsymbol{A}$ in the basis $\{(\hat{e}_a, 0), (0, 1)\}$ are the same (by (2.22)) as the components of g in the basis $\{e_a, \zeta\}$, it follows directly from (2.5) that the contravariant metric $g^\sharp$ can be expressed in the basis $\{\zeta, e_a\}$ as

$$g^{\mu\nu} \overset{\phi(\mathcal{N})}{=} n^{(2)}\zeta^\mu\zeta^\nu + n^c \left(\zeta^\mu e^\nu_c + \zeta^\nu e^\mu_c\right) + P^{cd} e^\mu_c e^\nu_d$$

$$\Longleftrightarrow \qquad g^{\mu\nu} \overset{\phi(\mathcal{N})}{=} e^\mu_c \theta^{c\nu} + \zeta^\mu \nu^\nu, \tag{2.27}$$

where the equivalence is a consequence of (2.25)-(2.26). Observe that this implies

$$g(\nu, \nu) \overset{\phi(\mathcal{N})}{=} g^\sharp(\boldsymbol{\nu}, \boldsymbol{\nu}) \overset{\phi(\mathcal{N})}{=} n^{(2)}. \tag{2.28}$$

Thus, in the embedded case $n^{(2)}$ is just the square norm of the normal $\boldsymbol{\nu}$ along $\phi(\mathcal{N})$, while $\phi_\star n = \nu - n^{(2)}\zeta$ (cf. (2.25)). In particular, at null points ν coincides with $\phi_\star n$.

We conclude the section of metric hypersurface data by quoting Lemma 3 in [58]. This result, which will be key later in many situations, establishes under which conditions it is possible to construct a vector field from a given pair consisting of a covector and a scalar function.

Lemma 2.2.8. *Let $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ be metric hypersurface data. Given a covector field $\boldsymbol{\varrho} \in \Gamma(T^\star\mathcal{N})$ and a scalar function $u_0 \in \mathcal{F}(\mathcal{N})$, there exists a vector field $W \in \Gamma(T\mathcal{N})$ satisfying $\gamma(W, \cdot) = \boldsymbol{\varrho}$, $\boldsymbol{\ell}(W) = u_0$ if and only if $\boldsymbol{\varrho}(n) + n^{(2)}u_0 = 0$. Such W is unique and reads $W = P(\boldsymbol{\varrho}, \cdot) + u_0 n$.*

2.2.1.1 *Gauge structure*

As the reader may have noticed, the choice of a rigging vector field along an embedded hypersurface is highly non-unique. This fact is captured within the formalism of hypersurface data by means of a built-in gauge freedom, whose main properties are summarized next [59].

Let $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ be metric hypersurface data, $z \in \mathcal{F}^\star(\mathcal{N})$ and $V \in \Gamma(T\mathcal{N})$. We define the gauge-transformed metric data

$$\mathcal{G}_{(z,V)}(\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}) \stackrel{\text{def}}{=} \left\{\mathcal{N}, \mathcal{G}_{(z,V)}(\gamma), \mathcal{G}_{(z,V)}(\boldsymbol{\ell}), \mathcal{G}_{(z,V)}\left(\ell^{(2)}\right)\right\} \tag{2.29}$$

as

$$\mathcal{G}_{(z,V)}(\gamma) \stackrel{\text{def}}{=} \gamma, \tag{2.30}$$

$$\mathcal{G}_{(z,V)}(\boldsymbol{\ell}) \stackrel{\text{def}}{=} z\left(\boldsymbol{\ell} + \gamma(V,\cdot)\right), \tag{2.31}$$

$$\mathcal{G}_{(z,V)}\left(\ell^{(2)}\right) \stackrel{\text{def}}{=} z^2\left(\ell^{(2)} + 2\boldsymbol{\ell}(V) + \gamma(V,V)\right). \tag{2.32}$$

The induced transformations of P, n, $n^{(2)}$ are

$$\mathcal{G}_{(z,V)}(P) = P + n^{(2)} V \otimes V - 2V \otimes_s n, \tag{2.33}$$

$$\mathcal{G}_{(z,V)}(n) = z^{-1}(n - n^{(2)} V), \tag{2.34}$$

$$\mathcal{G}_{(z,V)}\left(n^{(2)}\right) = z^{-2} n^{(2)}, \tag{2.35}$$

while the gauge behaviour of the metric hypersurface data connection $\overset{\circ}{\nabla}$ is given by the following proposition [59, Prop. 4.6].

Proposition 2.2.9. *Let $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ be metric hypersurface data, $z \in \mathcal{F}^\star(\mathcal{N})$ and $V \in \Gamma(T\mathcal{N})$. Then, the difference tensor between the connection $\overset{\circ}{\nabla}$ and $\mathcal{G}_{(z,V)}(\overset{\circ}{\nabla})$ is*

$$\begin{aligned}\mathcal{G}_{(z,V)}(\overset{\circ}{\nabla}) - \overset{\circ}{\nabla} &= \frac{1}{2z} V \otimes \left(\pounds_{zn}\gamma - n^{(2)}\pounds_{zV}\gamma + 2z\boldsymbol{\ell} \otimes_s dn^{(2)}\right) \\ &+ \frac{1}{2z} n \otimes \left(\pounds_{zV}\gamma + 2\boldsymbol{\ell} \otimes_s dz\right).\end{aligned} \tag{2.36}$$

As proven in [59, Lem. 3.3], the set of all possible transformations $\{\mathcal{G}_{(z,V)}\}$ constitutes a group with the following properties.

Proposition 2.2.10. *The set of transformations $\{\mathcal{G}_{(z,V)}\}$ forms a group*

$$\mathcal{G} = \mathcal{F}(\mathcal{N}) \times \Gamma(T\mathcal{N})$$

with composition law $\mathcal{G}_{(z_2,V_2)} \circ \mathcal{G}_{(z_1,V_1)} = \mathcal{G}_{(z_1 z_2, V_2 + z_2^{-1} V_1)}$, identity element $\mathcal{G}_{\mathbb{I}} \stackrel{\text{def}}{=} \mathcal{G}_{(1,0)}$ and inverse element $\mathcal{G}^{-1}_{(z,V)} \stackrel{\text{def}}{=} \mathcal{G}_{(z^{-1}, -zV)}$.

This justifies the terminology of calling $\mathcal{G}$ *gauge group*, each element $\mathcal{G}_{(z,V)}$ *gauge transformation* (also *gauge group element*) and the quantities $\{z, V\}$ *gauge parameters*.

Observe that the gauge freedom associated to any metric hypersurface data is an intrinsic property of the data, independently of whether it is embedded in an ambient space. When the data is embedded, the connection between the freedom in the choice of a rigging vector field and the gauge structure can be established as follows [59, Prop. 3.4].

Proposition 2.2.11. *Let* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ *be metric hypersurface data embedded in a semi-Riemannian manifold* $(\mathcal{M}, g)$ *with embedding* ϕ *and rigging vector field* ζ*. Then* $\mathcal{G}_{(z,V)}(\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\})$ *is also embedded in the same space with the same embedding* ϕ *but with a different rigging* $\mathcal{G}_{(z,V)}(\zeta)$ *defined by*

$$\mathcal{G}_{(z,V)}(\zeta) \stackrel{\text{def}}{=} z(\zeta + \phi_\star V). \tag{2.37}$$

Since the normal vector ν is fixed for any choice of rigging according to the condition $\boldsymbol{\zeta}(\nu) = 1$, Proposition 2.2.11 forces

$$\mathcal{G}_{(z,V)}(\nu) = \frac{1}{z}\nu. \tag{2.38}$$

The highly non-uniqueness of the rigging could seem a priori to be a drawback of the formalism. However, the possibility of fixing the gauge at will when studying hypersurfaces at the abstract level actually constitutes a huge advantage in many situations because one can adjust the gauge freedom to the specific problem one wants to solve.

As proved in [59, Lemma 3.6], given metric hypersurface data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ and a point $p \in \mathcal{N}$, the only elements of the gauge group $\mathcal{G}$ leaving $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ invariant at p are (i) $\mathcal{G}_{(1,0)}|_p$ if p is null and (ii) $\{\mathcal{G}_{(1,0)}|_p, \mathcal{G}_{(-1,-2\ell)}|_p\}$ if p is non-null (here the vector $\ell|_p$ is defined as $\ell|_p \stackrel{\text{def}}{=} \gamma^\sharp(\boldsymbol{\ell}, \cdot)|_p$, where $\gamma^\sharp|_p$ is the inverse of the metric $\gamma|_p$). Since the gauge parameters $\{z, V\}$ are smooth by definition, it follows that when $\mathcal{N}$ contains a null point, only the identity element of $\mathcal{G}$ leaves the whole metric hypersurface data invariant. On the contrary, when $\mathcal{N}$ consists exclusively of non-null points there exist two gauge elements which do not transform the metric data. In this last case, the rigging $\mathcal{G}_{(-1,-2\ell)}(\zeta)$ corresponds [59] to the reflection of ζ with respect to the tangent plane $T_q\phi(\mathcal{N})$ at each point $q \in \phi(\mathcal{N})$.

2.2.2 *Hypersurface data*

We have already introduced the notion of metric hypersurface data, which codifies all the intrinsic geometric information of a hypersurface. The next natural step is to encode the extrinsic geometry within the formalism. As mentioned before, this leads us to the concept of *hypersurface data*.

Definition 2.2.12. *(Hypersurface data) A five-tuple* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ *defines hypersurface data if* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ *is metric hypersurface data and* $\mathcal{N}$ *is equipped with an extra symmetric 2-covariant tensor* $\mathbf{Y}$*.*

We will frequently use the notation $\mathcal{D} \equiv \{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$. As happened before for the metric hypersurface data, the geometric interpretation of the tensor field $\mathbf{Y}$ comes from the definition of embedded hypersurface data, which we give next.

Definition 2.2.13. *(Embedded hypersurface data) A hypersurface data* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ *is said to be embedded in a semi-Riemannian manifold* $(\mathcal{M}, g)$ *with embedding* ϕ *and rigging vector field* ζ *if its metric part* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ *is embedded in* $(\mathcal{M}, g)$ *with same embedding and rigging and, in addition,*

$$\frac{1}{2}\phi^{\star}\left(\pounds_{\zeta} g\right) = \mathbf{Y}. \tag{2.39}$$

Definition 2.2.13 directly relates the tensor $\mathbf{Y}$ with transverse derivatives of the ambient metric g on $\phi(\mathcal{N})$, which automatically guarantees that $\mathbf{Y}$ encodes extrinsic information of the hypersurface $\phi(\mathcal{N})$ in the embedded case. The gauge transformation (2.37) of ζ determines the behaviour of $\mathbf{Y}$ (and hence of the whole hypersurface data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$) under the action of a gauge group element. This transformation must be introduced at the abstract level as a definition, namely

$$\mathcal{G}_{(z,V)}(\mathbf{Y}) \stackrel{\text{def}}{=} z\mathbf{Y} + \boldsymbol{\ell} \otimes_s dz + \frac{1}{2}\pounds_{zV}\gamma = z\mathbf{Y} + (\boldsymbol{\ell} + \gamma(V,\cdot)) \otimes_s dz + \frac{z}{2}\pounds_V \gamma, \tag{2.40}$$

and it is straightforward to prove [59] that (2.40) guarantees that $\mathcal{G}_{(z_2,V_2)} \circ \mathcal{G}_{(z_1,V_1)}(\mathbf{Y}) = \mathcal{G}_{(z_2,V_2)\circ(z_1,V_1)}(\mathbf{Y})$.

One can obtain the tangent covariant derivative of the rigging for given embedded hypersurface data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$. We perform this calculation in a given basis, namely by assuming Setup 2.2.7 (where the vector fields ν and θ^a are given by

(2.25)-(2.26) in terms of the basis $\{\zeta, e_a\}$. By defining $\boldsymbol{\zeta} \stackrel{\text{def}}{=} g(\zeta, \cdot)$ and using (2.10), (2.22), (2.39) and the fact that d commutes with the pull-back, one gets

$$\langle \nabla_{e_a} \zeta, e_b \rangle_g = \frac{1}{2} \left(\langle \nabla_{e_a} \zeta, e_b \rangle_g + \langle \nabla_{e_b} \zeta, e_a \rangle_g \right) + \frac{1}{2} \left(\langle \nabla_{e_a} \zeta, e_b \rangle_g - \langle \nabla_{e_b} \zeta, e_a \rangle_g \right)$$
$$= \frac{1}{2} (\pounds_\zeta g)(e_a, e_b) + \frac{1}{2} d\boldsymbol{\zeta}(e_a, e_b) = \mathrm{Y}_{ab} + \mathrm{F}_{ab}, \tag{2.41}$$
$$\langle \nabla_{e_a} \zeta, \zeta \rangle_g = \frac{1}{2} \nabla_{e_a} \langle \zeta, \zeta \rangle_g \stackrel{\mathcal{N}}{=} \frac{1}{2} \overset{\circ}{\nabla}_a \ell^{(2)}. \tag{2.42}$$

The combination of (2.41) and (2.42) yields

$$\nabla_{e_a} \zeta = \frac{1}{2} \overset{\circ}{\nabla}_a \ell^{(2)} \nu + (\mathrm{Y}_{ab} + \mathrm{F}_{ab}) \theta^b. \tag{2.43}$$

This identity will be of use in Section 5.1.

Given hypersurface data $\mathcal{D} = \{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$, it is useful to define the objects

$$\boldsymbol{r} \stackrel{\text{def}}{=} \mathbf{Y}(n, \cdot), \qquad \kappa_n \stackrel{\text{def}}{=} -\mathbf{Y}(n, n), \tag{2.44}$$
$$\mathbf{K} \stackrel{\text{def}}{=} n^{(2)} \mathbf{Y} + \mathbf{U}. \tag{2.45}$$

Observe that $\mathbf{K}$ is symmetric by construction. Moreover, under the action of a gauge group element, it transforms as [59, Lemma 3.5]

$$\mathcal{G}_{(z,V)}(\mathbf{K}) = \frac{1}{z} \mathbf{K}. \tag{2.46}$$

The simple gauge behaviour (2.46) can be easily understood in the embedded case. When $\mathcal{D}$ is embedded in an ambient space with embedding ϕ and rigging ζ, the tensor $\mathbf{K}$ defined in (2.45) coincides [59] with the second fundamental form of $\phi(\mathcal{N})$ with respect to the unique normal $\boldsymbol{\nu}$ satisfying $\boldsymbol{\nu}(\zeta) = 1$, i.e.

$$\mathbf{K} = \phi^\star (\nabla \boldsymbol{\nu}). \tag{2.47}$$

This, together with the fact that $\phi^\star \boldsymbol{\nu} = 0$, explains the transformation behaviour of $\mathbf{K}$ at the abstract level. Expression (2.46), however, holds independently of whether the data is embedded or not, and it is a consequence of definition (2.45). Observe that $\mathbf{K}|_p = \mathbf{U}|_p$ for null points $p \in \mathcal{N}$. In such case, all extrinsic information drops from $\mathbf{K}$ and therefore we recover the well-known property that the second fundamental form only codifies extrinsic geometric information for non-null points. This also means that at a null point p, $\mathbf{U}|_p$ is the second fundamental form of $\phi(\mathcal{N})$ with respect to $\boldsymbol{\nu}$, which will be relevant later, as many of our results will depend strongly on this tensor.

Another remarkable result is that, for given hypersurface data embedded in a semi-Riemannian manifold $(\mathcal{M}, g)$, the relation between the metric hypersurface connection $\overset{\circ}{\nabla}$ and the ambient Levi-Civita connection ∇ of $(\mathcal{M}, g)$ is [58]

$$\nabla_X Y = \overset{\circ}{\nabla}_X Y - \mathbf{Y}(X, Y)\nu - \mathbf{U}(X, Y)\zeta, \tag{2.48}$$

where $X, Y \in \Gamma(T\mathcal{N})$ and $\nu = n^{(2)}\zeta + \phi_\star n$ (recall (2.25)).

The behaviour of the tensor field $\mathbf{Y}$ under the action of the gauge elements $\mathcal{G}_{(1,0)}$, $\mathcal{G}_{(-1,-2\boldsymbol{\ell})}$ also deserves a brief comment. Of course, for the former we find $\mathcal{G}_{(1,0)}(\mathbf{Y}) = \mathbf{Y}$ (cf. (2.40)) and hence $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ still remains invariant under the identity element. For the latter, on non-null points one can prove that $\mathcal{G}_{(-1,-2\boldsymbol{\ell})}(\mathbf{Y}) = \mathbf{Y} - \frac{2}{n^{(2)}}\mathbf{K}$ and hence the whole hypersurface data is not invariant unless $\mathbf{K} = 0$. Observe that, in the case when $\mathbf{K} = 0$, one can determine $\mathbf{Y}$ in terms of the metric hypersurface data (by means of (2.45)). Thus, the invariance of $\mathbf{Y}$ when $\mathbf{K} = 0$ is a direct consequence of $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ being invariant itself. Anyway, in general $\mathbf{Y}$ does not remain invariant under the transformation $\mathcal{G}_{(-1,-2\boldsymbol{\ell})}$. This is consistent with the fact that a reflection of the rigging affects the transverse Lie derivative of g unless the normal transversal derivative vanishes (and hence so does $\mathbf{K}$).

2.2.2.1 *Hypersurface connection* $\overline{\nabla}$

The metric hypersurface connection $\overset{\circ}{\nabla}$ is not the only useful covariant derivative that can be constructed from given hypersurface data. In fact, from $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ one can define another torsion free connection $\overline{\nabla}$, which is called *hypersurface connection* or *rigging connection*. The simplest form of defining it is by providing its relation with $\overset{\circ}{\nabla}$.

Definition 2.2.14. *(Hypersurface connection $\overline{\nabla}$) Let $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ be hypersurface data and $\overset{\circ}{\nabla}$ the corresponding metric hypersurface connection defined in Theorem 2.2.4. For any $X, Z \in \Gamma(T\mathcal{N})$, $\overline{\nabla}$ is uniquely defined by*

$$\overline{\nabla}_X Z = \overset{\circ}{\nabla}_X Z - \mathbf{Y}(X, Z)n. \tag{2.49}$$

Unlike $\overset{\circ}{\nabla}$, the rigging connection depends on both the metric hypersurface data and the extrinsic part $\mathbf{Y}$. Depending on the situation it is advantageous to use $\overset{\circ}{\nabla}$ or $\overline{\nabla}$, so it is useful to keep both connections in mind. However, for the purposes of this thesis $\overset{\circ}{\nabla}$ will definitely be of more use. The only situation in which $\overline{\nabla}$ is independent of $\mathbf{Y}$ occurs when n vanishes identically. In such case, (2.6) entails that either $n^{(2)}$ or $\boldsymbol{\ell}$ has to be zero. The requirement of $\boldsymbol{A}$ being non-degenerate excludes the former, which means that $\boldsymbol{\ell} = 0$. We have already discussed that this only happens when (i) γ is a metric (which only holds for non-null points) and (ii) the rigging is normal to $\mathcal{N}$. The importance of the covariant derivative $\overline{\nabla}$ relies on the fact that it coincides with the connection induced from the Levi-Civita covariant derivative of the ambient space in the embedded case. In fact, combining (2.25), (2.45) and (2.48)-(2.49) yields

$$\nabla_X Y = \overline{\nabla}_X Y - \mathbf{K}(X, Y)\zeta, \qquad \forall X, Y \in \Gamma(T\mathcal{N}). \tag{2.50}$$

In the embedded case, this connection was first introduced by Schouten [62] and studied in detail in [64].

2.2.2.2 *Curvature of the metric hypersurface connection* $\overset{\circ}{\nabla}$

We conclude this section with some results involving the curvature and Ricci tensors $\overset{\circ}{\mathbf{Riem}}$, $\overset{\circ}{\mathbf{Ric}}$ (or $\overset{\circ}{R}{}^{d}{}_{abc}$ and $\overset{\circ}{R}{}_{ab}$ in abstract index notation) of the metric hypersurface connection $\overset{\circ}{\nabla}$. The first proposition is a general identity for $\overset{\circ}{\mathbf{Ric}}$ [59, Prop. 5.1], while the second provides *all* the components of the curvature tensor $R_{\alpha\beta\gamma\delta}$ of $(\mathcal{M}, g)$ that are computable in terms of the hypersurface data [64] (see also [58, Prop. 6]).

Proposition 2.2.15. *Given metric hypersurface data* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$*, the curvature tensor* $\overset{\circ}{\mathbf{Ric}}$ *verifies*

$$\overset{\circ}{\mathbf{Ric}}(X,Z) - \overset{\circ}{\mathbf{Ric}}(Z,X) = \left(ds - \frac{1}{2} dn^{(2)} \wedge d\ell^{(2)} \right)(X,Z), \qquad X, Z \in \Gamma(T\mathcal{N}). \tag{2.51}$$

Proposition 2.2.16. *Let* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ *be hypersurface data embedded in a semi-Riemannian manifold* $(\mathcal{M}, g)$ *with embedding* ϕ *and rigging* ζ*. Let* $\{\hat{e}_a\}$ *be a (local) basis of* $\Gamma(T\mathcal{N})$ *and* $e_a \stackrel{\text{def}}{=} \phi_\star(\hat{e}_a)$*. Then, the Riemann tensor* $R_{\alpha\beta\gamma\delta}$ *of* g *satisfies*

$$R_{\alpha\beta\gamma\delta}\zeta^\alpha e^\beta_b e^\gamma_c e^\delta_d \stackrel{\phi(\mathcal{N})}{=} \ell_a \overset{\circ}{R}{}^a{}_{bcd} + 2\overset{\circ}{\nabla}_{[d} Y_{c]b} + 2\ell^{(2)}\overset{\circ}{\nabla}_{[d} U_{c]b} + U_{b[c}\overset{\circ}{\nabla}_{d]}\ell^{(2)} + Y_{b[d}\left(2\left(F_{c]f} + Y_{c]f}\right) n^f + n^{(2)}\overset{\circ}{\nabla}_{c]}\ell^{(2)} \right), \tag{2.52}$$

$$R_{\alpha\beta\gamma\delta}e^\alpha_a e^\beta_b e^\gamma_c e^\delta_d \stackrel{\phi(\mathcal{N})}{=} \gamma_{af}\overset{\circ}{R}{}^f{}_{bcd} + 2\ell_a\overset{\circ}{\nabla}_{[d} U_{c]b} + 2Y_{b[c} U_{d]a} + 2U_{b[c} Y_{d]a} + 2n^{(2)} Y_{b[c} Y_{d]a} + 2U_{b[c} F_{d]a}. \tag{2.53}$$

2.2.3 *Matter-hypersurface data and constraint equations*

We have presented the formalism of hypersurface data, firstly by introducing the metric data and secondly by studying the extrinsic component of the data. However, the information concerning the matter content of the would-be ambient space cannot be extracted from neither of these two data sets. For instance, in the standard Cauchy problem of General Relativity (where one prescribes data on a spacelike hypersurface), the matter data corresponds to the energy density ρ and the energy current $\mathbf{J}$. These quantities are defined by

$$\rho \stackrel{\text{def}}{=} T(v,v)|_{\overline{\mathcal{N}}} \qquad \text{and} \qquad \mathbf{J}(X) \stackrel{\text{def}}{=} -T(v,X)|_{\overline{\mathcal{N}}},$$

where T is the energy-momentum tensor of the spacetime, $\overline{\mathcal{N}}$ is the spacelike hypersurface of initial data, v is the observer orthogonal to $\overline{\mathcal{N}}$ at all of its points and $X \in \Gamma(T\overline{\mathcal{N}})$. At the abstract level, one should prescribe a function ρ and a covector $\mathbf{J}$ so that a posteriori (i.e. once the initial value problem has been solved and a spacetime and a spacelike hypersurface $\overline{\mathcal{N}}$ have been constructed) they get their corresponding physical meaning. In fact, in order to be able to accommodate other geometric theories of gravity, it is advantageous to define ρ and $\mathbf{J}$ not by means of the energy-momentum tensor but in terms of the ambient Einstein tensor. Note that even in General Relativity the two definitions are not the same when the spacetime has a non-zero cosmological constant. We keep using the term “matter" to refer to these quantities due to its close relationship with the energy-momentum tensor in General Relativity, but we always work with purely geometric objects independent of any field equations.

In order for ρ, $\mathbf{J}$ to become the suitable physical/geometrical quantities at the spacetime level, we need to impose a set of restrictions to them, usually known as *constraint equations*. Apart from ρ, $\mathbf{J}$ themselves, the constraint equations involve the first and second fundamental forms. These equations are well-known in the spacelike case (see e.g. [105]), and have been generalized for hypersurfaces of arbitrary causal character in [58].

The constraint equations are closely related to the concept of matter-hypersurface data [58], which we present next. As usual, we first introduce the corresponding definitions at the abstract level and then we endow them with a physical and geometrical interpretation through the notion of embedded data.

Definition 2.2.17. *(Matter-Hypersurface data) A tuple* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}, \rho_\ell, \mathbf{J}\}$ *formed by hypersurface data* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$*, a scalar* $\rho_\ell \in \mathcal{F}(\mathcal{N})$ *and a one-form* $\mathbf{J} \in \Gamma(T^\star\mathcal{N})$

defines matter-hypersurface data if, under the action of the gauge group element $\mathcal{G}_{(z,V)}$, $\{\rho_\ell, \mathbf{J}\}$ *transform as*

$$\mathcal{G}_{(z,V)}(\rho_\ell) = \rho_\ell + \mathbf{J}(V), \qquad \mathcal{G}_{(z,V)}(\mathbf{J}) = z^{-1}\mathbf{J} \tag{2.54}$$

and the following identities, called constraint equations, hold:

$$\begin{aligned}
\rho_\ell = {} & \frac{1}{2}\overset{\circ}{R}{}^{c}{}_{bcd}P^{bd} + \frac{1}{2}\ell_a \overset{\circ}{R}{}^{a}{}_{bcd}P^{bd}n^c + \overset{\circ}{\nabla}_d\Big((P^{bd}n^c - P^{bc}n^d)\mathrm{Y}_{bc}\Big) \\
& + n^{(2)}P^{bd}P^{ac}\mathrm{Y}_{b[c}\mathrm{Y}_{d]a} + \frac{1}{2}\big(P^{bd}n^c - P^{bc}n^d\big)\Big(\ell^{(2)}\overset{\circ}{\nabla}_d\mathrm{U}_{bc} \\
& + (\mathrm{U}_{bc} + n^{(2)}\mathrm{Y}_{bc})\overset{\circ}{\nabla}_d\ell^{(2)} + 2\mathrm{Y}_{bc}(\mathrm{F}_{df} - \mathrm{Y}_{df})n^f\Big),
\end{aligned} \tag{2.55}$$

$$\begin{aligned}
\mathrm{J}_c = {} & \ell_a \overset{\circ}{R}{}^{a}{}_{bcd}n^b n^d - 2\overset{\circ}{\nabla}_f\Big((n^{(2)}P^{bd} - n^b n^d)\mathrm{Y}_{b[c}\delta^f_{d]}\Big) + 2(P^{bd} - \ell^{(2)}n^b n^d)\overset{\circ}{\nabla}_{[c}\mathrm{U}_{d]b} \\
& - (n^{(2)}P^{bd} - n^b n^d)\Big((\mathrm{U}_{b[c} + n^{(2)}\mathrm{Y}_{b[c})\overset{\circ}{\nabla}_{d]}\ell^{(2)} + 2\mathrm{Y}_{b[c}\mathrm{F}_{d]f}n^f\Big) \\
& - (P^{bd}n^f - P^{bf}n^d)\mathrm{Y}_{bd}\mathrm{U}_{cf} - 2P^{bd}n^f\mathrm{U}_{b[c}\mathrm{F}_{d]f}.
\end{aligned} \tag{2.56}$$

The next theorem, which constitutes one of the main results in [58], justifies both condition (2.54) on the gauge behaviour of $\{\rho_\ell, \mathbf{J}\}$ as well as the explicit form of (2.55) and (2.56).

Theorem 2.2.18. *Let* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}, \rho_\ell, \mathbf{J}\}$ *be matter-hypersurface data and assume that the hypersurface data* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ *is embedded in a semi-Riemannian manifold* $(\mathcal{M}, g)$ *with embedding* ϕ *and rigging* ζ*. Then,*

$$-\rho_\ell = \phi^\star\big(\mathbf{Ein}_g(\zeta, \nu)\big), \tag{2.57}$$

$$-\mathbf{J} = \phi^\star\big(\mathbf{Ein}_g(\cdot, \nu)\big), \tag{2.58}$$

where $\mathbf{Ein}_g$ *is the* (2*-covariant) Einstein tensor of* $(\mathcal{M}, g)$ *and* ν *the (unique) normal vector field along* $\phi(\mathcal{N})$ *satisfying* $g(\zeta, \nu) = 1$.

Definition 2.2.17 generalizes the notion of abstract spacelike initial data by including some matter-content information in addition to the hypersurface data. On the other hand, Theorem 2.2.18 provides the constraint equations for the case of fully general hypersurfaces in semi-Riemannian spaces. We emphasize that the constraint equations have been written in terms of the metric hypersurface connection $\overset{\circ}{\nabla}$, which means that the dependence on the extrinsic part of the data $\mathbf{Y}$ is fully explicit. This turns out to be highly useful in certain contexts, as we shall see later.

2.3 geometry of submanifolds

The geometry and properties of embedded null hypersurfaces, which constitute the main object of this thesis, will be analyzed in detail in the next section. However, prior to this discussion it is convenient to revisit some fundamental aspects on submanifolds of arbitrary codimension. The underlying reason why this is helpful is that in sufficiently local domains within any null hypersurface there always exists a Riemannian submanifold embedded in the hypersurface itself, so the geometry of submanifolds will therefore play a fundamental role also. General references on this topic are e.g. [106], [107].

Consider a semi-Riemannian manifold $(\mathcal{M}, g)$ and a manifold $\mathcal{X}$ of codimension greater or equal than one. Let $\iota : \mathcal{X} \longhookrightarrow \mathcal{M}$ be the embedding of $\mathcal{X}$ in $\mathcal{M}$ and define $\widetilde{\mathcal{X}} \stackrel{\text{def}}{=} \iota(\mathcal{X}) \subset \mathcal{M}$. It is a general fact (see e.g. [108]) that the Lie bracket of two vectors tangent to a submanifold is also tangent, namely

$$X, Y \in \Gamma\left(T\widetilde{\mathcal{X}}\right) \qquad \Longrightarrow \qquad [X, Y] \in \Gamma\left(T\widetilde{\mathcal{X}}\right). \tag{2.59}$$

The *first fundamental form* of $\mathcal{X}$ is the tensor field γ defined by

$$\gamma \stackrel{\text{def}}{=} \iota^{\star} g \tag{2.60}$$

and $\mathcal{X}$ is called degenerate whenever the radical of γ has dimension greater or equal to one, otherwise it is referred to as non-degenerate or Riemannian because γ defines a metric.

A fundamental concept in the context of submanifolds is the notion of normal vector field. Any vector field N along $\widetilde{\mathcal{X}}$ is said to be normal to $\widetilde{\mathcal{X}}$ if it satisfies

$$g(N, X) \stackrel{\widetilde{\mathcal{X}}}{=} 0 \qquad \forall X \in \Gamma(T\widetilde{\mathcal{X}}). \tag{2.61}$$

Observe that the combination of (2.59) and (2.61) entails that

$$g(N, [X, Y]) \stackrel{\widetilde{\mathcal{X}}}{=} 0. \tag{2.62}$$

Given a vector field N normal to a submanifold, one can define (see e.g. [106]) the so-called *second fundamental form* $\mathbf{K}^N$ of $\mathcal{X}$ with respect to N as

$$\mathbf{K}^N \stackrel{\text{def}}{=} \iota^{\star}(\nabla N), \tag{2.63}$$

a fact that was already used in (2.47). We have defined the second fundamental form as a tensor intrinsic to X. However, a closely related definition can also be made on its image $\bar{X}$. We therefore define the second fundamental form $\bar{\mathbf{K}}^N$ of $\bar{X}$ as the tensor field

$$\bar{\mathbf{K}}^N(X, Y) \stackrel{\text{def}}{=} g(\nabla_X N, Y) \qquad \forall X, Y \in \Gamma(T\bar{X}), \tag{2.64}$$

which of course satisfies that $\mathbf{K}^N = \iota^\star \bar{\mathbf{K}}^N$. By virtue of (2.61)-(2.62), it is immediate to prove that $\bar{\mathbf{K}}^N$ (and hence $\mathbf{K}^N$) is symmetric, since

$$\begin{aligned} \bar{\mathbf{K}}^N(X, Y) &= X\big(g(N, Y)\big) - g(N, \nabla_X Y) = -g(N, [X, Y]) - g(N, \nabla_Y X) \\ &= -g(N, \nabla_Y X) = -Y\big(g(N, X)\big) + g(\nabla_Y N, X) = \bar{\mathbf{K}}^N(Y, X). \end{aligned}$$

Moreover, $\bar{\mathbf{K}}^N$ is directly related to the Lie derivative of the metric g along N according to

$$(\pounds_N g)(X, Y) = 2\bar{\mathbf{K}}^N(X, Y) \qquad \forall X, Y \in \Gamma(T\bar{X}). \tag{2.65}$$

To prove this fact explicitly we need to extend X, Y off $\bar{X}$, although the final result (2.65) obviously does not depend on the extension. Using that $\bar{\mathbf{K}}^N$ is symmetric, we find

$$\begin{aligned} (\pounds_N g)(X, Y) &\stackrel{\bar{X}}{=} \pounds_N(g(X, Y)) - g(\pounds_N X, Y) - g(X, \pounds_N Y) \\ &\stackrel{\bar{X}}{=} g(\nabla_X N, Y) + g(X, \nabla_Y N) \stackrel{\bar{X}}{=} 2\bar{\mathbf{K}}^N(X, Y). \end{aligned}$$

Observe that rescaling N in (2.64) by a non-zero function $f \in \mathcal{F}^\star(\bar{X})$ (i.e. performing the change $N \longrightarrow fN$) simply multiplies $\bar{\mathbf{K}}^N$ by f. Up to transformations of this type, one can define as many second fundamental forms as directions normal to $\bar{X}$ happen to exist.

For the rest of this section, we make the harmless abuse of notation of identifying vector fields on $\Gamma(TX)$ with their counterparts along $\bar{X}$.

A particular case of the above occurs when X, $\bar{X}$ are Riemannian. Then, as we have already mentioned, the first fundamental form γ constitutes a metric and hence one can define its corresponding Levi-Civita connection, which we denote by $\nabla^{(\gamma)}$. In these circumstances, it is a general fact that the tangent space $T_p\mathcal{M}$ at a point $p \in \bar{X}$ decomposes as

$$T_p\mathcal{M} = T_p\bar{X} \oplus (T_p\bar{X})^\perp, \quad \text{where} \quad (T_p\bar{X})^\perp \stackrel{\text{def}}{=} \{X \in T_p\mathcal{M} \mid g(X, Y) = 0 \ \forall Y \in T_p\bar{X}\}.$$

An immediate consequence of the above is that the covariant derivative $\nabla_X Y$ with $X, Y \in \Gamma(T\widetilde{X})$ can be split in a tangent and an orthogonal part, namely

$$\nabla_X Y = D_X Y - K(X,Y), \tag{2.66}$$

where $D_X Y \in \Gamma(T\widetilde{X})$ and $K(X,Y)$ is a normal vector field, i.e. it satisfies

$$g(K(X,Y), Z) = 0 \qquad \forall Z \in \Gamma(T\widetilde{X}). \tag{2.67}$$

Observe that the vector field $K(X,Y)$ verifies $g(K(X,Y), N) = \mathbf{K}^N(X,Y)$ because

$$g(K(X,Y),N) = g(D_X Y - \nabla_X Y, N) = -g(\nabla_X Y, N) = g(\nabla_X N, Y) \stackrel{\text{def}}{=} \mathbf{K}^N(X,Y).$$

It is well-known that D defines a covariant derivative on $\widetilde{X}$. Even more,

$$\begin{aligned}(D_X\gamma)(Y,Z) &= X(\gamma(Y,Z)) - \gamma(D_X Y, Z) - \gamma(Y, D_X Z)\\ &= X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = (\nabla_X g)(Y,Z) = 0,\end{aligned}$$

so D actually coincides with the Levi-Civita covariant derivative $\nabla^{(\gamma)}$ of γ and therefore (2.66) can be rewritten as

$$\nabla_X Y = \nabla^{(\gamma)}_X Y - K(X,Y) \qquad \forall X, Y \in \Gamma(T\widetilde{X}). \tag{2.68}$$

One of the main results of non-degenerate submanifolds is the so-called Gauss equation or Gauss identity, which we derive next. We then apply this result to prove an identity relating the pull-back to $\widetilde{X}$ of the Ricci tensor of M with the Ricci tensor of $\widetilde{X}$. This latter result will be required in Section 2.5.

Lemma 2.3.1. *Consider a semi-Riemannian manifold* (M, g) *and a non-degenerate submanifold* $\widetilde{X} \subset M$*. Let* R*,* $R^{(\gamma)}$ *be the curvature operators of the Levi-Civita covariant derivatives* ∇*,* $\nabla^{(\gamma)}$ *of* M *and* $\widetilde{X}$ *respectively. Then, for any* $X, Y, Z, W \in \Gamma(T\widetilde{X})$*, it holds*

$$\begin{aligned} g\big(W, R(X,Y)Z\big) &= g\big(W, R^{(\gamma)}(X,Y)Z\big)\\ &\quad - g\big(K(X,W), K(Y,Z)\big) + g\big(K(Y,W), K(X,Z)\big). \end{aligned} \tag{2.69}$$

Proof. By direct computation one obtains

$$\begin{aligned} R(X,Y)Z &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\\ &= \nabla_X\big(\nabla^{(\gamma)}_Y Z - K(Y,Z)\big) - \nabla_Y\big(\nabla^{(\gamma)}_X Z - K(X,Z)\big)\\ &\quad - \nabla^{(\gamma)}_{[X,Y]} Z + K([X,Y],Z) \end{aligned}$$

$$
\begin{aligned}
&= \nabla^{(\gamma)}_X \nabla^{(\gamma)}_Y Z - K\left(X, \nabla^{(\gamma)}_Y Z\right) - \nabla_X \left(K(Y,Z)\right) - \nabla^{(\gamma)}_Y \nabla^{(\gamma)}_X Z \\
&\quad + K\left(Y, \nabla^{(\gamma)}_X Z\right) + \nabla_Y \left(K(X,Z)\right) - \nabla^{(\gamma)}_{[X,Y]} Z + K([X,Y],Z) \\
&= R^{(\gamma)}(X,Y)Z - K\left(X, \nabla^{(\gamma)}_Y Z\right) + K\left(Y, \nabla^{(\gamma)}_X Z\right) \\
&\quad + K([X,Y],Z) - \nabla_X \left(K(Y,Z)\right) + \nabla_Y \left(K(X,Z)\right). \qquad (2.70)
\end{aligned}
$$

Computing the scalar product $g(W, R(X,Y)Z)$ then yields (recall (2.67))

$$
g\left(W, R(X,Y)Z\right) = g\left(W, R^{(\gamma)}(X,Y)Z\right) + g\left(\nabla_X W, K(Y,Z)\right) - g\left(\nabla_Y W, K(X,Z)\right).
$$

from where equation (2.69) follows at once after using (2.67) and (2.68). □

We can particularize the setup above to the case when $(\mathcal{M}, g)$ is Lorentzian and $\widetilde{X}$ is Riemannian and has codimension two. In these circumstances, we can always select two linearly independent vector fields $N_1, N_2 \in \Gamma(T\mathcal{M})|_{\widetilde{X}}$ satisfying (2.61) and define the corresponding second fundamental forms $\mathbf{K}^{N_1}, \mathbf{K}^{N_2}$, one with respect to each normal vector. If in addition N_1, N_2 both happen to be null, then $\mu \stackrel{\text{def}}{=} g(N_1, N_2) \neq 0$ and

$$
K(X,Y) = \frac{1}{\mu}\left(\mathbf{K}^{N_2}(X,Y)N_1 + \mathbf{K}^{N_1}(X,Y)N_2\right) \qquad \forall X, Y \in \Gamma(T\widetilde{X}). \qquad (2.71)
$$

This transforms (2.69) into

$$
\begin{aligned}
g\left(W, R(X,Y)Z\right) &= g\left(W, R^{(\gamma)}(X,Y)Z\right) + \frac{1}{\mu}\Big(-\mathbf{K}^{N_1}(Y,Z)\mathbf{K}^{N_2}(X,W) \\
&- \mathbf{K}^{N_1}(X,W)\mathbf{K}^{N_2}(Y,Z) + \mathbf{K}^{N_1}(X,Z)\mathbf{K}^{N_2}(Y,W) + \mathbf{K}^{N_1}(Y,W)\mathbf{K}^{N_2}(X,Z)\Big). \qquad (2.72)
\end{aligned}
$$

The next identity is a consequence of the Gauss equation as written in (2.72). It relates the Ricci tensor of $\widetilde{X}$, the pull-back to $\widetilde{X}$ of the Ricci tensor of $\mathcal{M}$ and additional normal-tangent-normal-tangential components of the ambient curvature tensor.

Lemma 2.3.2. *Consider a Lorentzian manifold $(\mathcal{M}, g)$, a non-degenerate submanifold $\widetilde{X} \subset \mathcal{M}$ of codimension two and two linearly independent null vector fields $N_1, N_2 \in \Gamma(T\mathcal{M})|_{\widetilde{X}}$ which are normal to $\widetilde{X}$. Define the scalar $\mu \stackrel{\text{def}}{=} g(N_1, N_2) \neq 0$ and the second fundamental forms $\mathbf{K}^{N_1}, \mathbf{K}^{N_2}$ according to (2.64). Let $\{v_A\}$ be a basis of $\Gamma(T\widetilde{X})$, R_{AB} the pull-back to $\widetilde{X}$ of the Ricci tensor of $(\mathcal{M}, g)$, γ the first fundamental form of $\widetilde{X}$ and $R^{(\gamma)}_{AB}$ the Ricci tensor of $\widetilde{X}$. Then,*

$$
\frac{1}{\mu} N_1^\mu N_2^\nu R_{\mu\alpha\beta\nu}\left(v_A^\alpha v_B^\beta + v_A^\beta v_B^\alpha\right) =
$$

$$R^{(\gamma)}_{AB} - R_{AB} + \frac{\gamma^{IJ}}{\mu}\Big(-\mathbf{K}^{N_1}_{BA}\mathbf{K}^{N_2}_{JI} - \mathbf{K}^{N_1}_{JI}\mathbf{K}^{N_2}_{BA} + \mathbf{K}^{N_1}_{JA}\mathbf{K}^{N_2}_{BI} + \mathbf{K}^{N_1}_{BI}\mathbf{K}^{N_2}_{JA}\Big). \qquad (2.73)$$

In particular, if any of the second fundamental forms $\mathbf{K}^{N_1}$, $\mathbf{K}^{N_2}$ *is identically zero, it holds*

$$\frac{1}{\mu} N^{\mu}_1 N^{\nu}_2 R_{\mu\alpha\beta\nu}\Big(v^{\alpha}_A v^{\beta}_B + v^{\beta}_A v^{\alpha}_B\Big) = R^{(\gamma)}_{AB} - R_{AB}. \qquad (2.74)$$

Proof. Clearly $\{N_1, N_2, v_A\}$ constitutes a basis of $\Gamma(TM)|_X$, so we can decompose the contravariant metric $g^{\mu\nu}$ as

$$g^{\mu\nu} = \frac{1}{\mu}\Big(N^{\mu}_1 N^{\nu}_2 + N^{\nu}_1 N^{\mu}_2\Big) + \gamma^{IJ} v^{\mu}_I v^{\nu}_J, \qquad (2.75)$$

where γ^{IJ} are the components of the contravariant metric $\gamma^{\#}$ of X. Thus,

$$\begin{aligned} R_{AB} \stackrel{\text{def}}{=} R_{\alpha\beta} v^{\alpha}_A v^{\beta}_B = g^{\mu\nu} R_{\mu\alpha\nu\beta} v^{\alpha}_A v^{\beta}_B &= \Big(\gamma^{IJ} v^{\mu}_I v^{\nu}_J + \frac{1}{\mu}\Big(N^{\mu}_1 N^{\nu}_2 + N^{\nu}_1 N^{\mu}_2\Big)\Big) R_{\mu\alpha\nu\beta} v^{\alpha}_A v^{\beta}_B \\ &= \gamma^{IJ} R_{\mu\alpha\nu\beta} v^{\mu}_I v^{\alpha}_A v^{\nu}_J v^{\beta}_B - \frac{1}{\mu} N^{\mu}_1 N^{\nu}_2 R_{\mu\alpha\beta\nu}\Big(v^{\alpha}_A v^{\beta}_B + v^{\beta}_A v^{\alpha}_B\Big). \end{aligned} \qquad (2.76)$$

Now, equation (2.72) in index notation reads

$$R_{\mu\alpha\nu\beta} v^{\mu}_I v^{\alpha}_A v^{\nu}_J v^{\beta}_B = R^{(\gamma)}_{IAJB} + \frac{1}{\mu}\Big(-\mathbf{K}^{N_1}_{BA}\mathbf{K}^{N_2}_{JI} - \mathbf{K}^{N_1}_{JI}\mathbf{K}^{N_2}_{BA} + \mathbf{K}^{N_1}_{JA}\mathbf{K}^{N_2}_{BI} + \mathbf{K}^{N_1}_{BI}\mathbf{K}^{N_2}_{JA}\Big). \qquad (2.77)$$

Combining (2.76) and (2.77), the result (2.73) follows at once after noticing that $R^{(\gamma)}_{AB} \stackrel{\text{def}}{=} \gamma^{IJ} R^{(\gamma)}_{IAJB}$. The second part of the lemma is immediate from (2.73). □

The existence of two null normal vector fields also allows us to define, at any point $p \in X$, the so-called *torsion one-form* $s|_p : T_p X \longrightarrow \mathbb{R}$ by

$$s(X)|_p \stackrel{\text{def}}{=} -\frac{g(\nabla_X N_1, N_2)}{g(N_1, N_2)}\Big|_p \qquad \forall X \in T_p X. \qquad (2.78)$$

Clearly the torsion one-form depends on the two normal vector fields N_1 and N_2. However, for notational simplicity we refrain ourselves from explicitly showing this dependence on the notation. Observe that definition (2.78) is insensitive to the rescaling of N_2 by a non-zero function, while the change $N_1 \longrightarrow f N_1$ with $f \in \mathcal{F}^{\star}(X)$ induces

$$s(X) \longrightarrow s(X) - X(\ln f). \qquad (2.79)$$

As we shall see next, the notions of second fundamental form and torsion one-form can be generalized to tensor fields constructed from everywhere transverse (instead of normal) vector fields. In any case, they together with the first fundamental form codify the essential geometric information about a submanifold.

2.4 geometry of null hypersurfaces

As anticipated before, we devote this section to study the geometry of null hypersurfaces. We provide several definitions and results to be used later on. General references for the topic are [109], [98], [110].

Our starting point is the notion of embedded null hypersurface.

Definition 2.4.1. *(Embedded null hypersurface) Let* $(\mathcal{M}, g)$ *be an* $(n+1)$*-dimensional semi-Riemannian manifold and* $\mathcal{N}$ *a manifold of dimension* n*. An embedded null hypersurface is a subset* $\overline{\mathcal{N}} \subset \mathcal{M}$ *satisfying that there exists an embedding* $\phi : \mathcal{N} \longrightarrow \mathcal{M}$ *such that* $\phi(\mathcal{N}) = \overline{\mathcal{N}}$ *and that the first fundamental form* γ *of* $\mathcal{N}$*, defined by* $\gamma \stackrel{\text{def}}{=} \phi^{\star} g$*, is degenerate.*

We let $T_p\overline{\mathcal{N}}^{\perp}$ denote the space of vectors at $p \in \overline{\mathcal{N}}$ that are orthogonal to $\overline{\mathcal{N}}$. It is well-known (see e.g. [98], [110]) that there exists only one degenerate direction in $\overline{\mathcal{N}}$. This means that at any point $p \in \overline{\mathcal{N}}$ there exist a normal, non-zero vector field $k|_p \in T_p\overline{\mathcal{N}}$, i.e. satisfying

$$k|_p \neq 0, \qquad \langle k, X\rangle_g|_p = 0 \qquad \forall \; X|_p \in T_p\overline{\mathcal{N}}. \tag{2.80}$$

Thus, $k|_p \in T_p\overline{\mathcal{N}} \cap T_p\overline{\mathcal{N}}^{\perp}$. Since $\phi_\star$ is of maximal rank, the dimension of $T_p\overline{\mathcal{N}}$ is n so the dimension of $T_p\overline{\mathcal{N}}^{\perp}$ is 1. It follows that $T_p\overline{\mathcal{N}}^{\perp} \subset T_p\overline{\mathcal{N}}$ and hence $\langle k|_p\rangle = T_p\overline{\mathcal{N}}^{\perp}$ and all the vectors in $T_p\overline{\mathcal{N}}^{\perp}$ are null. We let $T\overline{\mathcal{N}}^{\perp} = \bigsqcup_p (T_p\overline{\mathcal{N}})^{\perp}$, and it is clear that this constitutes a subbundle to $T\overline{\mathcal{N}}$.

A *null generator* k of $\overline{\mathcal{N}}$ is defined to be a nowhere zero section $k \in \Gamma\left(T\overline{\mathcal{N}}^{\perp}\right)$. Since they are by construction non-zero, their integral curves are likewise referred to as *null generators of* $\overline{\mathcal{N}}$.

All null generators are necessarily proportional to each other. Moreover, they are geodesic (not necessarily affinely parametrized) vector fields (see e.g. equation (2.21) in [98]). In fact, any vector field η which is null and everywhere tangent to $\overline{\mathcal{N}}$ is necessarily geodesic, since for any $X \in \Gamma(T\overline{\mathcal{N}})$ it holds

$$\begin{aligned} g(X, \nabla_\eta \eta) &\stackrel{\overline{\mathcal{N}}}{=} \nabla_\eta \left(g\left(X, \eta\right)\right) - g(\nabla_\eta X, \eta) \stackrel{\overline{\mathcal{N}}}{=} -g(\pounds_\eta X, \eta) - g\left(\nabla_X \eta, \eta\right) \\ &\stackrel{\overline{\mathcal{N}}}{=} -g(\pounds_\eta X, \eta) - \frac{1}{2} X\left(g\left(\eta, \eta\right)\right) \stackrel{\overline{\mathcal{N}}}{=} -g(\pounds_\eta X, \eta) \stackrel{\overline{\mathcal{N}}}{=} 0, \end{aligned}$$

where in the last step we particularized (2.59) for $Y = \eta$ and $X = \mathcal{N}$. Thus, $\nabla_\eta \eta \in \Gamma\left(T\mathcal{N}^\perp\right)$, hence it is proportional to η itself, i.e. geodesic. Denoting the subset of zeroes of η on $\mathcal{N}$ by S, we can define a function $\kappa \in \mathcal{F}(\mathcal{N} \setminus S)$ (usually referred to as the *surface gravity* of η) as the proportionality function between $\nabla_\eta \eta$ and η, i.e.

$$\nabla_\eta \eta = \kappa \eta \qquad \text{on} \qquad \mathcal{N} \setminus S. \tag{2.81}$$

In particular, for a null generator k of $\mathcal{N}$ (which by definition is such that $S = \emptyset$), we define its surface gravity κ_k on the whole $\mathcal{N}$ as

$$\nabla_k k = \kappa_k k. \tag{2.82}$$

Any two null generators k, k' with respective surface gravities $\kappa_k, \kappa_{k'}$ are related by $k' = \alpha k$ where $\alpha \in \mathcal{F}^\star(\mathcal{N})$. Therefore, their surface gravities verify

$$\kappa_{k'} k' = \nabla_{k'} k' = (k(\alpha) + \alpha\kappa_k)k' \qquad \Longrightarrow \qquad \kappa_{k'} = k(\alpha) + \alpha\kappa_k. \tag{2.83}$$

It is natural to ask whether one can prescribe $\kappa_{k'}$ freely, i.e. if it is possible to find a k' with a specific surface gravity $\kappa_{k'}$ of our choice. Given any smooth function $\kappa_{k'}$ and having fixed k (and hence also κ_k) equation (2.83) is a linear inhomogeneous ODE for α along each generator. This equation can be solved globally along any generator. However, the resulting function α need not be smooth on $\mathcal{N}$. One possible reason is that generators can come very close to themselves (or even be dense in $\mathcal{N}$). Then, there is no reason why the value of α integrated along the curve at the infinitely close point should stay close to the value of α that we started with. Thus, generically equation (2.83) does not admit a globally well-defined smooth solution α unless some global hypotheses are made on $\mathcal{N}$. We shall have more to say on this later on.

An embedded null hypersurface $\mathcal{N}$ is a codimension-one degenerate submanifold of $\mathcal{M}$ to which any null generator k is normal. Therefore, one can define the second fundamental form $\mathbf{K}^k$ of $\mathcal{N}$ with respect to k according to (2.64), i.e.

$$\mathbf{K}^k \stackrel{\text{def}}{=} g(\nabla_X k, Y) \qquad \forall X, Y \in \Gamma(T\mathcal{N}). \tag{2.84}$$

Then, equation (2.65) transforms into

$$(\pounds_k g)(X, Y) \stackrel{\mathcal{N}}{=} 2\mathbf{K}^k(X, Y) \qquad \forall X, Y \in \Gamma(T\mathcal{N}). \tag{2.85}$$

Although we have seen that equation (2.85) is satisfied for general submanifolds admitting a normal vector field, in the context of null hypersurfaces this fact becomes even more relevant. The underlying reason is that null generators are tangent to the hypersurface everywhere, which makes (2.85) intrinsic to $\widetilde{\mathcal{N}}$. More concretely, using the well-known property

$$\phi^\star \left(\pounds_{\phi_\star Z} T \right) = \pounds_Z(\phi^\star T), \tag{2.86}$$

valid for any vector field $Z \in \Gamma(T\mathcal{N})$ and any covariant tensor field T along $\widetilde{\mathcal{N}}$, one can compute the pull-back of (2.85) to $\mathcal{N}$ and obtain

$$\pounds_n \gamma \overset{\mathcal{N}}{=} 2\mathbf{K}^{k}, \qquad \text{where} \qquad \phi_\star n \overset{\text{def}}{=} k, \quad \text{and} \quad \mathbf{K}^{k} \overset{\text{def}}{=} \phi^\star \mathrm{K}^{k}. \tag{2.87}$$

Remark 2.4.2. *In what follows, given an* $(n+1)$*-dimensional semi-Riemannian manifold* $(\mathcal{M}, g)$ *and a vector field* $X \in \Gamma(T\mathcal{M})$*, we shall use the notation* $\boldsymbol{X} \overset{\text{def}}{=} g(X, \cdot)$*. For instance, we let* $\boldsymbol{k} \overset{\text{def}}{=} g(k, \cdot)$*.*

Observe that $\mathrm{K}^{k} = \phi^\star(\nabla \boldsymbol{k})$. More details on the second fundamental form K^{k} will be provided later in Section 2.4.1. However, for the time being it is worth stressing that (2.87) is a well-known relation (see e.g. [98]) between the rate of change of the first fundamental form of $\widetilde{\mathcal{N}}$ along the degenerate direction and the second fundamental form of $\widetilde{\mathcal{N}}$.

One of the main purposes of this thesis is to address the problem of matching two spacetimes across a null hypesurface. In this context $\widetilde{\mathcal{N}}$ will be two-sided and $\mathcal{M}$ will be oriented and Lorentzian, so for simplicity we take this assumptions for the rest of this section. In these circumstances, γ is semi-positive definite (which in particular means that all non-null directions of $\widetilde{\mathcal{N}}$ are spacelike) and $\widetilde{\mathcal{N}}$ always admits (see Lemma 1 in [58]) an everywhere transversal vector field L_0, i.e. satisfying $L_0 \notin T_p\widetilde{\mathcal{N}}\ \forall p \in \widetilde{\mathcal{N}}$.

We now introduce the notions of transverse submanifold, cross-section and foliation of a null hypersurface.

Definition 2.4.3. *(Transverse submanifold, cross-section) Let* $\widetilde{\mathcal{N}}$ *be an embedded null hypersurface and* k *a choice of null generator. A transverse submanifold of* $\widetilde{\mathcal{N}}$ *is any submanifold* $S \subset \widetilde{\mathcal{N}}$ *to which* k *is everywhere transverse. If, in addition, each integral curve of* k *crosses* S *exactly once then* S *is a cross-section.*

Definition 2.4.4. *(Foliation of a null hypersurface) Let* $\widetilde{\mathcal{N}}$ *be an embedded null hypersurface and* k *a choice of null generator. Assume further that there exists a function* $\lambda \in \mathcal{F}(\widetilde{\mathcal{N}})$*,* *called foliation function, such that each subset* $S_{\lambda_0} \stackrel{\text{def}}{=} \{p \in \mathcal{N} \mid \lambda(p) = \lambda_0 \in \mathbb{R}\}$ *defines a cross-section of* $\mathcal{N}$*. Then, the family of cross-sections* $\{S_\lambda\}$ *define a foliation of* $\mathcal{N}$*.*

Remark 2.4.5. *Any null generator is not only transverse to a given a transverse submanifold* $S \subset \mathcal{N}$ *but also normal to it. This means that one can define the corresponding second fundamental form of* S *with respect to* k *according to* (2.64)*. This tensor field is simply a restriction to* S *(and to vector fields tangent to* S*) of* (2.84)*, so in the following we denote the second fundamental forms of* $\mathcal{N}$ *and* S *with the same symbol, namely* $\mathbf{K}_k$ *.*

Remark 2.4.6. *The existence of a cross-section on* $\mathcal{N}$ *is a non-trivial global assumption on the hypersurface.*

Remark 2.4.7. *In what follows, we shall use the names cross-section and section indistinctly.*

A remarkable property of null hypersurfaces admitting a section is that one can always find an affine null generator (see e.g. [98]). Indeed, given a choice of null generator k, one can always construct another null generator $k' = \alpha k$ by solving the first-order differential equation $k(\ln|\alpha|) = -\kappa_k$ for $\alpha \in \mathcal{F}^\star(\mathcal{N})$ along the integral curves of k. For that it suffices to provide nowhere vanishing initial data $\alpha|_S$ on S. In this case, there are no obstructions in solving globally this infinite collection of ODEs. In particular, note that now no generator can come close to itself because of the presence of a section. Once α and k' have been constructed, it follows at once that $\kappa_{k'} = 0$ as a direct consequence of (2.83).

Remark 2.4.8. *If* $\mathcal{N}$ *admits a cross-section then any null vector field* η *which is tangent to* $\mathcal{N}$ *can be written as*

$$\eta \stackrel{\mathcal{N}}{=} \alpha k, \tag{2.88}$$

for an affine null generator k *of* $\mathcal{N}$ *and a function* $\alpha \in \mathcal{F}(\mathcal{N})$*. Defining* $\mathcal{S} \stackrel{\text{def}}{=} \{p \in \mathcal{N} \mid \alpha(p) = 0\}$ *and particularizing* (2.83) *for* $k' = \eta$*,* $\kappa_k = 0$*, one obtains*

$$\kappa \stackrel{\mathcal{N}\setminus\mathcal{S}}{=} k(\alpha), \tag{2.89}$$

where κ *is the surface gravity of* η*. If, in addition,* κ *is constant along the null generators of* $\mathcal{N}\setminus\mathcal{S}$*, then the general solution of* (2.89) *is* $\alpha = f + \kappa v$*, where the integration function* $f \in \mathcal{F}(\mathcal{N}\setminus\mathcal{S})$ *satisfies* $k(f) = 0$ *and* v *is any scalar function on* $\mathcal{N}\setminus\mathcal{S}$ *verifying* $k(v) = 1$*. Summarizing, in these circumstances* η *will be given by*

$$\eta \stackrel{\mathcal{N}\setminus\mathcal{S}}{=} (f + \kappa v)\, k, \quad \text{where} \quad k(f) = 0, \quad k(v) = 1, \quad \nabla_k k = 0, \quad k(\kappa) = 0. \tag{2.90}$$

Observe that in the case when $\mathcal{N}\setminus S$ is dense in $\mathcal{N}$, both η and k being well defined along $\mathcal{N}$ means that $\eta = (f + \kappa v)\, k$ everywhere on $\mathcal{N}$.

Whenever $\mathcal{N}$ can be foliated by spacelike sections $\{S_\lambda\}$, the fact that k is transverse to any leaf S_λ implies that $k(\lambda) \neq 0$ everywhere on $\mathcal{N}$. Observe that in general the global existence of λ in Definition 2.4.4 entails a strong topological restriction upon $\mathcal{N}$. We emphasize, however, that the existence of a foliation function can always be granted in sufficiently local domains of $\mathcal{N}$.

2.4.1 *Quotient space in a null hypersurface*

As already mentioned, the fact that the first fundamental form of a null hypersurface is degenerate represents a geometric difficulty, as one can define neither a metric on $\mathcal{N}$ nor its inverse tensor. Therefore, there is no natural way of raising and lowering indices of tensors on $\mathcal{N}$. The standard way of dealing with this difficulty is to introduce a quotient structure (see e.g. [109]) by defining for any $Z, W \in T_p\mathcal{N}$ the equivalence relation $\sim$ as

$$Z \sim W \quad \stackrel{\text{def}}{\Longleftarrow} \quad Z - W = ak, \tag{2.91}$$

where $a \in \mathbb{R}$.

Definition 2.4.9. *Let $\mathcal{N}$ be an embedded null hypersurface and $p \in \mathcal{N}$. Then the quotient vector space $T_p\mathcal{N}/k$ is defined as*

$$T_p\mathcal{N}/k \stackrel{\text{def}}{=} \left\{ \overline{Z} : Z \in T_p\mathcal{N} \right\}, \tag{2.92}$$

where $\overline{Z} \stackrel{\text{def}}{=} \left\{ X \in T_p\mathcal{N} : X \sim Z \right\}$. The fiber bundle on $\mathcal{N}$ is the collection of all quotient spaces

$$T\mathcal{N}/k \stackrel{\text{def}}{=} \bigcup_{p\in\mathcal{N}} T_p\mathcal{N}/k. \tag{2.93}$$

This quotient structure of $T_p\mathcal{N}$ allows to construct a metric $\hat{h}$ and a symmetric 2-covariant tensor $\hat{K}^k$ (closely related to the second fundamental form of $\mathcal{N}$) on the quotient bundle. The metric, denoted by h, is the symmetric 2-covariant tensor defined by

$$\hat{h}\left(\overline{Z},\overline{W}\right)\big|_p \stackrel{\text{def}}{=} g(Z,W)\big|_p, \tag{2.94}$$

where $\overline{Z},\overline{W} \in T_p\mathcal{N}/k$. The tensor is well-defined because the right-hand side is independent of the representatives $Z \in \overline{Z}, W \in \overline{W}$. Besides, for any non-zero $\overline{Y} \in$

$T_p\mathcal{N}/k$ (i.e. classes associated to spacelike directions Y), it holds that $h\left(\overline{Y},\overline{Y}\right)|_p = \langle Y, Y\rangle_g|_p > 0$. Thus, h is a positive definite metric.

At any transverse hypersurface $S \subset \mathcal{N}$ (in particular if S were a cross section), h is isometric to the induced metric h of S at any point $p_0 \in S$. Indeed, the map $T_{p_0}S \longrightarrow T_{p_0}\mathcal{N}/k$ defined by $X \longmapsto \overline{X}$ is an isomorphism[1] and for any two vectors $Z, W \in T_{p_0}S$ it holds

$$h\left(\overline{Z},\overline{W}\right)|_{p_0} = \langle Z + ak, W + bk\rangle_g|_{p_0} = \langle Z, W\rangle_g|_{p_0} \equiv h\,(Z, W)\,|_{p_0}. \tag{2.95}$$

Thus, h is also positive definite and we denote by $h^{\#}$ its associated contravariant metric. Their components in a basis $\{v_I|_{p_0}\}$ of $T_{p_0}S_0$ and its corresponding dual $\{\boldsymbol{\theta}^I|_{p_0}\}$ will be denoted by h_{IJ} and h^{IJ} respectively. We will use these tensors to lower and raise capital indices, irrespectively of whether they are tensorial or identify elements in a basis.

As already mentioned, one can also define the so-called *quotient second fundamental form* with respect to k, denoted by $\overline{\mathbf{K}}^k$, as the 2-covariant tensor on $T_p\mathcal{N}/k$ given by

$$\overline{\mathbf{K}}^k\left(\overline{Z},\overline{W}\right)|_p \stackrel{\text{def}}{=} g(\nabla_Z k, W)|_p, \qquad \overline{Z},\overline{W} \in T_p\mathcal{N}/k. \tag{2.96}$$

As before, this tensor is well-defined in the sense that it is independent of the choice of representatives. In fact, given $a, b \in \mathbb{R}$ it holds

$$\begin{aligned}
\overline{\mathbf{K}}^k\left(\overline{Z+ak},\overline{W+bk}\right)|_p &= \langle\nabla_{Z+ak}k, W + bk\rangle_g|_p \\
&= \left(\langle\nabla_Z k, W\rangle_g + b\langle\nabla_Z k, k\rangle_g + a\langle\nabla_k k, W\rangle_g + ab\langle\nabla_k k, k\rangle_g\right)\big|_p \\
&= \langle\nabla_Z k, W\rangle_g = \overline{\mathbf{K}}^k\left(\overline{Z},\overline{W}\right)|_p,
\end{aligned}$$

where we have used that $2\langle\nabla_Z k, k\rangle_g = Z(\langle k, k\rangle_g) = 0$ and (2.82).

It is immediate from definitions (2.84) and (2.96) that $\overline{\mathbf{K}}^k\left(\overline{Z},\overline{W}\right)|_p \stackrel{\text{def}}{=} \mathbf{K}^k(Z, W)|_p$. Besides, it is straightforward to conclude that the quotient tensor $\overline{\mathbf{K}}^k$ and the second fundamental form $\mathbf{K}^k$ of S with respect to the normal k (recall the notational considerations in Remark 2.4.5) obey the relation

$$\overline{\mathbf{K}}^k\left(\overline{Z},\overline{W}\right)|_{p_0} = \mathbf{K}^k\,(Z, W)\,|_{p_0} \qquad \forall Z, W \in T_{p_0}S. \tag{2.97}$$

[1] Both spaces have the same dimension and the kernel is obviously $\{0\}$.

Observe that, as a direct consequence of (2.65), when there exists a transverse hypersurface S of $\mathcal{N}$, any two vector fields $Z, W \in \Gamma(TS)$ which are extended off S by requiring $[k, Z] = [k, W] = 0$ satisfy

$$k\left(g\left(Z, W\right)\right) \stackrel{S_0}{=} 2\mathbf{K}^k\left(Z, W\right). \tag{2.98}$$

As exposed in Section 2.3, the fact that $S \subset \mathcal{M}$ is a codimension-two transverse submanifold allows us to select another linearly independent normal direction and define their corresponding second fundamental form and torsion one-form. We also mentioned that these two tensors could in fact be generalized for arbitrary transverse (non-necessarily normal) directions. This is more convenient for the purposes of this thesis, so we define, for any choice of vector field L everywhere transverse to $\mathcal{N}$ (which always exists, as we already discussed), the 2-covariant tensor $\boldsymbol{\Theta}^L$ and the covector $\boldsymbol{\sigma}_L$ at $p \in S$ by

$$\boldsymbol{\Theta}^L\left(Z, W\right)|_p \stackrel{\mathrm{def}}{=} \langle \nabla_Z L, W\rangle_g|_p, \qquad \boldsymbol{\sigma}_L\left(Z\right)|_p \stackrel{\mathrm{def}}{=} -\frac{1}{g(L,k)}\langle \nabla_Z k, L\rangle_g|_p. \tag{2.99}$$

Had we chosen L to be null and orthogonal to S then $\boldsymbol{\Theta}^L$ and $\boldsymbol{\sigma}_L$ would be the second fundamental form of S with respect to L and the torsion one-form according to (2.64) and (2.78). However, for our purposes on this thesis it is convenient to allow L to be unrelated to the sections. In the general case $\boldsymbol{\sigma}_L$ and $\boldsymbol{\Theta}^L$ are generalizations of those tensors and still encode extrinsic information of the sections. However, we emphasize that $\boldsymbol{\Theta}^L$ is *not* symmetric in general.

2.4.2 *Raychaudhuri equation on a null hypersurface*

A fundamental result of null hypersurfaces is the so-called Raychaudhuri equation (see e.g. [98], [111]). In Chapter 4, we shall prove that for abstract hypersurfaces it is possible to define a tensor field (which we call *constraint tensor*) which in the null, embedded case coincides with the pull-back of the ambient Ricci tensor. Among many other results, we will recover the Raychaudhuri equation at a purely abstract level. In order to compare such abstract identity with the standard Raychaudhuri equation, we recall the latter here.

Let $(\mathcal{M}, g)$ be a semi-Riemannian manifold endowed with a metric g of Lorentzian signature. Consider an embedded null hypersurface $\mathcal{N} \subset \mathcal{M}$. Then, there always exists a suitable neighbourhood $\mathcal{O}$ of $\mathcal{N}$ which can be foliated by null hypersurfaces $\{\mathcal{N}_u\}$. We take a null generator k of $\mathcal{N}$ and extend it to $\mathcal{O}$ so that it is a null generator of all leaves $\{\mathcal{N}_u\}$. This vector field k hence verifies $\nabla_k k = \kappa_k k$ on each N_u (cf. (2.82)). It also defines a null congruence on $\mathcal{O}$ and, since $\boldsymbol{k} \stackrel{\text{def}}{=} g(k, \cdot)$ is normal to all leaves $\{N_u\}$, it satisfies the irrotationality condition

$$k_{[\mu}\nabla_\nu k_{\alpha]} = 0 \tag{2.100}$$

on $\mathcal{O}$. Now we let $L \in \Gamma(T\mathcal{O})$ be a null vector satisfying $g(L, k) = 1$. We define the projector $\Pi \stackrel{\text{def}}{=} g - 2\boldsymbol{L} \otimes_s \boldsymbol{k}$ to each leaf N_u. Observe that Π is symmetric and satisfies $\Pi(k, \cdot) = 0$, $\Pi(L, \cdot) = 0$ everywhere on $\mathcal{O}$. In these circumstances, it is a general fact that the derivative $\nabla_\mu k_\nu$ can be decomposed as [111]

$$\nabla_\mu k_\nu = q_{\mu\nu} + F_\mu k_\nu + k_\mu a_\nu, \qquad \text{where} \tag{2.101}$$
$$q_{\mu\nu} \stackrel{\text{def}}{=} \Pi_\mu{}^\alpha \Pi_\nu{}^\beta \nabla_\alpha k_\beta, \qquad F_\mu \stackrel{\text{def}}{=} \Pi_\mu{}^\alpha L^\beta \nabla_\alpha k_\beta + \kappa_k L_\mu, \qquad a_\mu \stackrel{\text{def}}{=} L^\beta(\nabla_\beta k_\mu - L_\mu k^\alpha \nabla_\beta k_\alpha).$$

Note that $k^\mu F_\mu = \kappa_k$ and $k^\mu a_\mu = k^\mu L^\beta(\nabla_\beta k_\mu - L_\mu k^\alpha \nabla_\beta k_\alpha) = 0$ because k is null everywhere on $\mathcal{O}$. We can decompose the tensor field $q_{\mu\nu}$ into its symmetric and antisymmetric parts $\varsigma_{\mu\nu}$ and $\omega_{\mu\nu}$, i.e. $q_{\mu\nu} = \varsigma_{\mu\nu} + \omega_{\mu\nu}$. Inserting this decomposition into (2.100) yields $k_\mu\omega_{\nu\alpha} + k_\nu\omega_{\alpha\mu} + k_\alpha\omega_{\mu\nu} = 0$, which upon contracting with L^μ gives $\omega_{\nu\alpha} = 0$ as a consequence of $2\omega_{\mu\nu}L^\mu = (q_{\mu\nu} - q_{\nu\mu})L^\mu = 0$. Thus, $q_{\mu\nu}$ is symmetric. This property is just a manifestation of the fact that the second fundamental forms of the leaves N_u are all symmetric.

The (now symmetric) tensor $q_{\mu\nu}$ can also be separated into a trace part and a traceless part with respect to $\Pi_\mu{}^\nu$ as

$$q_{\mu\nu} = \varsigma_{\mu\nu} + \frac{\theta}{n-1}\Pi_{\mu\nu}, \tag{2.102}$$

where $\theta \stackrel{\text{def}}{=} \Pi^{\mu\nu} q_{\mu\nu}$ or, more explicitly, $\theta = \Pi^{\alpha\beta}\nabla_\alpha k_\beta = \nabla_\alpha k^\alpha - \kappa_k$, where we again used that $g(k, k)|_{\mathcal{O}} = 0$. Now the Ricci identity for k yields

$$\nabla_\lambda\nabla_\mu k_\nu - \nabla_\mu\nabla_\lambda k_\nu = R^\alpha{}_{\nu\mu\lambda} k_\alpha \quad \Longrightarrow \quad \nabla_\lambda\nabla_\mu k^\lambda - \nabla_\mu\nabla_\lambda k^\lambda = R^\lambda{}_{\alpha\lambda\mu} k^\alpha = R_{\alpha\mu} k^\alpha.$$

Contracting with k^μ and using (2.101) twice and $\Pi^{\mu\nu}\varsigma_{\mu\nu} = 0$, one gets

$$\begin{aligned}
R_{\alpha\mu}k^\alpha k^\mu &= k^\mu\nabla_\lambda\nabla_\mu k^\lambda - k^\mu\nabla_\mu\nabla_\lambda k^\lambda \\
&= \nabla_\lambda(k^\mu\nabla_\mu k^\lambda) - (\nabla_\lambda k^\mu)(\nabla_\mu k^\lambda) - k^\mu\nabla_\mu\nabla_\lambda k^\lambda \\
&= \nabla_\lambda\left(k^\mu F_\mu k^\lambda\right) - (\nabla_\lambda k^\mu)\left(\varsigma_\mu{}^\lambda + \frac{\theta}{n-1}\Pi_\mu{}^\lambda + F_\mu k^\lambda + k_\mu a^\lambda\right) - k(\theta + \kappa_k) \\
&= \kappa_k(\theta + \kappa_k) - \left(\varsigma^2 + \frac{\theta^2}{(n-1)^2}\Pi_\mu{}^\lambda\Pi_\lambda{}^\mu + \kappa_k^2\right) - k(\theta) \\
&= \kappa_k\theta - \varsigma^2 - \frac{\theta^2}{(n-1)} - k(\theta),
\end{aligned}$$

where we have defined $\varsigma^2 \stackrel{\text{def}}{=} \varsigma^{\mu\nu}\varsigma_{\mu\nu}$ and used that $\Pi^{\alpha\beta}\Pi_{\alpha\beta} = n-1$, $\varsigma^{\alpha\beta}\Pi_{\alpha\beta} = 0$. This gives the well-known Raychaudhuri equation

$$k(\theta) - \kappa_k\theta + \frac{\theta^2}{(n-1)} + \varsigma^2 + R(k,k) = 0, \tag{2.103}$$

where θ, $\varsigma_{\mu\nu}$ are the so-called expansion scalar and shear tensor respectively. In particular, this equation holds on $\mathcal{N}$. It is worth stressing that, although θ and ς have been defined by means of an extension of k off $\mathcal{N}$, these objects are in fact intrinsic to the null hypersurface.

2.4.3 *Totally geodesic null hypersurfaces*

For later use in the thesis and specially for the discussion on Killing and non-expanding horizons that we perform in the next two sections, it is convenient to revisit briefly some aspects of totally geodesic null hypersurfaces. We first provide the precise definition.

Definition 2.4.10. *(Totally geodesic null hypersurface) Let* $(\mathcal{M}, g)$ *be a semi-Riemannian manifold. An embedded null hypersurface* $\mathcal{N} \subset \mathcal{M}$ *with vanishing second fundamental form with respect to a null generator* k *of* $\mathcal{N}$ *is a totally geodesic null hypersurface.*

By Definition 2.4.10, a totally geodesic null hypersurface $\mathcal{N}$ satisfies (recall (2.84))

$$\mathbf{K}^k \stackrel{\text{def}}{=} g(\nabla_X k, Y) = 0 \qquad \forall X, Y \in \Gamma(T\mathcal{N}). \tag{2.104}$$

Even more, any null (not necessarily everywhere non-zero) vector field η which is tangent to $\mathcal{N}$ verifies

$$g(\nabla_X \eta, Y) = 0 \qquad \forall X, Y \in \Gamma(T\mathcal{N}) \tag{2.105}$$

as an immediate consequence of (2.104) and η being proportional to k on $\mathcal{N}$. Thus, the vector field $\nabla_X \eta|_{\mathcal{N}}$ is null and tangent to $\mathcal{N}$. Denoting again by $\mathcal{S}$ the set of points of $\mathcal{N}$ where η vanishes, we can define a one-form ϖ on $\mathcal{N} \setminus \mathcal{S}$ by

$$\nabla_X \eta \stackrel{\mathcal{N}\setminus\mathcal{S}}{=} \varpi(X)\eta \qquad \forall X \in \Gamma(T\mathcal{N} \setminus \mathcal{S}). \tag{2.106}$$

Observe that ϖ is only univocally defined when acting on tangential vectors to $\mathcal{N} \setminus \mathcal{S}$. This means that, as a spacetime one-form defined along $\mathcal{N} \setminus \mathcal{S}$ it is not uniquely defined. One can always add a multiple of the normal one-form, i.e.

$\varpi' = \varpi + f\boldsymbol{\eta}$, $f \in \mathcal{F}(\mathcal{N}\setminus \mathcal{S})$, which also verifies (2.106). Imposing $X = \eta$ in (2.106) yields

$$\varpi(\eta) \stackrel{\mathcal{N}\setminus\mathcal{S}}{=} \kappa, \tag{2.107}$$

so the surface gravity of η is directly related to ϖ.

In particular, $\mathcal{S} = \emptyset$ in the case of a null generator k of $\widetilde{\mathcal{N}}$, so its corresponding one-form ϖ is defined by (2.106) on the whole $\widetilde{\mathcal{N}}$, while its surface gravity is given by (2.107).

The formalism of hypersurface data can be applied, in particular, to the study of totally geodesic null hypersurfaces. Since clearly the one-form ϖ constitutes a fundamental object in such context, it is convenient to provide the connection of ϖ with the tensor fields of the hypersurface data. This is done in the next remark.

Remark 2.4.11. *Let us define hypersurface data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ embedded in $(\mathcal{M}, g)$ with embedding ϕ and rigging ζ (see (2.22), (2.39)) so that $\phi(\mathcal{N}) = \widetilde{\mathcal{N}}$. Then, the fact that $\widetilde{\mathcal{N}}$ is everywhere null means that $\mathcal{N}$ consists of null points exclusively. Therefore, $\phi_\star n$ is a null generator of $\widetilde{\mathcal{N}}$ and $n^{(2)} = 0$ on the whole $\mathcal{N}$ (recall (2.6)-(2.9)).*

In these circumstances, any null vector field η which is tangent to $\widetilde{\mathcal{N}}$ can be written as $\eta = \alpha\phi_\star n$ for a function $\alpha \in \mathcal{F}(\widetilde{\mathcal{N}})$ that vanishes only at the zero set $\mathcal{S}$ of η. It turns out that the pull-back $\phi^\star\varpi$ can be expressed in terms of the one-forms $\boldsymbol{s}$, $\boldsymbol{r}$ (defined by (2.11), (2.44) respectively) and $d\alpha$. Indeed, combining (2.20), (2.25), (2.48) and (2.106) it follows

$$\varpi(X)\eta \stackrel{\mathcal{N}\setminus\mathcal{S}}{=} \nabla_X \eta \stackrel{\mathcal{N}\setminus\mathcal{S}}{=} \phi_\star\left(\overset{\circ}{\nabla}_X(\alpha n) - \alpha\boldsymbol{r}(X)n\right) \stackrel{\mathcal{N}\setminus\mathcal{S}}{=} \left(X(\alpha) + \alpha\left(\boldsymbol{s}(X) - \boldsymbol{r}(X)\right)\right)\phi_\star n$$

because $\widetilde{\mathcal{N}}$ (or $\mathcal{N}$) is totally geodesic, which means that the second fundamental form $\mathbf{K}$, defined by (2.45), vanishes (and hence so does $\mathbf{U}$). Thus,

$$\phi^\star\varpi = \frac{d\alpha}{\alpha} + \boldsymbol{s} - \boldsymbol{r} = d(\ln|\alpha|) + \boldsymbol{s} - \boldsymbol{r} \tag{2.108}$$

at those points of $\mathcal{N}$ where $\alpha \neq 0$. Observe that, although ϖ is not univocally defined, its pull-back to $\mathcal{N}$ is unique, as $\phi^\star\boldsymbol{\eta} = 0$.

The reason why the one-form ϖ cannot defined everywhere on $\widetilde{\mathcal{N}}$ is because at points where η vanishes the right part of (2.106) vanishes while the derivative $\nabla_X\eta$ does not need to be zero in general. In fact, from (2.108) it follows that ϖ must diverge at points where $\alpha = 0$.

Another remarkable property of the one-form ϖ associated to a null generator k of $\widetilde{\mathcal{N}}$ is that it is related to the tensor field $\boldsymbol{\sigma}_L$ defined in (2.99). In fact, given a transverse submanifold $S \subset \overline{\mathcal{N}}$ and a vector field L along $\overline{\mathcal{N}}$ which is everywhere transverse to it, (2.106) entails

$$\varpi(X) \overset{S}{=} \frac{1}{g(L,k)} g(\nabla_X k, L) \overset{\text{def}}{=} -\boldsymbol{\sigma}_L(X) \qquad \forall X \in \Gamma(TS). \tag{2.109}$$

We conclude the section with a fundamental result of totally geodesic null hypersurfaces, namely that it is possible to define a covariant derivative operator $\overline{\nabla}$ on $\overline{\mathcal{N}}$ by

$$\nabla_X Y = \overline{\nabla}_X Y, \qquad \forall X, Y \in \Gamma(T\overline{\mathcal{N}}). \tag{2.110}$$

The derivative $\overline{\nabla}$ is the induced connection on $\overline{\mathcal{N}}$ and it is obviously torsion-free and intrinsic to the horizon. Observe that it is automatically compatible with the (degenerate) first fundamental form γ of $\overline{\mathcal{N}}$, i.e. $\overline{\nabla}_X \gamma = 0$ for all $X \in \Gamma(T\overline{\mathcal{N}})$.

Remark 2.4.12. *It is consistent to use the same symbol for the derivative* $\overline{\nabla}$ *introduced in* (2.110) *and for the hypersurface connection (recall* (2.49)-(2.50)*). Indeed, as we have seen in Remark 2.4.11, the embedded hypersurface data corresponding to a totally geodesic null hypersurface is such that the tensors* $\mathbf{U}$ *and* $\mathbf{K}$ *defined in* (2.12) *and* (2.45) *are both zero. In these circumstances,* (2.50) *becomes* (2.110)*.*

One of the most common examples of totally geodesic null hypersurfaces are non-expanding, (weakly) isolated and Killing horizons. All these sort of horizons will be studied in detail in the next sections, so that the results above will become helpful by then.

2.5 non-expanding, weakly isolated and isolated horizons

Among all types of horizons that have been studied in the literature, isolated horizons have been proven to be of great use in many interesting physical situations. These horizons are introduced in General Relativity with the idea of being able to mimic some properties of an event horizon of a black hole but without requiring the existence of a Killing vector field in a neighbourhood of the hypersurface. Since the concept of isolated horizon is less restrictive than that of Killing horizon, it makes sense to construct the former by successively adding conditions to a minimally restricted notion of horizon. This process gives rise to non-expanding horizons and weakly isolated horizons.

In this section, we revisit several properties of all these horizons. We also obtain the so-called *master equation*, i.e. an identity involving a second fundamental form, curvature terms and the torsion one-form of a section. This equation is known to hold for isolated horizons and for multiple Killing horizons (we discuss this in Section 2.6.1). In Chapter 6, we will generalize this equation to any null hypersurface admitting an extra null and tangent vector field. For further details on non-expanding and (weakly) isolated horizons, we refer to [95], [96], [97], [1], [98], [99] and references therein.

We start with the concept of non-expanding horizon as originally presented in [1].

Definition 2.5.1. *(Non-expanding horizon) Let* $\mathcal{N}$ *be a null hypersurface embedded on an* $(n+1)$*-dimensional spacetime* $(\mathcal{M}, g)$*. Then,* $\mathcal{N}$ *is a non-expanding horizon if*

(i) $\mathcal{N}$ *is diffeomorphic to* $\mathbb{S}^{n-1} \times \mathbb{R}$*, where* $\mathbb{S}^{n-1}$ *is the* $(n-1)$*-sphere and the null generators of* $\mathcal{N}$ *are along* $\mathbb{R}$*.*

(ii) The expansion of any null generator of $\mathcal{N}$ *vanishes.*

(iii) The Einstein field equations hold on $\mathcal{N}$ *and* $-T^{\alpha}{}_{\beta}k^{\beta}$*, where* $T_{\alpha\beta}$ *is the energy-momentum tensor of* $(\mathcal{M}, g)$*, is causal and future on* $\mathcal{N}$ *for whatever choice of future null generator* k *of* $\mathcal{N}$*.*

Observe that Definition 2.5.1 does not depend on the choice of future null generator k.

We next summarize some implications of Definition 2.5.1. The first three are presented in separate remarks, for which we also provide proof.

Remark 2.5.2. *A non-expanding horizon* $\mathcal{N}$ *is shear-free.*

Proof. For any future null generator k, the Raychaudhuri equation (2.103) for a null congruence without twist (and with no expansion, cf. (*ii*)) reads $\varsigma_{\alpha\beta}\varsigma^{\alpha\beta} + T(k,k) = 0$ after using the Einstein equations (note that the cosmological constant is allowed to take any value). Now, $T(k,k) \geq 0$ necessarily because of (*iii*), while $\varsigma_{\alpha\beta}\varsigma^{\alpha\beta} \geq 0$ because the induced metric on the sections is positive-definite. Thus, both terms in the Raychaudhuri equation must vanish and hence $\boldsymbol{\varsigma} = 0$. □

Remark 2.5.3. *A non-expanding horizon* $\mathcal{N}$ *is totally geodesic.*

Proof. The shear $\boldsymbol{\varsigma}$, the twist $\boldsymbol{\omega}$ and the expansion θ are zero in the present case, so the tensor field $q_{\mu\nu}$ defined by (2.101) is identically zero. This immediately proves the claim. □

Remark 2.5.4. *There exists a function* $\mathfrak{G} \in \mathcal{F}(\mathcal{N})$ *such that* $\boldsymbol{Ric}(k,\cdot) = \mathfrak{G}k$ *on* $\mathcal{N}$*. In particular,* $\boldsymbol{Ric}(k,X)|_{\mathcal{N}} = 0$ *for any* $X \in \Gamma(T\mathcal{N})$*.*

Proof. By the Einstein equations $\mathbf{Ric}(k,\cdot)$ and $T(k,\cdot)$ differ by a vector proportional to k. Since by hypothesis $-T(k,\cdot)$ is causal future and perpendicular to k (because $T(k,k) = 0$, as already shown in the previous remark), the only possibility is that $T(k,\cdot)$ is proportional to k and the result follows. □

Observe that $\mathcal{N}$ being totally geodesic means that all results in Section 2.4.3 apply, as we anticipated therein. In particular, for each null generator k there exists a one-form ϖ along $\mathcal{N}$ defined by (2.106) and the surface gravity of k is given by (2.107). We also have the derivative operator $\overline{\nabla}$ introduced in (2.110) at hand. We continue with the definition of weakly isolated horizon.

Definition 2.5.5. *(Weakly isolated horizon) A weakly isolated horizon is a non-expanding horizon* $\mathcal{N}$ *endowed with a null generator* η *such that*

$$(\pounds_\eta \varpi)(X) \stackrel{\mathcal{N}}{=} 0, \qquad \forall X \in \Gamma(T\mathcal{N}), \tag{2.111}$$

where ϖ *is a one-form field along* $\mathcal{N}$ *satisfying* (2.106).

Given a non-expanding horizon $\mathcal{N}$, the fact that η is null and normal along H allows one to define its associated surface gravity $\kappa \in \mathcal{F}(\mathcal{N})$ according to (2.81). We also let $\iota : \mathcal{N} \hookrightarrow \mathcal{M}$ be the embedding of $\mathcal{N}$ into $\mathcal{M}$ such that $\iota(\mathcal{N}) = \mathcal{N}$. Condition (i) in Definition 2.5.1 means that any non-expanding horizon admits a foliation by spacelike cross-sections. Thus, one can always select one such cross-section $S \subset \mathcal{N}$ and let ψ, h be the corresponding embedding $\psi : S \hookrightarrow \mathcal{N}$ of S into $\mathcal{N}$ and the induced metric on S respectively. For simplicity, we identify S, $X \in \Gamma(TS)$ with their counterparts $\psi(S)$, $\psi_\star X$. We also construct a foliation function v by fixing $v|_S$ and solving the equation $\eta(v)|_{\mathcal{N}} = 1$. In these circumstances, the leaves $\{S_v\}$ of the foliation are defined by $S_{v_0} \stackrel{\text{def}}{=} \{p \in \mathcal{N} \mid v(p) = v_0 \in \mathbb{R}\}$.
Any given vector field $X \in \Gamma(TS)$ can be extended uniquely to $\mathcal{N}$ by solving the evolution equation $\pounds_\eta X = 0$. Moreover, the extension $X \in \Gamma(T\mathcal{N})$ is everywhere tangent to the foliation (i.e. such that $X(v) = 0$), since

$$\begin{aligned} 0 \stackrel{\mathcal{N}}{=} (\pounds_\eta X)(v) \stackrel{\mathcal{N}}{=} \eta(X(v)) - X(\eta(v)) \stackrel{\mathcal{N}}{=} \eta(X(v)) \\ \Longrightarrow \quad X(v)|_{\mathcal{N}} = X(v)|_S = 0. \end{aligned}$$

This allows us to define $L \in \Gamma(T\mathcal{M})|_{\mathcal{N}}$ as the unique vector field satisfying

$$\begin{gathered} g(L,\eta) = 1, \qquad g(L,L) = 0, \qquad g(L,X) = 0, \\ \forall X \in \Gamma(T\mathcal{N}) \quad \text{satisfying} \quad \pounds_\eta X = 0, \quad X(v) = 0. \end{gathered} \tag{2.112}$$

Observe that (2.112) immediately entails (recall the notational considerations of Remark 2.4.2)

$$\boldsymbol{\ell} \stackrel{\text{def}}{=} \iota^\star \boldsymbol{L} = dv \quad \text{and} \quad \psi^\star \boldsymbol{\ell} = 0. \tag{2.113}$$

Given any two vector fields $X, Y \in \Gamma(T\mathcal{N})$ tangent to the foliation and commuting with η, one can also construct a covariant derivative on each leaf S_v by means of the second fundamental form $\mathbf{K}^L(X,Y) \stackrel{\text{def}}{=} g(\nabla_X L, Y)$ of S_v along the transverse normal L (recall (2.64)). Indeed, the derivative D given by

$$D_X Y \stackrel{\text{def}}{=} \nabla_X Y + \mathbf{K}^L(X,Y)\eta \tag{2.114}$$

defines a torsion-free connection on the leaves $\{S_v\}$. Moreover, it is immediate to prove that $D_X Y$ is tangent to the foliation, as (recall (2.112))

$$(D_X Y)(v) \stackrel{\mathcal{N}}{=} g(dv, \nabla_X Y) + \mathbf{K}^L(X,Y) \stackrel{\mathcal{N}}{=} g(dv, \nabla_X Y) - g(L, \nabla_X Y) \stackrel{\mathcal{N}}{=} 0,$$

and that D coincides with the Levi-Civita connection ∇^h corresponding to the induced metric h of each leaf S_v, because

$$\begin{aligned}(D_V h)(W, Z) &\stackrel{S_v}{=} D_V(g(W,Z)) - h(D_V W, Z) - h(W, D_V Z)\\ &\stackrel{S_v}{=} \nabla_V(g(W,Z)) - g(\nabla_V W, Z) - g(W, \nabla_V Z) \stackrel{S_v}{=} 0 \quad \forall V, W, Z \in \Gamma(TS_v).\end{aligned}$$

In particular, the considerations above apply for the section S, so we can write

$$\nabla_X W = \overline{\nabla}_X W = \nabla^h_X W - \mathbf{K}^L(X, W)\eta \qquad \forall X, W \in \Gamma(TS), \tag{2.115}$$

where $\overline{\nabla}$ is the induced connection on $\mathcal{N}$ (recall (2.110)).

Remark 2.5.6. *In index notation, the tensor* $\mathbf{K}^L$ *reads*

$$\mathbf{K}^L_{ab} = \overline{\nabla}_a \boldsymbol{\ell}_b \tag{2.116}$$

because, given any local basis $\{e_a\}$ *of* $\Gamma(T\mathcal{N})$*, one gets*

$$\mathbf{K}^L(e_a, e_b) = g(\nabla_{e_a} L, e_b) = e_a(g(L, e_b)) - g(\nabla_{e_a} e_b, L) = e_a(\boldsymbol{\ell}_b) - g(\overline{\nabla}_{e_a} e_b, L) = \overline{\nabla}_a \boldsymbol{\ell}_b.$$

To derive the master equation for isolated horizons, one of the key steps is to know the explicit form of the Lie derivative of $\mathbf{K}^L$ along η, which we compute next.

Lemma 2.5.7. *Consider a spacetime* (M, g) *and an embedded weakly isolated horizon* $\mathcal{N} \subset \mathrm{M}$ *with respect to a null generator* η*. Let* $S \subset \mathcal{N}$ *be a cross-section and* ψ*,* h *be* *the corresponding embedding $\psi : S \hookrightarrow \overline{N}$ of S into $\overline{N}$ and the induced metric on S respectively. Denote by ∇^h, R^h the Levi-Civita covariant derivative and the Ricci tensor of S. Construct a function $v \in \mathcal{F}(\overline{N})$ by solving $\eta(v)|_{\overline{N}} = 1$ with $v|_S = 0$, so that $\{S_v\}$ with*

$$S_{v_0} \overset{\mathrm{def}}{=} \{p \in \overline{N} \mid v(p) = v_0 \in \mathbb{R}\}$$

constitutes a foliation of $\overline{N}$ by spacelike cross-sections. Then, the Lie derivative of the tensor field $\mathbf{K}^L$ defined in (2.116) along η is given by

$$\psi^\star\left(\pounds_\eta \mathbf{K}^L\right)_{BC} \overset{S}{=} -\kappa \mathbf{K}^L_{BC} - \nabla^h_{(B}\varpi_{C)} - \varpi_B\varpi_C - \frac{1}{2}\left(R_{BC} - R^h_{BC}\right), \tag{2.117}$$

where R_{BC}, ϖ_A are the pull-back to S of the Ricci tensor of $(\mathcal{M}, g)$ and the one-form ϖ defined by (2.106) respectively.

Proof. Consider a local basis $\{e_A\}$ of $\Gamma(TS)$ and extend its vectors off S by requiring $\pounds_\eta e_A = 0$ on $\overline{N}$. Then, $0 = (\pounds_\eta e_A)(v) = \eta(e_A(v)) - e_A(\eta(v)) = \eta(e_A(v))$, so $\{e_A\}$ are tangent to the leaves $\{v = \text{const.}\}$. We let $\{e_1 \overset{\mathrm{def}}{=} \eta, e_A\}$ be a basis of $\Gamma(T\overline{N})$, and construct a unique vector field L by enforcing (2.112). Observe that $\pounds_\eta e_a = 0$ and $g(L, e_a) = \text{const.}$ by construction.

In these circumstances, we find

$$\begin{aligned} e^\alpha_a e^\beta_b e^\gamma_c \nabla_\alpha \nabla_\beta L_\gamma &= e^\alpha_a \nabla_\alpha\left(e^\beta_b e^\gamma_c \nabla_\beta L_\gamma\right) - (\nabla_\beta L_\gamma) e^\gamma_c e^\alpha_a \nabla_\alpha e^\beta_b - (\nabla_\beta L_\gamma) e^\beta_b e^\alpha_a \nabla_\alpha e^\gamma_c \\ &= e_a\left(\mathbf{K}^L_{bc}\right) - (\nabla_\beta L_\gamma) e^\gamma_c \left(\overline{\nabla}_{e_a} e_b\right)^\beta - (\nabla_\beta L_\gamma) e^\beta_b \left(\overline{\nabla}_{e_a} e_c\right)^\gamma = \overline{\nabla}_a \mathbf{K}^L_{bc}, \end{aligned}$$

which, together with the Ricci identity $R_{\alpha\beta\gamma\rho} L^\rho = \nabla_\alpha \nabla_\beta L_\gamma - \nabla_\beta \nabla_\alpha L_\gamma$ for L, entails

$$R_{\alpha\beta\gamma\rho} e^\beta_b e^\gamma_c L^\rho \overset{\overline{N}}{=} \overline{\nabla}_a \mathbf{K}^L_{bc} - \overline{\nabla}_b \mathbf{K}^L_{ac}. \tag{2.118}$$

The multiplication of (2.118) with η^α gives[2]

$$R_{\alpha\beta\gamma\rho} \eta^\alpha e^\beta_b e^\gamma_c L^\rho \overset{\overline{N}}{=} \eta^a \left(\overline{\nabla}_a \mathbf{K}^L_{bc} - \overline{\nabla}_b \mathbf{K}^L_{ac}\right). \tag{2.119}$$

Now by (2.106) we know that for any $X \in \Gamma(T\overline{N})$ it holds $\overline{\nabla}_X \eta = \varpi(X)\eta$ on $\overline{N}$. Particularizing this for $X = e_a$ allows us to conclude that

$$\overline{\nabla}_a \eta^b \overset{\overline{N}}{=} \varpi_a \eta^b. \tag{2.120}$$

[2]As usual, we are making the harmless abuse of notation of calling $\iota_\star\eta$ still as η.

Moreover, the tensor $\mathbf{K}^L$ and the one-form $\boldsymbol{\varpi}$ are related by

$$\mathbf{K}^L_{ab}\eta^a = -\varpi_b \tag{2.121}$$

because

$$\begin{aligned}\mathbf{K}^L(\eta, e_a) &\stackrel{\mathcal{N}}{=} g(\nabla_\eta L, e_a) \stackrel{\mathcal{N}}{=} \nabla_\eta\left(g(L, e_a)\right) - g(L, \nabla_\eta e_a) \stackrel{\mathcal{N}}{=} -g(L, \nabla_\eta e_a)\\ &\stackrel{\mathcal{N}}{=} -g(L, £_\eta e_a + \nabla_{e_a}\eta) \stackrel{\mathcal{N}}{=} -\boldsymbol{\varpi}(e_a)g(L, \eta) \stackrel{\mathcal{N}}{=} -\boldsymbol{\varpi}(e_a).\end{aligned}$$

Taking into account (2.120)-(2.121), the derivative $£_\eta \mathbf{K}^L_{bc}$ can be expressed as

$$\begin{aligned}£_\eta \mathbf{K}^L_{bc} &= \eta^a\overline{\nabla}_a \mathbf{K}^L_{bc} + (\overline{\nabla}_b\eta^a)\mathbf{K}^L_{ac} + (\overline{\nabla}_c\eta^a)\mathbf{K}^L_{ba}\\ &= \eta^a\left(\overline{\nabla}_a \mathbf{K}^L_{bc} - \overline{\nabla}_b \mathbf{K}^L_{ac}\right) + \overline{\nabla}_b(\eta^a\mathbf{K}^L_{ac}) + (\overline{\nabla}_c\eta^a)\mathbf{K}^L_{ba}\\ &= \eta^a\left(\overline{\nabla}_a \mathbf{K}^L_{bc} - \overline{\nabla}_b \mathbf{K}^L_{ac}\right) - \overline{\nabla}_b\varpi_c - \varpi_c\varpi_b.\end{aligned} \tag{2.122}$$

Thus, upon inserting (2.122) into (2.119) one gets

$$\frac{1}{2}R_{\alpha\beta\gamma\rho}\eta^\alpha(e^\beta_b e^\gamma_c + e^\beta_c e^\gamma_b)L^\rho \stackrel{\mathcal{N}}{=} £_\eta \mathbf{K}^L_{bc} + \overline{\nabla}_{(b}\varpi_{c)} + \varpi_b\varpi_c. \tag{2.123}$$

Equation (2.123) already provides the Lie derivative of $\mathbf{K}^L$ along η in terms of $\boldsymbol{\varpi}$ and a curvature term. We are interested in deriving an equation on the section S, so we need to compute the pull-back of (2.123) to such section. The whole calculation is based on the two identities (we identify vector fields on N with their images on $\mathcal{N}$)

$$R_{\alpha\beta\gamma\rho}\eta^\alpha(e^\beta_B e^\gamma_C + e^\beta_C e^\gamma_B)L^\rho \stackrel{S}{=} R^h_{BC} - R_{BC}, \tag{2.124}$$

$$e^a_A e^b_B\overline{\nabla}_a\varpi_b = e^a_A\overline{\nabla}_a(e^b_B\varpi_b) - \varpi_b e^a_A\overline{\nabla}_a e^b_B \stackrel{S}{=} \stackrel{h}{\nabla}_A\varpi_B + \kappa\mathbf{K}^L_{AB}. \tag{2.125}$$

Equation (2.124) can be obtained by particularizing (2.74) for $N_1 = \eta$, $N_2 = L$ and $R^{(\gamma)} = R^h$, while (2.125) is a direct consequence of (2.115) together with $(\iota^\star\boldsymbol{\varpi})(\eta) = \kappa$. By (2.124)-(2.125), it is immediate to check that the pull-back of (2.123) to S is given by (2.117). □

With (2.117) at hand, we can now present the definition of isolated horizon and obtain its corresponding master equation.

Definition 2.5.8. *(Isolated horizon) An isolated horizon is a non-expanding horizon* $\mathcal{N}$ — *endowed with a null generator* η *such that*

$$[\pounds_\eta, \overline{\nabla}_a] \overset{\mathcal{N}}{=} 0. \tag{2.126}$$

An important consequence of this definition is that $\pounds_\eta \mathrm{K}^L = 0$. Indeed,

$$\begin{aligned}
\pounds_\eta \mathrm{K}^L_{ab} &\overset{\mathcal{N}}{=} \pounds_\eta \overline{\nabla}_a \ell_b \overset{\mathcal{N}}{=} \overline{\nabla}_a(\pounds_\eta \ell_b) \overset{\mathcal{N}}{=} \overline{\nabla}_a \left(\eta^c \overline{\nabla}_c \overline{\nabla}_b v + (\overline{\nabla}_b \eta^c)(\overline{\nabla}_c v) \right) \\
&\overset{\mathcal{N}}{=} \overline{\nabla}_a \left(\eta^c \overline{\nabla}_c \overline{\nabla}_b v + \overline{\nabla}_b(\eta^c \overline{\nabla}_c v) - \eta^c \overline{\nabla}_b \overline{\nabla}_c v \right) \overset{\mathcal{N}}{=} \overline{\nabla}_a \overline{\nabla}_b (\eta^c \overline{\nabla}_c v) \overset{\mathcal{N}}{=} 0,
\end{aligned} \tag{2.127}$$

where we have used (2.126), $\eta(v) = 1$ and $\boldsymbol{\ell} = dv$. Observe that the combination of (2.121) and (2.127) automatically implies $\pounds_\eta \boldsymbol{\varpi} = 0$, so *any isolated horizon is also a weakly isolated horizon*. Setting $\pounds_\eta \mathrm{K}^L = 0$ into (2.117) yields

$$0 \overset{S}{=} \kappa \mathrm{K}^L_{BC} + \nabla^h_{(B} \varpi_{C)} + \varpi_B \varpi_C + \frac{1}{2} \left(R_{BC} - R^h_{BC} \right), \tag{2.128}$$

which is the master equation for isolated horizons. In particular, the so-called *extremal case* that takes place whenever $\kappa = 0$ transforms (2.128) into

$$0 \overset{S}{=} \nabla^h_{(B} \varpi_{C)} + \varpi_B \varpi_C + \frac{1}{2} \left(R_{BC} - R^h_{BC} \right). \tag{2.129}$$

Equation (2.129), valid for isolated horizons with respect to a generator η with vanishing surface gravity, is known as the *near horizon equation*.

2.6 killing horizons

Killing horizons are a special type of null hypersurface and constitute one of the main objects of this thesis. In particular, in Chapter 5 we introduce the notions of Killing horizons of order zero and one; in Chapter 6, as we already mentioned, we obtain a *generalized master equation* which relates properties of the generator of the horizon with the intrinsic and extrinsic geometry of such hypersurface; and finally in Chapter 7 we study the matching of two given spacetimes whose boundaries are Killing horizons (of order zero). It is therefore convenient to provide several definitions and fundamental properties of these sort of hypersurfaces. For further details on the topic see e.g. [112], [113].

We start with the basic notions of Killing horizon, bifurcation surface and bifurcate Killing horizon.

Definition 2.6.1. *(Killing horizon) Let η be a Killing vector in a spacetime* $(\mathcal{M}, g)$*. An embedded null hypersurface $\mathcal{H}$ where η is null, nowhere zero and to which η is tangent defines a Killing horizon of η.*

Definition 2.6.2. *(Bifurcation surface, bifurcate Killing horizon) Let η be a non-trivial Killing vector in a spacetime* $(\mathcal{M}, g)$*. A bifurcation surface is a connected spacelike codimension-two submanifold $\mathcal{S}$ of fixed points, i.e. where $\eta|_{\mathcal{S}} = 0$. The set of points along all null geodesics orthogonal to $\mathcal{S}$ comprises a bifurcate Killing horizon with respect to η.*

A Killing horizon $\mathcal{H}$ may have one or several connected components. It would therefore be natural to restrict ourselves to one of them. However, it turns out to be more advantageous to select those connected components that share a common topological boundary. More precisely, we can always restrict $\mathcal{H}$ so that its topological closure $\overline{\mathcal{H}}$ is a smooth connected (necessarily null) hypersurface without boundary. When dealing with Killing horizons, this shall be always assumed. We will denote by $\mathcal{S}$ the (possibly empty) set of fixed points of η within $\overline{\mathcal{H}}$, i.e. $\mathcal{S} \stackrel{\text{def}}{=} \{p \in \overline{\mathcal{H}} \mid \eta|_p = 0\}$. The set of fixed points of a Killing vector η is either the empty set or the union of connected totally geodesic closed submanifolds of even codimension (this is proven in [114], [106] for the Riemannian and pseudo-Riemannian cases respectively). Therefore, the Killing vector η cannot vanish on open subsets of $\overline{\mathcal{H}}$.

As we did in Section 2.5, we define the surface gravity $\kappa \in \mathcal{F}(\mathcal{H})$ of η according to (2.81) (this can always be done because η is null and normal along $\mathcal{H}$, see the discussion in Section 2.4). We emphasize that κ can only be defined on $\mathcal{H}$, as at those points of $\overline{\mathcal{H}}$ where $\eta = 0$ both sides of (2.81) vanish. For the rest of the thesis, we use the standard terminology of calling $\mathcal{H}$, $\overline{\mathcal{H}}$ *non-degenerate* if κ is not identically zero on $\mathcal{H}$, and *degenerate* otherwise.

A fundamental property of Killing horizons is that they are totally geodesic null hypersurfaces (see Definition 2.4.10). This is a consequence of (2.65), which in the present case yields $g(\nabla_X \eta, Y) = 0$ for any two vector fields $X, Y \in \Gamma(T\mathcal{H})$. This, as we know from Section 2.4.3, means that there exists a one-form ϖ along $\mathcal{H}$ defined by (2.106), and that the surface gravity of η is directly related to ϖ by (2.107). In the case of Killing horizons, the two-form $\nabla_\mu \eta_\nu$ at points in $\mathcal{H}$ also adopts a simple form in terms of ϖ [112], as we prove next.

Lemma 2.6.3. *Consider a spacetime* $(\mathcal{M}, g)$ *admitting a non-trivial Killing vector field η which defines a Killing horizon $\mathcal{H}$. Let ϖ be a one-form along $\mathcal{H}$ satisfying* (2.106)*. Then,*

$$\nabla_\mu \eta_\nu \stackrel{\mathcal{H}}{=} \varpi_\mu \eta_\nu - \varpi_\nu \eta_\mu. \tag{2.130}$$

Proof. We start by taking a basis $\{L, \eta, X_A\}$ of $\Gamma(TM)|_H$, where $\{X_A\}$ are tangent vector fields and L is everywhere transverse to H and satisfies $g(L, X_A) = 0$ and $g(L, \eta) = 1$. Now, we define the two-forms $F_{\mu\nu} \stackrel{\text{def}}{=} \frac{1}{2}(\nabla_\mu \eta_\nu - \nabla_\nu \eta_\mu) = \nabla_\mu \eta_\nu$ and $\Omega_{\mu\nu} \stackrel{\text{def}}{=} \varpi_\mu \eta_\nu - \varpi_\nu \eta_\mu$, the latter only along H. We want to show that both tensors are the same. We prove this by showing that all their contractions with the basis vectors are the same. We start with $F_{\mu\nu}$.

H being totally geodesic entails $g(\nabla_{X_A} \eta, X_B)|_H = 0$, hence $F_{\mu\nu} X_A^\mu X_B^\nu|_H = 0$. Moreover, $g(\nabla_\eta \eta, X_A)|_H = \kappa g(\eta, X_A)|_H = 0$, so $F_{\mu\nu}\eta^\mu X_A^\nu|_H = 0$ also. On the other hand, $F_{\mu\nu}$ being antisymmetric means $F_{\mu\nu}L^\mu L^\nu|_H = 0$ and $F_{\mu\nu}\eta^\mu\eta^\nu|_H = 0$.

Finally, for any $Y \in \Gamma(TH)$ it holds that $F_{\mu\nu} Y^\mu L^\nu = \varpi_\mu Y^\mu$ on H. Therefore, the only non-zero contractions are

$$F_{\mu\nu}\eta^\mu L^\nu \stackrel{H}{=} \varpi_\mu \eta^\mu, \qquad F_{\mu\nu} X_A^\mu L^\nu \stackrel{H}{=} \varpi_\mu X_A^\mu.$$

It is immediate to prove that the only non-zero contractions of Ω with the basis vectors $\{L, \eta, X_A\}$ are

$$\Omega_{\mu\nu}\eta^\mu L^\nu \stackrel{H}{=} \varpi_\mu \eta^\mu \qquad \text{and} \qquad \Omega_{\mu\nu} X_A^\mu L^\nu \stackrel{H}{=} \varpi_\mu X_A^\mu,$$

so necessarily $\Omega_{\mu\nu} = F_{\mu\nu} = \nabla_\mu \eta_\nu$ on H, and (2.130) follows. □

Remark 2.6.4. *The freedom in the definition of ϖ is compatible with (2.130) as this combination is insensitive to terms of the form $f\boldsymbol{\eta}$.*

We now provide a well-known identity involving the Riemann tensor of (M, g), the Killing vector η and the one-form ϖ (see e.g. [2]).

Proposition 2.6.5. *Let (M, g) be a spacetime admitting a non-trivial Killing vector field η which defines a Killing horizon H. Consider a vector field $X \in \Gamma(TH)$ and let ϖ be a one-form along H satisfying (2.106). Then,*

$$R^\nu{}_{\sigma\rho\mu}\eta_\nu X^\sigma \stackrel{H}{=} \eta_\mu \left(X^\sigma \nabla_\sigma \varpi_\rho + \varpi_\rho \varpi_\sigma X^\sigma \right) - \eta_\rho \left(X^\sigma \nabla_\sigma \varpi_\mu + \varpi_\mu \varpi_\sigma X^\sigma \right). \tag{2.131}$$

Proof. Any Killing vector field satisfies $\nabla_\sigma \nabla_\rho \eta_\mu = R^\nu{}_{\sigma\rho\mu}\eta_\nu$. Combining this with (2.130) yields (note that X is tangential to H so we are allowed to take the derivative of (2.130) along this vector)

$$\begin{aligned} R^\nu{}_{\sigma\rho\mu}\eta_\nu X^\sigma &\stackrel{H}{=} X^\sigma \nabla_\sigma \nabla_\rho \eta_\mu \stackrel{H}{=} X^\sigma \nabla_\sigma \left(\varpi_\rho \eta_\mu - \varpi_\mu \eta_\rho \right) \\ &\stackrel{H}{=} \eta_\mu \left(X^\sigma \nabla_\sigma \varpi_\rho + \varpi_\rho \varpi_\sigma X^\sigma \right) - \eta_\rho \left(X^\sigma \nabla_\sigma \varpi_\mu + \varpi_\mu \varpi_\sigma X^\sigma \right) \end{aligned}$$

after using that $X^\sigma \eta_\sigma = 0$ on H. □

In particular, setting $X = \eta$ in (2.131) and taking into account (2.107) as well as the symmetries of the Riemann tensor, one obtains

$$0 \stackrel{\overline{H}}{=} \eta_\mu \left(\eta^\sigma \nabla_\sigma \varpi_\rho + \kappa \varpi_\rho \right) - \eta_\rho \left(\eta^\sigma \nabla_\sigma \varpi_\mu + \kappa \varpi_\mu \right), \tag{2.132}$$

from where it follows that there exists a function $G \in \mathcal{F}(\overline{H})$ such that

$$\nabla_\eta \varpi + \kappa \varpi \stackrel{\overline{H}}{=} G\boldsymbol{\eta}. \tag{2.133}$$

Equation (2.133) have several consequences. First, for any vector field $X \in \Gamma(T\overline{H})$ one finds (recall (2.106))

$$(\pounds_\eta \varpi)(X) = (\nabla_\eta \varpi)(X) + \varpi(\nabla_X \eta) = (\nabla_\eta \varpi)(X) + \kappa\varpi(X) = G\boldsymbol{\eta}(X) = 0, \tag{2.134}$$

which means that the property (2.111) also holds in full Killing horizons. Secondly, (2.133) implies the well-known property (see e.g. [112]) that κ is constant along the null generators of $\overline{H}$. Indeed, its contraction with η gives

$$0 \stackrel{\overline{H}}{=} (\nabla_\eta \varpi)(\eta) + \kappa^2 \stackrel{\overline{H}}{=} \nabla_\eta \kappa - \varpi(\nabla_\eta \eta) + \kappa^2 \stackrel{\overline{H}}{=} \eta(\kappa). \tag{2.135}$$

As we discussed before (see Section 2.4), when $\overline{H}$ admits a cross-section it is possible to find an affine null generator k (i.e. with $\kappa_k = 0$) of $\overline{H}$. Combining (2.90), (2.135) and the fact that η does not vanish on open subsets of $\overline{H}$, it follows that

$$\eta \stackrel{\overline{H}}{=} (f + \kappa v)\, k, \tag{2.136}$$

where $f, v \in \mathcal{F}(\overline{H})$ are functions satisfying $k(f) = 0$, $k(v) = 1$. Observe that if α happens to vanish at some point p along a null generator, the surface gravity κ could in principle be a different constant before and after the fixed point p. This, however, is impossible because η and k are well defined on $\overline{H}$ and k vanishes nowhere, which implies that α is smooth along the curve, and since f is constant on the generator, κ must be the same constant on each side of the point p.

2.6.1 *Multiple Killing horizons*

An interesting particular case happens when the same null hypersurface $\overline{H}$ admits more than one null, tangent Killing vector field. Such hypersurface is named multiple Killing horizon (see e.g. [2], [93], [94]) and it is defined as follows.

Definition 2.6.6. *(Multiple Killing horizon) Assume that an* $(n+1)$*-dimensional spacetime* $(\mathcal{M}, g)$ *admits* $m \geq 2$ *non-trivial Killing vector fields* $\{\eta_r\}$ *($r = 1, \ldots, m$) which define respective Killing horizons* $\{\mathcal{H}_r\}$*. A null hypersurface* $\mathcal{H}$ *embedded in* $(\mathcal{M}, g)$ *is a multiple Killing horizon of order* m *if*

$$\overline{\mathcal{H}} = \overline{\mathcal{H}}_1 = \ldots = \overline{\mathcal{H}}_m. \tag{2.137}$$

A fundamental property of multiple Killing horizons [2, Prop. 4.3] is that the surface gravities $\{\kappa_r\}$ of the Killing vectors $\{\eta_r\}$ are all constant everywhere on their respective horizons $\{\mathcal{H}_r\}$. This, in particular, allows one to extend each them trivially to the whole closure $\overline{\mathcal{H}}$.

One can also prove that the set $\mathcal{L}_{\mathcal{H}} \stackrel{\text{def}}{=} \operatorname{span}\{\eta_r, r = 1, \ldots, m\}$ of Killing vectors null and tangent to the multiple Killing horizon is an m-dimensional Lie subalgebra of the Killing Lie algebra of $(\mathcal{M}, g)$ [2, Thm. 2]. Moreover, there always exists [2, Thm. 3] an Abelian subalgebra $\mathcal{L}^{\text{deg}}_{\mathcal{H}}$ of $\mathcal{L}_{\mathcal{H}}$ with at least dimension $m-1$ and composed only by Killing vectors for which the multiple Killing horizon is degenerate. For the rest of this section we let $\{\eta_2, \ldots, \eta_m\}$ be degenerate Killing vector fields (i.e. with $\underline{\kappa}_2 = \ldots = \underline{\kappa}_m = 0$) and consider η_1 as a Killing vector field with arbitrary surface gravity $\underline{\kappa}_1$. Observe that in this notation $\{\eta_1, \eta_2, \ldots, \eta_m\}$ is a basis of $\mathcal{L}_{\mathcal{H}}$ such that $\{\eta_2, \ldots, \eta_m\}$ commute with each other. We also let $r = 2, \ldots, m$ from now on, e.g. by writing $\{\eta_1, \eta_r\}$ or $\{\mathcal{H}_1, \mathcal{H}_r\}$.

A multiple Killing horizon $\mathcal{H}$ is called *fully degenerate* if $\mathcal{L}_{\mathcal{H}} = \mathcal{L}^{\text{deg}}_{\mathcal{H}}$, i.e. when $\mathcal{L}_{\mathcal{H}}$ is Abelian and all Killing vectors are degenerate. Otherwise, $\mathcal{H}$ is called *non-fully degenerate* and it has a unique non-degenerate Killing vector. The subalgebra $\mathcal{L}^{\text{deg}}_{\mathcal{H}}$ can be at most n-dimensional [2, Cor. 1], and hence the maximum possible order of a multiple Killing horizon is $m = n$ (resp. $m = n+1$) for fully (resp. non-fully) degenerate horizons. When $\dim \mathcal{L}^{\text{deg}}_{\mathcal{H}} = m-1$, the remaining non-degenerate independent Killing vector η_1 verifies

$$[\eta_1, \eta_r] = -\underline{\kappa}_1 \eta_r \qquad \forall \eta_r \in \mathcal{L}^{\text{deg}}_{\mathcal{H}}. \tag{2.138}$$

Since all Killing vectors are null and tangent everywhere along $\overline{\mathcal{H}}$, they are all necessarily proportional to each other therein. In particular, there exists a set of functions $\{\alpha_r\} \in \mathcal{F}(\mathcal{H}_1)$ defined as the proportionality functions between any Killing vector field η_r and η_1, i.e.

$$\eta_r = \alpha_r \eta_1. \tag{2.139}$$

In these circumstances, it holds (see equation (8) in [2])

$$\alpha_r \overset{H_1}{=} \hat{\alpha} e^{-\kappa_1 v}, \tag{2.140}$$

where $\hat{\alpha}, v \in \mathcal{F}(H_1)$ are functions satisfying $\eta_1(v) = 1$ and $\eta_1(\hat{\alpha}) = 0$ and v is not univocally defined, as any other choice $v' = v + h$ with $h \in \mathcal{F}(H_1)$ satisfying $\eta_1(h) = 0$ also verifies $\eta_1(v') = 1$.

For the rest of the section we let $H_{1,r}$ be the subset of $\overline{H}$ where both η_1 and η_r are everywhere non-zero, i.e. $H_{1,r} \stackrel{\mathrm{def}}{=} H_1 \cap H_r$.

We have seen before that it is possible to define a one-form ϖ along any Killing horizon (recall (2.106)). This applies, in particular, to the case of multiple Killing horizons. More concretely, there exist m one-forms[3] $\{\varpi, \varpi^r\}$, respectively defined on the horizons $\{H_1, H_r\}$ and satisfying (2.106) for its corresponding Killing vectors $\{\eta_1, \eta_r\}$. Observe that on $H_{1,r}$, both ϖ and ϖ^r exist and are well-defined. This, together with (2.139), means that for any vector field $X \in \Gamma(TH_{1,r})$ it holds

$$\begin{aligned} \varpi^r(X)\alpha_r\eta_1 &\overset{H_{1,r}}{=} \varpi^r(X)\eta_r \overset{H_r}{=} \nabla_X\eta_r \overset{H_{1,r}}{=} \alpha_r\nabla_X\eta_1 + X(\alpha_r)\eta_1 \overset{H_{1,r}}{=} \left(\alpha_r\varpi(X) + X(\alpha_r)\right)\eta_1, \\ &\iff \quad d\alpha_r + \alpha_r\left(\varpi - \varpi^r\right) \overset{H_{1,r}}{=} 0, \end{aligned} \tag{2.141}$$

where d should be understood as the exterior derivative on $H_{1,r}$ as manifold. Inserting (2.140) into (2.141) yields

$$d(\ln|\hat{\alpha}|) + \varpi - \varpi^r - \kappa_1 dv \overset{H_{1,r}}{=} 0 \tag{2.142}$$

after using that $\hat{\alpha}$ vanishes no-where on $H_{1,r}$.

The identity (2.131), which was obtained for a general Killing horizon, is also true in this context for all Killing vectors $\{\eta_1, \eta_r\}$. Particularizing (2.131) for η_1 and η_r gives two equations that hold on H_1 and H_r respectively. Since on $H_{1,r}$ these two identities are valid, one can subtract one from the other to find

$$0 \overset{H_{1,r}}{=} X^\sigma Y^\rho \left(\nabla_\sigma\varpi^r_\rho + \varpi^r_\rho\varpi^r_\sigma - \nabla_\sigma\varpi_\rho - \varpi_\rho\varpi_\sigma\right) \qquad \forall X, Y \in \Gamma(TH_{1,r}). \tag{2.143}$$

Substituting ϖ^r in terms of ϖ by means of (2.142) gives

$$0 \overset{H_{1,r}}{=} X^\sigma Y^\rho \Big(\nabla_\sigma\nabla_\rho \ln|\alpha| + (\nabla_\rho \ln|\alpha|)(\nabla_\sigma \ln|\alpha|) + 2(\nabla_{(\rho} \ln|\hat{\alpha}|)\left(\varpi_{\sigma)} - \kappa_1\nabla_{\sigma)} v\right)$$

[3] For shortness, instead of using ϖ^1 to denote the one-form associated to η_1, we simply write ϖ.

$$
- \kappa_1 \Big(\nabla_\sigma \nabla_\rho v + 2\varpi_{(\rho}\nabla_{\sigma)} v - \kappa_1 (\nabla_\rho v)(\nabla_\sigma v) \Big)^{1} \tag{2.144}
$$

after using that κ_1 is constant on $H_{1,r}$.

Equation (2.144) (already obtained in [2]) is usually called *master equation*. It constitutes a fully covariant identity for multiple Killing horizons, and it relates derivatives of the functions α and v (cf. (2.140)) with the one-form ϖ and the surface gravity κ_1. Observe that since the terms inside the parenthesis are contracted with vector fields X, Y which are everywhere tangent to the horizon, (2.144) can be understood as an identity of *abstract* nature (i.e. as an equation on $H_{1,r}$ as a manifold).

By combining (2.106)-(2.107) (recall that here $\kappa_r = 0$) and (2.133), it is straightforward to prove that the right hand side of (2.143) vanishes whenever $X = \eta_1$ or $Y = \eta_1$. The same occurs in (2.144), so it makes sense to study the case when both X and Y are tangent to $H_{1,r}$ but non-null everywhere. In [2], this is done by assuming that $H_{1,r}$ admits a cross-section S and that it can be foliated by diffeomorphic spacelike surfaces. In these circumstances, one can construct a (unique) foliation function $v \in \mathcal{F}(H_{1,r})$ by fixing it at S, i.e. by choosing $v|_S$, and solving $\eta_1(v) = 1$. Then, the set of spacelike submanifolds $S_v \stackrel{\text{def}}{=} \{p \in H_{1,} \mid v(p) = v_0 \in \mathbb{R}\}$ define a foliation of $H_{1,r}$.

As discussed in Section 2.5, we can construct vector fields everywhere tangent to the leaves $\{S_v\}$ by extending any vector field $X \in \Gamma(TS)$ uniquely to $H_{1,r}$ according to $\pounds_{\eta_1} X = 0$. For the rest of this section, we let $X, Y \in \Gamma(TH_{1,r})$ be any two vector fields tangent to the foliation and commuting with η_1. We also define $L \in \Gamma(TM)|_{H_{1,r}}$ as the unique vector field verifying (2.112) for $\eta = \eta_1$. In these circumstances, if we denote by ι the embedding of $H_{1,r}$ into M, we know by (2.113) that $\boldsymbol{\ell} \stackrel{\text{def}}{=} \iota^\star \boldsymbol{L} = dv$. We can also define the Levi-civita covariant derivative on each leaf S_v by particularizing (2.115) for $S = S_v$.

Using that

$$
\begin{aligned}
X^\sigma Y^\rho \nabla_\sigma \nabla_\rho v &= X^\sigma \nabla_\sigma (Y^\rho \nabla_\rho v) - (X^\sigma \nabla_\sigma Y^\rho)(\nabla_\rho v) = -(X^\sigma \nabla_\sigma Y^\rho)(\nabla_\rho v) \\
&= -((\overset{h}{\nabla}_X Y)^\rho - \mathrm{K}^L(X,Y)\eta_1^\rho)(\nabla_\rho v) = \mathrm{K}^L(X,Y),
\end{aligned} \tag{2.145}
$$

together with all considerations above, one can rewrite (2.144) as

$$
0 \stackrel{H_{1,r}}{=} \overset{h}{\nabla}_A \overset{h}{\nabla}_B \ln|\alpha| + (\overset{h}{\nabla}_A \ln|\alpha|)(\overset{h}{\nabla}_B \ln|\alpha|) + 2(\overset{h}{\nabla}_{(A} \ln|\alpha|)\varpi_{B)} - \kappa_1 \mathrm{K}^{L}_{AB}, \tag{2.146}
$$

where we have chosen a basis $\{e_A\}$ of vector fields tangent to the leaves of the foliation and commuting with η_1 and we have enforced $X = e_A$, $Y = e_B$. In particular, (2.146) holds at S, and can be rewritten in terms of derivatives of α as

$$0 \overset{S}{=} \overset{h}{\nabla}_A \overset{h}{\nabla}_B \alpha + 2(\overset{h}{\nabla}_{(A}\alpha)\varpi_{B)} - \alpha\kappa_1 \overset{L}{\mathrm{K}}_{AB}. \tag{2.147}$$

We want to rewrite this equation in a way that does not involve $\overset{L}{\mathrm{K}}_{AB}$. The strategy will be to use (2.117) together with the fact that the components $\overset{L}{\mathrm{K}}_{AB}$ are constant along the null generators. We start by proving this last claim. First, we use the well-known identities $[\nabla_\mu, \nabla_\alpha]L_\beta = R_{\mu\alpha\beta\lambda}L^\lambda$ and $\nabla_\alpha\nabla_\beta\eta_{1\lambda} = \eta_1^\mu R_{\mu\alpha\beta\lambda}$ to obtain

$$\begin{aligned}
\pounds_{\eta_1}(\nabla_\alpha L_\beta) &= \eta_1^\mu\nabla_\mu\nabla_\alpha L_\beta + (\nabla_\alpha\eta_1^\mu)(\nabla_\mu L_\beta) + (\nabla_\beta\eta_1^\mu)(\nabla_\alpha L_\mu) \\
&= \eta_1^\mu[\nabla_\mu, \nabla_\alpha]L_\beta + \nabla_\alpha(\eta_1^\mu\nabla_\mu L_\beta) + (\nabla_\beta\eta_1^\mu)(\nabla_\alpha L_\mu) \\
&= \eta_1^\mu R_{\mu\alpha\beta\lambda}L^\lambda + \nabla_\alpha(\pounds_{\eta_1} L_\beta) - \nabla_\alpha(L_\mu\nabla_\beta\eta_1^\mu) + (\nabla_\beta\eta_1^\mu)(\nabla_\alpha L_\mu) \\
&= L_\mu\nabla_\alpha\nabla_\beta\eta_1^\mu + \nabla_\alpha(\pounds_{\eta_1} L_\beta) - L_\mu\nabla_\alpha\nabla_\beta\eta_1^\mu = \nabla_\alpha(\pounds_{\eta_1} L_\beta).
\end{aligned} \tag{2.148}$$

Next we show that the pull-back of $\pounds_{\eta_1}L_\beta$ to $H_{1,r}$ vanishes identically. It is clear that this quantity does not depend on how we extend L_β off $H_{1,r}$, so we extend the function v arbitrarily and define $L_{beta} = \nabla_\beta v$. Then, for any arbitrary vector W^β tangent to $H_{1,r}$,

$$\begin{aligned}
W^\beta\pounds_{\eta_1}L_\beta &= W^\beta\left(\eta_1^\mu\nabla_\mu L_\beta + (\nabla_\beta\eta_1^\mu)L_\mu\right) = W^\beta\left(\eta_1^\mu\nabla_\mu\nabla_\beta v + (\nabla_\beta\eta_1^\mu)(\nabla_\mu v)\right) \\
&= W^\beta\left(\eta_1^\mu\nabla_\mu\nabla_\beta v + \nabla_\beta(\eta_1(v)) - \eta_1^\mu\nabla_\beta\nabla_\mu v\right) = W(\eta_1(v)) = 0.
\end{aligned} \tag{2.149}$$

This together with (2.115) (particularized for $\eta = \eta_1$) gives

$$e_A^\alpha e_B^\beta\nabla_\alpha(\pounds_{\eta_1}L_\beta) = -e_A^\alpha(\nabla_\alpha e_B^\beta)(\pounds_{\eta_1}L_\beta) = \left(\overset{L}{\mathrm{K}}(e_A, e_B)\eta_1^\beta - (\overset{h}{\nabla}_{e_A}e_B)^\beta\right)(\pounds_{\eta_1}L_\beta) = 0. \tag{2.150}$$

The combination of (2.148) and (2.150) (recall that $\{e_A, e_B\}$ commute with η_1) yields

$$\pounds_{\eta_1}(\overset{L}{\mathrm{K}}_{AB}) = 0, \tag{2.151}$$

which proves the claim. At this point we note that the proof of (2.117) only requires that $\mathcal{N}$ can be foliated by spacelike sections and that $(\pounds_\eta\varpi)(X) = 0$ holds for any $X \in \Gamma(T\mathcal{N})$. Since in the present case this is also true (recall (2.134)), enforcing (2.151) in (2.117) leads to

$$0 \overset{S}{=} R_{AB} - \overset{h}{R}_{AB} + 2\overset{h}{\nabla}_{(A}\varpi_{B)} + 2\varpi_A\varpi_B + 2\kappa_1\overset{L}{\mathrm{K}}_{AB}, \tag{2.152}$$

where, as before, R_{AB} and R^h_{AB} are the pull-back to S of the Ricci tensor of $\mathcal{M}$ and the Ricci tensor of the induced metric h on S respectively. Equation (2.152) can be inserted into (2.147) to obtain [2, Eq. (60)]

$$0 \stackrel{S}{=} \nabla^h_A \nabla^h_B \alpha + 2\nabla^h_{(A}\alpha\, \varpi_{B)} + \frac{\alpha}{2}\left(2\nabla^h_{(A}\varpi_{B)} + 2\varpi_A\varpi_B + R_{AB} - R^h_{AB}\right). \qquad (2.153)$$

Since L is null and normal to S, we know from (2.78) and (2.109) that the pull-back ϖ_A of $\boldsymbol{\varpi}$ to S coincides (up to a sign) with the so-called torsion one-form $\boldsymbol{\sigma}_L$ of S. Consequently, (2.153) is an identity relating derivatives of the function α at S, curvature terms (namely R_{AB} and R^h_{AB}) and the torsion one-form of S.

2.6.2 *Constancy of the surface gravity*

The constancy of the surface gravity is not guaranteed everywhere on a Killing horizon $\mathcal{H}$. However, it holds in many situations of physical interest, namely when (*i*) the Einstein tensor of the spacetime $(\mathcal{M}, g)$ satisfies the dominant energy condition [115], [113]; (*ii*) the Killing vector field η is integrable, i.e. it verifies $\boldsymbol{\eta} \wedge d\boldsymbol{\eta} = 0$ [101]; (*iii*) for any bifurcate Killing horizon [116], [101], [112]; and, as we have already mentioned, (*iv*) for multiple Killing horizons [2], [93], [94]. Nevertheless, we emphasize that the constancy of the surface gravity restricts the class of horizons under consideration (see e.g. [117] for a situation where a non-constant surface gravity implies a rather different behaviour of the properties of the horizon).

When the surface gravity κ is everywhere constant (e.g. in any of the situations above) and $\overline{\mathcal{H}}$ admits a cross-section (which, as we have discussed, allows us to take an affine null generator k), one can trivially extend κ to the whole closure $\overline{\mathcal{H}}$ as the same constant (again because $\eta = (f + \kappa v)k$ everywhere on $\overline{\mathcal{H}}$ and η, k and f are smooth, see (2.90)). This will be used later in Chapter 7 when studying matching across Killing horizons of order zero.

2.7 matching of spacetimes

In the Introduction, we have seen that the problem of matching two spacetimes across a hypersurface plays a fundamental role in any theory of gravity. We have also mentioned that the standard way of approaching the problem of matching is to consider two spacetimes with boundary and then construct a resulting differentiable spacetime by establishing an identification between the boundary points and between the full tangent spaces at the boundaries (this must be done in such a way that a rigging vector field pointing inwards (resp. outwards) on one side is identified with a rigging pointing outwards (resp. inwards) on the other side).

Of course, the matching between two whatever given spacetimes will be impossible in general. The complete set of necessary and sufficient conditions that allow for the matching has never been presented in terms of abstract metric hypersurface data (this notion was introduced later in [58]). However, the results of [67], [64], [65] can be collected in the following result.

Theorem 2.7.1. *Let* $(\mathcal{M}^{\pm}, g^{\pm})$ *be two spacetimes with boundaries* $\widetilde{\mathcal{N}}^{\pm}$*. Assume that the dimension of* $(\mathcal{M}^{\pm}, g^{\pm})$ *is* $n+1 \geq 2$*. Then,* $(\mathcal{M}^{\pm}, g^{\pm})$ *can be matched across* $\widetilde{\mathcal{N}}^{\pm}$ *and give rise to a resulting spacetime* $(\mathcal{M}, g)$ *with continuous metric* g *(in a suitable differentiable atlas) if and only if*

(i) There exist metric hypersurface data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ *that can be embedded in both spacetimes* $(\mathcal{M}^{\pm}, g^{\pm})$ *with embeddings* $\phi^{\pm}$ *such that* $\phi^{\pm}(\mathcal{N}) = \widetilde{\mathcal{N}}^{\pm}$ *and riggings* $\zeta^{\pm}$*.*

(ii) One rigging vector field must point inwards with respect to their corresponding boundary while the other must point outwards.

We have also explained before that if the boundaries are everywhere non-null and (*i*) is fulfilled, then (*ii*) can always be satisfied provided that one chooses the riggings to be unit, normal and with suitable orientation (observe that in this case $\boldsymbol{\ell} = 0$, $|\ell^{(2)}| = 1$ and therefore the whole metric data is easily identified). This, as we know, is not so when the boundaries contain a null point. In such case, finding embeddings and riggings for which a metric hypersurface data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ is embedded in $(\mathcal{M}^{\pm}, g^{\pm})$ is not sufficient to guarantee the matching [65], and condition (*ii*) needs to be included. The underlying reason why one needs the extra condition (*ii*) on null points is that the only gauge transformation which leaves a metric data with null points invariant is the identity, i.e. one cannot adjust the orientations of the two riggings at will. Observe that, in the language of the formalism of hypersurface data, (*i*)-(*ii*) mean that both boundaries must define hypersurface data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}^{\pm}\}$ with the same metric part and (possibly) different extrinsic part.

Another point that has been stressed in the Introduction is that, when there exists a jump in the second fundamental forms of each boundary, then it appears a singular part in the Einstein tensor of the resulting spacetime. This singular part is interpreted as the energy-momentum tensor of the thin shell (cf. (1.2)), and must satisfy the Israel equations (which in the non-null case are (1.3)).

The Israel equations, however, can be obtained more directly for the case with arbitrary causal character as an application of the formalism of hypersurface data. It suffices to take the difference between the constraint equations (2.55)-(2.56) at each side of the matching hypersurface. Since the constraint equations have been introduced at the abstract level, the argument is valid for any pair of hypersurface data (in any gauge), independently of whether they contain null points or not. This procedure is more convenient for several reasons. First, the derivation of the Israel equations is conceptually easier as there is no need to make use of the distributional theory. Secondly, one can obtain the shell equations without using the subatlas where the metric g is continuous (this choice is essential within the context of distributional calculus). Even more, the computations are much more general in the sense that the two hypersurface data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}^{\pm}\}$ corresponding to the thin shell do not need to be embedded at all. This may seem superfluous but in fact it is not because while the spacelike initial value problem defines a well-posed Cauchy problem, in completely general circumstances it is not to be expected that a hypersurface data gives rise to a spacetime (specially if the data contains a point where γ has Lorentzian signature). Even in situations where one could expect, for physical or geometric reasons, that the initial value problem is well-posed (e.g. whenever all points are null or spacelike) it could well happen that the result has not been proven yet.

Anyway, the set of all possible abstract hypersurface data is, a priori, more general than the set of spacetimes satisfying the Einstein equations and with matter content prescribed by the data. It is therefore convenient to find the Israel equations for thin shells at a fully abstract level. This problem was addressed in [58] as one of the first applications of the formalism of hypersurface data. The outcome was the first derivation of the Israel equations for general spacetimes containing thin shells of arbitrary causal character. We describe these results next.

2.7.1 *Thin shells: formalism of hypersurface data*

Let us now explore the physics and geometry of thin shells from a completely abstract viewpoint, i.e. by means of the formalism of hypersurface data. We start with the abstract notion of thin shell.

Definition 2.7.2. *(Thin shell) A thin shell is a pair of matter-hypersurface data with same metric hypersurface data, i.e. of the form* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}^{\pm}, \rho_{\ell}{}^{\pm}, \mathbf{J}^{\pm}, \epsilon\}$*, where* ϵ *is a sign with gauge behaviour:*

$$\mathcal{G}_{(z,V)}(\epsilon) = \frac{z}{|z|}\epsilon. \tag{2.154}$$

Given a hypersurface data set $\{\gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}^{\pm}\}$, we write $Q^{\pm}$ to refer to any geometric quantity constructed from it. We also use the notation $[Q] \stackrel{\text{def}}{=} Q^{+} - Q^{-}$.

One of the main properties of thin shells is that one can define an energy-momentum tensor encoding their matter-energy content. In the spacelike case, as we have seen, the singular part of the Einstein tensor of the matched spacetime is given by (1.2). In that situation it is therefore to be expected that the energy-momentum tensor coincides with T . In a completely general case, the energy-momentum tensor is defined as follows.

Definition 2.7.3. *(Energy-momentum tensor of a thin shell) For a thin shell* $\{N , \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}^{\pm}, \rho_{\ell}{}^{\pm}, \mathbf{J}^{\pm}, \epsilon\}$ *the energy-momentum tensor is the symmetric 2-covariant tensor τ defined by*

$$
\begin{aligned}
\tau^{df} \stackrel{\text{def}}{=} \epsilon \Big(& (P^{af} n^d + P^{ad} n^f) n^b \\
& - (n^{(2)} P^{af} P^{bd} + P^{df} n^a n^b) + P^{ab}(n^{(2)} P^{df} - n^d n^f) \Big) [\mathrm{Y}_{ab}].
\end{aligned} \tag{2.155}
$$

Remark 2.7.4. *The sign ϵ in Definitions 2.7.2 and 2.7.3 is necessary for the energy-momentum tensor τ to be well-defined. This is so because a change in the orientation of the one-form $\boldsymbol{\ell}$ (or of rigging in the embedded picture) introduces a sign in* $[\mathbf{Y}]$ *(recall* (2.40)*), and τ must be invariant under this type of transformations.*

Concretely, when embedding a thin shell data $\{N , \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}^{\pm}, \rho_{\ell}{}^{\pm}, \mathbf{J}^{\pm}\}$ *in two semi-Riemannian manifolds* $(M^{\pm}, g^{\pm})$ *with embeddings $\phi^{\pm}$ and riggings $\zeta^{\pm}$ respectively, the sign ϵ must be chosen positive if ζ^{-} points outwards with respect to* (M^{-}, g^{-}) *and negative otherwise.*

When evaluating the difference between the constraint equations (2.55)-(2.56) for the plus and the minus matter-hypersurface data, all purely metric terms cancel out (recall that $\overset{\circ}{\nabla}$ only depends on the metric part of the data). As proven in [58][4], the result of this operation is

$$
\epsilon[\rho_{\ell}] = \overset{\circ}{\nabla}_f T^f + T^d \left(\frac{1}{2} n^{(2)} \overset{\circ}{\nabla}_d \ell^{(2)} + \mathrm{F}_{df} n^f \right) - \tau^{df} \overline{\mathrm{Y}}_{df}, \tag{2.156}
$$

$$
\epsilon[J_c] = \overset{\circ}{\nabla}_f \tau^f{}_c + \tau^f{}_c \left(\frac{1}{2} n^{(2)} \overset{\circ}{\nabla}_f \ell^{(2)} + \mathrm{F}_{fa} n^a \right) + T^f \mathrm{U}_{cf}, \tag{2.157}
$$

where we have defined the tensor fields

$$
\overline{\mathbf{Y}} \stackrel{\text{def}}{=} \frac{1}{2}(\mathbf{Y}^{+} + \mathbf{Y}^{-}), \qquad \tau^f{}_c \stackrel{\text{def}}{=} \tau^{fa}\gamma_{ac}, \qquad T^f \stackrel{\text{def}}{=} \tau^{fa} \ell_a. \tag{2.158}
$$

[4] Observe that in [58] the sign ϵ has not been considered in the definition of τ. However, introducing ϵ in Definitions 2.7.2 and 2.7.3 is necessary for the reasons explained above.

Under the action of a gauge group element $\mathcal{G}_{(z,V)}$, the tensor field $[\mathbf{Y}]$ transforms as $\mathcal{G}_{(z,V)}([\mathbf{Y}]) = z[\mathbf{Y}]$ [58] and, as a consequence,

$$\mathcal{G}_{(z,V)}(\boldsymbol{\tau}) = \frac{\boldsymbol{\tau}}{|z|}. \tag{2.159}$$

Moreover, the tensor $\boldsymbol{\tau}$ coincides with the energy-momentum tensor T of (1.2) when γ is non-degenerate and the hypersurface data is expressed in the normal gauge, i.e. with $\boldsymbol{\ell} = 0$ and $\ell^{(2)} = \pm 1$ [58]. Consequently, the tensor field $\boldsymbol{\tau}$, which has the symmetries of an energy-momentum tensor, coincides with the energy-momentum tensor of the shell whenever it does not contain null points. Moreover, for null thin shells, it is straightforward to check that the definition of energy-momentum tensor provided in [63, Eq. (31)] by Barrabés and Israel yields precisely $\boldsymbol{\tau}$. Given a basis $\{e_a\}$ of $\Gamma(T\widetilde{\mathcal{N}})$ (recall that $\widetilde{\mathcal{N}}$ is the matching hypersurface of the resulting spacetime $(\mathcal{M}, g)$), one can also check that the quantity $\tau^{ab}e^{\mu}_a e^{\nu}_b$ gives the singular part of the Einstein tensor of $(\mathcal{M}, g)$, as it is written in [64, Eq. (71)]. The gauge behaviour of $\boldsymbol{\tau}$ turns out to be essential in the embedded case, as it ensures that the singular part of the Einstein tensor of the matched spacetime remains invariant under rescaling the normal vector ν.

All the reasons above justify the Definition 2.7.3 for the energy-momentum tensor on a thin shell [58], irrespectively of whether data is embedded in any space.

Observe that the dependence on $\boldsymbol{\tau}$ on the surface layer equations (2.156)-(2.157) is linear. This supports the interpretation of $\boldsymbol{\tau}$ as the energy-momentum tensor of the shell, and also implies that $\boldsymbol{\tau} = 0$ is always a solution of the shell equations with $\{[\rho_\ell] = 0, [\mathrm{J}] = 0\}$. It is also worth stressing that at null points (and only there), $\boldsymbol{\tau} = 0$ does not necessarily imply $[\mathbf{Y}] = 0$. Indeed, in order to get $\boldsymbol{\tau} = 0$ when $n^{(2)} = 0$, it suffices to require $[\mathbf{Y}](n, \cdot) = 0$ and $\mathrm{tr}_P[\mathbf{Y}] = 0$, which does not mean that the whole tensor $[\mathbf{Y}]$ vanishes identically. It is precisely this property that allows us to conclude that the solution $\{[\rho_\ell] = 0, [\mathrm{J}] = 0, \boldsymbol{\tau} = 0\}$ does not necessarily correspond to the situation in which both matter-hypersurface data defining the shell are identical, but to scenarios in which a gravitational field with no contribution to the energy-momentum tensor and with support on the thin shell appears. Physically this type of thin shells describe pure impulsive gravitational waves and they can only exist on null points, as at non-null points we would require, in addition, that $P^{af}P^{bd}[\mathrm{Y}]_{ab} = 0$ for $\boldsymbol{\tau}$ to be zero, and this would entail that $0 = \gamma_{fi}\gamma_{dj}P^{af}P^{bd}[\mathrm{Y}]_{ab} = (\delta^a_i - n^a\ell_i)(\delta^b_j - n^b\ell_j)[\mathrm{Y}]_{ab} = [\mathrm{Y}]_{ij}$ (cf. (2.9)). This eventually means that non-trivial thin shells with vanishing energy-momentum tensor can only exist on null points.

We have already mentioned that the gauge behaviour of P, $[\rho_{\ell}]$ and $[J]$ is remarkably simple. This, however, does not occur for the metric hypersurface connection $\overset{\circ}{\nabla}$. A natural question is whether one can rewrite the shell equations in such a way that the gauge-dependence is explicit. For that it is convenient to express the surface layer equations (2.156)-(2.157) in an arbitrary coordinate frame, obtaining [58]

$$\epsilon[\rho_{\ell}] = \frac{1}{\sqrt{|\det \boldsymbol{A}|}}\partial_a\left(\sqrt{|\det \boldsymbol{A}|}\,P^{ab}\ell_b\right) - \frac{1}{2}P^{ab}\left(Y^{+}_{ab} + Y^{-}_{ab}\right), \tag{2.160}$$

$$\epsilon[J_a] = \frac{1}{\sqrt{|\det \boldsymbol{A}|}}\partial_b\left(\sqrt{|\det \boldsymbol{A}|}\,P^{bc}\gamma_{ca}\right) - \frac{1}{2}P^{bd}\partial_a\gamma_{bd}, \tag{2.161}$$

where we recall that $\boldsymbol{A}$ is defined in terms of the metric hypersurface data as in (2.4). Under the action of a gauge group element $\mathcal{G}_{(z,V)}$, $|\det \boldsymbol{A}|$ behaves as $\mathcal{G}_{(z,V)}(\det \boldsymbol{A}) = z^2 \det \boldsymbol{A}$ [58]. Using this it is straightforward to show that the shell equations satisfy the gauge-covariance properties described above.

For later purposes, we conclude this section by providing the definition of null thin shell.

Definition 2.7.5. *(Null thin shell) A null thin shell is a thin shell* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}^{\pm}, \rho_{\ell}{}^{\pm}, \mathbf{J}^{\pm}, \epsilon\}$ *such that* $\mathcal{N}$ *consists of null points exclusively.*

2.7.2 *Cut-and-paste method*

Thin shells have been traditionally constructed à là Darmois, i.e. by using the previously described distributional formalism defined on a spacetime endowed with a metric g which is globally continuous and differentiable (away from the thin shell hypersurface). In this framework, one defines a distribution associated to the metric and, since the metric is continuous, one can introduce a Riemann tensor in distributional form.

As we anticipated in the Introduction, Penrose [3], [85], [86], [87] presented an alternative procedure to construct explicit examples of null thin shells. This method relies on a distributional metric with a Dirac delta with support on a null hypersurface. The distributional Einstein field equations formally still make sense because the coordinates are selected so that the Einstein tensor depends linearly on the metric coefficients.

This so-called *cut-and-paste* construction method works for very specific spacetimes. Given one such special spacetime $(\mathcal{M}, g)$ and an embedded null hypersurface $\widetilde{\mathcal{N}}$, one first chooses appropriate lightlike coordinates adapted to $\overline{N}$. Then one "cuts" the spacetime, obtaining two separated regions that are later "pasted" again after some reorganization of points on one of the sides. The resulting spacetime contains a null thin shell (generically with no pressure and no energy flux) located on the hypersurface where the cut of the initial spacetime has taken place. In the original coordinates, the resulting spacetime presents a Dirac delta with support on the thin shell.

In their seminal works, Penrose introduced this idea in the spacetime of Minkowski, proposing a specific reorganization of points. With his construction, he could generate impulsive gravitational waves (or shells of null dust, as we shall see in Section 7.3.3) with plane and spherical topology propagating in the Minkowski spacetime. More complicated spacetimes have been successfully studied in later works. Among the many contributions in this regard, we stress [88], [89], [90], [91], [92], [4], [5], [6], [7] and references therein.

A natural question that arises is what is the relation between the cut-and-paste constructions and the matching conditions prescribed by the formalism of matching introduced before. Before the work that has lead to this thesis, there did not exist any systematic analysis of the connection between them. This problem, which we address in Chapters 7, 8, 9, constitutes the starting point of this thesis.

In the language of the formalism of hypersurface data, the matching conditions rely on the metric hypersurface data from both boundaries being the same. These metric hypersurface data depend on the spacetime geometry and on the embedding of the hypersurface on such spacetime. In the cut-and-paste construction, the ambient spacetime and the hypersurface (understood as a set of points) are the same on both regions. On the other hand, the redistribution of points that takes place on one of the spacetime regions forces the embeddings from both sides to be different. The requirement of the new embedding defining the same metric data restricts the set of all possible redistributions of points. Thus, the cut-and-paste formalism will be compatible with the matching procedure à là Darmois if and only if the redistribution of points leaves the metric hypersurface data invariant. As we shall see in Chapter 7, this is in fact the case in all cut-and-paste constructions. In that chapter, both formalisms will be connected, and later in Chapter 9 the matching à là cut-and-paste will be even described at a fully abstract level.

For its later use, it is convenient that we now examine the cut-and-paste construction of the plane-fronted wave in the 4-dimensional spacetime of Minkowski. We start by writing down the metric of any plane-fronted wave (see e.g. [118], [119]):

$$ds^2 = -2\left(dV + P\left(U, x, z\right) dU\right) dU + dx^2 + dz^2. \tag{2.162}$$

The spacetimes describing purely gravitational waves, i.e. solutions of the vacuum Einstein field equations, are those for which the function P verifies the condition

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) \mathrm{P} = 0. \tag{2.163}$$

In [86], Penrose addresses the impulsive case of (2.162) by setting $\mathrm{P}\,(\mathrm{U}, x, z)$ to zero except on the hypersurface defined by $\mathrm{U} = 0$, i.e. by enforcing

$$\mathrm{P}\,(\mathrm{U}, x, z) = \delta\,(\mathrm{U})\,\mathrm{H}\,(x, z)\,, \tag{2.164}$$

where δ denotes Dirac delta distribution and $\mathrm{H}\,(x, z)$ is any real function. Under these circumstances, the metric becomes

$$ds^2 = -2\,(d\mathrm{V} + \delta\,(\mathrm{U})\,\mathrm{H}\,(x, z)\,d\mathrm{U})\,d\mathrm{U} + dx^2 + dz^2. \tag{2.165}$$

The possibility to perform a coordinate change which turns (2.165) into a C^0-form is already mentioned by Penrose in [86], [87]. In fact, by writing (2.165) in the coordinates

$$\mathrm{U}, \mathrm{V}, \eta \overset{\text{def}}{=} \frac{1}{\sqrt{2}}(x + iz)\,, \bar{\eta} \overset{\text{def}}{=} \frac{1}{\sqrt{2}}(x - iz)\quad , \tag{2.166}$$

which yields

$$ds^2 = -2\,(d\mathrm{V} + \delta\,(\mathrm{U})\,\mathrm{H}\,(\eta, \bar{\eta})\,d\mathrm{U})\,d\mathrm{U} + 2d\eta d\bar{\eta}, \tag{2.167}$$

Podolský et al. [88], [5] found the suitable coordinate transformation, namely

$$\mathrm{U} = U, \qquad \mathrm{V} = V + \Theta(U)h + U_+(U)h_{,Z}\,h_{,\bar{Z}}, \qquad \eta = Z + U_+(U)h_{,\bar{Z}}, \tag{2.168}$$

where the comma denotes partial derivative, $\Theta(U)$ is the Heaviside step function, $U_+ \overset{\text{def}}{=} U\Theta(U)$ is the so-called kink function and $h\left(Z, \bar{Z}\right) \overset{\text{def}}{=} \mathrm{H}\,(\eta, \bar{\eta})\,\big|_{\mathrm{U}=0}$ is a real-valued function. Inserting (2.168) into (2.167), one obtains the following continuous metric[5]:

$$ds^2 = 2\left|dZ + U_+(U)\left(h_{,ZZ}\,dZ + h_{,Z\bar{Z}}\,d\bar{Z}\right)\right|^2 - 2dUdV. \tag{2.169}$$

The transformation (2.168) immediately shows that the lightlike coordinate V is discontinuous across the hypersurface $\mathrm{U} = 0$ and that the presence of the δ-function on (2.165) is due to this jump. More precisely, the discontinuous coordinates $\{\mathrm{U}, \mathrm{V}, \eta, \bar{\eta}\}$, chosen to preserve the Minkowski form of (2.165) on $\mathrm{U} \gtrless 0$, produce discontinuities on the metric, while with the continuous coordinates $\{U, V, Z, \bar{Z}\}$ the metric tensor becomes C^0 but loses the Minkowski form for $U > 0$. Never-

[5]As pointed out in [5], to obtain (2.169) one needs to use the distributional identities $\frac{d\Theta}{dU} = \delta$, $\frac{d(U\Theta)}{dU} = \Theta$, $\Theta^2 = \Theta$, which in general may lead to mathematical inconsistencies.

theless, the coordinates $\{U, V, \eta, \bar{\eta}\}$ are useful to understand this spacetime as the outcome of the disjoint union of $U > 0$ and $U < 0$ with a jump on V when crossing the hypersurface $U = 0$.

When applying the cut-and-paste method to the case of a plane-fronted impulsive wave in Minkowski, Penrose proposes a jump on the lightlike coordinate V of the form

$$V_{+}|_{U_{+}=0} = V_{-} + H(x_{-}, z_{-})|_{U_{-}=0}, \tag{2.170}$$

where $\{V_{\pm}, x_{\pm}, z_{\pm}\}$ refer to the coordinates $\{V, x, z\}$ on the regions $U \gtrless 0$ of (2.165) respectively. This jump follows directly from the coordinate transformation (2.168). It is also worth mentioning that the jump (2.170) is not exclusive of this specific cut-and-paste construction. A jump of this type also gives rise to null thin shells in the cut-and-paste construction corresponding non-expanding impulsive gravitational waves propagating in the spacetimes of (anti-)de Sitter (see e.g. [6]).

3

NEW RESULTS ON THE FORMALISM OF HYPERSURFACE DATA

Once we have presented the preliminary results that we shall need throughout this thesis, we can start with its actual developments. Concretely, in this chapter we concentrate on developing the formalism of hypersurface data itself.

The chapter is divided in three sections. In Section 3.1, we work with completely general metric hypersurface data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$. We start by providing the relation between the signatures of the ambient metric $\boldsymbol{A}$ and γ, by proving a result analogous to Lemma 2.2.8 taylored to the embedded case and by deriving useful identities concerning the metric hypersurface connection $\overset{\circ}{\nabla}$ and the Lie derivative of the data tensor γ. Then, we introduce the tensor "Lie derivative of a connection D along a vector field Z", and examine its properties in detail. Finally, we conclude with several results concerning the curvature tensor $\overset{\circ}{R}{}^{a}{}_{bcd}$ of $\overset{\circ}{\nabla}$.

In Section 3.2 we restrict ourselves to null hypersurface data. We first introduce the notion of null (metric) hypersurface data together with the explicit decomposition of γ, P and the energy-momentum tensor τ defined in (2.7.3) in a given basis. Then, we include several gauge-fixing results as well as several useful identities concerning the curvature tensors $\overset{\circ}{R}{}^{a}{}_{bcd}$ and $\overset{\circ}{R}_{ab}$. Afterwards, we study the particular case when the manifold $\mathcal{N}$ admits a submanifold S to which n is everywhere transverse. Under this assumption, we again provide gauge-fixing results and explore the relation between the metric hypersurface connection $\overset{\circ}{\nabla}$ and a general torsion-free connection ∇^{S} on S. This allows us to identify under which circumstances ∇^{S} coincides with the Levi-Civita connection of S. We also give a Gauss-type identity on S and derive explicit expressions for the pull-back to S of covariant and Lie derivatives of tensor fields. We conclude the section by analyzing what occurs if $\mathcal{N}$ admits a cross-section (i.e. a submanifold S which is intersected for each integral curve of n exactly once). In this context, the flexibility associated to the gauge freedom turns out to be a great advantage. We also recall the definition of characteristic hypersurface data, originally presented in [60], and compare it with the concept of null (metric) hypersurface data.

The final part of the chapter is devoted to analyzing some consequences of having a gauge-invariant vector field along the degenerate direction of $\mathcal{N}$.

3.1 general hypersurface data

As anticipated, we start by studying the relation between the signatures of the tensor fields $\boldsymbol{A}$ (cf. (2.4)) and γ. For non-null points, this aspect was discussed in [59, Lem. 2.7]. Here we give the corresponding result for null points. As in that reference, we view the signature of a quadratic form q as the (unordered) set $\operatorname{sign}(q) = \{0, ..., 0, -1, ..., -1, +1, ..., +1\}$ of diagonal entries corresponding to the canonical form of q.

Lemma 3.1.1. *Let* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ *be metric hypersurface data and* $p \in \mathcal{N}$ *a null point, i.e.* $\operatorname{Rad}(\gamma)|_p \neq \{0\}$. *Then the signatures of* $\gamma|_p$ *and* $\boldsymbol{A}|_p$ *are related by*

$$\operatorname{sign}(\boldsymbol{A}|_p) = \{-1, 1\} \sqcup (\operatorname{sign}(\gamma|_p) \setminus \{0\}). \tag{3.1}$$

where $\sqcup$ *is the disjoint union. In particular,* $\boldsymbol{A}|_p$ *has Lorentzian signature if and only if* $\gamma|_p$ *is semi-positive definite.*

Proof. Assume $n \stackrel{\text{def}}{=} \dim(\mathcal{N}) \geq 2$ (if $n = 1$ the proof is the same with small changes of notation). Since $\operatorname{Rad}(\gamma)|_p \neq \{0\}$, it must be one-dimensional. Let $\{e_a\}$ be a canonical basis of $\gamma|_p$ with $e_1 \in \operatorname{Rad}(\gamma)|_p$ and define $\epsilon_a \stackrel{\text{def}}{=} \gamma|_p(e_a, e_a)$, $s_A \stackrel{\text{def}}{=} \boldsymbol{\ell}|_p(e_A)$. Observe that $\epsilon_1 = 0$ and $\epsilon_A^2 = 1$. Then, the vectors

$$E_0 \stackrel{\text{def}}{=} (V, 1), \qquad E_a \stackrel{\text{def}}{=} (e_a, 0), \qquad \text{with} \qquad V \stackrel{\text{def}}{=} -\sum_{B=2}^{n} \epsilon_B s_B e_B \in T_p\mathcal{N} \tag{3.2}$$

form a basis of $T_p\mathcal{N} \times \mathbb{R}$. From (2.4) we get

$$\boldsymbol{A}|_p(E_0, E_0) = \gamma|_p(V, V) + 2\boldsymbol{\ell}|_p(V) + \ell^{(2)} \stackrel{\text{def}}{=} C, \qquad \boldsymbol{A}|_p(E_0, E_1) = \boldsymbol{\ell}|_p(e_1), \tag{3.3}$$

$$\boldsymbol{A}|_p(E_0, E_A) = \gamma|_p(V, e_A) + \boldsymbol{\ell}|_p(e_A), \qquad \boldsymbol{A}|_p(E_1, E_1) = 0, \tag{3.4}$$

$$\boldsymbol{A}|_p(E_A, E_B) = \gamma|_p(e_A, e_B) = \delta_{AB}\epsilon_A, \qquad \boldsymbol{A}|_p(E_1, E_A) = 0. \tag{3.5}$$

Since $\boldsymbol{A}|_p$ is non-degenerate, $\boldsymbol{\ell}|_p(e_1) \neq 0$ and we can introduce the vectors

$$\hat{E}_0 \stackrel{\text{def}}{=} E_0 - \frac{1+C}{2(\boldsymbol{\ell}|_p(e_1))} E_1, \qquad \hat{E}_1 \stackrel{\text{def}}{=} -E_0 - \frac{1-C}{2(\boldsymbol{\ell}|_p(e_1))} E_1, \qquad \hat{E}_A \stackrel{\text{def}}{=} E_A \tag{3.6}$$

A simple computation yields

$$\boldsymbol{A}|_p(\hat{E}_0, \hat{E}_0) = -1, \qquad \boldsymbol{A}|_p(\hat{E}_0, \hat{E}_1) = 0, \qquad \boldsymbol{A}|_p(\hat{E}_0, \hat{E}_A) = 0, \tag{3.7}$$

$$\boldsymbol{A}|_p(\hat{E}_1, \hat{E}_1) = 1, \qquad \boldsymbol{A}|_p(\hat{E}_1, \hat{E}_A) = 0, \qquad \boldsymbol{A}|_p(\hat{E}_A, \hat{E}_B) = \delta_{AB}\epsilon_A. \tag{3.8}$$

Thus, $\{\hat{E}_0, \hat{E}_a\}$ is a canonical basis of $\boldsymbol{A}|_p$ and $\text{sign}(\boldsymbol{A}|_p) = \{-1, 1, \epsilon_2, ..., \epsilon_n\}$, which proves (3.1). The last claim is immediate. □

In the previous chapter we have recalled Lemma 2.2.8, which finds a unique vector field from a covector and a scalar function on $\mathcal{N}$ satisfying suitable restrictions. This result will be of much use in this thesis. In addition we will also require a related result which applies to the embedded case. This analogous statement and its proof are as follows.

Lemma 3.1.2. *Let* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ *be hypersurface data embedded on a semi-riemannian manifold* $(\mathcal{M}, g)$ *with embedding* ϕ *and rigging* ζ. *Given a covector* $\boldsymbol{\beta}$ *along* $\mathcal{N}$ *and a scalar function* $v_0 \in \mathcal{F}(\mathcal{N})$, *only the vector field* $V = (\boldsymbol{\beta}(n) + n^{(2)}v_0)\zeta + \phi_\star(P(\boldsymbol{\beta}, \cdot) + v_0 n) \in \Gamma(T\mathcal{M})|_{\phi(\mathcal{N})}$ *solves the equations* $\boldsymbol{\beta} = \phi^\star(g(V, \cdot))$ *and* $v_0 = \phi^\star(g(\zeta, V))$.

Proof. Consider a local basis $\{\hat{e}_a\}$ of $\Gamma(T\mathcal{N})$ and its image basis $\{e_a\}$ of $\Gamma(T\phi(\mathcal{N}))$. Then $\{\zeta, e_a\}$ form a basis of $\Gamma(T\mathcal{M})|_{\phi(\mathcal{N})}$ so that any $V \in \Gamma(T\mathcal{M})|_{\phi(\mathcal{N})}$ can be decomposed as $V = \alpha\zeta + W^a e_a$ for suitable scalar functions $\alpha, W^a \in \mathcal{F}(\phi(\mathcal{N}))$. Thus, by defining $\boldsymbol{V} \stackrel{\text{def}}{=} g(V, \cdot)$, it follows

$$(\phi^\star(\boldsymbol{V}))_b = \alpha\ell_b + \gamma_{ab}W^a, \qquad \phi^\star(\boldsymbol{V}(\zeta)) = \alpha\ell^{(2)} + W^a\ell_a. \tag{3.9}$$

Therefore, V is a solution of the equations $\boldsymbol{\beta} = \phi^\star(\boldsymbol{V})$, $v_0 = \phi^\star(g(\zeta, V))$ if and only if $\gamma_{ab}W^a = \beta_b - \alpha\ell_b$ and $\ell_a W^a = v_0 - \alpha\ell^{(2)}$. By Lemma 2.2.8, there exists a unique solution $W^a = P^{ab}(\beta_b - \alpha\ell_b) + (v_0 - \alpha\ell^{(2)})n^a$ if and only if $(\beta_b - \alpha\ell_b)n^b + n^{(2)}(v_0 - \alpha\ell^{(2)})$ vanishes identically. This is equivalent to

$$0 = \beta_b n^b - \alpha(1 - n^{(2)}\ell^{(2)}) + n^{(2)}v_0 - n^{(2)}\ell^{(2)}\alpha = \beta_b n^b - \alpha + n^{(2)}v_0,$$

and hence $\alpha = \beta_b n^b + n^{(2)}v_0$. Using (2.8), it follows $W^a = P^{ab}\beta_b - \alpha P^{ab}\ell_b + (v_0 - \alpha\ell^{(2)})n^a = P^{ab}\beta_b + v_0 n^a$. □

There will be somewhat heavy computations below involving the connection $\overset{\circ}{\nabla}$ defined in Theorem 2.2.4. The derivations will be aided by several identities that will be used repeatedly. We start by linking the $\overset{\circ}{\nabla}$ derivatives of one-forms or symmetric and antisymmetric tensor fields with Lie derivatives and exterior derivatives.

Lemma 3.1.3. *Let $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ be metric hypersurface data and θ_a, S_{ab}, A_{ab} tensor fields on $\mathcal{N}$ with the symmetries $S_{ab} = S_{(ab)}$ and $A_{ab} = A_{[ab]}$. Then,*

$$n^b\left(\mathring{\nabla}_b\theta_d - \mathring{\nabla}_d\theta_b\right) = \pounds_n\theta_d - \mathring{\nabla}_d(\boldsymbol{\theta}(n)), \tag{3.10}$$

$$\begin{aligned} n^b\left(\mathring{\nabla}_b\theta_d + \mathring{\nabla}_d\theta_b\right) &= \pounds_n\theta_d + \mathring{\nabla}_d(\boldsymbol{\theta}(n)) \\ &\quad - 2\left(\boldsymbol{\theta}(n)\left(s_d - n^{(2)}\mathring{\nabla}_d\ell^{(2)}\right) + P^{ab}\theta_b\left(\mathrm{U}_{da} - n^{(2)}\mathrm{F}_{da}\right)\right), \end{aligned} \tag{3.11}$$

$$n^c\left(\mathring{\nabla}_d S_{cb} - \mathring{\nabla}_c S_{db}\right) = \mathring{\nabla}_d\left(S_{bc}n^c\right) - \pounds_n S_{bd} + S_{cd}\mathring{\nabla}_b n^c \tag{3.12}$$

$$n^c\left(\mathring{\nabla}_d A_{cb} - \mathring{\nabla}_c A_{db}\right) = \mathring{\nabla}_{(b}a_{d)} - A_{c(b}\mathring{\nabla}_{d)}n^c + \frac{1}{2}n^c(dA)_{dcb} - \frac{1}{2}n^c\mathring{\nabla}_c A_{db} \tag{3.13}$$

$$n^c\left(\mathring{\nabla}_d A_{cb} - \mathring{\nabla}_c A_{db}\right) = n^c(dA)_{dcb} + \mathring{\nabla}_b a_d - A_{cd}\mathring{\nabla}_b n^c \tag{3.14}$$

where $dA_{abc} \stackrel{\text{def}}{=} 3\mathring{\nabla}_{[a}A_{bc]}$ *and* $a_a \stackrel{\text{def}}{=} n^c A_{ca}$.

Proof. Since the connection $\mathring{\nabla}$ has no torsion, the Lie derivative of any covariant tensor $T_{a_1\cdots a_p}$ along any direction V is

$$(\pounds_V T)_{a_1\cdots a_p} = V^b\mathring{\nabla}_b T_{a_1\cdots a_p} + \sum_{i=1}^{p} T_{a_1\cdots a_{i-1}ba_{i+1}\cdots a_p}\mathring{\nabla}_{a_i}V^b. \tag{3.15}$$

We will use this repeatedly. Equation (3.10) follows from $n^b(\mathring{\nabla}_b\theta_d - \mathring{\nabla}_d\theta_b) = n^b\mathring{\nabla}_b\theta_d + \theta_b\mathring{\nabla}_d n^b - \mathring{\nabla}_d(\boldsymbol{\theta}(n))$. Moreover,

$$n^b\left(\mathring{\nabla}_b\theta_d + \mathring{\nabla}_d\theta_b\right) = n^b\mathring{\nabla}_b\theta_d + \mathring{\nabla}_d(\boldsymbol{\theta}(n)) - \theta_b\mathring{\nabla}_d n^b = \pounds_n\theta_d + \mathring{\nabla}_d(\boldsymbol{\theta}(n)) - 2\theta_b\mathring{\nabla}_d n^b,$$

which yields (3.11) after using (2.20). For the symmetric tensor S, (3.15) gives

$$\begin{aligned} n^c\left(\mathring{\nabla}_d S_{cb} - \mathring{\nabla}_c S_{db}\right) &= n^c\mathring{\nabla}_d S_{cb} + S_{cb}\mathring{\nabla}_d n^c + S_{dc}\mathring{\nabla}_b n^c - \pounds_n S_{bd} \\ &= \mathring{\nabla}_d\left(S_{bc}n^c\right) - \pounds_n S_{db} + S_{cd}\mathring{\nabla}_b n^c, \end{aligned}$$

which is (3.12). For the antisymmetric tensor A, we use $(dA)_{dcb} \stackrel{\text{def}}{=} 3\mathring{\nabla}_{[d}A_{cb]} = \mathring{\nabla}_d A_{cb} - \mathring{\nabla}_b A_{cd} - \mathring{\nabla}_c A_{db}$ and find

$$\begin{aligned} n^c\left(\mathring{\nabla}_d A_{cb} - \mathring{\nabla}_c A_{db}\right) &= \frac{1}{2}n^c\left(\mathring{\nabla}_d A_{cb} + \mathring{\nabla}_b A_{cd} + \mathring{\nabla}_d A_{cb} - \mathring{\nabla}_b A_{cd} - \mathring{\nabla}_c A_{db} - \mathring{\nabla}_c A_{db}\right) \\ &= \frac{1}{2}\Big(\mathring{\nabla}_d(A_{cb}n^c) + \mathring{\nabla}_b(A_{cd}n^c) - A_{cb}\mathring{\nabla}_d n^c - A_{cd}\mathring{\nabla}_b n^c \end{aligned}$$

$$
\left. + n^c (dA)_{dcb} - n^c \overset{\circ}{\nabla}_c A_{db} \right) = \overset{\circ}{\nabla}_{(b} a_{d)} - A_{c(b} \overset{\circ}{\nabla}_{d)} n^c + \frac{1}{2} n^c (dA)_{dcb} - \frac{1}{2} n^c \overset{\circ}{\nabla}_c A_{db},
$$

which is (3.13). Moreover, we also find

$$
\begin{aligned}
n^c \left(\overset{\circ}{\nabla}_d A_{cb} - \overset{\circ}{\nabla}_c A_{db} \right) &= n^c \left(\overset{\circ}{\nabla}_d A_{cb} + \overset{\circ}{\nabla}_c A_{bd} + \overset{\circ}{\nabla}_b A_{dc} + \overset{\circ}{\nabla}_b A_{cd} \right) \\
&= n^c (dA)_{dcb} + \overset{\circ}{\nabla}_b \left(n^c A_{cd} \right) - A_{cd} \overset{\circ}{\nabla}_b n^c,
\end{aligned}
$$

which is the alternative form (3.14). □

Next we provide the relation between the Lie derivative of the "metric" tensor γ along a general direction V and $\overset{\circ}{\nabla}$ covariant derivatives of a covector geometrically constructed from V.

Lemma 3.1.4. *Let $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ be metric hypersurface data, V^a any vector field and w_a any covector field. Define $\check{V}_a \stackrel{\text{def}}{=} \gamma_{ab} V^b$ and $\hat{w}^a \stackrel{\text{def}}{=} P^{ab} w_b$. Then the following identities hold*

$$
\frac{1}{2} \pounds_V \gamma_{ab} = \boldsymbol{\ell}(V) \mathrm{U}_{ab} + \overset{\circ}{\nabla}_{(a} \check{V}_{b)}, \tag{3.16}
$$

$$
\frac{1}{2} \pounds_{\hat{w}} \gamma_{ab} = \overset{\circ}{\nabla}_{(a} w_{b)} - \ell_{(a} \overset{\circ}{\nabla}_{b)} w(n). \tag{3.17}
$$

Proof. We first note that $\overset{\circ}{\nabla}_c \gamma_{ab} - \overset{\circ}{\nabla}_a \gamma_{bc} - \overset{\circ}{\nabla}_b \gamma_{ac} = 2 \ell_c \mathrm{U}_{ab}$ as a direct consequence of (2.18). Applying (3.15) to $T = \gamma$ we get

$$
\begin{aligned}
\pounds_V \gamma_{ab} &= V^c \overset{\circ}{\nabla}_c \gamma_{ab} + \gamma_{bc} \overset{\circ}{\nabla}_a V^c + \gamma_{ac} \overset{\circ}{\nabla}_b V^c \\
&= V^c \left(\overset{\circ}{\nabla}_c \gamma_{ab} - \overset{\circ}{\nabla}_a \gamma_{bc} - \overset{\circ}{\nabla}_b \gamma_{ac} \right) + \overset{\circ}{\nabla}_a \check{V}_b + \overset{\circ}{\nabla}_b \check{V}_a = 2 \boldsymbol{\ell}(V) \mathrm{U}_{ab} + 2 \overset{\circ}{\nabla}_{(a} \check{V}_{b)}
\end{aligned}
$$

which is (3.16). To prove the second identity we apply (3.16) to $V = \hat{w}$. Since by (2.9) we have $\gamma_{ab} P^{bc} w_c = w_a - w(n) \ell_a$, identity (3.16) gives

$$
\frac{1}{2} \pounds_{\hat{w}} \gamma_{ab} = \boldsymbol{\ell}(\hat{w}) \mathrm{U}_{ab} + \overset{\circ}{\nabla}_{(a} \left(w_{b)} - w(n) \ell_{b)} \right).
$$

From (2.8), we find $\boldsymbol{\ell}(\hat{w}) = -\ell^{(2)} w(n)$. Inserting above yields

$$
\frac{1}{2} \pounds_{\hat{w}} \gamma_{ab} = -\ell^{(2)} w(n) \mathrm{U}_{ab} + \overset{\circ}{\nabla}_{(a} w_{b)} - \ell_{(a} \overset{\circ}{\nabla}_{b)} w(n) - w(n) \overset{\circ}{\nabla}_{(a} \ell_{b)},
$$

which simplifies to (3.17) after taking into account (2.19). □

We stress that all the results so far in this section are valid for general metric hypersurface data. Note also that the extrinsic part $\mathbf{Y}$ of the data has played no role in any of them. This of course is a consequence of the fact that $\overset{\circ}{\nabla}$ is completely independent of $\mathbf{Y}$.

3.1.1 *The Lie derivative of the connection $\overset{\circ}{\nabla}$ along n*

As anticipated, we now introduce the tensor field "Lie derivative of a connection". This tensor carries useful information on the curvature and, as we will see in Chapters 5 and 6, plays a key role in the study of the geometry of horizons. We first provide some basic general results and then we derive its explicit form in the case when the connection is precisely the metric hypersurface connection $\overset{\circ}{\nabla}$. For general properties of this tensor we refer to [120].

Given any smooth manifold $\mathcal{M}$ endowed with an affine connection D and a vector field Z, the tensor field Lie derivative of D along Z, denoted by Σ_Z, is defined by

$$\Sigma_Z(X,W) \overset{\text{def}}{=} \pounds_Z D_X W - D_X \pounds_Z W - D_{\pounds_Z X} W \tag{3.18}$$

for any pair of vector fields $X, W \in \Gamma(T\mathcal{M})$. This tensor only depends on the vector field Z and on the connection D, so we will use the notation $\Sigma_Z = \pounds_Z D$ in the following.

When D is torsion-free, this tensor is symmetric. Indeed, using $\pounds_{X_1} X_2 = D_{X_1} X_2 - D_{X_2} X_1$ a few times, one gets

$$\begin{aligned}\Sigma_Z(X,W) - \Sigma_Z(W,X) &= \pounds_Z \pounds_X W - D_X \pounds_Z W + D_W \pounds_Z X - D_{\pounds_Z X} W + D_{\pounds_Z W} X \\ &= \pounds_Z \pounds_X W + \pounds_X \pounds_W Z + \pounds_W \pounds_Z X = 0,\end{aligned}$$

the last equality being a consequence of the Jacobi identity.

The tensor field Σ_Z plays an important role whenever the Lie derivative $\pounds_Z$ of a covariant derivative of a tensor field needs to be computed. In the case of one-forms $\boldsymbol{\theta}$, the fact that

$$\pounds_Z D_X\left(\boldsymbol{\theta}(W)\right) - D_X \pounds_Z\left(\boldsymbol{\theta}(W)\right) - D_{\pounds_Z X}\left(\boldsymbol{\theta}(W)\right) = \left(\pounds_Z X\right)\left(\boldsymbol{\theta}(W)\right) + \left(\pounds_X Z\right)\left(\boldsymbol{\theta}(W)\right) = 0$$

combined with (3.18) gives

$$\begin{aligned}0 &= \pounds_Z\left((D_X\boldsymbol{\theta})(W) + \boldsymbol{\theta}(D_X W)\right) - D_X\left(\boldsymbol{\theta}(\pounds_Z W) + (\pounds_Z\boldsymbol{\theta})(W)\right) - D_{\pounds_Z X}\left(\boldsymbol{\theta}(W)\right) \\ &= \pounds_Z\left(D_X\boldsymbol{\theta}\right)(W) - D_X\left(\pounds_Z\boldsymbol{\theta}\right)(W) - D_{\pounds_Z X}\boldsymbol{\theta}\,(W) + \boldsymbol{\theta}(\Sigma_Z(X,W))\end{aligned}$$

$$\Longleftrightarrow \quad (\pounds_Z D_X \boldsymbol{\theta})(W) = (D_X \pounds_Z \boldsymbol{\theta})(W) + \left(D_{\pounds_Z X}\boldsymbol{\theta}\right)(W) - \boldsymbol{\theta}(\Sigma_Z(X,W)), \quad (3.19)$$

or, in abstract index notation,

$$\pounds_Z\, D_\alpha\theta_\beta = D_\alpha\, \pounds_Z\theta_\beta - (\Sigma_Z)^\mu{}_{\alpha\beta}\theta_\mu. \tag{3.20}$$

This identity can be extended to any p-covariant tensor field T. Specifically, it holds

$$\pounds_Z D_\alpha T_{\beta_1\cdots\beta_p} = D_\alpha \pounds_Z T_{\beta_1\cdots\beta_p} - \sum_{i=1}^{p}(\Sigma_Z)^\mu{}_{\alpha\beta_i} T_{\beta_1\cdots\beta_{i-1}\mu\beta_{i+1}\cdots\beta_p}. \tag{3.21}$$

Let us now study several properties of the tensor Σ_Z, first for general manifolds equipped with a torsion-free connection and then for general hypersurface data and for the metric hypersurface connection $\overset{\circ}{\nabla}$. These results will be helpful later.

Lemma 3.1.5. [120] *Let $\mathcal{M}$ be a manifold endowed with a torsion-free connection D. Then, for any $X, W, Z \in \Gamma(T\mathcal{M})$, it holds*

$$\Sigma_Z(X,W) = D_X D_W Z - D_{D_X W} Z + R^D(Z,X)W \quad \textit{or, in index notation,}$$
$$(\Sigma_Z)^\mu{}_{\alpha\beta} = D_\alpha D_\beta Z^\mu + R^D{}^\mu{}_{\beta\nu\alpha} Z^\nu. \tag{3.22}$$

Proof. We will use $\pounds_X Y = D_X Y - D_Y X$ throughout, in particular we expand each Lie bracket of (3.18). Combining (3.18) with the definition $R^D(Z,X)W = D_Z D_X W - D_X D_Z W - D_{\pounds_Z X} W$ for the curvature tensor R^D of D yields

$$\begin{aligned}\Sigma_Z(X,W) &= D_Z D_X W - D_{D_X W} Z - D_X D_Z W + D_X D_W Z - D_{\pounds_Z X} W\\ &= D_X D_W Z - D_{D_X W} Z + R^D(Z,X)W,\end{aligned}$$

which is (3.22). Now from the fact that

$$(D_X D_W Z)^\mu = (X^\alpha D_\alpha W^\beta)(D_\beta Z^\mu) + X^\alpha W^\beta D_\alpha D_\beta Z^\mu = (D_{D_X W} Z)^\mu + X^\alpha W^\beta D_\alpha D_\beta Z^\mu$$

the second equation in (3.22) follows at once. □

Lemma 3.1.6. *Let $\mathcal{M}$ be a manifold, D a torsion-free connection, $Z \in \Gamma(T\mathcal{M})$ a vector field and $S_{\alpha\beta}$ a symmetric 2-covariant tensor field. Define $\overset{Z}{Q}_{\alpha\mu\nu} \stackrel{\text{def}}{=} D_\alpha \pounds_Z S_{\mu\nu} - \pounds_Z D_\alpha S_{\mu\nu}$. Then,*

$$(\Sigma_Z)^\lambda{}_{\alpha\mu} S_{\lambda\nu} = \frac{1}{2}\left(\overset{Z}{Q}_{\alpha\mu\nu} + \overset{Z}{Q}_{\mu\nu\alpha} - \overset{Z}{Q}_{\nu\alpha\mu}\right). \tag{3.23}$$

In particular, if $S_{\alpha\beta}$ verifies $D_\mu S_{\alpha\beta} = 0$, it holds

$$(\Sigma_Z)^\lambda{}_{\alpha\mu} S_{\lambda\nu} = \frac{1}{2}\left(D_\alpha \pounds_Z S_{\mu\nu} + D_\mu \pounds_Z S_{\nu\alpha} - D_\nu \pounds_Z S_{\alpha\mu}\right). \tag{3.24}$$

Proof. Since D is torsion free, $(\Sigma_Z)^\mu{}_{\alpha\beta}$ is symmetric in α, β. Particularizing (3.21) for $S_{\alpha\beta}$ yields

$$0 = \mathrm{Q}^Z_{\alpha\mu\nu} - (\Sigma_Z)^\lambda{}_{\alpha\mu} S_{\lambda\nu} - (\Sigma_Z)^\lambda{}_{\alpha\nu} S_{\mu\lambda}, \tag{3.25}$$

$$0 = \mathrm{Q}^Z_{\mu\nu\alpha} - (\Sigma_Z)^\lambda{}_{\mu\nu} S_{\lambda\alpha} - (\Sigma_Z)^\lambda{}_{\mu\alpha} S_{\nu\lambda}, \tag{3.26}$$

$$0 = \mathrm{Q}^Z_{\nu\alpha\mu} - (\Sigma_Z)^\lambda{}_{\nu\alpha} S_{\lambda\mu} - (\Sigma_Z)^\lambda{}_{\nu\mu} S_{\alpha\lambda}. \tag{3.27}$$

where (3.26)-(3.27) arise from the cyclic permutation of the indices α, μ, ν. Substracting (3.27) to the sum of (3.25)-(3.26) gives (3.23) because $(\Sigma_Z)^\mu{}_{\alpha\beta}$, $S_{\alpha\beta}$ are symmetric in α, β. When $S_{\alpha\beta}$ is covariantly constant we have $\mathrm{Q}^Z_{\alpha\mu\nu} \stackrel{\text{def}}{=} D_\alpha \pounds_Z S_{\mu\nu}$ and equation (3.24) follows at once. ☐

Corollary 3.1.7. *If $S_{\alpha\beta}$ is non-degenerate everywhere on* M*, then we can define its inverse tensor $S_\sharp^{\alpha\beta}$ by $S_\sharp^{\alpha\mu} S_{\mu\beta} = \delta^\alpha_\beta$ and it follows that*

$$(\Sigma_Z)^\lambda{}_{\alpha\mu} = \frac{1}{2} S_\sharp^{\nu\lambda} \left(\mathrm{Q}^Z_{\alpha\mu\nu} + \mathrm{Q}^Z_{\mu\nu\alpha} - \mathrm{Q}^Z_{\nu\alpha\mu} \right). \tag{3.28}$$

In particular, when $S_{\alpha\beta}$ verifies $D_\mu S_{\alpha\beta} = 0$ then

$$(\Sigma_Z)^\lambda{}_{\alpha\mu} = \frac{1}{2} S_\sharp^{\nu\lambda} \left(D_\alpha \pounds_Z S_{\mu\nu} + D_\mu \pounds_Z S_{\nu\alpha} - D_\nu \pounds_Z S_{\alpha\mu} \right). \tag{3.29}$$

A metric hypersurface data set gives rise to a privileged vector field n on N . In the null case, this vector is even more privileged, as the direction (but not the scale) of n remains unchanged by arbitrary gauge transformations. It therefore makes sense to study the properties of the tensor $\mathring{\Sigma} \stackrel{\text{def}}{=} \pounds_n \mathring{\nabla}$ which, for completely general data $\{\mathrm{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$, is defined by

$$\mathring{\Sigma}(X, W) \stackrel{\text{def}}{=} \pounds_n \mathring{\nabla}_X W - \mathring{\nabla}_X \pounds_n W - \mathring{\nabla}_{\pounds_n X} W. \tag{3.30}$$

Our next aim is to provide the explicit form of $\mathring{\Sigma}$, for which we shall use the fact that $\mathring{\nabla}$ is uniquely determined by the properties (2.18)-(2.19). For simplicity, for calculations at the abstract level we no longer reflect the fact that the tensors $\mathring{\Sigma}$ and Q depend on n.

Lemma 3.1.8. *Let $\{\mathrm{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ be metric hypersurface data and define $\mathring{\Sigma} \stackrel{\text{def}}{=} \pounds_n \mathring{\nabla}$ by* (3.30)*. Then, $\mathring{\Sigma}$ is explicitly given by*

$$\mathring{\Sigma}^d{}_{ab} = n^d \left(2\mathring{\nabla}_{(a} s_{b)} - n^{(2)} \mathring{\nabla}_a \mathring{\nabla}_b \ell^{(2)} - 2\mathring{\nabla}_{(a} n^{(2)} \mathring{\nabla}_{b)} \ell^{(2)} + n(\ell^{(2)}) \mathrm{U}_{ab} \right. \tag{3.31}$$

$$+ P^{dc}\Big(\overset{\circ}{\nabla}_a \mathrm{U}_{bc} + \overset{\circ}{\nabla}_b \mathrm{U}_{ca} - \overset{\circ}{\nabla}_c \mathrm{U}_{ab} + \Big(2s_c - n^{(2)}\overset{\circ}{\nabla}_c \ell^{(2)}\Big)\mathrm{U}_{ab} + 2F_{c(a}\overset{\circ}{\nabla}_{b)} n^{(2)}\Big).$$

Proof. Particularizing (3.21) and (3.23) for $D = \overset{\circ}{\nabla}$, $T = \boldsymbol{\ell}$, $S = \gamma$ and $Z = n$ gives

$$\ell_f \overset{\circ}{\Sigma}{}^f{}_{ab} = \overset{\circ}{\nabla}_a \pounds_n \ell_b - \pounds_n \overset{\circ}{\nabla}_a \ell_b \stackrel{\text{def}}{=} Q_{ab}, \tag{3.32}$$

$$\gamma_{cf}\overset{\circ}{\Sigma}{}^f{}_{ab} = \frac{1}{2}\Big(\mathcal{Q}_{abc} + \mathcal{Q}_{bca} - \mathcal{Q}_{cab}\Big), \tag{3.33}$$

where $\mathcal{Q}_{abc} \stackrel{\text{def}}{=} \overset{\circ}{\nabla}_a \pounds_n \gamma_{bc} - \pounds_n \overset{\circ}{\nabla}_a \gamma_{bc}$. From these we will retrieve the explicit form of the tensor $\overset{\circ}{\Sigma}$. We start by computing Q_{ab} and $\mathcal{Q}_{abc}$ explicitly. For the first we recall (2.13) and (2.19), namely $\pounds_n \boldsymbol{\ell} = 2\boldsymbol{s} - d(n^{(2)}\ell^{(2)})$ and $\overset{\circ}{\nabla}_a \ell_b = \mathrm{F}_{ab} - \ell^{(2)}\mathrm{U}_{ab}$, and use $\pounds_n \mathbf{F} = \frac{1}{2}\pounds_n d\boldsymbol{\ell} = \frac{1}{2} d\pounds_n \boldsymbol{\ell} = d\boldsymbol{s}$, i.e. $\pounds_n \mathrm{F}_{ab} = \overset{\circ}{\nabla}_a s_b - \overset{\circ}{\nabla}_b s_a$. Then

$$\begin{aligned} Q_{ab} &= \overset{\circ}{\nabla}_a \pounds_n \ell_b - \pounds_n \overset{\circ}{\nabla}_a \ell_b = 2\overset{\circ}{\nabla}_a s_b - \overset{\circ}{\nabla}_a \overset{\circ}{\nabla}_b \Big(n^{(2)}\ell^{(2)}\Big) - \pounds_n\Big(\mathrm{F}_{ab} - \ell^{(2)}\mathrm{U}_{ab}\Big) \\ &= \overset{\circ}{\nabla}_a s_b + \overset{\circ}{\nabla}_b s_a - \overset{\circ}{\nabla}_a \overset{\circ}{\nabla}_b \Big(n^{(2)}\ell^{(2)}\Big) + n(\ell^{(2)})\mathrm{U}_{ab} + \ell^{(2)}\pounds_n \mathrm{U}_{ab}. \end{aligned} \tag{3.34}$$

For the second we recall (2.12) and (2.18), namely $\pounds_n \gamma_{bc} = 2\mathrm{U}_{bc} - 2\ell_{(b}\overset{\circ}{\nabla}_{c)} n^{(2)}$ and $\overset{\circ}{\nabla}_a \gamma_{bc} = -\ell_b \mathrm{U}_{ac} - \ell_c \mathrm{U}_{ab}$, so

$$\begin{aligned} \mathcal{Q}_{abc} &= \overset{\circ}{\nabla}_a\Big(2\mathrm{U}_{bc} - 2\ell_{(b}\overset{\circ}{\nabla}_{c)} n^{(2)}\Big) + \pounds_n\left(\ell_b \mathrm{U}_{ac} + \ell_c \mathrm{U}_{ab}\right) \\ &= 2\overset{\circ}{\nabla}_a \mathrm{U}_{bc} + 2\Big(-\mathrm{F}_{a(b} + \ell^{(2)}\mathrm{U}_{a(b}\Big)\overset{\circ}{\nabla}_{c)} n^{(2)} \\ &\quad - 2\ell_{(b}\overset{\circ}{\nabla}_{c)}\overset{\circ}{\nabla}_a n^{(2)} + 2(\pounds_n \ell_{(b})\mathrm{U}_{c)a} + 2\ell_{(b}\pounds_n \mathrm{U}_{c)a}, \end{aligned}$$

where in the second equality we inserted (2.19). Using the expression (2.13) for $\pounds_n \boldsymbol{\ell}$ (recalled just before (3.34)) yields, after simple cancellations,

$$\begin{aligned} \mathcal{Q}_{abc} &= 2\overset{\circ}{\nabla}_a \mathrm{U}_{bc} - 2\mathrm{F}_{a(b}\overset{\circ}{\nabla}_{c)} n^{(2)} \\ &\quad + 2\ell_{(b}\Big(\pounds_n \mathrm{U}_{c)a} - \overset{\circ}{\nabla}_{c)}\overset{\circ}{\nabla}_a n^{(2)}\Big) + 2\Big(2s_{(b} - n^{(2)}\overset{\circ}{\nabla}_{(b}\ell^{(2)}\Big)\mathrm{U}_{c)a}. \end{aligned} \tag{3.35}$$

Now, any $(0,3)$-tensor of the form $t_{abc} = 2u_{(b}S_{c)a}$ with S_{ca} symmetric satisfies $t_{abc} + t_{bca} - t_{cab} = 2u_c S_{ab}$. If $t_{abc} = 2A_{a(b}u_{c)}$ with A_{ca} antisymmetric then $t_{abc} + t_{bca} - t_{cab} = -4A_{c(a}u_{b)}$. Inserting (3.35) into (3.33) and using these properties gives

$$\begin{aligned} \gamma_{cf}\overset{\circ}{\Sigma}{}^f{}_{ab} &= \overset{\circ}{\nabla}_a \mathrm{U}_{bc} + \overset{\circ}{\nabla}_b \mathrm{U}_{ca} - \overset{\circ}{\nabla}_c \mathrm{U}_{ab} + \ell_c\Big(\pounds_n \mathrm{U}_{ab} - \overset{\circ}{\nabla}_a \overset{\circ}{\nabla}_b n^{(2)}\Big) \\ &\quad + \Big(2s_c - n^{(2)}\overset{\circ}{\nabla}_c \ell^{(2)}\Big)\mathrm{U}_{ab} + 2\mathrm{F}_{c(a}\overset{\circ}{\nabla}_{b)} n^{(2)}. \end{aligned} \tag{3.36}$$

To conclude the proof we use

$$\mathring{\Sigma}^d{}_{ab} = \delta^d_f \mathring{\Sigma}^f{}_{ab} \overset{(2.9)}{=} \left(P^{dc}\gamma_{cf} + n^d \ell_f \right) \mathring{\Sigma}^f{}_{ab} = P^{dc} \left(\gamma_{cf}\mathring{\Sigma}^f{}_{ab} \right) + n^d Q_{ab}.$$

Replacing here (3.36) and (3.34) yields (3.31) after using

$$\mathring{\nabla}_a\mathring{\nabla}_b \left(n^{(2)}\ell^{(2)}\right) = n^{(2)}\mathring{\nabla}_a\mathring{\nabla}_b \ell^{(2)} + 2\mathring{\nabla}_{(a} n^{(2)} \mathring{\nabla}_{b)} \ell^{(2)} + \ell^{(2)}\mathring{\nabla}_a\mathring{\nabla}_b n^{(2)}$$

as well as $P^{dc}\ell_c = -\ell^{(2)} n^d$. □

In this thesis we will mostly use the metric hypersurface connection $\mathring{\nabla}$ rather than the hypersurface connection $\overline{\nabla}$ (see Definition 2.2.14). However, in a later section we will need the tensor $\overline{\Sigma} \stackrel{\text{def}}{=} \pounds_n \overline{\nabla}$ associated to $\overline{\nabla}$, so it is convenient to provide the corresponding relation between $\overline{\Sigma}$ and $\mathring{\Sigma}$. This is done in the following lemma.

Lemma 3.1.9. *Let* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ *be hypersurface data,* $\mathring{\Sigma} \stackrel{\text{def}}{=} \pounds_n \mathring{\nabla}$ *and* $\overline{\Sigma} \stackrel{\text{def}}{=} \pounds_n \overline{\nabla}$. *Then,*

$$\overline{\Sigma} = \mathring{\Sigma} - n \otimes \pounds_n \mathbf{Y}. \tag{3.37}$$

Proof. Consider two vector fields $X, W \in \Gamma(T\mathcal{N})$. Then, the combination of (2.49) and (3.30) yields

$$\begin{aligned}
\overline{\Sigma}(X,W) &\stackrel{\text{def}}{=} \pounds_n(\overline{\nabla}_X W) - \overline{\nabla}_X(\pounds_n W) - \overline{\nabla}_{\pounds_n X} W \\
&= \pounds_n\left(\mathring{\nabla}_X W - \mathbf{Y}(X,W)n\right) - \mathring{\nabla}_X(\pounds_n W) + \mathbf{Y}(X,\pounds_n W)n - \mathring{\nabla}_{\pounds_n X} W + \mathbf{Y}(\pounds_n X, W)n \\
&= \mathring{\Sigma}(X,W) - (\pounds_n \mathbf{Y})(X,W)n,
\end{aligned}$$

which proves (3.37). □

3.1.2 *Curvature of the metric hypersurface connection* $\mathring{\nabla}$

We conclude Section 3.1 by analyzing various properties of the curvature tensor of $\mathring{\nabla}$. On a smooth manifold $\mathcal{M}$ endowed with a connection D, the curvature tensor is the 3-covariant, 1-contravariant tensor defined by

$$\text{Riem}^D(\boldsymbol{\alpha}, Z, X, W)) \stackrel{\text{def}}{=} \boldsymbol{\alpha}\left(R^D(X,W)Z\right), \tag{3.38}$$

where $\boldsymbol{\alpha} \in \Gamma(T^\star\mathcal{M})$ and $Z, X, W \in \Gamma(T\mathcal{M})$. The Ricci tensor $\mathbf{Ric}^D$ is the contraction of this tensor in the first and third indices. We recall two notational conventions that we already presented. First, when $\mathcal{M}$ is equipped with a metric g, we write simply R to refer to the curvature operator of the Levi-Civita connection ∇ of g (see Section 2.1). Secondly, for the metric hypersurface connection $\mathring{\nabla}$, we denote the curvature and Ricci tensors by $\mathring{\mathbf{Riem}}$, $\mathring{\mathbf{Ric}}$ except when using abstract index notation where we just write $\mathring{R}^a{}_{bcd}$, $\mathring{R}_{ab}$.

The contractions of $\mathring{R}^d{}_{cab}$ with ℓ_d and with n^c have already been derived in [59] by simply applying the Ricci identity to ℓ_d and to n^d respectively. Later in this thesis, it will be of relevance to have an explicit expression for the contraction $\mathring{R}^a{}_{bcd}n^c$. This quantity cannot be computed by means of the Ricci identity. For this reason, we follow a different path based on the Lie derivative of $\mathring{\nabla}$ that we just computed in the previous section.

From (3.22), we know that the calculation of $\mathring{R}^a{}_{bcd}n^c$ only requires the explicit expression of the tensors $\Sigma^d{}_{ab}$ and $\mathring{\nabla}_a\mathring{\nabla}_b n^d$. The former has already been computed in Section 3.1.1 (see (3.31)). In the next lemma we calculate the latter.

Lemma 3.1.10. *Let* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ *be metric hypersurface data. Then*

$$
\begin{aligned}
\mathring{\nabla}_a\mathring{\nabla}_b n^d = n^d \Big[& \mathring{\nabla}_a s_b + s_a s_b - P^{cf}F_{af}(U_{bc} - n^{(2)}F_{bc}) \\
& - n^{(2)}\mathring{\nabla}_a\mathring{\nabla}_b \ell^{(2)} - \frac{1}{2}\mathring{\nabla}_a \ell^{(2)}\mathring{\nabla}_b n^{(2)} - \mathring{\nabla}_a n^{(2)}\mathring{\nabla}_b \ell^{(2)} \\
& - 2n^{(2)} s_{(a}\mathring{\nabla}_{b)}\ell^{(2)} + \frac{1}{2}(n^{(2)})^2\mathring{\nabla}_a \ell^{(2)}\mathring{\nabla}_b \ell^{(2)} \Big] \\
& + P^{dc}\Big[\mathring{\nabla}_a \big(U_{cb} - n^{(2)}F_{cb} \big) + U_{ac}\big(s_b + n^{(2)}\mathring{\nabla}_b \ell^{(2)} \big) \\
& + F_{ac}\Big(-n^{(2)}s_b - \frac{1}{2}\mathring{\nabla}_b n^{(2)} + \frac{1}{2}(n^{(2)})^2\mathring{\nabla}_b \ell^{(2)} \Big) \Big].
\end{aligned}
\tag{3.39}
$$

Proof. The proof is a direct computation based on the derivatives (2.20) and (2.21). However, since the expressions are rather involved, it is advantageous to define

$$
t_b \stackrel{\text{def}}{=} s_b - n^{(2)}\mathring{\nabla}_b \ell^{(2)}, \qquad T_{bc} \stackrel{\text{def}}{=} U_{bc} - n^{(2)}F_{bc} \qquad G^c{}_a \stackrel{\text{def}}{=} -P^{cf}F_{af} - \frac{1}{2}n^c\mathring{\nabla}_a \ell^{(2)}.
$$

and write (2.20)-(2.21) as

$$
\mathring{\nabla}_b n^d = n^d t_b + P^{dc}T_{bc}, \qquad \mathring{\nabla}_a P^{dc} = 2n^{(d}G^{c)}{}_a.
$$

Thus,

$$
\mathring{\nabla}_a\mathring{\nabla}_b n^d = t_b\mathring{\nabla}_a n^d + n^d \mathring{\nabla}_a t_b + 2n^{(d}G^{c)}{}_a T_{bc} + P^{dc}\mathring{\nabla}_a T_{bc}
$$

$$= n^d \left(\overset{\circ}{\nabla}_a t_b + t_a t_b + G^c{}_a T_{bc} \right) + P^{dc} \left(T_{ac} t_b + \overset{\circ}{\nabla}_a T_{bc} \right) + n^c T_{bc} G^d{}_a. \quad (3.40)$$

Now, (2.11) and (2.14) imply

$$n^c T_{bc} = \frac{1}{2} \overset{\circ}{\nabla}_b n^{(2)} + \frac{1}{2} (n^{(2)})^2 \overset{\circ}{\nabla}_b \ell^{(2)}, \quad (3.41)$$

which in turn gives

$$\begin{aligned} G^c{}_a T_{bc} &= \left(-P^{cf} F_{af} - \frac{1}{2} n^c \overset{\circ}{\nabla}_a \ell^{(2)} \right) T_{bc} \\ &= -P^{cf} F_{af} T_{bc} - \frac{1}{4} \overset{\circ}{\nabla}_a \ell^{(2)} \left(\overset{\circ}{\nabla}_b n^{(2)} + (n^{(2)})^2 \overset{\circ}{\nabla}_b \ell^{(2)} \right). \end{aligned}$$

Inserting this and (3.41) into (3.40) we can write

$$\begin{aligned} \overset{\circ}{\nabla}_a \overset{\circ}{\nabla}_b n^d &= n^d \left(\overset{\circ}{\nabla}_a t_b + t_a t_b - P^{cf} F_{af} T_{bc} - \frac{1}{4} \overset{\circ}{\nabla}_a \ell^{(2)} \overset{\circ}{\nabla}_b n^{(2)} - \frac{1}{4} (n^{(2)})^2 \overset{\circ}{\nabla}_a \ell^{(2)} \overset{\circ}{\nabla}_b \ell^{(2)} \right) \\ &\quad + P^{dc} \left(\overset{\circ}{\nabla}_a T_{bc} + T_{ac} t_b \right) - \frac{1}{2} \left(\overset{\circ}{\nabla}_b n^{(2)} + (n^{(2)})^2 \overset{\circ}{\nabla}_b \ell^{(2)} \right) \left(P^{dc} F_{ac} + \frac{1}{2} n^d \overset{\circ}{\nabla}_a \ell^{(2)} \right) \\ &= n^d \underbrace{\left(\overset{\circ}{\nabla}_a t_b + t_a t_b - P^{cf} F_{af} T_{bc} - \frac{1}{2} \overset{\circ}{\nabla}_a \ell^{(2)} \overset{\circ}{\nabla}_b n^{(2)} - \frac{1}{2} (n^{(2)})^2 \overset{\circ}{\nabla}_a \ell^{(2)} \overset{\circ}{\nabla}_b \ell^{(2)} \right)}_{\stackrel{\text{def}}{=} I} \\ &\quad + P^{dc} \underbrace{\left(\overset{\circ}{\nabla}_a T_{bc} + T_{ac} t_b - \frac{1}{2} F_{ac} \left(\overset{\circ}{\nabla}_b n^{(2)} + (n^{(2)})^2 \overset{\circ}{\nabla}_b \ell^{(2)} \right) \right)}_{\stackrel{\text{def}}{=} II}. \end{aligned}$$

To conclude we just need to elaborate each parenthesis. For the first one we note

$$\begin{aligned} \overset{\circ}{\nabla}_a t_b + t_a t_b = {} & \overset{\circ}{\nabla}_a s_b - n^{(2)} \overset{\circ}{\nabla}_a \overset{\circ}{\nabla}_b \ell^{(2)} - \overset{\circ}{\nabla}_a n^{(2)} \overset{\circ}{\nabla}_b \ell^{(2)} \\ & + s_a s_b - 2 n^{(2)} s_{(a} \overset{\circ}{\nabla}_{b)} \ell^{(2)} + (n^{(2)})^2 \overset{\circ}{\nabla}_a \ell^{(2)} \overset{\circ}{\nabla}_b \ell^{(2)}, \end{aligned}$$

from where it follows

$$\begin{aligned} I = {} & \overset{\circ}{\nabla}_a s_b + s_a s_b - P^{cf} F_{af} (U_{bc} - n^{(2)} F_{bc}) \\ & - n^{(2)} \overset{\circ}{\nabla}_a \overset{\circ}{\nabla}_b \ell^{(2)} - \frac{1}{2} \overset{\circ}{\nabla}_a \ell^{(2)} \overset{\circ}{\nabla}_b n^{(2)} - \overset{\circ}{\nabla}_a n^{(2)} \overset{\circ}{\nabla}_b \ell^{(2)} \\ & - 2 n^{(2)} s_{(a} \overset{\circ}{\nabla}_{b)} \ell^{(2)} + \frac{1}{2} (n^{(2)})^2 \overset{\circ}{\nabla}_a \ell^{(2)} \overset{\circ}{\nabla}_b \ell^{(2)}. \end{aligned}$$

From the definition of T_{bc} and t_b one gets

$$II = \overset{\circ}{\nabla}_a \left(U_{cb} - n^{(2)} F_{cb} \right) + U_{ac} \left(s_b + n^{(2)} \overset{\circ}{\nabla}_b \ell^{(2)} \right)$$

$$+ F_{ac}\left(-n^{(2)}s_b - \frac{1}{2}\overset{\circ}{\nabla}_b n^{(2)} + \frac{1}{2}(n^{(2)})^2 \overset{\circ}{\nabla}_b \ell^{(2)}\right),$$

and the validity of (3.39) is proved. □

We can now find the components $\overset{\circ}{\mathbf{Riem}}(\cdot,\cdot,n,\cdot)$ of the curvature tensor.

Proposition 3.1.11. *Let* $\{\mathcal{N},\gamma,\boldsymbol{\ell},\ell^{(2)}\}$ *be metric hypersurface data. Then the curvature tensor* $\overset{\circ}{\mathbf{Riem}}$ *satisfies the following identity*

$$\begin{aligned}
\overset{\circ}{R}{}^d{}_{bca} n^c = {} & n^d \Big\{ \overset{\circ}{\nabla}_b s_a - s_a s_b + 2n^{(2)} s_{(b}\overset{\circ}{\nabla}_{a)} \ell^{(2)} + n(\ell^{(2)}) U_{ba} \\
& + P^{cf}F_{af}\left(U_{bc} - n^{(2)}F_{bc}\right) - \frac{1}{2}\overset{\circ}{\nabla}_b n^{(2)} \overset{\circ}{\nabla}_a \ell^{(2)} - \frac{1}{2}(n^{(2)})^2 \overset{\circ}{\nabla}_b \ell^{(2)} \overset{\circ}{\nabla}_a \ell^{(2)} \Big\} \\
& + P^{dc}\Big\{ \overset{\circ}{\nabla}_b U_{ca} - \overset{\circ}{\nabla}_c U_{ba} + 2 s_c U_{ba} - s_b U_{ac} + 2F_{cb}\overset{\circ}{\nabla}_a n^{(2)} + \frac{1}{2}F_{ca}\overset{\circ}{\nabla}_b n^{(2)} \\
& + n^{(2)}\left(- U_{ba}\overset{\circ}{\nabla}_c \ell^{(2)} - U_{ac}\overset{\circ}{\nabla}_b \ell^{(2)} + \overset{\circ}{\nabla}_a F_{cb} + F_{ac}s_b - \frac{1}{2}n^{(2)} \overset{\circ}{\nabla}_b \ell^{(2)} \right)\Big\}.
\end{aligned} \tag{3.42}$$

Proof. The result follows immediately after inserting Lemmas 3.1.8 and 3.1.10 into the identity $\overset{\circ}{R}{}^d{}_{bca} n^c = \Sigma^d{}_{ab} - \overset{\circ}{\nabla}_a \overset{\circ}{\nabla}_b n^d$ (cf. (3.22)). □

Again, observe that the extrinsic part $\mathbf{Y}$ of the data has played no role so far in this chapter.

3.2 null hypersurface data

The foundations of the formalism of hypersurface data were fully established in [58], [59]. In these two works, most of the results therein apply for completely general hypersurface data, so the existence of null and/or non-null points in the hypersurface is always allowed. In general, this actually constitutes a great advantage. However, for the purposes of this thesis it becomes necessary to expand the formalism of hypersurface data in the case when the abstract hypersurface is null. For this reason, in this section we collect all results of abstract null hypersurfaces that will be helpful afterwards.

We start with the notions of "null (metric) hypersurface data".

Definition 3.2.1. *(Null metric hypersurface data) A metric hypersurface data set* $\{\mathcal{N},\gamma,\boldsymbol{\ell},\ell^{(2)}\}$ *is called null if the scalar field* $n^{(2)}$ *defined by* (2.6)-(2.9) *is everywhere zero on* $\mathcal{N}$.

Definition 3.2.2. *(Null hypersurface data) A hypersurface data set* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ *is called null if* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ *defines null metric hypersurface data.*

The notion of null metric hypersurface data can also be constructed purely in terms of the data $\{\gamma, \boldsymbol{\ell}, \ell^{(2)}\}$, i.e. without making any reference to the scalar field $n^{(2)}$.

Lemma 3.2.3. *Let* $\mathcal{N}$ *be a smooth manifold. The collection* $\{\gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ *where* γ *is a symmetric* $(0, 2)$*-tensor,* $\boldsymbol{\ell}$ *a covector and* $\ell^{(2)}$ *a scalar field defines null metric hypersurface data if and only if*

(i) The radical $\mathrm{Rad}\gamma|_p$ *of* $\gamma|_p$ *is one-dimensional at every point* $p \in \mathcal{N}$*.*

(ii) For all $p \in \mathcal{N}$ *and any non-zero vector* $e_1 \in \mathrm{Rad}\gamma|_p$ *the contraction* $\boldsymbol{\ell}|_p(e_1) \neq 0$*.*

Proof. It is clear that condition (ii) is independent of the element $e_1 \in \mathrm{Rad}\gamma$ one chooses. If $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ is null metric hypersurface data, we may take $e_1|_p = n|_p$ and conditions (i) and (ii) are satisfied (recall (2.6)-(2.7)). To prove the converse, we only need to make sure that the symmetric 2-covariant tensor $\mathcal{A}|_p$ on $T_p\mathcal{N} \oplus \mathbb{R}$ defined in (3.1.1) is non-degenerate (observe that (i) together with (2.6) already imply that $n^{(2)} = 0$). The proof of Lemma 3.1.1 only uses that $\gamma|_p$ has one-dimensional radical, that $\mathrm{span}\{e_1\} = \mathrm{Rad}\gamma|_p$ and that $\boldsymbol{\ell}|_p(e_1) \neq 0$. Thus, under conditions (i) and (ii) the signature of $\mathcal{A}|_p$ is given by (3.1), hence $\mathcal{A}|_p$ is non-degenerate. □

Remark 3.2.4. *Note that condition* (ii) *needs to be added only because the tensors* $\{\gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ *considered in Lemma 3.2.3 are completely general (i.e. they do not define metric hypersurface data a priori). Had we let* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ *define metric hypersurface data, then only* (i) *would be necessary, as the ambient tensor* $\boldsymbol{\mathcal{A}}$ *would be non-degenerate by definition and hence* (ii) *would be automatic.*

Let us study some direct consequences of $n^{(2)} = 0$. Firstly, as already mentioned, $\mathrm{Rad}\gamma = \langle n \rangle$ and hence $\gamma(n, \cdot) = 0$. On the other hand, the tensor $\mathbf{U}$ introduced in (2.12) acquires a particularly prominent role. It is given by $\mathbf{U} = \frac{1}{2}\pounds_n\gamma$ (see (2.12)), hence it satisfies $\mathbf{U}(n, \cdot) = 0$ (by (2.14)). Moreover, when $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ is embedded on an ambient space $(\mathcal{M}, g)$ with embedding ϕ and rigging ζ, it coincides with the second fundamental form $\mathbf{K}$ (cf. (2.45)) with respect to the null normal $\nu \in \Gamma(T\phi(\mathcal{N}))$ satisfying $g(\zeta, \nu)|_{\phi(\mathcal{N})} = 1$.

For later use, we particularize (2.13) and (2.20) for $n^{(2)} = 0$, which gives

$$s = \frac{1}{2}\pounds_n \boldsymbol{\ell}, \tag{3.43}$$

$$\overset{\circ}{\nabla}_b n^c = n^c s_b + P^{ac}\mathrm{U}_{ab}, \tag{3.44}$$

and we stress that any vector field $\bar{\eta} \in \mathrm{Rad}\gamma$ satisfies

$$\pounds_{\bar{\eta}} \boldsymbol{\ell} = 2\alpha s + d\alpha, \tag{3.45}$$

where $\alpha \in \mathcal{F}(\mathcal{N})$ is defined by $\bar{\eta} = \alpha n$ and we have used (3.43). It is also worth mentioning that the combination of $\mathbf{U}(n, \cdot) = 0$, $s(n) = \mathbf{F}(n, n) = 0$ and (3.44) entails

$$\overset{\circ}{\nabla}_n n = 0, \tag{3.46}$$

which together with (2.44) and $\nu = \phi_\star n$ (recall (2.25)) yields

$$\nabla_\nu \nu \overset{(2.48)}{=} \phi_\star \left(\overset{\circ}{\nabla}_n n - \mathbf{Y}(n, n) n \right) \overset{(3.46)}{=} \kappa_n \phi_\star n = \kappa_n \nu. \tag{3.47}$$

Consequently, κ_n can be interpreted as the surface gravity of the null normal vector ν on $\phi(\mathcal{N})$.

We have already discussed that the vector field n is privileged in any null hypersurface data. This often makes it convenient to decompose tensors on $\mathcal{N}$ in terms of a basis $\{n, e_A\}$ of $\Gamma(T\mathcal{N})$ and its corresponding dual basis. The next lemma provides such a decomposition for the tensors γ and P.

Lemma 3.2.5. *Consider null metric hypersurface data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$. Let $\{n, e_A\}$ be a basis of $\Gamma(T\mathcal{N})$ and $\{\mathbf{q}, \boldsymbol{\theta}^A\}$ be its corresponding dual, i.e.*

$$\mathbf{q}(n) = 1, \qquad \mathbf{q}(e_A) = 0, \qquad \boldsymbol{\theta}^A(n) = 0, \qquad \boldsymbol{\theta}^A(e_B) = \delta^A_B \tag{3.48}$$

Define the functions $\psi_A \in \mathcal{F}(\mathcal{N})$ as $\psi_A \stackrel{\text{def}}{=} \boldsymbol{\ell}(e_A)$. Then, the tensor fields γ and P decompose as

$$\gamma = h_{AB} \boldsymbol{\theta}^A \otimes \boldsymbol{\theta}^B, \tag{3.49}$$

$$P = h^{AB} e_A \otimes e_B - h^{AB} \psi_B (n \otimes e_A + e_A \otimes n) - \left(\ell^{(2)} - h^{AB} \psi_A \psi_B \right) n \otimes n, \tag{3.50}$$

where $h_{AB} \stackrel{\text{def}}{=} \gamma(e_A, e_B)$ is a metric and h^{AB} denotes its inverse.

Proof. First, we notice that $\boldsymbol{\ell}$ decomposes in the basis $\{\mathbf{q}, \boldsymbol{\theta}^A\}$ as $\boldsymbol{\ell} = \mathbf{q} + \psi_A \boldsymbol{\theta}^A$ because $\boldsymbol{\ell}(n) = 1$ (cf. (2.7)) and $\psi_A \stackrel{\text{def}}{=} \boldsymbol{\ell}(e_A)$. Equation (3.49) is an immediate consequence of $\gamma(n, \cdot) = 0$. This, together with the fact that $\mathrm{Rad}\gamma$ is one-dimensional, means that h_{AB} defines a metric. On the other hand, since P is symmetric it decomposes in the basis $\{n, e_A\}$ as

$$P = P(\boldsymbol{\theta}^A, \boldsymbol{\theta}^B) e_A \otimes e_B + P(\mathbf{q}, \boldsymbol{\theta}^A)(n \otimes e_A + e_A \otimes n) + P(\mathbf{q}, \mathbf{q}) n \otimes n. \tag{3.51}$$

The fact that $P(\boldsymbol{\theta}^A, \boldsymbol{\theta}^B) = h^{AB}$ follows from

$$\delta^B_A = \delta^b_a \theta^B_b e^a_A \overset{(2.9)}{=} (P^{bf}\gamma_{fa} + n^b \ell_a)\theta^B_b e^a_A = P^{bf}\gamma_{fa}\theta^B_b e^a_A \overset{(3.49)}{=} h_{AC} P(\boldsymbol{\theta}^B, \boldsymbol{\theta}^C), \tag{3.52}$$

while for $P(\mathbf{q}, \cdot)$ one finds

$$\begin{aligned} P(\mathbf{q}, \cdot) &= P(\boldsymbol{\ell} - \psi_A \boldsymbol{\theta}^A, \cdot) \overset{(2.8)}{=} -\ell^{(2)} n - \psi_A P(\boldsymbol{\theta}^A, \cdot) \\ &= -\left(\ell^{(2)} + \psi_A P(\boldsymbol{\theta}^A, \mathbf{q})\right) n - h^{AB}\psi_A e_B \end{aligned}$$

and hence $P(\mathbf{q}, \boldsymbol{\theta}^C) = -h^{AC}\psi_A$ and $P(\mathbf{q}, \mathbf{q}) = -\left(\ell^{(2)} - h^{AB}\psi_A\psi_B\right)$. □

The concept of null thin shell follows immediately from Definitions 2.7.2 and 3.2.1. *A thin shell* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}^{\pm}, \rho_{\ell}^{\pm}, J^{\pm}\}$ *is said to be null if* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ *defines null metric hypersurface data*. As anticipated, the energy-momentum tensor τ of a null thin shell can be decomposed in the basis $\{\mathbf{q}, \boldsymbol{\theta}^A\}$ as well, and it turns out that the components of τ take a very simple form, as we see next.

Corollary 3.2.6. *In the setup of Lemma 3.2.5, let* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}^{\pm}, \rho_{\ell}^{\pm}, J^{\pm}\}$ *be a null thin shell. Then, the components of the energy-momentum tensor* τ *in the basis* $\{\mathbf{q}, \boldsymbol{\theta}^A\}$ *read*

$$\tau(\mathbf{q}, \mathbf{q}) = -\epsilon h^{AB}[\mathbf{Y}](e_A, e_B), \tag{3.53}$$

$$\tau(\mathbf{q}, \boldsymbol{\theta}^A) = \epsilon h^{AB}[\mathbf{Y}](n, e_B), \tag{3.54}$$

$$\tau(\boldsymbol{\theta}^A, \boldsymbol{\theta}^B) = -\epsilon h^{AB}[\mathbf{Y}](n, n). \tag{3.55}$$

Proof. Inserting the decomposition (3.50) into Definition 2.7.3 yields

$$\tau^{df} = -\epsilon h^{AB}\left([\mathbf{Y}](e_A, e_B)n^d n^f - [\mathbf{Y}](n, e_A)(n^d e^f_B + e^d_B n^f) + [\mathbf{Y}](n, n)e^d_A e^f_B\right)$$

after a simple but somewhat long computation in which many terms cancel out. Contracting with $\{\mathbf{q}, \boldsymbol{\theta}^A\}$ it is immediate to get (3.53)-(3.55). □

3.2.1 *Gauge-fixing results*

Later on, we shall introduce several geometric quantities that are invariant under the gauge transformations with gauge parameters $\{z = 1, V\}$. In particular, these quantities will play a fundamental role in the study of abstract Killing horizons. In order to motivate their definitions, we first need to know the gauge behaviour of various tensor fields defined before. We devote this section to this task.

For arbitrary gauge parameters $\{z, V\}$ we introduce

$$w \stackrel{\text{def}}{=} \gamma(V,\cdot), \qquad f \stackrel{\text{def}}{=} \boldsymbol{\ell}(V), \tag{3.56}$$

from where it immediately follows that (recall Lemma 2.2.8)

$$V^a = f n^a + P^{ab} w_b. \tag{3.57}$$

In terms of $\{w, f\}$, the gauge transformations (2.31)-(2.32) take the form

$$\mathcal{G}_{(z,V)}(\boldsymbol{\ell}) = z(\boldsymbol{\ell} + w), \tag{3.58}$$

$$\mathcal{G}_{(z,V)}\left(\ell^{(2)}\right) = z^2\left(\ell^{(2)} + 2f + P(w,w)\right). \tag{3.59}$$

In the next lemma, we obtain the gauge behaviour of $\mathbf{U}$, $\mathbf{F}$, $\boldsymbol{s}$, $\boldsymbol{r}$ and κ_n.

Lemma 3.2.7. *Let* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ *be null hypersurface data. Consider arbitrary gauge parameters* $\{z, V\}$ *and define the covector* w *and the function* f *according to* (3.56)*. Then, the following gauge transformations hold:*

$$\mathcal{G}_{(z,V)}(\mathbf{U}) = \frac{1}{z}\mathbf{U}, \tag{3.60}$$

$$\mathcal{G}_{(z,V)}(\mathbf{F}) = z\left(\mathbf{F} + \frac{1}{2}dw\right) + \frac{1}{2}dz \wedge (\boldsymbol{\ell} + w), \tag{3.61}$$

$$\mathcal{G}_{(z,V)}(\boldsymbol{s}) = \boldsymbol{s} + \frac{1}{2}\pounds_n w + \frac{n(z)}{2z}(\boldsymbol{\ell} + w) - \frac{1}{2z}dz, \tag{3.62}$$

$$\mathcal{G}_{(z,V)}(\boldsymbol{r}) = \boldsymbol{r} + \frac{1}{2z}dz + \frac{n(z)}{2z}(\boldsymbol{\ell} + w) + \frac{1}{2}\pounds_n w - \mathbf{U}(V,\cdot), \tag{3.63}$$

$$\mathcal{G}_{(z,V)}(\kappa_n) = \frac{1}{z}\left(\kappa_n - \frac{n(z)}{z}\right). \tag{3.64}$$

Proof. For notational simplicity we write a prime to denote a gauge transformed quantity. The first three expressions are obtained as follows (recall (2.30), (2.31), (2.34))

$$\mathbf{U}' = \frac{1}{2}\pounds_{n'}\gamma = \frac{1}{2}\pounds_{z^{-1}n}\gamma = \frac{1}{2z}\pounds_n\gamma = z^{-1}\mathbf{U}.$$

$$\mathbf{F}' = \frac{1}{2}d\boldsymbol{\ell}' = \frac{z}{2}(d\boldsymbol{\ell} + dw) + \frac{1}{2}dz \wedge (\boldsymbol{\ell} + w) = z\left(\mathbf{F} + \frac{1}{2}dw\right) + \frac{1}{2}dz \wedge (\boldsymbol{\ell} + w),$$

$$\boldsymbol{s}' = i_{n'}\mathbf{F}' = z^{-1}i_n\mathbf{F}' = \boldsymbol{s} + \frac{1}{2}i_n dw + \frac{n(z)}{2z}(\boldsymbol{\ell} + w) - \frac{1}{2z}dz,$$

where i_n denotes interior contraction in the first index and in the last equality we used $w(n) = 0$. Using Cartan's formula $\pounds_n w = i_n dw + d i_n w = i_n dw$ yields (3.62).

For the transformation of $\boldsymbol{r}$ we contract the first equality in (2.40) with $z^{-1}n$ to get

$$\begin{aligned} \boldsymbol{r}' &= \boldsymbol{r} + \frac{1}{2z}dz + \frac{n(z)}{2z}\boldsymbol{\ell} - \frac{1}{2z}\gamma\left(\pounds_{zV}n, \cdot\right) = \boldsymbol{r} + \frac{1}{2z}dz + \frac{n(z)}{2z}\boldsymbol{\ell} + \frac{1}{2z}\gamma\left(\pounds_{n}(zV), \cdot\right) \\ &= \boldsymbol{r} + \frac{1}{2z}dz + \frac{n(z)}{2z}\boldsymbol{\ell} + \frac{1}{2z}\pounds_{n}\left(\gamma(zV,\cdot)\right) - \frac{1}{2z}\left(\pounds_{n}\gamma\right)(zV,\cdot) \end{aligned}$$

where we used the antisymmetry of the Lie bracket and "integrated by parts". Expression (3.63) follows after using $\gamma(V,\cdot) = \boldsymbol{w}$. The last transformation follows at once from the previous one and the definition $\kappa_n = -\boldsymbol{r}(n)$. $\square$

Lemma 3.2.7 admits the following immediate corollary.

Corollary 3.2.8. *The covector $\boldsymbol{s} - \boldsymbol{r}$ has the following simple gauge behaviour*

$$\mathcal{G}_{(z,V)}\left(\boldsymbol{s} - \boldsymbol{r}\right) = \boldsymbol{s} - \boldsymbol{r} + \mathbf{U}(V,\cdot) - \frac{1}{z}dz.$$

One of the main results in the context of null metric data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ is that by means of a gauge transformation one can always adapt the one-form $\boldsymbol{\ell}$ and the scalar $\ell^{(2)}$ to whatever pair $\{u \in \mathcal{F}(\mathcal{N}), \boldsymbol{\vartheta} \in \Gamma(T^\star\mathcal{N})\}$ as long as $\boldsymbol{\vartheta}(n) \neq 0$ everywhere on $\mathcal{N}$. We prove this in the following lemma.

Lemma 3.2.9. *Let $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ be null metric hypersurface data. Let u be a function on $\mathcal{N}$ and $\boldsymbol{\vartheta} \in \Gamma(T^\star\mathcal{N})$ be a covector satisfying $\boldsymbol{\vartheta}(n) \neq 0$ everywhere. Then there exists a unique gauge transformation $\mathcal{G}_{(z,V)}$ satisfying*

$$\mathcal{G}_{(z,V)}(\boldsymbol{\ell}) = \boldsymbol{\vartheta}, \qquad \mathcal{G}_{(z,V)}\left(\ell^{(2)}\right) = u. \tag{3.65}$$

Moreover, the gauge group element $\mathcal{G}_{(z,V)}$ is given by

$$z = \boldsymbol{\vartheta}(n), \qquad V = \frac{1}{\boldsymbol{\vartheta}(n)}P(\boldsymbol{\vartheta},\cdot) + \frac{u - P(\boldsymbol{\vartheta},\boldsymbol{\vartheta})}{2\left(\boldsymbol{\vartheta}(n)\right)^2}n. \tag{3.66}$$

Remark 3.2.10. *The condition $\boldsymbol{\vartheta}(n) \neq 0$ is necessary. Observe that if at any point p it occurs that $\boldsymbol{\vartheta}(n)|_p = 0$, then $\boldsymbol{\vartheta}$ cannot correspond to $\boldsymbol{\ell}$ in any gauge, as*

$$1 = \left(\mathcal{G}_{(z,V)}(\boldsymbol{\ell})\right)\left(\mathcal{G}_{(z,V)}(n)\right)\big|_p = z^{-1}\left(\mathcal{G}_{(z,V)}(\boldsymbol{\ell})\right)(n)\big|_p.$$

Thus, $\left(\mathcal{G}_{(z,V)}(\boldsymbol{\ell})\right)(n) \neq 0$ must hold for all possible gauge parameters.

Proof. We first assume that the gauge transformation exists and restrict its form up to a function yet to be determined. We then restrict to group elements of such a form and show that there exists one and only one them that satisfies (3.65), namely (3.66). This will prove both the existence and uniqueness claims of the lemma. For the first part we impose (3.65):

$$z\left(\boldsymbol{\ell}+\gamma(V,\cdot)\right)=\boldsymbol{\vartheta},\qquad z^2\left(\ell^{(2)}+2\boldsymbol{\ell}(V)+\gamma(V,V)\right)=u. \tag{3.67}$$

Contracting the first with n gives $z=\boldsymbol{\vartheta}(n)$, so

$$\boldsymbol{w}\stackrel{\text{def}}{=}\gamma(V,\cdot)=\frac{1}{\boldsymbol{\vartheta}(n)}\boldsymbol{\vartheta}-\boldsymbol{\ell}.$$

Observe that $\boldsymbol{w}(n)=0$. Moreover, the vector $V-P(\boldsymbol{w},\cdot)$ lies in the kernel of γ because $\gamma_{ab}\left(V^b-P^{bc}w_c\right)=w_a-(\delta^c_a-n^c\ell_a)\,w_c=0$. Therefore, there exists $f\in\mathcal{F}(\mathcal{N})$ such that

$$V^a=P^{ab}w_b+fn^b=\frac{1}{\boldsymbol{\vartheta}(n)}P^{ab}\vartheta_b+\left(\ell^{(2)}+f\right)n^a.$$

Thus, it suffices to restrict oneself to gauge parameters in the class

$$\left(z=\boldsymbol{\vartheta}(n),\ V=\frac{1}{\boldsymbol{\vartheta}(n)}P(\boldsymbol{\vartheta},\cdot)+qn\ ,q\in\mathcal{F}(\mathcal{N})\right). \tag{3.68}$$

We now start anew and prove that there is precisely one function q such that the corresponding (z,V) in (3.68) fulfills conditions (3.65). For V as in (3.68) we get

$$\begin{aligned}
\boldsymbol{\vartheta}(V)&=\frac{1}{\boldsymbol{\vartheta}(n)}P(\boldsymbol{\vartheta},\boldsymbol{\vartheta})+q\boldsymbol{\vartheta}(n),\\
\boldsymbol{\ell}(V)&=-\ell^{(2)}+q,\\
\gamma(V,\cdot)&=\frac{1}{\boldsymbol{\vartheta}(n)}\gamma(P(\boldsymbol{\vartheta},\cdot),\cdot)=\frac{1}{\boldsymbol{\vartheta}(n)}\boldsymbol{\vartheta}-\boldsymbol{\ell},\\
\gamma(V,V)&=\frac{1}{\boldsymbol{\vartheta}(n)}\boldsymbol{\vartheta}(V)-\boldsymbol{\ell}(V)=\frac{P(\boldsymbol{\vartheta},\boldsymbol{\vartheta})}{\boldsymbol{\vartheta}(n)^2}+\ell^{(2)}.
\end{aligned}$$

The first condition in (3.67) is satisfied for all q. The second is satisfied if and only if

$$\boldsymbol{\vartheta}(n)^2\left(2q+\frac{P(\boldsymbol{\vartheta},\boldsymbol{\vartheta})}{\boldsymbol{\vartheta}(n)^2}\right)=u\qquad\Longleftrightarrow\qquad q=\frac{u-P(\boldsymbol{\vartheta},\boldsymbol{\vartheta})}{2\boldsymbol{\vartheta}(n)^2}.$$

which ends the proof. □

In particular, Lemma 3.2.9 means that two given null metric hypersurface data sets are related by a gauge transformation if and only if they both have the same data tensor γ. We prove this in the following corollary.

Corollary 3.2.11. *Let* $\mathcal{D} \stackrel{\text{def}}{=} \{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$, $\underline{\mathcal{D}} \stackrel{\text{def}}{=} \{\mathcal{N}, \underline{\gamma}, \underline{\boldsymbol{\ell}}, \underline{\ell}^{(2)}\}$ *be two null metric hypersurface data. Then there is a gauge group element* $\mathcal{G}_{(z,V)} \in \mathcal{F}^{\star}(\mathcal{N}) \times \Gamma(T\mathcal{N})$ *such that* $\mathcal{G}_{(z,V)}(\mathcal{D}) = \underline{\mathcal{D}}$ *if and only if* $\gamma = \underline{\gamma}$*. This gauge element is given by*

$$z = \underline{\boldsymbol{\ell}}(n), \qquad V = \frac{1}{\underline{\boldsymbol{\ell}}(n)} P(\underline{\boldsymbol{\ell}}, \cdot) + \frac{\underline{\ell}^{(2)} - P(\underline{\boldsymbol{\ell}}, \underline{\boldsymbol{\ell}})}{2\,(\underline{\boldsymbol{\ell}}(n))^2}\, n. \tag{3.69}$$

Proof. The necessity is obvious from the fact that γ remains unchanged by a gauge transformation. Sufficiency is a direct application of Lemma 3.2.9 to $\boldsymbol{\vartheta} = \underline{\boldsymbol{\ell}}$ and $u = \underline{\ell}^{(2)}$. □

Lemma 3.2.9 and Corollary 3.2.11 are remarkable because they suggest that in the null case one can codify all the metric hypersurface data information exclusively in the tensor γ, and that $\boldsymbol{\ell}$ and $\ell^{(2)}$ are pure gauge. This fact will become specially important in Chapter 9 when studying the matching of spacetimes with null boundaries from an abstract point of view (i.e. in a detached way from the actual two spacetimes to be matched). As we will see then, apart from a condition upon the orientation of the rigging vector fields that are identified in the matching process, the matching will be possible if the tensors γ, $\underline{\gamma}$ of two given null metric hypersurface data sets satisfy $\varphi^{\star}\underline{\gamma} = \gamma$ for a diffeomorphism $\varphi : \mathcal{N} \longrightarrow \underline{\mathcal{N}}$.

3.2.2 *Curvature of the metric hypersurface connection* $\overset{\circ}{\nabla}$*: null case*

In this section we compute several contractions involving the curvature tensor $\overset{\circ}{R}{}^{d}{}_{bca}$ and the Ricci tensor $\overset{\circ}{R}_{ab}$. These identities will be necessary later on.

We start with the contractions with ℓ_d and with γ_{fd}. Both of them will follow from the general identity obtained in Proposition 3.1.11 for any hypersurface data.

Proposition 3.2.12. *Any null metric hypersurface data* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ *satisfies*

$$\ell_d \overset{\circ}{R}{}^{d}{}_{bca} n^c = \overset{\circ}{\nabla}_b s_a - s_b s_a + n(\ell^{(2)}) \mathrm{U}_{ba} + \ell^{(2)} (\pounds_n \mathrm{U})_{ba} + (\mathrm{F}_{af} - \ell^{(2)} \mathrm{U}_{af}) P^{cf} \mathrm{U}_{bc}, \tag{3.70}$$

$$\gamma_{fd} \overset{\circ}{R}{}^{d}{}_{bca} n^c = \overset{\circ}{\nabla}_b \mathrm{U}_{fa} - \overset{\circ}{\nabla}_f \mathrm{U}_{ba} + 2 s_f \mathrm{U}_{ba} - s_b \mathrm{U}_{af} + \ell_f (\pounds_n \mathrm{U})_{ba} - \ell_f P^{cd} \mathrm{U}_{bc} \mathrm{U}_{ad}. \tag{3.71}$$

Proof. Setting $n^{(2)} = 0$ in (3.42) simplifies the expression to

$$\overset{\circ}{R}{}^{d}{}_{bca} n^c = n^d H_{ba} + P^{dc} L_{bca}, \qquad \text{with} \qquad L_{bca} \stackrel{\text{def}}{=} \overset{\circ}{\nabla}_b \mathrm{U}_{ca} - \overset{\circ}{\nabla}_c \mathrm{U}_{ba} + 2 s_c \mathrm{U}_{ba} - s_b \mathrm{U}_{ac}$$

$$\text{and} \qquad H_{ba} \stackrel{\text{def}}{=} \overset{\circ}{\nabla}_b s_a - s_b s_a + n(\ell^{(2)}) \mathrm{U}_{ba} + P^{cf} \mathrm{U}_{cb} \mathrm{F}_{af}. \tag{3.72}$$

Hence, from (2.6)-(2.9),

$$\ell_d \overset{\circ}{R}{}^d{}_{bca} n^c = H_{ba} - \ell^{(2)} n^c L_{bca}, \qquad \gamma_{fd} \overset{\circ}{R}{}^d{}_{bca} n^c = L_{bfa} - \ell_f n^c L_{bca}.$$

The proof will be complete once we establish that $n^c L_{bca} = -\pounds_n \mathrm{U}_{ba} + P^{cf} \mathrm{U}_{fa} \mathrm{U}_{cb}$. This expression holds true because, from (3.12) together with $\mathbf{U}(n, \cdot) = 0$ and $s(n) = 0$,

$$\begin{aligned} n^c L_{bca} &= n^c \left(\overset{\circ}{\nabla}_b \mathrm{U}_{ca} - \overset{\circ}{\nabla}_c \mathrm{U}_{ba} + 2 s_c \mathrm{U}_{ba} - s_b \mathrm{U}_{ac} \right) \\ &= -\pounds_n \mathrm{U}_{ba} + \mathrm{U}_{cb} \overset{\circ}{\nabla}_a n^c = -\pounds_n \mathrm{U}_{ba} + P^{cf} \mathrm{U}_{cb} \mathrm{U}_{fa} \end{aligned}$$

where in the last equality we inserted (3.44). □

We also need an expression for $\overset{\circ}{R}{}^d{}_{acb} n^a$. This was computed for general hypersurface data in [59] by using the Ricci identity applied to n^a. With the expressions above the result can be obtained as a simple consequence of the first Bianchi identity. The method of proof is valid for general data, but we restrict ourselves to the null case.

Lemma 3.2.13. *Let* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ *be null metric hypersurface data. Then*

$$\overset{\circ}{R}{}^d{}_{acb} n^a = 2 n^d \left(\overset{\circ}{\nabla}_{[c} s_{b]} + P^{af} U_{a[c} F_{b]f} \right) + 2 P^{df} \left(\overset{\circ}{\nabla}_{[c} U_{b]f} - s_{[c} U_{b]f} \right). \qquad (3.73)$$

Proof. Since $\overset{\circ}{\nabla}$ has no torsion, the first Bianchi identity takes the form $\overset{\circ}{R}{}^d{}_{acb} + \overset{\circ}{R}{}^d{}_{cba} + \overset{\circ}{R}{}^d{}_{bac} = 0$. Contracting with n^a and using the antisymmetry of the curvature tensor on the last two indices one gets

$$\overset{\circ}{R}{}^d{}_{acb} n^a = n^a \left(\overset{\circ}{R}{}^d{}_{cab} - \overset{\circ}{R}{}^d{}_{bac} \right) = 2 n^d H_{[cb]} + P^{df} (L_{cfb} - L_{bfc}),$$

which gives (3.73) upon inserting the expressions for H_{bc} and L_{cfb} provided in (3.72). □

Finally, we compute some contractions of $\overset{\circ}{\mathbf{R}}\mathbf{ic}$ with n.

Lemma 3.2.14. *Let* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ *be null metric hypersurface data. The following identities hold:*

$$\overset{\circ}{R}_{ab} n^a = \pounds_n s_b - 2 P^{af} \mathrm{U}_{ab} s_f + P^{cf} \overset{\circ}{\nabla}_c \mathrm{U}_{bf} - \overset{\circ}{\nabla}_b (\mathrm{tr}_P \mathbf{U}) + (\mathrm{tr}_P \mathbf{U}) s_b, \qquad (3.74)$$

$$\overset{\circ}{R}_{(ab)} n^a = \frac{1}{2} \pounds_n s_b - 2 P^{af} \mathrm{U}_{ab} s_f + P^{cf} \overset{\circ}{\nabla}_c \mathrm{U}_{bf} - \overset{\circ}{\nabla}_b (\mathrm{tr}_P \mathbf{U}) + (\mathrm{tr}_P \mathbf{U}) s_b, \qquad (3.75)$$

$$\overset{\circ}{R}_{(ab)} n^a n^b = -P^{ab} P^{cd} \mathrm{U}_{ac} \mathrm{U}_{bd} - n(\mathrm{tr}_P \mathbf{U}). \tag{3.76}$$

Proof. To prove (3.74) we contract the indices d and c in (3.73). Identity (3.10) (for $\boldsymbol{\theta} = s$ together with $s(n) = 0$) gives

$$2n^c \overset{\circ}{\nabla}_{[c} s_{b]} = \pounds_n s_b \tag{3.77}$$

and hence

$$\overset{\circ}{R}_{ab} n^a \overset{\text{def}}{=} \overset{\circ}{R}{}^c{}_{acb} n^a = \pounds_n s_b - P^{af} \mathrm{U}_{ab} s_f + P^{cf} \left(\overset{\circ}{\nabla}_c \mathrm{U}_{bf} - s_c \mathrm{U}_{bf} \right) - P^{cf} \left(\overset{\circ}{\nabla}_b \mathrm{U}_{cf} - s_b \mathrm{U}_{cf} \right).$$

The validity of (3.74) follows because $P^{cf} \overset{\circ}{\nabla}_b \mathrm{U}_{cf} = \overset{\circ}{\nabla}_b (\mathrm{tr}_P \mathbf{U}) - (\overset{\circ}{\nabla}_b P^{cf}) \mathrm{U}_{cf} = \overset{\circ}{\nabla}_b (\mathrm{tr}_P \mathbf{U})$, the last equality being a consequence of (2.21). Imposing (2.51) in $n^{(2)} = 0$ it follows $\overset{\circ}{R}_{(ab)} = \overset{\circ}{R}_{ab} - \overset{\circ}{\nabla}_{[a} s_{b]}$. Contracting with n^a and using (3.77) and (3.74) gives (3.75). To obtain (3.76), it suffices to notice that $n^b P^{cd} \overset{\circ}{\nabla}_c \mathrm{U}_{bd} = -\mathrm{U}_{bd} P^{cd} \overset{\circ}{\nabla}_c n^b = -P^{ab} P^{cd} \mathrm{U}_{ac} \mathrm{U}_{bd}$. □

3.2.3 *Transverse submanifolds*

In Sections 2.3 and 2.4, we have discussed several fundamental notions and results concerning the geometry of submanifolds. In particular, we have defined transverse submanifold of a null hypersurface embedded on a Lorentzian manifold (see Definition 2.4.3), and we have seen that these submanifolds are of relevance to understand the geometry of null hypersurfaces. It is therefore natural to introduce and study this notion in the context of hypersurface data. In this subsection we analyze the geometric properties of a given null metric hypersurface data set with a transverse submanifold S.

By definition a *transverse submanifold is a codimension one embedded submanifold of* $\mathcal{N}$ *to which n is everywhere transverse*. Existence of such S is always guaranteed in sufficiently local domains of any null metric hypersurface data. Note that we are not assuming that S is a global section of $\mathcal{N}$, i.e. there can be generators of $\mathcal{N}$ that do not cross S. What we actually enforce is that generators intersecting S do it only once.

We have several purposes in mind. First, we will derive an explicit relation between the covariant derivative $\overset{\circ}{\nabla}$ and its induced covariant derivative ∇^S on S. Secondly, we will find an identity between ∇^S and the Levi-Civita covariant derivative on S. Then, we shall obtain a version of the Gauss identity (see e.g. [106]) by particularizing the results from Appendix B (see Theorem B.0.1 and equation (B.7)). Finally, we conclude with several lemmas to be used later.

In the following, whenever we consider null metric hypersurface data plus a transverse submanifold S, our setup will be the following.

Setup 3.2.15. *We let $\mathcal{D} = \{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ be null hypersurface data and S an $(n-1)$-dimensional smooth submanifold of $\mathcal{N}$, everywhere transversal to n. We denote by ψ the corresponding embedding $\psi : S \longhookrightarrow \mathcal{N}$ of S in $\mathcal{N}$. We define $\boldsymbol{\ell}_\parallel \overset{\text{def}}{=} \psi^\star \boldsymbol{\ell}$ (with components ℓ_A), $\ell^{(2)}_\parallel \overset{\text{def}}{=} h^\sharp(\boldsymbol{\ell}_\parallel, \boldsymbol{\ell}_\parallel)$ and let $\mathbf{q}$ be the only normal covector along $\psi(S)$ satisfying $\mathbf{q}(n) = 1$. We take a basis $\{\hat{v}_A\}$ of $\Gamma(TS)$ and construct the basis $\{n, v_A \overset{\text{def}}{=} \psi_\star(\hat{v}_A)\}$ of $\Gamma(T\mathcal{N})|_{\psi(S)}$.*

In the present setup, Lemmas 3.2.5 and 3.2.9 admit the following two corollaries respectively.

Corollary 3.2.16. *Assume Setup 3.2.15. Then,*

$$P^{cf}\mathrm{U}_{fa} = h^{IJ} v_J^f (v_I^c - \ell_I n^c)\mathrm{U}_{fa}, \qquad P^{cd}\mathrm{U}_{ac}\mathrm{U}_{bd} = h^{IJ} v_I^c v_J^d \mathrm{U}_{ac}\mathrm{U}_{bd}, \tag{3.78}$$

$$\mathrm{tr}_P \mathbf{Y} = \mathrm{tr}_h \mathbf{Y}_\parallel - 2\ell^A r_A + \kappa_n(\ell^{(2)} - \ell^{(2)}_\parallel), \qquad \mathrm{tr}_P \mathbf{U} = \mathrm{tr}_h \mathbf{U}_\parallel. \tag{3.79}$$

Proof. Recall that $\mathbf{U}(n,\cdot) = 0$. By adapting Lemma 3.2.5 to the basis $\{n, v_A\}$ introduced in Setup 3.2.15, it follows at once that the tensor field P decomposes as

$$P^{cf} = h^{AB} v_A^c v_B^f - h^{AB}\ell_B(n^c v_A^f + n^f v_A^c) + (\ell^{(2)}_\parallel - \ell^{(2)}) n^c n^f \tag{3.80}$$

because $h_{AB} = h_{AB}$ and $\psi_A = \ell_A$. Equations (3.78) automatically follow from the decomposition (3.80). Expressions (3.79) can be computed by inserting (3.80) into $\mathrm{tr}_P \mathbf{Y}$ and $\mathrm{tr}_P \mathbf{U}$. For the former we find

$$\begin{aligned} \mathrm{tr}_P \mathbf{Y} = P^{cd}\mathrm{Y}_{cd} &= (h^{CD} v_C^c v_D^d - \ell^D(n^c v_D^d + n^d v_D^c) - (\ell^{(2)} - \ell^{(2)}_\parallel) n^c n^d)\mathrm{Y}_{cd} \\ &= \mathrm{tr}_h \mathbf{Y}_\parallel - 2\ell^D r_D + \kappa_n(\ell^{(2)} - \ell^{(2)}_\parallel), \end{aligned}$$

while the latter is given by $\mathrm{tr}_P \mathbf{U} = P^{cd}\mathrm{U}_{cd} = h^{CD} v_C^c v_D^d \mathrm{U}_{cd} = \mathrm{tr}_h \mathbf{U}_\parallel$. □

Corollary 3.2.17. *Consider null metric hypersurface data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ and let S be an embedded hypersurface of $\mathcal{N}$ everywhere transversal to n, $\psi : S \longhookrightarrow \mathcal{N}$ the corresponding embedding and $u_S \in \mathcal{F}(S)$ be arbitrary. Then, there exists a choice of gauge such that*

$$\psi^\star \boldsymbol{\ell} = 0, \qquad \ell^{(2)}|_S = u_S. \tag{3.81}$$

Proof. Let $\boldsymbol{\vartheta}_0$ be a normal covector to S. By transversality of S and n it follows $\boldsymbol{\vartheta}_0(n)|_p \neq 0$ for all $p \in \psi(S)$. Extend smoothly $\boldsymbol{\vartheta}_0$ to a covector $\boldsymbol{\vartheta} \in \Gamma(T^\star\mathcal{N})$ satisfying $\boldsymbol{\vartheta}(n) \neq 0$ everywhere. Extend u_S to a smooth function $u \in \mathcal{F}(\mathcal{N})$. The result follows immediately from Lemma 3.2.9. □

Remark 3.2.18. *Note that in this case, the family of gauges satisfying* (3.81) *is highly non-unique.*

In Setup 3.2.15, the induced metric $h \stackrel{\text{def}}{=} \psi^\star\gamma$ is non-degenerate everywhere on S. Indeed, a vector $X \in T_pS$ which is h-orthogonal to all T_pS satisfies also that $\psi_\star|_p(X)$ is γ-orthogonal to all $T_p\mathcal{N}$ (here we use that $T_p\mathcal{N} = T_pS \oplus \langle n|_p\rangle$ and $\gamma(n|_p, \cdot) = 0$). Thus, $\psi_\star|_p(X) \in \operatorname{Rad}(\gamma|_p)$ and hence it must be proportional to $n|_p$. This can only occur if $X = 0$.

The contravariant metric of h will be denoted by $h^\sharp$, and we will simplify notation by identifying S, $X \in \Gamma(TS)$, $f \in \mathcal{F}(\psi(S))$ with their respective counterparts $\psi(S)$, $\psi_\star X$ and $\psi^\star f$. Moreover, for any general p-covariant tensor $\boldsymbol{T}$ along S, we define $\boldsymbol{T}_\parallel \stackrel{\text{def}}{=} \psi^\star\boldsymbol{T}$ and write $T_{A_1\ldots A_p} \stackrel{\text{def}}{=} \boldsymbol{T}_\parallel(\hat{v}_{A_1}, \ldots, \hat{v}_{A_p})$ (without the parallel symbol) for its components.

Given vector fields $X, Y \in \Gamma(TS)$, the derivative $\overset{\circ}{\nabla}_X Y$ can be decomposed on S as

$$\overset{\circ}{\nabla}_X Y = \nabla^S_X Y + \Omega(X,Y)n, \tag{3.82}$$

with $\nabla^S_X Y \in \Gamma(TS)$. It is well-known that $\overset{\circ}{\nabla}$ being torsion-free entails that the two-covariant tensor field Ω is symmetric and that ∇^S is a torsion-free connection on S. Specifically, the tensor Ω is given by

$$\begin{aligned} 2\Omega(X,Y)n &= \overset{\circ}{\nabla}_X Y - \nabla^S_X Y + \overset{\circ}{\nabla}_Y X - \nabla^S_Y X \\ \Longrightarrow \quad 2\Omega(X,Y) &= \mathbf{q}\left(\overset{\circ}{\nabla}_X Y + \overset{\circ}{\nabla}_Y X\right). \end{aligned} \tag{3.83}$$

We can elaborate this in terms of $\mathbf{U}_\parallel \stackrel{\text{def}}{=} \psi^\star\mathbf{U}$ and derivatives of $\boldsymbol{\ell}_\parallel$. To do this, we first note that

$$\boldsymbol{\ell} = \boldsymbol{\ell}_\parallel + \mathbf{q} \tag{3.84}$$

everywhere on S (because both sides agree when acting on the vector n as well as on a tangential vector X). Taking into account (2.19) we compute

$$\begin{aligned} \mathbf{q}(\overset{\circ}{\nabla}_X Y) &= \boldsymbol{\ell}(\overset{\circ}{\nabla}_X Y) - \boldsymbol{\ell}_\parallel(\nabla^S_X Y) = X(\boldsymbol{\ell}(Y)) - (\overset{\circ}{\nabla}_X \boldsymbol{\ell})(Y) - \boldsymbol{\ell}_\parallel(\nabla^S_X Y) \\ &= X(\boldsymbol{\ell}_\parallel(Y)) - \mathbf{F}(X,Y) + \ell^{(2)}\mathbf{U}_\parallel(X,Y) - \boldsymbol{\ell}_\parallel(\nabla^S_X Y) \\ &= (\nabla^S_X \boldsymbol{\ell}_\parallel)(Y) - \mathbf{F}(X,Y) + \ell^{(2)}\mathbf{U}_\parallel(X,Y). \end{aligned} \tag{3.85}$$

Inserting this into (3.83) and using that $\mathbf{F}$ is antisymmetric yields

$$\Omega(X,Y) = \frac{1}{2}\Big((\nabla^S_X \boldsymbol{\ell}_\parallel)(Y) + (\nabla^S_Y \boldsymbol{\ell}_\parallel)(X) \Big) + \ell^{(2)} \mathbf{U}_\parallel(X,Y). \tag{3.86}$$

We now obtain the explicit relation between the connections $\overset{\circ}{\nabla}$, ∇^S and the Levi-Civita covariant derivative ∇^h on S.

Lemma 3.2.19. *In the Setup 3.2.15, let ∇^S and ∇^h be the torsion-free connection given by (3.82) and the Levi-Civita covariant derivative on S respectively. Then,*

$$\nabla^h = \nabla^S - h^\sharp(\boldsymbol{\ell}_\parallel, \cdot) \otimes \mathbf{U}_\parallel, \tag{3.87}$$

$$\nabla_X Y = \nabla^h_X Y + h^\sharp(\boldsymbol{\ell}_\parallel, \cdot)\mathbf{U}_\parallel(X,Y) + \Omega(X,Y)n \qquad \forall X, Y \in \Gamma(TS), \tag{3.88}$$

where Ω is given by

$$\Omega(X,Y) = \frac{1}{2}\Big(\nabla^h_X \boldsymbol{\ell}_\parallel(Y) + \nabla^h_Y \boldsymbol{\ell}_\parallel(X) \Big) + \left(\ell^{(2)} - \ell^{(2)}_\parallel \right) \mathbf{U}_\parallel(X,Y). \tag{3.89}$$

Proof. It is well-known that any torsion-free connection D on S relates to ∇^h according to

$$D_X Y = \nabla^h_X Y - \Xi(X,Y), \qquad \text{where}$$
$$\Xi^A_{BC} \stackrel{\text{def}}{=} \frac{1}{2} h^{AJ} \left(D_B h_{CJ} + D_C h_{BJ} - D_J h_{BC} \right). \tag{3.90}$$

In order to apply this for ∇^S we compute $(\nabla^S_X h)(Y,W)$ as follows:

$$\begin{aligned}
(\nabla^S_X h)(Y,W) &= \nabla^S_X(h(Y,W)) - h(\nabla^S_X Y, W) - h(\nabla^S_X W, Y) \\
&= \overset{\circ}{\nabla}_X(\gamma(Y,W)) - \gamma(\overset{\circ}{\nabla}_X Y, W) - \gamma(\overset{\circ}{\nabla}_X W, Y) \\
&= \psi^\star(\overset{\circ}{\nabla}_X \gamma)(Y,W) = -\boldsymbol{\ell}_\parallel(Y)\mathbf{U}_\parallel(X,W) - \boldsymbol{\ell}_\parallel(W)\mathbf{U}_\parallel(X,Y),
\end{aligned} \tag{3.91}$$

where in the second equality we used $\gamma(n,\cdot) = 0$ and in the last step we inserted (2.18). The tensor Ξ corresponding to $D = \nabla^S$ is therefore

$$\Xi^A{}_{BC} \stackrel{\text{def}}{=} \frac{1}{2} h^{AD} \Big(\nabla^S_B h_{CD} + \nabla^S_C h_{BD} - \nabla^S_D h_{BC} \Big) = -h^{AD} \ell_D \mathrm{U}_{BC},$$

which establishes (3.87). Equation (3.89) follows at once by combining (3.86) and (3.87). Equation (3.88) is an immediate consequence of inserting (3.87) into (3.82). □

Equation (3.87) means that ∇^S coincides with ∇^h if either (i) $\boldsymbol{\ell}_\parallel = 0$ or (ii) $\mathrm{U}_\parallel = 0$. Moreover, (ii) is equivalent to $\mathbf{U} = 0$ because $\mathbf{U}(n, \cdot) = 0$ (cf. (2.14)). Observe that ∇^h is a gauge independent quantity, but ∇^S is not. In fact, as proven in Corollary 3.2.17, the one-form $\boldsymbol{\ell}_\parallel$ can be made zero by an appropriate choice of gauge. The tensor $\mathbf{U}$ is a property of the data and in general it is non-zero (this is a gauge invariant statement, as in the null case $\mathcal{G}_{(z,V)}(\mathbf{U}) = z^{-1}\mathbf{U}$, see (3.60)). Therefore, generically ∇^S will coincide with ∇^h only in case (i).

Our next aim is to relate the tangential components of the curvature tensor of $\mathring{\nabla}$ to the curvature tensor of the induced metric h. The key ingredient that allows us to do this is a generalized Gauss identity that we derive in Appendix B. Recall that on a semi-Riemannian ambient manifold, the Gauss identity is an equation relating the curvature tensor of the Levi-Civita connection along tangential directions of a non-degenerate hypersurface with the curvature tensor of the induced metric and the second fundamental form. In Appendix B, we have extended this result to the more general case when the connection of the space and of the hypersurface are completely general, except for the condition that they are both torsion-free. By particularizing Theorem B.0.1 (more specifically the abstract index notation form (B.7)) to the case of null hypersurface data, we get to the following result.

Lemma 3.2.20. *Consider null metric hypersurface data* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ *and assume Setup 3.2.15. Let* R^h *the Riemann tensor of* S*. Then,*

$$
\begin{aligned}
v^a_A \gamma_{af} \mathring{R}^f{}_{bcd} v^b_B v^c_C v^d_D = {} & R^h_{ABCD} + 2\nabla^h_{[C|}\left(\ell_A \mathrm{U}_{B|D]}\right) + \ell_A \ell^F \left(\mathrm{U}_{BD}\mathrm{U}_{CF} - \mathrm{U}_{BC}\mathrm{U}_{DF}\right) \\
& + \mathrm{U}_{AC}\left(\left(\ell^{(2)} - \ell^{(2)}_\parallel\right)\mathrm{U}_{BD} + \nabla^h_{(B}\ell_{D)}\right) \\
& - \mathrm{U}_{AD}\left(\left(\ell^{(2)} - \ell^{(2)}_\parallel\right)\mathrm{U}_{BC} + \nabla^h_{(B}\ell_{C)}\right).
\end{aligned}
\tag{3.92}
$$

Proof. We particularize Theorem B.0.1 for $\mathcal{M} = \mathcal{N}$, $\widehat{\nabla} = \mathring{\nabla}$, $\widehat{D} = \nabla^h$, $\widehat{\gamma}_C = \gamma$. In such case, $\widehat{h} = h$ and (3.88)-(3.89) hold, which means that $A^C{}_{AB} = \ell^C \mathrm{U}_{AB}$, $A_{hCAB} = \ell_C \mathrm{U}_{AB}$ and $\Omega_{AB} = \nabla^h_{(A}\ell_{B)} + \left(\ell^{(2)} - \ell^{(2)}_\parallel\right)\mathrm{U}_{AB}$. The only term that needs further evaluation is $v^d_D v^a_A \left(\mathring{\nabla}_d \gamma_{af}\right)P^f{}_{BC}$. This is straightforward from (2.7) and (2.18), namely

$$
\begin{aligned}
v^d_D v^a_A \left(\mathring{\nabla}_d \gamma_{af}\right)P^f{}_{BC} & = -v^d_D v^a_A \left(\ell_a \mathrm{U}_{df} + \ell_f \mathrm{U}_{da}\right)\left(v^f_F A^F{}_{BC} + n^f \Omega_{BC}\right) \\
& = -\ell_A \ell^F \mathrm{U}_{DF}\mathrm{U}_{BC} - \ell^{(2)}_\parallel \mathrm{U}_{DA}\mathrm{U}_{BC} - \mathrm{U}_{DA}\Omega_{BC} \\
& = -\ell_A \ell^F \mathrm{U}_{DF}\mathrm{U}_{BC} - \mathrm{U}_{DA}\left(\nabla^h_{(B}\ell_{C)} + \ell^{(2)}\mathrm{U}_{BC}\right).
\end{aligned}
\tag{3.93}
$$

Equation ([3.92](#)) follows at once after inserting ([3.93](#)) into ([B.7](#)) and using $\gamma(n,n) = 0$. □

We conclude this section by providing the pull-back to S of the $\overset{\circ}{\nabla}$ derivative of any p-covariant tensor field T and of the Lie derivative of a general symmetric $(0,2)$-tensor $\boldsymbol{T}$ satisfying $\boldsymbol{T}(n,\cdot) = 0$.

Lemma 3.2.21. *Consider null metric hypersurface data* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ *and assume Setup [3.2.15](#). Any p-covariant tensor field* T *along S verifies*

$$
\begin{aligned}
v^{a_1}_{A_1}\cdots v^{a_p}_{A_p} v^b_B \overset{\circ}{\nabla}_b \mathrm{T}_{a_1\cdots a_p} = {}& \nabla^h_B \mathrm{T}_{A_1\cdots A_p} - \sum_{i=1}^{p} \ell^J \mathrm{T}_{A_1\cdots A_{i-1}JA_{i+1}\cdots A_p} \mathrm{U}_{A_iB} \\
& - \sum_{i=1}^{p} \mathrm{T}_{a_1\cdots a_p} v^{a_1}_{A_1}\cdots v^{a_{i-1}}_{A_{i-1}} n^{a_i} v^{a_{i+1}}_{A_{i+1}}\cdots v^{a_p}_{A_p}\left(\nabla^h_{(A_i}\ell_{B)} + (\ell^{(2)} - \ell^{(2)}_{\parallel})\mathrm{U}_{A_iB}\right),
\end{aligned}
\tag{3.94}
$$

where $\mathrm{T}_{\parallel} \stackrel{\mathrm{def}}{=} \psi^\star \mathrm{T}$.

Proof. We prove it for covectors. The case of covariant tensors with more indices is analogous. From ([3.88](#))-([3.89](#)), we obtain

$$
\begin{aligned}
v^a_A v^b_B \overset{\circ}{\nabla}_b \mathrm{T}_a &= v_B(\mathrm{T}_A) - \mathrm{T}_a v^b_B \overset{\circ}{\nabla}_b v^a_A = v_B(\mathrm{T}_A) - \mathrm{T}_J\left(\nabla^h_{v_B} v^J_A + \ell^J \mathrm{U}_{AB}\right) - \mathrm{T}_a n^a \Omega_{AB} \\
&= \nabla^h_B \mathrm{T}_A - \ell^J \mathrm{T}_J \mathrm{U}_{AB} - \mathrm{T}_a n^a \left(\nabla^h_{(A}\ell_{B)} + (\ell^{(2)} - \ell^{(2)}_{\parallel})\mathrm{U}_{AB}\right),
\end{aligned}
$$

where in the last step we used that $\Omega_{AB} = \nabla^h_{(A}\ell_{B)} + (\ell^{(2)} - \ell^{(2)}_{\parallel})\mathrm{U}_{AB}$ (cf. ([3.89](#))). □

Lemma 3.2.22. *Assume Setup [3.2.15](#) and let $\boldsymbol{T}$ be a symmetric $(0,2)$-tensor on $\mathcal{N}$ satisfying $\boldsymbol{T}(n,\cdot) = 0$. Consider a smooth function q and a covector field $\boldsymbol{\beta} \in \Gamma(T^\star\mathcal{N})|_{\psi(S)}$ satisfying $\boldsymbol{\beta}(n) = 0$ and define $t^a \stackrel{\mathrm{def}}{=} qn^a + P^{ab}\beta_b$. Then,*

$$
(\pounds_t T)_{AB} = (q - \ell^C\beta_C)|_S(\pounds_n T)_{AB} + \beta^C\nabla^h_C T_{AB} + T_{AC}\nabla^h_B\beta^C + T_{CB}\nabla^h_A\beta^C. \tag{3.95}
$$

Proof. Using the decomposition ([3.80](#)) of P^{ab} and the fact that $\beta_a n^a = 0$ we write

$$
t^a = qn^a + h^{AB}v^a_A v^b_B\beta_b - h^{AB}\ell_B n^a v^b_A\beta_b = (q - \ell^A\beta_A)n^a + \beta^A v^a_A.
$$

For any function f we have $\pounds_{fn}\mathbf{T} = f\pounds_n\mathbf{T}$ because $\mathbf{T}(n,\cdot) = \mathbf{T}(\cdot,n) = 0$. On the other hand, for any vector field W tangent to S (i.e. such that there exists $\overline{W} \in \Gamma(TS)$ such that $W|_S = \psi_\star\overline{W}$) it holds $\psi^\star(\pounds_W\mathbf{T}) = \pounds_{\overline{W}}(\psi^\star\mathbf{T})$. Thus,

$$
\psi^\star(\pounds_t\mathbf{T}) = \psi^\star(\pounds_{(q-\ell^C\beta_C)n}\mathbf{T}) + \pounds_{\beta^\sharp}(\psi^\star\mathbf{T}) = (q - \ell^C\beta_C)|_S\psi^\star(\pounds_n\mathbf{T}) + \pounds_{\beta^\sharp}(\psi^\star\mathbf{T}),
$$

where $\beta^\sharp$ is the vector field in S with abstract index components β^A. Since ∇^h is torsion-free the last term can be expanded in terms of the covariant derivative and (3.95) follows. □

3.2.4 *Null metric hypersurface data admitting a cross-section*

In the same spirit as in the case of embedded null hypersurfaces, it is natural to define *a cross-section* S (or simply a section) *of a metric hypersurface data* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ *to be a codimension-one embedded hypersurface in* $\mathcal{N}$ *which is intersected precisely once by each integral curve of* n. Although the existence of a cross-section clearly imposes global topological restrictions on the data, these submanifolds are present in many situations of physical interest. Thus, it makes sense to pay special attention to this case.

A cross-section is by definition a transverse submanifold, so all the results from the previous section apply in this context. As we shall see, one of the most important consequences of having a section is that one can obtain stronger gauge-fixing results. Specifically, it is possible to select the one-form $\boldsymbol{s}$ and the scalar $\ell^{(2)}$ at will.

Proposition 3.2.23. *Consider null hypersurface data* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ *and assume Setup 3.2.15 with the additional condition that* S *is intersected precisely once by each integral curve of* n*. Let* $u \in \mathcal{F}(\mathcal{N})$ *and* $\boldsymbol{\sigma} \in \Gamma(T^\star\mathcal{N})$ *be arbitrary, with the only restriction that* $\boldsymbol{\sigma}$ *verifies* $\boldsymbol{\sigma}(n) = 0$ *everywhere. Then, for any choice of gauge parameter* z *on* $\mathcal{N}$*, there exists a gauge parameter* V*, unique up to the choice of* $\gamma(V,\cdot)$ *on* S*, such that*

$$\mathcal{G}_{(z,V)}(\boldsymbol{s}) = \boldsymbol{\sigma}, \qquad \mathcal{G}_{(z,V)}(\ell^{(2)}) = u. \tag{3.96}$$

Proof. For notational simplicity we denote $\mathcal{G}_{(z,V)}$-transformed quantities with a prime. Consider any gauge parameters $\{z, V\}$ and define $\boldsymbol{w} \stackrel{\text{def}}{=} \gamma(V,\cdot)$. Then, $V - P(\boldsymbol{w},\cdot)$ lies in the kernel of γ (this is a consequence of (2.9) together with $\boldsymbol{w}(n) = 0$). Thus, there exists a function $f \in \mathcal{F}(\mathcal{N})$ such that $V = P(\boldsymbol{w},\cdot) + fn$. This decomposition combined with (2.6)-(2.9) implies $\gamma(V,V) = \boldsymbol{w}(V) = P(\boldsymbol{w},\boldsymbol{w})$ and $\boldsymbol{\ell}(V) = f$. In these circumstances, (2.32) and (3.62) give

$$f = \frac{1}{2}\left(\frac{1}{z^2}\ell'^{(2)} - \ell^{(2)} - P(\boldsymbol{w},\boldsymbol{w})\right), \qquad \pounds_n \boldsymbol{w} = 2(\boldsymbol{s}' - \boldsymbol{s}) + \frac{1}{z}\left(dz - n(z)(\boldsymbol{\ell} + \boldsymbol{w})\right).$$

Thus, (3.96) holds if and only if

$$V = P(\boldsymbol{w},\cdot) + \frac{1}{2}\left(\frac{u}{z^2} - \ell^{(2)} - P(\boldsymbol{w},\boldsymbol{w})\right) n \qquad \text{and} \tag{3.97}$$

$$\pounds_n \boldsymbol{w} = 2(\boldsymbol{\sigma} - \boldsymbol{s}) + \frac{1}{z}\left(dz - n(z)(\boldsymbol{\ell} + \boldsymbol{w})\right). \tag{3.98}$$

Since S is a cross-section, (3.98) gives a unique solution $\boldsymbol{w}$ on $\mathcal{N}$ for given initial data $\boldsymbol{w}|_S$. However, we still need to prove that the solution satisfies $\boldsymbol{w}(n) = 0$, as this has been assumed in the derivation of the equation. Contracting (3.98) with n yields $\pounds_n(\boldsymbol{w}(n)) = -\frac{n(z)}{z}\boldsymbol{w}(n)$. This is a linear homogeneous ODE and by uniqueness of the solution it follows that $\boldsymbol{w}(n) = 0$ on $\mathcal{N}$ if and only of $\boldsymbol{w}(n)|_S = 0$. We therefore conclude that (3.96) is satisfied if and only if V is of the form (3.97) and $\boldsymbol{w}$ is any covector along S satisfying $\boldsymbol{w}(n)|_S = 0$ and extended (uniquely) off S by means of (3.98). Observe that the gauge parameter z can be prescribed freely and that $\boldsymbol{w}$ is fixed up to its value at S, hence V is unique up to the choice $\gamma(V,\cdot)|_S$. □

We now prove that one can always select $z|_{\mathcal{N}}$ and $\gamma(V,\cdot)|_S$ in Proposition 3.2.23 so that $\mathcal{G}_{(z,V)}(\kappa_n) = 0$ and $\mathcal{G}_{(z,V)}(\boldsymbol{\ell}_\parallel)$ is any covector of our choice.

Lemma 3.2.24. *Assume the hypotheses and setup of Proposition 3.2.23 and let κ_n be given by (2.44). Then, the scalar ODE*

$$\pounds_n x - x\kappa_n = 0 \tag{3.99}$$

admits a unique global solution for x provided initial data $x|_S$. Moreover, if $x|_S \neq 0$ then the solution is everywhere non-zero. In particular, if $x|_S \neq 0$ and $z = x$ then

$$\mathcal{G}_{(z,V)}(\kappa_n) = 0 \tag{3.100}$$

and the remaining freedom in the choice of $\{z|_{\mathcal{N}}, \gamma(V,\cdot)|_S\}$ reduces to selecting z and $\gamma(V,\cdot)$ at S.

Proof. Equation (3.99) is a linear homogeneous ODE, hence it admits a global unique solution for x for given initial data $x|_S$. The fact that $x \neq 0$ whenever $x|_S \neq 0$ can be argued as follows. Suppose that $x|_S \neq 0$ and that there exists a point $p \in \mathcal{N}$ where $x|_p = 0$. Then $n(x)|_p = 0$ and hence $x = 0$ on the whole integral curve $\mathcal{C}_p$ of n containing p, in particular at $\mathcal{C}_p \cap S$. Since by hypothesis $x|_S \neq 0$, this means that $x \neq 0$ everywhere.

Setting $z = x$, (3.64) yields $\mathcal{G}_{(z,V)}(\kappa_n) = 0$. The remaining freedom in the choice of $\{z, V\}$ follows from Proposition 3.2.23 and the fact that z needs to satisfy (3.99), so it is fixed up to its value at S. □

Lemma 3.2.25. *Assume the hypotheses and setup of Proposition 3.2.23 and let $\boldsymbol{\varrho}$ be any covector on S. Then,*

$$\mathcal{G}_{(z,V)}\,\boldsymbol{\ell}_\parallel \stackrel{S}{=} \boldsymbol{\varrho} \quad \textit{if and only if} \tag{3.101}$$

$$V \stackrel{S}{=} P(w,\cdot) + \frac{1}{2}\left(\frac{u}{z^2} - \ell^{(2)} - P(w,w)\right) n, \quad \textit{where} \quad w = \frac{1}{z}\boldsymbol{\varrho} + \mathbf{q} - \boldsymbol{\ell}.$$

Moreover, the freedom in the choice of $\{z|_{\mathcal{N}}, \gamma(V,\cdot)|_S\}$ *reduces to selecting* $z|_{\mathcal{N}}$ *at will.*

Proof. First, recall that $\mathbf{q}$ is the unique covector normal to S and such that $\mathbf{q}(n) = 1$. Under a change of gauge we have $n' = z^{-1}n$, so $\mathbf{q}' \stackrel{S}{=} z\mathbf{q}$ is forced. We now use the decomposition (3.84) and write $\boldsymbol{\ell}_\parallel = \boldsymbol{\ell} - \mathbf{q}$. Then,

$$\boldsymbol{\ell}'_\parallel \stackrel{S}{=} z(\boldsymbol{\ell} + w - \mathbf{q}) \stackrel{S}{=} z(\boldsymbol{\ell}_\parallel + w).$$

Thus, condition $\boldsymbol{\ell}'_\parallel \stackrel{S}{=} \boldsymbol{\varrho}$ is satisfied if and only if $w = z^{-1}\boldsymbol{\varrho} + \mathbf{q} - \boldsymbol{\ell}$ on S. This, together with (3.97)-(3.98), forces V to be given by (3.101) at S. Note that we can fulfill (3.96) and (3.101) for any choice of gauge parameter function z. □

Remark 3.2.26. *We have presented the results in a way that will allow us to apply Proposition 3.2.23 either on its own, or in any combination with Lemmas 3.2.24 and 3.2.25.*

Remark 3.2.27. *As a particular case, the function u and the covectors $\boldsymbol{\sigma}, \boldsymbol{\varrho}$ can be set to zero.*

Analogously as in the case of null hypersurfaces, whenever there exists a section S of $\mathcal{N}$ one can always build a foliation of $\mathcal{N}$ by a family of sections. This follows directly from the fact that one can always construct a foliation function λ by solving the first-order ODE $n(\lambda) = \mu \in \mathcal{F}^{\star}(\mathcal{N})$ for some initial data $\lambda \in \mathcal{F}(S)$. It is convenient to introduce a different name when the null metric hypersurface data is restricted to satisfy such a global topological restriction. The terminology was introduced in [60] with the aim of studying the characteristic initial value problem. We adopt the same name and identical definition here.

Definition 3.2.28. *[60], [61] (Characteristic hypersurface data) A hypersurface data set $\mathcal{D} \stackrel{\text{def}}{=} \{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ is called characteristic hypersurface data if*

(i) $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ *is NMD and γ is semi-positive definite.*

(ii) There exists a "foliation function", i.e. a function $\lambda \in \mathcal{F}(\mathcal{N})$ satisfying $n(\lambda)|_p \neq 0$ $\forall p \in \mathcal{N}$.

(iii) The leaves $S_\lambda \stackrel{\text{def}}{=} \{p \in \mathcal{N} : \lambda(p) = \lambda\}$ are all diffeomorphic.

It is worth discussing the differences between Definitions 3.2.2 and 3.2.28. In the former imposing $n^{(2)} = 0$ means that $\operatorname{Rad}(\gamma) = \langle n \rangle$, while the notion of characteristic hypersurface data requires, in addition, that (i) γ is semi-positive definite (which by Lemma 3.1.1 means that $\boldsymbol{A}$ has Lorentzian signature), (*ii*) that there exist a so-called foliation function everywhere on $\mathcal{N}$ and (*iii*) the topological restriction of all leaves $\{\lambda = \text{const.}\}$ being diffeomorphic. In particular, the absence of the topological conditions (*ii*) and (*iii*) makes the notion of null hypersurface data far more general. An example of a null hypersurface data which is not characteristic hypersurface data is a null hypersurface with topology $\mathbb{S}^3$ embedded in a spacetime. It is clear that $\mathbb{S}^3$ cannot be globally foliated by two-dimensional surfaces of identical topology. In more generality, closed (i.e. compact and without boundary) null metric hypersurface data will typically not be charateristic hypersurface data. Such hypersurfaces play a key role e.g. when discussing spacetimes admitting compact Cauchy horizons (see e.g. [121], [122], [123]).

3.3 gauge-invariant vector along the degenerate direction

As already mentioned, Killing horizons of order zero and one are studied later in this thesis by means of the formalism of hypersurface data. Much in the same way as standard Killing horizons, they also involve a privileged vector field which is null and tangent to the hypersurface. This vector field, in addition, is a property of the hypersurface (or of the ambient space where it is embedded) and hence it is gauge-invariant. Therefore, we conclude this chapter by discussing the case when a completely general null hypersurface data set $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ admits an extra gauge-invariant vector field $\overline{\eta}$ in the radical of γ.

In these circumstances, $\overline{\eta}$ is proportional to n and hence there exists a function $\alpha \in \mathcal{F}(\mathcal{N})$ given by $\overline{\eta} = \alpha n$. We denote by $\mathcal{S}$ the submanifold of $\mathcal{N}$ where $\overline{\eta}$ vanishes, i.e. $\mathcal{S} \stackrel{\text{def}}{=} \{p \in \mathcal{N} \mid \alpha(p) = 0\}$. In the following lemma we prove that one can define an associated gauge-invariant scalar function κ on $\mathcal{N}$ for each given gauge-invariant vector $\overline{\eta} \in \operatorname{Rad}\gamma$.

Lemma 3.3.1. *Consider null hypersurface data* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ *equipped with a gauge-invariant vector field* $\overline{\eta} \in \operatorname{Rad}\gamma$ *and let* $\alpha \in \mathcal{F}(\mathcal{N})$ *be given by* $\overline{\eta} = \alpha n$*. Then, the function* $\kappa \in \mathcal{F}(\mathcal{N})$ *defined by*

$$\kappa \stackrel{\text{def}}{=} d\alpha(n) - \alpha\mathbf{Y}(n, n) \tag{3.102}$$

is gauge-invariant.

Proof. By hypothesis $\bar{\eta} = \mathcal{G}_{(z,V)}(\bar{\eta})$ for any pair $\{z \in \mathcal{F}^\star(\mathcal{N}), V \in \Gamma(T\mathcal{N})\}$. This, together with (2.34), implies that the proportionality function α transforms as

$$\mathcal{G}_{(z,V)}(\alpha) = z\alpha. \tag{3.103}$$

To check that $\alpha\mathbf{Y}(n,n) - d\alpha(n)$ is gauge-invariant we start with the first term:

$$\begin{aligned}
\mathcal{G}_{(z,V)}\big(\alpha\mathbf{Y}(n,n)\big) &= z\alpha\left(z\mathbf{Y} + \boldsymbol{\ell}\otimes_s dz + \frac{1}{2}\pounds_{zV}\gamma\right)\left(z^{-1}n, z^{-1}n\right)\\
&= \alpha\mathbf{Y}(n,n) + z^{-1}\alpha\boldsymbol{\ell}(n)dz(n) + \frac{1}{2}z^{-1}\alpha(\pounds_{zV}\gamma)(n,n)\\
&= \alpha\mathbf{Y}(n,n) + z^{-1}\alpha dz(n),
\end{aligned} \tag{3.104}$$

where in the last step we used $\boldsymbol{\ell}(n) = 1$ (cf. (2.7)) and $(\pounds_{zV}\gamma)(n,n) = \pounds_{zV}(\gamma(n,n)) - 2\gamma(\pounds_{zV}n, n) = 0$. Now, the term $d\alpha(n)$ transforms as

$$\mathcal{G}_{(z,V)}(d\alpha(n)) = d\left(\mathcal{G}_{(z,V)}\alpha\right)\left(z^{-1}n\right) = \frac{1}{z}(\alpha dz + z d\alpha)(n) = \frac{\alpha}{z}dz(n) + d\alpha(n). \tag{3.105}$$

From (3.104) and (3.105), the gauge invariance of κ follows at once. □

In later sections, we shall use that (3.102) can be rewritten as $n(\alpha) = \kappa - \alpha\kappa_n$ by means of (2.44).

Whenever $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ happens to be embedded on a semi-Riemannian space $(\mathcal{M}, g)$ with embedding ϕ and rigging ζ, we can define a vector field $\eta \in \Gamma(T\phi(\mathcal{N}))$ by $\eta \stackrel{\text{def}}{=} \phi_\star\bar{\eta}$. By (2.25) and (2.28), the vector field η is null on $\phi(\mathcal{N})$. This, as mentioned in Section 2.6, allows one to define on $\phi(\mathcal{N} \setminus \mathcal{S})$ the so-called surface gravity $\bar{\kappa}$ of η according to (2.81). The pullback of this function to $\mathcal{N}$ is precisely the function κ introduced in Lemma 3.3.1, as we show next. The interesting fact is that expression (3.102) does not require α to be non-zero. By construction κ is smooth and well-defined everyhere on $\mathcal{N}$. It is not obvious a priori that the spacetime function $\bar{\kappa}$, which in general is defined only on an open subset of $\phi(\mathcal{N})$, extends smoothly to all the hypersurface. This is an interesting corollary of the following result.

Proposition 3.3.2. *In the setup of Lemma 3.3.1, define $\eta \stackrel{\text{def}}{=} \phi_\star\bar{\eta} \in \Gamma(T\phi(\mathcal{N}))$ and let $\bar{\kappa}$ be the function defined by (2.81) on $\phi(\mathcal{N} \setminus \mathcal{S})$. Then, $\bar{\kappa} \circ \phi = \kappa$ on $\mathcal{N} \setminus \mathcal{S}$.*

Proof. As usual we identify scalars on $\mathcal{N}$ with their counterparts on $\phi(\mathcal{N})$. The combination of (2.48), (2.81), (3.46) and and the fact that $\mathbf{U}(\bar{\eta}, \bar{\eta}) = 0$ yields

$$\kappa\phi_\star\bar{\eta} = \bar{\kappa}\eta = \nabla_\eta\eta = \phi_\star\left(\overset{\circ}{\nabla}_{\bar{\eta}}\bar{\eta} - \mathbf{Y}(\bar{\eta},\bar{\eta})n\right)$$

$$= \alpha\phi_\star \left(\alpha \overset{\circ}{\nabla}_n n + \left(d\alpha(n) - \alpha \mathbf{Y}(n,n)\right) n \right) = \alpha\kappa\phi_\star\, n = \kappa\phi_\star\overline{\eta}$$

$$\Longleftrightarrow \qquad \kappa = \kappa \qquad \text{on} \qquad N \setminus S, \tag{3.106}$$

where we inserted the definition of κ. □

This result justifies calling the abstract function κ *surface gravity* also. This function is always well-defined everywhere on N and in the embedded case it agrees with the usual definition in the domain where the latter is defined.

4 THE CONSTRAINT TENSOR

A natural question that arises when exploiting the formalism of hypersurface data is whether it is possible to capture some curvature information of the ambient manifold at the abstract level. We have already mentioned (see Proposition 2.2.16) that for any embedded hypersurface data one can determine some components of the ambient Riemann tensor purely in terms of the data. One may also wonder whether it is also possible to codify some components of the ambient Ricci tensor at the abstract level. If this turns out to be possible, then it will make sense to introduce new abstract definitions that encode precisely this information, so that one can work with them without requiring the existence of any ambient space.

It is in this context where the so-called constraint tensor plays a crucial role. This tensor, that can be defined at the abstract level, captures information about a certain combination of the pull-back to the hypersurface of the ambient Ricci tensor and the transverse-tangent-transverse-tangent components of the ambient Riemann tensor. Moreover, in the null and embedded case, it coincides with the pull-back to the hypersurface of the ambient Ricci tensor.

In this chapter, we motivate the definition of the constraint tensor for general hypersurface data. Then, we focus on the null case, where we study its properties and derive some important identities. Since the ambient Ricci tensor is gauge-independent, it is to be expected that the constraint tensor in the null case is gauge-invariant. This is precisely the case, as proven in [60]. The gauge-invariant character of the constraint tensor in the null case allows us to construct several new gauge-invariant quantities. These quantities will play a fundamental role in the description of horizons and their properties that we shall make in Chapters 5 and 6.

4.1 definition and first properties

In this section, we will motivate the (purely abstract) definition of the constraint tensor. First, we show that a certain linear combination of the tangential components of the ambient Ricci tensor and of the transverse-tangential-transverse-tangential components of the ambient Riemann tensor can be computed exclusively in terms of the hypersurface data (whenever it is embedded). This will lead naturally to the definition, on any hypersurface data, of a symmetric 2-covariant tensor that encodes at the purely abstract level this combination of the ambient Riemann tensor. This construction is done for general data, although, as we shall see next, the most interesting case arises at null points because then this tensor encodes precisely the information of the tangential components of the ambient Ricci tensor.

Consider hypersurface data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ embedded on an ambient space $(\mathcal{M}, g)$ with embedding ϕ and rigging vector ζ and assume Setup 2.2.7. The first decomposition in (2.27) can be used to compute the ambient Ricci tensor along tangential directions to $\phi(\mathcal{N})$, i.e. $R_{\alpha\beta} e^\alpha_b e^\beta_d$. From $R_{\alpha\beta} \stackrel{\text{def}}{=} g^{\mu\nu} R_{\mu\alpha\nu\beta}$, it follows

$$
\begin{aligned}
R_{\alpha\beta} e^\alpha_b e^\beta_d &\stackrel{\phi(\mathcal{N})}{=} \left(n^{(2)} \zeta^\mu \zeta^\nu + n^c \left(\zeta^\mu e^\nu_c + \zeta^\nu e^\mu_c \right) + P^{cd} e^\mu_c e^\nu_d \right) R_{\mu\alpha\nu\beta} e^\alpha_b e^\beta_d \\
&\stackrel{\phi(\mathcal{N})}{=} n^{(2)} R_{\mu\alpha\nu\beta} \zeta^\mu e^\alpha_b \zeta^\nu e^\beta_d \\
&\quad + n^c \left(R_{\mu\alpha\nu\beta} \zeta^\mu e^\alpha_b e^\nu_c e^\beta_d + R_{\nu\beta\mu\alpha} \zeta^\nu e^\beta_d e^\mu_c e^\alpha_b \right) + P^{cd} R_{\mu\alpha\nu\beta} e^\mu_c e^\alpha_b e^\nu_d e^\beta_d
\end{aligned}
$$

By Proposition 2.2.16 we know that the contractions $R_{\mu\alpha\nu\beta} \zeta^\mu e^\alpha_b e^\nu_c e^\beta_d$ and $R_{\mu\alpha\nu\beta} e^\mu_c e^\alpha_b e^\nu_d e^\beta_d$ can be written in terms of the hypersurface data. However, in general this is not true for the components $R_{\mu\alpha\nu\beta} \zeta^\mu e^\alpha_b \zeta^\nu e^\beta_d$. We thus write the previous identity as (recall (2.28))

$$
\begin{aligned}
\mathbf{Ric}(e_b, e_d) - g(\nu, \nu) \mathbf{Riem}(\zeta, e_b, \zeta, e_d) &\stackrel{\phi(\mathcal{N})}{=} \\
& 2 n^c \mathbf{Riem}(\zeta, e_{(b|}, e_c, e_{|d)}) + P^{cd} \mathbf{Riem}(e_c, e_b, e_d, e_d),
\end{aligned} \tag{4.1}
$$

where $\mathbf{Ric}$ and $\mathbf{Riem}$ are respectively the Ricci and Riemann tensors of $(\mathcal{M}, g)$. Note that at null points (where $n^{(2)} = g(\nu, \nu) = 0$) the left-hand side simplifies and reduces to the tangential components of the ambient Ricci tensor alone. At non-null points, it is precisely that combination of tangential Ricci and tangential-transverse Riemann tensor that can be computed in terms of the hypersurface data.

It therefore makes sense to obtain the explicit expressions in the right-hand side of (4.1) in terms of $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$. To do that there is no need to assume any longer that the data is embedded. We work at the abstract level by introducing two tensors A_{bcd} and B_{abcd} on $\mathcal{N}$, which correspond to the hypersurface data counterparts of $R_{\mu\alpha\nu\beta}\zeta^{\mu}e^{\alpha}_{b}e^{\nu}_{c}e^{\beta}_{d}$ and $R_{\mu\alpha\nu\beta}e^{\mu}_{a}e^{\alpha}_{b}e^{\nu}_{c}e^{\beta}_{d}$ respectively (as given in Proposition 2.2.16). The right-hand side of (4.1) can then be elaborated at the abstract level by computing the contractions $n^c(A_{bcd} + A_{dcd})$ and $P^{ac}B_{abcd}$. As already mentioned, we start with the definitions of the tensors A_{bcd} and B_{abcd} as dictated by Proposition 2.2.16.

Definition 4.1.1. *(Tensors A and B) Given hypersurface data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$, the tensors A and B are defined as*

$$
\begin{aligned}
A_{bcd} \stackrel{\text{def}}{=} {} & \ell_a \overset{\circ}{R}{}^a{}_{bcd} + \overset{\circ}{\nabla}_d \mathrm{Y}_{cb} - \overset{\circ}{\nabla}_c \mathrm{Y}_{db} \\
& + \ell^{(2)} \left(\overset{\circ}{\nabla}_d \mathrm{U}_{cb} - \overset{\circ}{\nabla}_c \mathrm{U}_{db} \right) + \frac{1}{2} \left(\mathrm{U}_{cb} \overset{\circ}{\nabla}_d \ell^{(2)} - \mathrm{U}_{db} \overset{\circ}{\nabla}_c \ell^{(2)} \right) \\
& + \mathrm{Y}_{bd} \left(\left(\mathrm{F}_{cf} + \mathrm{Y}_{cf} \right) n^f + \frac{1}{2} n^{(2)} \overset{\circ}{\nabla}_c \ell^{(2)} \right) \\
& - \mathrm{Y}_{bc} \left(\left(\mathrm{F}_{df} + \mathrm{Y}_{df} \right) n^f + \frac{1}{2} n^{(2)} \overset{\circ}{\nabla}_d \ell^{(2)} \right),
\end{aligned}
\tag{4.2}
$$

$$
\begin{aligned}
B_{abcd} \stackrel{\text{def}}{=} {} & \gamma_{af} \overset{\circ}{R}{}^f{}_{bcd} + \ell_a \left(\overset{\circ}{\nabla}_d \mathrm{U}_{cb} - \overset{\circ}{\nabla}_c \mathrm{U}_{db} \right) + \mathrm{Y}_{bc} \mathrm{U}_{da} - \mathrm{Y}_{bd} \mathrm{U}_{ca} + \mathrm{U}_{bc} \mathrm{Y}_{da} - \mathrm{U}_{bd} \mathrm{Y}_{ca} + \\
& + n^{(2)} \left(\mathrm{Y}_{bc} \mathrm{Y}_{da} - \mathrm{Y}_{bd} \mathrm{Y}_{ca} \right) + \mathrm{U}_{bc} \mathrm{F}_{da} - \mathrm{U}_{bd} \mathrm{F}_{ca}.
\end{aligned}
\tag{4.3}
$$

We proceed with the evaluation of $n^c(A_{bcd} + A_{dcd})$ and $P^{ac}B_{abcd}$. Our guiding principle for the computation is to let as many derivatives of $\mathbf{Y}$ as possible appear in the form of $\pounds_n \mathbf{Y}$, i.e. as evolution terms along the direction n. This will be particularly useful in the null case, where n is the degeneration direction of γ. The result, however, holds in full generality.

Proposition 4.1.2. *Let $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ be hypersurface data and $\mathbf{r}$, κ_n be given by (2.44). Then, the tensors A and B introduced in Definition 4.1.1 satisfy the following identities:*

$$
\begin{aligned}
P^{ac} B_{abcd} = {} & \overset{\circ}{R}_{(bd)} - \overset{\circ}{\nabla}_{(b} s_{d)} + s_b s_d - \mathrm{U}_{bd} n(\ell^{(2)}) \\
& + \left(-2 n^{(2)} s_{(b} + \frac{1}{2} \overset{\circ}{\nabla}_{(b} n^{(2)} + \frac{1}{2} (n^{(2)})^2 \overset{\circ}{\nabla}_{(b} \ell^{(2)} \right) \overset{\circ}{\nabla}_{d)} \ell^{(2)} \\
& + P^{ac} \Big(n^{(2)} \mathrm{F}_{ba} \mathrm{F}_{dc} + \mathrm{U}_{ba} \mathrm{Y}_{dc} + \mathrm{U}_{da} \mathrm{Y}_{bc} - \mathrm{U}_{bd} \mathrm{Y}_{ac} - \mathrm{Y}_{bd} \mathrm{U}_{ac} \\
& + n^{(2)} \mathrm{Y}_{ba} \mathrm{Y}_{dc} - n^{(2)} \mathrm{Y}_{bd} \mathrm{Y}_{ac} \Big),
\end{aligned}
\tag{4.4}
$$

$$
\begin{aligned}
n^c \left(A_{bcd} + A_{dcb} \right) = {} & -2 \pounds_n \mathrm{Y}_{bd} + 2 \overset{\circ}{\nabla}_{(b} \left(s_{d)} + r_{d)} \right) - 2 \kappa_n \mathrm{Y}_{bd} - 2 \left(r_b - s_b \right) \left(r_d - s_d \right) \\
& + \left(3 n^{(2)} s_{(b} - 3 n^{(2)} r_{(b} - \frac{1}{2} \overset{\circ}{\nabla}_{(b} n^{(2)} - \frac{1}{2} (n^{(2)})^2 \overset{\circ}{\nabla}_{(b} \ell^{(2)} \right) \overset{\circ}{\nabla}_{d)} \ell^{(2)}
\end{aligned}
$$

$$+\left(U_{bd} + n^{(2)}Y_{bd}\right) n(\ell^{(2)}) + 2P^{ac}\left(Y_{c(b} - F_{c(b}\right)\left(U_{d)a} - n^{(2)}F_{d)a}\right). \tag{4.5}$$

Moreover, it also holds

$$\begin{aligned}
n^c\left(A_{bcd} + A_{dcb}\right) + P^{ac}B_{abcd} = {} & \mathring{R}_{(bd)} - 2\pounds_n Y_{bd} \\
& - \left(2\kappa_n + \mathrm{tr}_P \mathbf{U} - n^{(2)}\left(n(\ell^{(2)}) - \mathrm{tr}_P \mathbf{Y}\right)\right) Y_{bd} \\
& + \mathring{\nabla}_{(b}\left(s_{d)} + 2r_{d)}\right) - 2r_b r_d + 4r_{(b}s_{d)} - s_b s_d - (\mathrm{tr}_P \mathbf{Y})\, U_{bd} \\
& + 2P^{ac}U_{a(b}\left(2Y_{d)c} + F_{d)c}\right) + n^{(2)}\Big(\left(s_{(b} - 3r_{(b}\right)\mathring{\nabla}_{d)}\ell^{(2)} \\
& + P^{ac}\left(Y_{ab} + F_{ab}\right)\left(Y_{cd} + F_{cd}\right)\Big).
\end{aligned} \tag{4.6}$$

Proof. Since the connection $\mathring{\nabla}$ has vanishing torsion, we can write the Ricci identity for $\boldsymbol{\ell}$ in the usual form

$$\ell_f \mathring{R}^f{}_{bcd} = \mathring{\nabla}_d\mathring{\nabla}_c \ell_b - \mathring{\nabla}_c\mathring{\nabla}_d \ell_b \overset{(2.19)}{=} \mathring{\nabla}_d\left(F_{cb} - \ell^{(2)}U_{cb}\right) - \mathring{\nabla}_c\left(F_{db} - \ell^{(2)}U_{db}\right). \tag{4.7}$$

We now contract this with n^c and use (3.14) applied to $A \longrightarrow \mathbf{F}$ and $a \longrightarrow s$. Using that $d\mathbf{F} = 0$ (which follows from $\mathbf{F} = \frac{1}{2}d\boldsymbol{\ell}$), we get

$$\ell_f \mathring{R}^f{}_{bcd}n^c = \mathring{\nabla}_b s_d - F_{cd}\mathring{\nabla}_b n^c - 2\ell^{(2)}n^c\mathring{\nabla}_{[d}U_{c]b} + U_{bd}n(\ell^{(2)}) - n^cU_{cb}\mathring{\nabla}_d\ell^{(2)}. \tag{4.8}$$

By Proposition 2.2.15 the Ricci tensor $\mathring{R}_{ab}$ can be written as

$$\mathring{R}_{bd} = \mathring{R}_{(bd)} + \mathring{R}_{[bd]} = \mathring{R}_{(bd)} + \mathring{\nabla}_{[b}s_{d]} - \frac{1}{2}\mathring{\nabla}_{[b}n^{(2)}\mathring{\nabla}_{d]}\ell^{(2)}. \tag{4.9}$$

Recalling (2.8)-(2.9) we then obtain, from (4.7) and (4.9),

$$\begin{aligned}
P^{ac}\left(Y_{af}\mathring{R}^f{}_{bcd} + 2\ell_a\mathring{\nabla}_{[d}U_{c]b}\right) &= \mathring{R}_{bd} - n^c\ell_f\mathring{R}^f{}_{bcd} - 2\ell^{(2)}n^c\mathring{\nabla}_{[d}U_{c]b} \\
&= \mathring{R}_{(bd)} - \mathring{\nabla}_{(b}s_{d)} - \frac{1}{2}\mathring{\nabla}_{[b}n^{(2)}\mathring{\nabla}_{d]}\ell^{(2)} - U_{bd}n(\ell^{(2)}) + F_{cd}\mathring{\nabla}_b n^c + n^cU_{cb}\mathring{\nabla}_d\ell^{(2)},
\end{aligned} \tag{4.10}$$

We elaborate the last two terms by taking into account (2.14) and (2.20). This yields

$$\begin{aligned}
F_{cd}\mathring{\nabla}_b n^c + n^cU_{cb}\mathring{\nabla}_d\ell^{(2)} = {} & -2n^{(2)}s_{(b}\mathring{\nabla}_{d)}\ell^{(2)} + P^{ac}\left(n^{(2)}F_{ba}F_{dc} - U_{ba}F_{dc}\right) \\
& + s_b s_d + \frac{1}{2}\mathring{\nabla}_b n^{(2)}\mathring{\nabla}_d\ell^{(2)} + \frac{1}{2}\left(n^{(2)}\right)^2\mathring{\nabla}_b\ell^{(2)}\mathring{\nabla}_d\ell^{(2)}.
\end{aligned} \tag{4.11}$$

We have all the ingredients to compute $P^{ac}B_{abcd}$. Contracting the right hand side of (4.3) with P^{ac} and replacing (4.10) and (4.11), expression (4.4) follows after simple manipulations.

For (4.5) we start by substituting (4.7) in (4.2), which gives

$$\begin{aligned} A_{bcd} = {} & \mathring{\nabla}_d \mathrm{F}_{cb} - \mathring{\nabla}_c \mathrm{F}_{db} + \mathring{\nabla}_d \mathrm{Y}_{cb} - \mathring{\nabla}_c \mathrm{Y}_{db} - \frac{1}{2}\left(\mathrm{U}_{cb} + n^{(2)}\mathrm{Y}_{cb}\right)\mathring{\nabla}_d \ell^{(2)} \\ & + \frac{1}{2}\left(\mathrm{U}_{db} + n^{(2)}\mathrm{Y}_{db}\right)\mathring{\nabla}_c \ell^{(2)} + \mathrm{Y}_{bd}\left(\mathrm{F}_{cf} + \mathrm{Y}_{cf}\right)n^f - \mathrm{Y}_{bc}\left(\mathrm{F}_{df} + \mathrm{Y}_{df}\right)n^f. \end{aligned} \tag{4.12}$$

We now contract with n^c and use (3.12) with $S \longrightarrow \mathbf{Y}$ and (3.13) with $A \longrightarrow \mathbf{F}$ to get

$$\begin{aligned} n^c A_{bcd} = {} & \mathring{\nabla}_{(b} s_{d)} - \mathrm{F}_{c(b}\mathring{\nabla}_{d)} n^c - \frac{1}{2} n^c \mathring{\nabla}_c \mathrm{F}_{db} + \mathring{\nabla}_d r_b - \pounds_n \mathrm{Y}_{bd} + \mathrm{Y}_{cd}\mathring{\nabla}_b n^c \\ & - \frac{1}{2} n^c \left(\mathrm{U}_{cb} + n^{(2)}\mathrm{Y}_{cb}\right)\mathring{\nabla}_d \ell^{(2)} + \frac{1}{2}\left(\mathrm{U}_{db} + n^{(2)}\mathrm{Y}_{db}\right) n\left(\ell^{(2)}\right) \\ & - \kappa_n \mathrm{Y}_{bd} + r_b s_d - r_b r_d, \end{aligned} \tag{4.13}$$

where we have taken into account the definitions (2.44). Taking the symmetric part one obtains

$$\begin{aligned} n^c\left(A_{bcd} + A_{dcb}\right) = {} & 2\mathring{\nabla}_{(b}\left(s_{d)} + r_{b)}\right) + 2\left(\mathrm{Y}_{c(b} - \mathrm{F}_{c(b}\right)\mathring{\nabla}_{d)} n^c \\ & - 2\pounds_n \mathrm{Y}_{bd} - 2\kappa_n \mathrm{Y}_{bd} + 2 r_{(b} s_{d)} - 2 r_b r_d \\ & - \left(n^c \mathrm{U}_{c(b} + n^{(2)} r_{(b}\right)\mathring{\nabla}_{d)} \ell^{(2)} + \left(\mathrm{U}_{bd} + n^{(2)}\mathrm{Y}_{bd}\right) n\left(\ell^{(2)}\right). \end{aligned} \tag{4.14}$$

By virtue of (2.20), we finally find

$$\begin{aligned} 2\left(\mathrm{Y}_{cb} - \mathrm{F}_{cb}\right)\mathring{\nabla}_d n^c = {} & 2P^{ac}\left(\mathrm{Y}_{cb} - \mathrm{F}_{cb}\right)\left(\mathrm{U}_{da} - n^{(2)}\mathrm{F}_{da}\right) \\ & + 2\left(r_b - s_b\right)\left(-n^{(2)}\mathring{\nabla}_d \ell^{(2)} + s_d\right), \end{aligned}$$

which together with (2.14) yields (4.5) when inserted into (4.14). Finally, equation (4.6) follows at once after simple index manipulations. □

Note that the right hand side of (4.4) is explicitly symmetric in the indices b, d. This property is consistent with the fact that, in the embedded case, the left-hand side of (2.53) is symmetric under the interchange of the first and second pair of indices. This provides a non-trivial consistency check for (4.4).

As explained above, expression (4.6) motivates introducing a symmetric tensor R on N that we call **constraint tensor**.

Definition 4.1.3. *(Constraint tensor* $\mathcal{R}$*) Given hypersurface data* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$*, the constraint tensor* $\mathcal{R}$ *tensor is the symmetric* 2*-covariant tensor*

$$\begin{aligned}\mathcal{R}_{bd} \stackrel{\text{def}}{=}\ & \overset{\circ}{R}_{(bd)} - 2\pounds_n \mathrm{Y}_{bd} - \Big(2\kappa_n + \operatorname{tr}_P \mathbf{U} - n^{(2)}\big(n(\ell^{(2)}) - \operatorname{tr}_P \mathbf{Y}\big)\Big)\mathrm{Y}_{bd} \\ & + \overset{\circ}{\nabla}_{(b}\big(s_{d)} + 2r_{d)}\big) - 2r_b r_d + 4r_{(b}s_{d)} - s_b s_d \\ & - (\operatorname{tr}_P \mathbf{Y})\mathrm{U}_{bd} + 2P^{ac}\mathrm{U}_{a(b}\big(2\mathrm{Y}_{d)c} + \mathrm{F}_{d)c}\big) \\ & + n^{(2)}\Big(\big(s_{(b} - 3r_{(b}\big)\overset{\circ}{\nabla}_{d)}\ell^{(2)} + P^{ac}(\mathrm{Y}_{ab} + \mathrm{F}_{ab})(\mathrm{Y}_{cd} + \mathrm{F}_{cd})\Big). \end{aligned} \tag{4.15}$$

where κ_n *and* r_a *are defined by* (2.44).

The whole construction has been performed so that the following result holds.

Proposition 4.1.4. *Let* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ *be hypersurface data embedded in* $(\mathcal{M}, g)$ *with embedding* ϕ *and rigging* ζ*. Let* $\boldsymbol{\nu}$ *be the unique normal covector along* $\phi(\mathcal{N})$ *satisfying* $\boldsymbol{\nu}(\zeta) = 1$ *and define* $\nu \stackrel{\text{def}}{=} g(\boldsymbol{\nu}, \cdot)$*. Consider the symmetric* 2*-covariant tensor* $\boldsymbol{R} \stackrel{\text{def}}{=} \mathbf{Ric} - g(\nu, \nu)\mathbf{Riem}(\zeta, \cdot, \zeta, \cdot)$ *along* $\phi(\mathcal{N})$*. Then*

$$\phi^\star \boldsymbol{R} = \mathcal{R}. \tag{4.16}$$

In particular at any point p *where the hypersurface* $\phi(\mathcal{N})$ *is null, it holds*

$$\phi^\star \mathbf{Ric}|_p = \mathcal{R}|_p. \tag{4.17}$$

At null points the expression for the constraint tensor simplifies, as one has $n^{(2)} = 0$. It is worth stressing that the expression for the tangential components of the ambient Ricci tensor in the null case has been obtained in a fully covariant way.

In the case $n^{(2)} = 0$, the conditions $\mathcal{R} = 0$ can be thought of as the vacuum constraint equations (with vanishing cosmological constant) on a null hypersurface. Such constraints have always appeared in the literature in a decomposed form adapted to a foliation by spacelike slices. To the best of our knowledge, the only exception to this is [60, Eq. (50)] (see also [61, Eq. (34)]), where the tensors A_{abc}, B_{abcd} and $\mathcal{R}_{ab} = n^c(A_{bcd} + A_{dcb}) + P^{ac}B_{abcd}$ were defined (only in the null case) in terms of the hypersurface connection $\overline{\nabla}$ introduced in Section 2.2.2.1 before. Recall that the torsion-free derivative $\overline{\nabla}$ coincides in the embedded case with the connection induced from the Levi-Civita covariant derivative of the ambient space. In [60], the expression of $\mathcal{R}$ is not fully explicit in the tensor $\mathbf{Y}$, as the connection $\overline{\nabla}$ and $\overline{R}$ depend on it. Definition (4.15), on the other hand, shows the full dependence on $\mathbf{Y}$ (in the terms involving $\mathbf{Y}$, $\boldsymbol{r}$ and κ_n), as both $\overset{\circ}{\nabla}$ and $\overset{\circ}{R}$ depend only on the metric part of the data. Moreover, the tensor $\mathcal{R}$ on [60] was not expanded in terms of the data, as we have done here in expression (4.15). Instead, it was decomposed in terms of a foliation by spacelike hypersurfaces, in analogy with other forms of the constraint equations that have appeared in the literature. The result above involves no decomposition with respect to any foliation. In fact, it makes no assumption on whether such foliation exists. The result is fully covariant on $\mathcal{N}$, even though this manifold admits no metric. It is by use of the hypersurface data formalism (in particular thanks to the existence of the connection $\overset{\circ}{\nabla}$) that such compact and unified form of the vacuum constraint equations in the null case becomes possible.

Given its interpretation in the embedded case, it is to be expected that the constraint tensor is gauge invariant at a null point. This was already proven in [60, Theorem 4.6] in the case of characteristic hypersurface data which (recall Definition 3.2.2 in the previous chapter) is null hypersurface data that can be foliated by diffeomorphic sections with positive definite induced metric. However, the proof of Theorem 4.6 in [60] does not rely on these global restrictions, so the gauge invariance of the constraint tensor $\mathcal{R}$ holds for general null hypersurface data[1]. In particular, this means that in the null case we can compute $\mathcal{R}$ in any gauge, which gives a lot of flexibility to adjust the gauge to the problem at hand. At non-null points gauge invariance does not hold since the spacetime tensor $\boldsymbol{R}$ depends on the rigging vector ζ.

At non-null points, Propositions 2.2.16 and 4.1.4 admit the following immediate corollary.

Corollary 4.1.5. *Let* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ *be hypersurface data embedded in* $(\mathcal{M}, g)$ *with embedding* ϕ *and rigging* ζ. *Assume that the tangential components of the Ricci tensor* **Ric** *along* $\phi(\mathcal{N})$ *are known, then the whole Riemann tensor* **Riem** *at any non-null point* $p \in \mathcal{N}$ *can be determined explicitly in terms of the hypersurface data.*

4.2 constraint tensor: null case

For the rest of the chapter, we shall focus on the null case, so we assume that $n^{(2)} = 0$ everywhere on $\mathcal{N}$. Since the definition (4.15) of the constraint tensor $\mathcal{R}$ simplifies remarkably in this context, it is convenient to write it down as a definition.

[1] It should actually be true that gauge invariance holds even at isolated null points. We do not attempt proving this fact here.

Definition 4.2.1. *Let* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ *be null hypersurface data. The constraint tensor* $\mathcal{R}$ *is the symnmetric tensor defined by*

$$\mathcal{R}_{ab} \stackrel{\text{def}}{=} \overset{\circ}{R}_{(ab)} - 2\pounds_n \mathrm{Y}_{ab} - (2\kappa_n + \mathrm{tr}_P \mathbf{U})\, \mathrm{Y}_{ab} + \overset{\circ}{\nabla}_{(a}\Big(s_{b)} + 2r_{b)}\Big)$$
$$- 2r_a r_b + 4r_{(a}s_{b)} - s_a s_b - (\mathrm{tr}_P \mathbf{Y})\mathrm{U}_{ab} + 2P^{cd}\mathrm{U}_{d(a}\Big(2\mathrm{Y}_{b)c} + \mathrm{F}_{b)c}\Big). \tag{4.18}$$

In the next section we will evaluate the contraction of $\mathcal{R}$ with the null direction n. After that, in Section 4.2.2 we will obtain contractions of the constraint tensor along directions tangent to a given transverse submanifold S (non-necessarily a cross-section) of $\mathcal{N}$.

4.2.1 *Constraint tensor along the null direction n*

As already mentioned several times before, the degeneration vector n defines a privileged direction on any null hypersurface data. It therefore makes sense to compute explicitly all the independent contractions of the constraint tensor with this vector. We emphasize that the result does not require any topological assumption whatsoever. In particular, the null hypersurface data does not need to be foliated by sections.

Theorem 4.2.2. *Consider null hypersurface data* $\mathcal{D} = \{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ *and let* $\mathcal{R}$ *be the constraint tensor. Then,*

$$\mathcal{R}_{ab}n^a = -\overset{\circ}{\nabla}_b \kappa_n - \pounds_n (r_b - s_b)$$
$$- (\mathrm{tr}_P \mathbf{U})(r_b - s_b) - \nabla_b(\mathrm{tr}_P \mathbf{U}) + P^{cd}\overset{\circ}{\nabla}_c \mathrm{U}_{bd} - 2P^{cd}\mathrm{U}_{bd}\, s_c, \tag{4.19}$$

$$\mathcal{R}_{ab}n^a n^b = -n(\mathrm{tr}_P \mathbf{U}) + (\mathrm{tr}_P \mathbf{U})\kappa_n - P^{ab}P^{cd}\mathrm{U}_{ac}\mathrm{U}_{bd}. \tag{4.20}$$

Proof. Recall the facts $\mathbf{U}(n,\cdot) = 0$, $\boldsymbol{s}(n) = 0$, $\mathbf{Y}(n,\cdot) = \boldsymbol{r}$, $\mathbf{Y}(n,n) = \boldsymbol{r}(n) = -\kappa_n$, $\mathbf{F}(n,\cdot) = \boldsymbol{s}$. Particularizing (3.11) for $n^{(2)} = 0$ and $\boldsymbol{\theta} = \boldsymbol{s} + 2\boldsymbol{r}$ we get

$$n^a \overset{\circ}{\nabla}_{(a}\big(s_{b)} + 2r_{b)}\big) = \frac{1}{2}\pounds_n s_b + \pounds_n r_b - \nabla_b \kappa_n + 2\kappa_n s_b - P^{ac}\mathrm{U}_{bc}(s_a + 2r_a). \tag{4.21}$$

The contraction of (4.18) with n^a gives (4.19) after inserting (3.75), (4.21) and $\pounds_n r_b = n^a \pounds_n \mathrm{Y}_{ab}$. Contracting (4.19) with n^b and using that $n^b P^{cd}\overset{\circ}{\nabla}_c \mathrm{U}_{bd} = -P^{cd}\mathrm{U}_{bd}\overset{\circ}{\nabla}_c n^b = -P^{ab}P^{cd}\mathrm{U}_{ac}\mathrm{U}_{bd}$ as well as $n^b \pounds_n (s_b - r_b) = n(\kappa_n)$ yields (4.20). □

Observe that the identity (4.20) corresponds to the Raychaudhuri equation (2.103) that we derived before in the context of null hypersurfaces. From the comparison between (4.20) and (2.103), it is straightforward to conclude that at the abstract level $\operatorname{tr}_P \mathbf{U}$ plays the role of the expansion θ while $P^{ab}P^{cd}\mathrm{U}_{ac}\mathrm{U}_{bd}$ stands for the term $(n-1)^{-1}\theta^2 + \varsigma^2$.

4.2.2 *Constraint tensor on a transverse submanifold S*

Let us now assume Setup 3.2.15 and analyze the case when we have selected a codimension one submanifold S of N to which n is everywhere transverse. In particular, all results from Section 3.2.3 will apply here. Our main aim is to derive an explicit expression for the pull-back to S of the constraint tensor, i.e. $\psi^\star \mathrm{R}$, in terms of the Ricci tensor of the Levi-Civita connection ∇^h (see Section 3.2.3). By (4.18), this task requires relating the pull-back $\psi^\star \overset{\circ}{\mathbf{R}}\mathbf{ic}$ with the Ricci tensor of ∇^h. Now, computing the pull-back $\psi^\star \overset{\circ}{\mathbf{R}}\mathbf{ic}$ amounts to calculating $\overset{\circ}{R}_{AB} \stackrel{\text{def}}{=} \overset{\circ}{R}{}^c{}_{acb} v^a_A v^b_B$. This trace can be obtained by means of (2.9) and (3.80) as follows:

$$\begin{aligned}
\overset{\circ}{R}_{AB} = \delta^c_f \overset{\circ}{R}{}^f{}_{acb} v^a_A v^b_B &= \left(P^{cd}\gamma_{df} + n^c \boldsymbol{\ell}_f \right) \overset{\circ}{R}{}^f{}_{acb} v^a_A v^b_B \\
&= \left(h^{CD} v^c_C v^d_D \gamma_{df} + n^c(\boldsymbol{\ell}_f - h^{CD}\boldsymbol{\ell}_C v^d_D \gamma_{df}) \right) \overset{\circ}{R}{}^f{}_{acb} v^a_A v^b_B
\end{aligned} \tag{4.22}$$

Thus, we need to evaluate both

$$h^{CD} v^d_D \gamma_{df} \overset{\circ}{R}{}^f{}_{acb} v^a_A v^c_C v^b_B \qquad \text{and} \qquad n^c(\boldsymbol{\ell}_f - h^{CD}\boldsymbol{\ell}_C v^d_D \gamma_{df}) \overset{\circ}{R}{}^f{}_{acb} v^a_A v^b_B$$

The first one is obtained by contracting (3.92) with h^{CD}. For the second one, substituting (3.78)-(3.79) into (3.70) and (3.71) yields

$$\begin{aligned}
n^c \left(\boldsymbol{\ell}_f - h^{CD}\boldsymbol{\ell}_C v^d_D \gamma_{df} \right) \overset{\circ}{R}{}^f{}_{acb} v^a_A v^b_B = v^a_A v^b_B \overset{\circ}{\nabla}_a s_b - s_A s_B + \left(n(\boldsymbol{\ell}^{(2)}) - 2\boldsymbol{\ell}^D s_D \right) \mathrm{U}_{AB} \\
+ (\boldsymbol{\ell}^{(2)} - \boldsymbol{\ell}^{(2)}_{\parallel})(\pounds_n \mathbf{U})_{AB} \\
+ 2\boldsymbol{\ell}^D \left(s_{(A}\mathrm{U}_{B)D} - v^d_D v^a_A v^b_B \overset{\circ}{\nabla}_{[a}\mathrm{U}_{d]b} \right) \\
- h^{CD}\mathrm{U}_{AC} \left(\mathrm{F}_{DB} + (\boldsymbol{\ell}^{(2)} - \boldsymbol{\ell}^{(2)}_{\parallel})\mathrm{U}_{BD} \right).
\end{aligned} \tag{4.23}$$

We elaborate (4.23) by particularizing (3.94) for T $= s$, T $= \mathbf{U}$ and T $= \boldsymbol{\ell}$. Since $s(n) = \mathbf{U}(n,\cdot) = 0$ they give, respectively,

$$v^a_A v^b_B \overset{\circ}{\nabla}_a s_b = \nabla^h_A s_B - \boldsymbol{\ell}^C s_C \mathrm{U}_{AB}, \tag{4.24}$$

$$2 v^d_D v^a_A v^b_B \overset{\circ}{\nabla}_{[a}\mathrm{U}_{d]b} = \nabla^h_A \mathrm{U}_{BD} - \nabla^h_D \mathrm{U}_{AB} - \boldsymbol{\ell}^C \mathrm{U}_{CD}\mathrm{U}_{AB} + \boldsymbol{\ell}^C \mathrm{U}_{AC}\mathrm{U}_{BD}, \tag{4.25}$$

$$v^a_A v^b_B \overset{\circ}{\nabla}_a \boldsymbol{\ell}_b = \nabla^h_A \boldsymbol{\ell}_B - \boldsymbol{\ell}^{(2)}_{\parallel} \mathrm{U}_{AB} \quad \Longrightarrow \quad \mathrm{F}_{AB} = v^a_A v^b_B \overset{\circ}{\nabla}_{[a}\boldsymbol{\ell}_{b]} = \nabla^h_{[A}\boldsymbol{\ell}_{B]}, \tag{4.26}$$

with which (4.23) becomes

$$
\begin{aligned}
n^c\Big(\ell_f - h^{CD}\ell_C v^d_D\gamma_{df}\Big)\mathring{R}^f{}_{acb}v^a_A v^b_B = {} & \nabla^h_A s_B - s_A s_B \\
& + \Big(n(\ell^{(2)}) + \ell^C\ell^D\mathrm{U}_{CD} - 3\ell^C s_C\Big)\mathrm{U}_{AB} \\
& + (\ell^{(2)} - \ell^{(2)}_{\parallel})(\pounds_n\mathbf{U})_{AB} + 2\ell^C s_{(A}\mathrm{U}_{B)C} + \ell^C\nabla^h_C\mathrm{U}_{AB} \\
& - \ell^C\nabla^h_A\mathrm{U}_{CB} - \Big(h^{CD}(\ell^{(2)} - \ell^{(2)}_{\parallel}) + \ell^C\ell^D\Big)\mathrm{U}_{AC}\mathrm{U}_{BD} \\
& - \frac{1}{2}h^{CD}(\nabla^h_D\ell_B - \nabla^h_B\ell_D)\mathrm{U}_{AC}.
\end{aligned}
\tag{4.27}
$$

The Ricci tensor $\mathring{R}_{AB}$ follows by substituting (4.27) and (3.92) (contracted with h^{CD}) into (4.22):

$$
\begin{aligned}
\mathring{R}_{AB} = {} & R^h_{AB} + \nabla^h_A s_B - s_A s_B + \Big(n(\ell^{(2)}) + 2\ell^C\ell^D\mathrm{U}_{CD} - 3\ell^C s_C + \nabla^h_C\ell^C \\
& + (\mathrm{tr}_h\mathbf{U}_{\parallel})(\ell^{(2)} - \ell^{(2)}_{\parallel})\Big)\mathrm{U}_{AB} + (\ell^{(2)} - \ell^{(2)}_{\parallel})(\pounds_n\mathbf{U})_{AB} + (\mathrm{tr}_h\mathbf{U}_{\parallel})\nabla^h_{(A}\ell_{B)} \\
& + 2\ell^C\Big(\nabla^h_C\mathrm{U}_{AB} + s_{(A}\mathrm{U}_{B)C} - \nabla^h_{(A}\mathrm{U}_{B)C}\Big) - \Big(2h^{CD}(\ell^{(2)} - \ell^{(2)}_{\parallel}) \\
& + \ell^C\ell^D\Big)\mathrm{U}_{AC}\mathrm{U}_{BD} - h^{CD}\Big(\mathrm{U}_{DB}\nabla^h_{(A}\ell_{C)} + \mathrm{U}_{DA}\nabla^h_{(B}\ell_{C)}\Big).
\end{aligned}
\tag{4.28}
$$

Observe that all terms in (4.28) except from $\nabla^h_A s_B$ are symmetric. This implies that $\mathring{R}_{AB} - \mathring{R}_{BA} = \nabla^h_A s_B - \nabla^h_B s_A$, which is in agreement with equation (2.51) and provides a consistency check to (4.28). The symmetrized tensor is

$$
\begin{aligned}
\mathring{R}_{(AB)} = {} & R^h_{AB} + \nabla^h_{(A}s_{B)} - s_A s_B + \Big(n(\ell^{(2)}) + 2\ell^C\ell^D\mathrm{U}_{CD} - 3\ell^C s_C + \nabla^h_C\ell^C \\
& + (\mathrm{tr}_h\mathbf{U}_{\parallel})(\ell^{(2)} - \ell^{(2)}_{\parallel})\Big)\mathrm{U}_{AB} + (\ell^{(2)} - \ell^{(2)}_{\parallel})(\pounds_n\mathbf{U})_{AB} + (\mathrm{tr}_h\mathbf{U}_{\parallel})\nabla^h_{(A}\ell_{B)} \\
& + 2\ell^C\Big(\nabla^h_C\mathrm{U}_{AB} + s_{(A}\mathrm{U}_{B)C} - \nabla^h_{(A}\mathrm{U}_{B)C}\Big) - \Big(2h^{CD}(\ell^{(2)} - \ell^{(2)}_{\parallel}) \\
& + \ell^C\ell^D\Big)\mathrm{U}_{AC}\mathrm{U}_{BD} - h^{CD}\Big(\mathrm{U}_{DB}\nabla^h_{(A}\ell_{C)} + \mathrm{U}_{DA}\nabla^h_{(B}\ell_{C)}\Big).
\end{aligned}
\tag{4.29}
$$

Having obtained (4.29), we can now write down the relation between the pull-back to S of the constraint tensor and the Ricci tensor of the induced metric h.

Theorem 4.2.3. *Consider null hypersurface data* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ *and assume the Setup 3.2.15. Let* R^h_{AB} *be the Ricci tensor of the Levi-Civita connection* ∇^h *on* S*. Then, the pull-back to* S *of the constraint tensor* R *defined by (4.15) is given by*

$$
\begin{aligned}
\mathrm{R}_{AB} = {} & R^h_{AB} + 2\nabla^h_{(A}\Big(s_{B)} + r_{B)}\Big) - 2(r_A - s_A)(r_B - s_B) \\
& + (\ell^{(2)} - \ell^{(2)}_{\parallel})(\pounds_n\mathbf{U})_{AB} - 2(\pounds_n\mathbf{Y})_{AB} - \Big(2\kappa_n + \mathrm{tr}_h\mathbf{U}_{\parallel}\Big)\Big(\mathrm{Y}_{AB} - \nabla^h_{(A}\ell_{B)}\Big)
\end{aligned}
$$

$$+\Big(n(\ell^{(2)}) + 2\ell^C\ell^D U_{CD} - 4\ell^C s_C + \big(\mathrm{tr}_h \mathbf{U}_\parallel + \kappa_n\big)\big(\ell^{(2)} - \ell^{(2)}_\parallel\big) - \mathrm{tr}_h \mathbf{Y}_\parallel + \nabla^h_C \ell^C \Big) U_{AB}$$
$$+ 2\ell^C\big(\nabla^h_C U_{AB} - \nabla^h_{(A} U_{B)C}\big) - 2\big(r_{(A} - s_{(A} + \ell^D U_{D(A}\big) U_{B)C}$$
$$+ 2h^{CD}\Big(2Y_{D(A} - \nabla^h_D \ell_{(A} - \big(\ell^{(2)} - \ell^{(2)}_\parallel\big)U_{D(A}\Big) U_{B)C}. \tag{4.30}$$

Proof. We need to multiply (4.18) by $v^a_A v^b_B$. We come across a term $v^a_A v^b_B \overset{\circ}{\nabla}_{(a}(s_{b)} + 2r_{b)})$ which we elaborate by using (3.94) for $\mathrm{T} = \boldsymbol{s}$ and $\mathrm{T} = \boldsymbol{r}$ (recall that $\boldsymbol{s}(n) = 0$, $\boldsymbol{r}(n) = -\kappa_n$), thus obtaining

$$v^a_A v^b_B \overset{\circ}{\nabla}_{(a}\big(s_{b)} + 2r_{b)}\big) = \nabla^h_{(A}\big(s_{B)} + 2r_{B)}\big) + 2\kappa_n \nabla^h_{(A}\ell_{B)} - \Big(\ell^J(s_J + 2r_J) - 2\kappa_n\big(\ell^{(2)} - \ell^{(2)}_\parallel\big)\Big) U_{AB}.$$

Since $F_{Bc}n^c = -s_B$ and $F_{AB} = \nabla^h_{[A}\ell_{B]}$ (by (4.26)), inserting (3.78)-(3.79) into (4.18) yields

$$\begin{aligned} R_{AB} \overset{\text{def}}{=} \; & \overset{\circ}{R}_{(AB)} - 2(\pounds_n \mathbf{Y})_{AB} - \big(2\kappa_n + \mathrm{tr}_h \mathbf{U}_\parallel\big) Y_{AB} + \nabla^h_{(A}\big(s_{B)} + 2r_{B)}\big) + 2\kappa_n \nabla^h_{(A}\ell_{B)} \\ & - 2r_A r_B + 4r_{(A}s_{B)} - s_A s_B - \Big(\mathrm{tr}_h \mathbf{Y}_\parallel - \kappa_n\big(\ell^{(2)} - \ell^{(2)}_\parallel\big) + \ell^J s_J\Big) U_{AB} \\ & + 2h^{CD} U_{D(A}\Big(2Y_{B)C} + \ell_C\big(s_{B)} - 2r_{B)}\big) + \frac{1}{2}\big(\nabla^h_{B)}\ell_C - \nabla^h_C \ell_{B)}\big)\Big). \end{aligned} \tag{4.31}$$

Substituting expression (4.29) for $\overset{\circ}{R}_{(AB)}$ and reorganizing terms, one easily arrives at (4.30). □

4.3 gauge invariant quantities on a transverse submanifold S

Equation (4.30) is rather complicated. The main reason behind this is that it has been written in a completely arbitrary gauge. This is clearly advantageous since the gauge can be adjusted to the problem at hand. However, the equation involves several quantities that are gauge invariant, namely the constraint tensor R and the metric h_{AB} together with all its derived objects, such as the Levi-Civita covariant derivative ∇^h and the Ricci tensor R^h_{AB}. A natural question arises as to whether one can find additional objects with simple gauge behaviour so that one can write down (4.30) fully in terms of gauge invariant quantities. There is an obvious answer to this, namely that the sum of all terms in the right-hand side of (4.30) except for the first one must necessarily be a gauge invariant quantity. While this must be true, it is clearly not very helpful. However, the idea behind it is useful. If we can find simple gauge invariant quantities that can then be substituted in the equation, then the reminder must also be gauge invariant. This procedure can lead to the determination of gauge invariant objects that would have been very hard to guess otherwise. Furthermore, showing explicitly that such object is indeed gauge invariant would provide a highly non-trivial independent test on the validity of equation (4.30). This is the task we set up to do in the present section.

To study this issue, we will rely on the following previous results. First, we will require the expressions for the gauge transformations of $\mathbf{U}$, $\mathbf{F}$, $\boldsymbol{s}$, $\boldsymbol{r}$ and κ_n that were derived in Lemma 3.2.7 and Corollary 3.2.8. Secondly, we will use the additional structure that comes from the existence of the transverse submanifold S. In particular, we will exploit the results and notation introduced in Section 3.2.3. Finally, we will need Lemma 3.2.22, which allows us to compute the pull-back of Lie derivatives of γ and $\mathbf{U}$ along arbitrary directions.

In the next lemma we write down two quantities on S with very simple gauge behaviour. The underlying reason why such objects behave in this way comes from the notion of normal pair and the associated geometric quantities on S defined and studied in [61]. However, for the purposes of this thesis we simply put forward the definitions and find explicitly how they transform under an arbitrary gauge.

Lemma 4.3.1. *Assume Setup 3.2.15 and define on S the covector $\boldsymbol{\omega}_\parallel$ and the symmetric $(0,2)$-tensor $\mathbf{P}_\parallel$ by*

$$\boldsymbol{\omega}_\parallel \stackrel{\mathrm{def}}{=} \psi^\star(\boldsymbol{s}-\boldsymbol{r}) - \mathbf{U}_\parallel(\ell^\sharp_\parallel, \cdot), \qquad \mathbf{P}_\parallel \stackrel{\mathrm{def}}{=} \psi^\star\mathbf{Y} + \frac{1}{2}\left(\ell^{(2)}_\parallel - \ell^{(2)}|_S\right)\mathbf{U}_\parallel - \frac{1}{2}\pounds_{\ell^\sharp_\parallel} h.$$

Under an arbitrary gauge transformation with gauge parameters $\{z, V\}$ they transform as

$$\mathcal{G}_{(z,V)}(\boldsymbol{\omega}_\parallel) = \boldsymbol{\omega}_\parallel - \frac{1}{\hat{z}} d\hat{z}, \qquad \mathcal{G}_{(z,V)}(\mathbf{P}_\parallel) = \hat{z}\mathbf{P}_\parallel, \tag{4.32}$$

where $\hat{z} \stackrel{\mathrm{def}}{=} \psi^\star z$.

Proof. From (3.58) and (3.60) we have the transformations (we again use prime to denote a gauge transformed object)

$$\boldsymbol{\ell}'_\parallel = \hat{z}(\boldsymbol{\ell}_\parallel + w_\parallel), \qquad \mathbf{U}'_\parallel = \hat{z}^{-1}\mathbf{U}_\parallel. \tag{4.33}$$

Thus $\ell^{A\prime} = \hat{z}(\ell^A + w^A)$ and $\left(\mathrm{U}_{AB}\ell^B\right)' = \mathrm{U}_{AB}\left(\ell^B + w^B\right)$. The transformation law of $\boldsymbol{\omega}_\parallel$ follows at once from this and Corollary 3.2.8 (recall that $\mathbf{U}(n,\cdot) = 0$, $\gamma(n,\cdot) = 0$).

Concerning $\mathbf{P}_\parallel$ we use the decomposition $V^a = fn + P^{ab}w_b$ (cf. (3.57)) and apply Lemma 3.2.22 to the transformation law (2.40) of $\mathbf{Y}$. This gives

$$\psi^\star\mathbf{Y}' = \hat{z}\psi^\star\mathbf{Y} + \boldsymbol{\ell}_\parallel \otimes_s d\hat{z} + \hat{z}\left(f|_S - \ell^C w_C\right)\mathbf{U}_\parallel + \frac{1}{2}\pounds_{\hat{z}w^\sharp} h.$$

Since

$$\ell_\parallel^{(2)\prime} = \hat{z}^2\left(\ell_\parallel^{(2)} + 2\,\ell^C w_C + w^C w_C\right), \qquad \ell^{(2)\prime}|_S = \hat{z}^2\left(\ell^{(2)}|_S + 2f|_S + w^C w_C\right), \tag{4.34}$$

the first because of definition $\ell_\parallel^{(2)} \overset{\mathrm{def}}{=} h^\sharp(\boldsymbol{\ell}_\parallel, \boldsymbol{\ell}_\parallel)$ and the second being a consequence of (3.59) together with (3.80), one finds

$$\left((\ell_\parallel^{(2)} - \ell^{(2)}|_S)\mathbf{U}_\parallel\right)' = \hat{z}(\ell_\parallel^{(2)} - \ell^{(2)}|_S)\mathbf{U}_\parallel + 2\hat{z}\left(\ell^C w_C - f|_S\right)\mathbf{U}_\parallel.$$

Given that $(\ell_\parallel^\sharp)' = \hat{z}\ell_\parallel^\sharp + \hat{z}w^\sharp$ and $\pounds_{\hat{z}\ell_\parallel^\sharp} h = \hat{z}\pounds_{\ell_\parallel^\sharp} h + 2\boldsymbol{\ell}_\parallel \otimes_s d\hat{z}$ all terms involving $w^\sharp$ and $d\hat{z}$ in $\mathbf{P}_\parallel'$ cancel out and the transformation law $\mathbf{P}_\parallel' = \hat{z}\mathbf{P}_\parallel$ follows. □

The result states in particular that $\boldsymbol{\omega}_\parallel$ and $\mathbf{P}_\parallel$ are nearly gauge invariant and, in fact, that they are exactly gauge invariant under the subgroup

$$\mathcal{G}_1 \overset{\mathrm{def}}{=} \{1, V\} \subset \mathcal{G} = \mathcal{F}^\star(\mathcal{N}) \times \Gamma(T\mathcal{N}).$$

The fact that $\mathcal{G}_1$ is a subgroup of $\mathcal{G}$ is immediate from the composition law of Proposition 2.2.10.

As already indicated, it makes sense to write the constraint tensor $\mathcal{R}$ on the submanifold S in terms of these quantities. We still need to decide which objects are to be replaced. For $\mathbf{P}_\parallel$ there is only one natural choice, namely $\psi^\star\mathbf{Y}$. For $\boldsymbol{\omega}_\parallel$, we could replace either $\boldsymbol{s}$ or $\boldsymbol{r}$, but the second choice is preferable because $\boldsymbol{\omega}_\parallel$ is not at the level of metric hypersurface data since it involves some components of the tensor $\mathbf{Y}$ as well.

The following result is obtained by a simple computation whereby $\boldsymbol{r}$ and $\psi^\star\mathbf{Y}$ are replaced in terms of $\boldsymbol{\omega}_\parallel$ and $\mathbf{P}_\parallel$ respectively in (4.30).

Proposition 4.3.2. *Assume Setup 3.2.15. The pull-back to S of the constraint tensor* $\mathcal{R}$ *reads*

$$\begin{aligned}\mathcal{R}_{AB} = {} & R^h_{AB} - 2\nabla^h_{(A}\,\omega_{B)} - 2\omega_A\omega_B - \left(2\kappa_n + \mathrm{tr}_h\mathbf{U}_\parallel\right)P_{AB} \\ & - (\mathrm{tr}_h P)\mathrm{U}_{AB} + 4P^C{}_{(A}\mathrm{U}_{B)C} - 2S_{AB},\end{aligned} \tag{4.35}$$

where

$$\begin{aligned} S_{AB} \stackrel{\text{def}}{=} {} & (\pounds_n \mathbf{Y})_{AB} - \frac{1}{2}\left(\ell^{(2)} - \ell^{(2)}_{\parallel}\right)(\pounds_n \mathbf{U})_{AB} - 2\nabla^h_{(A} s_{B)} \\ & + \left(-\frac{1}{2} n(\ell^{(2)}) - \ell^C \ell^D \mathrm{U}_{CD} + 2\ell^C s_C \right) \mathrm{U}_{AB} \\ & + \ell^C \left(-\nabla^h_C \mathrm{U}_{AB} + 2\nabla^h_{(A} \mathrm{U}_{B)C} \right). \end{aligned} \tag{4.36}$$

The definition of the symmetric (0, 2)-tensor $\mathbf{S}_{\parallel}$ is not artificial. As mentioned above, the fact that the tensors $\psi^{\star}\mathrm{R}$ and $\mathbf{Ric}^h$ are gauge invariant, together with the simple gauge behaviour of $\boldsymbol{\omega}_{\parallel}$, $\mathbf{P}_{\parallel}$, $\mathbf{U}_{\parallel}$ and κ_n, imply that $\mathbf{S}_{\parallel}$ must also have a simple gauge behaviour. To conclude this section, we determine the gauge transformation of $\mathbf{S}_{\parallel}$ as a simple consequence of expression (4.35). However, in Appendix C we provide a direct and completely independent proof of this property. This serves as a stringent consistency test for the various expressions above.

We emphasize that while the existence and explicit form of the G_1-gauge invariant quantities $\boldsymbol{\omega}_{\parallel}$ and $\mathbf{P}_{\parallel}$ can be justified by the use of normal pairs and their associated geometric objects [61], the existence of the G_1-gauge invariant quantity $\mathbf{S}_{\parallel}$ could not be anticipated and comes as an interesting by-product of the constraint tensor. The tensor $\mathbf{S}_{\parallel}$ contains information on the first order variation of the extrinsic curvature $\mathbf{Y}$ along the null direction n.

This quantity has several interesting features that would deserve further investigation. Here we shall only mention that this object is not only G_1-gauge invariant and it has a simple full G-gauge behaviour (which makes it computable in any gauge) but it is also intrinsic to the submanifold S. By "intrinsic" we mean that it encodes geometric information of S as a submanifold of N (or of the ambient space (M, g) in case the data is embedded), independently of S belonging or not to any foliation of N . This information is at the level of second derivatives (curvature) unlike $\boldsymbol{\omega}_{\parallel}$ or $\mathbf{P}_{\parallel}$ which involve only first derivatives (extrinsic curvature).

The gauge behaviour of $\mathbf{S}_{\parallel}$ is obtained next as a consequence of Proposition 4.3.2.

Corollary 4.3.3. *Under a gauge transformation with gauge parameters $\{z, V\}$ the tensor $\mathbf{S}_{\parallel}$ transforms as*

$$\mathcal{G}_{(z,V)}\left(S\right)_{AB} = S_{AB} + \frac{1}{\hat{z}}\nabla^h_A \nabla^h_B \hat{z} - \frac{2}{\hat{z}^2}\nabla^h_A \hat{z} \nabla^h_B \hat{z} + \frac{2}{\hat{z}}\omega_{(A}\nabla^h_{B)}\hat{z} + \frac{\hat{z}_n}{\hat{z}} P_{AB},$$

where $\hat{z} \stackrel{\text{def}}{=} z|_S$ and $\hat{z}_n \stackrel{\text{def}}{=} n(z)|_S$. In particular $\mathbf{S}_{\parallel}$ is invariant under the subgroup G_1.

Proof. We apply a gauge transformation with gauge parameters $\{z, V\}$ to (4.35) and subtract the equation itself. Using, as usual, a prime to denote gauge transformed objects one has

$$0 = -2\nabla^h_{(A}\left(\omega'_{B)} - \omega_{B)}\right) - 2\omega'_A\omega'_B + 2\omega_A\omega_B - 2\left(\kappa'_n \hat{z} - \kappa_n\right) P_{AB} - 2S'_{AB} + 2S_{AB},$$

where we used the gauge invariance of $\psi^\star \mathrm{R}$, h, ∇^h and $\mathbf{Ric}^h$, as well as the fact that $\mathbf{U}_\parallel$ scales with $\hat{z}^{-1}$ while $\mathbf{P}_\parallel$ scales with $\hat{z}$, so their product is gauge invariant. Using the definition $\hat{z}_n \stackrel{\text{def}}{=} n(z)|_S$ and inserting $\boldsymbol{\omega}'_\parallel = \boldsymbol{\omega}_\parallel - \hat{z}^{-1}d\hat{z}$, as well as (3.64), the result follows after simple cancellations. □

As we shall see in Section 5.4.2, the quantity $\mathbf{S}_\parallel$ is of particular relevance in the study of Killing horizons of order one containing a submanifold S. The underlying reason is that $\mathbf{S}_\parallel$ is related to the pull-back to S of the tensor field $\mathring{\Sigma} - n \otimes \pounds_n \mathbf{Y}$, which vanishes at a horizon in the gauge where the Killing vector coincides with n (recall that $\mathbf{S}_\parallel$ is only $\mathcal{G}_1$-invariant). The next lemma provides the corresponding relation between $\mathbf{S}_\parallel$ and $\mathring{\Sigma} - n \otimes \pounds_n \mathbf{Y}$.

Lemma 4.3.4. *Assume Setup 3.2.15, where* $\mathbf{q}$ *is the unique normal covector field to* $\psi(S)$ *satisfying* $\mathbf{q}(n) = 1$ *and* $\ell^\sharp_\parallel \stackrel{\text{def}}{=} h^\sharp(\boldsymbol{\ell}_\parallel, \cdot)$*. Then,* $\mathbf{S}_\parallel$ *and the tensor* $\mathring{\Sigma}$ *defined by* (3.31) *verify the following identity:*

$$\begin{aligned}\mathbf{S}_\parallel \stackrel{S}{=} &-\psi^\star\left(\mathbf{q}(\mathring{\Sigma} - n \otimes \pounds_n \mathbf{Y})\right) + \frac{1}{2}(\ell^{(2)} - \ell^{(2)}_\parallel)(\pounds_n \mathbf{U})_\parallel \\ &+ \left(\frac{1}{2}n(\ell^{(2)}) + \mathbf{U}_\parallel(\ell^\sharp_\parallel, \ell^\sharp_\parallel) - 2s_\parallel(\ell^\sharp_\parallel)\right)\mathbf{U}_\parallel.\end{aligned} \tag{4.37}$$

In particular, if $\mathbf{U} = 0$ *everywhere on* $\mathcal{N}$*, it follows*

$$\mathbf{S}_\parallel \stackrel{S}{=} -\psi^\star\left(\mathbf{q}(\mathring{\Sigma} - n \otimes \pounds_n \mathbf{Y})\right). \tag{4.38}$$

Proof. We first use (3.94) to obtain the contraction $v^a_A v^b_B v^c_C \mathring{\nabla}_a \mathrm{U}_{bc}$:

$$v^a_A v^b_B v^c_C \mathring{\nabla}_a \mathrm{U}_{bc} = \nabla^h_A \mathrm{U}_{BC} - \ell^D \mathrm{U}_{CD}\mathrm{U}_{BA} - \ell^D \mathrm{U}_{BD}\mathrm{U}_{CA}. \tag{4.39}$$

This, in turn, allows us to conclude

$$\begin{aligned}v^a_A v^b_B v^c_C\left(\mathring{\nabla}_a \mathrm{U}_{bc} + \mathring{\nabla}_b \mathrm{U}_{ca} - \nabla_c \mathrm{U}_{ab} + 2s_c \mathrm{U}_{ab}\right) &= \nabla^h_A \mathrm{U}_{BC} + \nabla^h_B \mathrm{U}_{AC} - \nabla^h_C \mathrm{U}_{AB} \\ &\quad - 2\ell^D \mathrm{U}_{CD}\mathrm{U}_{AB} + 2s_C \mathrm{U}_{AB}\end{aligned} \tag{4.40}$$

after using (4.39) thrice. On the other hand, one finds

$$
\begin{aligned}
n^c\left(\overset{\circ}{\nabla}_a \mathrm{U}_{bc} + \overset{\circ}{\nabla}_b \mathrm{U}_{ca} - \overset{\circ}{\nabla}_c \mathrm{U}_{ab} + 2s_c \mathrm{U}_{ab}\right) &= -\left(n^c \overset{\circ}{\nabla}_c \mathrm{U}_{ab} + \mathrm{U}_{bc}\overset{\circ}{\nabla}_a n^c + \mathrm{U}_{ca}\overset{\circ}{\nabla}_b n^c\right)\\
&= -(\pounds_n \mathbf{U})_{ab},
\end{aligned}
$$

and hence

$$
v^a_A v^b_B n^c\left(\overset{\circ}{\nabla}_a \mathrm{U}_{bc} + \overset{\circ}{\nabla}_b \mathrm{U}_{ca} - \nabla_c \mathrm{U}_{ab} + 2s_c \mathrm{U}_{ab}\right) = -(\pounds_n \mathbf{U})_{AB}. \qquad (4.41)
$$

Now, by (3.80) the tensor P can be decomposed as $P^{dc} = v^c_C\left(h^{CD}v^d_D - \ell^C n^d\right) + n^c\left((\ell^{(2)}_{\parallel} - \ell^{(2)})n^d - \ell^D v^d_D\right)$. Thus,

$$
\begin{aligned}
v^a_A v^b_B P^{dc}&\left(\overset{\circ}{\nabla}_a \mathrm{U}_{bc} + \overset{\circ}{\nabla}_b \mathrm{U}_{ca} - \nabla_c \mathrm{U}_{ab} + 2s_c \mathrm{U}_{ab}\right) = \\
&v^a_A v^b_B v^c_C\left(h^{CD}v^d_D - \ell^C n^d\right)\left(\overset{\circ}{\nabla}_a \mathrm{U}_{bc} + \overset{\circ}{\nabla}_b \mathrm{U}_{ca} - \nabla_c \mathrm{U}_{ab} + 2s_c \mathrm{U}_{ab}\right)\\
&+ v^a_A v^b_B n^c\left((\ell^{(2)}_{\parallel} - \ell^{(2)})n^d - \ell^D v^d_D\right)\left(\overset{\circ}{\nabla}_a \mathrm{U}_{bc} + \overset{\circ}{\nabla}_b \mathrm{U}_{ca} - \nabla_c \mathrm{U}_{ab} + 2s_c \mathrm{U}_{ab}\right). \qquad (4.42)
\end{aligned}
$$

This means that (4.42) can be elaborated by inserting (4.40)-(4.41). Since the tensor $\overset{\circ}{\Sigma}$ in the null case reads (recall (3.31)):

$$
\overset{\circ}{\Sigma}{}^d{}_{ab} = n^d\left(2\overset{\circ}{\nabla}_{(a} s_{b)} + n(\ell^{(2)})\mathrm{U}_{ab}\right) + P^{dc}\left(\overset{\circ}{\nabla}_a \mathrm{U}_{bc} + \overset{\circ}{\nabla}_b \mathrm{U}_{ca} - \overset{\circ}{\nabla}_c \mathrm{U}_{ab} + 2s_c \mathrm{U}_{ab}\right), \qquad (4.43)
$$

it is straightforward to conclude that its contraction with $v^a_A v^b_B$ is

$$
\begin{aligned}
\overset{\circ}{\Sigma}{}^d{}_{ab} v^a_A v^b_B = n^d &\Big(2\nabla^h_{(A} s_{B)} - 2\ell^C s_C \mathrm{U}_{AB} + n(\ell^{(2)})\mathrm{U}_{AB} + (\ell^{(2)} - \ell^{(2)}_{\parallel})(\pounds_n \mathbf{U})_{AB}\\
&- \ell^C\left(2\nabla^h_{(A} \mathrm{U}_{B)C} - \nabla^h_C \mathrm{U}_{AB} - 2\ell^D \mathrm{U}_{CD}\mathrm{U}_{AB} + 2s_C \mathrm{U}_{AB}\right)\Big)\\
&+ v^d_D\Big(h^{CD}\left(2\nabla^h_{(A} \mathrm{U}_{B)C} - \nabla^h_C \mathrm{U}_{AB} - 2\ell^D \mathrm{U}_{CD}\mathrm{U}_{AB} + 2s_C \mathrm{U}_{AB}\right)\\
&+ \ell^D(\pounds_n \mathbf{U})_{AB}\Big) \qquad (4.44)
\end{aligned}
$$

after using $v^a_A v^b_B \overset{\circ}{\nabla}_{(a} s_{b)} = \nabla^h_{(A} s_{B)} - \ell^C s_C \mathrm{U}_{AB}$ (cf. (3.94)). Equation (4.37) follows from (4.44) after taking into account $\mathbf{q}(v_D) = 0$, $\mathbf{q}(n) = 1$, definition (4.36) and the fact that $\psi^\star\left(\mathbf{q}(n \otimes \pounds_n \mathbf{Y})\right) = (\pounds_n \mathbf{Y})_{\parallel}$. □

5

EMBEDDED HYPERSURFACE DATA AND AMBIENT VECTOR FIELDS

In many interesting situations a spacetime has a privileged vector field. One simple (although important) example occurs when the spacetime admits a Killing vector, but there are many more indeed. Besides the natural generalization of $(\mathcal{M}, g)$ admitting a less restrictive type of symmetry such as a homothety or a conformal Killing vector, it can also happen that there is one observer (modelled, as usual, by a unit future timelike Killing vector) that is physically or geometrically privileged e.g by being geodesic, or shear-free, or irrotational or any combination thereof. Privileged null vector fields are also commonplace, e.g. when a spacetime is algebraically special so that the Weyl tensor admits a multiple principle null direction, or when the spacetime admits a Kerr-Schild decomposition. The examples are endless.

In principle, we do not want to restrict ourselves to any particular situation (at least from the beginning). Thus, we will start by assuming that the spacetime admits a privileged vector field y in a neighbourhood $\mathcal{O}$ of a general hypersurface $\widetilde{\mathcal{N}}$. Given such vector field, we can always define a symmetric 2-covariant tensor field that encodes the relationship between the metric g and the vector itself. This is the so-called deformation tensor. Our first aim is to compute the explicit expression for the Lie bracket $[y, \zeta]|_{\widetilde{\mathcal{N}}}$ of y with any extension to $\mathcal{O}$ of a rigging ζ of $\widetilde{\mathcal{N}}$. With this result at hand, we shall be able to obtain an identity on $\widetilde{\mathcal{N}}$ for the Lie derivative of the data tensor $\mathbf{Y}$ along y in terms of the deformation tensor, its first transversal derivative and the metric part of the data. This identity will be essential in Chapter 6 when we derive a fully general form of master equation.

The rest of the chapter is devoted to the case when the hypersurface $\widetilde{\mathcal{N}}$ is null and y is null and tangent to it. In such case we use the symbol η (instead of y) to refer to the privileged vector field. We have several purposes in mind. First, we derive another identity for the Lie derivative of $\mathbf{Y}$ along η, but now in terms of the proportionality function between η and a null generator of the hypersurface, the deformation tensor and the tangent components of its first transversal derivative. Secondly, we study in depth the tensor field $\Sigma_\eta \stackrel{\text{def}}{=} \pounds_\eta \nabla$ introduced in Section 3.1.1. This tensor field plays a basic role in the geometry of Killing horizons of order zero and one, so it becomes necessary to codify it at the abstract level. With this goal in mind, we derive the pull-back to the abstract hypersurface of the contraction Σ_η with a general covector. This leads us to a natural definition of a tensor field, called $\sqcap^\eta$, that encodes geometric properties of Σ_η at the abstract level. The analysis of its gauge behaviour reveals that its contraction with the null generator n of the data is gauge-invariant, and this allows us to detect another $\mathcal{G}_1$-invariant quantity related to the second fundamental form $\mathbf{U}$ and its Lie derivative $\pounds_n \mathbf{U}$. We also obtain a general expression for the vector field $\Sigma_\eta(X, Y)$, $X, Y \in \Gamma(T\mathcal{N})$.

We emphasize that the results of this chapter hold in full generality, as we are not imposing any a priori condition of the deformation tensor of η. We expect that the results obtained here will have many different uses besides the ones we concentrate on later in this thesis. To mention just one, conformal infinity is a null hypersurface in the case of vanishing cosmological constant, and the conformal compactification introduces a privileged vector field, namely the gradient of the conformal factor. It is quite certain that the result here will be of relevance in that context.

We conclude the chapter by motivating and presenting the definitions of Killing horizons of order zero and one, as well as connecting them with the notions of non-expanding and (weakly) isolated horizons introduced in Section 2.5.

5.1 lie derivative of $\mathbf{Y}$

As mentioned above, this section is divided in two parts. First, we consider a general vector field y and compute the Lie bracket $[y, \zeta]$ on a (non-necessarily null) hypersurface. As a prior step, this requires that we know the transverse covariant derivative of y at the hypersurface and the pull-back to the hypersurface of the deformation tensor of y. We then focus on the case when y is tangent to such hypersurface and compute the Lie derivative of $\mathbf{Y}$ along y. We emphasize that all results here are valid for hypersurfaces of arbitrary causal character.

Consider completely general hypersurface data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ embedded on a semi-Riemannian manifold $(\mathcal{M}, g)$ with embedding ϕ and rigging ζ. Define $\boldsymbol{\zeta} \stackrel{\text{def}}{=} g(\zeta, \cdot)|_{\phi(\mathcal{N})}$ and assume the notation introduced in Setup 2.2.7. In these circumstances, the vector fields ν and θ^a are given by (2.25)-(2.26) in terms of the basis $\{\zeta, e_a\}$. As elsewhere in this thesis, we make no distinction between scalar functions on $\phi(\mathcal{N})$ and their pullbacks to $\mathcal{N}$.

Given a vector field y in a neighbourhood of $\phi(\mathcal{N})$, one can define the so-called deformation tensor K^y of $(\mathcal{M}, g)$ by

$$\mathrm{K}^y \stackrel{\mathrm{def}}{=} \pounds_y g. \tag{5.1}$$

The next proposition finds identities that relate K^y along $\phi(\mathcal{N})$ with the hypersurface data and the transversal covariant derivative of y on $\phi(\mathcal{N})$.

Proposition 5.1.1. *Consider hypersurface data* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ *embedded in a semi-Riemannian manifold* $(\mathcal{M}, g)$ *with embedding* ϕ *and rigging* ζ *and assume the notation in Setup 2.2.7. Let* y *be any vector field in a neighbourhood of* $\phi(\mathcal{N})$ *and define* $\beta \in \mathcal{F}(\mathcal{N})$, $\bar{y} \in \Gamma(T\mathcal{N})$ *by*

$$y \stackrel{\phi(\mathcal{N})}{=} \beta\zeta + \phi_\star(\bar{y}). \tag{5.2}$$

Then, the deformation tensor K^y *of* y *satisfies the following identities on* $\phi(\mathcal{N})$*:*

$$\phi^\star(\mathrm{K}^y)_{ab} = 2\beta \mathrm{Y}_{ab} + \ell_a \overset{\circ}{\nabla}_b \beta + \ell_b \overset{\circ}{\nabla}_a \beta + \pounds_{\bar{y}} \gamma_{ab}, \tag{5.3}$$

$$\begin{aligned} \nabla_\zeta y = {}& \frac{1}{2}\mathrm{K}^y(\zeta,\zeta)\nu + \Big(\mathrm{K}^y(\zeta, e_a) - \ell^{(2)} \overset{\circ}{\nabla}_a \beta \\ & - \frac{\beta}{2} \overset{\circ}{\nabla}_a \ell^{(2)} - \ell_b \overset{\circ}{\nabla}_a \bar{y}^b + (\mathrm{Y}_{ab} + \ell^{(2)} \mathrm{U}_{ab}) \bar{y}^b \Big) \theta^a. \end{aligned} \tag{5.4}$$

Proof. First we observe that

$$\langle \nabla_{e_a} \zeta, y \rangle_g \stackrel{\mathcal{N}}{=} \beta \langle \nabla_{e_a} \zeta, \zeta \rangle_g + \langle \nabla_{e_a} \zeta, \bar{y} \rangle_g \stackrel{\mathcal{N}}{=} \frac{1}{2}\beta \overset{\circ}{\nabla}_a \ell^{(2)} + \mathbf{Y}(\hat{e}_a, \bar{y}) + \mathbf{F}(\hat{e}_a, \bar{y}) \tag{5.5}$$

where we used (2.41)-(2.42). Identity (5.3) is based of the fact that for any embedding $\phi : \mathcal{N} \hookrightarrow \mathcal{M}$, vector field $X \in \Gamma(T\mathcal{N})$ and covariant tensor field T on $\mathcal{M}$, the Lie derivative satisfies (2.86). Therefore

$$\begin{aligned} \phi^\star(\mathrm{K}^y) = \phi^\star(\pounds_y g) &= \phi^\star\left(\pounds_{\beta\zeta} g + \pounds_{\phi_\star(\bar{y})} g\right) = \phi^\star\left(\beta \pounds_\zeta g + d\beta \otimes \boldsymbol{\zeta} + \boldsymbol{\zeta} \otimes d\beta + \pounds_{\phi_\star(\bar{y})} g\right) \\ &= 2\beta \mathbf{Y} + \boldsymbol{\ell} \otimes d\beta + d\beta \otimes \boldsymbol{\ell} + \pounds_{\bar{y}} \gamma, \end{aligned} \tag{5.6}$$

where in the last equality we applied identity (2.86) and (2.22)-(2.39), and in the previous identity we used the simple property

$$\pounds_{fX} S = f \pounds_X S + df \otimes i_X S + i_X S \otimes df, \qquad i_X S \stackrel{\mathrm{def}}{=} S(X, \cdot),$$

valid for any symmetric 2-covariant tensor S. To show (5.4) we recall that $g^{\mu\nu} = e_c^\mu \theta^{c\nu} + \zeta^\mu \nu^\nu$ on $\phi(\mathcal{N})$ (see (2.27)) and compute $\nabla_\zeta y$ on $\phi(\mathcal{N})$ as follows (all equalities take place at $\phi(\mathcal{N})$)

$$
\begin{aligned}
\nabla_\zeta y^\beta &= g^{\alpha\beta}\zeta^\mu \nabla_\mu y_\alpha \\
&= \left(e_a^\alpha \theta^{a\beta} + \zeta^\alpha \nu^\beta \right) \zeta^\mu \nabla_\mu y_\alpha \\
&= \theta^{a\beta} e_a^\alpha \zeta^\mu \nabla_\mu y_\alpha + \frac{1}{2}\zeta^\alpha \zeta^\mu \left(\nabla_\mu y_\alpha + \nabla_\alpha y_\mu \right) \nu^\beta \\
&= \theta^{a\beta} \left(\mathrm{K}^y(\zeta, e_a) - \langle \zeta, \nabla_{e_a} y \rangle_g \right) + \frac{1}{2}\mathrm{K}^y(\zeta,\zeta)\nu^\beta,
\end{aligned} \tag{5.7}
$$

where in the last equality we used $\mathrm{K}^y_{\alpha\beta} = \nabla_\alpha y_\beta + \nabla_\beta y_\alpha$ twice. We elaborate the second term using (5.5) and the fact that $\langle \zeta, y\rangle_g \stackrel{\mathcal{N}}{=} \beta \ell^{(2)} + \boldsymbol{\ell}(\bar{y})$ (cf. (5.2)), which yields

$$
\begin{aligned}
\langle \zeta, \nabla_{e_a} y \rangle_g &= \nabla_{e_a} \langle \zeta, y \rangle - \langle \nabla_{e_a}\zeta, y \rangle \\
&= \mathring{\nabla}_a \left(\beta \ell^{(2)} + \ell_b \bar{y}^b \right) - \frac{1}{2}\beta \mathring{\nabla}_a \ell^{(2)} - \left(\mathrm{Y}_{ab} + \mathrm{F}_{ab} \right) \bar{y}^b \\
&= \ell^{(2)} \mathring{\nabla}_a \beta + \frac{1}{2}\beta \mathring{\nabla}_a \ell^{(2)} + \ell_b \mathring{\nabla}_a \bar{y}^b - \left(\mathrm{Y}_{ab} + \ell^{(2)} \mathrm{U}_{ab} \right) \bar{y}^b,
\end{aligned}
$$

where in the last step we inserted (2.19). Substituting into (5.7) yields (5.4). □

Proposition 5.1.1 allows us to find a general identity for the commutator $[y, \zeta]$.

Lemma 5.1.2. *In the setup of Proposition 5.1.1, let ζ be any extension of the rigging off $\phi(\mathcal{N})$. Define $a_\zeta \stackrel{\text{def}}{=} \nabla_\zeta \zeta \,|_{\phi(\mathcal{N})}$. Then,*

$$
\begin{aligned}
[y, \zeta] \stackrel{\phi(\mathcal{N})}{=}\; & \beta a_\zeta + \frac{1}{2}\left(\bar{y}(\ell^{(2)}) - \mathrm{K}^y(\zeta,\zeta) \right) \nu \\
& + \left((\pounds_{\bar{y}} \boldsymbol{\ell})_a + \ell^{(2)} \mathring{\nabla}_a \beta + \frac{1}{2}\beta \mathring{\nabla}_a \ell^{(2)} - \mathrm{K}^y(\zeta, e_a) \right) \theta^a .
\end{aligned} \tag{5.8}
$$

Proof. We compute $\nabla_y \zeta$ by means of the decomposition (5.2). One obtains

$$
\begin{aligned}
\nabla_y \zeta &\stackrel{\phi(\mathcal{N})}{=} \beta \nabla_\zeta \zeta + \nabla_{\phi_\star(\bar{y})} \zeta \stackrel{\phi(\mathcal{N})}{=} \beta a_\zeta + \bar{y}^b \nabla_{e_b} \zeta \stackrel{\phi(\mathcal{N})}{=} \beta a_\zeta + \frac{1}{2}\bar{y}(\ell^{(2)})\nu + \bar{y}^b \left(\mathrm{Y}_{ba} + \mathrm{F}_{ba} \right) \theta^a \\
&\stackrel{\phi(\mathcal{N})}{=} \beta a_\zeta + \frac{1}{2}\bar{y}(\ell^{(2)})\nu + \bar{y}^b \left(\mathring{\nabla}_b \ell_a + \mathrm{Y}_{ba} + \ell^{(2)} \mathrm{U}_{ab} \right) \theta^a,
\end{aligned} \tag{5.9}
$$

where in the second equality we inserted (2.43) and in the final step we used (2.19). Combining this with (5.4) yields

$$
[y, \zeta] = \nabla_y \zeta - \nabla_\zeta y
$$

$$= \beta a_\zeta + \frac{1}{2}\Big(\bar{y}(\ell^{(2)}) - K^y(\zeta,\zeta)\Big)\nu + \Big(\bar{y}^b \overset{\circ}{\nabla}_b \ell_a + \ell_b \overset{\circ}{\nabla}_a \bar{y}^b + \ell^{(2)} \overset{\circ}{\nabla}_a \beta + \frac{1}{2}\beta \overset{\circ}{\nabla}_a \ell^{(2)} - K^y(\zeta, e_a)\Big)\theta^a,$$

which can be written as (5.8) after using (3.15). □

A case of particular interest occurs when the vector field y is tangent to $\phi(\mathcal{N})$, i.e. when $\beta = 0$ and $y = \phi_\star \bar{y}$. In these circumstances, one can find an explicit expression for $\pounds_{\bar{y}}\mathbf{Y}$ in terms of the deformation tensor of y.

Proposition 5.1.3. *In the setup of Proposition 5.1.1, let ζ be any extension of the rigging off $\phi(\mathcal{N})$. Assume that y is tangent to $\phi(\mathcal{N})$, i.e. such that $\beta = 0$ and $y = \phi_\star(\bar{y})$ for some $\bar{y} \in \Gamma(T\mathcal{N})$. Define the scalar $A_y \in \mathcal{F}(\mathcal{M})$ and the vector $X_y \in \Gamma(T\mathcal{N})$ by*

$$A_y \stackrel{\text{def}}{=} \frac{1}{2}K^y(\zeta, n^{(2)}\zeta - 2\nu) + \frac{1}{2}n^{(2)}\bar{y}(\ell^{(2)}) + n^a(\pounds_{\bar{y}}\boldsymbol{\ell})_a, \tag{5.10}$$

$$X_y \stackrel{\text{def}}{=} \Big(\frac{1}{2}K^y(\zeta, n^a\zeta - 2\theta^a) + \frac{1}{2}n^a\bar{y}(\ell^{(2)}) + P^{ab}(\pounds_{\bar{y}}\boldsymbol{\ell})_b\Big)\hat{e}_a. \tag{5.11}$$

Then the derivative $\pounds_{\bar{y}}\mathbf{Y}$ is given by

$$\pounds_{\bar{y}}\mathbf{Y} = A_y\mathbf{Y} + \boldsymbol{\ell}\otimes_s dA_y + \frac{1}{2}\pounds_{X_y}\gamma + \frac{1}{2}\phi^\star\big(\pounds_\zeta K^y\big). \tag{5.12}$$

Proof. The identity is based on the commutation property $[\pounds_X, \pounds_W] = \pounds_{[X,W]}$. Applying this to the ambient metric g and to the vectors y and ζ, one obtains

$$\pounds_y\pounds_\zeta g = \pounds_{[y,\zeta]}g + \pounds_\zeta K^y. \tag{5.13}$$

Note that this expression requires that the rigging is extended off $\phi(\mathcal{N})$, but the final result is independent of the extension, as one can easily check from (5.12). By Lemma 5.1.2 with $\beta = 0$, the commutator $[y,\zeta]$ is (all equalities are on $\phi(\mathcal{N})$)

$$\begin{aligned}
[y,\zeta] &= \frac{1}{2}\Big(\bar{y}(\ell^{(2)}) - K^y(\zeta,\zeta)\Big)\nu + \Big((\pounds_{\bar{y}}\boldsymbol{\ell})_a - K^y(\zeta,e_a)\Big)\theta^a \\
&= \Big(\frac{1}{2}n^{(2)}\Big(\bar{y}(\ell^{(2)}) - K^y(\zeta,\zeta)\Big) + n^a(\pounds_{\bar{y}}\boldsymbol{\ell})_a - n^aK^y(\zeta,e_a)\Big)\zeta \\
&\quad + \Big(\frac{1}{2}n^a\Big(\bar{y}(\ell^{(2)}) - K^y(\zeta,\zeta)\Big) + P^{ab}(\pounds_{\bar{y}}\boldsymbol{\ell})_b - P^{ab}K^y(\zeta,e_b)\Big)e_a,
\end{aligned}$$

where in the second step we inserted (2.25)-(2.26). Using again (2.25)-(2.26) in the entries of K^y yields

$$[y,\zeta] = \Big(\frac{1}{2}K^y\Big(\zeta, n^{(2)}\zeta - 2\nu\Big) + \frac{1}{2}n^{(2)}\bar{y}(\ell^{(2)}) + n^a(\pounds_{\bar{y}}\boldsymbol{\ell})_a\Big)\zeta$$

$$
\begin{aligned}
&+ \left(\frac{1}{2} K^{y}\left(\zeta, n^{a}\zeta - 2\theta^{a}\right) + \frac{1}{2} n^{a}\overline{y}\left(\ell^{(2)}\right) + P^{ab}\left(\pounds_{\overline{y}}\boldsymbol{\ell}\right)_{b} \right) e_{a} \\
&= A_{y}\zeta + \phi_{\star}(X_{y}),
\end{aligned} \tag{5.14}
$$

We now take the pullback of (5.13) on $\mathcal{N}$. For the left-hand side we use identity (2.86) and the definition of $\mathbf{Y}$. For the first term in the right-hand side we apply identity (5.3) with y replaced by the commutator $[y, \zeta]$, so that β and $\overline{y}$ get replaced by A_y and X_y respectively. Identity (5.12) follows at once. □

5.2 vector field along the degenerate direction

For the rest of the chapter we consider null hypersurface data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ embedded on a semi-Riemannian manifold $(\mathcal{M}, g)$ with embedding ϕ and rigging ζ and we let y not only be tangent to $\phi(\mathcal{N})$ but also null therein. Following our previous notation, we use η (instead of y) to denote such null, tangent to $\phi(\mathcal{N})$ vector field and $\overline{\eta}$ for its counterpart on $\mathcal{N}$, i.e. $\eta \stackrel{\text{def}}{=} \phi_\star\overline{\eta}$. In these circumstances, $\overline{\eta}$ is not only gauge-invariant[1] (because η is a fixed spacetime vector field) but also belongs to the radical of γ, which in particular means that all results from Section 3.3 apply. As we did there, we let $\alpha \in \mathcal{F}(\mathcal{N})$ be defined by $\eta \stackrel{\text{def}}{=} \alpha n$.

In the context above, it is helpful to introduce the functions ש, ק, the covector field ף and the symmetric 2-covariant tensor field ד defined by[2]

$$
\text{ש} \stackrel{\text{def}}{=} \phi^{\star}\left(K^{\eta}(\zeta, \nu)\right), \qquad \text{ק} \stackrel{\text{def}}{=} \phi^{\star}\left(K^{\eta}(\zeta, \zeta)\right), \qquad \text{ף} \stackrel{\text{def}}{=} \phi^{\star}\left(K^{\eta}(\zeta, \cdot)\right), \tag{5.15}
$$

$$
\text{ד} \stackrel{\text{def}}{=} \frac{1}{2}\phi^{\star}\left(\pounds_{\zeta} K^{\eta}\right). \tag{5.16}
$$

Although these objects depend on η, for simplicity we do not reflect this dependence in the notation.

Definitions (5.15) only involve transverse components of the deformation tensor K^{η}. This is because there is no need to introduce symbols for the tangential components, as they are given by (recall (2.86))

$$
\phi^{\star} K^{\eta} = \pounds_{\overline{\eta}}\gamma. \tag{5.17}
$$

Observe that (5.17) is consistent with (5.3), since here η is tangent to $\phi(\mathcal{N})$. The pullback $\phi^{\star}K^{\eta}$ can actually be related to the tensor $\mathbf{U}$ by (cf. (2.12))

[1] The gauge invariance of $\overline{\eta}$ is true even if $\overline{\eta}$ was not along the degeneration direction.

[2] The letters ש (/Jin/), ק (/kuf/, /kof/), ף (/fe sofit/, /fej sofit/) and ד (/’dalɛt/, /’dalɛd/) are respectively the twenty-first, the nineteenth, the seventeenth and the fourth of the Hebrew alphabet.

$$\phi^\star \mathrm{K}^\eta = \pounds_{\alpha n}\gamma = \alpha \pounds_n \gamma = 2\alpha \mathbf{U}, \tag{5.18}$$

where we have used that

$$\pounds_{\varrho n}\gamma = 2\varrho \mathbf{U} \qquad \forall \varrho \in \mathcal{F}(\mathcal{N}) \tag{5.19}$$

because $\gamma(n,\cdot) = 0$.

From definitions (5.15), it is also immediate to check that $\mathrm{i}(n) = \mathrm{w}$ (recall (2.25)). The function w, in addition, turns out to be gauge-invariant.

Lemma 5.2.1. *For null hypersurface data* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$*, the function* w *is gauge-invariant.*

Proof. As usual, we use a prime to denote gauge-transformed quantities. Using (2.37), (5.18) and the fact that $\nu' = \phi_\star(n') = z^{-1}\phi_\star(n) = z^{-1}\nu$, we get

$$\begin{aligned} \mathrm{w}' \stackrel{\mathrm{def}}{=} \phi^\star \left(\mathrm{K}^\eta(\zeta', \nu') \right) &= \phi^\star \left(\mathrm{K}^\eta(\zeta + \phi_\star V, \nu) \right) \\ &= \mathrm{w} + (\phi^\star \mathrm{K}^\eta)(V, n) = \mathrm{w} + 2\alpha \mathbf{U}(V,n) = \mathrm{w}, \end{aligned}$$

as claimed. □

In the following lemma we derive completely general identities that relate the variation of the tensor $\mathbf{Y}$ along the degenerate direction defined by $\overline{\eta}$ and the deformation tensor of η. This gives a kind of evolution equation of $\mathbf{Y}$ along the generators, sourced by the ambient properties of η (and the proportionality function α). These identities have many potential applications. For instance, they will play a key role later in Chapter 6 when we derive a generalized form of the master equations (2.144), (2.153)) and (2.128).

Lemma 5.2.2. *Consider null hypersurface data* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ *embedded on a semi-Riemannian manifold* $(\mathcal{M}, g)$ *with embedding* ϕ *and rigging* ζ*. Assume further that* $\mathcal{N}$ *admits a gauge-invariant vector field* $\overline{\eta} \in \mathrm{Rad}\gamma$ *and let* $\kappa \in \mathcal{F}(\mathcal{N})$ *be its surface gravity according to Lemma 3.3.1 and* $\alpha \in \mathcal{F}(\mathcal{N})$ *be the function given by* $\overline{\eta} = \alpha n$*. Extend* $\phi_\star\overline{\eta}$ *to a vector field* η *on a neighbourhood* $\mathcal{O}$ *of* $\phi(\mathcal{N})$ *and define its deformation tensor* K^η*,* w*,* p*,* i *and* $\daleth$ *as in* (5.1)*,* (5.15)*-*(5.16) *respectively. Then,*

$$\begin{aligned} \pounds_{\overline{\eta}} Y_{bd} = {} & \overset{\circ}{\nabla}_b \overset{\circ}{\nabla}_d \alpha + 2\alpha \overset{\circ}{\nabla}_{(b} s_{d)} + 2 s_{(b} \overset{\circ}{\nabla}_{d)} \alpha + (n(\alpha) - \mathrm{w}) Y_{bd} \\ & - \overset{\circ}{\nabla}_{(b} \mathrm{i}_{d)} + \frac{1}{2}\left(\overline{\eta}(\ell^{(2)}) - \mathrm{p} \right) \mathrm{U}_{bd} + \daleth_{bd}, \end{aligned} \tag{5.20}$$

$$\alpha \pounds_n r_d = n^b \left(\overset{\circ}{\nabla}_b \overset{\circ}{\nabla}_d \alpha + 2\alpha \overset{\circ}{\nabla}_{(b} s_{d)} + 2 s_{(b} \overset{\circ}{\nabla}_{d)} \alpha - \psi Y_{bd} - \overset{\circ}{\nabla}_{(b} \mathbf{i}_{d)} + \daleth_{bd} \right) + \kappa_n \overset{\circ}{\nabla}_d \alpha. \tag{5.21}$$

Proof. The proof relies on two preliminary expressions that will be established first. Define the vectors $V \overset{\text{def}}{=} P(\pounds_\eta \boldsymbol{\ell}, \cdot)$ and $W \overset{\text{def}}{=} -P(\mathbf{i}, \cdot) - \frac{\mathfrak{p}}{2} n$ in N. We want to prove that the following two expressions hold:

$$\frac{1}{2}(\pounds_V \gamma)_{bd} = \overset{\circ}{\nabla}_b \overset{\circ}{\nabla}_d \alpha + 2\alpha \overset{\circ}{\nabla}_{(b} s_{d)} + 2 s_{(b} \overset{\circ}{\nabla}_{d)} \alpha - \ell_{(b} \overset{\circ}{\nabla}_{d)} n(\alpha), \tag{5.22}$$

$$\frac{1}{2}(\pounds_W \gamma)_{bd} = -\overset{\circ}{\nabla}_{(b} \mathbf{i}_{d)} + \ell_{(b} \overset{\circ}{\nabla}_{d)} \psi - \frac{\mathfrak{p}}{2} \mathrm{U}_{bd}. \tag{5.23}$$

To establish (5.22) we particularize (3.17) for $w = \pounds_\eta \boldsymbol{\ell}$ and use (3.45) to compute

$$\overset{\circ}{\nabla}_{(b} (\pounds_\eta \boldsymbol{\ell})_{d)} = \overset{\circ}{\nabla}_b \overset{\circ}{\nabla}_d \alpha + 2\alpha \overset{\circ}{\nabla}_{(b} s_{d)} + 2 s_{(b} \overset{\circ}{\nabla}_{d)} \alpha \quad \text{and} \quad (\pounds_\eta \boldsymbol{\ell})(n) = n(\alpha). \tag{5.24}$$

For (5.23), we use (5.19) and find

$$(\pounds_W \gamma)_{cd} = -\left(\pounds_{P(\mathbf{i},\cdot)} \gamma\right)_{cd} - \mathfrak{p} \mathrm{U}_{cd}. \tag{5.25}$$

Expression (5.23) follows by particularizing (3.17) to $w = \mathbf{i}$ and using $\mathbf{i}(n) = \psi$. Once we have (5.23) and (5.24), the identity (5.20) and its corollary (5.21) will be a consequence of Proposition 5.1.3. Therefore we need to compute the function A_η and the vector X_η. First, the second expression in (5.24) entails

$$A_\eta = n(\alpha) - \psi \quad \Longrightarrow \quad (\boldsymbol{\ell} \otimes_s dA_\eta)_{bd} = \ell_{(b} \overset{\circ}{\nabla}_{d)} n(\alpha) - \ell_{(b} \overset{\circ}{\nabla}_{d)} \psi. \tag{5.26}$$

Secondly, substituting (2.26) into $\mathrm{K}^\eta(\zeta, n^a \zeta - 2\theta^a)$ leads to

$$\frac{1}{2} \mathrm{K}^\eta(\zeta, n^a \zeta - 2\theta^a) \hat{e}_a = \left(-P^{ab} \mathbf{i}_b - \frac{\mathfrak{p}}{2} n^a \right) \hat{e}_a = W. \tag{5.27}$$

Consequently,

$$X_\eta = W + \frac{1}{2}\eta(\ell^{(2)}) n + V \quad \Longrightarrow \quad \frac{1}{2}\pounds_{X_\eta} \gamma = \frac{1}{2}\pounds_W \gamma + \frac{1}{2}\eta(\ell^{(2)}) \mathbf{U} + \frac{1}{2}\pounds_V \gamma, \tag{5.28}$$

where the implication is a consequence of (5.19). Inserting (5.26) and (5.28) into (5.12) gives (5.20) after using (5.22)-(5.23). Finally, (5.21) follows by contracting the decomposition (recall the definitions (2.44))

$$\pounds_\eta \mathbf{Y} = d\alpha \otimes \boldsymbol{r} + \boldsymbol{r} \otimes d\alpha + \alpha \pounds_n \mathbf{Y} \tag{5.29}$$

with n. This yields $n^b \pounds_\eta \mathrm{Y}_{bd} = -\kappa\, n \overset{\circ}{\nabla}_d\, \alpha + n(\alpha) r_d + \alpha \pounds_n r_d$ and hence (5.21) after inserting (5.20) and using $\mathrm{U}_{ab} n^a = 0$. □

5.3 the tensor $\Sigma_\eta \stackrel{\text{def}}{=} \pounds_\eta \nabla$

In the context of embedded null hypersurfaces admitting a null and tangent vector field η with surface gravity κ, the tensor $\Sigma_\eta \stackrel{\text{def}}{=} \pounds_\eta \nabla$ (defined according to (3.18) for the Levi-Civita connection of the ambient space) plays a fundamental role. As we shall see, at the ambient level Σ_η is closely related to first derivatives of the deformation tensor K^η of η, which automatically endows it with a great geometrical importance. Its influence at the abstract level is also remarkable for the following reasons. First, one can compute explicitly the pull-back of Σ_η (contracted with a general one-form) to the abstract hypersurface, and this reveals a new abstract tensor field that takes a fundamental part in (*a*) determining the constancy κ on the whole hypersurface and (*b*) determining whether κ is constant along the null generators (see Theorem 6.1.1 in Chapter 6). Secondly, the analysis of Σ_η rises, in addition, another tensor field with significantly simple gauge behaviour. Finally, the tensor Σ_η is also of great use in the context of Killing horizons of order one, and in fact it is precisely its study that will allow us to introduce an abstract notion of these sort of horizons (see Section 5.4). Consequently, we devote this section to study the properties of Σ_η.

Our first aim is to derive an explicit expression for the pull-back $\phi^\star\big(g(W, \Sigma_\eta)\big)$, where W is any vector field along $\phi(\mathcal{N})$ (not necessarily tangential). This will allow us to define a new tensor on $\mathcal{N}$ which encodes information about the deformation tensor K^η and its first transversal derivative on $\phi(\mathcal{N})$. Then, we shall obtain the gauge transformation of the pull-back $\phi^\star\big(\boldsymbol{\zeta}(\Sigma_\eta)\big)$, first for general null hypersurface data and then for the case when $\mathbf{U} = 0$. This process will reveal a new $\mathcal{G}_1$-invariant tensor field on $\mathcal{N}$ and a full gauge-invariant covector on $\mathcal{N}$ in the case with $\mathbf{U} = 0$. Finally, we compute the explicit form of the vector $\Sigma_\eta(\phi_\star Y, \phi_\star Z)$ for any pair of vector fields $Y, Z \in \Gamma(T\mathcal{N})$.

In the present context, an explicit form of Σ_η in terms of the deformation tensor of η can be obtained by particularizing Corollary 3.1.7 for $D = \nabla$ (recall that ∇ is the Levi-Civita derivative of g), $Z = \eta$ and $S_{\alpha\beta} = g_{\alpha\beta}$. This gives

$$(\Sigma_\eta)^\lambda{}_{\alpha\beta} = \frac{1}{2} g^{\mu\lambda} \left(\nabla_\alpha \mathrm{K}^\eta{}_{\beta\mu} + \nabla_\beta \mathrm{K}^\eta_{\mu\alpha} - \nabla_\mu \mathrm{K}^\eta_{\alpha\beta} \right). \tag{5.30}$$

The following lemma provides the explicit form of the pull-back $\phi^\star g(W, \Sigma_\eta)$ in terms of the corresponding hypersurface data and the various components of K^η, $\pounds_\zeta \mathrm{K}^\eta$ introduced in (5.15)-(5.16).

Lemma 5.3.1. *Consider null hypersurface data* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ *embedded on a semi-Riemannian manifold* $(\mathcal{M}, g)$ *with embedding* ϕ *and rigging* ζ*. Let* ∇ *be the Levi-Civita connection of* g *and assume that we have selected a vector field* η *on a neighbourhood* $\mathcal{O} \subset \mathcal{M}$ *of* $\phi(\mathcal{N})$ *with the properties of being null and tangent to* $\phi(\mathcal{N})$ *everywhere. Define* $\alpha \in \mathcal{F}(\mathcal{N})$ *by* $\eta|_{\phi(\mathcal{N})} = \alpha \phi_\star n$*. Let* $\mathrm{K}^\eta \stackrel{\mathrm{def}}{=} \pounds_\eta g$ *be the deformation tensor of* η*,* $\{\text{ש}, \text{ק}, \mathrm{i}, \text{ד}\}$ *be given by* (5.15)-(5.16) *and* $\Sigma_\eta = \pounds_\eta \nabla$*. Take any vector field* W *defined along* $\phi(\mathcal{N})$ *(not-necessarily tangent) and decompose it as*

$$W = \beta\zeta + \phi_\star \overline{W}, \tag{5.31}$$

where $\beta \in \Gamma(T\mathcal{N})$ *and* $\overline{W} \in \Gamma(T\mathcal{N})$*. Then,*

$$\begin{aligned}\phi^\star\left(W(\Sigma_\eta)\right)_{ab} = {} & \beta\left(\mathring{\nabla}_{(a}\mathrm{i}_{b)} + \text{ש}\, \mathrm{Y}_{ab} + \text{ק}\, \mathrm{U}_{ab} - \text{ד}_{ab}\right) + \overline{W}^c\Big((\mathring{\nabla}_a\alpha)\mathrm{U}_{bc} + (\mathring{\nabla}_b\alpha)\mathrm{U}_{ac} \\ & + \mathrm{U}_{ab}(\mathrm{i}_c - \mathring{\nabla}_c\alpha) + \alpha(\mathring{\nabla}_a \mathrm{U}_{bc} + \mathring{\nabla}_b \mathrm{U}_{ca} - \mathring{\nabla}_c \mathrm{U}_{ab})\Big).\end{aligned} \tag{5.32}$$

Proof. We shall use the notation introduced in Setup 2.2.7. From the decomposition (5.31), we have

$$\phi^\star(W(\Sigma_\eta))_{ab} = \beta\zeta_\lambda(\Sigma_\eta)^\lambda{}_{\alpha\beta}e^\alpha_a e^\beta_b + \overline{W}^c(e_c)_\lambda(\Sigma_\eta)^\lambda{}_{\alpha\beta}e^\alpha_a e^\beta_b. \tag{5.33}$$

Therefore, to prove the lemma it suffices to compute the contractions (recall (5.30))

$$(e_c)_\lambda(\Sigma_\eta)^\lambda{}_{\alpha\beta}e^\alpha_a e^\beta_b = \frac{1}{2}e^\alpha_a e^\beta_b e^\mu_c\left(\nabla_\alpha \mathrm{K}^\eta{}_{\beta\mu} + \nabla_\beta \mathrm{K}^\eta_{\mu\alpha} - \nabla_\mu \mathrm{K}^\eta_{\alpha\beta}\right), \tag{5.34}$$

$$\zeta_\lambda(\Sigma_\eta)^\lambda{}_{\alpha\beta}e^\alpha_a e^\beta_b = \frac{1}{2}e^\alpha_a e^\beta_b \zeta^\mu\left(\nabla_\alpha \mathrm{K}^\eta{}_{\beta\mu} + \nabla_\beta \mathrm{K}^\eta_{\mu\alpha} - \nabla_\mu \mathrm{K}^\eta_{\alpha\beta}\right). \tag{5.35}$$

The first only requires the calculation of $e^\alpha_a e^\beta_b e^\mu_c \nabla_\mu \mathrm{K}^\eta{}_{\alpha\beta}$, which is obtained as follows:

$$\begin{aligned} e^\alpha_a e^\beta_b e^\mu_c \nabla_\mu \mathrm{K}^\eta{}_{\alpha\beta} &= e^\mu_c\nabla_\mu(e^\alpha_a e^\beta_b \mathrm{K}^\eta{}_{\alpha\beta}) - \mathrm{K}^\eta{}_{\alpha\beta}e^\beta_b e^\mu_c \nabla_\mu e^\alpha{}_a - \mathrm{K}^\eta{}_{\alpha\beta}e^\alpha_a e^\mu_c\nabla_\mu e^\beta{}_b \\ &= \hat{e}_c(2\alpha \mathrm{U}_{ab}) - 2\alpha \mathrm{U}_{db}\mathring{\Gamma}^d_{ac} + \mathrm{U}_{ac}\mathrm{i}_b - 2\alpha \mathrm{U}_{da}\mathring{\Gamma}^d_{bc} + \mathrm{U}_{bc}\mathrm{i}_a \\ &= 2(\mathring{\nabla}_c\alpha)\mathrm{U}_{ab} + 2\alpha\mathring{\nabla}_c \mathrm{U}_{ab} + \mathrm{U}_{ac}\mathrm{i}_b + \mathrm{U}_{bc}\mathrm{i}_a, \end{aligned}$$

where we have used (2.48), definitions (5.15), (5.18) and the fact that $\mathrm{K}^\eta(\phi_\star n, \phi_\star X) = \mathbf{U}(n, X) = 0$ for any $X \in \Gamma(TN)$. Replacing this in (5.34) and cancelling terms gives

$$(e_c)_\lambda (\Sigma_\eta)^\lambda{}_{\alpha\beta} e^\alpha_a e^\beta_b = (\mathring{\nabla}_a \alpha)\mathrm{U}_{bc} + (\mathring{\nabla}_b \alpha)\mathrm{U}_{ac} + \mathrm{U}_{ab}(\mathrm{i}_c - \mathring{\nabla}_c \alpha) + \alpha(\mathring{\nabla}_a \mathrm{U}_{bc} + \mathring{\nabla}_b \mathrm{U}_{ca} - \mathring{\nabla}_c \mathrm{U}_{ab}). \tag{5.36}$$

For (5.35), we first use (2.48) and $\zeta^\mu \nabla_\mu \mathrm{K}^\eta{}_{\alpha\beta} = \pounds_\zeta \mathrm{K}^\eta_{\alpha\beta} - \mathrm{K}^\eta_{\mu\beta}\nabla_\alpha \zeta^\mu - \mathrm{K}^\eta_{\mu\alpha}\nabla_\beta \zeta^\mu$, which allows us to rewrite

$$\boldsymbol{\zeta}_\lambda (\Sigma_\eta)^\lambda{}_{\alpha\beta} e^\alpha_a e^\beta_b = \frac{1}{2}\Big(e^\alpha_a \nabla_\alpha (\mathrm{K}^\eta{}_{\beta\mu} e^\beta_b \zeta^\mu) - \mathrm{K}^\eta_{\beta\mu}\zeta^\mu e^\alpha_a \nabla_\alpha e^\beta_b + e^\beta_b \nabla_\beta (\mathrm{K}^\eta_{\mu\alpha} e^\alpha_a \zeta^\mu) - \mathrm{K}^\eta_{\mu\alpha}\zeta^\mu e^\beta_b \nabla_\beta e^\alpha_a - e^\alpha_a e^\beta_b \pounds_\zeta \mathrm{K}^\eta_{\alpha\beta}\Big).$$

Inserting now the decomposition (2.48) and using (5.15)-(5.16) gives

$$\begin{aligned}\boldsymbol{\zeta}_\lambda (\Sigma_\eta)^\lambda{}_{\alpha\beta} e^\alpha_a e^\beta_b &= \frac{1}{2}\Big(\hat{e}_a(\mathrm{i}_b) - \mathrm{K}^\eta_{\beta\mu}\zeta^\mu (\mathring{\Gamma}^d_{ba} e^\beta_d - \mathrm{Y}_{ab}\nu^\beta - \mathrm{U}_{ab}\zeta^\beta) \\ &\quad + \hat{e}_b(\mathrm{i}_a) - \mathrm{K}^\eta_{\mu\alpha}\zeta^\mu (\mathring{\Gamma}^d_{ab} e^\alpha_d - \mathrm{Y}_{ab}\nu^\alpha - \mathrm{U}_{ab}\zeta^\alpha) - e^\alpha_a e^\beta_b \pounds_\zeta \mathrm{K}^\eta_{\alpha\beta}\Big) \\ &= \mathring{\nabla}_{(a}\mathrm{i}_{b)} + \text{ש}\,\mathrm{Y}_{ab} + \text{ק}\,\mathrm{U}_{ab} - \text{ד}_{ab}.\end{aligned} \tag{5.37}$$

Equation (5.32) follows from substituting (5.36) and (5.37) into (5.33). □

We can particularize (5.32) to the case when the vector W is chosen to be the rigging. In such case $\beta = 1$ and $\overline{W} = 0$, so (5.32) simplifies to (recall that $\boldsymbol{\zeta} := g(\zeta, \cdot)$)

$$\big(\phi^\star\, \boldsymbol{\zeta}(\Sigma_\eta)\big)_{ab} = \mathring{\nabla}_{(a}\mathrm{i}_{b)} + \text{ש}\,\mathrm{Y}_{ab} + \text{ק}\,\mathrm{U}_{ab} - \text{ד}_{ab}. \tag{5.38}$$

We therefore find that the combination $\mathring{\nabla}_{(a}\mathrm{i}_{b)} + \text{ש}\,\mathrm{Y}_{ab} + \text{ק}\,\mathrm{U}_{ab} - \text{ד}_{ab}$ appears naturally. One can expect this quantity to be of relevance, so we give it a name for it, namely[3]

$$\text{ה}^\eta_{ab} \stackrel{\text{def}}{=} \mathring{\nabla}_{(a}\mathrm{i}_{b)} + \text{ש}\,\mathrm{Y}_{ab} + \text{ק}\,\mathrm{U}_{ab} - \text{ד}_{ab}, \tag{5.39}$$

so that

$$\text{ה}^\eta = \phi^\star\big(\boldsymbol{\zeta}(\Sigma_\eta)\big). \tag{5.40}$$

We now provide the gauge transformations of ה^η and its contraction with n, first for arbitrary $\mathbf{U}$ and then for $\mathbf{U} = 0$. In the latter case, we prove that ה^η is $\mathcal{G}_1$-invariant and hence that $\text{ה}^\eta(n, \cdot)$ is fully gauge-invariant.

[3] The letter ה (/he/, /hej/) is the fifth of the Hebrew alphabet.

Lemma 5.3.2. *Under the action of a gauge group element $\mathcal{G}_{(z,V)}$, the tensor fields Π^η and $\Pi^\eta(n,\cdot)$ transform as*

$$\mathcal{G}_{(z,V)}\left(\Pi^\eta_{ab}\right) = z\Pi^\eta_{ab} + zV^c\Big((\overset{\circ}{\nabla}_a\alpha)\mathrm{U}_{bc} + (\overset{\circ}{\nabla}_b\alpha)\mathrm{U}_{ac} + \mathrm{U}_{ab}(\mathbf{i}_c - \nabla_c\alpha) + \alpha(\overset{\circ}{\nabla}_a\mathrm{U}_{bc} + \overset{\circ}{\nabla}_b\mathrm{U}_{ca} - \overset{\circ}{\nabla}_c\mathrm{U}_{ab})\Big), \tag{5.41}$$

$$\mathcal{G}_{(z,V)}\left(\Pi^\eta_{ab}n^a\right) = \Pi^\eta_{ab}n^a + V^c\Big(n(\alpha)\mathrm{U}_{bc} + \alpha\big((\pounds_n\mathbf{U})_{bc} - 2P^{ad}\mathrm{U}_{ac}\mathrm{U}_{bd}\big)\Big). \tag{5.42}$$

In particular, if $\mathbf{U} = 0$ then

$$\mathcal{G}_{(z,V)}\left(\Pi^\eta\right) = z\Pi^\eta, \qquad \mathcal{G}_{(z,V)}\left(\Pi^\eta(n,\cdot)\right) = \Pi^\eta(n,\cdot). \tag{5.43}$$

Proof. The proof relies on (2.37), which implies $\mathcal{G}_{(z,V)}(\boldsymbol{\zeta}) \stackrel{\text{def}}{=} z(\boldsymbol{\zeta} + g(\phi_\star V,\cdot))$. Since the tensor field Σ_η is gauge-invariant (because it depends only on η and the Levi-Civita connection of g), it holds

$$\mathcal{G}_{(z,V)}\left(\phi^\star\big(\boldsymbol{\zeta}(\Sigma_\eta)\big)\right) = \phi^\star\big(\mathcal{G}_{(z,V)}(\boldsymbol{\zeta})(\Sigma_\eta)\big) = z\phi^\star\big(\boldsymbol{\zeta}(\Sigma_\eta) + g(\phi_\star V, \Sigma_\eta)\big).$$

Particularizing (5.32) for $\beta = z$ and $W = zV$ and using (5.39), equation (5.41) follows at once. In order to obtain (5.42), it suffices to contract (5.41) with $\mathcal{G}_{(z,V)}(n^a) = z^{-1}n^a$ (cf. (2.34)). Using $\mathbf{U}(n,\cdot) = 0$, the fact that $n^a\overset{\circ}{\nabla}_a\mathrm{U}_{bc} = (\pounds_n\mathbf{U})_{bc} - (\overset{\circ}{\nabla}_b n^a)\mathrm{U}_{ac} - (\overset{\circ}{\nabla}_c n^a)\mathrm{U}_{ab}$ and (3.44), one gets

$$\begin{aligned}
\mathcal{G}_{(z,V)}\left(\Pi^\eta_{ab}n^a\right) &= \Pi^\eta_{ab}n^a + V^c\Big(n(\alpha)\mathrm{U}_{bc} + \alpha(n^a\overset{\circ}{\nabla}_a\mathrm{U}_{bc} - \mathrm{U}_{ca}\overset{\circ}{\nabla}_b n^a + \mathrm{U}_{ab}\overset{\circ}{\nabla}_c n^a)\Big) \\
&= \Pi^\eta_{ab}n^a + V^c\Big(n(\alpha)\mathrm{U}_{bc} + \alpha\big((\pounds_n\mathbf{U})_{bc} - 2(\overset{\circ}{\nabla}_b n^a)\mathrm{U}_{ac}\big)\Big) \\
&= \Pi^\eta_{ab}n^a + V^c\Big(n(\alpha)\mathrm{U}_{bc} + \alpha\big((\pounds_n\mathbf{U})_{bc} - 2P^{ad}\mathrm{U}_{ac}\mathrm{U}_{bd}\big)\Big),
\end{aligned}$$

which is (5.42). Expressions (5.43) are immediate from (5.41)-(5.42). □

The gauge transformation (5.42) introduces in a natural way a symmetric 2-covariant tensor. It is worth exploring its gauge behaviour.

Lemma 5.3.3. *The tensor field Ψ, defined by*

$$\Psi_{bc} \stackrel{\text{def}}{=} n(\alpha)\mathrm{U}_{bc} + \alpha\big((\pounds_n\mathbf{U})_{bc} - 2P^{ad}\mathrm{U}_{ac}\mathrm{U}_{bd}\big), \tag{5.44}$$

transforms under the action of a gauge group element $\mathcal{G}_{(z,V)}$ *according to*

$$\mathcal{G}_{(z,V)}(\Psi) = \frac{1}{z}\Psi. \tag{5.45}$$

Proof. The gauge transformations (2.33)-(2.34) and (3.60) entail (recall that $\mathbf{U}(n,\cdot) = 0$)

$$\mathcal{G}_{(z,V)}\left(P^{ad}\mathrm{U}_{ac}\mathrm{U}_{bd}\right) = \frac{1}{z^2}P^{ad}\mathrm{U}_{ac}\mathrm{U}_{bd} \qquad \text{and} \qquad \mathcal{G}_{(z,V)}(\pounds_n\mathbf{U}) = \frac{1}{z^2}\pounds_n\mathbf{U} - \frac{n(z)}{z^3}\mathbf{U},$$

while from $\mathcal{G}_{(z,V)}(\alpha) = z\alpha$ (see (3.103)) it follows

$$\mathcal{G}_{(z,V)}(n(\alpha)) = n(\alpha) + \frac{\alpha n(z)}{z}.$$

Thus,

$$\mathcal{G}_{(z,V)}(\Psi_{bc}) = \frac{1}{z}n(\alpha)\mathrm{U}_{bc} + \frac{\alpha n(z)}{z^2}\mathrm{U}_{bc} + z\alpha\left(\frac{1}{z^2}(\pounds_n\mathbf{U})_{bc} - \frac{n(z)}{z^3}\mathrm{U}_{bc} - \frac{2}{z^2}P^{ad}\mathrm{U}_{ac}\mathrm{U}_{bd}\right),$$

which yields (5.45) after simple cancellations. □

The following lemma finds the remarkable result that ח^η can be written solely in terms of ק and hypersurface data quantities. In particular, all dependence on the transverse derivatives of the deformation tensor drop off. This result is one of the several interesting applications of the identities in Lemma 5.2.2.

Lemma 5.3.4. *Assume the definitions and hypotheses in Lemma 5.3.1 and let* $\eta \stackrel{\text{def}}{=} \alpha n$. *The tensor* ח^η *defined in* (5.39) *admits the alternative expression*

$$\begin{aligned}\text{ח}^\eta_{ab} = {} & \overset{\circ}{\nabla}_a\overset{\circ}{\nabla}_b\alpha + \alpha\ell_f\overset{\circ}{\Sigma}{}^f{}_{ab} - \alpha(\pounds_n\mathbf{Y})_{ab} + 2(s_{(a} - r_{(a})\overset{\circ}{\nabla}_{b)}\alpha + n(\alpha)\mathrm{Y}_{ab} \\ & - \frac{\alpha}{2}n(\ell^{(2)})\mathrm{U}_{ab} - \alpha\ell^{(2)}\pounds_n\mathrm{U}_{ab} + \frac{1}{2}\text{ק}\,\mathrm{U}_{ab}.\end{aligned} \tag{5.46}$$

Proof. We first particularize (3.34) for $n^{(2)} = 0$. This gives

$$2\overset{\circ}{\nabla}_{(a}s_{b)} = \ell_f\overset{\circ}{\Sigma}{}^f{}_{ab} - n(\ell^{(2)})\mathrm{U}_{ab} - \ell^{(2)}\pounds_n\mathrm{U}_{ab}. \tag{5.47}$$

Inserting (5.47) and the identity (5.20) into the definition (5.39), the alternative expression (5.46) follows easily after using $(\pounds_\eta\mathbf{Y})_{ab} = \alpha(\pounds_n\mathbf{Y})_{ab} + 2r_{(a}\overset{\circ}{\nabla}_{b)}\alpha$. □

To conclude the section, we obtain a completely general expression for $\Sigma_\eta(\phi_\star Y, \phi_\star Z)$, $Y, Z \in \Gamma(T\mathcal{N})$ under the only condition that the ambient vector field η is null and tangent to the hypersurface.

Lemma 5.3.5. *Under the same hypotheses as in Lemma 5.3.1, consider any two vector fields $Y, Z \in \Gamma(T\mathcal{N})$. Then $\Sigma_\eta(\phi_\star Y, \phi_\star Z)$ is given by*

$$\begin{aligned}\Sigma_\eta(\phi_\star Y, \phi_\star Z) = &\Big((\psi - n(\alpha))\mathbf{U}(Y,Z) - \alpha(\pounds_n \mathbf{U})(Y,Z) \Big)\zeta \\ &+ \phi_\star\Big(\alpha \overset{\circ}{\Sigma}(Y,Z) + \Big(\overset{\circ}{\mathrm{Hess}}(\alpha) - \alpha \pounds_n \mathbf{Y} + n(\alpha)\mathbf{Y} + 2(s-r)\otimes_s d\alpha \Big)(Y,Z)\, n \\ &+ \Big(P(\mathrm{i} - d\alpha - 2\alpha s, \cdot) + \frac{1}{2}\Big(\mathrm{p} - \alpha n(\ell^{(2)})\Big) n \Big)\mathbf{U}(Y,Z) \\ &+ P(\mathbf{U}(Z,\cdot),\cdot)Y(\alpha) + P(\mathbf{U}(Y,\cdot),\cdot)Z(\alpha) \Big). \end{aligned} \tag{5.48}$$

where $\overset{\circ}{\mathrm{Hess}}$ is the Hessian of $\overset{\circ}{\nabla}$. In particular, if $\mathbf{U} = 0$ then

$$\begin{aligned}\Sigma_\eta(\phi_\star Y, \phi_\star Z) = \phi_\star\Big(&\alpha \overset{\circ}{\Sigma}(Y,Z) + \Big(\overset{\circ}{\mathrm{Hess}}(\alpha) - \alpha \pounds_n \mathbf{Y} \\ &+ n(\alpha)\mathbf{Y} + 2(s-r)\otimes_s d\alpha \Big)(Y,Z)\, n \Big)\end{aligned} \tag{5.49}$$

or, equivalently,

$$\Sigma_\eta(\phi_\star Y, \phi_\star Z) = \Pi^\eta(Y,Z)\nu \tag{5.50}$$

Remark 5.3.6. *The expressions in Lemma 5.3.5 look rather complicated, mainly for the notation that we have used. In index notation and assuming Setup 2.2.7, (5.48) can be written in a somewhat simpler form*

$$\begin{aligned}\Sigma_\eta(e_a, e_b) = &\Big((\psi - n(\alpha))\mathrm{U}_{ab} - \alpha \pounds_n \mathrm{U}_{ab} \Big)\zeta + \alpha\phi_\star\Big(\overset{\circ}{\Sigma}(\hat{e}_a, \hat{e}_b) \Big) \\ &+ \Big(\overset{\circ}{\nabla}_a \overset{\circ}{\nabla}_b \alpha - \alpha \pounds_n \mathrm{Y}_{ab} + n(\alpha)\mathrm{Y}_{ab} + 2(s-r)_{(a}\overset{\circ}{\nabla}_{b)}\alpha + \frac{1}{2}(\mathrm{p} - \alpha n(\ell^{(2)}))\mathrm{U}_{ab} \Big)\nu \\ &+ P^{cd}\Big((\mathrm{i}_c - \overset{\circ}{\nabla}_c \alpha - 2\alpha s_c)\mathrm{U}_{ab} + 2\mathrm{U}_{c(a}\overset{\circ}{\nabla}_{b)}\alpha \Big)\phi_\star \hat{e}_d. \end{aligned} \tag{5.51}$$

Proof. We will identify vector fields on $\mathcal{N}$ with their push-forwards through $\phi_\star$ and let the context determine the meaning. The proof relies on Lemma 3.1.2 and equations (5.32) and (5.46) Let us define the function $v_0 \in \mathcal{F}(\mathcal{N})$ and the covector $\boldsymbol{\mu}$ on $\mathcal{N}$ as

$$v_0 \overset{\text{def}}{=} \phi^\star\big(g(\zeta,\Sigma_\eta(Y,Z))\big), \qquad \boldsymbol{\mu}(X) \overset{\text{def}}{=} \phi^\star\big(g(X,\Sigma_\eta(Y,Z))\big) \tag{5.52}$$

for any $X, Y, Z \in \Gamma(T\mathcal{N})$. Obviously v_0 and $\boldsymbol{\mu}$ depend on Y, Z but for simplicity we do not reflect this dependence. Particularizing (5.32) first for $W = \zeta$ and then for $W = e_c$ leads to

$$v_0 = \boldsymbol{\Pi}^{\eta}_{ab} Y^a Z^b, \tag{5.53}$$

$$\begin{aligned}
\mu_c &= Y^a Z^b \Big((\overset{\circ}{\nabla}_a \alpha)\mathbf{U}_{bc} + (\overset{\circ}{\nabla}_b \alpha)\mathbf{U}_{ac} + \mathbf{U}_{ab}(\mathbf{i}_c - \overset{\circ}{\nabla}_c \alpha) + \alpha(\overset{\circ}{\nabla}_a \mathbf{U}_{bc} + \overset{\circ}{\nabla}_b \mathbf{U}_{ca} - \overset{\circ}{\nabla}_c \mathbf{U}_{ab}) \Big) \\
&= Y^a Z^b \Big((\overset{\circ}{\nabla}_a \alpha)\mathbf{U}_{bc} + (\overset{\circ}{\nabla}_b \alpha)\mathbf{U}_{ac} + \mathbf{U}_{ab}(\mathbf{i}_c - \overset{\circ}{\nabla}_c \alpha) \\
&\quad + \alpha(\gamma_{cf}\overset{\circ}{\Sigma}{}^f{}_{ab} - \boldsymbol{\ell}_c \pounds_n \mathbf{U}_{ab} - 2 s_c \mathbf{U}_{ab}) \Big).
\end{aligned} \tag{5.54}$$

where in the last equality we inserted (3.36). Since $\Sigma_\eta(Y,Z)$ is a vector field along $\phi(\mathcal{N})$, by Lemma 3.1.2 and (5.52) it must hold

$$\Sigma_\eta(Y,Z) = \boldsymbol{\mu}(n)\zeta + \phi_\star\big(P(\boldsymbol{\mu},\cdot) + v_0 n\big). \tag{5.55}$$

Thus, to complete the proof it suffices to compute the scalar $\boldsymbol{\mu}(n)$ and the vector $P(\boldsymbol{\mu},\cdot) + v_0 n$. To obtain $\boldsymbol{\mu}(n)$ we contract (5.54) with n^c and use $\boldsymbol{\ell}(n) = 1$, $\gamma(n,\cdot) = 0$, $\mathbf{U}(n,\cdot) = 0$, $s(n) = 0$ and $\mathbf{i}(n) = \psi$ to get

$$\boldsymbol{\mu}(n) = (\psi - n(\alpha))\mathbf{U}(Y,Z) - \alpha(\pounds_n \mathbf{U})(Y,Z). \tag{5.56}$$

For the vector $P(\boldsymbol{\mu},\cdot) + v_0 n$, we first contract (5.54) with P^{cd} and use $P^{cd}\boldsymbol{\ell}_c = -\ell^{(2)} n^d$ and $P^{cd}\gamma_{cf} = \delta^d_f - n^d \boldsymbol{\ell}_f$, so that

$$\begin{aligned}
P^{cd}\mu_c = Y^a Z^b \Big(& P^{cd}\big((\overset{\circ}{\nabla}_a \alpha)\mathbf{U}_{bc} + (\overset{\circ}{\nabla}_b \alpha)\mathbf{U}_{ac} + \mathbf{U}_{ab}(\mathbf{i}_c - \overset{\circ}{\nabla}_c \alpha)\big) \\
& + \alpha\overset{\circ}{\Sigma}{}^d{}_{ab} - \alpha n^d \boldsymbol{\ell}_f \overset{\circ}{\Sigma}{}^f{}_{ab} + \alpha \ell^{(2)} n^d \pounds_n \mathbf{U}_{ab} - 2\alpha P^{cd} s_c \mathbf{U}_{ab} \Big).
\end{aligned} \tag{5.57}$$

Combining (5.46), (5.53) and (5.57) it is now straightforward to conclude that

$$\begin{aligned}
P^{cd}\mu_c + v_0 n^d = Y^a Z^b \Big(& P^{cd}\big((\overset{\circ}{\nabla}_a \alpha)\mathbf{U}_{bc} + (\overset{\circ}{\nabla}_b \alpha)\mathbf{U}_{ac} + \mathbf{U}_{ab}(\mathbf{i}_c - \overset{\circ}{\nabla}_c \alpha - 2\alpha s_c)\big) \\
& + \alpha\big(\overset{\circ}{\Sigma}{}^d{}_{ab} - n^d(\pounds_n \mathbf{Y})_{ab}\big) + n^d\big(\overset{\circ}{\nabla}_a\overset{\circ}{\nabla}_b \alpha + 2(s_{(a} - r_{(a})\overset{\circ}{\nabla}_{b)} \alpha + n(\alpha)\mathbf{Y}_{ab}
\end{aligned}$$

$$+\frac{1}{2}\big(\mathfrak{p}-\alpha n(\ell^{(2)})\big)\mathrm{U}_{ab}\Big)\Bigg] . \tag{5.58}$$

Equation (5.48) is obtained by inserting (5.56) and (5.58) into (5.55). The particularization (5.49) to the case $\mathbf{U}=0$ is immediate. To prove that this can be written in the equivalent form (5.50), we note that when $\mathbf{U}=0$ the tensor $\overset{\circ}{\Sigma}$ is proportional to n (by (3.31) with $n^{(2)}=0, \mathbf{U}=0$), so it satisfies the relation $\overset{\circ}{\Sigma}(Y,Z)=\boldsymbol{\ell}(\overset{\circ}{\Sigma}(Y,Z))n$. From this and identity (5.46) particularized to $\mathbf{U}=0$, the equivalence between (5.49) and (5.50) follows. □

Remark 5.3.7. *An interesting consequence of this lemma is that whenever* $\mathbf{U}=0$*, the vector field* $\Sigma_\eta(\phi_\star Y,\phi_\star Z)$*,* $Y,Z\in\Gamma(T\mathcal{N})$ *can be codified entirely by the function* α *and the abstract objects* n*,* s*,* $\overset{\circ}{\nabla}$*,* $\overset{\circ}{\Sigma}$ *and* $\mathbf{Y}$*, as expression* (5.49) *immediately shows. This means in particular that the vector field* $\Sigma_\eta(\phi_\star Y,\phi_\star Z)$ *is completely independent of how* η *behaves off* $\phi(\mathcal{N})$*.*

5.4 abstract killing horizons of order zero and one

As mentioned in Section 2.6, Killing horizons have played a fundamental role in General Relativity, mainly because its close relation with black holes in equilibrium through Hawking's rigidity theorem (see e.g. [124]). They are characterized, as we already know, by a Killing vector field η which becomes null and tangent at the hypersurface (a typical situation is when the Killing changes its causal character from timelike to spacelike across the hypersurface, but this is by no means the only possibility). The deformation tensor K^η of η is identically zero everywhere, so in particular it vanishes together with all its derivatives on the hypersurface. However, it turns out that some of the most relevant properties of Killing horizons can be fully recovered by only requiring that a few derivatives of K^η vanish on the horizon. It is in these circumstances that the notions of Killing horizons of order zero/one arise naturally.

By definition, a Killing horizon of order m (embedded on a semi-Riemannian manifold) corresponds to a null hypersurface together with a vector field η, defined in a neighbourhood thereof, that has the properties of (i) being null and tangent to the hypersurface and (ii) the transverse derivatives up to order m of the deformation tensor K^η vanish on the hypersurface. The purpose of this section is to provide **abstract** definitions of Killing horizons of order zero and one. The idea is to be able to describe these sort of null hypersurfaces in a detached way from any space where they may be embedded.

It is to be expected that the abstract notions of Killing horizons of order zero and one rely on hypersurface data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ satisfying certain extra conditions. However, this already raises the question of how much geometric information from the embedded picture can be codified only in terms of $\{\gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$. For the order zero, ideally one would like the definition to enforce $\mathcal{K}^{\eta} = 0$ everywhere on the hypersurface. However, as we already know, only the pull-back of $\mathcal{K}^{\eta}$ can be expressed solely in terms of the data, namely by means of $\mathbf{U}$ (see (5.17)). The remaining components of $\mathcal{K}^{\eta}$ are given by $\mathfrak{i}$, $\mathfrak{p}$ (cf. (5.15)) and cannot be encoded in the tensor fields $\{\gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$.

For this reason, we split the definition in two different levels. We start with a weaker definition which only restricts the metric hypersurface data and which is truly at the abstract level, in the sense that no embedding into an ambient space is required. In a second stage we assume the data to be embedded and add extra restrictions so as to enforce also that the remaining components of the deformation tensor vanish on the hypersurface.

Obviously, to define abstractly the notion of Killing horizon of order zero we need a privileged vector field $\overline{\eta}$ on the data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$. This field can and will be restricted to be along the degeneration direction of γ. Since in general Killing vectors can have zeroes, we want to allow for the possibility that $\overline{\eta}$ vanishes somewhere on the abstract hypersurface. However, we are definitely not interested in the case when $\overline{\eta}$ vanishes on open subsets of $\mathcal{N}$, so we need to make sure that this situation is excluded. In these circumstances, the following definition arises naturally.

Definition 5.4.1. *(Abstract Killing horizon of order zero, AKH$_0$) Consider null hypersurface data* $\mathcal{D} \stackrel{\text{def}}{=} \{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ *admitting a gauge-invariant vector field* $\overline{\eta} \in \operatorname{Rad}\gamma$. *Define* $\mathcal{S} \stackrel{\text{def}}{=} \{p \in \mathcal{N} \mid \overline{\eta}|_p = 0\}$. *Then* $\mathcal{D}$ *is an abstract Killing horizon of order zero if*

(i) $\mathcal{S}$ *is a finite union of smooth connected closed submanifolds of dimension* $n-1$*,*

(ii) $\pounds_{\overline{\eta}}\gamma = 0$.

Condition (*i*) in Definition 5.4.1 certainly ensures that $\mathcal{N} \setminus \mathcal{S}$ is dense in $\mathcal{N}$ (hence that $\overline{\eta}$ does not vanish on open subsets of $\mathcal{N}$). Moreover, it mimics the possible behaviour of the zeros of a Killing vector, so the definition is indeed justified. Combining this with the fact that $\overline{\eta}$ is proportional to n, it follows that (*ii*) is equivalent to $\mathbf{U} = 0$ everywhere on $\mathcal{N}$, as (cf. (2.12))

$$\overline{\eta} = \alpha n \qquad \Longrightarrow \qquad 0 = \pounds_{\overline{\eta}}\gamma = \alpha \pounds_{n}\gamma = 2\alpha \mathbf{U} \qquad \Longleftrightarrow \qquad \mathbf{U} = 0.$$

For a better understanding of Definition 5.4.1, it is convenient to consider the embedded picture, so we embed $\mathcal{D}$ on a semi-Riemannian manifold $(\mathcal{M}, g)$ with embedding ϕ and rigging ζ. As we already know, in these circumstances the tensor $\mathbf{U}$ coincides with the second fundamental form with respect to the null normal vector field $\phi_\star n$. Thus, condition (ii) means that $\phi(\mathcal{N})$ is totally geodesic. In other words, embedding an abstract Killing horizon of order zero $\mathcal{D}$ yields a totally geodesic null hypersurface equipped with an extra vector field $\eta|_{\phi(\mathcal{N})} \stackrel{\text{def}}{=} \phi_\star\overline{\eta}$ with a restriction on its set of zeroes. For the rest of the thesis, we call the vector fields $\overline{\eta}$, $\eta|_{\phi(\mathcal{N})}$ and the submanifolds $\mathcal{S}$, $\phi(\mathcal{S})$ *symmetry generators* and *fixed points sets* respectively.

As already discussed, neither the abstract nor the embedded levels of Definition 5.4.1 restrict the components $\mathbf{i}$ and $\mathfrak{p}$ of the deformation tensor. In this sense, $\mathcal{D}$ does not correspond to a *full* Killing horizon of order zero. To capture the full notion we are forced to restrict ourselves to the embedded case. The corresponding concept is naturally called *Killing horizon of order zero*.

Definition 5.4.2. *(Killing horizon of order zero, KH₀) Consider an abstract Killing horizon of order zero* $\mathcal{D} \stackrel{\text{def}}{=} \{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ *with symmetry generator* $\overline{\eta} \in \operatorname{Rad}\gamma$ *and assume that* $\mathcal{D}$ *is embedded in a semi-Riemannian manifold* $(\mathcal{M}, g)$ *with embedding* ϕ *and rigging* ζ*. Then,* $\mathcal{H}_0 \stackrel{\text{def}}{=} \phi(\mathcal{N})$ *is a Killing horizon of order zero if there exists at least one extension* η *of* $\phi_\star\overline{\eta}$ *to a neighbourhood* $\mathcal{O} \subset \mathcal{M}$ *of* $\phi(\mathcal{N})$ *such that*

$$\phi^\star(g(\pounds_\zeta\eta, \zeta)) = -\frac{1}{2}\pounds_{\overline{\eta}}\ell^{(2)}, \qquad \phi^\star(g(\pounds_\zeta\eta, \cdot)) = -\pounds_{\overline{\eta}}\boldsymbol{\ell} \quad on \quad \mathcal{N}. \tag{5.59}$$

Remark 5.4.3. *The Lie derivative* $\pounds_\zeta\eta|_{\phi(\mathcal{N})}$ *is given by* $\pounds_\zeta\eta|_{\phi(\mathcal{N})} = \nabla_\zeta\eta - \nabla_\eta\zeta|_{\phi(\mathcal{N})}$*. Since there are no transverse derivatives of the rigging (because* $\eta|_{\phi(\mathcal{N})}$ *is tangent to* $\phi(\mathcal{N})$*), there is no need to extend the rigging vector field* ζ *off* $\phi(\mathcal{N})$ *in Definition 5.4.2.*

Let us prove that Definition 5.4.2 indeed guarantees that $\mathcal{K}^\eta = 0$ on $\phi(\mathcal{N})$.

Proposition 5.4.4. *The deformation tensor* $\mathcal{K}^\eta$ *is everywhere zero on any KH₀.*

Proof. Firstly, since an AKH₀ satisfies that $\pounds_{\overline{\eta}}\gamma = 0$ and by (5.17) we know that $\phi^\star\mathcal{K}^\eta = \pounds_{\overline{\eta}}\gamma$, the *tangent-tangent* components of $\mathcal{K}^\eta$ are automatically zero. Concerning $\mathcal{K}^\eta(\zeta, \zeta)$, we find

$$\mathcal{K}^\eta(\zeta, \zeta) = 2g(\nabla_\zeta\eta, \zeta) = 2g(\pounds_\zeta\eta + \nabla_\eta\zeta, \zeta) = 2g(\pounds_\zeta\eta, \zeta) + \eta(g(\zeta, \zeta)). \tag{5.60}$$

Using now Definition 5.4.2, we obtain $\phi^\star(\mathcal{K}^\eta(\zeta, \zeta)) = -\pounds_{\overline{\eta}}\ell^{(2)} + \overline{\eta}(\ell^{(2)}) = 0$ and hence $\mathcal{K}^\eta(\zeta, \zeta) = 0$. Now let X be a vector field tangent to $\phi(\mathcal{N})$. Combining $\mathbf{U} = 0$

and $F_{ab} = \overset{\circ}{\nabla}_a \ell_b$ (cf. (2.19)) with (5.9) (here $\beta = 0$ and ν is normal to $\phi(\mathcal{N})$), one obtains

$$\phi^\star(\nabla_\eta \boldsymbol{\zeta}) = \mathbf{F}(X, \cdot) + \mathbf{Y}(X, \cdot), \qquad \text{where} \qquad \boldsymbol{\zeta} \stackrel{\text{def}}{=} g(\zeta, \cdot). \tag{5.61}$$

Then, from the fact that $K^\eta(\zeta, X) = g(\nabla_\zeta \eta, X) + g(\nabla_X \eta, \zeta)$ it follows

$$\begin{aligned}
\phi^\star\left(K^\eta(\zeta, X)\right) &= \phi^\star\left(g(\pounds_\zeta \eta, X)\right) + (\nabla_\eta \boldsymbol{\zeta})(X) + X(\alpha) - (\nabla_X \boldsymbol{\zeta})(\eta) \\
&= \phi^\star\left(g(\pounds_\zeta \eta, X)\right) + \alpha(\boldsymbol{s}(X) + \boldsymbol{r}(X)) + X(\alpha) - \alpha(\mathbf{F}(X, n) + \boldsymbol{r}(X)) \\
&= \phi^\star\left(g(\pounds_\zeta \eta, X)\right) + 2\alpha \boldsymbol{s}(X) + X(\alpha) \\
&= \phi^\star\left(g(\pounds_\zeta \eta, \cdot)\right)(X) + (\pounds_T \boldsymbol{\ell})(X) = 0,
\end{aligned}$$

where in the first line we noticed that $g(\zeta, \eta)|_{\phi(\mathcal{N})} = \alpha g(\zeta, \nu)|_{\phi(\mathcal{N})} = \alpha$, in the second line we have inserted (5.61) and in the third and fourth lines we used (3.43), (3.45), the fact that $\mathbf{F}$ is antisymmetric and Definition 5.4.2. □

Definitions 5.4.1 and 5.4.2 establish both the abstract and the embedded levels of a Killing horizon of order zero. Concerning the characterization of the first order, there is one object that is particularly useful, namely the tensor $\Sigma_\eta \stackrel{\text{def}}{=} \pounds_\eta \nabla$. Its relevance comes from the fact that it encodes both extrinsic and intrinsic properties of the hypersurface. Specifically, on (5.48) we have proven that for any two vector fields $X, W \in \Gamma(T\mathcal{N})$, $\Sigma_\eta(\phi_\star X, \phi_\star W)$ can be entirely constructed (on the hypersurface) from the data tensors $\{\gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ (recall that $\overset{\circ}{\Sigma}$ can be fully built from the metric part of the data), the quantities i, ק (note that $\mathrm{i}(n) =$ ש), the function α and the rigging ζ. Even more, when it comes to defining an abstract Killing horizon of first order, one would want that all conditions from the order zero (in particular that $\mathbf{U} = 0$) are fulfilled, and this means that (5.48) simplifies to (5.49), so $\Sigma_\eta(\phi_\star X, \phi_\star W)$ can be written in terms of $\{\gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ and α exclusively. On the other hand, by (5.40)-(5.39) we know that the pull-back $\phi^\star(\boldsymbol{\zeta}(\Sigma_\eta))$ contains information about the zeroth and the *first* order derivatives (because of the presence of the tensor ד) of K^η. These two ingredients can be combined to set up a sensible definition of abstract Killing horizon of order one.

There is yet another reason that justifies the importance of the tensor Σ_η in characterizing Killing horizons of first order, namely the identity (3.22). A Killing vector on a semi-Riemannian manifold $(\mathcal{M}, g)$ equipped with the Levi-Civita connection ∇ satisfies the well-known property $0 = \nabla_\alpha \nabla_\beta \eta^\mu + R^\mu{}_{\beta\nu\alpha} \eta^\nu$. This, together with (3.22), suggest that the first order can be codified by requiring that some components of Σ_η vanish on the hypersurface. All the above considerations, combined with Lemma 5.3.5, naturally leads us to the following definition for an abstract Killing horizon of order one.

Definition 5.4.5. *(Abstract Killing horizon of order one, AKH1) Consider null hypersurface data* $\mathcal{D} \stackrel{\mathrm{def}}{=} \{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ *admitting a gauge-invariant vector field* $\overline{\eta} \in \mathrm{Rad}\gamma$*. Let* $\alpha \in \mathcal{F}(\mathcal{N})$ *be given by* $\overline{\eta} = \alpha n$ *and* $\boldsymbol{r}$, $\mathring{\Sigma}$ *be defined by* (2.44), (3.30) *respectively. Then,* $\mathcal{D}$ *defines an abstract Killing horizon of order one if it is an abstract Killing horizon of order zero and for any two vector fields* $X, W \in \Gamma(T\mathcal{N})$ *it holds*

$$0 = \alpha\left(\mathring{\Sigma}(X,W) - (\pounds_n \mathbf{Y})(X,W)n\right) + \Big(X(W(\alpha)) - (\mathring{\nabla}_X W)(\alpha) \\ + \mathbf{Y}(X,W)n(\alpha) + W(\alpha)(\boldsymbol{s}(X) - \boldsymbol{r}(X)) + X(\alpha)(\boldsymbol{s}(W) - \boldsymbol{r}(W))\Big)n. \tag{5.62}$$

Remark 5.4.6. *An abstract Killing horizon of order one embedded on a semi-Riemannian manifold* $(\mathcal{M}, g)$ *with embedding* ϕ *and rigging* ζ *does not need to satisfy* (5.59)*, so in general it does not define a Killing horizon of order zero according to Definition* 5.4.2*. This may be confusing at first sight, but the key to understand the terminology is the word "abstract". Whenever it appears, the related notion must be fully insensitive to the data being embedded and hence to any kind of extension of* $\overline{\eta}$*. Since a Killing horizon of order zero is embedded and requires an extension of* $\overline{\eta}$ *it makes sense that abstract Killing horizons of order one need not be Killing horizons of order zero.*

Definition 5.4.5 establishes two restrictions on the hypersurface data (namely $\mathbf{U} = 0$ and (5.62)), so at this point the reader may wonder how these two conditions are related to the first transverse derivative of the deformation tensor $\mathcal{K}^{\eta}$, which in the end is what it is expected to vanish on a Killing horizon of order one. The answer to this question of course requires assuming embeddedness of the data, and should be addressed in various separate stages. First, we need to prove that when an AKH1 $\mathcal{D}$ is embedded on a semi-Riemannian manifold $(\mathcal{M}, g)$ with embedding ϕ and rigging ζ, the vector field $\Sigma_{\eta}(\phi_\star X, \phi_\star W)$, $X, W \in \Gamma(T\mathcal{N})$ vanishes everywhere on $\phi(\mathcal{N})$. With this result at hand and using (5.50) (which holds in this context because $\mathbf{U} = 0$), we will be able to find an identity involving the objects $\{\mathfrak{i}, \mathfrak{w}, \daleth\}$ defined in (5.15)-(5.16). The tensor $\daleth$ encodes precisely the pull-back to $\mathcal{N}$ of the first transverse derivative of $\mathcal{K}^{\eta}$, which is therefore restricted by such identity. Finally, we will see that enforcing $\mathfrak{i} = 0$ implies $\daleth = 0$. This extra restriction of $\mathfrak{i}$ being zero will be connected with the notion of KH0 introduced above and will allow us to introduce the new notion of *Killing horizon of order ½* to refer to a KH0 which also satisfies (5.62).

In the following lemma we address the first two stages of the procedure above.

Lemma 5.4.7. *Consider a Killing horizon of order one* $\mathcal{D} = \{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ *embedded on a semi-Riemannian manifold* $(\mathcal{M}, g)$ *with embedding* ϕ *and rigging* ζ*. Let* η *be an* *extension of* $\phi_\star T$ *off* $\phi(\mathcal{N})$ *and* $\Sigma_\eta \stackrel{\text{def}}{=} \pounds_\eta \nabla$ *be given by* (3.18). *Define* $\{\mathrm{i}, \text{ש}, \text{ד}\}$ *according to* (5.15)-(5.16). *Then,*

$$\Sigma_\eta(\phi_\star X, \phi_\star W) = 0, \qquad \forall X, W \in \Gamma(T\mathcal{N}). \tag{5.63}$$

and

$$\text{ה}^{\eta}_{\ ab} = \mathring{\nabla}_{(a}\mathrm{i}_{b)} + \text{ש}\,\mathrm{Y}_{ab} - \text{ד}_{ab} = 0. \tag{5.64}$$

Proof. Imposing $\mathbf{U} = 0$ and (5.62) in (5.48) immediately proves (5.63), while (5.64) follows from combining (5.50) (which gives $\text{ח}^{\eta} = 0$) and (5.39). □

Remark 5.4.8. *It is precisely the fact that* $\mathbf{U} = 0$ *that allows for an abstract notion of Killing horizon of order one because, in such case, the vector field* $\Sigma(\phi_\star Y, \phi_\star Z)$, $Y, Z \in \Gamma(T\mathcal{N})$ *can be entirely codified abstractly, as we have discussed in Remark* 5.3.7. *If in the right hand side of* (5.49) *appeared any combination of the tensor fields* $\{\mathrm{i}, \text{ש}, \text{ק}, \text{ד}\}$ *depending on the behaviour of* η *off* $\phi(\mathcal{N})$, *then it would be impossible to establish a condition on the data* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ *so that in the embedded picture* $\Sigma(\phi_\star Y, \phi_\star Z) = 0$.

Equation (5.64) relates the abstract Definitions 5.4.1 and 5.4.5 with (the pull-back to $\mathcal{N}$ of) the first transverse Lie derivative of the deformation tensor. It is now immediate to see that whenever $\mathrm{i} = 0$ everywhere on $\mathcal{N}$ (and hence $\text{ש} = \mathrm{i}(n) = 0$), then $\text{ד} = 0$. We capture this fact in the following corollary.

Corollary 5.4.9. *Under the hypotheses of Lemma* 5.4.7, *whenever* $\mathrm{i} = 0$ *it holds*

$$\phi^\star(\pounds_\zeta \mathrm{K}^\eta) = 0. \tag{5.65}$$

In particular, if an embedded AKH₁ $\mathcal{D}$ happens to be in addition a Killing horizon of order zero (i.e. it satisfies (5.59) for at least one extension η of $\phi_\star\overline{\eta}$), then $\mathrm{K}^\eta|_{\phi(\mathcal{N})} = 0$ (by Proposition 5.4.4), hence $\mathbf{i} = 0$ and the pull-back $\phi^\star(\pounds_\zeta \mathrm{K}^\eta)$ vanishes everywhere on $\mathcal{N}$. Note that, as in the case of order zero (where only $\phi^\star \mathrm{K}^\eta$ was restricted), here the transverse components of $\pounds_\zeta \mathrm{K}^\eta$ are totally unfixed. This fact suggests that we introduce the notion of *Killing horizon of order ½* as follows.

Definition 5.4.10. *(Killing horizon of order ½, KH½) Consider an abstract Killing horizon of order one* $\mathcal{D} \stackrel{\text{def}}{=} \{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ *with symmetry generator* $\overline{\eta} \in \mathrm{Rad}\gamma$ *and assume that* $\mathcal{D}$ *is embedded in a semi-Riemannian manifold* $(\mathcal{M}, g)$ *with embedding* ϕ *and rigging* ζ. *Then,* $\mathcal{H}_{1/2} \stackrel{\text{def}}{=} \phi(\mathcal{N})$ *is a Killing horizon of order ½ if, in addition, it is Killing horizon of order zero.*

Summarizing, in this section we have introduced the two fully abstract notions of Killing horizons of order zero and one (Definitions 5.4.1 and 5.4.5) as well as the concepts of Killing horizons of order zero and ½ (Definitions 5.4.2 and 5.4.10), which apply at the embedded level. The whole construction has been performed so that

(a) $\phi^\star \mathrm{K}^\eta = 0$ for an **embedded** AKH0 (which does not necessarily define a KH0),

(b) $\mathrm{K}^\eta|_{\phi(\mathcal{N})} = 0$ for a KH0,

(c) $\mathrm{K}^\eta|_{\phi(\mathcal{N})} = 0$ and $\phi^\star(\pounds_\zeta \mathrm{K}^\eta)|_{\phi(\mathcal{N})} = 0$ for a KH½.

5.4.1 *Some aspects of abstract Killing horizons of order zero*

In this section, we consider an abstract Killing horizon of order zero and obtain a result concerning the causal nature of the set of fixed points S of the symmetry generator $\overline{\eta}$ whenever its surface gravity κ is constant along the null generators. This is done in the following lemma.

Lemma 5.4.11. *Let* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ *be an abstract Killing horizon of order zero with symmetry generator* η *and fixed points set* S*. Assume further that* $\mathcal{N}$ *admits a cross-section and that the surface gravity* κ *of* $\overline{\eta}$ *is constant along the null generators of* $\mathcal{N}$*. Then, there exists a choice of gauge for which*

$$\overline{\eta} = (f + \kappa\lambda)n, \tag{5.66}$$

where $f, \lambda \in \mathcal{F}(\mathcal{N})$ *are functions satisfying* $k(\lambda) = 1$, $k(f) = 0$*. Also in that gauge,*

(i) *if* $\kappa \neq 0$ *and* $\mathrm{S} \neq \emptyset$*, then* S *is defined by the implicit equation* $\lambda = -\kappa^{-1} f$ *and it is a non-degenerate submanifold.*

(ii) *if* $\kappa = 0$*,* S *is either empty or is the union of smooth connected codimension-two degenerate submanifolds of* $\mathcal{N}$ *given by the zeros of* f*.*

Proof. Since $\mathcal{N}$ admits a cross-section, we know by Lemma 3.2.24 that one can always select the gauge so that $\kappa_n = 0$, which we enforce for the rest of the proof. This, together with (2.44) and (3.102) means that

$$n(\alpha) = \kappa. \tag{5.67}$$

Since κ is constant along the null generators of $\mathcal{N}$, the general solution of (5.67) for α is $\alpha = f + \kappa\lambda$, where λ, f are functions satisfying $n(\lambda) = 1$, $n(f) = 0$. This proves (5.66).

Now let $\{\lambda, u^I\}$ be coordinates on $\mathcal{N}$. Then $n = \partial_\lambda$, $f(u^I)$ and $\kappa(u^I)$. By definition of AKH$_0$, α cannot vanish on open subsets of $\mathcal{N}$. Moreover, from (5.66) it follows that the symmetry generator vanishes at points where $\kappa\lambda = -f$. When $\kappa \neq 0$ this implies (i) at once. When $\kappa = 0$, $\overline{\eta} = fn$ and either f vanishes no-where on $\mathcal{N}$ (hence $\mathcal{S} = \emptyset$) or there exist several smooth connected codimension-two subsets $\{F_{(i)}\} \subset \mathcal{N}$ $(i = 1, 2, ...)$ where f vanishes (hence $\mathcal{S} \equiv \bigsqcup_i F_{(i)}$). The fact that each connected component $F_{(i)}$ is a degenerate submanifold is a consequence of f depending only on $\{u^I\}$ and not on λ. □

Lemma 5.4.11 will play an important role later in Chapter 7 when we study the matching of spacetimes across Killing horizons of order zero. In that context we shall assume constancy of the surface gravity everywhere in the horizon, and this result will allow us to obtain all possible matchings explicitly in a simple way.

5.4.2 *Some aspects of abstract Killing horizons of order one*

In this section, we discuss briefly some aspects of abstract Killing horizons of order one for which the symmetry generator $\overline{\eta}$ is everywhere non-zero. For that purpose, we consider null hypersurface data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ defining an AKH$_1$ according to Definition 5.4.5. Since $\overline{\eta}$ is no-where zero on $\mathcal{N}$, it is convenient to fix the gauge so that the null generator n of the data coincides with $\overline{\eta}$, so we enforce $\overline{\eta} = n$. In these circumstances, $\mathcal{N}$ cannot contain fixed points and the proportionality function α is equal to one. By Definition 5.4.5 and because in the present case $\alpha = 1$, it follows

$$\overset{\circ}{\Sigma} - n \otimes \pounds_n \mathbf{Y} = 0, \tag{5.68}$$

which after inserting (3.31) takes the form

$$\pounds_n \mathrm{Y}_{ab} - 2\overset{\circ}{\nabla}_{(a} s_{b)} = 0. \tag{5.69}$$

Observe that while condition $\mathbf{U} = 0$ is fully-gauge invariant ($\mathbf{U}$ simply rescales under the action of a gauge group element, see (3.60)), condition (5.69) is not. Actually, the tensor $\pounds_n \mathrm{Y}_{ab} - 2\overset{\circ}{\nabla}_{(a} s_{b)}$ itself turns out to be gauge invariant under the action of the subgroup $\mathcal{G}_1$.

Lemma 5.4.12. *For any null hypersurface data* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ *with* $\mathbf{U} = 0$*, the quantity* $\overset{\circ}{\Sigma} - n \otimes \pounds_n \mathbf{Y}$ *or, what is the same* $\pounds_n \mathrm{Y}_{ab} - 2\overset{\circ}{\nabla}_{(a} s_{b)}$*, is invariant under the action of the subgroup* $\mathcal{G}_1$.

Proof. Let us denote G_1-transformed quantities with a prime symbol. Particularizing expressions (2.34), (2.40), (3.62) as well as Proposition 2.2.9 for $z = 1$, $n^{(2)} = 0$ and $\mathbf{U} = 0$ yields

$$n' = n, \qquad \mathbf{Y}' = \mathbf{Y} + \frac{1}{2}\pounds_V \gamma, \qquad s' = s + \frac{1}{2}\gamma(\pounds_n V, \cdot), \qquad \mathring{\nabla}' - \mathring{\nabla} = \frac{1}{2} n \otimes \pounds_V \gamma.$$

Firstly, the gauge behaviour of n and $\mathbf{Y}$ entails

$$\begin{aligned}
(\pounds_{n'}\mathbf{Y}')_{ab} &= (\pounds_n \mathbf{Y}')_{ab} = (\pounds_n \mathbf{Y})_{ab} + \frac{1}{2}(\pounds_n \pounds_V \gamma)_{ab} = (\pounds_n \mathbf{Y})_{ab} + \frac{1}{2}(\pounds_{[n,V]}\gamma)_{ab} \\
&= (\pounds_n \mathbf{Y})_{ab} + \frac{1}{2}\left(\gamma_{bc}\mathring{\nabla}_a [n,V]^c + \gamma_{ac}\mathring{\nabla}_b [n,V]^c\right)
\end{aligned} \tag{5.70}$$

after using that $\mathring{\nabla}_c \gamma_{ab} = 0$ (cf. (2.18)) and the well-known property $\pounds_{[X,W]}T = \pounds_X \pounds_W T - \pounds_W \pounds_X T$. Secondly, the gauge transformations of s and $\mathring{\nabla}$ imply

$$\begin{aligned}
\mathring{\nabla}'_a s'_b - \mathring{\nabla}_a s_b &= \mathring{\nabla}'_a \left(s_b + \frac{1}{2}\gamma_{bc}[n,V]^c \right) - \mathring{\nabla}_a s_b \\
&= -\frac{1}{2}(\pounds_V \gamma)_{ab}\, n^c s_c + \frac{1}{2}\mathring{\nabla}'_a (\gamma_{bc}[n,V]^c) \\
&= \frac{1}{2}\mathring{\nabla}_a (\gamma_{bc}[n,V]^c) - \frac{1}{4}(\pounds_V \gamma)_{ab}\, n^c \gamma_{cd}[n,V]^d \\
&= \frac{1}{2}\mathring{\nabla}_a (\gamma_{bc}[n,V]^c) \overset{(2.18)}{=} \frac{1}{2}\gamma_{bc}\mathring{\nabla}_a ([n,V]^c),
\end{aligned} \tag{5.71}$$

from where it follows

$$2\mathring{\nabla}'_{(a} s'_{b)} = 2\mathring{\nabla}_{(a} s_{b)} + \frac{1}{2}\left(\gamma_{bc}\mathring{\nabla}_a [n,V]^c + \gamma_{ac}\mathring{\nabla}_b [n,V]^c\right). \tag{5.72}$$

The combination of (5.70) and (5.72) ensures that $(\pounds_n \mathbf{Y})_{ab} - 2\mathring{\nabla}_{(a} s_{b)}$ is G_1-invariant, and hence so it is $\mathring{\Sigma} - n \otimes \pounds_n \mathbf{Y}$ (recall (3.31)). □

Remark 5.4.13. *Any AKH$_1$ $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ admits a submanifold S to which n is everywhere transverse. If, in addition, $\overline{\eta}$ is everywhere non-zero and one selects the gauge so that $\overline{\eta} = n$, the combination of Lemma 5.4.12 with (5.68) ensures that the gauge-invariant quantity $\mathbf{S}_{\parallel}$, defined by (4.36), is identically zero on S (because of (4.38)).*

5.4.3 *Connection with non-expanding and isolated horizons*

From the considerations above, it is immediate to check that a full Killing horizon is by definition a KH$_{1/2}$. A natural question that arises now is how the previous definitions are connected to the notions of non-expanding horizons, weakly isolated horizons and isolated horizons introduced in Section 2.5 (see Definitions 2.5.1, 2.5.5 and 2.5.8). We devote this section to address this matter.

We start by stressing two important differences between Killing horizons of order zero/one and non-expanding and (weakly) isolated horizons. Firstly, Definitions 5.4.1 and 5.4.5 are purely abstract, and do not assume any spacetime nor any embedding. Secondly, and perhaps more important, these definitions do not make any global assumptions on $\mathcal{N}$, while the non-expanding and (weakly) isolated horizons require (at least in most cases) that $\mathcal{N}$ has a product topology $\mathcal{N} = S \times \mathbb{R}$, where $\mathbb{R}$ is along the null generators.

Having pointed out this fact, we now connect the notion of non-expanding horizon with Definitions 5.4.1 and 5.4.5. As discussed in Remark 2.5.3, a non-expanding horizon $\widetilde{\mathcal{N}}$ is a totally geodesic null hypersurface. This means that $\widetilde{\mathcal{N}}$ constitutes an embedded Killing horizon of order zero, since condition (i) in Definition 5.4.1 is always verified by null generators of $\widetilde{\mathcal{N}}$. A weakly isolated horizon is then an embedded Killing horizon of order zero with symmetry generator η, satisfying the additional restriction (2.111) for the one-form ϖ defined by (2.106). Finally, condition (2.111) can be written in the language of the present section as $\Sigma_\eta(\phi_\star Y, \phi_\star Z) = 0$, $Y, Z \in \Gamma(T\mathcal{N})$ (recall (3.21)). Thus, an isolated horizon constitutes an embedded Killing horizon of order ½.

6

GENERALIZED MASTER EQUATION

In Chapter 2, we have presented the so-called master equation and near horizon equation in the contexts of multiple Killing horizons ((2.144) and (2.153)) and isolated horizons ((2.128) and (2.129)) respectively. These identities relate second derivatives of the proportionality function between the generator η of the horizon and one of its null generators, the one-form ϖ associated to η (cf. (2.106)) and curvature terms.

These master equations, however, hold under very specific conditions. For multiple Killing horizons, one needs that two Killing vectors share the same Killing horizon and that the horizon can be foliated by spacelike cross-sections, and even in these circumstances the master equation is only valid at points when both Killing vectors are non-zero. Isolated horizons, on the other hand, are totally geodesic null hypersurfaces with product topology $\mathbb{S}^{n-1} \times \mathbb{R}$ (see Definition 2.5.1), without expansion and satisfying an energy condition as well as Einstein field equations. Moreover, both the one-form ϖ and the second fundamental form of the horizon with respect to a null, transverse vector field L must be constant along the null generators, and again the master equation applies wherever η is non-zero.

A natural question that arises is whether these equations can be generalized. For instance, one may be interested in horizons with much more general topologies, in null hypersurfaces containing points where the generator η vanishes, or even in less restrictive notions of horizons.

In this chapter, we exploit the formalism of hypersurface data and prove that the master equation can indeed be generalized for any null hypersurface $\mathcal{N}$ equipped with an extra vector field η which is everywhere null and tangent on $\mathcal{N}$. We will obtain a new, fully covariant equation (called generalized master equation) which is valid on the whole $\mathcal{N}$ and that generalizes (2.144). The contractions of such equation with the data vector field n will provide useful information concerning the constancy of the surface gravity of η, as we shall see.

We will also particularize these results to the case when the deformation tensor K^{η} of η is proportional to the metric. In these circumstances, if in addition $K^{\eta} \neq 0$ on $\mathcal{N}$, we shall be able to provide several identities relating the surface gravity of η, the deformation tensor and the constraint tensor (see Chapter 4).

The generalized master equation will also be analyzed for abstract Killing horizons of order zero and one, for which we will prove that if the surface gravity is non-constant at some point of $\mathcal{N}$, then $\mathcal{N}$ cannot be geodesically complete.

The last part of the chapter is devoted to computing the generalized master equation on a transverse submanifold S of $\mathcal{N}$ and to recover the master equations (2.153) and (2.128) from the generalized master equation.

We conclude with an application in the case of vacuum degenerate Killing horizons of order one.

6.1 covariant master equation on a general null hypersurface

We start by deriving a generalized form of the master equation (2.144), valid for a completely general null hypersurface endowed with an extra null, tangent, gauge-invariant vector field $\overline{\eta}$. We do this in the following theorem, in which we also provide the contractions of such generalized master equation with the vector field n.

Theorem 6.1.1. *Consider null hypersurface data* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ *embedded on a semi-Riemannian manifold* $(\mathcal{M}, g)$ *with embedding* ϕ *and rigging* ζ. *Assume further that* $\mathcal{N}$ *admits a gauge-invariant vector field* $\overline{\eta} \in \mathrm{Rad}\gamma$ *and let* $\kappa \in \mathcal{F}(\mathcal{N})$ *be its surface gravity (cf. Lemma 3.3.1) and* $\alpha \in \mathcal{F}(\mathcal{N})$ *be the function given by* $\eta = \alpha n$. *Extend* $\phi_\star\overline{\eta}$ *to a vector field* η *on a neighbourhood* $\mathcal{O}$ *of* $\phi(\mathcal{N})$ *and define its deformation tensor* K^{η}, ק *and* ח^{η} *as in* (5.1), (5.15) *and* (5.40) *respectively. Using the notation* $\boldsymbol{\omega} \stackrel{\mathrm{def}}{=} \boldsymbol{s} - \boldsymbol{r}$, *it holds*

$$
\begin{aligned}
0 = {} & \overset{\circ}{\nabla}_b \overset{\circ}{\nabla}_d \alpha + 2\omega_{(b} \overset{\circ}{\nabla}_{d)} \alpha + \frac{\alpha}{2}\left(2\overset{\circ}{\nabla}_{(b}\omega_{d)} + 2\omega_b\omega_d + R_{bd} - \overset{\circ}{R}_{(bd)} \right) + \kappa Y_{bd} \\
& + \frac{\alpha}{2}\left(\overset{\circ}{\nabla}_{(b} s_{d)} - s_b s_d \right) + \frac{1}{2}\overline{\eta}(\ell^{(2)}) U_{bd} - \alpha W_{bd} - \text{ח}^{\eta}_{bd} + \frac{\text{ק}}{2} U_{bd},
\end{aligned} \tag{6.1}
$$

$$
0 = \overset{\circ}{\nabla}_d \kappa - P^{bf} \overset{\circ}{\nabla}_b (\alpha U_{df}) + \alpha \left(-(\mathrm{tr}_P \mathbf{U})\, \omega_d + \overset{\circ}{\nabla}_d (\mathrm{tr}_P \mathbf{U}) + n^b R_{bd} \right) - \text{ח}^{\eta}_{bd} n^b, \tag{6.2}
$$

$$
0 = \pounds_n \kappa - \text{ח}^{\eta}_{bd} n^b n^d, \tag{6.3}
$$

where W_{bd} is defined by

$$W_{bd} \stackrel{\text{def}}{=} -\frac{1}{2}(\operatorname{tr}_P \mathbf{U})\mathrm{Y}_{bd} - \frac{1}{2}(\operatorname{tr}_P \mathbf{Y})\mathrm{U}_{bd} + P^{ac}\mathrm{U}_{a(b}\left(2\mathrm{Y}_{d)c} + \mathrm{F}_{d)c}\right). \tag{6.4}$$

Remark 6.1.2. *Observe that the notation for the covector $\boldsymbol{\omega}$ defined above is consistent with the one introduced in Lemma 4.3.1 because clearly $\psi^\star \boldsymbol{\omega} = \boldsymbol{\omega}_\parallel$ whenever Setup 3.2.15 holds.*

Proof. Expression (4.18) in Definition 4.2.1 can be rewritten in terms of W_{bd} as

$$\begin{aligned}\pounds_n \mathrm{Y}_{bd} = {}& \overset{\circ}{\nabla}_{(b} r_{d)} - \kappa_n \mathrm{Y}_{bd} - (r_b - s_b)(r_d - s_d) \\ & + \frac{1}{2}\overset{\circ}{R}_{(bd)} - \frac{1}{2}\mathrm{R}_{bd} + \frac{1}{2}\overset{\circ}{\nabla}_{(b}s_{d)} + \frac{1}{2}s_b s_d + W_{bd}.\end{aligned} \tag{6.5}$$

Inserting (6.5) into (5.29) yields

$$\begin{aligned}\pounds_{\overline{\eta}} \mathrm{Y}_{bd} = {}& 2r_{(b}\overset{\circ}{\nabla}_{d)}\alpha + \frac{\alpha}{2}\Big(2\overset{\circ}{\nabla}_{(b} r_{d)} - 2\kappa_n \mathrm{Y}_{bd} - 2(r_b - s_b)(r_d - s_d) \\ & + \overset{\circ}{R}_{(bd)} - \mathrm{R}_{bd} + \overset{\circ}{\nabla}_{(b}s_{d)} + s_b s_d + 2W_{bd}\Big).\end{aligned} \tag{6.6}$$

By comparison with (5.20) one easily obtains (6.1) after reorganizing some of the terms and using (5.39), (6.4) and $n(\alpha) = \kappa - \alpha\kappa_n$. To demonstrate (6.2)-(6.3), we firstly note that (2.20)-(2.21) give $\overset{\circ}{\nabla}_c n^c = \operatorname{tr}_P \mathbf{U}$, $\overset{\circ}{\nabla}_c P^{cf} = -P^{fb}s_b - n^f n(\ell^{(2)})$. Moreover, by (2.44) and (3.11), we write $-n^b\overset{\circ}{\nabla}_{(b}r_{d)}$ as

$$-n^b\overset{\circ}{\nabla}_{(b} r_{d)} = -\frac{1}{2}\pounds_n r_d + \frac{1}{2}\overset{\circ}{\nabla}_d \kappa_n - \kappa_n s_d + P^{bf}\mathrm{U}_{df} r_b. \tag{6.7}$$

Multiplying (6.7) by α and inserting (5.21) gives (recall that $\mathbf{U}(n,\cdot) = 0$)

$$\begin{aligned}-\alpha n^b\overset{\circ}{\nabla}_{(b}r_{d)} = {}& -\frac{1}{2}n^b\Big(2\alpha\overset{\circ}{\nabla}_{(b}s_{d)} + 2s_{(b}(\overset{\circ}{\nabla}_{d)}\alpha) + \overset{\circ}{\nabla}_b\overset{\circ}{\nabla}_d\alpha + n(\alpha)\mathrm{Y}_{bd} - \text{ה}^{\eta}_{bd} + \frac{\text{ק}}{2}\mathrm{U}_{bd}\Big) \\ & - \frac{\kappa_n}{2}\overset{\circ}{\nabla}_d\alpha + \frac{1}{2}n(\alpha)r_d + \frac{\alpha}{2}\overset{\circ}{\nabla}_d\kappa_n - \alpha\kappa_n s_d + \alpha P^{bf}\mathrm{U}_{df} r_b.\end{aligned} \tag{6.8}$$

Now, contracting (6.1) with n^b and inserting (6.8) yields

$$\begin{aligned}0 = {}& n^b\overset{\circ}{\nabla}_b\overset{\circ}{\nabla}_d\alpha - 2n^b(r_{(b} - s_{(b})\overset{\circ}{\nabla}_{d)}\alpha - \alpha n^b\overset{\circ}{\nabla}_{(b}r_{d)} + \frac{\alpha}{2}\Big(2\kappa_n s_d \\ & + n^b\mathrm{R}_{bd} - n^b\overset{\circ}{R}_{(bd)}\Big) + n(\alpha)r_d + \frac{3\alpha}{2}n^b\overset{\circ}{\nabla}_{(b}s_{d)} - \alpha n^b W_{bd} - n^b\text{ה}^{\eta}_{bd} \\ = {}& \frac{1}{2}n^b\overset{\circ}{\nabla}_b\overset{\circ}{\nabla}_d\alpha + \frac{\kappa_n}{2}\overset{\circ}{\nabla}_d\alpha + \frac{1}{2}s_d n(\alpha) + \frac{\alpha}{2}\overset{\circ}{\nabla}_d\kappa_n + \alpha P^{bf}\mathrm{U}_{df} r_b\end{aligned}$$

$$+\frac{\alpha}{2}\left(n^b \mathrm{R}_{bd} - n^b \mathring{R}_{(bd)}\right) + \frac{\alpha}{2} n^b \mathring{\nabla}_{(b} s_{d)} - \alpha n^b W_{bd} - \frac{1}{2} n^b \Pi^{\eta}_{bd}. \tag{6.9}$$

Finally, using the identities (recall (2.20), (6.4), $n(\alpha) = \kappa - \alpha\kappa_n$ and $s(n) = 0$)

$$\begin{aligned}
n^b \mathring{\nabla}_b \mathring{\nabla}_d \alpha = n^b \mathring{\nabla}_d \mathring{\nabla}_b \alpha &= \mathring{\nabla}_d (n^b \mathring{\nabla}_b \alpha) - (\mathring{\nabla}_d n^b) \mathring{\nabla}_b \alpha \\
&= \mathring{\nabla}_d (n(\alpha)) - n(\alpha) s_d - P^{bf} \mathrm{U}_{df} \mathring{\nabla}_b \alpha \\
&= \mathring{\nabla}_d (\kappa - \alpha\kappa_n) - (\kappa - \alpha\kappa_n) s_d - P^{bf} \mathrm{U}_{df} \mathring{\nabla}_b \alpha \\
&= \mathring{\nabla}_d \kappa - \alpha \mathring{\nabla}_d \kappa_n - \kappa_n \mathring{\nabla}_d \alpha - (\kappa - \alpha\kappa_n) s_d - P^{bf} \mathrm{U}_{df} \mathring{\nabla}_b \alpha, && (6.10) \\
n^b \mathring{\nabla}_{(b} s_{d)} &= \frac{1}{2} n^b \left(\mathring{\nabla}_b s_d + \mathring{\nabla}_d s_b \right) \overset{(3.11)}{=} \frac{1}{2} \pounds_n s_d - P^{bf} \mathrm{U}_{df} s_b, && (6.11) \\
n^b W_{bd} &= -\frac{1}{2} (\mathrm{tr}_P \mathbf{U}) r_d + P^{cf} \mathrm{U}_{df} r_c + \frac{1}{2} P^{cf} \mathrm{U}_{df} s_c, && (6.12)
\end{aligned}$$

equation (6.9) becomes (6.2). Contracting (6.2) with n^d immediately yields (6.3) after using (4.20) and $n^d P^{cf} \mathring{\nabla}_c \mathrm{U}_{df} = -\mathrm{U}_{df} P^{cf} \mathring{\nabla}_c n^d = -P^{cf} P^{db} \mathrm{U}_{df} \mathrm{U}_{cb}$. □

Equation (6.1) is a new, fully covariant identity that involves hypesurface data, derivatives of the function α, curvature terms (i.e. R and $\mathring{R}$), the surface gravity κ of $\overline{\eta}$ and the ambient objects {ק, Π^{η}}. Although in fact the appearance of {ק, Π^{η}} in (6.1) makes the whole identity non-purely abstract, it is however remarkable that its whole non-abstract part can be entirely codified only by the tensor Π^{η} and a term in ק$\mathbf{U}$. Observe that precisely the tensor fields $\mathbf{U}$ and Π^{η} are those vanishing for abstract Killing horizons of order zero and one (see Section 5.4).

As claimed before, (6.1) generalizes in several directions the already known forms of near horizon and master equations (see (2.128), (2.144) and (2.153)) that have been previously obtained in the literature. First, (6.1) holds everywhere on the hypersurface and not only on a specific section (in fact, here such a section does not even need to exist).

Secondly, (6.1) does not require any topological assumption on the hypersurface apart from the existence of an everywhere non-zero, smooth vector field n. This also makes a significant difference with respect to the works on isolated and multiple Killing horizons cited in Sections 2.6.1 and 2.5 before. In all those works, the topology of the hypersurface is assumed to be a product of the form $S \times \mathbb{R}$, where S is a cross-section and the null generators are along $\mathbb{R}$. The result (6.1), however, is fully general in this sense and applies for any topology of $\phi(\mathcal{N})$.

Regarding the vector field $\overline{\eta}$ we have also kept maximum generality. We have allowed $\overline{\eta}$ to vanish anywhere on $\mathcal{N}$ and we have enforced neither any specific extension of $\overline{\eta}$ off $\phi(\mathcal{N})$ nor any specific form of the deformation tensor of K^{η} (or of its pull-back to N, given by $\mathbf{U}$). We have neither restricted the one-form $\boldsymbol{\omega}$ or the tensor $\mathbf{Y}$ to satisfy any restriction (observe that in Section 2.5 the Lie derivatives of ϖ and $\mathbf{K}$ along η had to vanish).

Finally, note that (6.1) has been obtained for a general ambient semi-Riemannian manifold (M, g), so the constrain tensor R (and hence the pull-back to N of the ambient Ricci tensor) is fully arbitrary. We have imposed neither energy conditions nor field equations. Moreover, equation (6.1) is valid in any gauge. Later on we will particularize (6.1) for the case of abstract Killing horizons of order zero and one and we will have more to say concerning the comparison of (6.1) with the master equations from Sections 2.6.1 and 2.5.

Equations (6.2)-(6.3) also reinforce the geometric relevance of (6.1), as we shall see next. Specifically, (6.3) allows us to know under which conditions the surface gravity κ remains constant along the null generators of N. This is the content of the following corollary.

Corollary 6.1.3. *Assume the hypotheses of Theorem 6.1.1 and let* $\Sigma_\eta \overset{\text{def}}{=} \pounds_\eta \nabla$ *and* ש *and* ד *be defined according to* (5.15) *and* (5.16) *respectively. Then, the surface gravity* κ *is constant along the integral curves of n if and only if any of the three equivalent conditions hold true:*

$$\Sigma_\eta(\phi_\star n, \phi_\star n) = 0, \tag{6.13}$$

$$\text{ח}^\eta(n,n) = 0, \tag{6.14}$$

$$\pounds_n \text{ש} - \kappa_n \text{ש} - \text{ד}(n,n) = 0. \tag{6.15}$$

Proof. In view of (6.3), it is obvious that κ is constant along the null generators of N if and only if $\text{ח}^\eta(n,n) = 0$, so it suffices to check whether the three conditions (6.13)-(6.15) are indeed equivalent. We first prove this for the last two. Combining (3.11) with the fact that $\mathbf{i}(n) = \text{ש}$, it follows

$$n^b \overset{\circ}{\nabla}_{(b} \mathbf{i}_{d)} = \frac{1}{2}\pounds_n \mathbf{i}_d + \frac{1}{2}\overset{\circ}{\nabla}_d \text{ש} - \text{ש} s_d - P^{bf}\mathrm{U}_{df}\mathbf{i}_b, \tag{6.16}$$

which together with (5.39) allows us to write

$$\text{ח}^\eta_{bd}\, n^b = \frac{1}{2}\pounds_n \mathbf{i}_d - P^{bf}\mathrm{U}_{df}\mathbf{i}_b + \frac{1}{2}\overset{\circ}{\nabla}_d \text{ש} - \text{ש}\omega_d - \text{ד}_{bd}\, n^b \quad \text{and} \tag{6.17}$$

$$\text{ח}^\eta_{bd}\, n^b n^d = \pounds_n \text{ש} + \text{ש}\kappa_n - \text{ד}_{bd} n^b n^d, \tag{6.18}$$

after using $\mathbf{U}(n,\cdot) = 0$. Finally, for any vector field W along $\phi(\mathcal{N})$ decomposed as in (5.31) we can define, as usual, a covector $\boldsymbol{W} \stackrel{\mathrm{def}}{=} g(W,\cdot)$. Then, the combination of (5.32) and (5.39) gives

$$\boldsymbol{W}(\Sigma_\eta(\phi_\star n, \phi_\star n)) = \phi^\star\left(\boldsymbol{W}(\Sigma_\eta)\right)(n,n) = \beta \Pi^\eta(n,n),$$

since the terms in $\overline{W}^c$ vanish because $\mathbf{U}(n,\cdot) = 0$ and (3.46). Since W is a completely general vector field, $\Pi^\eta(n,n) = 0$ is equivalent to $\Sigma_\eta(\phi_\star n, \phi_\star n) = 0$, as claimed. □

The behaviour of the surface gravity κ along the null generators is therefore governed by the tensor Σ_η. In particular, for an abstract Killing horizon of order one according to Definition 5.4.5, $\pounds_n \kappa$ is automatically zero. Observe also that, when the source term $\daleth(n,n)$ is known, (6.15) constitutes a first-order ODE for ψ along the integral curves of n. Concretely, if $\mathcal{N}$ admits a cross-section S then there exists a (unique) solution ψ of (6.15) provided initial data $\psi|_S$. Later on we shall obtain the explicit form of this equation in the specific case when the deformation tensor of η is proportional to the metric.

So far we have considered only constancy of the surface gravity along the generators. It is natural to enquire about its constancy everywhere on the manifold. In that context, the geometric relevance of (6.2) is clear. It allows one to determine precisely under which conditions the surface gravity κ is not only constant along the null generators but everywhere on the hypersurface. In Section 6.3 we will study this identity in detail for the case of abstract Killing horizons of order zero and one. However, for the moment we just include the following general result.

Corollary 6.1.4. *Under the hypotheses of Theorem 6.1.1, the surface gravity κ is constant everywhere on $\mathcal{N}$ if and only if any of the two equivalent equations are satisfied:*

$$0 = P^{bf} \mathring{\nabla}_b(\alpha \mathrm{U}_{df}) + \alpha\Big((\mathrm{tr}_P\, \mathbf{U})\, \omega_d - \nabla_d(\mathrm{tr}_P \mathbf{U}) - \mathrm{R}_{bd}\, n^b \Big) + \Pi^\eta_{bd} n^b, \tag{6.19}$$

$$\begin{aligned} 0 = {} & \frac{1}{2}\pounds_n \mathrm{i}_d + \frac{1}{2}\mathring{\nabla}_d \psi - \psi \omega_d - \daleth_{bd} n^b \\ & + P^{bf}\Big(\mathring{\nabla}_b(\alpha \mathrm{U}_{df}) - \mathrm{U}_{df}\, \mathrm{i}_b \Big) + \alpha\Big((\mathrm{tr}_P\, \mathbf{U})\, \omega_d - \nabla_d(\mathrm{tr}_P \mathbf{U}) - \mathrm{R}_{bd}\, n^b \Big). \end{aligned} \tag{6.20}$$

Proof. Equation (6.19) follows at once from (6.2), while (6.20) is a consequence of combining (6.17) and (6.19). □

Equations (6.1)-(6.2) can be rewritten in such a way that Lie derivatives of the tensors $\mathbf{Y}$ and $\boldsymbol{r} \stackrel{\mathrm{def}}{=} \mathbf{Y}(n,\cdot)$ appear explicitly. We include the corresponding result below, together with a comment on its usefulness.

Lemma 6.1.5. *Equations (6.1)-(6.2) are respectively equivalent to the following two identities:*

$$
\begin{aligned}
0 = {} & \overset{\circ}{\nabla}_b \overset{\circ}{\nabla}_d \alpha + 2\left(s_{(b} - r_{(b} \right) \overset{\circ}{\nabla}_{d)} \alpha - \alpha \pounds_n \mathrm{Y}_{bd} + (\kappa - \alpha \kappa_n) \mathrm{Y}_{bd} \\
& + 2\alpha \overset{\circ}{\nabla}_{(b} s_{d)} + \frac{1}{2}\left(\overline{\eta}(\ell^{(2)}) + \text{ק} \right) \mathrm{U}_{bd} - \Pi^{\eta}_{bd},
\end{aligned} \tag{6.21}
$$

$$
0 = \overset{\circ}{\nabla}_d \kappa + \alpha \left(\pounds_n (s_d - r_d) - \overset{\circ}{\nabla}_d \kappa_n \right) - P^{bc} \mathrm{U}_{cd} \left(\overset{\circ}{\nabla}_b \alpha + 2\alpha s_b \right) - \Pi^{\eta}_{bd} n^b, \tag{6.22}
$$

Proof. Combining (6.1)-(6.2) with (4.18)-(4.19) and (6.4), one gets (6.21)-(6.22) after using that $\boldsymbol{\omega} \stackrel{\text{def}}{=} \boldsymbol{s} - \boldsymbol{r}$. □

In the regions of $\mathcal{N}$ where $\alpha \neq 0$, equations (6.21)-(6.22) are evolution equations for all components of the tensor $\mathbf{Y}$ except $\mathbf{Y}(n, n)$. At first one could think that this term also appears in (6.21) (or in (6.22)). However, this is not the case. Contracting (6.22) with n yields

$$
0 = n(\kappa) - \Pi^{\eta}(n, n) \tag{6.23}
$$

after using $\mathbf{U}(n, \cdot) = 0$, $\boldsymbol{s}(n) = 0$, $\boldsymbol{r}(n) = -\kappa_n$. Equation (6.23) is just (6.1) and does not involve the component $\mathbf{Y}(n, n)$, which therefore cannot be determined.

Equations (6.21)-(6.22) are useful in many situations. The problem of matching two spacetimes across null hypersurfaces offers a clear example of this. As we have discussed in Chapter 2, when the matching between two given spacetimes across their null boundaries is possible, the matter content of the null shell of the resulting spacetime happens to be given by the jump of the tensor fields $\mathbf{Y}$ from each side, so it becomes helpful to be able to compute these tensors. Even more, as we shall discuss later, sometimes more than one matching is allowed (e.g. when the boundaries are totally geodesic) and in that case (6.21)-(6.22) allow one to determine all possible matchings (i.e. all possible matter-contents) at once.

6.2 deformation tensor K^{η} proportional to the metric

We now particularize to the case when the deformation tensor K^{η} of η is proportional to the ambient metric. In the setup of Theorem 6.1.1, this means that in a neighbourhood $\mathcal{O}$ of the null hypersurface $\phi(\mathcal{N})$ it holds

$$
K^{\eta} = 2\chi g, \qquad \text{where} \qquad 2\chi \in \mathcal{F}(\mathcal{O}). \tag{6.24}
$$

In these circumstances, η is a conformal Killing vector on $\mathcal{O}$ (in particular a homothetic vector field or a Killing vector field if $\chi = \text{const.} \neq 0$ and $\chi = 0$ respectively). The function χ necessarily takes the form

$$\chi = \frac{\operatorname{tr}_g \mathrm{K}^{\eta}}{2(n+1)}, \tag{6.25}$$

which follows immediately from taking the trace in (6.24). When evaluating (6.24) on $\phi(\mathcal{N})$ one obtains (cf. (5.15), (5.18))

$$\text{ש} = 2\chi g(\zeta, \nu) = 2\chi, \tag{6.26}$$

$$\text{ק} = 2\chi \ell^{(2)}, \tag{6.27}$$

$$\mathbf{i} = 2\chi \boldsymbol{\ell}, \tag{6.28}$$

$$\alpha \mathbf{U} = \chi \gamma. \tag{6.29}$$

Some important aspects of (6.26)-(6.29) are worth mentioning. First, observe that (6.26) is consistent with the fact that ש is a gauge-invariant function (recall Lemma 5.2.1). Secondly, ק, $\mathbf{i}$ are proportional to $\ell^{(2)}$ and $\boldsymbol{\ell}$ respectively, which means that the choice of gauge plays a fundamental role in the study of the geometry of these sort of hypersurfaces. In particular, we know by Lemma 3.2.9 that in the null case $\{\ell^{(2)}, \ell^{(2)}\}$ can be chosen freely (one can for instance enforce $\ell^{(2)} = 0$). Finally, the combination of (6.29) with the fact that both γ and $\mathbf{U}$ are well-defined and regular tensor fields on $\mathcal{N}$ bring us to the following proposition.

Proposition 6.2.1. *Under the hypotheses of Theorem 6.1.1, assume further that* K^{η} *satisfies* (6.24) *for a function* $\chi \in \mathcal{F}(\mathcal{O})$ *and define* $\mathcal{S} \stackrel{\text{def}}{=} \{p \in \mathcal{N} \mid \alpha(p) = 0\}$, $\mathcal{Z}_{\chi} \stackrel{\text{def}}{=} \{p \in \mathcal{N} \mid \chi(p) = 0\}$. *Then, the following two compatibility conditions must hold:*

$$\textit{(i)}\quad \mathcal{S} \subseteq \mathcal{Z}_{\chi}, \qquad \textit{and} \qquad \textit{(ii)}\quad \chi \textit{ has a zero of at least order } \alpha \textit{ on the whole } \mathcal{S} \tag{6.30}$$

In particular, if $\chi \in \mathbb{R} - \{0\}$ *then* $\mathcal{S} = \emptyset$ *is forced.*

Proof. The manifold $\mathcal{N}$ and the data tensor fields $\{\gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ are assumed to be everywhere smooth, so the tensor field $\mathbf{U}$, which is constructed from the data, is necessarily smooth as well. The proof relies on the fact that (6.29) can be rewritten as

$$\mathbf{U} = \frac{\chi}{\alpha}\gamma \tag{6.31}$$

(again because both γ and $\mathbf{U}$ are regular). Equation (6.31) implies that $\mathbf{U}$ becomes non-smooth at any point $p \in \mathcal{N}$ where $\alpha(p) = 0$ and $\chi(p) \neq 0$ (i.e. where (i) does not hold) or where $\frac{\alpha(p)}{\chi(p)} = 0$ (hence where (ii) is not satisfied). When χ is constant and non-zero everywhere on $\mathcal{N}$ then $Z_\chi = \emptyset$, from where the second part of the lemma follows at once. □

Remark 6.2.2. *Proposition 6.2.1 entails that the function $\chi\alpha^{-1}$ must be smooth everywhere on $\mathcal{N}$. Thus, we define*

$$X \stackrel{\text{def}}{=} \frac{\chi}{\alpha}. \tag{6.32}$$

Remark 6.2.3. *In the null, embedded case, we know that the constraint tensor $\mathcal{R}$ coincides with the pull-back to $\mathcal{N}$ of the ambient Ricci tensor (recall (4.17)). On the other hand, in the abstract definition of the constraint tensor $\mathcal{R}$ for the null case (namely (4.18)), the tensor $\mathbf{U}$ appears on the terms*

$$(\mathrm{tr}_P\mathbf{U})\mathrm{Y}_{ab}, \qquad (\mathrm{tr}_P\mathbf{Y})\mathrm{U}_{ab}, \qquad \mathrm{U}_{d(a}\left(2\mathrm{Y}_{b)c} + \mathrm{F}_{b)c}\right). \tag{6.33}$$

Consequently, if the data tensors $\mathbf{Y}$ and $\mathbf{F}$ are non-zero on $\mathcal{N}$ and any of the compatibility conditions from Proposition 6.2.1 is not satisfied, then $\mathcal{R}$ would become non-smooth at some point on $\mathcal{N}$, and hence there would exist a singularity in the ambient manifold itself.

Remark 6.2.4. *From Proposition 6.2.1 means, it follows that a (smooth) homothetic Killing horizon cannot admit fixed points (i.e. points where the homothetic Killing vector vanishes).*

Our next aim is to particularize the expressions of Theorem 6.1.1 for the present case. For that purpose, we first compute the explicit form of some basic quantities. We start by deriving $\daleth$ explicitly, for which we extend the rigging ζ arbitrarily to $\mathcal{O}$ and compute the derivative $\pounds_\zeta K^\eta|_{\mathcal{O}}$ as

$$\frac{1}{2}\pounds_\zeta K^\eta = \zeta(\chi)g + \chi\pounds_\zeta g. \tag{6.34}$$

Note that the pull back of this quantity to $\mathcal{N}$ is independent of the extension of ζ off $\phi(\mathcal{N})$. Defining[1] $\aleph \stackrel{\text{def}}{=} \phi^\star(\zeta(\chi))$ and computing the pull-back to $\mathcal{N}$ of (6.34) yields

$$\daleth = \aleph\gamma + 2\chi\mathbf{Y}, \quad \Longrightarrow \quad \daleth(n,\cdot) = 2\chi\boldsymbol{r} \quad \Longrightarrow \quad \daleth(n,n) = -2\chi\kappa_n. \tag{6.35}$$

after using (2.39) and (5.16). Now, the combination of (6.29) and (2.8)-(2.9) yields $\alpha(\mathrm{tr}_P\mathbf{U}) = \chi(\mathrm{tr}_P\gamma) = \chi(n-1)$. This, in turn, implies

$$\alpha\mathrm{tr}_P\mathbf{U} = \chi(n-1) \quad \Longrightarrow \quad \alpha\overset{\circ}{\nabla}_d(\mathrm{tr}_P\mathbf{U}) = (n-1)\overset{\circ}{\nabla}_d\chi - (\mathrm{tr}_P\mathbf{U})\overset{\circ}{\nabla}_d\alpha. \tag{6.36}$$

[1] The letter $\aleph$ (/alɛf/) is the first of the Hebrew alphabet.

The following lemma provides three more expressions valid in this context.

Lemma 6.2.5. *In the setup above, the following identities are satisfied:*

$$\alpha n(\mathrm{tr}_P \mathbf{U}) = (n-1)n(\chi) - \kappa(\mathrm{tr}_P\mathbf{U}) + \kappa_n \chi (n-1), \tag{6.37}$$

$$\alpha W_{bd} = -\frac{\chi}{2}\Big((n-5)\mathrm{Y}_{bd} + (\mathrm{tr}_P \mathbf{Y})\gamma_{bd} + 2\,\ell_{(b}s_{d)} - 4\,\ell_{(b}\omega_{d)}\Big), \tag{6.38}$$

$$-P^{bf}\mathring{\nabla}_b(\alpha \mathrm{U}_{df}) = -\mathring{\nabla}_d\chi + \ell_d\big(n(\chi) + \chi\mathrm{tr}_P\mathbf{U}\big). \tag{6.39}$$

Proof. Equation (6.37) follows automatically from (6.36) when using that $n(\alpha) = \kappa - \alpha\kappa_n$. To obtain (6.38), we start from (6.4) and insert (6.29) and (6.36). This yields

$$\begin{aligned}
\alpha W_{bd} &= -\frac{\chi}{2}\Big((n-1)\mathrm{Y}_{bd} + (\mathrm{tr}_P\mathbf{Y})\gamma_{bd} - 2P^{ac}\gamma_{a(b}\big(2\mathrm{Y}_{d)c} + \mathrm{F}_{d)c}\big)\Big)\\
&= -\frac{\chi}{2}\Big((n-1)\mathrm{Y}_{bd} + (\mathrm{tr}_P\mathbf{Y})\gamma_{bd} - 2(\delta^c_{(b} - n^c\,\ell_{(b})\big(2\mathrm{Y}_{d)c} + \mathrm{F}_{d)c}\big)\Big)\\
&= -\frac{\chi}{2}\Big((n-5)\mathrm{Y}_{bd} + (\mathrm{tr}_P\mathbf{Y})\gamma_{bd} + 4\,\ell_{(b}r_{d)} - 2\,\ell_{(b}s_{d)}\Big),\\
&= -\frac{\chi}{2}\Big((n-5)\mathrm{Y}_{bd} + (\mathrm{tr}_P\mathbf{Y})\gamma_{bd} + 2\,\ell_{(b}s_{d)} - 4\,\ell_{(b}\omega_{d)}\Big),
\end{aligned} \tag{6.40}$$

where we used (2.9) and $\boldsymbol{\omega} \stackrel{\mathrm{def}}{=} \boldsymbol{s} - \boldsymbol{r}$ in the second and fourth steps respectively. To demonstrate (6.39), we first note that $-P^{bf}\mathring{\nabla}_b(\alpha\mathrm{U}_{df}) = -P^{bf}(\gamma_{df}\mathring{\nabla}_b\chi + \chi\mathring{\nabla}_b\gamma_{df})$ because of (6.29). Now we use (2.8)-(2.9), (2.18) and the fact that $\mathbf{U}(n,\cdot) = 0$ and find

$$\begin{aligned}
-P^{bf}\mathring{\nabla}_b(\alpha\mathrm{U}_{df}) &= -P^{bf}\Big(\gamma_{df}\mathring{\nabla}_b\chi + \chi(-\ell_d\mathrm{U}_{bf} - \ell_f\mathrm{U}_{bd})\Big)\\
&= -\nabla_d\chi + n(\chi)\,\ell_d + \chi(\mathrm{tr}_P\,\mathrm{U})\,\ell_d,
\end{aligned} \tag{6.41}$$

which is (6.39). □

With the identities above at hand, it is straightforward to particularize (6.1)-(6.3) for the present case.

Lemma 6.2.6. *Under the hypotheses of Theorem 6.1.1, assume further that, on the neighbourhood* $\mathcal{O}$ *of* $\phi(\mathcal{N})$*, it holds* $\mathrm{K}^{\eta} = 2\chi g$ *with* $\chi \in \mathcal{F}(\mathcal{O})$*. Then, equations (6.1)-(6.3) read*

$$\begin{aligned}
0 = {}& \mathring{\nabla}_b\mathring{\nabla}_d\alpha + 2\omega_{(b}\mathring{\nabla}_{d)}\alpha + \frac{\alpha}{2}\Big(2\mathring{\nabla}_{(b}\omega_{d)} + 2\omega_b\omega_d + \mathrm{R}_{bd} - \mathring{R}_{(bd)}\Big)\\
& + \Big(\kappa + \frac{(n-5)\chi}{2}\Big)\mathrm{Y}_{bd} + \bigg(\aleph + \chi\Big(\frac{1}{2}n(\ell^{(2)}) + \frac{1}{2}(\mathrm{tr}_P\mathbf{Y}) + \frac{\chi}{\alpha}\ell^{(2)}\Big)\bigg)\gamma_{bd}\\
& + \frac{\alpha}{2}\Big(\mathring{\nabla}_{(b}s_{d)} - s_bs_d\Big) + \Big(\chi\big(2r_{(b} - s_{(b}\big) - 2\mathring{\nabla}_{(b}\chi\Big)\ell_{d)},
\end{aligned} \tag{6.42}$$

$$0 = \mathring{\nabla}_d\kappa - \chi(n-1)\omega_d + (n-3)\mathring{\nabla}_d\chi + \frac{\chi}{\alpha}\left(\chi\ell_d - \mathring{\nabla}_d\alpha\right)(n-1) + \alpha n^b R_{bd}, \quad (6.43)$$

$$0 = n\,(\kappa - 2\chi)\,. \quad (6.44)$$

Proof. By (2.19) we know that $\mathring{\nabla}_{(b}\ell_{d)} = -\ell^{(2)}U_{bd}$, which together with (6.28) gives

$$\mathring{\nabla}_{(b}\mathrm{i}_{d)} = 2\left(\mathring{\nabla}_{(b}\chi)\,\ell_{d)} - \chi\,\ell^{(2)}U_{bd}\right). \quad (6.45)$$

Taking into account (6.26), (6.27), (6.31) and (6.35), the combination of (6.45) and (5.39) yields

$$\Pi^\eta_{bd} = 2(\mathring{\nabla}_{(b}\chi)\,\ell_{d)} - \aleph\gamma_{bd}, \qquad \Pi^\eta_{bd}\,n^b = n(\chi)\,\ell_d + \mathring{\nabla}_d\chi, \qquad \Pi^\eta_{bd}\,n^b n^d = 2n(\chi). \quad (6.46)$$

Thus, (6.42) follows from inserting (6.38), (6.46) into (6.1) and using again (6.27), (6.31); equation (6.43) is obtained after substituting (6.36), (6.39) and (6.46) into (6.2); and (6.44) is immediate from (6.3) and (6.46). □

Observe that in the present case $\pounds_n\kappa = 0$ if and only if (cf. (6.44))

$$n(\chi) \stackrel{\mathcal{N}}{=} 0, \quad (6.47)$$

i.e. on a conformal Killing horizon the surface gravity κ remains constant along the null generators if and only if χ is also constant along the generators.

Another remarkable consequence of (6.43)-(6.44) is the following algebraic equation for the surface gravity κ.

Proposition 6.2.7. *Assume the hypotheses of Theorem 6.1.1 and suppose further that on a suitable neighbourhood $\mathcal{O}$ of $\phi(\mathcal{N})$ it holds $K^\eta = 2\chi g$ for a function $\chi \in \mathcal{F}(\mathcal{O})$. Define $\tilde{\chi} := \chi\alpha^{-1}$. Then,*

$$\tilde{\chi}\kappa = n(\chi) + \tilde{\chi}\chi + \frac{\alpha}{n-1}R(n,n). \quad (6.48)$$

In particular, if $R(n,n) = 0$ and $n(\chi) = 0$, then $\kappa = \chi$ at any point $p \in \mathcal{N}$ where $\tilde{\chi}(p)$ takes a non-zero value, and hence κ is everywhere constant on $\mathcal{N} \setminus Z_\chi$ if and only if χ is also constant therein.

Proof. Contracting (6.43) with n^d and using $n(\alpha) = \kappa - \alpha\kappa_n$ and (6.44) gives

$$0 = (n-1)n(\chi) + (n-1)\tilde{\chi}(\chi - \kappa) + \alpha R_{bd}n^b n^d$$

which upon dividing by $(n-1)$ becomes (6.48). The second part of the proposition is immediate. □

Observe that if $\chi = 0$ everywhere on $\mathcal{N}$ then (6.48) gives $\mathrm{R}(n, n) = 0$. Since in this case $\mathbf{U} = 0$ (by (6.31)), we simply recover the Raychaudhuri equation (4.20).

An interesting case occurs when χ vanishes no-where on $\mathcal{N}$. Then, by Proposition 6.2.1 we know that $\alpha \neq 0$ everywhere, which allows us to rewrite (6.48) as

$$\kappa = \frac{\alpha n(\chi)}{\chi} + \chi + \frac{\alpha^2 \mathrm{R}(n, n)}{(n-1)\chi} \qquad \Longleftrightarrow \qquad \kappa = \frac{\bar{\eta}(\chi)}{\chi} + \chi + \frac{\mathrm{R}(\bar{\eta}, \bar{\eta})}{(n-1)\chi}. \tag{6.49}$$

Note how the gauge behaviour of the surface gravity κ, the constraint tensor and the function ψ (which in this case coincides with χ, cf. (6.26)) is consistent in (6.49). Actually, every term in (6.49) is gauge-invariant separately (recall that $\bar{\eta}$ is gauge-invariant by hypothesis).

It is also worth stressing that (6.49) entails that *the surface gravity κ of a homothetic vector field $\bar{\eta}$ is everywhere constant on the horizon if and only if* $\mathrm{R}(\bar{\eta}, \bar{\eta})$ *is also constant therein.*

For a better understanding of Proposition 6.2.7 and (6.49), next we include an example of a situation where they apply.

Example 6.2.8. Consider the four dimensional spacetime of Minkowski $(\mathcal{M}, g_{\mathrm{Mk}})$, with metric $g = -dt^2 + dx^2 + dy^2 + dz^2$ (and hence vanishing Ricci tensor **Ric**). In this flat coordinates, the null cone of the origin $\{t = 0, x = 0, y = 0, z = 0\}$ is the null hypersurface $\widetilde{\mathcal{N}}$ defined by

$$\widetilde{\mathcal{N}} \equiv \{0 = -t^2 + x^2 + y^2 + z^2\} \setminus \{t = 0, x = 0, y = 0, z = 0\}.$$

It is straightforward to prove that the vector field $\eta = t\partial_t + x\partial_x + y\partial_y + z\partial_z$ satisfies $\pounds_\eta g = 2g$ everywhere on $\mathcal{M}$ (in particular at $\widetilde{\mathcal{N}}$) and that η is non-zero, null and tangent to $\widetilde{\mathcal{N}}$ at any of its points. The constant χ takes the value $\chi = 1$ so we know (without doing any computation) that the surface gravity is $\kappa = 1$. Indeed, it is straightforward to check that in the present case $\nabla_\eta \eta = \eta$ holds everywhere on $\mathcal{M}$ (in particular on $\phi(\mathcal{N})$). Thus, η is a null generator of $\widetilde{\mathcal{N}}$ with surface gravity $\kappa = 1$. ■

6.3 abstract killing horizons of order zero and one

A particular case, yet of physical and mathematical interest, happens when $\mathcal{N}$ defines an abstract Killing horizon of order zero or one according to Definitions 5.4.1 and 5.4.5. An AKH_0 satisfies that $\mathbf{U} = 0$, which simplifies (6.1)-(6.2) to

$$0 = \mathring{\nabla}_b \mathring{\nabla}_d \alpha + 2\omega_{(b} \mathring{\nabla}_{d)} \alpha + \frac{\alpha}{2}\left(2\mathring{\nabla}_{(b}\omega_{d)} + 2\omega_b\omega_d + R_{bd} - \mathring{R}_{(bd)} \right) + \kappa Y_{bd} - \Pi^{\eta}_{bd}$$
$$+ \frac{\alpha}{2}\left(\mathring{\nabla}_{(b} s_{d)} - s_b s_d \right), \tag{6.50}$$

$$0 = (d\kappa)_d + \alpha R_{bd} n^b - \Pi^{\eta}{}_{bd} n^b. \tag{6.51}$$

Observe that (6.51) gives an alternative proof for the property proved Lemma 5.3.2 that $\Pi^{\eta}(n,\cdot)$ is gauge-invariant whenever $\mathbf{U} = 0$ because κ and R are gauge invariant and α, n change according to $\alpha' = z\alpha$, $n' = z^{-1}n$ (see (2.34) and (3.103)). The simple form of (6.51) is remarkable and leads us to the following result.

Proposition 6.3.1. *Under the hypotheses of Theorem 6.1.1, assume that $\mathcal{N}$ defines an abstract Killing horizon of order zero (cf. Definition 5.4.1) and let $S \overset{\mathrm{def}}{=} \{p \in \mathcal{N} \mid \bar{\eta}|_p = 0\}$. Then, the surface gravity κ of $\bar{\eta}$, given by (3.102), is everywhere constant on $\mathcal{N}$ if and only if*

$$0 \overset{\mathcal{N}}{=} \alpha R(n,\cdot) - \Pi^{\eta}(n,\cdot). \tag{6.52}$$

If, in addition, $\mathcal{N}$ is an abstract Killing horizon of order one (cf. Definition 5.4.5), then κ is everywhere constant on $\mathcal{N}$ if and only if

$$0 = R(n,\cdot) \quad \text{on} \quad \mathcal{N} \setminus S. \tag{6.53}$$

Proof. The first part of the lemma is immediate from (6.51). For the second, it suffices to notice that Π^{η} vanishes on an abstract Killing horizon of order one, as we proved in Lemma 5.4.7. □

Observe that although (6.53) constitutes a strong restriction, it holds in several situations of physical interest, e.g. in vacuum or for non-expanding horizons (see Remark 2.5.4). Our statement on constancy of the surface gravity whenever (6.53) holds extends to the much more general case of Killing horizons of order zero a well-known property of (full) Killing horizons (see e.g. equation (12.5.30) in [113]).

In the next lemma we prove that whenever $\mathcal{N}$ admits a cross-section, if $d\kappa \neq 0$ at some point of $\mathcal{N}$ then $\mathcal{N}$ cannot be geodesically complete.

Proposition 6.3.2. *Under the hypotheses of Theorem 6.1.1, assume that* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ *defines an abstract Killing horizon of order one according to Definition 5.4.5 and that* $\mathcal{N}$ *admits a cross-section* $S \subset \mathcal{N}$*, i.e. a codimension-one embedded hypersurface intersected precisely once by each integral curve of* n*. Then, if* $d\kappa|_p \neq 0$ *at some point* $p \in \mathcal{N}$*, the horizon* $\mathcal{N}$ *cannot be geodesically complete.*

Proof. Recall that the constraint tensor $\mathrm{R}(n, \cdot)$ is smooth everywhere on $\mathcal{N}$ (cf. (4.18)). For an abstract Killing horizon of order one, (6.51) becomes (recall that $\Pi^{\eta} = 0$)

$$0 = d\kappa + \alpha \mathrm{R}(n, \cdot). \tag{6.54}$$

On the other hand, since $\mathcal{N}$ admits a cross-section, we know by Proposition 3.2.23 and Lemma 3.2.24 that there always exists a gauge where $\kappa_n = 0$. We therefore make this choice of gauge and prove the statement by contradiction.

Assume that $\mathcal{N}$ is geodesically complete and that there exists a point $p \in \mathcal{N}$ where $d\kappa|_p \neq 0$. We call C_p the null generator (i.e. the integral curve of n) containing p. Since $n(\kappa) = 0$ (by (6.3)), the value of κ at a null generator is given by its value at one of its points. This, in turn, entails that $\pounds_n(d\kappa) = d(\pounds_n\kappa) = 0$, so $d\kappa$ is also constant along the null generators. Now $\kappa_n = 0$ together with (2.44) and (3.102) yield $n(\alpha) = \kappa$, which is an ODE for α along the null generators of $\mathcal{N}$. The general solution to this equation is

$$\alpha = \kappa\lambda + \alpha_0, \quad \text{where} \quad \lambda, \alpha_0 \in \mathcal{F}(\mathcal{N}) \quad \text{satisfy} \quad n(\lambda) = 1, \quad n(\alpha_0) = 0.$$

We split the analysis in two cases, namely $\kappa|_p \neq 0$ and $\kappa|_p = 0$. If $\kappa|_p \neq 0$, because $\mathcal{N}$ is geodesically complete there exists a point $q \in C_p$ where $\alpha|_q = 0$ (namely the point q where $\lambda|_q = -\kappa^{-1}\alpha_0|_q$). This means that $d\kappa|_q \neq 0$ and $\alpha|_q = 0$, which contradicts (6.54). If, on the contrary, $\kappa|_p = 0$ while $d\kappa|_p \neq 0$, then there exists a point $p' \in \mathcal{N}$ (sufficiently close to p) where $\kappa|_{p'} \neq 0$ while $d\kappa|_{p'} \neq 0$, so we can apply the reasoning above and arrive at the same contradiction. □

Several aspects of Proposition 6.3.2 are worth mentioning. First, although the existence of a cross-section constitutes a global topological restriction (as discussed in Section 2.4), it still allows for physically interesting situations. Secondly, the hypesurface $\mathcal{N}$ does not need to be a "full" Killing horizon but only an abstract Killing horizon of order one, and the symmetry generator η is allowed to vanish anywhere on $\mathcal{N}$. These are in fact two relevant advantages of Proposition 6.3.2, compared with similar results in the literature. For instance, in [101] it was proven that, given a spacetime with a Killing vector η which defines a Killing horizon $\mathcal{H}$, if H contains a null geodesic $\mathcal{C} \subset H$ where $\kappa|_{\mathcal{C}} \neq 0$ and $d\kappa|_{\mathcal{C}} \neq 0$ then necessarily $\mathcal{C}$ terminates in a curvature singularity. However, the proof requires that H has topology $S \times \mathbb{R}$ and that η is no-where zero on H. These two requirements are dropped in Proposition 6.3.2.

6.4 generalized master equation on a transverse submanifold S

One of the main results of this chapter is a generalized form of the master equation on a codimension one smooth submanifold $S \subset \mathcal{N}$ to which n is everywhere transverse. We devote this section to derive this equation and to compare it with the master equations described in Chapter 2 (see (2.153) and (2.128)).

Our starting point is Setup 3.2.15 (where S does not need to be a cross-section) and we use the notation and results of Section 3.2.3. In particular, we identify S, $X \in \Gamma(TS)$ with their respective images $\psi(S)$, $\psi_\star X$ and denote by $\boldsymbol{T}_\parallel$ the pull-back to S of any general p-covariant tensor $\boldsymbol{T}$ along S and by $T_{A_1 \dots A_p}$ its components. As before, we let h be a metric on S, ∇^h be its corresponding Levi-Civita covariant derivative and R^h its Riemann tensor.

The process of deriving the generalized master equation is divided in two parts. In the first one, we compute the pull-back to S of (6.1)-(6.2). However, since (6.1) is written in terms of the curvature tensor $\overset{\circ}{R}_{ab}$, we will still need to rewrite its pull-back equation in terms of the Riemann tensor R^h of h. This constitutes the second step. The following lemma collects the results for the first part.

Lemma 6.4.1. *Under the hypotheses of Theorem 6.1.1, let $S \subset \mathcal{N}$ be a codimension one embedded hypersurface to which n is everywhere transverse. Then, the pull-backs to S of* (6.1)-(6.2) *read*

$$
\begin{aligned}
0 = {} & \nabla^h_A \nabla^h_B \alpha + 2\omega_{(A} \nabla^h_{B)} \alpha + \frac{\alpha}{2}\left(2\nabla^h_{(A}\omega_{B)} + 2\omega_A\omega_B + \mathrm{R}_{AB} - \overset{\circ}{R}_{(AB)} \right) \\
& + \kappa\left(\mathrm{Y}_{AB} - \nabla^h_{(A}\ell_{B)}\right) + \frac{\alpha}{2}\left(\nabla^h_{(A} s_{B)} - s_A s_B \right) - \alpha W_{AB} - \Pi^\eta_{AB} + \frac{\mathcal{P}}{2}\mathrm{U}_{AB} \\
& + \left(\frac{\alpha}{2} n(\ell^{(2)}) - \ell^C \left(\frac{\alpha}{2} s_C + \alpha\omega_C + \nabla^h_C \alpha \right) - \left(\ell^{(2)} - \ell^{(2)}_\parallel\right)\kappa \right) \mathrm{U}_{AB},
\end{aligned} \tag{6.55}
$$

$$
\begin{aligned}
0 = {} & \nabla^h_A \kappa + \alpha\left((\pounds_n \boldsymbol{\omega})_A - \nabla^h_A \kappa_n \right) \\
& + \left(2\alpha s_C + \ell^B(\kappa - \alpha\kappa_n) - h^{BC}\nabla^h_C \alpha \right) \mathrm{U}_{AB} - \psi^\star\left(\Pi^\eta(n,\cdot)\right)_A.
\end{aligned} \tag{6.56}
$$

Proof. We recall that $\boldsymbol{r}(n) = -\kappa_n$, $\mathbf{U}(n,\cdot) = 0$, $\boldsymbol{s}(n) = 0$ and $n(\alpha) = \kappa - \alpha\kappa_n$. Particularizing (3.94) for $\mathrm{T}_a = \mathring{\nabla}_a\alpha$, $\mathrm{T}_a = r_a$ and $\mathrm{T}_a = s_a$ gives

$$v^a_A v^b_B \mathring{\nabla}_a \mathring{\nabla}_b \alpha = \nabla^h_A \nabla^h_B \alpha - n(\alpha)\nabla^h_{(A}\ell_{B)} - \left((\ell^{(2)} - \ell^{(2)}_{\parallel}) n(\alpha) + \ell^C \nabla^h_C \alpha \right) \mathrm{U}_{AB}, \tag{6.57}$$

$$v^a_A v^b_B \mathring{\nabla}_{(a} r_{b)} = \nabla^h_{(A} r_{B)} + \kappa_n \nabla^h_{(A}\ell_{B)} + \left((\ell^{(2)} - \ell^{(2)}_{\parallel}) \kappa_n - \ell^C r_C \right) \mathrm{U}_{AB}, \tag{6.58}$$

$$v^a_A v^b_B \mathring{\nabla}_{(a} s_{b)} = \nabla^h_{(A} s_{B)} - \ell^C s_C \mathrm{U}_{AB}, \tag{6.59}$$

while the combination of (6.58)-(6.59) entails, in turn,

$$v^a_A v^b_B \mathring{\nabla}_{(a} \omega_{b)} = \nabla^h_{(A} \omega_{B)} - \kappa_n \nabla^h_{(A}\ell_{B)} - \left((\ell^{(2)} - \ell^{(2)}_{\parallel}) \kappa_n + \ell^C \omega_C \right) \mathrm{U}_{AB}. \tag{6.60}$$

Inserting (6.57) and (6.59)-(6.60) into (6.1) yields (6.55). The proof of (6.56) is based on computing the pull-back of (6.22) to S. The only non-trivial term in (6.22) is $P^{bc}\mathrm{U}_{cd}\left(\nabla_b\alpha + 2\alpha s_b\right)$, so it suffices to elaborate its pull-back. This is done by means of the decomposition (3.50) of P (here $h = h$ and $\psi_A = \ell_A$), from where one obtains

$$\begin{aligned} v^d_A P^{bc}\mathrm{U}_{cd}\left(\mathring{\nabla}_b\alpha + 2\alpha s_b\right) &= v^d_A v^c_B \mathrm{U}_{cd}\left(h^{CB} v^b_C - \ell^B n^b\right)\left(\mathring{\nabla}_b\alpha + 2\alpha s_b\right) \\ &= \mathrm{U}_{AB}\left(h^{BC}\nabla^h_C\alpha - 2\alpha s_C - \ell^B n(\alpha)\right). \end{aligned}$$

With the identity above, it is straightforward to get (6.56) from (6.22). $\square$

Equation (6.55) already constitutes a generalized form of master equation on the transverse submanifold S. We now write (6.55) in terms of the Ricci tensor R^h of the metric h on S.

Theorem 6.4.2. *Under the hypotheses of Theorem 6.1.1, let $S \subset \mathcal{N}$ be a codimension one embedded hypersurface to which n is everywhere transverse. Then, (6.55) can be written in terms of the curvature tensor R^h of ∇^h as*

$$\begin{aligned} 0 = {} & \nabla^h_A \nabla^h_B \alpha + 2\omega_{(A}\nabla^h_{B)}\alpha + \frac{\alpha}{2}\left(2\nabla^h_{(A}\omega_{B)} + 2\omega_A\omega_B + \mathrm{R}_{AB} - R^h_{AB}\right) \\ & + \left(\kappa + \frac{\alpha}{2}(\mathrm{tr}_h \mathbf{U}_{\parallel})\right)\left(\mathrm{Y}_{AB} - \nabla^h_{(A}\ell_{B)}\right) - \text{ח}^{\eta}_{AB} + \frac{\text{ק}}{2}\mathrm{U}_{AB} - \left(\alpha\, \ell^C \ell^D \mathrm{U}_{CD} + \frac{1}{2}\nabla^h_C \ell^C\right. \\ & \text{¥} \\ & + \ell^C\nabla^h_C\alpha + \left(\kappa + \frac{\alpha}{2}\mathrm{tr}_h\mathbf{U}_{\parallel} - \frac{\alpha}{2}\kappa_n\right)\left(\ell^{(2)} - \ell^{(2)}_{\parallel}\right) - \left.\frac{\alpha}{2}\mathrm{tr}_h\mathbf{Y}_{\parallel}\right)\mathrm{U}_{AB} \\ & - \frac{\alpha}{2}(\ell^{(2)} - \ell^{(2)}_{\parallel})(\pounds_n\mathbf{U})_{AB} - \alpha\,\ell^C\left(\nabla^h_C\mathrm{U}_{AB} + 2\omega_{(A}\mathrm{U}_{B)C} - \nabla^h_{(A}\mathrm{U}_{B)C}\right) \\ & + \alpha\left(h^{CD}(\ell^{(2)} - \ell^{(2)}_{\parallel}) + \ell^C\ell^D\right)\mathrm{U}_{AC}\mathrm{U}_{BD} + \frac{\alpha}{2}h^{CD}\left(2(\nabla^h_C\ell_{(A})\mathrm{U}_{B)D} - 4\mathrm{U}_{D(A}\mathrm{Y}_{B)C}\right). \end{aligned} \tag{6.61}$$

Proof. The proof of (6.61) is based on two ingredients, namely the substitution of (4.29) into (6.55) and the computation of W_{AB}. Inserting (3.78)-(3.79) into (6.4), we get

$$
\begin{aligned}
W_{AB} = & -\frac{1}{2}(\mathrm{tr}_h \mathbf{U}_{\parallel})\mathrm{Y}_{AB} - \frac{1}{2}\left(\mathrm{tr}_h \mathbf{Y}_{\parallel} - 2\,\ell^{C} r_C + \kappa_n\left(\ell^{(2)} - \ell^{(2)}_{\parallel}\right)\right)\mathrm{U}_{AB} \\
& + h^{CD}\mathrm{U}_{D(A}\left(2\mathrm{Y}_{B)C} + \mathrm{F}_{B)C} + (s_{B)} - 2r_{B)})\,\ell_C\right).
\end{aligned}
\tag{6.62}
$$

Moreover, the combination of (2.19) and Lemma 3.2.21 one obtains

$$
\mathrm{F}_{AB} = v^a_A v^b_B \mathrm{F}_{ab} = v^a_A v^b_B \overset{\circ}{\nabla}_a \ell_b + \ell^{(2)}\mathrm{U}_{AB} = \nabla^h_{[A}\ell_{B]}.
$$

Using the results above in the process of substituting (4.29) into (6.55) yields (6.61) after a cumbersome but straightforward calculation. □

We emphasize that the derivation of (6.61) does not require S to be a cross-section (in fact, such cross-section does not need to exist in this context). This, together with the reasons exposed in Section 6.1, already makes equation (6.61) a remarkable generalization of the master equations (2.153) and (2.128).

Our next aim is to establish a comparison between (6.61) and the previous forms of master equations valid for isolated horizons and multiple Killing horizons. Since both cases give rise, at the abstract level, to abstract Killing horizons of order one (see the discussion in Section 5.4.3), it is convenient to particularize (6.61) to this case by enforcing $\mathbf{U} = 0$ and $\Pi^{\eta} = 0$. This yields

$$
\begin{aligned}
0 \stackrel{S}{=} & \nabla^h_A \nabla^h_B \alpha + 2\omega_{(A}\nabla^h_{B)}\alpha + \frac{\alpha}{2}\left(2\nabla^h_{(A}\omega_{B)} + 2\omega_A\omega_B + \mathrm{R}_{AB} - R^h_{AB}\right) \\
& + \kappa\left(\mathrm{Y}_{AB} - \nabla^h_{(A}\ell_{B)}\right).
\end{aligned}
\tag{6.63}
$$

The corresponding comparison of (6.63) with (2.153) and (2.128) is collected in the following two remarks.

Remark 6.4.3. *In Section 2.6.1, the derivation of the master equation (2.153) requires two Killing vector fields η_1, η_r which are null, non-zero and tangent to the horizon $H_{1,r}$, i.e. they are null generators of $H_{1,r}$. These vector fields, in addition, have constant surface gravities κ_1, $\kappa_r = 0$ and are related by $\eta_r = \alpha_r \eta_1$ on $H_{1,r}$ (see (2.139)-(2.140)).*

Equation (2.153) is presented in terms of (i) the one-form $\boldsymbol{\varpi}$ associated to η_1 (cf. (2.106)) and (ii) the scalar function α given by (2.140). Thus, in order to recover (2.153) from (6.63) we need to find the explicit expression of $\boldsymbol{\varpi}$ and α in terms of tensor fields and functions at the abstract level.

In the present case the embedding ϕ *and the abstract hypersurface* $\mathcal{N}$ *need to satisfy* $\phi(\mathcal{N}) = H_{1,r}$*, and it turns out that the pull-back of* ϖ *to* $\mathcal{N}$ *coincides precisely with the one-form* $\boldsymbol{\omega}$*. Indeed, the choice of transverse vector field* L *made in* (2.112) *corresponds, in the formalism of hypersurface data, to select the gauge so that* $\eta_1 = \phi_\star n$ *and* $\boldsymbol{\ell}_A = 0$*. This, together with Remark* 2.4.11 *(note that the function* α *in that remark is the proportionality function between* $\phi_\star n$ *and* η_1*, so it is equal to one), means that the one-form* ϖ *associated to* η_1 *satisfies*

$$\phi^\star \varpi = \boldsymbol{s}_{-r} \stackrel{\text{def}}{=} \boldsymbol{\omega}. \tag{6.64}$$

The function α*, on the other hand, must be the proportionality function between* η_r *and* $\phi_\star n$ *(which in this case is just* η_1*). Thus,* $\alpha = \mathfrak{a} e^{-\kappa_1^{-1} v}$ *(cf.* (2.140)*), and since the derivatives of* $e^{-\kappa_1^{-1} v}$ *along directions tangent to* S *are zero, one can write* (6.63) *as*

$$0 \stackrel{S}{=} \nabla^h_A \nabla^h_B \mathfrak{a} + 2\omega_{(A} \nabla^h_{B)} \mathfrak{a} + \frac{\mathfrak{a}}{2}\left(2\nabla^h_{(A}\omega_{B)} + 2\omega_A\omega_B + R_{AB} - R^h_{AB} \right), \tag{6.65}$$

where we have taken into account the fact that η_r *is degenerate which, together with Proposition* 3.3.2 *means that* $\kappa = \phi^\star \kappa_r = 0$*. Equation* (6.65) *is precisely* (2.153) *since the constraint tensor in the null case is simply the pull-back to* $\mathcal{N}$ *of the ambient Ricci tensor.*

Remark 6.4.4. *The master equation* (2.128) *from Section* 2.5 *only requires a privileged null generator* η *of the horizon* $\overline{\mathcal{N}}$*, with surface gravity* κ*. As before, the embedding* ϕ *and the abstract hypersurface* $\mathcal{N}$ *need to satisfy* $\phi(\mathcal{N}) = \overline{\mathcal{N}}$*, and the pull-back of* ϖ *to* $\mathcal{N}$ *coincides precisely with the one-form* $\boldsymbol{\omega}$*. This, again, is a consequence of the choice of transverse vector field* L *made in* (2.112)*, which amounts to enforce* $\eta = \phi_\star n$ *and* $\boldsymbol{\ell}_A = 0$*. Since in this case the master equation is for the generator* η*, here the function* α *of* (6.63) *is equal to one, so it does not appear in the master equation. Finally, the restriction of* $\mathbf{K}^L$ *to* S *gives the second fundamental form of* S *with respect to* L *(recall* (2.64)*). This, in combination with* (2.65)*, means that* $\mathbf{K}^L|_S$ *is just the pull-back* $\psi^\star \mathbf{Y}$*, so we can rewrite* (6.63) *as*

$$0 \stackrel{S}{=} \kappa Y_{AB} + \nabla^h_{(A}\omega_{B)} + \omega_A\omega_B + \frac{1}{2}\left(R_{AB} - R^h_{AB} \right). \tag{6.66}$$

after reorganizing some terms. Equation (6.66) *coincides with* (2.128)*, again because* R *is the pull-back to* $\mathcal{N}$ *of the ambient Ricci tensor.*

For its use in Chapter 8, we now obtain the explicit form of the components of $\mathbf{Y}$ in the case when $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ defines a non-degenerate abstract Killing horizon of order one with constant surface gravity κ.

Proposition 6.4.5. *Assume the hypotheses and setup of Theorem* 6.1.1 *and let* $S \stackrel{\text{def}}{=} \{p \in \mathcal{N}$ *such that* $\bar{\eta}|_p = 0\}$*. Suppose further that* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ *defines an abstract Killing* *horizon of order one with non-zero constant surface gravity κ, and that $\mathcal{N}$ can be foliated by a family of diffeomorphic cross-sections. Take one such cross-section S and construct a basis $\{n, v_A\}$ of $\Gamma(T\mathcal{N})$ by taking a basis $\{v_A\}$ of $\Gamma(TS)$ and requiring $\pounds_n v_A = 0$. If, in addition, $\kappa_n = 0$, $\boldsymbol{s} = 0$ and $\boldsymbol{\ell}_A \stackrel{\text{def}}{=} \boldsymbol{\ell}(v_A) = 0$, the components of the tensor $\mathbf{Y}$ in the basis $\{n, v_A\}$ are given by*

$$
\begin{aligned}
\mathbf{Y}(n,n) &= 0, && (6.67)\\
\mathbf{Y}(n,v_A) &= \mathbf{Y}(n,v_A)|_S, && (6.68)\\
\mathbf{Y}(v_A,v_B) &= -\frac{1}{\kappa}\left(\nabla^h_A\nabla^h_B\alpha - 2r_{(A}\nabla^h_{B)}\alpha\right)\\
&\quad+\frac{\alpha}{2\kappa}\left(R^h_{AB} - \mathrm{R}_{AB} + 2\nabla^h_{(A}r_{B)} - 2r_Ar_B\right). && (6.69)
\end{aligned}
$$

Remark 6.4.6. *We know from Proposition 3.2.23, Lemmas 3.2.24 and 3.2.25 and Remark 3.2.26 that one can select the gauge so that $\boldsymbol{s} = 0$, $\kappa_n = 0$ and $\boldsymbol{\ell}_A|_S = 0$. Moreover, since the basis $\{n, v_A\}$ is constructed so that $\pounds_n v_A = 0$, it holds that $\pounds_n\boldsymbol{\ell}_A = (\pounds_n\boldsymbol{\ell})(v_A) = 2\boldsymbol{s}(v_A) = 0$ (cf. (3.43)), so in these circumstances we can find a gauge where $\{\boldsymbol{s} = 0, \kappa_n = 0, \boldsymbol{\ell}_A = 0\}$ everywhere on $\mathcal{N}$.*

Proof. Let $\lambda \in \mathcal{F}(\mathcal{N})$ be the unique foliation function satisfying $\lambda|_S = 0$, $n(\lambda) = 1$. Then, the vector fields $\{v_A\}$ are tangent to the leaves $S_\lambda \stackrel{\text{def}}{=} \{\lambda = \text{const.}\}$, since

$$0 = (\pounds_n v_A)(\lambda) = n(v_A(\lambda)) - v_A(n(\lambda)) = n(v_A(\lambda)) \quad \Longrightarrow v_A(\lambda) = v_A(\lambda)|_S = 0.$$

Equation (6.67) follows at once because $\kappa_n = 0$ (recall (2.44)). On the other hand, from (6.22) and using that $\kappa = \text{const.}$, $\mathbf{U} = 0$ and $\boldsymbol{\Pi}^\eta = 0$, it follows

$$\alpha\pounds_n\boldsymbol{r} = 0 \qquad \Longrightarrow \qquad \pounds_n\boldsymbol{r} = 0, \tag{6.70}$$

where the implication is a consequence of the fact that $\mathcal{N}\setminus S$ is dense in $\mathcal{N}$. Since $\pounds_n v_A = 0$, it is immediate that (6.70) implies $\boldsymbol{r}(v_A) = \boldsymbol{r}(v_A)|_S$, which proves (6.68) after again taking into account (2.44). Finally, in the present case equation (6.63) must hold for all leaves $\{S_\lambda\}$, hence everywhere on $\mathcal{N}$. Particularizing it for $\boldsymbol{\ell}_A = 0$ and constant non-zero κ gives (6.69) (recall that $\omega_A \stackrel{\text{def}}{=} s_A - r_A = -r_A$). □

6.5 vacuum degenerate killing horizons of order one

As an application of the results of Chapters 5 and 6, we consider the particular case of $\mathcal{N}$ defining a vacuum degenerate abstract Killing horizon of order one without assuming that the horizon can be foliated by spacelike sections. The analysis will reveal a version of the near horizon equation in a quotient space.

Our setup is the following. We consider an abstract Killing horizon of order one $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ with everywhere non-zero symmetry generator $\overline{\eta}$ (cf. Definition 5.4.5). We assume that $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ is embedded in a spacetime $(\mathcal{M}, g)$ with embedding ϕ and rigging ζ, and we let η be an extension of $\eta \stackrel{\text{def}}{=} \phi_\star \overline{\eta}$ to a neighbourhood $\mathcal{O}$ of $\phi(\mathcal{N})$. The fact that $\overline{\eta}$ vanishes no-where on $\mathcal{N}$ means that $\mathcal{N}$ cannot contain fixed points and that the proportionality function between $\overline{\eta}$ and n, which we have called α before, is equal to one. Moreover, it allows us to select the gauge so that $\overline{\eta} = n$, which automatically forces the rigging ζ to satisfy $g(\eta, \zeta) = 1$. Observe that since $\mathcal{N}$ is an AKH$_1$ it holds that $\mathbf{U} = 0$ and that the tensor $\Sigma_\eta \stackrel{\text{def}}{=} \pounds_\eta \nabla$ satisfies $\Sigma_\eta(\phi_\star X, \phi_\star W) = 0$, $X, W \in \Gamma(T\mathcal{N})$.

We let $(\mathcal{M}, g)$ be a spacetime admitting a Killing vector field η which defines a Killing horizon $\mathcal{H} \subset \mathcal{M}$ according to Definition 2.6.1.

As indicated elsewhere in this thesis, one can define the surface gravity $\underline{\kappa}$ of $\eta|_{\phi(\mathcal{N})}$ according to (2.81). Moreover, from Proposition 3.3.2 we know that the (abstract) surface gravity κ of $\overline{\eta}$ coincides with the pull-back of $\underline{\kappa}$ to $\mathcal{N}$, i.e. $\kappa = \phi^\star \underline{\kappa}$. Since in the present case $\overline{\eta} = n$, from (3.102) it also holds (recall that $\alpha = 1$)

$$\kappa = -\mathbf{Y}(n, n) \stackrel{\text{def}}{=} \kappa_n.$$

When no assumptions on the topology of $\mathcal{N}$ are made, it is not possible in general to select a global cross-section. However, one can can always introduce a quotient space $\mathcal{N}/\sim$ of equivalent classes under the equivalence relation

$$p, q \in \mathcal{N}, \quad p \sim q \quad \Longleftrightarrow \quad q \text{ and } p \text{ belong to the same integral curve of } n. \tag{6.71}$$

In general $\mathcal{N}/\sim$ is not a smooth manifold. Nevertheless, the case when $\mathcal{N}/\sim$ is smooth is of special interest because, while still allowing for a simple treatment, it includes not only all cases where a global section exists but also topologically non-trivial cases. Since our aim in this section is to provide an example where our previous results apply, it makes sense to restrict ourselves to this situation. We therefore assume from now that $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ constitutes a degenerate[2] abstract Killing horizon of order one such that $\mathcal{N}/\sim$ is a smooth manifold. We moreover assume that the data is *vacuum, i.e. that the constraint tensor* $\mathcal{R}_{ab}$ *vanishes identically.*

[2] Recall the terminology introduced in Section 2.6 according to which degenerate Killing horizons are those for which $\underline{\kappa}$ (and hence κ) vanishes identically.

It is well-known that covariant tensor fields in the quotient are in one to one-correspondence with covariant tensor fields on $\mathcal{N}$ that are completely orthogonal to n and are Lie constant along n. For a tensor $T_{a_1\cdots a_p}$ satisfying $T(\cdot\cdot\cdot, n, \cdot\cdot\cdot) = 0$ and $\pounds_n T = 0$, we shall denote with a hat the corresponding tensor in the quotient. The only exception in this notation is γ which indeed satisfies $\gamma(n,\cdot) = 0$ and $\pounds_n\gamma = 0$ (because $\mathbf{U} = 0$), but for which the corresponding tensor on $\mathcal{N}/\sim$ will be denoted with h. A reasoning analogous to that of Section 2.4.1 allows one to prove that h is a positive definite metric on $\mathcal{N}/\sim$, so we also define its associated covariant derivative ∇^{h} and Ricci tensor R^{h}.

By Lemma 3.2.7 the tensor $\boldsymbol{\omega} \stackrel{\mathrm{def}}{=} \boldsymbol{s} - \boldsymbol{r}$ is gauge invariant under the subgroup $\mathcal{G}_1$. This tensor satisfies $\boldsymbol{\omega}(n) = 0$ (because $\boldsymbol{\omega}(n) = \kappa_n$). By (4.19) together with $R_{ab} = 0$ and $\kappa_n = \kappa = 0$ we also have $\pounds_n\boldsymbol{\omega} = 0$. Thus, this tensor descends to the quotient. The following result determines the field equations that h and $\hat{\boldsymbol{\omega}}$ need to satisfy.

Proposition 6.5.1. *Let* $\{\mathcal{N},\gamma,\boldsymbol{\ell},\ell^{(2)},\mathbf{Y}\}$ *be a vacuum degenerate abstract Killing horizon of order one and assume that the quotient* $\mathcal{N}/\sim$ *is a smooth manifold. Then the metric* h *and the covector* $\hat{\boldsymbol{\omega}}$ *satisfy the near horizon equation, namely*

$$R^{h}_{AB} - \nabla^{h}_{A}\hat{\omega}_B - \nabla^{h}_{B}\hat{\omega}_A - 2\hat{\omega}_A\hat{\omega}_B = 0. \tag{6.72}$$

Proof. The projection $\pi : \mathcal{N} \longrightarrow \mathcal{N}/\sim$ is a submersion. Consider any point $\hat{p} \in \mathcal{N}/\sim$ and select a point $p \in \mathcal{N}$ satisfying $\pi(p) = \hat{p}$. There always exists a local section S_p of $\mathcal{N}$ near p (i.e. a non-degenerate embedded hypersurface S containing p). We let $\psi : S_p \hookrightarrow \mathcal{N}$ denote the corresponding embedding. Define $U_{\hat{p}} \stackrel{\mathrm{def}}{=} \pi(S_p)$ and $\hat{\pi} \stackrel{\mathrm{def}}{=} \pi \circ \psi$. This map is a diffeomorphism between S_p and $U_{\hat{p}}$ and satisfies $\hat{\pi}^{\star} h = h$. Moreover, $\hat{\pi}^{\star}\hat{\boldsymbol{\omega}} = \boldsymbol{\omega}_{\parallel}$, which follows at once from $\pi^{\star}\hat{\boldsymbol{\omega}} = \boldsymbol{\omega}$ and $\psi^{\star}\boldsymbol{\omega} = \boldsymbol{\omega}_{\parallel}$. Moreover, the tensor $\mathbf{S}_{\parallel}$ vanishes identically as a consequence of (4.37) together with (5.62) (recall that $\alpha = 1$). Proposition 4.3.2 in the present context yields the equation

$$R^{h}_{AB} - \nabla^{h}_{A}\omega_B - \nabla^{h}_{B}\omega_A - 2\omega_A\omega_B = 0. \tag{6.73}$$

Applying $(\hat{\pi}^{-1})^{\star}$ to this equation gives (6.72) on $U_{\hat{p}}$. Since the point $\hat{p}$ is arbitrary, the equation holds everywhere on $\mathcal{N}/\sim$. □

7

MATCHING FROM A SPACETIME VIEWPOINT

In this chapter, we start addressing the problem of matching two completely general spacetimes across a null hypersurface. The analyisis of this problem consists of two distinct parts. In the first (corresponding to Chapters 7 and 8) we approach the problem from a spacetime perspective, i.e. without considering the matching hypersurfaces in a detached way from the ambient spaces. The main assumption that we shall require is that the boundaries of the spacetimes to be matched can be foliated by a family of spacelike cross-sections. Chapter 7 focuses on the matching problem across null hypersurfaces in a general context while Chapter 8 is devoted to the problem of matching across embedded abstract Killing horizons of order zero.

In the second part (namely Chapter 9), we adopt a fully abstract approach in order to provide a completely abstract formulation of the matching problem. We do this in a much more general framework, e.g. by refraining ourselves from making any topological assumption on the boundaries. This abstract viewpoint, as we shall see, is advantageous for various reasons.

The structure of the chapter is as follows. In Section 7.1 we obtain some prior results that are needed later in the chapter. In Section 7.2, we include a brief discussion on the problem of matching in the general case, namely for boundaries of arbitrary causal character. Section 7.3 constitutes the main part of the chapter and focuses on the null case. We first rewrite the standard matching conditions in terms of a basis of vector fields. Then we identify the necessary and sufficient conditions that allow for the matching. In Section 7.3.1, we demonstrate that all the information about the matching is encoded in a scalar function called *step function* and in a diffeomorphism between the set of null generators of each side. We also study the circumstances in which an infinite number of matchings are feasible, which occurs in particular whenever the boundaries are totally geodesic. In Section 7.3.2 we obtain explicit expressions for the matter-energy content of the most general null shell resulting from the matching. We conclude the chapter by particularizing the results to the matching of two regions of the spacetime of Minkowski across a null hyperplane. This allows us to connect the matching formalism with the cut-and-paste constructions (see e.g. [3], [85], [86], [87], [5]).

7.1 prior considerations and setup

So far, all the results we have presented in Chapters 3, 4, 5 and 6 (except those in Section 6.5) apply to arbitrary signature either of the tensor $\boldsymbol{A}$ in the purely abstract setting or of the semi-Riemannian ambient manifold $(\mathcal{M}, g)$ in the embedded case. The only restriction of course is that this signature cannot be positive or negative definite whenever we deal with null (metric) hypersurface data (cf. Lemma 3.1.1).

From now on we shall concentrate on the matching problem of two spacetimes across null hypersurfaces. Thus, for the purposes of this chapter we shall assume that $(\mathcal{M}, g)$ is a spacetime with a null boundary $\widetilde{\mathcal{N}}$. As discussed in Section 2.4, in such case $\widetilde{\mathcal{N}}$ is two-sided. We follow the notation in Definition 2.4.1 and let $\phi : \mathcal{N} \hookrightarrow \mathcal{M}$ be the embedding of the corresponding abstract hypersurface $\mathcal{N}$ in $\mathcal{M}$ (i.e. $\phi(\mathcal{N}) = \widetilde{\mathcal{N}}$) and γ be the first fundamental form (i.e. $\gamma = \phi^\star g$). By construction, γ is semi-positive definite[1] (non-null directions tangent to $\widetilde{\mathcal{N}}$ are all spacelike) and $\widetilde{\mathcal{N}}$ always admits an everywhere transversal vector field $L_0 \in \Gamma(T\mathcal{M})|_{\widetilde{\mathcal{N}}}$.

The vector field L_0 defines a rigging of $\widetilde{\mathcal{N}}$, and it can always be taken to be null everywhere. Indeed, given a null generator k of $\widetilde{\mathcal{N}}$, L_0 being transversal means that $g(L_0, k) \neq 0$ everywhere. Thus,

$$L \stackrel{\text{def}}{=} L_0 - \frac{g(L_0, L_0)}{2g(L_0, k)} k \tag{7.1}$$

is both transversal (because $g(L, k)|_p \neq 0$ at every point $p \in \widetilde{\mathcal{N}}$) and null (which follows from squaring (7.1)). We emphasize, however, that the choice of a transversal vector field is non-unique, even when we fix it to be null.

As mentioned above, in this chapter we shall assume that the boundaries of the spacetimes to be matched have product topology $S \times \mathbb{R}$ with the null generators along $\mathbb{R}$. This implies that they can be foliated by diffeomorphic spacelike sections (see Section 2.4). In these circumstances, it becomes helpful to introduce a basis of $\Gamma(T\mathcal{M})|_{\widetilde{\mathcal{N}}}$ adapted to the foliation. Given a spacelike cross-section $S \subset \widetilde{\mathcal{N}}$, we

[1] Of course this is in agreement with Lemma 3.1.1 because in the present case g has Lorentzian signature, and hence so does the ambient metric $\boldsymbol{A}$.

construct a foliation function $v \in \mathcal{F}(\mathcal{N})$ (see Definition 2.4.4) and a basis $\{L, k, v_I\}$ of $\Gamma(TM)|_{\mathcal{N}}$ adapted to the foliation as follows:

$$\begin{aligned}
&\text{(A)} \quad k \text{ is a future null generator with surface gravity } \kappa_k. \\
&\text{(B)} \quad v \in \mathcal{F}(\mathcal{N}) \text{ is the only foliation function satisfying } v \stackrel{S}{=} 0,\ k(v) \stackrel{\mathcal{N}}{=} 1. \\
&\text{(C)} \quad \text{Each vector field } v_I \text{ is tangent to the foliation, i.e. } v_I(v) = 0. \\
&\text{(D)} \quad \text{The basis vectors } \{k, v_I\} \text{ are such that } [k, v_I] = 0 \text{ and } [v_I, v_J] = 0. \\
&\text{(E)} \quad L \text{ is a past null vector field everywhere transversal to } \mathcal{N}.
\end{aligned} \tag{7.2}$$

Remark 7.1.1. *The basis vector fields $\{k, v_I\}$ can always be constructed. Indeed, given a cross-section $S \subset \mathcal{N}$ and a choice of null generator k, one can take a basis $\{v_I\}$ of $\Gamma(TS)$ and extend its vector fields uniquely to $\mathcal{N}$ by enforcing $\pounds_k v_I = 0$. Then*

$$0 = (\pounds_k v_I)(v) = k(v_I(v)) - v_I(k(v)) = k(v_I(v)) \quad \Longrightarrow \quad v_I(v)|_{\mathcal{N}} = v_I(v)|_S = 0,$$

hence the vector fields $\{v_I\}$ are tangent to the foliation and satisfy $[k, v_I] = 0$. If, in addition, we take $\{v_I\}$ on S so that $[v_I, v_J]|_S = 0$, it follows that

$$\pounds_k(\pounds_{v_I} v_J) = \pounds_{v_I}(\pounds_k v_J) + \pounds_{[k,v_I]} v_J = 0, \tag{7.3}$$

so $[v_I, v_J] = 0$ holds everywhere on $\mathcal{N}$.

As usual, we denote the leaves of the foliation by $\{S_v\}$, i.e. we let $S_{v_0} \stackrel{\text{def}}{=} \{p \in \mathcal{N} \mid v(p) = v_0 \in \mathbb{R}\}$. Although the topology of $\mathcal{N}$ allows us to take k affine (see the discussion in Section 2.4), for the moment we refrain ourselves from enforcing $\kappa_k = 0$. Observe that all vector fields $\{v_A\}$ are by costruction spacelike and that $\{k, v_I\}$ constitutes a basis of $\Gamma(T\mathcal{N})$. The fact that k is future, together with $k(v)|_{\mathcal{N}} = 1$, means that the foliation function v increases towards the future of $\mathcal{N}$.

Following the notation of 2.4, we let h be the induced metric on the leaves $\{S_v\}$, and denote its components in the basis $\{v_I\}$ by h_{IJ}, i.e.

$$h_{IJ} \stackrel{\text{def}}{=} g(v_I, v_J). \tag{7.4}$$

The components of the inverse metric $h^\sharp$ of h in the dual basis $\{\boldsymbol{\theta}^I\}$ of $\{v_I\}$ are h^{IJ} and, just as before, we use h^{IJ}, h_{IJ} to lower and raise Capital Latin indices. We also define the second fundamental form $\mathbf{K}^k$ of $\mathcal{N}$ with respect to the normal k

according to (2.84). By the property $[k, v_I] = 0$ satisfied by the basis vectors $\{v_I\}$, (2.85) can be rewritten in the present case as

$$k\big(h(v_I, v_J)\big) \stackrel{\mathcal{N}}{=} 2\mathbf{K}^k(v_I, v_J). \tag{7.5}$$

As we did in (2.99), we introduce the tensor field $\boldsymbol{\Theta}^L$ and the one-form $\boldsymbol{\sigma}_L$ on the leaves $\{S_v\}$ (recall that $\boldsymbol{\Theta}^L_{\pm}$ is not symmetric in general because L does not need to be normal to the leaves $\{S_v\}$). For any basis $\{L, k, v_I\}$ verifying (7.2), we also define n scalar functions $\{\mu_a\} \subset \mathcal{F}(\mathcal{N})$ as

$$\mu_1(p) \stackrel{\text{def}}{=} g(L,k)|_p, \qquad \mu_I(p) \stackrel{\text{def}}{=} g(L, v_I)|_p \qquad \forall p \in \mathcal{N}. \tag{7.6}$$

Although clearly the functions $\{\mu_a\}$ depend on the choice of the basis vectors $\{L, k, v_I\}$, for the sake of simplicity we do not reflect this dependence in the notation. Observe that necessarily $\mu_1 \neq 0$ everywhere on $\mathcal{N}$ (in fact, since L is past and k is future $\mu_1 > 0$ is forced, recall (A), (E) in (7.2)).

It may seem strange not to restrict L to satisfy $\mu_I = 0$, i.e. to be orthogonal to the leaves of the foliation. The reason is that there are many cases where the most convenient choice of L (e.g. to simplify the computations) does not verify $\mu_I = 0$. An explicit example where choosing L non-orthogonal to the leaves turned out to be useful appears in [5]. The functions μ_I in that paper happen to be the currents $J(U, \eta, \bar{\eta})$ and $\bar{J}(U, \eta, \bar{\eta})$, which play a fundamental role in the physical description of the impulsive gravitational wave associated to the matching.

For later purposes, it is convenient to provide the explicit form of several covariant derivatives with respect to the vector fields k and v_I. Since $\nabla_k k$ is given by (2.82) and $\nabla_k v_I = \nabla_{v_I} k$ (cf. (7.2)), we only require $\nabla_{v_I} v_J$, $\nabla_{v_I} k$, $\nabla_k k$, $\nabla_{v_I} L$ and $\nabla_k L$. When L is normal to the sections, the corresponding expressions can be found e.g. in [125]. The general form when L need not be orthogonal to the leaves was obtained for the first time in our paper [103]. These expressions can be regarded as an expanded form of equations (19) and (21) in [64].

Lemma 7.1.2. *Let $\mathcal{N}$ be an embedded null hypersurface admitting a foliation $\{S_v\}$ given by $v \in \mathcal{F}(\mathcal{N})$. Consider a basis $\{L, k, v_I\}$ of $\Gamma(TM)|_{\mathcal{N}}$ satisfying conditions (7.2). Then, the tangential derivatives of the basis vectors read:*

$$\nabla_{v_I} v_J = -\frac{1}{\mu_1}\mathbf{K}_k(v_I, v_J)\, L + \frac{1}{\mu_1}\Big(v_I(\mu_J) - \boldsymbol{\Theta}^L(v_I, v_J) - \Xi^K_{JI}\,\mu_K \Big) k + \Xi^K_{JI}\, v_K, \tag{7.7}$$

$$\nabla_k v_I = \nabla_{v_I} k = -\left(\boldsymbol{\sigma}_L(v_I) + \frac{\mu^J}{\mu_1}\mathbf{K}^k(v_I, v_J) \right) k + \mathbf{K}^k(v_I, v^J) v_J, \tag{7.8}$$

$$\nabla_k k = \kappa_k k \tag{7.9}$$

$$
\begin{aligned}
\nabla_{v_I} L &= \eta_I L - \frac{\mu^J}{\mu_1}\left(\boldsymbol{\Theta}^L(v_I, v_J) - \eta_I \mu_J\right) k + \left(\boldsymbol{\Theta}^L(v_I, v^J) - \eta_I \mu^J\right) v_J, && (7.10)\\
\nabla_k L &= \left(\frac{k(\mu_1)}{\mu_1} - \kappa_k\right)\left(L + \frac{\mu^I \mu_I}{\mu_1} k - \mu^I v_I\right) \\
&\quad - \left(k(\mu_I) + \mu_1 \boldsymbol{\sigma}_L(v_I)\right)\left(\frac{\mu^I}{\mu_1} k - v^I\right), && (7.11)
\end{aligned}
$$

where Ξ^K_{JI} *and* η_I *are defined by*

$$
\begin{aligned}
\Xi^K_{JI} &\stackrel{\text{def}}{=} \left(g(v^K, \nabla_{v_I} v_J) + \frac{\mu^K}{\mu_1} \mathbf{K}^k(v_I, v_J)\right), && (7.12)\\
\eta_I &\stackrel{\text{def}}{=} \left(\frac{v_I(\mu_1)}{\mu_1} + \boldsymbol{\sigma}_L(v_I)\right). && (7.13)
\end{aligned}
$$

Remark 7.1.3. *The vector field* $L + \frac{\mu^I \mu_I}{\mu_1} k - \mu^I v_I$ *is orthogonal to both* v_J *and* L*, whereas* $\frac{\mu^I}{\mu_1} k - v^I$ *is orthogonal to* L*.*

Proof. We start with $\nabla_{v_I} v_J$. For suitable scalar functions α_{IJ}, β_{IJ} and Ξ^K_{JI} this derivative can be expressed as $\nabla_{v_I} v_J = \alpha_{IJ} L + \beta_{IJ} k + \Xi^K_{JI} v_K$. Using (7.6), it follows that

$$
\begin{aligned}
\langle k, \nabla_{v_I} v_J\rangle_g &= \alpha_{IJ}\mu_1, &&\Longleftrightarrow & \alpha_{IJ} &= \frac{1}{\mu_1}\langle k, \nabla_{v_I} v_J\rangle_g, \\
\langle v_L, \nabla_{v_I} v_J\rangle_g &= \alpha_{IJ}\mu_L + \Xi^K_{JI} h_{KL}, &&\Longleftrightarrow & \Xi^K_{JI} &= \langle v^K, \nabla_{v_I} v_J\rangle_g - \alpha_{IJ}\mu^K, && (7.14)\\
\langle L, \nabla_{v_I} v_J\rangle_g &= \beta_{IJ}\mu_1 + \Xi^K_{JI}\mu_K, &&\Longleftrightarrow & \beta_{IJ} &= \frac{1}{\mu_1}\left(\langle L, \nabla_{v_I} v_J\rangle_g - \Xi^K_{JI}\mu_K\right),
\end{aligned}
$$

which, together with (2.97) and (2.99), gives

$$
\begin{aligned}
\alpha_{IJ} &= -\frac{1}{\mu_1}\langle v_J, \nabla_{v_I} k\rangle_g = -\frac{1}{\mu_1}\mathbf{K}^k(v_I, v_J), \\
\Xi^K_{JI} &= \langle v^K, \nabla_{v_I} v_J\rangle_g + \frac{\mu^K}{\mu_1}\mathbf{K}^k(v_I, v_J), && (7.15)\\
\beta_{IJ} &= \frac{1}{\mu_1}\left(v_I(\mu_J) - \langle \nabla_{v_I} L, v_J\rangle_g - \Xi^K_{JI}\mu_K\right) = \frac{1}{\mu_1}\left(v_I(\mu_J) - \boldsymbol{\Theta}^L(v_I, v_J) - \Xi^K_{JI}\mu_K\right)
\end{aligned}
$$

and hence (7.7). We can repeat the process for the derivative $\nabla_{v_I} k$ and decompose it as $\nabla_{v_I} k = \alpha_I L + \beta_I k + \xi_I^L v_L$. Then,

$$
\begin{aligned}
\alpha_I &= \frac{1}{\mu_1}\langle k, \nabla_{v_I} k\rangle_g = 0, \\
\xi_I^L &= \langle v^L, \nabla_{v_I} k\rangle_g = \mathbf{K}^k(v_I, v^L), && (7.16)
\end{aligned}
$$

$$\beta_I = \frac{1}{\mu_1}\Big(\langle L, \nabla_{v_I} k \rangle_g - \varepsilon^K_I \mu_K \Big) = - \Big(\boldsymbol{\sigma}_L (v_I) + \frac{\mu^J}{\mu_1} \mathbf{K}^k (v_I, v_J) \Big) .$$

Substituting in $\nabla_{v_I} k$ gives (7.8). Equation (7.9) is simply (2.82). On the other hand, decomposing $\nabla_{v_I} L = \vartheta_I L + \nu_I k + \rho^L_I v_L$ one obtains

$$\begin{aligned}
\vartheta_I &= \frac{1}{\mu_1}\langle k, \nabla_{v_I} L\rangle_g = \frac{1}{\mu_1}\Big(v_I(\mu_1) - \langle \nabla_{v_I} k, L\rangle_g \Big) = \frac{v_I(\mu_1)}{\mu_1} + \boldsymbol{\sigma}_L(v_I), \\
\rho^L_I &= \langle v^L, \nabla_{v_I} L\rangle_g - \mu^L \vartheta_I = \boldsymbol{\Theta}^L(v_I, v^L) - \frac{\mu^L}{\mu_1} v_I(\mu_1) - \mu^L \boldsymbol{\sigma}_L(v_I), \qquad (7.17)\\
\nu_I &= \frac{1}{\mu_1}\Big(\langle L, \nabla_{v_I} L\rangle_g - \rho^J_I \mu_J \Big) = -\frac{\rho^J_I \mu_J}{\mu_1} = -\frac{\mu_J}{\mu_1}\Big(\boldsymbol{\Theta}^L(v_I, v_J) - \frac{\mu^J}{\mu_1} v_I(\mu_1) - \mu^J \boldsymbol{\sigma}_L(v_I) \Big) .
\end{aligned}$$

Using the definitions (2.99) and (7.13) and inserting the results into $\nabla_{v_I} L$ proves (7.10). Finally, writing $\nabla_k L = aL + bk + c^I v_I$ yields

$$a\mu_1 = \langle k, \nabla_k L\rangle_g = k(\mu_1) - \kappa_k \mu_1, \qquad (7.18)$$

$$b\mu_1 + c^I \mu_I = \langle L, \nabla_k L\rangle_g = 0, \qquad (7.19)$$

$$a\mu_J + c_J = \langle v_J, \nabla_k L\rangle_g = k(\mu_J) - \langle L, \nabla_k v_J\rangle_g = k(\mu_J) + \mu_1 \boldsymbol{\sigma}_L(v_J) . \qquad (7.20)$$

Equation (7.18) immediately provides a, while from (7.19) one gets $b = -\frac{c^I \mu_I}{\mu_1}$. Multiplying (7.20) by h^{JK} gives $c^I = \Big(-\kappa_k - \frac{k(\mu_1)}{\mu_1}\Big)\mu^I + h^{IJ} k(\mu_J) + \mu_1 \boldsymbol{\sigma}_L(v^I)$, and the substitution of a, b, c^I on $\nabla_k L$ proves (7.11). □

Remark 7.1.4. *A straightforward calculation yields*

$$\Xi^K_{JI} = \frac{1}{2} h^{KA}\Big(v_I(h_{AJ}) + v_J(h_{AI}) - v_A(h_{IJ}) \Big) + \frac{\mu^K}{\mu_1} \mathbf{K}^k (v_I, v_J) . \qquad (7.21)$$

Proof. The proof is based on the fact that $[v_I, v_J] = 0$. A direct computation gives

$$\begin{aligned}
2g(v_A, \nabla_{v_I} v_J) &= g(v_A, \nabla_{v_I} v_J) + g(v_A, \nabla_{v_J} v_I) \\
&= v_I(h_{AJ}) - g(v_J, \nabla_{v_I} v_A) + v_J(h_{AI}) - g(v_I, \nabla_{v_J} v_A) \\
&= v_I(h_{AJ}) + v_J(h_{AI}) - g(v_J, \nabla_{v_A} v_I) - g(v_I, \nabla_{v_A} v_J) \\
&= v_I(h_{AJ}) + v_J(h_{AI}) - v_A(h_{IJ})
\end{aligned}$$

from where (7.21) follows at once after using (7.12). □

7.2 matching of spacetimes with boundaries of any causality

We devote the rest of the chapter to the problem of matching two spacetimes with boundary. In particular, in this section we discuss briefly the general case (namely when the boundaries have any causal character) while in later sections we focus on the matching across null boundaries, which constitutes the core topic of the second part of this thesis.

Consider two given spacetimes $(\mathcal{M}^{\pm}, g^{\pm})$ with boundaries $\widetilde{\mathcal{N}}^{\pm}$ of any causal character. As we mentioned in Section 2.7, the matching of $(\mathcal{M}^{\pm}, g^{\pm})$ across $\widetilde{\mathcal{N}}^{\pm}$ is possible if and only if the so-called *junction conditions* or *matching conditions* are satisfied. In the language of the formalism of hypersurface data, we already know from Theorem 2.7.1 that the junction conditions require that there exists a metric hypersurface data set that can be embedded in both spacetimes so that the riggings $\zeta^{\pm}$ to be identified in the matching process satisfy an orientation condition.

It should be emphasized however, that this formulation of the matching conditions in terms of metric hypersurface data is a reformulation of the traditional one, which can be called "standard" or "*à là Darmois*" matching conditions. In the standard *à là Darmois* matching procedure, the junction conditions constitute a set of equalities that provide information about the identification between points of the boundaries $\widetilde{\mathcal{N}}^{\pm}$ and between the tangent spaces $T_{\widetilde{\mathcal{N}}^{\pm}}\mathcal{M}^{\pm}$, together with the mentioned restriction upon the orientation of two transverse vector fields that are identified in the matching process. Specifically, the junction conditions in the traditional way are formulated as follows (see e.g. [64]).

Standard Junction Conditions. *The matching of two given spacetimes* $(\mathcal{M}^{\pm}, g^{\pm})$ *with boundaries* $\widetilde{\mathcal{N}}^{\pm}$ *can be performed if and only if*

(i) *There exist two riggings* $\zeta^{\pm}$ *along* $\widetilde{\mathcal{N}}^{\pm}$ *and a diffeomorphism* $\Phi : \widetilde{\mathcal{N}}^{-} \longrightarrow \widetilde{\mathcal{N}}^{+}$ *such that*

$$\begin{aligned} \Phi^{\star}(g^{+}) &= g^{-}, \\ \Phi^{\star}(g^{+}(\zeta^{+}, \cdot)) &= g^{-}(\zeta^{-}, \cdot), \\ \Phi^{\star}(g^{+}(\zeta^{+}, \zeta^{+})) &= g^{-}(\zeta^{-}, \zeta^{-}). \end{aligned} \tag{7.22}$$

(ii) *One rigging must point inwards with respect to its boundary and the other outwards.*

In the following, two riggings $\zeta^{\pm}$ satisfying (7.22) for a diffeomorphism Φ will be called *matching riggings*. The diffeomorphism Φ, on the other hand, will be referred to as *matching map*.

Observe that if (7.22) holds for one pair of riggings $\zeta^{\pm}$ then, for any other choice of rigging on one of the sides, conditions (7.22) are also fulfilled. Indeed, one can take a function $z \in \mathcal{F}^{\star}(\mathcal{N}^{-})$ and a vector $V \in \Gamma(T\mathcal{N}^{-})$ and construct another rigging $\zeta'^{-} = z(\zeta^{-} + V)$ on the minus side, and it is straightforward to check that the rigging $\zeta'^{+} \stackrel{\mathrm{def}}{=} \hat{z}(\zeta^{+} + \hat{V})$ with $\hat{z} \stackrel{\mathrm{def}}{=} (\Phi^{-1})^{\star} z$ and $\hat{V} \stackrel{\mathrm{def}}{=} \Phi_{\star} V$ also verifies (7.22). Since the same logic obviously applies if one decides to change the rigging on the plus side, it follows that the rigging can always be selected at will on one of the sides (although of course different choices of rigging on one side will correspond to different riggings on the other side). For the rest of the chapter, we shall make use of this freedom to fix ζ^{-} at our convenience. This entails no loss of generality, as one can always switch the names of the spacetimes to be matched.

As proven e.g. in Lemmas 2 and 3 of [65], given a rigging on one side (say ζ^{-}) and a diffeomorphism $\Phi : \mathcal{N}^{-} \longrightarrow \mathcal{N}^{+}$ satisfying $\Phi^{\star} g^{+} = g^{-}$, at non-null points the second and third equations of (7.22) yield either no solution for ζ^{+} (hence the matching is not possible) or two solutions for ζ^{+} with opposite orientation. At null points, on the other hand, if there exists a solution ζ^{+} then it is unique. This means that at non-null points one can always make a suitable choice of rigging ζ^{+} so that the junction condition (*ii*) is fulfilled (as long as (7.22) provide a solution). In the null case, however, this is not so. Since both riggings must be identified in the matching process (which in particular force them to point into the same side of the matching hypersurface on the resulting spacetime), it could well happen that there exists a solution ζ^{+} of (7.22) with unsuitable orientation, and then the matching could not be performed. Thus, at null points conditions (7.22) are necessary but not sufficient to guarantee that the matching can be performed. As we mentioned in Section 2.7, this can also be understood within the framework of the formalism of hypersurface data, and has to do with the fact that there exist two gauge group elements leaving a fully non-null metric hypersurface data set invariant, and only one when the data contains a null point.

When a matching of two spacetimes $(\mathcal{M}^{\pm}, g^{\pm})$ is possible, the associated matching map Φ turns out to be the key object upon which the whole matching depends. This is so because once the point-to-point identification of the boundaries $\mathcal{N}^{\pm}$ (ruled by Φ) is known, one matching rigging can be selected at will (as we have seen) and the other is simply the unique solution that arises from enforcing both (7.22) and (*ii*). This fact is relevant, since it implies that all the information about the resulting thin shell (e.g. the matter-energy or the purely gravitational content) is fully codified in Φ (in fact, this is the underlying reason why e.g. in all cut-and-paste constructions the whole matching information is codified in a specific jump in the coordinates that takes place at the matching hypersurface).

We have stated that the standard junction conditions can also be rewritten in the language of the formalism of hypersurface data according to Theorem 2.7.1. For the sake of self-consistency, we now justify how Theorem 2.7.1 follows from the standard junction conditions.

For one of the boundaries, say $\tilde{N}^-$, we can consider an abstract manifold N and an embedding $\phi^- : N \hookrightarrow M^-$ such that $\phi^-(N) = \tilde{N}^-$. We can also take any rigging ζ^- along $\tilde{N}^-$ and define embedded metric hypersurface data $\{N, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ according to (2.22). Should the matching be possible, there must exist a matching map Φ. In that case, one can define yet another embedding $\phi^+ : N \hookrightarrow \tilde{N}^+$ by $\Phi \circ \phi^- \stackrel{\mathrm{def}}{=} \phi^+$. The condition on the existence of a matching map Φ is therefore equivalent to requiring existence of such extra embedding ϕ^+ of N in M^+. For a (still unknown) rigging vector field ζ^+ along $\tilde{N}^+$, conditions (7.22) can be expressed in terms of ϕ^+ as

$$(\phi^+)^\star(g^+) = \gamma, \qquad (\phi^+)^\star(g^+(\zeta^+, \cdot)) = \boldsymbol{\ell}, \qquad (\phi^+)^\star(g^+(\zeta^+, \zeta^+)) = \ell^{(2)} \quad (7.23)$$

after applying the pull-back $(\phi^-)^\star$ to (7.22) and using that $\{N, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ is embedded in (M^-, g^-) with embedding ϕ^- and rigging ζ^-. Thus, the matching requires $\{N, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ to be embedded also in (M^+, g^+) with embedding ϕ^+ and rigging ζ^+, as was claimed in Theorem 2.7.1. For the reasons explained above, one must require in addition that one matching rigging points inwards and the other outwards, with completes the justification of Theorem 2.7.1. For later purposes, we write (7.23) in terms of both ϕ^- and ϕ^+:

$$\begin{aligned}
\gamma &= (\phi^-)^\star(g^-) = (\phi^+)^\star(g^+), \\
\boldsymbol{\ell} &= (\phi^-)^\star(g^-(\zeta^-, \cdot)) = (\phi^+)^\star(g^+(\zeta^+, \cdot)), \\
\ell^{(2)} &= (\phi^-)^\star(g^-(\zeta^-, \zeta^-)) = (\phi^+)^\star(g^+(\zeta^+, \zeta^+)).
\end{aligned} \quad (7.24)$$

The same way as one of the riggings (in our case ζ^-) can always be chosen at will, it is always possible to select one of the embeddings freely. It suffices to adapt the abstract manifold N to one of the boundaries. In the following we shall make use of this freedom by fixing ϕ^- at our convenience. Thus, all the information about the matching (which was encoded in the matching map Φ in the spacetime picture) will be codified in the (unknown) embedding ϕ^+, which becomes the core object upon which the matching depends.

To summarize, given two spacetimes with boundary, determining whether they can be matched amounts to finding two embeddings of an abstract manifold N onto their respective boundaries, in such a way that the matching conditions are

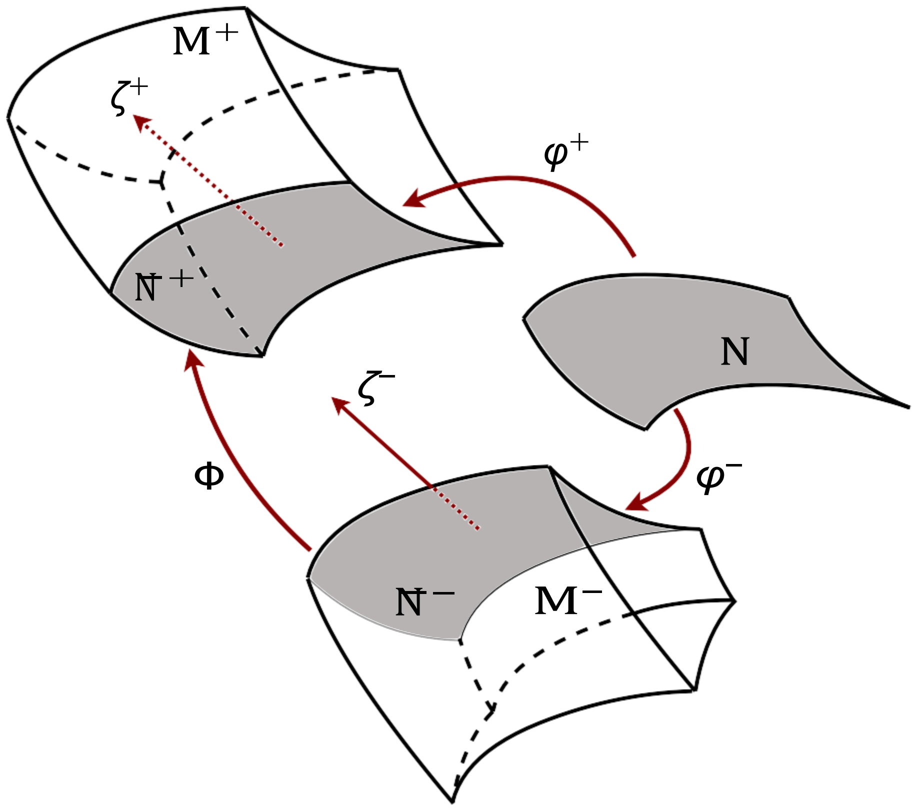


Figure 7.1: Matching of two spacetimes $(\mathcal{M}^{\pm}, g^{\pm})$ with boundaries $\mathcal{N}^{\pm}$ of any causality, where $\mathcal{N}$ is an abstract hypersurface embedded in $(\mathcal{M}^{\pm}, g^{\pm})$ with embeddings $\phi^{\pm}$, $\zeta^{\pm}$ are matching riggings and Φ is the corresponding matching map.

fulfilled (i.e. that the corresponding metric hypersurface data agree). We include Figure 7.1 for a schematic picture of the construction. The embeddings and the rigging vectors are not known or given a priori (although they can be freely chosen on one of the sides). In many circumstances such embeddings do not exist, and then the two spacetimes simply cannot be matched. In other cases, there exists several (even infinite) possible embeddings, giving rise to a certain number of joined spacetimes, which in general are different from each other (we discuss this later in Section 7.3.1 and in Chapters 8 and 9).

When the junction conditions are satisfied, the geometry of the shell is determined by the jump of the transverse tensors $\mathbf{Y}^{\pm}$ defined as (cf. (2.39))

$$\mathbf{Y}^{\pm} \stackrel{\mathrm{def}}{=} \frac{1}{2}(\phi^{\pm})^{\star} \pounds_{\zeta^{\pm}} g^{\pm}, \qquad \text{namely} \qquad [\mathbf{Y}] \stackrel{\mathrm{def}}{=} \mathbf{Y}^{+} - \mathbf{Y}^{-}. \tag{7.25}$$

7.3 matching of spacetimes across a null hypersurface

For the rest of the chapter we concentrate on the null case, i.e. we assume that the boundaries $\mathcal{N}^{\pm}$ consist of null points exclusively.

Later we shall make use of the freedom of selecting one of the matching riggings $\zeta^{\pm}$ by enforcing that ζ^{-} is null and past. We will also need to compute the explicit expression for $\zeta^{\pm}$ in terms of a basis of vector fields adapted to the boundaries. In order to avoid complications with the signs of the components of $\zeta^{\pm}$ in the adapted basis, we take the following precaution. If the null past rigging ζ^{-} points inwards (resp. outwards) with respect to $(\mathcal{M}^{-}, g^{-})$ and the null riggings along $\mathcal{N}^{+}$ which point outwards (resp. inwards) with respect to $(\mathcal{M}^{+}, g^{+})$ happen to be future, we change the time-orientation of $(\mathcal{M}^{+}, g^{+})$ before doing the matching. If this change of orientation was not done a priori, then it would have to be done a posteriori after the matching was performed. Otherwise the resulting spacetime would not have a well-defined notion of past and future at points on the matching hypersurface. In the already matched spacetime, we would be allowed to change the orientation to either the $\mathcal{M}^{-}$ or to the $\mathcal{M}^{-}$ side, so it may seem that the choice we make of changing the orientation of $\mathcal{M}^{+}$ before matching entails some loss of generality. However, this is not so because after we have performed the matching it is always possible to change the orientation everywhere. In summary, the choice we make is able to recover all matchings, and removes some spurious signs that would complicate unnecessarily the presentation. This change, in the end, is equivalent to assuming that *one of the boundaries lies in the future of its corresponding spacetime while the other lies in its spacetime past.* Diagrams (a)-(d) in the left part of Figure 7.2 depict all scenarios where a null past rigging ζ^{-} pointing inwards (resp. outwards) is identified with a null future rigging ζ^{+} pointing outwards (resp. inwards). Diagrams (a)-(d) in the right part of Figure 7.2, on the other hand, show the corresponding matchings once the time-orientation of $(\mathcal{M}^{+}, g^{+})$ has been changed.

For the rest of the section, our setup will be the following.

Setup 7.3.1. *We let $(\mathcal{M}^{\pm}, g^{\pm})$ be two spacetimes with null boundaries $\mathcal{N}^{\pm}$ that can be foliated by a family of diffeomorphic spacelike cross-sections. We construct respective foliation functions $v_{\pm} \in \mathcal{F}(\mathcal{N}^{\pm})$ and basis $\{L^{\pm}, k^{\pm}, v_{I}^{\pm}\}$ of $\Gamma(T\mathcal{M}^{\pm})|_{\mathcal{N}^{\pm}}$ according to* (7.2)*. The leaves of the foliations, as usual, are denoted by $\{S_{v_{\pm}}\}$, while their corresponding induced metrics are $h^{\pm}$. We also let $\mathbf{K}^{k}_{\pm}$ be the second fundamental forms of $\mathcal{N}^{\pm}$ with respect to the normals $k^{\pm}$ (cf.* (2.84)*), and introduce the tensor fields $\boldsymbol{\Theta}^{L}_{\pm}$ and the one-*

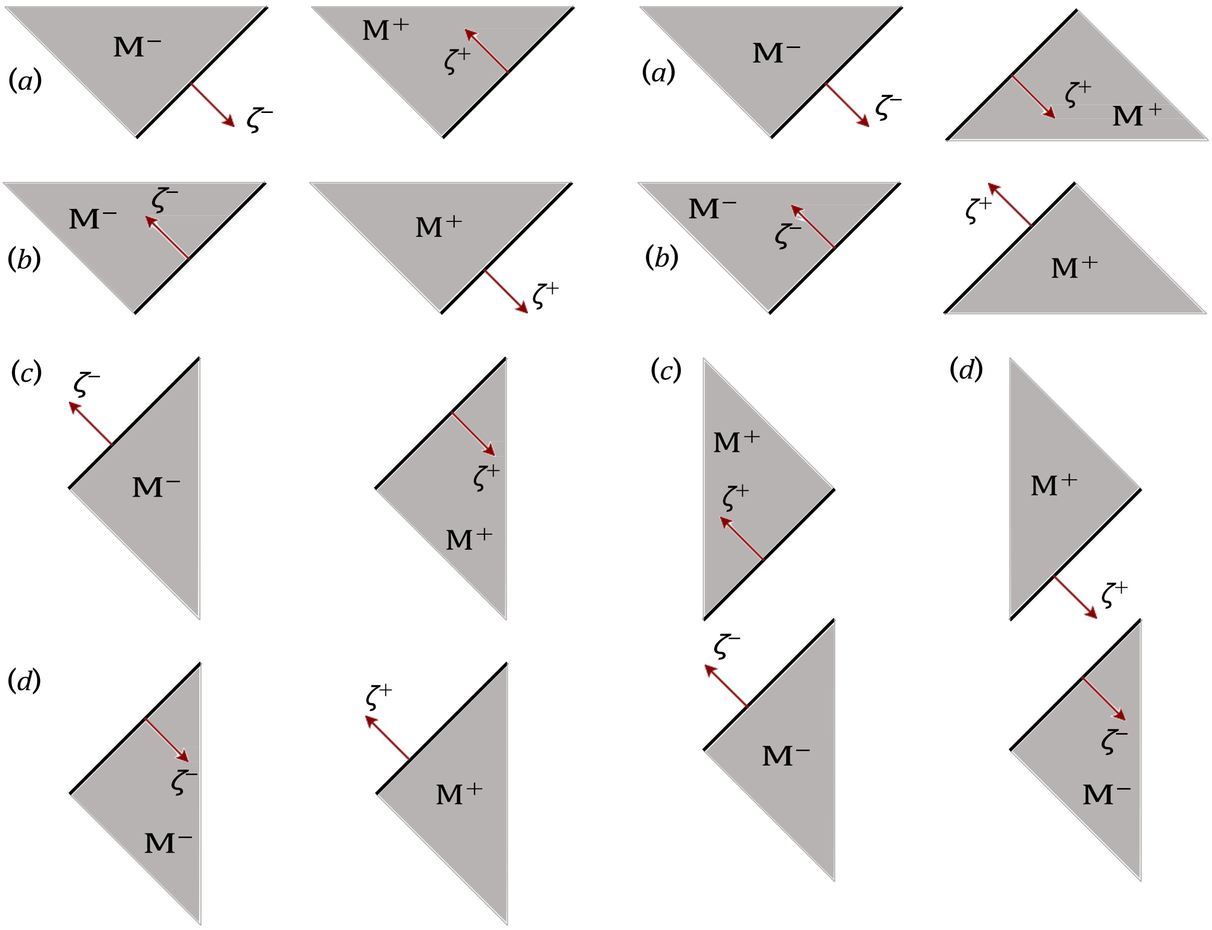


Figure 7.2: Left part: (a)-(d) depict the possible scenarios in which a null past rigging ζ^- along $\tilde{\mathcal{N}}^-$ points inwards (resp. outwards) and a null rigging ζ^+ along $\tilde{\mathcal{N}}^+$ pointing outwards (resp. inwards) happens to be future. Right part: (a)-(d) are the corresponding matchings after having changed the time-orientation of $(\mathcal{M}^+, g^+)$.

forms $\boldsymbol{\sigma}^{\mp}_{L}$ *on the leaves* $\{S^{\pm}_{v_{\pm}}\}$ *(cf. (2.99)). The scalar functions* $\{\mu^{\mp}_{a}\} \subset \mathcal{F}(\tilde{\mathcal{N}}^{\pm})$ *are defined by (7.6) with respect to the basis* $\{L^{\pm}, k^{\pm}, v^{\mp}_{I}\}$.

We have seen that the matching requires being able to embed a single metric hypersurface data set in both spacetimes, and that this metric hypersurface data can always be adapted to one of the boundaries, namely that the embedding and rigging on one side (in this case the $(\mathcal{M}^-, g^-)$ side) can always be selected at will. On the other hand, in Setup 7.3.1 we have already taken a basis $\{L^-, k^-, v^-_I\}$ of $\Gamma(T\mathcal{M}^-)|_{\tilde{\mathcal{N}}}$ We codify the freedom in the choice of $\{\phi^-, \zeta^-\}$ as follows. We first consider an abstract null hypersurface $\mathcal{N}$ and define coordinates $\{y^1 = \lambda, y^A\}$ therein. Then, we construct null embedded metric hypersurface data by enforcing that (a) the push-forwards of the vector fields $\{\partial_{y^a}\}$ coincide with the basis vectors $\{k^-, v_I^-\}$ (since we have full freedom in the choice of $\{k^-, v_I^-\}$, with this procedure we ensure that the embedding ϕ^- is built at our convenience) and (b) that the rigging ζ^- coincides with the basis vector L^- (with which we ensure that ζ^- is selected freely). This amounts to impose

$$e_1^- = k^-, \qquad e_I^- = v_I^-, \qquad \zeta^- = L^-, \qquad \text{where} \qquad e_a^- \stackrel{\text{def}}{=} \phi_\star^-(\partial_{y^a}). \tag{7.26}$$

Observe that an immediate consequence of (7.26) is that $\gamma(\partial_\lambda, \cdot) = 0$, i.e. λ is a coordinate along the degenerate direction of $\mathcal{N}$.

For the matching of $(\mathcal{M}^\pm, g^\pm)$ to be possible, there must exist another pair $\{\phi^+, \zeta^+\}$ so that (7.24) hold (and the orientations of $\zeta^\pm$ are suitable). If that is the case, we can build another basis $\{e_a^+ \stackrel{\text{def}}{=} \phi_\star^+(\partial_{y^a})\}$ of $\Gamma(T\mathcal{N}^+)$ and rewrite the matching conditions (7.24) as[2]

$$\gamma_{ij} = g^-(e_i^-, e_j^-) = g^+(e_i^+, e_j^+), \tag{7.27}$$

$$\ell_i = g^-(e_i^-, \zeta^-) = g^+(e_i^+, \zeta^+), \tag{7.28}$$

$$\ell^{(2)} = g^-(\zeta^-, \zeta^-) = g^+(\zeta^+, \zeta^+). \tag{7.29}$$

In these circumstances, determining the matching amounts to finding the explicit form of the vector fields $\{e_a^+\}$, since they fully codify the embedding ϕ^+ and, as we have seen, it is precisely this map that encodes all the information about the matching.

One way of obtaining the vectors $\{\zeta^+, e_a^+\}$ explicitly is to derive their components in the basis $\{L^+, k^+, v_I^+\}$. As one can expect, the vector field e_1^+ takes a simple form. Indeed, particularizing (7.27)-(7.28) for $i = 1$ and using (7.26) yields

$$g^+(e_1^+, e_j^+) = 0, \qquad 0 \neq \ell_1 = g^+(e_1^+, \zeta^+),$$

which means that e_1^+ must be a null generator of $\mathcal{N}$, hence proportional to k^+. The vector fields $\{\zeta^+, e_i^+\}$ (still to be determined) can therefore be decomposed as

$$e_1^+ = f k^+, \qquad e_I^+ = a_I k^+ + b_I^J v_J^+, \qquad \zeta^+ = \frac{1}{A} L^+ + B k^+ + C^K v_K^+, \tag{7.30}$$

for suitable scalar functions $f, a_I, b_I^J, A, B, C^K \in \mathcal{F}(\mathcal{N}^+)$. Note that in (7.30) we have written $1/A$. This has been done for later convenience and to emphasize that this coefficient cannot vanish because the rigging ζ^+ is, by definition, transversal to

[2] Although the data tensor fields and the scalar products are evaluated on different points, whenever it is clear from the context we do not reflect this in the notation.

$\mathcal{N}^+$. Combining (7.2), condition (7.29) and the choice (7.26), it follows that both riggings $\zeta^\pm$ must be null. The sign of A, which cannot change because this would mean that A vanishes at some point, must be such that L^- and $A^{-1}L^+$ have the same causal character. Since $L^\pm$ are both past vector fields, $A > 0$ is required for the matching to be possible.

The basis vectors $k^\pm$ have been taken future in (7.2). This, together with the choice (7.26) and with the fact that the vector fields $e_1^\pm$ are to be identified in the matching process means that $e_1^\pm$ are also future. Concretely, this means that the coordinate λ must increase to the future along the null generators on both sides. Since by construction the foliation functions $v_\pm$ also increase to the future along the null generators, it follows that $v_\pm$ must grow with λ. More on the relation between the foliation functions $v_\pm$ and λ is discussed in the next section.

Equations (7.27)-(7.29) can also be written in terms of the components f, a_I, b^J_I, A, B, C^K and the induced metrics $h^\pm$. Indeed, inserting (7.30) into (7.27)-(7.29) and defining $f \stackrel{\text{def}}{=} (\mu_1^- \circ \Phi^{-1})/\mu_1^+$ leads us to

$$h^-_{IJ}|_p = b^L_I b^K_J h^+_{LK}|_{\Phi(p)} \quad , \tag{7.31}$$

$$\mu^-_1|_p = \frac{f\mu^+_1}{A}\Big|_{\Phi(p)} \quad \Longrightarrow \quad e^+_1 = fAk^+ \quad , \tag{7.32}$$

$$\mu^-_I|_p = \frac{1}{A}\left(a_I\mu^+_1 + b^J_I\mu^+_J\right) + C^K b^J_I h^+_{JK}\Big|_{\Phi(p)} \quad , \tag{7.33}$$

$$0 = 2B\mu^+_1 + 2C^J\mu^+_J + AC^I C^J h^+_{IJ}\Big|_{\Phi(p)} \quad , \tag{7.34}$$

where p is any point of $\mathcal{N}^-$. We recall that the spacetimes $(\mathcal{M}^\pm, g^\pm)$ and the basis $\{L^\pm, k^\pm, v^\pm_I\}$ are known, and hence so are the quantities $\{\mu^\pm_a\}$ and $h^\pm_{IJ}$. In these circumstances, equations (7.33)-(7.34) provide a unique solution for B and C^K in terms of the quantities A, a_I and b^K_I, yet to be determined. Consequently, by expressing the matching conditions as (7.31)-(7.34), we are reducing the problem of matching to finding explicit expressions for the functions a_I, b^J_I, A which, as we shall see in the next section, are given by a set of n scalar functions (namely the components of ϕ^+).

In order to simplify the notation from now on we make the slight abuse of notation of identifying functions on $\mathcal{N}$ with their counterparts in $\mathcal{N}^\pm$.

7.3.1 *The step function*

The fact that the quantities A, a_I and b^K_I can be (locally) written in terms of a set of n scalar functions will be proved by studying the properties of the vector fields $\{e_i^\pm\}$. From the standard property $\phi_\star^\pm[X,Y] = [\phi_\star{}^\pm X, \phi_\star{}^\pm Y]$, $X,Y \in \Gamma(T\mathcal{N})$ and given that $e_i^\pm$ are the push-forward of a coordinate basis, it must hold

$$[e_i^\pm, e_j^\pm] = 0. \tag{7.35}$$

For e_i^- these conditions are not helpful because $\{k^-, v_I^-\}$ verify item (D) in (7.2). On the other hand, the vectors $\{e_i^+\}$ are still unknown, so (7.35) provide useful information. Inserting (7.30) one easily finds

$$0 = [e_I^+, e_J^+] = \left(e_I^+(a_J) - e_J^+(a_I) \right) k^+ + \left(e_I^+(b^K_J) - e_J^+(b^K_I) \right) v^+_K, \tag{7.36}$$

$$0 = [e_1^+, e_J^+] = \left(e_1^+(a_J) - e_J^+(f) \right) k^+ + e_1^+(b^I_J) v^+_I. \tag{7.37}$$

Setting each component to zero and using that $\phi_\star^+(X)(u) = X(u \circ \phi^+)$, we get

$$e_I^+(a_J) = e_J^+(a_I), \quad e_I^+(b^K_J) = e_J^+(b^K_I), \quad \Longleftrightarrow \quad \frac{\partial a_J}{\partial y^I} = \frac{\partial a_I}{\partial y^J}, \quad \frac{\partial b^K_J}{\partial y^I} = \frac{\partial b^K_I}{\partial y^J}, \tag{7.38}$$

$$e_1^+(a_J) = e_J^+(f), \quad e_1^+(b^K_J) = 0, \quad \Longleftrightarrow \quad \frac{\partial a_J}{\partial \lambda} = \frac{\partial f}{\partial y^J}, \quad \frac{\partial b^K_J}{\partial \lambda} = 0. \tag{7.39}$$

It follows that, locally on $\mathcal{N}$, there exist functions $H(\lambda, y^A)$ and $h^I(\lambda, y^A)$ such that

$$a_I = \frac{\partial H(\lambda, y^A)}{\partial y^I}, \qquad \frac{\partial f}{\partial y^J} = \frac{\partial a_J}{\partial \lambda} = \frac{\partial^2 H(\lambda, y^A)}{\partial \lambda \partial y^J}, \tag{7.40}$$

$$b^K_I = \frac{\partial h^K(\lambda, y^A)}{\partial y^I}, \qquad b^K_I = b^K_I(y^A). \tag{7.41}$$

From (7.41) we conclude that $h^I(\lambda, y^A)$ must decompose as

$$h^I(\lambda, y^A) = h^I_\lambda(\lambda) + h^I_y(y^A). \tag{7.42}$$

The integration "constant" $h^I_\lambda(\lambda)$ is irrelevant because it does not change b^K_I or a_I, hence it affects neither q^+ nor the embedding ϕ^+. Thus we may set $h^I_\lambda(\lambda) = 0$ without loss of generality and conclude $h^I = h^I(y^A)$ and

$$b^K_I = \frac{\partial h^K(y^A)}{\partial y^I}. \tag{7.43}$$

On the other hand, substituting (7.32) into (7.40) yields

$$\frac{\partial}{\partial y^J}\left(fA - \frac{\partial H\left(\lambda, y^A\right)}{\partial \lambda}\right) = 0, \qquad \text{i.e.} \qquad \frac{\partial H\left(\lambda, y^A\right)}{\partial \lambda} = fA + \beta(\lambda)\ , \tag{7.44}$$

where $\beta(\lambda)$ is an arbitrary function of λ. Since the function $\beta(\lambda)$ does not affect e^+_I (by (7.30)), with a suitable redefinition of $H(\lambda, y^A)$ we ensure

$$\frac{\partial H(\lambda,\, y^A)}{\partial \lambda} = fA = f, \qquad \frac{\partial H(\lambda,\, y^A)}{\partial y^I} = a_I. \tag{7.45}$$

Now combining (7.2) and (7.30), it follows

$$e^+_1(v_+) = fAk^+(v_+) = fA = \frac{\partial H(\lambda,\, y^A)}{\partial \lambda}, \qquad e^-_1(v_-) = k^-(v_-) = 1, \tag{7.46}$$

$$e^+_I(v_+) = a_I k^+(v_+) = a_I = \frac{\partial H(\lambda,\, y^A)}{\partial y^I}, \qquad e^-_I(v_-) = v^-_I(v_-) = 0, \tag{7.47}$$

from where one concludes that the foliation functions $v_\pm$ verify

$$v_- \circ \phi^- = \lambda + \text{const.}, \qquad v_+ \circ \phi^+ = H(\lambda, y^A) + \text{const.} \tag{7.48}$$

on N. The constants are again irrelevant and can be absorbed in the coordinate λ and in H respectively, so we may set them to zero without loss of generality and write

$$v_- \circ \phi^- = \lambda, \qquad v_+ \circ \phi^+ = H(\lambda, y^A). \tag{7.49}$$

Given a point $p^\pm \in \mathrm{N}^\pm$, the value $v_\pm(p^\pm)$ indicates at what height (as measured by λ) the point $p^\pm$ is located along the null generator that contains it. In view of (7.49), the function $H(\lambda, y^A)$ measures the step on the null coordinate when crossing from M^- to M^+. For this reason, we call $H(\lambda, y^A)$ *step function*.

The existence of the step function immediately connects the cut-and-paste constructions with the matching formalism developed above. In the seminal construction by Penrose [86], [87], plane-fronted impulsive gravitational waves propagating in the Minkowski spacetime are constructed by cutting out Minkowski across a null hyperplane and reattaching the two regions after shifting the null coordinate of one of the regions. To be specific, using double null coordinates where the four-dimensional Minkowski metric is $g_{\text{Mink}} = -2dudv + dx^2 + dy^2$ and the impulsive wave is located at $u = 0$, the reattachment is performed after shifting v in $u = 0^+$ by $v \longmapsto v + h(x, y)$, where $h(x, y)$ is an arbitrary function. This jump is precisely of the form (7.49) with $H = v + h(x, y)$, provided we use $\{v = \lambda,\, x,\, y\}$ also as coordinates intrinsic to the null hyperplane, so that the embedding ϕ^- becomes the identity. Another example of the direct connection between the step function H and the cut-and-paste construction appears in [5], where expression (7.49) is equivalent to $H = \mathrm{V} - \mathrm{H}(\eta, \bar{\eta})$. More details about the connection between the matching formalism and the cut-and-paste construction are given in Section 7.3.3 below.

At this point, it is convenient to pause for a moment and summarize what we have found. Assuming that one matching rigging points inwards and the other outwards, the matching is possible if and only if the junction conditions (7.31)-(7.34) are satisfied. The last two are always solvable and determine uniquely the coefficients B and C^I (i.e. the tangential components of the rigging ζ^+) in terms of A, a_I and b^J_I, which in turn are given by $\{H(\lambda, y^J), h^I(y^J)\}$ according to (7.43) and (7.45). The functions $H(\lambda, y^J), h^I(y^J) \in \mathcal{F}(\mathcal{N})$ verify that $\partial_\lambda H > 0$ (because of (7.49), recall that $v_\pm$ and λ increase towards the future) and that the Jacobian matrix

$$\frac{\partial(h^2, ..., h^{n+1})}{\partial(y^2, ..., y^{n+1})}$$

has non-zero determinant (because $\{e^\pm_I\}$ must be spacelike necessarily).

Equation (7.32), on the other hand, allows us to conclude that the null generator e_1^- must be identified with another null generator e_1^+ in the process of matching. This equation also establishes the explicit form of e_1^+ in terms of k^+ and (the first derivative of) the step function $H(\lambda, y^A)$.

Finally, we come to equation (7.31), whose solvability constitutes the core problem for the existence of the matching. In order to understand the geometric meaning of this condition, we argue as follows. Firstly, note that for any $p \in \widetilde{\mathcal{N}}^-$, the section $S^-_{v_-(p)} = \{v_- = v_-(p)\} \subset \widetilde{\mathcal{N}}^-$ is mapped via Φ to the (necessarily) spacelike submanifold $\Phi(S^-_{v_-(p)}) \subset \widetilde{\mathcal{N}}^+$. In these circumstances, the combination of (7.31) and (7.43) implies that there exists an isometry between these two submanifolds. Even more, since the functions $\{h^B\}$ depend only on $\{y^A\}$, this isometry must be universal in the sense of being independent of the value $v_-(p)$. This fact was already observed in [126] (see equations (2.9)-(2.10)) and later in [127] when studying the coordinate changes leaving the first fundamental form γ invariant.

In order to describe this more explicitly, we transfer the coordinates $\{\lambda, y^I\}$ from $\mathcal{N}$ to $\widetilde{\mathcal{N}}^-$ so that the embedding ϕ^- takes the simple form

$$\begin{array}{rccccl} \phi^- : & \mathcal{N} & \hookrightarrow & \widetilde{\mathcal{N}}^- \subset \mathcal{M}^- & & \\ & (\lambda, y^I) & \longmapsto & \phi^-(\lambda, y^I) = & \left(v_- = \lambda, y^I\right), & \end{array} \tag{7.50}$$

and construct coordinates $\{v_+, u^I\}$ on $\mathcal{N}^+$ such that $v_I^+ = \partial_{u^I}$ (in particular, they are constant along the null generators). Then, the embedding ϕ^+ takes the form

$$\begin{aligned} \phi^+ : \quad \mathcal{N} &\longrightarrow \mathcal{N}^+ \subset \mathcal{M}^+ \\ (\lambda, y^I) &\longmapsto \phi^+(\lambda, y^I) = \left\{ v_+ = H(\lambda, y^I), u^I = h^I(y^J) \right\}. \end{aligned} \tag{7.51}$$

and the section $S^-_{v_-^0}$ in $\mathcal{N}^-$ is mapped to $\Phi(S^-_{v_-^0}) = \{v_+ = H(\lambda = v_-^0, y^J), u^I(y^J)\}$. A point $p \in \mathcal{N}^-$ can be identified uniquely by specifying the null generator to which it belongs together with its height $v_-(p)$ along the generator, and the same happens on $\mathcal{N}^+$. Thus, the matching is feasible if and only if there exists a diffeomorphism Ψ between the set of null generators of $\mathcal{N}^-$ and the set of null generators of $\mathcal{N}^+$ (defined locally by $u^I(y^J)$) such that, for each possible value of v_-^0, the map that takes each point at height v_-^0 along a generator σ in $\mathcal{N}^-$ to the point at height $H|_\sigma(v_-^0)$ in $\mathcal{N}^+$ along the generator $\Psi(\sigma)$, happens to be an isometry. This is of course a very strong restriction and generically it will not be possible to find H and Ψ verifying it (which simply means that the matching cannot be done). However, as we see next, there are situations where the matching is not only feasible but it even allows for an infinite number of possibilities, and other cases where there is at most one possible step function H for each admissible choice of Ψ.

7.3.1.1 *Second fundamental forms and multiple matchings*

We conclude this section on the step function by showing that, in any feasible matching, the second fundamental forms $\mathbf{K}^k_\pm$ are related to each other and to the step function. To prove this, we first recall that $e_1^- = k^-$ and $e_1^+ = (\partial_\lambda H)k^+$. Combining this with (7.5) one obtains

$$e_1^-(h^-_{IJ}) = \partial_\lambda h^-_{IJ} = 2\mathbf{K}^k_-(v_I^-, v_J^-), \qquad e_1^+(h^+_{IJ}) = \partial_\lambda h^+_{IJ} = 2(\partial_\lambda H)\,\mathbf{K}^k_+(v_I^+, v_J^+). \tag{7.52}$$

On the other hand, the partial derivative of (7.31) with respect to λ gives

$$\partial_\lambda h^-_{IJ} = \partial_\lambda(b^A_I b^B_J)h^+_{AB} + b^A_I b^B_J \partial_\lambda h^+_{AB} = b^A_I b^B_J \partial_\lambda h^+_{AB} \tag{7.53}$$

after using that the coefficients b^J_I do not depend on λ (cf. (7.39)). Putting (7.52)-(7.53) together yields

$$\mathbf{K}^k_-(v_I^-, v_J^-) = (\partial_\lambda H) b^A_I b^B_J \mathbf{K}^k_+(v_A^+, v_B^+). \tag{7.54}$$

Since we are assuming the geometry of $\mathcal{N}^{\pm}$ to be known and the basis $\{k^{\pm}, v_I^{\pm}\}$ have been chosen, this expression determines, for each possible choice of Ψ (i.e. of b^A_B fulfilling (7.31)), a unique value for $\partial_\lambda H$ unless the two second fundamental forms vanish simultaneously. If, on the other hand, there exist open sets $\mathcal{O}^{\pm} \subset \mathcal{N}^{\pm}$ related by $\mathcal{O}^{+} = \Phi(\mathcal{O}^{-})$ and such that

$$\mathbf{K}^{k}_{\pm}(v_I^{\pm}, v_J^{\pm})|_{\mathcal{O}^{\pm}} = 0, \tag{7.55}$$

then (7.54) is identically satisfied. Whenever (7.55) holds, all the spacelike sections in $\mathcal{O}^{\pm}$ are isometric to each other, and the same happens in $\mathcal{O}^{+}$. This is a consequence of (7.5) and the equality (2.95) between the quotient metric at any point $p \in \mathcal{N}^{\pm}$ and the metric $h^{\pm}$ of any spacelike section passing through this point. Thus, the set of null generators can be endowed with a positive definite metric. If there is an isometry Ψ between these two spaces, then *any* step function $H(\lambda, y^I)$ satisfying $\partial_\lambda H > 0$ defines a feasible matching. This means that a point $p \in \mathcal{N}^{-}$ lying on a null generator σ^{-} can be shifted arbitrarily along the null generator $\sigma^{+} \stackrel{\mathrm{def}}{=} \Psi(\sigma^{-})$ in $\mathcal{N}^{+}$ with the only condition that if q is to the future of p along σ^{-} then their images have the same causal relation along σ^{+}. The matching in these circumstances exhibits a large freedom. Two examples of this are the following cut-and-paste constructions: the plane-fronted impulsive wave [85], [86], [87] by Penrose and both the non-expanding impulsive wave in constant-curvature backgrounds [88], [6] and the impulsive wave with gyratons [5] by Podolský and collaborators.

7.3.2 *Energy-momentum tensor of the shell*

As mentioned in Chapter 2, the fundamental properties of the matter and energy content of a shell are encoded in its associated energy-momentum tensor, which we denote by τ. This tensor can be suitably defined within the formalism of hypersurface data (see Definition 2.7.3), so a natural question is whether one can find a more explicit form for it, now that we have been able to codify all the information about the matching in the set of functions $\{H(\lambda, y^A), h^B(y^A)\}$.

In this section, we assume that the two spacetimes $(\mathcal{M}^{\pm}, g^{\pm})$ can be matched and derive the energy-momentum tensor τ explicitly in terms of the functions $\{H(\lambda, y^A), h^B(y^A)\}$ and all the (known) geometric objects defined on the boundaries $\mathcal{N}^{\pm}$.

Our starting point is Definition 2.7.3 and Corollary 3.2.6. In terms of the basis $\{\partial_\lambda, \partial_{y^A}\}$ and its dual $\{d\lambda, dy^A\}$, it is straightforward to conclude from (7.26)-(7.29) that the data tensors $\{\gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ are given by

$$\gamma = h^-_{AB} dy^A \otimes dy^B, \qquad \boldsymbol{\ell} = \mu^-_1 d\lambda + \mu^-_A dy^A, \qquad \ell^{(2)} = 0. \tag{7.56}$$

Observe that the components γ_{IJ} coincide with those of the induced metric h^-_{IJ} on the leaves $\{\lambda = \text{const.}\}$. So, to simplify the notation, from now on we shall use γ_{IJ} instead of h^-_{IJ}. We shall also use γ^{IJ} with the understanding that this just means h^{IJ}_- (the tensor γ is degenerate and cannot be inverted).

The vector field n defined by (2.6) must be proportional to ∂_λ and such that $\boldsymbol{\ell}(n) = 1$, and hence it is given by $n = (\mu^-_1)^{-1}\partial_\lambda$. Moreover, particularizing Lemma 3.2.5 for the basis $\{n, v_A = \partial_{y^A}\}$, we get the following form for the tensor P:

$$P = \gamma^{AB}\partial_{y^A} \otimes \partial_{y^B} - \frac{2\gamma^{AB}\mu^-_B}{\mu^-_1}\partial_\lambda \otimes_s \partial_{y^A} + \frac{\gamma^{AB}\mu^-_A\mu^-_B}{(\mu^-_1)^2}\partial_\lambda \otimes \partial_\lambda. \tag{7.57}$$

Combining (2.155) with the explicit form of the tensors $\{P, n, n^{(2)}\}$ above (or simply using Corollary 3.2.6 for $\mathbf{q} = \mu^-_1 d\lambda$ and $\boldsymbol{\theta}^A = dy^A$), one gets

$$\tau(d\lambda, d\lambda) = -\epsilon \frac{\gamma^{IJ}[\mathbf{Y}](\partial_{y^I}, \partial_{y^J})}{(\mu^-_1)^2}, \tag{7.58}$$

$$\tau(d\lambda, dy^I) = \epsilon \frac{\gamma^{IJ}[\mathbf{Y}](\partial_\lambda, \partial_{y^J})}{(\mu^-_1)^2}, \tag{7.59}$$

$$\tau(dy^I, dy^J) = -\epsilon \frac{\gamma^{IJ}[\mathbf{Y}](\partial_\lambda, \partial_\lambda)}{(\mu^-_1)^2}. \tag{7.60}$$

In view of (7.58)-(7.60), once we have derived the specific form of the components of the tensor $[\mathbf{Y}]$, the calculation of the energy-momentum tensor follows at once. For this reason, from now on we focus on the calculation of $[\mathbf{Y}]$. As we will see, the fact that the rigging had been adapted to the basis vector L^- in the minus side makes the computation of $\mathbf{Y}^-$ considerably less involved than that of $\mathbf{Y}^+$. It is therefore convenient to compute $\mathbf{Y}^+$ and obtain $\mathbf{Y}^-$ as a particular case.

We start the calculations with some lemmas and results that will aid us along the way. In the first one, we provide the explicit form of the one-form $g^+(L^+, \cdot)$.

Lemma 7.3.2. *The one-form* $\boldsymbol{L}^+ \stackrel{\text{def}}{=} g^+(L^+, \cdot)$ *satisfies*

$$(\phi^+)^\star \boldsymbol{L}^+ = -\boldsymbol{\vartheta} \qquad \text{where} \qquad \boldsymbol{\vartheta} \stackrel{\text{def}}{=} -\mu^+_1 dH - \mu^+_J dh^J \in \Gamma(T^\star N). \tag{7.61}$$

Proof. For any vector field $Z \in \Gamma(T\mathcal{N})$, it holds

$$\begin{aligned}
(\phi^+)^\star \boldsymbol{L}^+ \left(Z^a \partial_{y^a} \right) &= \boldsymbol{L}^+ \left((\phi^+)_\star Z^a \partial_{y^a} \right) = \langle L^+, Z^a e_a^+ \rangle_{g^+} \\
&= Z^1 \langle L^+, e_1^+ \rangle_{g^+} + Z^A \langle L^+, e_A^+ \rangle_{g^+} \\
&= Z^1 \langle L^+, f A k^+ \rangle_{g^+} + Z^A \langle L^+, a_A k^+ + b^J_A v_J^+ \rangle_{g^+} \\
&= \mu_1^+ \left(f A Z^1 + Z^A a_A \right) + Z^A b^J_A \mu_J^+ \\
&= \mu_1^+ dH(Z) + \mu_J^+ dh^J(Z) = -\boldsymbol{\vartheta}(Z),
\end{aligned} \tag{7.62}$$

where we have used (7.32), (7.30) in the third equality, (7.6) in the fourth one and (7.41), (7.45) for the last step. □

Our next aim is to compute the explicit form of the matching rigging ζ^+ in the basis $\{L^+, k^+, v_I^+\}$, given in terms of the metric h^+ and the functions $\{H(\lambda, y^A), h^B(y^A)\}$ and $\{\mu_a^\pm\}$. For that it is convenient to introduce a vector field $X \in \Gamma(T\mathcal{N})$ which encodes the tangent part of ζ^+ in (7.30), namely

$$\zeta^+ = \frac{1}{A} \left(L^+ + \phi_\star^+ X \right). \tag{7.63}$$

It is also helpful to define the functions $X^a \in \mathcal{F}(\tilde{\mathcal{N}}^+)$ by the decomposition

$$\phi_\star^+ X = X^1 e_1^+ + X^A e_A^+. \tag{7.64}$$

In the next lemma, we provide explicit expressions for the one-form $\gamma(X, \cdot)$ and the components $\{X^a\}$.

Lemma 7.3.3. *The vector field $X = X^a \partial_{y^a}$ satisfies*

$$\gamma(X, \cdot) = \boldsymbol{\vartheta} + A \mu_I^- dy^I + \mu_1^+ (\partial_\lambda H) d\lambda \tag{7.65}$$

Moreover, its components X^a are given by

$$X^A = \gamma^{IA} \left(\vartheta_I + A \mu_I^- \right), \tag{7.66}$$

$$X^1 = \frac{\gamma^{IJ}}{2\mu_1^- A} \left(\vartheta_I + A\mu_I^- \right) \left(\vartheta_J - A\mu_J^- \right). \tag{7.67}$$

Proof. Recall that $\boldsymbol{\ell}_1 = \mu_1^-$, $\boldsymbol{\ell}_A = \mu_A^-$, cf. (7.56). Combining (7.28), (7.63) and Lemma 7.3.2 it follows that $\boldsymbol{\ell}_a = \langle \zeta^+, e_a^+ \rangle_{g^+} = \frac{1}{A} \left(\langle L^+, e_a^+ \rangle_{g^+} + X^b \gamma_{ab} \right) = \frac{1}{A} \left(-\vartheta_a + X^b \gamma_{ab} \right)$, i.e.

$$X^b \gamma_{ab} = \vartheta_a + A \boldsymbol{\ell}_a. \tag{7.68}$$

This proves (7.65)-(7.66) after using that $\gamma_{1a} = 0$, that γ_{IJ} is non-degenerate and

$$\mu_1^- A = \mu_1^+ f A = \mu_1^+ \partial_\lambda H \qquad \Longleftrightarrow \qquad A = \frac{\mu_1^+}{\mu_1^-} f A = \frac{\mu_1^+}{\mu_1^-} \partial_\lambda H. \tag{7.69}$$

On the other hand, condition (7.29) together with ζ^- and L^+ being null entails

$$0 = \langle \zeta^+, \zeta^+ \rangle_{g^+} = \frac{1}{A^2} \Big(2X^a \langle L^+, e_a^+ \rangle_{g^+} + X^a X^b \gamma_{ab} \Big) \quad \Longrightarrow \quad -2X^a \vartheta_a + X^a X^b \gamma_{ab} = 0.$$

Combining this with (7.68) and using $X^a \boldsymbol{\ell}_a = X^1 \mu_1^- + X^A \mu_A^-$ yields

$$X^a \vartheta_a = A \left(X^1 \mu_1^- + X^A \mu_A^- \right) \qquad \Longrightarrow \qquad 2X^1 \mu_1^+ \partial_\lambda H = X^A \left(\vartheta_A - A \mu_A^- \right),$$

which gives (7.67) after using again (7.69). □

We can now obtain the matching rigging ζ^+ as a corollary of the results above.

Corollary 7.3.4. *In the basis* $\{L^+, k^+, v_I^+\}$ *of* $\Gamma(TM^+|_{\mathcal{N}^+})$*, the matching rigging* ζ^+ *reads*

$$\zeta^+ = \frac{\mu_1^-}{\partial_\lambda H} \left(\frac{1}{\mu_1^+} L^+ - h_+^{AB} \left((b^{-1})^I{}_A \left(\partial_{y^I} H - \frac{1}{\mu_1^-} (\partial_\lambda H) \mu_I^- \right) + \frac{1}{\mu_1^+} \mu_A^+ \right) Z_B \right), \tag{7.70}$$

where $(b^{-1})^J{}_I \stackrel{\text{def}}{=} \partial_{h^I} y^J$ *and* $Z_B \stackrel{\text{def}}{=} \frac{1}{2} \left((b^{-1})^J{}_B \left(\partial_{y^J} H - \frac{1}{\mu_1^-} (\partial_\lambda H) \mu_J^- \right) - \frac{1}{\mu_1^+} \mu_B^+ \right) k^+ + v_B^+$.

Proof. The coefficients $(b^{-1})^J{}_I$ are of course inverse to $b^J{}_I$. From (7.31) one obtains

$$\begin{aligned} \delta^A_C = h_-^{AB} h^-_{BC} = h_-^{AB} b^I{}_B b^J{}_C h^+_{IJ} \quad &\Longrightarrow \quad (b^{-1})^A{}_K = h_-^{AB} b^I{}_B h^+_{IK} \\ \Longrightarrow \quad h_+^{KL} (b^{-1})^A{}_K = h_-^{AB} b^L{}_B \quad &\Longrightarrow \quad h_+^{KL} (b^{-1})^A{}_K (b^{-1})^J{}_L = h_-^{AJ} \equiv \gamma^{AJ}. \end{aligned} \tag{7.71}$$

The result (7.70) follows from (7.63) after inserting (7.66)-(7.67) and using (7.69), the definition (7.61) of ϑ and (7.71). □

Remark 7.3.5. *The result (7.70) is relevant for various reasons. First, it proves that indeed all the information about the matching can be fully codified in the embedding* ϕ^+*, since* ζ^+ *is given by the functions* $\{H(\lambda, y^A), h^B(y^A)\}$*, which are actually the "components" of* ϕ^+ *if one selects suitable coordinates on* $\mathcal{N}^+$ *(see the discussion in Section 7.3.1).*

Secondly, when the matching of $(\mathcal{M}^\pm, g^\pm)$ *is possible, (7.70) provides the explicit form of the matching rigging* ζ^+ *for whatever choice of* ***null*** *matching rigging* ζ^-*. In other words, in the spacetime* $(\mathcal{M}, g)$ *resulting from the matching of* $(\mathcal{M}^\pm, g^\pm)$*, a* C^1 *curve* $\mathcal{C}$ *that* *crosses the matching hypersurface* $\mathcal{N} \subset \mathcal{M}$ *with (arbitrary) direction* ζ^- *continues with direction* ζ^+ *given by* (7.70) *after passing through the shell.*

Finally, (7.70) *allows one to compute (for each choice of* ζ^-*) the matching riggings* ζ^+ *associated to each different matching by simply substituting the corresponding functions* $\{H(\lambda, y^A), h^B(y^A)\}$ *in* (7.70).

It is useful to define the quantities

$$\mu_I^+ \stackrel{\text{def}}{=} b_I^K \mu_K^+, \qquad W_I \stackrel{\text{def}}{=} b_I^K v_K^+, \tag{7.72}$$

which will appear in several expressions below.

Now that we know the explicit form of the matching rigging ζ^+, we can focus on the computation of the components $[\mathbf{Y}](\partial_\lambda, \partial_\lambda)$, $[\mathbf{Y}](\partial_\lambda, \partial_{y^A})$ and $[\mathbf{Y}](\partial_{y^A}, \partial_{y^B})$, for which we need the pull-backs $(\phi^\pm)^\star(\pounds_{\zeta^\pm} g^\pm)$ (cf. (2.39)). As a previous step, it is convenient to calculate the derivative $\partial_{y^a} A$ as well as several identities involving scalar products of the form $g^+(\nabla_{e_a^+} L^+, e_b^+)$. This is done in the following lemma.

Lemma 7.3.6. *The following identities hold:*

$$\frac{\partial_{y^a} A}{A} = \frac{\partial_{y^a}\partial_\lambda H}{\partial_\lambda H} + \frac{\partial_{y^a}\mu_1^+}{\mu_1^+} - \frac{\partial_{y^a}\mu_1^-}{\mu_1^-}, \tag{7.73}$$

$$\langle \nabla^+_{e_1^+} L^+, e_1^+ \rangle_{g^+} = \partial_\lambda H \left(\partial_\lambda \mu_1^+ - \mu_1^+ \kappa^+_{k^+} \partial_\lambda H \right), \tag{7.74}$$

$$\begin{aligned} \langle \nabla^+_{e_1^+} L^+, e_J^+ \rangle_{g^+} + \langle \nabla^+_{e_J^+} L^+, e_1^+ \rangle_{g^+} = {}& \partial_\lambda H \left(\frac{\partial_\lambda \mu_J^+}{\partial_\lambda H} - 2\mu_1^+ \kappa^+_{k^+} \partial_{y^J} H \right. \\ & \left. + 2\mu_1^+ \boldsymbol{\sigma}_{L^+}(W_J) + \partial_{y^J} \mu_1^+ \right) + \partial_{y^J} H \partial_\lambda \mu_1^+, \end{aligned} \tag{7.75}$$

$$\begin{aligned} \langle \nabla^+_{e_I^+} L^+, e_J^+ \rangle_{g^+} = {}& -\mu_1^+ \kappa^+_{k^+} \partial_{y^I} H \partial_{y^J} H + \frac{\partial_{y^I} H \partial_\lambda \mu_J^+}{\partial_\lambda H} + \partial_{y^J} H \partial_{y^I} \mu_1^+ \\ & + \mu_1^+ \left(\partial_{y^I} H \boldsymbol{\sigma}_{L^+}(W_J) + \partial_{y^J} H \boldsymbol{\sigma}_{L^+}(W_I) \right) \\ & + \boldsymbol{\Theta}^{L}_{+}(W_I, W_J). \end{aligned} \tag{7.76}$$

Proof. Throughout the proof, we shall use repeatedly the decompositions $e_1^+ = (\partial_\lambda H) k^+$, $e^+_I = (\partial_{y^I} H) k^+ + W_I$ which follow directly from (7.30)-(7.32) and (7.45). By direct computation, we get

$$\frac{\partial_{y^a} A}{A} = \frac{1}{A}\partial_{y^a}\left(\frac{fA}{f}\right) = \frac{1}{fA}\left(\partial_{y^a}\partial_\lambda H - A\partial_{y^a} f\right) = \frac{\partial_{y^a}\partial_\lambda H}{\partial_\lambda H} - \frac{\partial_{y^a} f}{f},$$

which leads to (7.73) by simply inserting $f \stackrel{\text{def}}{=} \mu_1^-/\mu_1^+$. By (7.11), one gets (7.74) as an immediate consequence of

$$\langle \nabla^+_{e_1^+} L^+, e_1^+ \rangle_{g^+} = (\partial_\lambda H)^2 \langle \nabla^+_{k^+} L^+, k^+ \rangle_{g^+} = (\partial_\lambda H)^2 \left(k^+(\mu_1^+) - \mu_1^+ \kappa^+_{k^+} \right). \tag{7.77}$$

In order to prove (7.75), we compute each term in the left-hand side separately. In both cases we use the covariant derivatives of $L^\pm$ given in Lemma 7.1.2. Firstly,

$$\begin{aligned}
\langle \nabla^+_{e_1^+} L^+, e_J^+ \rangle_{g^+} &= \partial_\lambda H \langle \nabla^+_{k^+} L^+, \partial_{y^J} H\, k^+ + W_J \rangle_{g^+} \\
&= \partial_{y^J} H \left(\partial_\lambda \mu_1^+ - \mu_1^+ \kappa^+_{k^+} \partial_\lambda H \right) + \partial_\lambda H \left(k^+ \left(\overline{\mu}_J^+ \right) + \mu_1^+ \boldsymbol{\sigma}^+_{L^+} (W_J) \right) \\
&= \partial_{y^J} H \left(\partial_\lambda \mu_1^+ - \mu_1^+ \kappa^+_{k^+} \partial_\lambda H \right) + \partial_\lambda \overline{\mu}_J^+ + \mu_1^+ \partial_\lambda H\, \boldsymbol{\sigma}^+_{L^+} (W_J),
\end{aligned} \tag{7.78}$$

where in the second equality we used $k^+(b^J_I) = 0$. Secondly,

$$\begin{aligned}
\langle \nabla^+_{e_J^+} L^+, e_1^+ \rangle_{g^+} &= \partial_\lambda H \left(\partial_{y^J} H \langle \nabla^+_{k^+} L^+, k^+ \rangle_{g^+} + \langle \nabla^+_{W_J} L^+, k^+ \rangle_{g^+} \right) \\
&= \partial_{y^J} H \left(\partial_\lambda \mu_1^+ - \mu_1^+ \kappa^+_{k^+} \partial_\lambda H \right) + \partial_\lambda H \left(W_J(\mu_1^+) + \mu_1^+ \boldsymbol{\sigma}^+_L (W_J) \right) \\
&= \partial_{y^J} H \left(\partial_\lambda \mu_1^+ - \mu_1^+ \kappa^+_{k^+} \partial_\lambda H \right) + \partial_\lambda H \left(\left(e_J^+ - \frac{\partial_{y^J} H}{\partial_\lambda H} e_1^+ \right)(\mu_1^+) + \mu_1^+ \boldsymbol{\sigma}^+_L (W_J) \right) \\
&= \partial_\lambda H \left(- \mu_1^+ \kappa^+_{k^+} \partial_{y^J} H + \partial_{y^J} \mu_1^+ + \mu_1^+ \boldsymbol{\sigma}^+_L (W_J) \right).
\end{aligned} \tag{7.79}$$

From (7.78)-(7.79) equation (7.75) follows at once. Finally, for the term $\langle \nabla^+_{e_I^+} L^+, e_J^+ \rangle_{g^+}$ one obtains

$$\begin{aligned}
\langle \nabla^+_{e_I^+} L^+, e_J^+ \rangle_{g^+} &= \frac{\partial_{y^I} H}{\partial_\lambda H} \left(\partial_{y^J} H \left(\partial_\lambda \mu_1^+ - \mu_1^+ \kappa^+_{k^+} \partial_\lambda H \right) + \partial_\lambda \overline{\mu}_J^+ + \mu_1^+ \partial_\lambda H\, \boldsymbol{\sigma}^+_L (W_J) \right) \\
&\quad + \partial_{y^J} H \langle \nabla^+_{W_I} L^+, k^+ \rangle_{g^+} + \langle \nabla^+_{W_I} L^+, W_J \rangle_{g^+} \\
&= \frac{\partial_{y^I} H}{\partial_\lambda H} \left(-\mu_1^+ \kappa^+_{k^+} \partial_{y^J} H\, \partial_\lambda H + \partial_\lambda \overline{\mu}_J^+ + \mu_1^+ \partial_\lambda H\, \boldsymbol{\sigma}^+_L (W_J) \right) \\
&\quad + \partial_{y^J} H \left(\partial_{y^I} \mu_1^+ + \mu_1^+ \boldsymbol{\sigma}^+_L (W_I) \right) + \boldsymbol{\Theta}_{L^+} (W_I, W_J),
\end{aligned}$$

from where one easily obtains (7.76). □

We are now ready to compute the pull-backs $(\phi^\pm)^\star \left(\pounds_{\zeta^\pm} g^\pm \right)$. As before, we derive the expression on the plus side (which is considerably more involved) and then we get $(\phi^-)^\star \left(\pounds_{\zeta^-} g^- \right)$ as a suitable specialization. The computation relies on the well-known fundamental property

$$(\phi^+)^\star \left(\pounds_{f \phi^+_\star Z} g^+ \right) = f \pounds_Z \gamma + 2 df \otimes_s \gamma(Z, \cdot) \tag{7.80}$$

satisfied by any function $f \in \mathcal{F}(\mathcal{N})$ and any vector field $Z \in \Gamma(T\mathcal{N})$. At some point in the calculations, we shall also need to use that

$$(\phi^\pm)^\star\left(\pounds_{L^\pm} g^\pm\right) = \left(\langle \nabla^\pm_{e^\pm_a} L^\pm, e^\pm_b\rangle_{g^\pm} + \langle \nabla^\pm_{e^\pm_b} L^\pm, e^\pm_a\rangle_{g^\pm}\right) dy^a \otimes dy^b. \tag{7.81}$$

From the decomposition (7.63) we get (recall (7.65))

$$\begin{aligned}
(\phi^+)^\star\left(\pounds_{\zeta^+} g^+\right) &= (\phi^+)^\star\left(\pounds_{\frac{1}{A}(L^+ + \phi^+_\star(X))} g^+\right)\\
&= (\phi^+)^\star\left(\frac{1}{A}\pounds_{(L^+ + \phi^+_\star(X))} g^+ - 2\frac{dA}{A^2} \otimes_s g^+\left(L^+ + \phi^+_\star(X), \cdot\right)\right)\\
&= \frac{1}{A}(\phi^+)^\star\left(\pounds_{L^+} g^+\right) + \frac{1}{A}\pounds_X\gamma - 2\frac{dA}{A^2} \otimes_s \left((\phi^+)^\star\left(\boldsymbol{L}^+\right) + \gamma(X,\cdot)\right)\\
&= \frac{1}{A}\left((\phi^+)^\star\left(\pounds_{L^+} g^+\right) + \pounds_X\gamma - 2\frac{dA}{A} \otimes_s \left(A\mu^-_I dy^I + \mu^+_1(\partial_\lambda H)d\lambda\right)\right)\\
&= \frac{1}{A}\left((\phi^+)^\star\left(\pounds_{L^+} g^+\right) + \pounds_X\gamma - 2dA \otimes_s \mu^-_I dy^I - 2\mu^-_1 dA \otimes_s d\lambda\right),
\end{aligned} \tag{7.82}$$

where we used (7.80) in the third line, Lemma 7.3.2 and (7.65) in the fourth and (7.69) in the last one. Next we elaborate the term $\pounds_X\gamma$. The first fundamental form γ is degenerate (i.e. $\gamma_{1A} = 0$), so in index notation $\pounds_X\gamma$ reads

$$(\pounds_X\gamma)_{ab} = X^c\partial_{y^c}\gamma_{ab} + \gamma_{aI}\partial_{y^b}X^I + \gamma_{Ib}\partial_{y^a}X^I. \tag{7.83}$$

The induced metric γ_{IJ} on the leaves $\{\lambda = \text{const.}\}$ (understanding γ_{IJ} as $h_{\overline{I}\,\overline{J}}$, as mentioned before) is positive definite. In the following we denote this metric by h (i.e. $h_{IJ} \stackrel{\text{def}}{=} \gamma_{IJ}$) so that we can define the corresponding Levi-Civita connection ∇^h on each leaf. Now, inserting (7.66)-(7.67) into (7.83) and using (7.5) yields

$$(\pounds_X\gamma)_{11} = 0, \tag{7.84}$$

$$(\pounds_X\gamma)_{1J} = \partial_\lambda(\gamma_{JL}X^L) - X^L\partial_\lambda\gamma_{JL} = \partial_\lambda(\vartheta_J + A\mu^-_J) - 2X^L\mathbf{K}^{k}_{-}(v^-_J, v^-_L), \tag{7.85}$$

$$\begin{aligned}
(\pounds_X\gamma)_{IJ} &= X^1\partial_\lambda\gamma_{IJ} + X^L\partial_{y^L}\gamma_{IJ} + \gamma_{IL}\partial_{y^J}X^L + \gamma_{LJ}\partial_{y^I}X^L\\
&= 2X^1\mathbf{K}^{k}_{-}(v^-_I, v^-_J) + \nabla^h_I X_J + \nabla^h_J X_I,
\end{aligned} \tag{7.86}$$

where $X_I \stackrel{\text{def}}{=} \gamma_{IL}X^L$. By (7.61) and (7.66), the derivative $\nabla^h_I X_J$ can be expanded to

$$\begin{aligned}
\nabla^h_I X_J &= \nabla^h_I\left(\vartheta_J + A\mu^-_J\right) = \nabla^h_I\left(-\mu^+_1\nabla^h_J H - \overline{\mu}^+_J + A\mu^-_J\right)\\
&= -\nabla^h_I\mu^+_1\nabla^h_J H - \mu^+_1\nabla^h_I\nabla^h_J H - \nabla^h_I\overline{\mu}^+_J + A\nabla^h_I\mu^-_J + \mu^-_J\nabla^h_I A.
\end{aligned} \tag{7.87}$$

We have now all the ingredients to compute $\mathbf{Y}^{\pm}$ and the energy-momentum tensor on the shell. The result is given in the next proposition (where brackets, as usual, denote symmetrization).

Proposition 7.3.7. *The tensor $\mathbf{Y}^{+}$ has the following components:*

$$\mathbf{Y}^{+}(\partial_\lambda, \partial_\lambda) = -\mu_1^- \left(\kappa^+_{k^+} \partial_\lambda H + \frac{\partial_\lambda \partial_\lambda H}{\partial_\lambda H} - \frac{\partial_\lambda \mu_1^-}{\mu_1^-} \right), \tag{7.88}$$

$$\begin{aligned}\mathbf{Y}^{+}(\partial_\lambda, \partial_{y^J}) = -\mu_1^- \Bigg(& \kappa^+_{k^+} \nabla^h_J H - \boldsymbol{\sigma}^+_L(W_J) + \frac{\partial_\lambda \partial_{y^J} H}{\partial_\lambda H} \\ & + \frac{X^L \mathbf{K}^k_-(q^-_J, v^-_L)}{\mu_1^+ \partial_\lambda H} - \frac{\nabla^h_J \mu_1^-}{2\mu_1^-} - \frac{\partial_\lambda \mu_J^-}{2\mu_1^-} \Bigg),\end{aligned} \tag{7.89}$$

$$\begin{aligned}\mathbf{Y}^{+}(\partial_{y^I}, \partial_{y^J}) = -\mu_1^- \Bigg(& \frac{\kappa^+_{k^+} \nabla^h_I H \nabla^h_J H}{\partial_\lambda H} - \frac{\nabla^h_{(I} H\, \partial_\lambda \bar{\mu}^+_{J)}}{\mu_1^+ (\partial_\lambda H)^2} - \frac{2\nabla^h_{(I} H\, \boldsymbol{\sigma}^+_L(W_{J)})}{\partial_\lambda H} \\ & - \frac{\boldsymbol{\Theta}^L_+(W_{(I}, W_{J)})}{\mu_1^+ \partial_\lambda H} - \frac{X^1 \mathbf{K}^k(q^-_I, v^-_J)}{\mu_1^+ \partial_\lambda H} + \frac{\nabla^h_I \nabla^h_J H}{\partial_\lambda H} + \frac{\nabla^h_{(I} \bar{\mu}^+_{J)}}{\mu_1^+ \partial_\lambda H} - \frac{\nabla^h_{(I} \mu^-_{J)}}{\mu_1^-} \Bigg),\end{aligned} \tag{7.90}$$

while those of $\mathbf{Y}^{-}$ are

$$\begin{aligned}\mathbf{Y}^{-}(\partial_\lambda, \partial_\lambda) &= -\mu_1^- \left(\kappa^-_{k^-} - \frac{\partial_\lambda \mu_1^-}{\mu_1^-} \right), \\ \mathbf{Y}^{-}(\partial_\lambda, \partial_{y^J}) &= \mu_1^- \left(\boldsymbol{\sigma}^-_L(v^-_J) + \frac{\nabla^h_J \mu_1^-}{2\mu_1^-} + \frac{\partial_\lambda \mu_J^-}{2\mu_1^-} \right), \\ \mathbf{Y}^{-}(\partial_{y^I}, \partial_{y^J}) &= \boldsymbol{\Theta}^L_-(v^-_{(I}, v^-_{J)}).\end{aligned} \tag{7.91}$$

Consequently, the components of the energy-momentum tensor of the shell are given by

$$\begin{aligned}\tau(d\lambda, d\lambda) = \epsilon \frac{\gamma^{IJ}}{\mu_1^-} \Bigg(& \frac{\kappa^+_{k^+} \nabla^h_I H \nabla^h_J H}{\partial_\lambda H} - \frac{\nabla^h_I H\, \partial_\lambda \bar{\mu}^+_J}{\mu_1^+ (\partial_\lambda H)^2} - \frac{2\nabla^h_I H\, \boldsymbol{\sigma}^+_L(W_J)}{\partial_\lambda H} - \frac{\boldsymbol{\Theta}^L_+(W_I, W_J)}{\mu_1^+ \partial_\lambda H} \\ & - \frac{X^1 \mathbf{K}^k(q^-_I, v^-_J)}{\mu_1^+ \partial_\lambda H} + \frac{\nabla^h_I \nabla^h_J H}{\partial_\lambda H} + \frac{\nabla^h_I \bar{\mu}^+_J}{\mu_1^+ \partial_\lambda H} - \frac{\nabla^h_I \mu^-_J}{\mu_1^-} + \frac{\boldsymbol{\Theta}^L_-(v^-_I, v^-_J)}{\mu_1^-} \Bigg),\end{aligned} \tag{7.92}$$

$$\begin{aligned}\tau(d\lambda, dy^I) = -\epsilon \frac{\gamma^{IJ}}{\mu_1^-} \Bigg(& \kappa^+_{k^+} \nabla^h_J H + \frac{\partial_\lambda \partial_{y^J} H}{\partial_\lambda H} \\ & + \frac{X^L \mathbf{K}^k_-(q^-_J, v^-_L)}{\mu_1^+ \partial_\lambda H} - \left(\boldsymbol{\sigma}^+_L(W_J) - \boldsymbol{\sigma}^-_L(v^-_J) \right) \Bigg),\end{aligned} \tag{7.93}$$

$$\tau(dy^I, dy^J) = \epsilon \frac{\gamma^{IJ}}{\mu_1^-} \left(\kappa^+_{k^+} \partial_\lambda H - \kappa^-_{k^-} + \frac{\partial_\lambda \partial_\lambda H}{\partial_\lambda H} \right). \tag{7.94}$$

Proof. Throughout the proof, we use the notation $\mathrm{Y}^{\pm}_{ab} \stackrel{\text{def}}{=} \mathbf{Y}^{\pm}(\partial_{y^a}, \partial_{y^b})$. Using (7.82) and the definition of $\mathbf{Y}^{+}$ one finds

$$\mathbf{Y}^{+} = \frac{1}{2A}\Big((\phi^{+})^{\star}\big(\pounds_{L^{+}} g^{+}\big) + \pounds_{X}\gamma - 2dA \otimes_s \mu^{-}_{I} dy^{I} - 2\mu^{-}_{1} dA \otimes_s d\lambda \Big), \qquad (7.95)$$

For the Y^{+}_{11} component, combining (7.74), (7.81) and (7.84) yields

$$\mathrm{Y}^{+}_{11} = \frac{1}{2A}\Big((\phi^{+})^{\star}\big(\pounds_{L^{+}} g^{+}\big)_{11} - 2\mu^{-}_{1}\partial_{\lambda}A \Big) = -\mu^{-}_{1}\left(\kappa^{+}_{k^{+}}\partial_{\lambda}H - \frac{\partial_{\lambda}\mu^{+}_{1}}{\mu^{+}_{1}} + \frac{\mu^{-}_{1}\,\partial_{\lambda}A}{\mu^{+}_{1}\,\partial_{\lambda}H} \right),$$

which is (7.88) after replacing $\partial_{\lambda}A$ as given in (7.73). The components Y^{+}_{1J} can be obtained from (7.95) by using (7.75), (7.81) to obtain $(\phi^{+})^{\star}\,(\pounds_{L^{+}} g^{+})\,(\partial_{\lambda}, \partial_{y^J})$ as well as the definition (7.61) of $\boldsymbol{\vartheta}$ and (7.85) to get $(\pounds_{X}\gamma)_{1J}$. This yields

$$\begin{aligned}
\mathrm{Y}^{+}_{1J} &= \frac{1}{2A}\Big((\phi^{+})^{\star}\big(\pounds_{L^{+}} g^{+}\big)_{1J} + (\pounds_{X}\gamma)_{1J} - \mu^{-}_{J}\partial_{\lambda}A - \mu^{-}_{1}\partial_{y^J}A \Big) \\
&= \frac{1}{2A}\Big(\partial_{\lambda}H\big(-2\mu^{+}_{1}\kappa^{+}_{k^{+}}\nabla^{h}_{J}H + 2\mu^{+}_{1}\boldsymbol{\sigma}^{+}_{L}(W_{J}) + \nabla^{h}_{J}\mu^{+}_{1}\big) \\
&\qquad - \mu^{+}_{1}\partial_{\lambda}\partial_{y^J}H + A\partial_{\lambda}\mu^{-}_{J} - 2X^{L}\mathbf{K}^{k}(v^{-}_{J}, v^{-}_{L}) - \mu^{-}_{1}\nabla^{h}_{J}A \Big), \qquad (7.96)
\end{aligned}$$

after some cancelling of terms occurs. Inserting (7.73) into (7.96) proves (7.89). Finally, for the components Y^{+}_{IJ} we combine (7.76), (7.81), (7.86), (7.87) and (7.95) to obtain

$$\begin{aligned}
\mathrm{Y}^{+}_{IJ} &= \frac{1}{2A}\Big((\phi^{+})^{\star}\big(\pounds_{L^{+}} g^{+}\big)_{IJ} + (\pounds_{X}\gamma)_{IJ} - \mu^{-}_{I}\partial_{y^J}A - \mu^{-}_{J}\partial_{y^I}A \Big) \\
&= \frac{1}{A}\Big(\langle \nabla^{+}_{e^{+}_{(I}} L^{+}, e^{+}_{J)} \rangle_{g^{+}} + X^{1}\mathbf{K}^{k}(v^{-}_{I}, v^{-}_{J}) + \nabla^{h}_{(I}X_{J)} - \mu^{-}_{(J}\nabla^{h}_{I)}A \Big) \\
&= \frac{1}{A}\Big(\langle \nabla^{+}_{e^{+}_{(I}} L^{+}, e^{+}_{J)} \rangle_{g^{+}} + X^{1}\mathbf{K}^{k}(v^{-}_{I}, v^{-}_{J}) - \nabla^{h}_{(I}\mu^{+}_{1}\nabla^{h}_{J)}H \\
&\qquad - \mu^{+}_{1}\nabla^{h}_{(I}\nabla^{h}_{J)}H - \nabla^{h}_{(I}\bar{\mu}^{+}_{J)} + A\nabla^{h}_{(I}\mu^{-}_{J)} \Big) \\
&= \frac{1}{A}\Big(-\mu^{+}_{1}\kappa^{+}_{k^{+}}\nabla^{h}_{(I}H\,\nabla^{h}_{J)}H + \frac{\nabla^{h}_{(I}H\,\partial_{\lambda}\mu^{+}_{J)}}{\partial_{\lambda}H} + 2\mu^{+}_{1}\nabla^{h}_{(I}H\,\boldsymbol{\sigma}^{+}_{L}(W_{J)}) + \boldsymbol{\Theta}^{L}_{+}(W_{(I}, W_{J)}) \\
&\qquad + X^{1}\mathbf{K}^{k}(v^{-}_{I}, v^{-}_{J}) - \mu^{+}_{1}\nabla^{h}_{(I}\nabla^{h}_{J)}H - \nabla^{h}_{(I}\bar{\mu}^{+}_{J)} + A\nabla^{h}_{(I}\mu^{-}_{J)} \Big),
\end{aligned}$$

which becomes (7.90) upon using (7.45) and $\nabla^{h}_{(I}\nabla^{h}_{J)}H = \nabla^{h}_{I}\nabla^{h}_{J}H$. To get $\mathbf{Y}^{-}$ it suffices to particularize (7.88)-(7.90) for $b^{J}_{I} = \delta^{J}_{I}$, $X^{a} = 0$ and $H(\lambda, y^{A}) = \lambda$, as well as replacing all $+$ superscripts by $-$. The components of the energy-momentum tensor are obtained from (7.58)-(7.60). □

Remark 7.3.8. *Proposition 7.3.7 provides explicit expressions for the tensor fields* $\mathbf{Y}^{\pm}$ *and* τ *arising from the matching of any two spacetimes in terms of known geometric objects (e.g.* $\mathbf{K}^{k}_{\pm}$, $\mathbf{\Theta}^{L}_{\pm}$*) plus the functions* $\{H(\lambda, y^A), h^B_A(y)\}$ *(note that the functions* $\{h^B_A(y)\}$ *are hidden in the coefficients* b^I_J*, which are present in* W_J *and* $\overline{\mu}^+_{(J)}$*). This result is fully general except by the restriction on the topology of the boundaries of the spacetimes to be matched.*

The relevance of Proposition 7.3.7 relies on the fact that, given two (matchable) spacetimes with null boundaries and once we know how the points of the boundaries are to be identified (i.e. given the step function $H(\lambda, y^A)$ *and the map* Ψ *which sends null generators to null generators), Proposition 7.3.7 automatically yields the gravitational and matter-energy content of the resulting null thin shell.*

Remark 7.3.9. *In the literature, the different components of the energy-momentum tensor are interpreted physically as an energy density* ρ*, an energy-flux* j *and a pressure* p *(see e.g. [128]). However, this is usually done in a context where the matching riggings are null and orthogonal to the leaves of the foliation (i.e. where* $\mu^-_A = 0$ *and* $\ell^{(2)} = 0$*). We propose the following geometric definitions for the physical quantities* $\{\rho, p, j\}$*:*

$$\rho \underset{\text{def}}{=} -\epsilon \operatorname{tr}_P[\mathbf{Y}], \qquad p \underset{\text{def}}{=} -\epsilon[\mathbf{Y}](n,n), \qquad j \underset{\text{def}}{=} \epsilon \left(P([\mathbf{Y}](n,\cdot),\cdot) - \epsilon \ell^{(2)} p n \right). \tag{7.97}$$

The underlying reason that justifies (7.97) is that the energy-momentum tensor (2.155) of a null thin shell $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}^{\pm}, \rho^{\pm}_{\ell}, \mathrm{J}^{\pm}, \epsilon\}$ *can be written in terms of* $\{\rho, p, j\}$ *as*

$$\tau = \rho n \otimes n + p\left(P + 2\ell^{(2)} n \otimes n \right) + 2j \otimes_s n. \tag{7.98}$$

As mentioned in Remark 2.7.4, the sign ϵ *must be taken positive when* ζ^- *points outwards with respect to* $(\mathcal{M}^-, g^-)$ *and vice versa. Here the necessity of introducing* ϵ *becomes clear, since it makes the definitions 7.97 invariant under a change in the orientation of the matching riggings (recall that* $[\mathbf{Y}]$ *changes its sign under a transformation of the form* $\zeta^{\pm} \longrightarrow -\zeta^{\pm}$*).*

The vector field j *satisfies* $\gamma(j, \cdot) = \epsilon[\mathbf{Y}](n, \cdot) + p\boldsymbol{\ell}$ *and* $\boldsymbol{\ell}(j) = 0$*, which makes the definitions (7.97) consistent since the one-form* $\boldsymbol{j} \stackrel{\text{def}}{=} \gamma(j, \cdot)$ *verifies* $\boldsymbol{j}(n) = 0$*. Moreover, a direct calculation that relies on (2.44) and (3.64) proves the following gauge behaviour for the pressure* p*:*

$$\mathcal{G}_{(z,V)}(p) = \frac{p}{|z|}. \tag{7.99}$$

Whenever $\{\mu_1^- = 1, \mu_A^- = 0\}$, $\ell^{(2)} = 0$, *it is straightforward to check that* (7.97) *becomes*

$$\rho = \tau(d\lambda, d\lambda), \qquad p\gamma^{AB} = \tau(dy^A, dy^B), \qquad j^a = \delta^a_A\, \tau(d\lambda, dy^A), \tag{7.100}$$

after using (7.57)-(7.60) *and the fact that* $n = (\mu_1^-)^{-1}\partial_\lambda$. *The combination of* (7.58)-(7.60) *and* (7.100) *allows one to recover the standard definitions for* $\{\rho,\ p,\ j\}$ *introduced e.g. in* [128], *namely*[3]

$$\rho = -\epsilon\gamma^{AB}[\mathbf{Y}](\partial_{y^A}, \partial_{y^B}), \quad p = -\epsilon[\mathbf{Y}](\partial_\lambda, \partial_\lambda), \quad j^a = \epsilon\delta^a_A\, \gamma^{AB}[\mathbf{Y}](\partial_\lambda, \partial_{y^B}). \tag{7.101}$$

7.3.2.1 *Gauge behaviour of the energy-momentum tensor*

As mentioned in Section 2.7.1 (recall (2.159)), the energy-momentum tensor on the shell depends on the choice of rigging solely by scale. Specifically, two energy-momentum tensors τ^{ab}, $\boldsymbol{\tau}^{ab}$ associated to two different choices of rigging ζ, $\tilde{\zeta} = z\zeta + V$, where z is a function on the matching hypersurface and V is a vector field tangent to it, are related by $\tilde{\tau}^{ab} = |z|^{-1}\tau^{ab}$. This is a consequence of the fact that the volume form (recall Definition 2.2.3) is also gauge dependent and transforms as $\mathcal{G}_{(z,V)}(\mathrm{W}_{\mathrm{vol}}^{(\ell)}) = |z|\mathrm{W}_{\mathrm{vol}}^{(\ell)}$ [59]. Under orientation preserving gauge transformations, the suitable gauge-invariant object is actually $\tau^{ab}\boldsymbol{\eta}$, which is consistent because in physical terms τ^{ab} is energy-momentum per unit volume. This fact can be used to perform a non-trivial consistency check on the expressions (7.92)-(7.94), as we show next.

Let us suppose that a matching of two spacetimes $(\mathcal{M}^{\pm}, g^{\pm})$ has been performed and that the rigging has been fixed by (7.26) after we have selected a null transverse vector field L^-. We may repeat the matching process using a different **null** transverse vector

$$\tilde{L}^- = aL^- + bk^- + c^I v_I^- \tag{7.102}$$

while still enforcing (7.26), i.e. $\tilde{\zeta}^- = \tilde{L}^-$. Let us use tilde for all objects constructed with $\tilde{L}$. Then, definitions (7.6) imply

$$\tilde{\mu}_1^- = a\mu_1^-, \qquad \tilde{\mu}_I^- = a\mu_I^- + c_I, \tag{7.103}$$

while the null character of $\tilde{L}^-$ imposes

$$-2a\left(b\mu_1^- + c^I\mu_I^-\right) = |q|^2_{h^-} \qquad \text{where} \qquad |q|^2_{h^-} \stackrel{\text{def}}{=} c^I c^J h^-_{IJ} \tag{7.104}$$

[3] Expressions (7.101) coincide with the definitions proposed by Poisson whenever $\epsilon = -1$ (i.e. when the rigging ζ^- points inwards with respect to $(\mathcal{M}^-, g^-)$).

Changing the rigging on the $(\mathcal{M}^-, g^-)$ side keeps the vector fields $e^\pm_a$ invariant, as they only depend on the embeddings $\phi^\pm$. This means that the functions f, a_I and b^J_I in (7.31)-(7.34) do not change either. On the other hand, the identification of the riggings of both sides implies that the rigging in the $(\mathcal{M}^+, g^+)$ side also gets modified. Let us decompose it in terms of L^+ and a vector field $\bar{X} \in \Gamma(T\bar{N}^+)$ as $\bar{\zeta}^+ = (1/\bar{A})\,L^+ + \bar{X}^a e^+_a$. The shell junction condition (7.32) forces

$$f = \frac{A\mu_1^-}{\mu_1^+} \qquad \text{as well as} \qquad f = \frac{\bar{A}\bar{\mu}_1^-}{\mu_1^+},$$

which together with (7.103) gives

$$f = \frac{\bar{A}a\mu_1^-}{\mu_1^+} \qquad \Longleftrightarrow \qquad \bar{A}a = A. \tag{7.105}$$

Since $\zeta^+ = (1/A)\,(L^+ + X^a e^+_a)$ (recall (7.63)), it follows that the two riggings $\bar{\zeta}^+$ and ζ^+ are related by

$$\bar{\zeta}^+ = a\zeta^+ + \frac{a}{A}\left(\bar{X}^a - X^a\right) e^+_a. \tag{7.106}$$

Inserting (7.106) into $\langle \bar{\zeta}^+, e^+_B\rangle_{g^+} = \bar{\mu}^-_B$, $\langle \bar{\zeta}^+, \bar{\zeta}^+\rangle_{g^+} = 0$ (which are simply the junction conditions (7.28)-(7.29)) and using (7.103)-(7.104) yields

$$\bar{X}^B = X^B + \frac{A}{a}c^B, \qquad \bar{X}^1 = X^1 + \frac{Ab}{a}. \tag{7.107}$$

Each component of the energy-momentum tensor (cf. (7.92)-(7.94)) is multiplied by $\epsilon(\mu_1^-)^{-1}$, and $\bar{\epsilon} = \text{sign}(a)\epsilon$ because of (2.154) and (7.102). Thus, $\bar{\epsilon}(\bar{\mu}_1^-)^{-1} = |a|^{-1}\epsilon(\mu_1^-)^{-1}$ and therefore the transformation law of the energy-momentum tensor will be guaranteed provided each bracket in (7.92)-(7.94) turns out to be invariant. The only parts that are not trivially invariant are

$$\gamma^{IJ} \cdot \left(\frac{X^1 \mathbf{K}^k_-(v^-_I, v^-_J)}{\mu_1^+ \partial_\lambda H} - \frac{\nabla_I^h \mu^-_J}{\mu_1^-} + \frac{\Theta^L_-(v^-_I, v^-_J)}{\mu_1^-} \right) \qquad \text{in} \qquad \tau(d\lambda, d\lambda), \tag{7.108}$$

$$\sigma_L^-(v^-_J) + \frac{X^B \mathbf{K}^k_-(v^-_J, v^-_B)}{\mu_1^+ \partial_\lambda H} \qquad \text{in} \qquad \tau(d\lambda, dy^J). \tag{7.109}$$

Combining (7.12), (7.26) and the definition $h^-_{IJ} \stackrel{\text{def}}{=} g^-(v^-_I, v^-_J)$, one finds

$$\begin{aligned}
\partial_{y^B} h^-_{IJ} &= e^-_B(h^-_{IJ}) = \nabla^-_{v^-_B}(h^-_{IJ}) \\
&= 2g^-(\nabla^-_{v^-_B} v_{(I}, v_{J)}) = 2\Xi^{-A}_{B(J}\, h^-_{I)A} - \frac{2}{\mu_1^-}\mu^-_{(I}\,\mathbf{K}^k_-(v^-_{J)}, v^-_B).
\end{aligned}$$

Therefore, the Christoffel symbols Γ^{hA}_{BI} of the Levi-Civita covariant derivative ∇^h of the metric h^-_{IJ} are given by

$$h^-_{JA}\Gamma^{hA}_{BI} = \frac{1}{2}\left(\partial_{y^B}h^-_{IJ} + \partial_{y^I}h^-_{BJ} - \partial_{y^J}h^-_{BI}\right) = \Xi^{-A}_{BI}h^-_{JA} - \frac{1}{\mu^-_1}\mu^-_J\,\mathbf{K}^{k^-}_{-}\left(v^-_I, v^-_B\right). \quad (7.110)$$

Now computing $\mathbf{\Theta}^{\underline{L}}_-(v^-_I, v^-_J)$ gives

$$\begin{aligned}
\mathbf{\Theta}^{\underline{L}}_-(v^-_I, v^-_J) &= \langle\nabla^-_{v^-_I}\underline{L}^-, v^-_J\rangle_{g^-}\\
&= \langle v^-_I(a)L^- + a\nabla^-_{v^-_I}L^- + b\nabla^-_{v^-_I}k^- + v^-_I(c^B)v^-_B + c^B\nabla^-_{v^-_I}v^-_B, v^-_J\rangle_{g^-}\\
&= \mu^-_J\partial_{y^I}a + a\mathbf{\Theta}^L_-(v^-_I, v^-_J) + b\mathbf{K}^k_-(v^-_I, v^-_J) + h^-_{BJ}\partial_{y^I}c^B\\
&\quad + c^B\left(\Xi^{-A}_{BI}h^-_{AJ} - \frac{1}{\mu^-_1}\mu^-_J\,\mathbf{K}^{k}_-\left(v^-_I, v^-_B\right)\right),\\
&= \mu^-_J\partial_{y^I}a + a\mathbf{\Theta}^L_-(v^-_I, v^-_J) + b\mathbf{K}^k_-(v^-_I, v^-_J) + \nabla^h_I c_J,
\end{aligned}$$

where in the third line we used Lemma 7.1.2. It follows directly from (7.103) that $\nabla^h_I\underline{\mu}^-_J = \mu^-_J\partial_{y^I}a + a\nabla^h_I\mu^-_J + \nabla^h_I c_J$. Thus, we conclude

$$\mathbf{\Theta}^{\underline{L}}_-(v^-_I, v^-_J) = a\left(\mathbf{\Theta}^L_-(v^-_I, v^-_J) - \nabla^h_I\mu^-_J\right) + b\mathbf{K}^k_-(v^-_I, v^-_J) + \nabla^h_I\underline{\mu}^-_J, \quad (7.111)$$

and the invariance of (7.108) follows from this expression and the second in (7.107). Concerning $\boldsymbol{\sigma}_{\underline{L}^-}(v^-_J)$, we easily find

$$\begin{aligned}
\boldsymbol{\sigma}_{\underline{L}^-}(v^-_J) &= -\frac{1}{\underline{\mu}^-_1}\langle\nabla^-_{v^-_J}k^-, \underline{L}^-\rangle_{g^-} = -\frac{1}{a\mu^-_1}\langle\nabla^-_{v^-_J}k^-, aL^- + c^B v^-_B\rangle_{g^-}\\
&= \boldsymbol{\sigma}_{L^-}(v^-_J) - \frac{c^B}{a\mu^-_1}\mathbf{K}^{k^-}_-(v^-_J, v^-_B).
\end{aligned} \quad (7.112)$$

Given that $\mu^+_1\partial_\lambda H = \mu^-_1 A$ (cf. (7.45)), one obtains the invariance of (7.109) by means of the first expression in (7.107). This eventually proves that indeed $\underline{\tau}^{ab} = \frac{1}{a}\tau^{ab}$ holds, and hence establishes a consistency check of (7.92)-(7.94).

7.3.3 *Matching of two Minkowski regions across a null hyperplane*

As we have discussed, one of the main benefits of using the previous formalism is that multiple sorts of matchings can be analysed at once. For instance, one may be interested in considering a family of energy-momentum tensors verifying the surface layer equations (2.160)-(2.161), or a set of step functions $H(\lambda, y^A)$ with certain properties. In the general case, this task is out of reach by means of the cut-and-paste method because these constructions only allow to match two regions of the same spacetime and, even if one is interested on macthings of this type, more complex shells will require more involved forms of the corresponding distributional metric.

In this section, we will exploit the matching formalism introduced before to analyse the most general matching of two regions of the spacetime of Minkowski across a null hyperplane. This was the first matching addressed with the cut-and-paste procedure [3], [85], [86], [87]. In these seminal works, Penrose was able to construct plane-fronted impulsive waves propagating in the spacetime of Minkowski by considering a metric with a Dirac delta distribution with support on the matching hypersurface. More recent research on the topic of plane-fronted impulsive gravitational waves in the spacetime of Minkowski can be found e.g. in [88], [90], [91], [92], [5], [6], [7] and references therein. In these latter publications, Penrose's construction has been generalized to a variety of more complex scenarios, e.g. to the de Sitter and anti-de Sitter spacetimes or to spacetimes with impulsive metrics containing so-called gyratonic terms.

Our aim in this section is two-fold. First, we will recover the results from cut-and-paste across a hyperplane in Minkowski within our matching formalism, establishing a connection between these two formalisms. Secondly, we will obtain the most general shell that can be generated by matching two regions of Minkowski across a null hypersplane. This will prove that one can produce more general shells with different values of energy, energy flux or pressure.

Consider the $(n+1)$-dimensional Minkowski spacetimes

$$\left(\mathrm{M}^{\pm}, g^{\pm}\right), \quad \text{with} \quad g^{\pm} = -2d\mathrm{U}_{\pm}d\mathrm{V}_{\pm} + \delta_{AB}dx^{A}_{\pm}dx^{B}_{\pm} \quad \text{and} \quad \mathrm{U}_{\pm} \gtrless 0. \tag{7.113}$$

The boundaries of $(\mathrm{M}^{\pm}, g^{\pm})$ are the null hypersurfaces defined by

$$\mathcal{N}^{\pm} \stackrel{\mathrm{def}}{=} \{\mathrm{U}_{\pm} = 0\}, \tag{7.114}$$

which clearly satisfy the topology condition that they can be foliated by diffeomorphic spacelike cross-sections.

The first step in order to apply the matching formalism presented above is to construct two basis $\{L^{\pm}, k^{\pm}, v^{\pm}_{I}\}$ of $\Gamma(T\mathrm{M}^{\pm})|_{\mathcal{N}^{\pm}}$ and two foliation functions $v_{\pm} \in \mathcal{F}(\mathcal{N}^{\pm})$ according to (7.2). For simplicity, we take $\{k^{\pm} = \partial_{\mathrm{V}_{\pm}}, v^{\pm}_{I} = \partial_{x^{I}_{\pm}}\}$ as a basis of $\Gamma(T\mathcal{N}^{\pm})$ and $v_{\pm} = \mathrm{V}_{\pm}$ as the foliation functions. These objects obviously verify the requirements in (7.2). As transverse null vector fields we select $L^{\pm} = -\partial_{\mathrm{U}_{\pm}}$

(note that $L^{\pm}$ are past vector fields while $k^{\pm}$ are future). These choices satisfy $\mu_1^{\pm} = 1$, $\mu_I^{\pm} = 0$ and it is straightforward to check that $\kappa^{\pm}$, $\sigma_L^{\pm}$, $p_k^{\pm}$, $\mathrm{K}^k_{\pm}(v_I^{\pm}, v_J^{\pm})$, $\Theta^L_{\pm}(v_I^{\pm}, v_J^{\pm})$ all vanish. Observe that (7.113) together with the choice of L^- (which points inwards with respect to $\mathcal{M}^-$) requires that we take $\epsilon = -1$ when computing the energy-momentum tensor (recall that with (7.26) we take L^- as a matching rigging).

Now, to construct null metric hypersurface data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ embedded in $(\mathcal{M}^-, g^-)$, we consider an embedding ϕ^- of the form

$$
\begin{array}{cccll}
\phi^- : & \mathcal{N} & \hookrightarrow & \widetilde{\mathcal{N}}^- \subset \mathcal{M}^- & \\
& (\lambda, y^I) & \longmapsto & \phi^-(\lambda, y^I) = & \left(U_- = 0, V_- = \lambda, x_-^I = y^I \right).
\end{array}
\tag{7.115}
$$

In these circumstances, the data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ being embedded in $(\mathcal{M}^-, g^-)$ with embedding ϕ^- and rigging $\zeta^- = L^-$ means (by (2.22))

$$
\gamma = \delta_{AB} dy^A \otimes dy^B, \qquad \boldsymbol{\ell} = d\lambda, \qquad \ell^{(2)} = 0. \tag{7.116}
$$

For the present case, equation (7.48) yields $V_+ = H(\lambda, y^A)$, while the vector fields $\{e_I^+\}$ are given by

$$
e_I^+ = (\partial_{y^I} H)\partial_{V_+} + (\partial_{y^I} h^J)\partial_{x_+^J} \tag{7.117}
$$

as an immediate consequence of combining (7.30) and (7.40)-(7.41). It is straightforward to conclude that for the matching of $(\mathcal{M}^{\pm}, g^{\pm})$ across $\widetilde{\mathcal{N}}^{\pm}$ to be possible the embedding ϕ^+ corresponding to the plus side must be necessarily of the form

$$
\begin{array}{cccll}
\phi^+ : & \mathcal{N} & \hookrightarrow & \widetilde{\mathcal{N}}^+ \subset \mathcal{M}^- & \\
& (\lambda, y^I) & \longmapsto & \phi^+(\lambda, y^I) = & \left(U_+ = 0, V_+ = H(\lambda, y^I), x_+^I = h^I(y^J) \right).
\end{array}
\tag{7.118}
$$

As we discussed in Section 7.3.1, the viability of the matching relies on the solvability of the isometry condition (7.31) because (7.32) only provides the form of the vector field e_1^+ (and hence of the function A which fixes the transverse part of the matching rigging ζ^+) while (7.33)-(7.34) determine the tangent part of ζ^+. Concretely, when the matching is possible the rigging ζ^+ will read (see (7.70) in Corollary 7.3.4)

$$
\zeta^+ = \frac{1}{\partial_\lambda H} \left(L^+ - \delta^{AB} (b^{-1})^I{}_A (\partial_{y^I} H) \left(\frac{1}{2} (b^{-1})^J{}_B (\partial_{y^J} H) k^+ + v_B^+ \right) \right). \tag{7.119}
$$

Let us therefore check whether condition (7.31) can indeed be satisfied by the boundaries $\mathcal{N}^{\pm}$. Particularizing it to the present case yields

$$\delta_{IJ} = b^L_I\, b^K_J\, \delta_{LK}, \qquad b^I_J \stackrel{\text{def}}{=} \frac{\partial h^I}{\partial y^J}, \tag{7.120}$$

which constitutes an isometry condition between the spatial cross-sections $S_\lambda \stackrel{\text{def}}{=} \{\lambda = \text{const.}\}$ of $\mathcal{N}$ and their images $\phi^+(S_\lambda)$ through the embedding ϕ^+. In particular, this isometry condition forces $\{S_\lambda\}$ and $\{\phi^+(S_\lambda)\}$ to be euclidean planes. The corresponding isometries are obviously translations and rotations, and it turns out that the symmetry properties of $(\mathcal{M}^+, g^+)$ allow one to perform the necessary combinations of rotations and translations so that the initial coordinates $\{U_+, V_+, x^I_+\}$ transform into new coordinates $\{U'_+, V'_+, x'^A_+\}$ verifying $b^J_I = \delta^J_I$. In other words, there always exists an isometry of $\mathcal{M}^+$ which turns (8.31) into an identically satisfied equation. Thus, the matching of $(\mathcal{M}^{\pm}, g^{\pm})$ is always possible and in fact, since $\mathcal{N}^{\pm}$ are totally geodesic null hypersurfaces (recall that $\mathbf{K}^k_{\pm} = 0$), an infinite number of matchings can be performed (see the discussion in Section 7.3.1.1). Observe that the reasoning above allows one to set $b^J{}_I = \delta^J{}_I$ whenever it is convenient. However, the results that follow are insensitive to $b^J{}_I$ so we refrain ourselves from doing this.

Now that we know that the matching is feasible, we can compute the matter-energy content of the resulting null shells. For that it suffices to particularize the results of Proposition 7.3.7. Using the notation $\mathrm{Y}^{\pm}_{ab} \stackrel{\text{def}}{=} \mathbf{Y}^{\pm}(\partial_{y^a}, \partial_{y^b})$, $\tau^{ab} \stackrel{\text{def}}{=} \tau(dy^a, dy^b)$, for the tensor fields $\mathbf{Y}^{\pm}$ one finds

$$\mathrm{Y}^-_{ab} = 0, \qquad \mathrm{Y}^+_{ab} = -\frac{\partial_{y^a}\partial_{y^b} H}{\partial_\lambda H}, \tag{7.121}$$

whereas the energy-momentum tensor of the shell in the present case reads

$$\tau^{11} = -\frac{\delta^{IJ}\partial_{y^I}\partial_{y^J} H}{\partial_\lambda H}, \qquad \tau^{1I} = \frac{\delta^{IJ}\partial_\lambda\partial_{y^J} H}{\partial_\lambda H}, \qquad \tau^{IJ} = -\frac{\delta^{IJ}\partial_\lambda\partial_\lambda H}{\partial_\lambda H}. \tag{7.122}$$

From (7.122) it is immediate to get

$$\delta_{IJ}\tau^{IJ} = -\frac{(n-1)\,\partial_\lambda\partial_\lambda H}{\partial_\lambda H} \quad \text{and} \quad \delta_{IJ}\tau^{1I} = \frac{\partial_\lambda\partial_{y^J} H}{\partial_\lambda H}, \tag{7.123}$$

which can be combined with (7.121)-(7.122) to obtain

$$\tau^{ab}\mathrm{Y}^+_{ab} = \frac{2\delta^{IJ}}{(\partial_\lambda H)^2}\Big((\partial_{y^I}\partial_{y^J} H)\,(\partial_\lambda\partial_\lambda H) - (\partial_\lambda\partial_{y^I} H)\,(\partial_\lambda\partial_{y^J} H) \Big)$$

$$= 2\delta_{IJ}\left(\frac{\tau^{11}\tau^{IJ}}{n-1} - \tau^{1I}\tau^{1J}\right). \tag{7.124}$$

This, together with $\mathbf{Y}^- = 0$, $|\det A| = 1$, $\tau^{ab}\ell_b = \tau^{1a}$, $\tau^{ba}\gamma_{ac} = \delta^B_c\gamma_{AB}\tau^{bA}$ and the vanishing of the Einstein tensor in Minkowski, brings the shell field equations (2.160)-(2.161) into the following form

$$0 = \partial_\lambda\tau^{11} + \partial_{y^A}\tau^{1A} - \delta_{IJ}\left(\frac{\tau^{11}\tau^{IJ}}{n-1} - \tau^{1I}\tau^{1J}\right), \quad 0 = \partial_\lambda\tau^{1A} + \partial_{y^B}\tau^{AB}. \tag{7.125}$$

A direct calculation shows that the expressions (7.122) indeed fulfil the surface layer equations (7.125).

Our next aim is to study different shells that can be generated from the matching of $(\mathcal{M}^\pm, g^\pm)$. As we shall see, for certain type of matchings we will recover the jump (2.170) proposed by Penrose, which corresponds to a step function of the form

$$H(\lambda, y^A) = \lambda + \mathrm{H}(y^A), \qquad \mathrm{H} \in \mathcal{F}(\mathcal{N}). \tag{7.126}$$

The matching of two Minkowski regions with the step function (7.126) will correspond to spacetimes describing plane-fronted impulsive waves (purely gravitational when $\mathrm{H}(y^A)$ is harmonic in the coordinates y^A). The framework introduced throughout Section 7.3, however, must provide all the possible matchings of two Minkowski regions and hence a more general set of step functions. This general matching, together with some interesting particular cases, are discussed below.

No-shell case

Let us start by considering no shell, i.e. $[\mathbf{Y}] = 0$. The results for this case should be viewed as a consistency check, since the absence of shell must give rise to the whole Minkowski spacetime. Since by (7.121), condition $[\mathbf{Y}] = 0$ is equivalent to $\mathbf{Y}^+ = 0$, we can integrate the right part of (7.121) and obtain $H(\lambda, y^A) = a\lambda + c_J y^J + d$, where $a, c_J, d \in \mathbb{R}$ and $a > 0$. In view of (7.115) and (7.118) this step function corresponds to the jump $V_+ = aV_- + c_J y^J + d$ when crossing the hypersurface $U_\pm = 0$. This means that the only possible isometries between the boundaries $\mathcal{N}^\pm$ are (besides the translations and rotations in the $\{x^I_+\}$ coordinates already discussed) null translations and null rotations in the $(\mathcal{M}^+, g^+)$ side . Since all of them are isometries of the Minkowski metric, the matching indeed recovers the global Minkowski spacetime.

Vacuum case

We next consider the vacuum case, i.e. $\tau^{ab} = 0$. Integrating (7.122) with the l.h.s. equal to zero gives the step function

$$\tau^{1J} = \tau^{IJ} = 0 \quad \Longleftrightarrow \quad H(\lambda, y^A) = a\lambda + \mathrm{H}(y^A), \qquad \text{where} \quad 0 < a \in \mathbb{R} \tag{7.127}$$

$$\tau^{11} = 0 \quad \Longleftrightarrow \quad \sum_{I=2}^{n} \frac{\partial^2 \mathrm{H}}{(\partial y^I)^2} = 0. \tag{7.128}$$

The freedom in a corresponds to a boost in the (M^+, g^+) spacetime so we may set $a = 1$ without loss of generality and hence recover Penrose's step (7.126). Note that setting $\tau^{ab} = 0$ automatically forces $\mathrm{H}(y^A)$ to be harmonic, which is consistent with the Dirac delta limit of $\mathrm{P}\,(\mathrm{U}, x, z)$ when the vacuum equations for (2.162) are imposed (see the discussion in Section 2.7.2). This type of matching also constitutes an example of the fact that $\tau^{ab} = 0$ does not necessarily mean $[\mathbf{Y}] = 0$, as we mentioned in Section 2.7.1. The non-vanishing jump $[\mathbf{Y}]$ encodes, in this case, the purely gravitational content of the shell.

Non-vanishing energy density

As a simple generalization of the previous example, one can consider non-zero energy, i.e. $\tau^{11} \neq 0$, while keeping $\tau^{1J} = \tau^{IJ} = 0$. This does not change the form of the step function, which is still given by (7.127). It follows that the step function (7.126) from Penrose's cases corresponds to absence of pressure and energy flux. Therefore, it describes either **purely gravitational waves** (when $\tau = 0$ but $[\mathbf{Y}] \neq 0$) or **shells of null dust** (when $\tau^{11} \neq 0$, $\tau^{1J} = \tau^{IJ} = 0$). The latter corresponds to a pressureless fluid of massless particles moving at the speed of light. Observe that (7.125) implies that τ^{11} must be λ-independent. By writing $\tau^{11} = \rho(y^A)$ (recall (7.100)) and using (7.122), one gets

$$\sum_{I=2}^{n} \frac{\partial^2 \mathrm{H}}{(\partial y^I)^2} = -a\rho(y^A). \tag{7.129}$$

Again, the constant a can be set to one by applying a boost in (M^+, g^+). Observe that the energy condition $\rho(y^A) \geq 0$ is equivalent to $H(\lambda, y^A)$ being a superharmonic function.

General null shell in the spacetime of Minkowski

Finally, let us keep both the energy and the energy flux of the shell completely free and consider a non-zero pressure $p\left(\lambda, y^A\right)$ (recall Remark 7.3.9). This case has not been covered in any of the cut-and-paste works cited above. Since $\partial_\lambda H > 0$, the pressure can be expressed as $p = -\partial_\lambda \left(\ln\left(\partial_\lambda H\right)\right)$, whose integration gives $\partial_\lambda H = \beta(y^A) \exp\left(-\int p\left(\lambda, y^A\right) d\lambda\right)$, where $\beta(y^A) > 0$ is the integration "constant". Therefore,

$$H(\lambda, y^A) = \beta(y^A) \int \exp\left(-\int p(\lambda, y^A) d\lambda\right) d\lambda + \mathrm{H}(y^A), \tag{7.130}$$

where $\mathrm{H}(y^A)$ is a second integration function.

In order to discuss the effect of the pressure in the matching, we start by noting the following simple consequences of $e_1^- = k^-$ and $e_1^+ = (\partial_\lambda H) k^+$. Since in the present case $k^\pm$ are geodesic and affinely parametrized, it follows

$$\begin{aligned} e_1^-(v_-) &= 1, & \nabla^-_{e_1^-} e_1^-(v_-) &= 0, \\ e_1^+(v_+) &= \partial_\lambda H, & \nabla^+_{e_1^+} e_1^+(v_+) &= \partial_\lambda \partial_\lambda H. \end{aligned} \tag{7.131}$$

Consider two null generators $\sigma^- \subset \widetilde{\mathcal{N}}^-$, $\sigma^+ = \Phi(\sigma^-) \subset \widetilde{\mathcal{N}}^+$. Both foliation functions $v_\pm$ have been built so that their rate of change measured by $k^\pm$ is equal to one, cf. (7.2). We call "velocity" the rate of change of $v_\pm$ along a null vector along $\widetilde{\mathcal{N}}^\pm$ and "acceleration" the rate of change of the velocity. The matching, however, does not identify the vectors $k^\pm$ but the vectors $e_1^\pm$. Therefore, when moving along $\sigma^\pm \subset \widetilde{\mathcal{N}}^\pm$, the velocity and acceleration associated to $e_1^\pm$ (i.e. as measured by λ) can be different, as shows (7.131).

Let us hence take λ as the measure parameter for both sides. This allows us to introduce the concepts of self-compression and self-stretching of points along any null generator $\sigma^\pm$. *There will exist self-compression (resp. self-stretching) whenever the acceleration measured by λ is strictly negative (resp. positive).* Accordingly, this effect will not take place on $\widetilde{\mathcal{N}}^-$ due to its identification with $\mathcal{N}$, but it may certainly occur in $\widetilde{\mathcal{N}}^+$. Equations (7.131) show that the velocity and the acceleration are respectively given by the first and second derivatives of $H(\lambda, y^A)$. Consequently, this effect is ruled by the pressure, as it essentially determines $\partial_\lambda \partial_\lambda H$ at each point $q \in \mathcal{N}$. Note that vanishing pressure entails constant velocity, which obviously gives no self-compression nor self-stretching. However, the velocity along the curves $\sigma^\pm$ can still be different (this is why we are not using the terms "stretching" or "compressing", which would still be occurring in this situation).

From the definition of the pressure, it follows that $\mathrm{sign}\, p\left(\lambda, y^A\right) = -\mathrm{sign}\,(\partial_\lambda \partial_\lambda H)$. Consequently, if the pressure is positive (resp. negative) (cf. (7.131)), then the acceleration along q^+ is negative (resp. positive) and there exists self-compression (resp. self-stretching) of points towards the future. Alternatively, one can conclude that *positive pressure pushes points towards lower values of* $H\left(\lambda, y^A\right)$ *(or* s^+*) and vice versa*.

For a better understanding of this behaviour, let us consider a pressure depending only on λ and write $p(\lambda) = -\frac{q''}{q'}$, where $'$ denotes derivative with respect to λ and $q(\lambda)$ is any regular function with $q'(\lambda) > 0\ \forall\lambda$. From (7.130), it follows that $\partial_\lambda H = \beta(y^A) q'(\lambda) > 0$ and $H(\lambda, y^A) = \beta(y^A) q(\lambda) + \mathrm{H}(y^A)$, after simple redefinitions of $\beta(y^A)$ and $\mathrm{H}(y^A)$. Note that necessary and sufficient conditions for the range of the embedding ϕ^+ to be the whole of $\mathcal{N}^+$ is that $\beta(y^A) > 0$ and that $\lim_{\lambda\to\pm\infty} q(\lambda) = \pm\infty$ (recall that $q(\lambda)$ is monotonically increasing). The components of the energy-momentum tensor are

$$\tau^{11} = -\frac{\delta^{IJ}}{\beta q'}\left(q\partial_{y^I}\partial_{y^J}\beta + \partial_{y^I}\partial_{y^J}\mathrm{H}\right), \quad \tau^{1I} = \frac{\delta^{IJ}\partial_{y^J}\beta}{\beta}, \quad \tau^{IJ} = -\delta^{IJ}\frac{q''}{q'}. \tag{7.132}$$

Observe that this setup is still fairly general in the sense that it allows for energy-momentum tensors with all components different from zero. The specific behaviour of the energy-momentum tensor is obviously ruled by $q(\lambda)$ and the particular form of the functions $\beta(y^A)$, $\mathrm{H}(y^A)$. It is now clear that fixing the pressure amounts to setting the form of $H(\lambda, y^A)$, which in turn contains the information about the effect of self-compression or self-stretching on the $\mathcal{N}^+$ boundary.

As an example, let us define the function $\bar{q}(\lambda) \stackrel{\text{def}}{=} \sqrt{(a+2)\lambda^2 + b^2}$ and consider

$$q(\lambda) \stackrel{\text{def}}{=} (a+1)\lambda - \sqrt{a}\,\bar{q}(\lambda) \tag{7.133}$$

with $a > 0$ and b real constants. As the inequality $a + 1 - \sqrt{a(a+2)} > 0$ holds for all positive a, this function satisfies $\lim_{\lambda\to\pm\infty} q(\lambda) = \pm\infty$. The previous expressions yield

$$q'(\lambda) = a + 1 - \frac{\sqrt{a}(a+2)\lambda}{\bar{q}(\lambda)} > 0, \qquad q''(\lambda) = -\frac{\sqrt{a}(a+2)b^2}{\bar{q}^3(\lambda)} \leq 0 \tag{7.134}$$

$$p(\lambda) = \frac{\sqrt{a}(a+2)b^2}{\bar{q}^2(\lambda)\left((a+1)\bar{q}(\lambda) - \sqrt{a}(a+2)\lambda\right)} \geq 0, \tag{7.135}$$

$$H(\lambda, y^A) = \beta(y^A)\left((a+1)\lambda - \sqrt{a}\,\bar{q}(\lambda)\right) + \mathrm{H}(y^A), \tag{7.136}$$

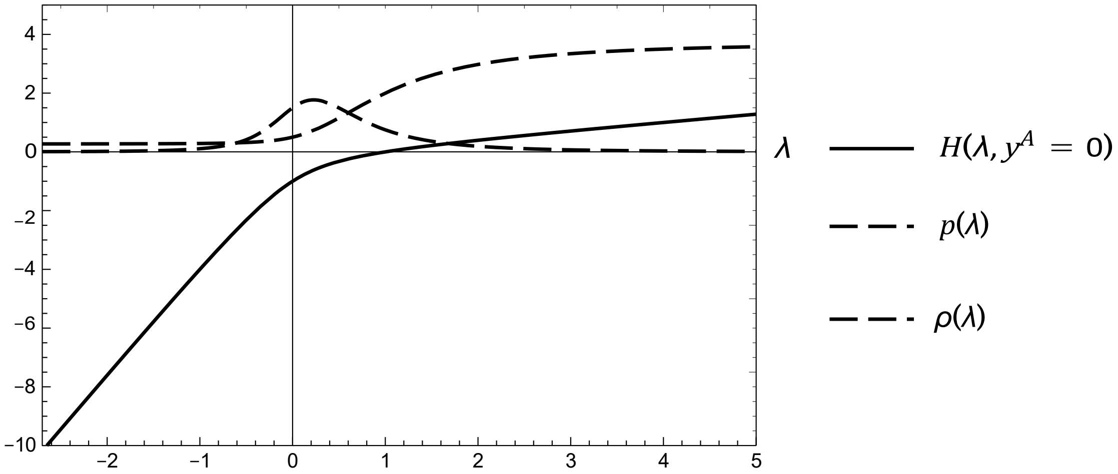


Figure 7.3: Matching of two regions of the spacetime of Minkowski: plot of the pressure p, step function H and energy density ρ given by (7.135)-(7.137) along the null generator $\{y^A = 0\}$ for the particular values $a = 1$, $b = 1$, $\beta(y^A) = 1$ and $\mathrm{H}(y^A) = \frac{1}{2(n-1)}\delta_{IJ}y^I y^J$.

and energy density of the shell is given by

$$\rho(\lambda, y^A) = -\frac{\bar{q}}{\beta}\,\frac{(a+1)\,\lambda - \sqrt{a}\,\bar{q} + \delta^{IJ}\partial_{y^I}\partial_{y^J}\beta + \delta^{IJ}\partial_{y^I}\partial_{y^J}\mathrm{H}}{(a+1)\,\bar{q} - \sqrt{a}\,(a+2)\,\lambda}. \tag{7.137}$$

This density diverges asymptotically at infinity (i.e. for $\lambda \to \pm\infty$) unless $\beta(y^A)$ is harmonic. If b vanishes we have zero pressure and we fall into a previous case (H linear in λ). When $b \neq 0$, the pressure is everywhere regular, positive and vanishes asymptotically at infinity. Under the restriction that $\beta(y^I)$ is harmonic, a plot of $p(\lambda)$, $H(\lambda, y^A)$ and $\rho(\lambda, y^A)$ along a null generator of $\overline{N}^{+}$ is depicted in Figure 7.3. For large negative values of λ, the step function exhibits a straight line behaviour which is a consequence of the fact that the pressure is negligibly small at past infinity. When $p(\lambda)$ starts increasing, the self-compression of points starts taking place and this forces the slope of $H(\lambda, y^A)$ to decrease until it reaches again an almost constant value in the late future, once the pressure becomes again negligible. The growth of the energy begins when the self-compression occurs and ends when the pressure approaches next-to-zero values. It tends to a finite positive value when the pressure vanishes, which suggests that *it only increases (resp. decreases) on regions where there exists self-compression (resp. self-stretching)*, showing an accumulative behaviour.

To illustrate that not all the choices for the pressure result in successful matchings, we consider one last case: positive constant pressure p (the negative case is completely analogous). In these circumstances, the integrals in (7.130) yield

$H(\lambda, y^A) = \mathrm{H}(y^A) - \frac{1}{p}\beta(y^A)e^{-p\lambda}$ and hence $\partial_\lambda\partial_\lambda H = -p\beta(y^A)e^{-p\lambda} < 0$. Combining (7.122) with Remark 7.3.9, it follows that the energy and energy flux of the shell are

$$\rho = \frac{\delta^{IJ}}{\beta}\left(\frac{1}{p}\partial_{y^I}\partial_{y^J}\beta - e^{p\lambda}\partial_{y^I}\partial_{y^J}\mathrm{H}\right), \qquad j^I = \delta^{IJ}\frac{\partial_J\beta}{\beta}. \tag{7.138}$$

In this situation, one finds that $\lim_{\lambda\to+\infty} H(\lambda, y^A) = \mathrm{H}(y^A)$. The positive pressure produces sustained and systematic self-compression of points for all values of λ, which eventually results in a positive upper bound for the step function. This spoils the matching, as all the points $p^+ \in \mathcal{N}^+$ with $v_+(p^+) > \mathrm{H}(y^A)$ cannot be identified with any point of $\mathcal{N}^-$ or, in other words, the hypersurface $\mathcal{N}^-$ is mapped onto the proper subset $\{v_+ < \mathrm{H}\} \subset \mathcal{N}^-$ via Φ.

This last example suggests that finding possible matchings with non-zero pressure may be a significantly complicated task, specially in non-flat spacetimes. In any case, the influence of the pressure producing a kind of *self-compression/self-stretching* of points along the matching and its associated energy storage is an interesting effect that, in our opinion, deserves further investigation.

7.3.3.1 C^0-*form of the metric on the resulting spacetime*

Now that we have analyzed the most general matching of the two Minkowski regions $(\mathrm{M}^\pm, g^\pm)$ defined in (7.113) and that we have proven that its corresponding step function $H(\lambda, y^A)$ is given by (7.130) for an arbitrary pressure $p(\lambda, y^A)$, a natural question that arises is how to construct a C^0 metric in the spacetime resulting from the matching. We devote this section to such matter. In particular, we shall construct coordinates in a neighbourhood of the matching hypersurface and prove that in such coordinates the metric in the resulting spacetime is Lipschitz-continuous.

The results exposed in this section are part of a bigger ongoing project in collaboration with Argam Ohanyan and Roland Steinbauer (University of Vienna), in which besides finding a C^0 form of the metric we intend to derive an associated distributional metric form for the most general matching of two Minkowski regions across a null hyperplane.

Let us call $\mathcal{N}$ the matching hypersurface embedded in the spacetime (M, g) resulting from the from the matching of $(\mathrm{M}^\pm, g^\pm)$ across $\mathcal{N}^\pm$. Then, $(\mathrm{M}^+ \cup \mathrm{M}^-)/\mathcal{N}$ where the quotient indicates that we are identifying the boundaries and this identification gives rise to a single null hypersurface, namely $\mathcal{N}$. The null shell is therefore located on $\mathcal{N}$. In these circumstances, there already exist two coordinate systems, namely the coordinates $\{U_\pm, V_\pm, x^A_\pm\}$ which cover the $(\mathcal{M}^\pm, g^\pm)$ regions respectively. In such coordinates, the matching hypersurface $\mathcal{N}$ is defined by $\mathcal{N} \equiv \{U_\pm = 0\}$.

Our aim is to build a new coordinate system[4] $\{u, v, z^A\}$ in a neighbourhood $\mathcal{O} \subset \mathcal{M}$ such that the metric $g|_{\mathcal{O}}$ takes a C^0 form. We shall perform this task in several separate steps. We will start by identifying $\{u, v, z^A\}$ with the coordinates $\{U_-, V_-, x^A_-\}$ on the $(\mathcal{M}^-, g^-)$ side. This will allow us to write the vector fields $\{\zeta^-, e^-_a\}$ in terms of the coordinate vectors $\{\partial_u, \partial_v, \partial_{z^A}\}$. Combining these expressions with the explicit form of $\{\zeta^+, e^+_a\}$ (which we know from previous sections), we will be able to provide the relations between the coordinate sets $\{U_+, V_+, x^A_+\}$ and $\{u, v, z^A\}$ on $(\mathcal{M}^\pm, g^\pm)$. The next step will be to write $g^\pm$ in the coordinates $\{u, v, z^A\}$, with which we will eventually find a C^0-form of the metric of the spacetime resulting from the matching. For the rest of the section, we enforce $b^J{}_I = \delta^J_I$ (which can always be done because of the symmetries of $(\mathcal{M}^\pm, g^\pm)$, see the discussion above).

To simplify the construction of $\{u, v, z^A\}$, it is convenient to enforce a trivial identification between them and $\{U_-, V_-, x^A_-\}$ on the region $\mathcal{M}^-$, i.e. to set

$$\{U_- = u, V_- = v, x^A_- = z^A\} \quad \text{on} \quad \mathcal{M}^-. \tag{7.139}$$

Then, (7.26) together with the choice $\{L^- = -\partial_{U_-}, k^- = \partial_{V_-}, v^-_I = \partial_{x^I_-}\}$ that we made before force the vector fields $\{\zeta^-, e^-_a\}$ to be given by

$$e_1^- = \partial_v, \qquad e^-_I = \partial_{z^I}, \qquad \zeta^- = -\partial_u, \tag{7.140}$$

in the basis $\{\partial_u, \partial_v, \partial_{z^A}\}$ of $\Gamma\, T\mathcal{M}\,|_{\mathcal{N}^-}$. The matching identifies the vector fields $\{e^\pm_a, \zeta^\pm\}$, so the rigging ζ^+ and the vector fields $\{e^+_a\}$ must verify (recall (7.30), (7.40)-(7.41) and (7.119))

$$e^+_1 = (\partial_\lambda H)\partial_{V_+} = \partial_v, \qquad e^+_I = (\partial_{y^I} H)\partial_{V_+} + \partial_{x^I_+} = \partial_{z^I}, \tag{7.141}$$

$$\zeta^+ = -\frac{1}{\partial_\lambda H}\left(\partial_{U_+} + \delta^{AB}(\partial_{y^A} H)\left(\frac{1}{2}(\partial_{y^B} H)\partial_{V_+} + \partial_{x^B_+}\right)\right) = -\partial_u, \tag{7.142}$$

Since only tangential derivatives of $H(\lambda, y^A)$ appear on (7.141)-(7.142) and $\{\lambda = V_- = v,\ y^A = x^A_- = z^A\}$ on $\mathcal{N}$ (cf. (7.115)), one can rewrite these equations as

$$e^+_1 = (\partial_v H)\partial_{V_+} = \partial_v, \qquad e^+_I = (\partial_{z^I} H)\partial_{V_+} + \partial_{x^I_+} = \partial_{z^I}, \tag{7.143}$$

[4] The coordinate v should not be confused with the foliation functions $v_\pm$ defined on the boundaries $\mathcal{N}^\pm$.

$$\zeta^{+} = -\frac{1}{\partial_v H}\left(\partial_{\mathrm{U}_+} + \delta^{AB}(\partial_{z^A} H)\left(\frac{1}{2}(\partial_{z^B} H)\partial_{\mathrm{V}_+} + \partial_{x^B_+}\right)\right) = -\partial_u. \tag{7.144}$$

The one-forms $d\mathrm{U}_+$, $d\mathrm{V}_+$ and dx^A_+ are covariantly constant on the $(\mathcal{M}^+, g^+)$ side (e.g. for $d\mathrm{U}_+$ one finds $\nabla^+_\alpha(d\mathrm{U}_+)_\beta = \nabla^+_\alpha\nabla^+_\beta \mathrm{U}_+ = \partial_\alpha\partial_\beta \mathrm{U}_+ = 0$, and the same argument applies to $d\mathrm{V}_+$ and dx^A_+). If we let d_+ denote any of U_+, V_+ or x^A_+ and require that $\xi \overset{\mathrm{def}}{=} \partial_u$ is null and affinely geodesic everywhere on $\mathcal{M}$, it follows that

$$\xi^\alpha \nabla^+_\alpha\left(\xi^\beta \nabla^+_\beta d_+\right) = \left(\xi^\alpha \nabla^+_\alpha \xi^\beta\right)\nabla^+_\beta d_+ + \xi^\alpha\xi^\beta\nabla^+_\alpha\nabla^+_\beta d_+ = 0 \iff d_+ = a + ub,$$

where $a(v, z^A)$ and $b(v, z^A)$ are scalar functions. Accordingly, the coordinate transformation on the $(\mathcal{M}^+, g^+)$ must be of the form

$$\mathrm{U}_+ = \mathrm{U}_0 + u\mathrm{U}_1, \qquad \mathrm{V}_+ = \mathrm{V}_0 + u\mathrm{V}_1, \qquad x^A_+ = x^A_0 + ux^A_1, \tag{7.145}$$

where U_0, U_1, V_0, V_1, x^A_0, x^A_1 only depend on the coordinates $\{v, z^A\}$. Moreover, the trivial identification on the $(\mathcal{M}^-, g^-)$ side entails that $\mathcal{N} \equiv \{u = 0\}$. This fact, together with $b^J_I = \delta^J_I$ and (7.118), forces $\mathrm{U}_0 = 0$, $\mathrm{V}_0 = H(v, z^A)$, $x^A_0 = z^A$. Therefore,

$$\begin{aligned}
\mathrm{U}_+ &= u\mathrm{U}_1, & \mathrm{V}_+ &= H + u\mathrm{V}_1, & x^A_+ &= z^A + ux^A_1, \\
d\mathrm{U}_+ &= \mathrm{U}_1 du + ud\mathrm{U}_1, & d\mathrm{V}_+ &= dH + \mathrm{V}_1 du + ud\mathrm{V}_1, & dx^A_+ &= dz^A + x^A_1 du + udx^A_1.
\end{aligned} \tag{7.146}$$

In the following, we extend any scalar function $f \in \mathcal{F}(\mathcal{N})$ to $\mathcal{M}^+ \subset \mathcal{M}$ by requiring that f is independent of the coordinate u. This, in particular, allows us to write $f|_{\mathcal{N}}$ as $f(\lambda, y^A)$ and $f|_{\mathcal{M}^+}$ as $f(v, z^A)$.

The scalar functions $\mathrm{U}_1, \mathrm{V}_1, x^A_1$ can be derived from (7.144) by decomposing ∂_u as

$$\partial_u = \frac{\partial \mathrm{U}_+}{\partial u}\partial_{\mathrm{U}_+} + \frac{\partial \mathrm{V}_+}{\partial u}\partial_{\mathrm{V}_+} + \frac{\partial x^A_+}{\partial u}\partial_{x^A_+} = \mathrm{U}_1\partial_{\mathrm{U}_+} + \mathrm{V}_1\partial_{\mathrm{V}_+} + x^A_1\partial_{x^A_+}. \tag{7.147}$$

Inserting (7.147) into (7.144), one obtains

$$\begin{aligned}
\mathrm{U}_1 &= \frac{1}{\partial_v H}, \qquad \mathrm{V}_1 = M\mathrm{U}_1, \qquad x^A_1 = q^A \mathrm{U}_1 \\
\text{where} \qquad M &:= \frac{1}{2}\delta^{AB}\partial_{z^A} H \partial_{z^B} H, \qquad q^A := \delta^{AB}\partial_{z^B} H.
\end{aligned} \tag{7.148}$$

Observe that the quantities q^A and U_1 verify

$$dq^A = \delta^{AB}\left(\partial_v\partial_{z^B} H dv + \partial_{z^C}\partial_{z^B} H dz^C\right), \tag{7.149}$$

$$d\mathrm{U}_1 = -\mathrm{U}_1^2\left(\partial_v\partial_v H dv + \partial_{z^B}\partial_v H dz^B\right). \tag{7.150}$$

The final coordinate transformation is therefore given by

$$U_+ = uU_1, \qquad V_+ = H + uMU_1, \qquad x^A_+ = z^A + uq^A U_1. \tag{7.151}$$

We are now ready to compute the metric g^+ in the coordinates $\{u, v, z^A\}$. From (7.146) it follows

$$\begin{aligned}
ds^2_+ &= -2dU_+ dV_+ + \delta_{AB} dx^A_+ dx^B_+ \\
&= -2\left(U_1 du + u dU_1\right)\left(dH + V_1 du + u dV_1\right) \\
&\quad + \delta_{AB}\left(dz^A + x^A_1 du + u dx^A_1\right)\left(dz^B + x^B_1 du + u dx^B_1\right) \\
&= du^2\left(-2U_1V_1 + \delta_{AB} x^A_1 x^B_1\right) \\
&\quad + 2du\left(u\left(-V_1 dU_1 - U_1 dV_1 + \delta_{AB} x^B_1 dx^A_1\right) - U_1 dH + \delta_{AB} x^A_1 dz^B\right) \\
&\quad - 2u dU_1 dH - 2u^2 dU_1 dV_1 + \delta_{AB}\left(dz^A + u dx^A_1\right)\left(dz^B + u dx^B_1\right).
\end{aligned} \tag{7.152}$$

The vector field ∂_u being null everywhere forces (this also follows directly from (7.148))

$$-2U_1V_1 + \delta_{AB} x^A_1 x^B_1 = 0 \qquad \Longrightarrow \qquad 0 = -V_1 dU_1 - U_1 dV_1 + \delta_{AB} x^A_1 dx^B_1. \tag{7.153}$$

Besides, since we have selected the rigging ζ^- to be orthogonal to the spacelike sections $\{V_- = \text{const.}\}$ of $\mathcal{N}^-$, the combination of (7.139) (where we imposed $V_- = v$) and (7.140) entails that ∂_u is orthogonal to the spacelike sections of $\{v = \text{const.}\}$ of $\mathcal{N}^-$ as well. In particular, this implies that (again this also follows from (7.148) by means of a straightforward calculation)

$$-U_1 \partial_{z^A} H + \delta_{AB} x^B_1 = 0 \qquad \Longleftrightarrow \qquad -\partial_{z^A} H + \delta_{AB}\delta^{BJ}\partial_{z^J} H = 0. \tag{7.154}$$

Since $dH = \partial_v H dv + \partial_{z^A} H dz^A$, it follows that

$$\begin{aligned}
-U_1 dH + \delta_{AB} x^A_1 dz^B &= U_1(-dH + \delta_{AB} q^A dz^B) \\
&= U_1(-\partial_v H dv - \partial_{z^A} H dz^A + \delta_{AB}\delta^{BJ}\partial_{z^J} H dz^A) = -dv,
\end{aligned} \tag{7.155}$$

where we have used (7.154) to cancel the last to terms and (7.148) to introduce the coefficient U_1. Inserting (7.153) and (7.155) into (7.152) yields

$$\begin{aligned}
ds^2_+ &= -2du dv - 2u dU_1 dH - 2u^2 dU_1 dV_1 + \delta_{AB}\left(dz^A + u dx^A_1\right)\left(dz^B + u dx^B_1\right) \\
&= -2du dv + \delta_{AB} dz^A dz^B + 2u\left(-dU_1 dH + \delta_{AB} dz^A dx^B_1\right) \\
&\quad + u^2\left(-2dU_1 dV_1 + \delta_{AB} dx^A_1 dx^B_1\right)
\end{aligned}$$

$$
\begin{aligned}
&= -2dudv + \delta_{AB}dz^A dz^B + 2u\left(-dU_1 dH + \delta_{AB}dz^A\left(U_1 dq^B + q^B dU_1\right)\right)\\
&\quad + u^2\left(-2dU_1\left(MdU_1 + U_1 dM\right) + \delta_{AB}\left(U_1 dq^A + q^A dU_1\right)\left(U_1 dq^B + q^B dU_1\right)\right).
\end{aligned}
$$

Using (7.155), as well as the equalities $-2M + \delta_{AB}q^A q^B = 0$, $-dM + \delta_{AB}q^A dq^B = 0$ that follow from (7.148), one gets

$$
\begin{aligned}
ds^2_+ &= -2dudv + \delta_{AB}dz^A dz^B + 2u\left(-\frac{1}{U_1}dvdU_1 + U_1\delta_{AB}dz^A dq^B\right)\\
&\quad + u^2\left(dU_1\left(-2U_1 dM + 2U_1\delta_{AB}q^A dq^B\right) + U_1^2\delta_{AB}dq^A dq^B\right)\\
&= -2dudv + \delta_{AB}dz^A dz^B + 2u\left(-\frac{1}{U_1}dvdU_1 + U_1\delta_{AB}dz^A dq^B\right) + u^2 U_1^2\delta_{AB}dq^A dq^B.
\end{aligned}
$$

Finally, inserting (7.149)-(7.150) and using (7.121) and the definition of the pressure (see Remark 7.3.9), it is straightforward to obtain the metric

$$
\begin{aligned}
ds^2_+ &= -2dudv + \delta_{AB}dz^A dz^B + udv^2\left(u\delta^{AB}[Y_{1A}][Y_{1B}] - 2p\right)\\
&\quad + 2u[Y_{1I}]dvdz^A\left(-2\delta^I_A + u\delta^{BI}[Y_{AB}]\right)\\
&\quad - 2udz^A dz^B\left([Y_{AB}] - \frac{u}{2}\delta^{IJ}[Y_{IA}][Y_{JB}]\right),
\end{aligned}
\tag{7.156}
$$

Given that $ds^2_- = -2dudv + \delta_{AB}dz^A dz^B$ because of the identification of $\{u, v, z^A\}$ and $\{U_-, V_-, x^A_-\}$ on the minus side, it follows that the metric g of $\mathcal{M}$ can be written in a C^0 form in terms of $\{u, v, z^A\}$ as

$$
\begin{aligned}
g &= -2dudv + \delta_{AB}dz^A dz^B + u\Theta(u)dv^2\left(u\delta^{AB}[Y_{1A}][Y_{1B}] - 2p\right)\\
&\quad + 2u\Theta(u)[Y_{1I}]dvdz^A\left(-2\delta^I_A + u\delta^{BI}[Y_{AB}]\right)\\
&\quad - 2u\Theta(u)dz^A dz^B\left([Y_{AB}] - \frac{u}{2}\delta^{IJ}[Y_{IA}][Y_{JB}]\right),
\end{aligned}
\tag{7.157}
$$

where $\Theta(u)$ is the Heaviside step function. We can summarize the results of this section in the following lemma.

Lemma 7.3.10. *Let $(\mathcal{M}^\pm, g^\pm)$ be two regions of the spacetime of Minkowski and suppose that their boundaries $\mathcal{N}^\pm$ are null hyperplanes. Assume that all the required conditions for the matching of $(\mathcal{M}^\pm, g^\pm)$ are satisfied and denote by $(\mathcal{M}, g)$ the resulting spacetime containing a null thin shell located on a null hyperplane $\mathcal{N}$. Then, there exists a set of continuous coordinates $\{u, v, z^A\}$ on a neighbourhood $\mathcal{O} \subset \mathcal{M}$ of $\mathcal{N}$ with the following properties:*

(i) $\{u, v, z^A\}$ *are Gaussian null coordinates*[5] *on both sides of* $\mathcal{N}$. *Moreover, the vector* $\partial_v|_{\mathcal{N}^-}$ *is a null generator of* $\mathcal{N}^-$ *while the vector* $\partial_u|_{\mathcal{N}^-}$ *is a rigging of* $\mathcal{N}^-$ *with the properties of being future-directed, null, orthogonal to the spacelike planes* $\mathcal{N}^- \cap \{v = \text{const.}\}$ *and satisfying* $g(\partial_u, \partial_v)|_{\mathcal{N}^-} = -1$.

(ii) *In the coordinates* $\{u, v, z^A\}$, *the metric* $g|_0$ *takes a* C^0*-form and it is given by*

$$\begin{aligned} g = &- 2dudv + \delta_{AB}dz^A dz^B + u\Theta(u)dv^2\left(u\delta^{AB}[Y_{1A}][Y_{1B}] - 2p\right) \\ &+ 2u\Theta(u)[Y_{1I}]dvdz^A\left(-2\delta^I_A + u\delta^{BI}[Y_{AB}]\right) \\ &- 2u\Theta(u)dz^A dz^B\left([Y_{AB}] - \frac{u}{2}\delta^{IJ}[Y_{IA}][Y_{JB}]\right), \end{aligned} \tag{7.158}$$

where $\Theta(u)$ *is the Heaviside step function.*

In particular, in the gravitational wave case (i.e. there is no energy density, no energy flux and no pressure on the shell) the step function $H(\lambda, y^A)$ *is given by* (7.127) *and hence the metric* $g|_0$ *becomes (cf.* (7.121)*)*

$$g = -2dudv + \delta_{AB}dz^A dz^B + \frac{2u\Theta(u)}{a}(\partial_{z^J}\partial_{z^B}\mathrm{H})\left(\delta^J_A + \frac{u}{2a}\delta^{IJ}\partial_{z^I}\partial_{z^A}\mathrm{H}\right)dz^A dz^B, \tag{7.159}$$

where a *is a positive real constant and* $\mathrm{H}(z^A)$ *is an arbitrary function.*

Remark 7.3.11. *The function* $u\Theta(u)$ *is Lipschitz continuous, so the metric* (7.158) *is Lipschitz in the variable* u*, even across the null hypersurface* $\mathcal{N}^- \equiv \{u = 0\}$.

Remark 7.3.12. *When* $(\mathcal{M}^{\pm}, g^{\pm})$ *are* 4*-dimensional, one can define spatial complex coordinates*

$$Z \stackrel{\mathrm{def}}{=} \frac{z^2 + iz^3}{\sqrt{2}}, \qquad \overline{Z} \stackrel{\mathrm{def}}{=} \frac{z^2 - iz^3}{\sqrt{2}}, \tag{7.160}$$

where i *denotes the imaginary unit, and then a straightforward calculation shows that the one-form*

$$\Omega \stackrel{\mathrm{def}}{=} dZ + \frac{u\Theta(u)}{a}\left((\partial_Z\partial_Z\mathrm{H})dZ + (\partial_{\overline{Z}}\partial_{\overline{Z}}\mathrm{H})d\overline{Z}\right) \tag{7.161}$$

satisfies (we let $\overline{\Omega}$ *be the conjugate of* Ω*)*

$$2\Omega \otimes_s \overline{\Omega} = (dz^2)^2 + (dz^3)^2 + \frac{2u\Theta(u)}{a}(\partial_{z^J}\partial_{z^B}\mathrm{H})\left(\delta^J_A + \frac{u}{2a}\delta^{IJ}\partial_{z^I}\partial_{z^A}\mathrm{H}\right)dz^A dz^B.$$

Thus, the metric (7.159) *can be expressed in terms of* Ω *as*

$$g = -2dudv + 2\Omega \otimes_s \overline{\Omega}, \tag{7.162}$$

[5] See e.g. [129], [100] or Appendix D for details on the construction of Gaussian null coordinates.

which is the standard form of writing the Lipschitz-continuous metric g when studying the matching of two regions of Minwkoski across a null hyperplane by means of the cut-and-paste method (note that (7.162) *becomes* (2.169) *when enforcing $a = 1$, $\mathrm{H} = h$). Recall that in these cut-and-paste constructions there is neither energy flux nor pressure, and that the constant a is usually set to one.*

8

MATCHING ACROSS ABSTRACT KILLING HORIZONS OF ORDER ZERO

In the previous chapter, we have seen that sometimes two spacetimes can be matched in more than one way, and we have provided an explicit situation in which this occurs, namely when the boundaries are totally geodesic null hypersurfaces. Although this is by no means the only scenario allowing for more than one matching, it is of particular geometric and physical interest because it applies whenever the boundaries of the spacetimes are horizons of the types we have introduced throughout this thesis, i.e. non-expanding horizons, (weakly) isolated horizons, (multiple) Killing horizons, abstract Killing horizons of order zero and one (as well as their embedded versions). In these circumstances, the enormous matching freedom can be exploited to consider a great amount of possibilities. For instance, one may match a non-expanding and a multiple Killing horizon, or a totally geodesic null hypersurface with a Killing horizon of order ½. The combinations are endless, and this facilitates the task of finding examples of matchable spacetimes.

Now, the problem of matching two spacetimes with boundary can involve several levels. For instance, one may be interested in finding the most general possible matching between them, in restricting the energy-momentum tensor type and study only those, or in preserving some additional geometric property that the two spacetimes might share. In this chapter we explore in detail a relevant example of the latter. All notions of horizons presented in this thesis (with the exception of non-expanding horizons) are defined in terms of a privileged vector field that we have denoted by η. On the horizon, this vector field has the properties of being null, non-zero almost everywhere (or everywhere) and tangent, and it defines a symmetry (up to whatever order) of the hypersurface under consideration. Since this vector field defines a certain kind of symmetry on each spacetime, it is of interest to study matchings for which the resulting spacetime is also endowed with a special vector field so that the final spacetime also possesses certain kind of symmetry. For this to happen it is necessary to restrict the matchings in such a way that the two vectors fields on each side are identified. By doing this, the matched spacetime will be equipped with a privileged continuous vector field. Note that in general we cannot expect more smoothness for this vector field, since the metric itself is only continuous across the shell.

Thus, in this chapter we study in detail the problem of matching when the symmetry generators from both spacetimes are identified in the process of matching. This, as we will see, restricts the set of all possible step functions and even the type of matching.

In order to keep as much generality as possible, we need to work with the weakest notion of horizon, so that our results are applicable to a variety of situations. Since abstract Killing horizons of order zero constitute the less restrictive horizons introduced in this thesis, it is sensible to consider the case when the boundaries of the spacetimes to be matched are **embedded** versions of this **abstract** horizons. Note the subtle difference between an embedded AKH$_0$ and a KH$_0$. As we discussed in Section 5.4, these two objects do not need to share the same properties. In particular, in an embedded AKH$_0$ the transverse components of the deformation tensor K^η do not need to vanish.

In this chapter, our setup will be the following. We consider two spacetimes $(\mathcal{M}^\pm, g^\pm)$ with boundaries $\widetilde{\mathcal{N}}^\pm$ which are embedded AKH$_0$. As we did in the previous section, we assume that $\widetilde{\mathcal{N}}^\pm$ can be foliated by a family of diffeomorphic spacelike cross-sections, and that one boundary lies in the future of its respective spacetime while the other lies in its spacetime past. We also suppose that $(\mathcal{M}^\pm, g^\pm)$ verify all matching conditions. Since AKH$_0$ are totally geodesic null hypersurfaces, this means that an infinite number of matchings of $(\mathcal{M}^\pm, g^\pm)$ across $\widetilde{\mathcal{N}}^\pm$ can be performed. Each of these possible matchings corresponds to a different step function $H(\lambda, y^A)$, and they all share the same identification between the set of null generators of both sides (ruled by the diffeomorphism Ψ). For each feasible matching, one can find metric hypersurface data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ that can be embedded in both spacetimes. As usual, we let $\{\phi^\pm, \zeta^\pm\}$ denote the corresponding matching embeddings and riggings, and we define the matching map $\Phi : \widetilde{\mathcal{N}}^- \longrightarrow \widetilde{\mathcal{N}}^+$ by $\Phi \circ \phi^- \stackrel{\text{def}}{=} \phi^+$. In these circumstances, the hypersurface data sets

$$\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}^\pm\} \quad \text{with} \quad \mathbf{Y}^\pm \stackrel{\text{def}}{=} \frac{1}{2}(\phi^\pm)^\star(\pounds_{\zeta^\pm} g^\pm) \quad (\text{cf. } (2.39)) \tag{8.1}$$

define abstract Killing horizons of order zero according to Definition 5.4.1. As already stressed, in this chapter we are interested in the case when the symmetry generators from the two sides are identified in the matching process. In the language of the formalism of hypersurface data, this means that $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}^{\pm}\}$ are AKH$_0$ with the same symmetry generator, which we denote by $\overline{\eta}$. We use $\mathcal{S}$ to refer to the fixed point set of $\overline{\eta}$. The symmetry generators $\eta^{\pm}$ on $(\mathcal{M}^{\pm}, g^{\pm})$ are given by

$$\eta^{-} \stackrel{\text{def}}{=} \phi_{\star}\overline{\eta}, \qquad a\eta^{+} \stackrel{\text{def}}{=} \phi_{\star}^{+}\overline{\eta}, \qquad a \in \mathbb{R} - \{0\}. \tag{8.2}$$

and we let $\mathcal{S}^{\pm} \stackrel{\text{def}}{=} \phi^{\pm}(\mathcal{S})$. We include the constant a in (8.2) in order to account for the freedom of rescaling the symmetry generators $\eta^{\pm}$ by a non-zero real constant. Finally, we assume that the surface gravities $\kappa^{\pm}$ of $\eta^{\pm}$ are everywhere constant on $\mathcal{N} \setminus \mathcal{S}^{\pm}$, and extend them trivially to $\mathcal{N}^{\pm}$ (this in turn means that the surface gravity κ of $\overline{\eta}$, defined in Lemma 3.3.1, is also constant on $\mathcal{N}$, see Proposition 3.3.2). In order to simplify later discussions and results, if it happens that $\kappa^{\pm} \neq 0$ then we select $\eta^{\pm}$ so that $\kappa^{\pm} > 0$. Since $\kappa^{\pm}$ are constant, this entails no loss of generality, as one can always take $-\eta^{\pm}$ as the Killing vector field whenever $\kappa^{\pm} < 0$ (cf. (2.81)). As elsewhere in the thesis, here we also identify functions on $\mathcal{N}$ with their counterparts on $\widetilde{\mathcal{N}}^{\pm}$.

In the previous chapter we have seen that each matching can be codified in a pair $\{H(\lambda, y^A), \Psi\}$, where $H(\lambda, y^A)$ is the step function of the matching and Ψ is a diffeomorphism between the set of null generators on both sides. An important aspect to bear in mind is that all the matching freedom that appears when the boundaries are totally geodesic is fully encoded in the step function. For this reason, we shall focus on this object, treating the diffeomorphism Ψ as known and concentrating our effort on finding the set of possible step functions. This is not problematic at all, since we are assuming that $(\mathcal{M}^{\pm}, g^{\pm})$ are always matchable, which automatically guarantees the existence of such a map Ψ.

In order to exploit the matching formalism introduced before, we need to construct two foliation functions $v^{\pm}$ and two basis $\{L^{\pm}, k^{\pm}, v_I^{\pm}\}$ of $\Gamma(T\mathcal{M}^{\pm})|_{\widetilde{\mathcal{N}}^{\pm}}$ according to (7.2). We recall that in (7.2) the basis vector $k^{\pm}$ is a choice of future null generator. For the purposes of this chapter, it is convenient that we select $k^{\pm}$ affine, i.e. such that $\kappa^{\pm}_{k^{\pm}} = 0$. This can always be done because both boundaries admit a cross-section (see the discussion in Section 2.4). The fact that $\widetilde{\mathcal{N}}^{\pm}$ are totally geodesic then means that the second fundamental forms $\mathbf{K}^{k}_{\pm}$ vanish everywhere on $\widetilde{\mathcal{N}}^{\pm}$.

The results (2.88)-(2.90) are fully valid in the present case because $\eta^{\pm}$ are null and tangent to $\widetilde{\mathcal{N}}^{\pm}$ everywhere therein and $\kappa^{\pm}_{k^{\pm}} = 0$, $\kappa^{\pm} \in \mathbb{R}$. Thus, we can write

$$\eta^{\pm} \stackrel{\widetilde{\mathcal{H}}}{=} \alpha^{\pm}k^{\pm}, \qquad \alpha^{\pm} \stackrel{\text{def}}{=} f^{\pm} + \kappa^{\pm}v^{\pm} \tag{8.3}$$

where $v_\pm$ is the foliation function constructed according to (7.2) (in particular $k^\pm(v_\pm) = 1$) and $f^\pm \in \mathcal{F}(\mathcal{N}^\pm)$ satisfy $k^\pm(f^\pm) = 0$.

At the spacetime level, the identification of $\{\eta^-, a\eta^+\}$ is guaranteed if and only if

$$\Phi_\star \eta^- \overset{\mathcal{N}^+}{=} a\eta^+ \tag{8.4}$$

holds. On the other hand, using $e_1^- = k^-$ (cf. (7.26)), $e_1^+ = \partial_\lambda H k^+$ (which follows from (7.30) and (7.45)) and $e_1^+ = \Phi_\star e_1^-$, one obtains

$$a\eta^+ \overset{\mathcal{N}^+}{=} a\alpha^+ k^+ \overset{\mathcal{N}^+}{=} \frac{a\alpha^+ e_1^+}{\partial_\lambda H}, \tag{8.5}$$

$$a\eta^+ \overset{\mathcal{N}^+}{=} \Phi_\star \eta^- \overset{\mathcal{N}^+}{=} \alpha^- \Phi_\star k^- \overset{\mathcal{N}^+}{=} \alpha^- \Phi_\star e_1^- \overset{\mathcal{N}^+}{=} \alpha^- e_1^+. \tag{8.6}$$

The combination of (8.5) and (8.6) hence yields

$$\alpha^- \overset{\mathcal{N}^+}{=} \frac{a\alpha^+}{\partial_\lambda H}. \tag{8.7}$$

Observe that equation (8.7) is consistent with the fact that the fixed point sets $\mathrm{S}^\pm$ (given by those points where $\alpha^\pm$ vanish) must be identified in the matching process. This, of course, was obvious from the abstract viewpoint, since there exists one unique symmetry generator $\overline{\eta}$ and by definition $\mathrm{S}^\pm \overset{\mathrm{def}}{=} \phi^\pm(\mathrm{S})$.

Equation (8.7) is of relevance in the sense that it constitutes an extra matching condition, namely that *matchings across embedded AKH0s in which the symmetry generators are to be identified cannot be performed unless the corresponding fixed point sets can be mapped to each other*. This restriction is important since, as we proved in Lemma 5.4.11, the causality of the fixed point set of an AKH0 strongly depends on the geometric properties of the symmetry generator (and the matching map sends null generators on one side to null generators on the other side, recall that $\Phi_\star e_1^- = e_1^+$). Thus, it could well happen that all matching conditions were satisfied but (8.7) was not, and hence the matching would be possible but *not* under the condition that the symmetry generators are identified in the matching process.

In Section 7.3, we proved that a general matching across totally geodesic null boundaries (without a priori identifying any pair of generators) allowed for an infinite set of possible step functions. This was a consequence of all leaves $\{v^\pm = \text{const.}\}$ being isometric to each other (recall (7.5)). Because of condition (8.7), this is no longer true here. Away from the zeroes of $\alpha^\pm$, the integration of (8.7) determines the step function $H(\lambda, y^A)$ up to an integration function. This solution, however, may be difficult to find in general. The problem becomes simpler under the assumptions of $k^\pm$ being affine (which implies no loss of generality) and $\kappa^\pm$ being constant (which is indeed a restriction that we are making). In fact, combining (7.49), (8.3) and (8.7) gives

$$f^- + \kappa^- \lambda \overset{\mathcal{N}}{=} \frac{a\left(f^+ + \kappa^+ H\right)}{\partial_\lambda H} \implies \partial_\lambda H \overset{\mathcal{N}\setminus\mathcal{S}}{=} \frac{a\left(f^+ + \kappa^+ H\right)}{f^- + \kappa^- \lambda} > 0, \tag{8.8}$$

where we have used that $\partial_\lambda H > 0$. The functions $f^\pm$ are constant along the null generators, so they are λ-independent. This makes it easier to integrate (8.8) in order to obtain the explicit form of $H(\lambda, y^A)$. Note that the second expression in (8.8) only holds away from the fixed points. The value of $H(\lambda, y^A)$ on $\mathcal{S}$ must be determined by continuity (recall that $\mathcal{N} \setminus \mathcal{S}$ is dense in $\mathcal{N}$ according to Definition 5.4.1). The right part of (8.8) can also be expressed as

$$\operatorname{sign}(a) \operatorname{sign}\left(f^+ + \kappa^+ H\right) = \operatorname{sign}\left(f^- + \kappa^- \lambda\right), \tag{8.9}$$

which geometrically means that both symmetry generators $\{\eta^-, a\eta^+\}$ must be simultaneously either future or past. This of course is consistent with the fact that $\{\eta^-, a\eta^+\}$ are to be identified. We now study separately the matching for the cases (a) $\kappa^\pm = 0$, (b) $\kappa^\pm \neq 0$ and (c) $\kappa^- = 0$, $\kappa^+ \neq 0$ or $\kappa^- \neq 0$, $\kappa^+ = 0$.

8.1 case of $\eta^\pm$ degenerate

When $\kappa^\pm$ vanish, we know by Lemma 5.4.11 that $\mathcal{S}^\pm$ are either empty or the union of smooth connected codimension-two null submanifolds of $\mathcal{N}^\pm$. The fact that the map Φ is a diffeomorphism forces both boundaries to have the same number of these connected components. On the other hand, enforcing $\kappa^\pm = 0$ in (8.8) yields the condition $\frac{af^+(y^A)}{f^-(y^A)} > 0$ as well as the explicit form of the step function, namely

$$H(\lambda, y^A) = \frac{af^+(y^A)}{f^-(y^A)}\lambda + \mathrm{H}(y^A), \tag{8.10}$$

where $\mathrm{H}(y^A)$ is an integration function.

Once we select the tuple $\{a, \eta^-, \eta^+\}$, the only remaining matching freedom is encoded in the function $\mathrm{H}(y^A)$ (the scalar functions $f^\pm$ are known beforehand as the spacetimes to be matched are assumed to be known, cf. (2.90)). In order to understand this freedom, recall that we have called "velocity" the rate of change of the foliation function $v_\pm$ along a null generator of $\mathcal{N}^\pm$. The velocity along the null generators of $\mathcal{N}^\pm$ is totally determined (outside of $\mathcal{S}^\pm$) by the identification of

$\{\eta^-, a\eta^+\}$. However, there still exist a freedom to select any pair of sections, one on each side, and force their identification via Φ. This is the freedom encoded in the arbitrary function $\mathrm{H}(y^A)$. Note that the step function (8.10) is linear in λ. This means, in particular, that the most general shell that can be generated under these circumstances has vanishing pressure (recall (7.94) and Remark 7.3.9).

8.2 case of $\eta^\pm$ non-degenerate

We now study the case when $\kappa^\pm \neq 0$. Again by Lemma 5.4.11, we know that $S^\pm$ are either empty or spacelike cross-sections defined by $S^\pm \stackrel{\mathrm{def}}{=} \{p \in \mathcal{N}^\pm \mid f^\pm + \kappa^\pm v_\pm|_p = 0\}$. We define the submanifolds $\mathcal{N}_p^\pm$, $\mathcal{N}_f^\pm$ by

$$
\begin{aligned}
\mathcal{N}_p^\pm &\stackrel{\mathrm{def}}{=} \left\{ p \in \mathcal{N}^\pm \mid f^\pm + \kappa^\pm v_\pm|_p < 0 \right\}, \\
\mathcal{N}_f^\pm &\stackrel{\mathrm{def}}{=} \left\{ p \in \mathcal{N}^\pm \mid f^\pm + \kappa^\pm v_\pm|_p > 0 \right\},
\end{aligned}
\tag{8.11}
$$

so that $\mathcal{N}^\pm \equiv \mathcal{N}_p^\pm \cup S^\pm \cup \mathcal{N}_f^\pm$. Since we are assuming nothing on the geodesic completeness of $\mathcal{N}^\pm$, we do not exclude the cases when any of $\mathcal{N}_p^\pm$, $\mathcal{N}_f^\pm$ and $S^\pm$ are empty. Note that, when $\mathcal{N}_p^\pm$, $\mathcal{N}_f^\pm$ are non-empty they are by definition embedded AKH0s. For later purposes, we also introduce the non-zero constant $\kappa \stackrel{\mathrm{def}}{=} a\kappa^+(\kappa^-)^{-1}$. Note that $\mathrm{sign}(\kappa) = \mathrm{sign}(a)$ because we have chosen the orientations of $\eta^\pm$ such that $\kappa^\pm > 0$.

We start by considering the case $S^\pm \neq \emptyset$ and show that in such case (8.8) forces κ to be equal to one. We apply the l'Hôpital rule to (8.8) and get

$$
\lim_{\lambda \to -f^-/\kappa^-} \partial_\lambda H = \lim_{\lambda \to -f^-/\kappa^-} \frac{a\kappa^+ \partial_\lambda H}{\kappa^-} = \kappa \lim_{\lambda \to -f^-/\kappa^-} \partial_\lambda H \qquad \Longleftrightarrow \qquad \kappa = 1. \tag{8.12}
$$

Thus, $a = \kappa^-(\kappa^+)^{-1} > 0$ and equation (8.8) becomes

$$
\partial_\lambda H = \left(\frac{f^+}{\kappa^+} + H \right) \left(\frac{f^-}{\kappa^-} + \lambda \right)^{-1} > 0, \tag{8.13}
$$

whose integration yields

$$
\ln \left| \frac{f^+(y^A)}{\kappa^+} + H(\lambda, y^A) \right| = \ln \left(\beta(y^A) \left| \frac{f^-(y^A)}{\kappa^-} + \lambda \right| \right), \tag{8.14}
$$

where $\beta(y^A)$ is a positive integration function. Putting (8.14), (8.9) and the fact that $a > 0$ together gives

$$H(\lambda, y^A) = \beta(y^A)\left(\lambda + \frac{f^-(y^A)}{\kappa^-}\right) - \frac{f^+(y^A)}{\kappa^+}, \qquad \beta(y^A) > 0. \tag{8.15}$$

In combination with the results in Section 8.1 we conclude that whenever $\mathcal{N}^{\pm}$ are degenerate or contain non-empty fixed point sets any matching of $(\mathcal{M}^{\pm}, g^{\pm})$ across $\mathcal{N}^{\pm}$ in which the symmetry generators $\{\eta^-|_{\mathcal{N}^-}, a\eta^+|_{\mathcal{N}^+}\}$ are identified requires the surface gravities $\{\kappa^-, a\kappa^+\}$ to coincide. Moreover, the step function must be linear in the coordinate λ, which excludes matchings giving rise to shells with non-vanishing pressure.

It is also physically interesting to study the matchings when no null generator of $\mathcal{N}^{\pm}$ crosses a fixed point set, i.e. when $S^{\pm}$ are both empty. Integrating (8.8) now leads to

$$\left|f^+(y^A) + \kappa^+ H(\lambda, y^A)\right| = \beta(y^A)\left|f^-(y^A) + \kappa^-\lambda\right|^{\kappa}, \tag{8.16}$$

where $\beta(y^A)$ is a non-zero positive integration function. We analyse the cases $a > 0$ (i.e. $\kappa > 0$) and $a < 0$ (i.e. $\kappa < 0$) separately. For the former, condition (8.9) gives $\mathrm{sign}(f^+ + \kappa^+ H) = \mathrm{sign}(f^- + \kappa^-\lambda)$, which only allows for the matchings (see (a), (b) in Figure 8.1)

(a)
- Boundaries: $\mathcal{N}^- = \mathcal{N}_f^-, \quad \mathcal{N}^+ = \mathcal{N}_f^+$,
- Matching map: $\Phi(\mathcal{N}_f^-) = \mathcal{N}_f^+$,
- Step function: $H(\lambda, y^A) = \dfrac{\beta(y^A)}{\kappa^+}\left|f^-(y^A) + \kappa^-\lambda\right|^{\kappa} - \dfrac{f^+(y^A)}{\kappa^+}$,

(b)
- Boundaries: $\mathcal{N}^- = \mathcal{N}_p^-, \quad \mathcal{N}^+ = \mathcal{N}_p^+$,
- Matching map: $\Phi(\mathcal{N}_p^-) = \mathcal{N}_p^+$,
- Step function: $H(\lambda, y^A) = -\dfrac{\beta(y^A)}{\kappa^+}\left|f^-(y^A) + \kappa^-\lambda\right|^{\kappa} - \dfrac{f^+(y^A)}{\kappa^+}$.

On the other hand, $a < 0$ together with (8.9) entail $\mathrm{sign}(f^+ + \kappa^+ H) = -\mathrm{sign}(f^- + \kappa^-\lambda)$, whence (see (c), (d) in Figure 8.1)

(c)
- Boundaries: $\mathcal{N}^- = \mathcal{N}_f^-, \quad \mathcal{N}^+ = \mathcal{N}_p^+$,
- Matching map: $\Phi(\mathcal{N}_f^-) = \mathcal{N}_p^+$,
- Step function: $H(\lambda, y^A) = -\dfrac{\beta(y^A)}{\kappa^+}\left|f^-(y^A) + \kappa^-\lambda\right|^{\kappa} - \dfrac{f^+(y^A)}{\kappa^+}$,

Boundaries: $\quad \partial\mathcal{N}^- = \mathcal{N}_p^-, \quad \partial\mathcal{N}^+ = \mathcal{N}_f^+,$

(d) Matching map: $\quad \Phi(\mathcal{N}_p^-) = \mathcal{N}_f^+,$

Step function: $\quad H(\lambda, y^A) = \dfrac{\beta(y^A)}{\kappa^+}\left(f^-(y^A) + \kappa^-\lambda\right)^{\kappa} - \dfrac{f^+(y^A)}{\kappa^+}.$

The function H can be written in a form that covers all cases at once by defining

$$\epsilon \stackrel{\mathrm{def}}{=} \operatorname{sign}(\kappa)\operatorname{sign}(f^- + \kappa^-\lambda) \tag{8.17}$$

and writing

$$H(\lambda, y^A) = \frac{1}{\kappa^+}\left(\epsilon\beta(y^A)\left|f^-(y^A) + \kappa^-\lambda\right|^{\kappa} - f^+(y^A)\right), \qquad \beta(y^A) > 0. \tag{8.18}$$

We emphasize that the expression (8.18) for the step function $H(\lambda, y^A)$ is only part of the matching, as the boundaries $\partial\mathcal{N}^{\pm}$ must correspond to any of the situations (a)-(d). Observe that in the present case the matchings (a)-(d) allow for shells with pressure, as the derivatives of (8.18) are given by

$$\partial_\lambda H = \beta(y^A)|a|\left|f^-(y^A) + \kappa^-\lambda\right|^{\kappa-1} > 0, \qquad \partial_\lambda\partial_\lambda H = \frac{\kappa^-(\kappa-1)}{f^-(y^A) + \kappa^-\lambda}\partial_\lambda H$$

which, together with (7.94) and Remark 7.3.9, implies

$$p(\lambda, y^A) = \frac{\epsilon}{\mu^{\pm}}\frac{\kappa^-(\kappa-1)}{f^-(y^A) + \kappa^-\lambda}. \tag{8.19}$$

As discussed in Section 7.3.3, the pressure accounts for the compression/stretching of points when crossing the matching hypersurface. This means, in particular, that this effect takes place whenever $\kappa \neq 1$.

The positive function $\beta(y^A)$ introduces a freedom in the matching that we analyse next. When $S^{\pm} = \emptyset$, it corresponds to the freedom of selecting a section on each side and impose their identification via Φ (this is the same that happened in the case with vanishing surface gravity of Section 8.1). The interpretation of this freedom is less obvious when $S^{\pm} \neq \emptyset$, because in such case the sections $S^{\pm}$ are forced to be mapped to each other. In order to understand this, we again call "velocity" the rate of change of $v_{\pm}$ along a null generator of $\mathcal{N}^{\pm}$. Both when $\mathcal{N}^{\pm}$ are degenerate and when $\mathcal{N}^{\pm}$ are non-degenerate with $S^{\pm} = \emptyset$, identifying two sections not only provides a mapping between their points, but also of the velocity along the null generators of $\mathcal{N}^{\pm}$ at those sections. This latter information is encoded in the symmetry generators to be identified. However, for non-degenerate $\mathcal{N}^{\pm}$ con-

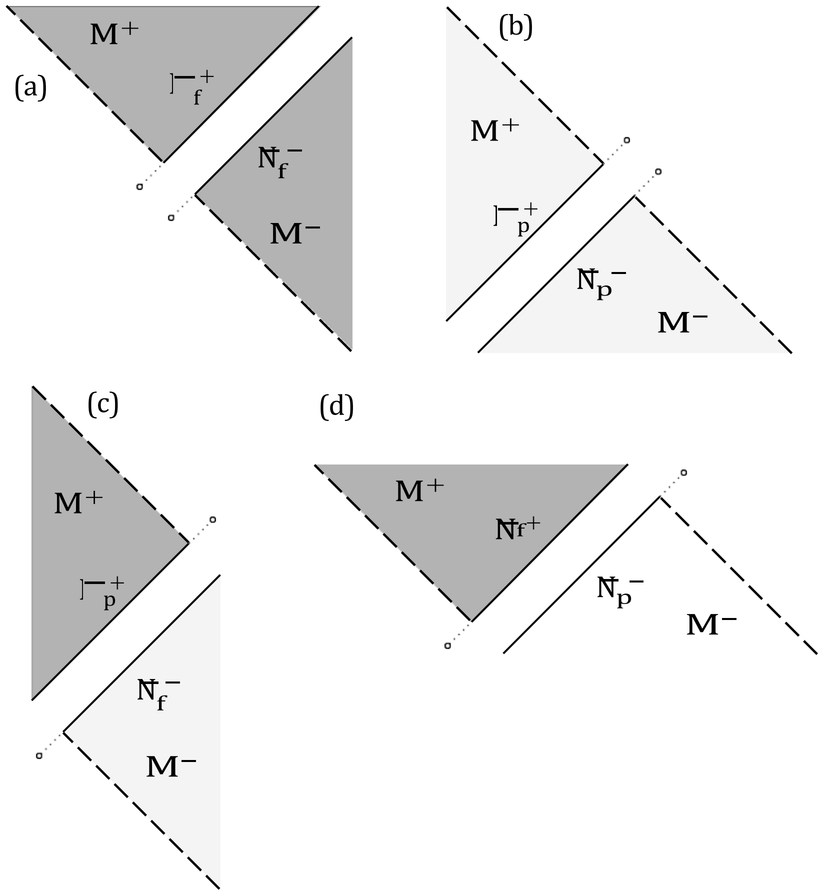


Figure 8.1: Possible matchings of two spacetimes $(M^{\pm}, g^{\pm})$ across their respective boundaries $\mathcal{N}^{\pm}$ in the case when these are non-degenerate embedded AKH_0s without a fixed points set. Here boundaries directly in front of each other are to be identified and the dot represents the point at which the fixed points set would be located if the horizons extended further.

taining fixed point sets $S^{\pm} /= \emptyset$, the map between the subsets $S^{\pm}$ only provides information on the identification of their points. The velocity along the null generators remains unfixed, as both symmetry generators vanish on $S^{\pm}$. The function $\beta(y^A)$ encodes precisely the freedom of selecting the initial velocities at S^{+} that rule the identifications off the fixed points set. Once we are off $S^{\pm}$, the velocity is determined by the identification of the symmetry generators themselves, just as in the previous cases.

8.3 case of η^- degenerate, η^+ non-degenerate

Now we address the case when one boundary is degenerate and the other is not. Since one symmetry generator is degenerate and the other is not, by Lemma 5.4.11 it follows that the matching (identifying $\{\eta^-, a\eta^+\}$) is only possible if $\eta^{\pm}$ are everywhere non-zero (again this is because Φ cannot send spacelike cross-sections to null submanifolds and vice versa). We therefore analyze the case $S^{\pm} = \emptyset$.

Without loss of generality, we take the degenerate symmetry generator to be future. First, we let $\mathcal{N}^-$ be the degenerate boundary. In that case, the causal character of η requires $f^- > 0$. Then (8.8) forces $a(f^+ + \kappa^+ H) > 0$ and can be integrated to get

$$H(\lambda, y^A) = \frac{1}{a\kappa^+}\left(\beta(y^A)\exp\left(\frac{a\kappa^+\lambda}{f^-(y^A)}\right) - af^+(y^A)\right), \qquad \beta(y^A) > 0. \tag{8.20}$$

The alternative case when $\mathcal{N}^+$ is the degenerate boundary is analogous. Now $f^+ > 0$ so $\operatorname{sign}(a) = \operatorname{sign}(f^- + \kappa^-\lambda)$ and integrating (8.8) yields

$$H(\lambda, y^A) = \frac{af^+(y^A)}{\kappa^-}\ln\left(\beta(y^A)\left|f^-(y^A) + \kappa^-\lambda\right|\right), \qquad \beta(y^A) > 0, \tag{8.21}$$

for whatever sign of a.

Summarizing, when $a > 0$ (resp. $a < 0$), a degenerate horizon $\mathcal{N}^-$ can be matched with a non-degenerate horizon $\mathcal{N}^+ \equiv \mathcal{N}_f^+$ (resp. $\mathcal{N}^+ \equiv \mathcal{N}_p^+$) with step function given by (8.20). On the other hand, a non-degenerate horizon $\mathcal{N}^- \equiv \mathcal{N}_f^-$ (resp. $\mathcal{N}^- \equiv \mathcal{N}_p^-$) can be matched with a degenerate horizon $\mathcal{N}^+$ with step function (8.21) and $a > 0$ (resp. $a < 0$). It is worth stressing that the step functions (8.20)-(8.21) are not linear, so the shell has non-zero pressure. Matchings of this type are allowed irrespectively of the extension of the degenerate horizon (which can even be geodesically complete) while the non-degenerate horizon is always limited by the fact that the would-be fixed point set must be absent. As before, from a physical point of view it is the presence of pressure, and its associated compression/stretching effect, that makes a matching of this type possible.

We collect the results from Sections 8.1, 8.2 and 8.3 in the following theorem.

Theorem 8.3.1. *The matching of* $(M^{\pm}, g^{\pm})$ *across the embedded AKH0s* $\mathcal{N}^{\pm}$ *in which the symmetry generators* $\{\eta^-, a\eta^+\}$, $a \in \mathbb{R} \setminus \{0\}$ *are to be identified is possible if the matching conditions are satisfied and the fixed points sets* $S^{\pm}$ *are identified via* Φ. *Moreover,*

(i) if $\mathcal{N}^{\pm}$ *are degenerate, the matching is possible with step function* (8.10)*;*

(ii) if $\mathcal{N}^{\pm}$ *are non-degenerate and* $S^{\pm} \neq \emptyset$*, the matching requires the surface gravities* $\{\kappa^-, a\kappa^+\}$ *to coincide and the step function is given by* (8.15)*;*

(iii) if $\mathcal{N}^{\pm}$ *are non-degenerate and* $S^{\pm} = \emptyset$*, the only possible matchings are (a)-(d) in Section* 8.2 *with step function* (8.18)*;*

(iv) *if* $\mathcal{N}^-$ *(resp.* $\mathcal{N}^+$*) is degenerate with future symmetry generator* $\eta^-|_{\overline{\mathcal{N}}^-} \neq 0$ *(resp.* $\eta^+|_{\overline{\mathcal{N}}^+} \neq 0$*) and* $\mathcal{N}^+$ *(resp.* $\mathcal{N}^-$*) is non-degenerate and with* $S^+ = \emptyset$ *(resp.* $S^- = \emptyset$*), the matching can be performed with step function* (8.20) *(resp.* (8.21)*);*

(v) *the matching between a degenerate and a non-degenerate boundaries is impossible when any of them contains fixed points.*

The resulting null shell has vanishing pressure in cases (i) and (ii) exclusively. Finally, the matching allows for the freedom of selecting a section on each side and imposing their identification via Φ *in (i), (iii) and (iv); and the freedom of setting the initial velocities at* $S^\pm$ *in (ii).*

8.4 killing horizons with bifurcation surfaces

From a physical point of view, perhaps one of the most interesting situations corresponds to non-degenerate Killing horizons with a bifurcation surface. This covers all black hole spacetimes with non-zero constant temperature and whose boundaries are geodesically complete Killing horizons, so it is sensible to analyze this case in more detail.

A Killing horizon (cf. Definition 2.6.1) satisfying the assumptions in Section 2.6 (in particular that its closure constitutes a smooth connected hypersurface without boundary) is also an embedded AKH$_0$. Thus, the matching across geodesically complete non-degenerate Killing horizons containing bifurcation surfaces falls into item (*ii*) in Theorem 8.3.1. However, since now we have much stronger conditions, namely the existence of a Killing vector on each side, we can restrict the matching far more, as we shall see next.

Consider two spacetimes $(\mathcal{M}^\pm, g^\pm)$ with smooth connected null boundaries $\overline{\mathcal{N}}^\pm$ containing bifurcation surfaces $S^\pm \subset \overline{\mathcal{N}}^\pm$ so that $\overline{\mathcal{N}}^\pm \setminus S^\pm$ are non-degenerate Killing horizons. As usual, we denote the corresponding Killing vectors by $\eta^\pm$. Our aim is to determine the matter content of the null shell arising from the matching of $(\mathcal{M}^\pm, g^\pm)$ whenever it happens to be possible. For that purpose and given the fact that the surface gravities $\kappa^\pm$ of $\eta^\pm$ are constant (see the discussion in Section 2.6.2) and non-zero on $\overline{\mathcal{N}}^\pm \setminus S^\pm$, it is convenient to take so-called Rácz-Wald coordinates $\{u_\pm, v_\pm, x^A_\pm\}$ (see Appendix D for details on the construction of this coordinates), which can be constructed so that

$$\overline{\mathcal{N}}^\pm = \{u_\pm = 0\}, \qquad S^\pm = \{u_\pm = 0, v_\pm = 0\} \tag{8.22}$$

and the Killing vectors $\eta^{\pm}$ and the spacetime metrics $g^{\pm}$ are given by

$$\eta^{\pm} = \kappa^{\pm}(-u_{\pm}\partial_{u_{\pm}} + v_{\pm}\partial_{v_{\pm}}), \tag{8.23}$$

$$g^{\pm} = -2G^{\pm}dv_{\pm}\left(du_{\pm} + u_{\pm}m^{\pm}_{A}dx^{A}_{\pm}\right) + \gamma^{\pm}_{AB}dx^{A}_{\pm}dx^{B}_{\pm}, \tag{8.24}$$

where as before we have extended $\kappa^{\pm}$ to $\mathcal{S}^{\pm}$ trivially as the same constants and $G^{\pm}, m^{\pm}_{A}, \gamma^{\pm}_{AB} \in \mathcal{F}(\mathcal{M})$ only depend on the product $u_{\pm}v_{\pm}$ and on the spatial coordinates $\{x^{C}_{\pm}\}$. The Rácz-Wald coordinates have a residual freedom that allows one to set $G^{\pm}|_{\mathcal{N}^{\pm}} = \text{const.} \neq 0$, which we enforce from now on. The non-zero components of the inverse metric are

$$g^{uv}_{\pm} = -\frac{1}{G^{\pm}}, \qquad g^{uu}_{\pm} = u^{2}_{\pm}\gamma^{AB}_{\pm}m^{\pm}_{A}m^{\pm}_{B}, \qquad g^{uA}_{\pm} = -u_{\pm}\gamma^{AB}_{\pm}m^{\pm}_{B}, \tag{8.25}$$

where $\gamma^{AB}_{\pm}$ is the inverse of $\gamma^{\pm}_{AB}$, i.e. $\gamma^{AB}_{\pm}\gamma^{\pm}_{BC} = \delta^{A}_{C}$.

In order to study the matching, we need to construct basis $\{L^{\pm}, k^{\pm}, v^{\pm}_{I}\}$ of $\Gamma(T\mathcal{M}^{\pm})|_{\mathcal{N}^{\pm}}$ according to (7.2). Our choice is

$$L^{\pm} = -\frac{1}{G^{\pm}}\partial_{u_{\pm}}, \quad k^{\pm} = \partial_{v_{\pm}}, \quad v^{\pm}_{I} = \partial_{x^{I}_{\pm}}, \tag{8.26}$$

and it is straightforward to check that $k^{\pm}$ are affine[1] (i.e. with $\kappa^{\pm}_{k^{\pm}} = 0$). We let the coordinates $v_{\pm}$ be the corresponding foliation functions satisfying $k^{\pm}(v_{\pm}) = 1$. Note that the choice (8.26) forces the functions $\{\mu^{\pm}_{1}, \mu^{\pm}_{I}\}$ defined in (7.6) to be given by $\mu^{\pm}_{1} = 1$, $\mu^{\pm}_{I} = 0$. However, we still cannot fix the sign ϵ of the energy-momentum tensor, as we do not know whether the boundaries $\mathcal{N}^{\pm}$ lie in the future or in the past with respect to $(\mathcal{M}^{\pm}, g^{\pm})$. The induced metrics on the sections $\{v_{\pm} = \text{const.}\}$ are $h^{\pm}_{IJ} \stackrel{\text{def}}{=} \gamma^{\pm}_{IJ}|_{u_{\pm}=0}$.

We are interested in matchings for which the vector fields $\eta^{\pm}$ are identified. In the language of the previous section, this means that $a = 1$, hence $\kappa^{+} = \kappa^{-}$ is forced (see item (*ii*) in Theorem 8.3.1). Since $\kappa^{\pm}$ must coincide, in the following we simply write κ to refer to both of them. Observe that the combination of (8.23) and (2.90) implies that $f^{\pm} = 0$.

[1] Indeed, one gets (we drop $\pm$ to ease the notation)

$$k^{\alpha}\nabla_{\alpha}k^{\beta} \stackrel{\mathcal{N}}{=} \Gamma^{\beta}_{vv} \stackrel{\mathcal{N}}{=} g^{\beta u}\partial_{v}g_{uv} \stackrel{\mathcal{N}}{=} \delta^{\beta}_{v}\frac{1}{G}\partial_{v}G \stackrel{\mathcal{N}}{=} \delta^{\beta}_{v}\frac{u}{G}\partial_{\varphi}G \stackrel{\mathcal{N}}{=} 0, \quad \text{where} \quad \varphi \stackrel{\text{def}}{=} uv.$$

We construct null metric hypersurface data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ embedded in $(\mathcal{M}^-, g^-)$ by considering an embedding ϕ^- of the form

$$\begin{aligned} \phi^- : \quad \mathcal{N} &\longrightarrow \widetilde{\mathcal{N}}^- \subset \mathcal{M}^- \\ (\lambda, y^I) &\longmapsto \phi^-(\lambda, y^I) = \left(u_- = 0, v_- = \lambda, x^I_- = y^I\right). \end{aligned} \tag{8.27}$$

In these circumstances, for the metric data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ to be embedded in $(\mathcal{M}^-, g^-)$ with embedding ϕ^- and rigging $\zeta^- = L^-$ (recall (2.22)), it must hold

$$\gamma = \mathring{\gamma}^-_{AB} dy^A \otimes dy^B, \qquad \boldsymbol{\ell} = d\lambda, \qquad \ell^{(2)} = 0. \tag{8.28}$$

As happened in Section 7.3.3, the matching is possible if and only if (a) $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ is also embedded in $(\mathcal{M}^+, g^+)$ with embedding ϕ^+ given by

$$\begin{aligned} \phi^+ : \quad \mathcal{N} &\longrightarrow \widetilde{\mathcal{N}}^+ \subset \mathcal{M}^- \\ (\lambda, y^I) &\longmapsto \phi^+(\lambda, y^I) = \left(u_+ = 0, v_+ = H(\lambda, y^I), x^I_+ = h^I(y^J)\right) \end{aligned} \tag{8.29}$$

and (b) the isometry condition (7.31) is satisfied. Note that $\eta^\pm$ being identified by the matching forces

$$H(\lambda, y^A) = \beta(y^A)\lambda, \tag{8.30}$$

where $\beta(y^A)$ is a strictly positive arbitrary function (this follows from combining $\eta^\pm|_{\widetilde{\mathcal{N}}^\pm} = \kappa v_\pm k^\pm$ with (8.15)). In the present context condition (7.31) reads

$$\mathring{\gamma}^-_{IJ}\big|_{u_-=0} = b^L_I b^K_J \mathring{\gamma}^+_{LK}\big|_{u_+=0}. \tag{8.31}$$

The functions $\mathring{\gamma}^\pm_{AB}$ depend on $\{u_\pm v_\pm ; x^A\}$, so condition (8.31) is λ-independent.

For the rest of the section we assume that $\{b^I_A, \mathring{\gamma}^\pm_{AB}\}$ are such that (8.31) holds. By Theorem 8.3.1, the bifurcation surfaces $S^\pm$ must be mapped to each other through the matching map Φ, so they must be isometric. This means that

$$\mathring{R}^-_{AB}|_p = b^I_A b^J_B \mathring{R}^+_{IJ}|_{\Phi(p)}, \qquad \forall p \in S^-, \quad \Phi(p) \in S^+, \tag{8.32}$$

where $\mathring{R}^\pm_{AB}$ are the Ricci tensors of the leaves $\{v_\pm = \text{const.}\}$. Obviously the tensors $\mathring{R}^\pm_{AB}$, which are constructed from the metrics $\mathring{\gamma}^\pm_{AB}|_{u_\pm=0}$, are independent of the coordinates $v_\pm$, i.e. they are constant along the null generators of $\widetilde{\mathcal{N}}^\pm$. The scalars b^J_I on $\widetilde{\mathcal{N}}^+$ do not depend on v_+ either (cf. (7.43)), so $\mathring{R}^-_{AB}$ and $b^I_A b^J_B \mathring{R}^+_{IJ}$ take the same value for all points of the null generators containing $p \in S^-$ and $\Phi(p) \in$

S^+ respectively. The fact that null generators must be identified by the matching entails that condition (8.33) holds everywhere, i.e.

$$\mathring{R}^-_{AB}\,|_p = b^I_A b^J_B \mathring{R}^+_{IJ}|_{\Phi(p)}, \qquad \forall p \in \overline{\mathcal{N}}^-, \quad \Phi(p) \in \overline{\mathcal{N}}^+. \tag{8.33}$$

In the following we shall remove the explicit writing of p and $\Phi(p)$ in this expression and similar ones. The trivial identification between $\overline{\mathcal{N}}^-$ and $\mathcal{N}$ ensures that the pull-back $(\phi^-)^*\mathring{R}^-$ coincides with the Ricci tensor R^h on the sections $\{\lambda = \text{const.}\} \subset \mathcal{N}$ (with metric $h_{AB} = \gamma_{AB}$). Consequently, it must hold that $R^h = (\phi^\pm)^*(\mathring{R}^\pm)$.

The following theorem determines the tensor fields $\mathbf{Y}^\pm$ and the energy-momentum tensor of the null shell arising from the matching of $(\mathcal{M}^\pm, g^\pm)$.

Theorem 8.4.1. *Consider two spacetimes $(\mathcal{M}^\pm, g^\pm)$ whose boundaries $\overline{\mathcal{N}}^\pm$ are closures of non-degenerate Killing horizons containing bifurcation surfaces $S^\pm$. Let $\eta^\pm$ be the corresponding Killing vector fields and assume that they have constant surface gravities $\kappa^\pm$ on $\overline{\mathcal{N}}^\pm$. Construct Rácz-Wald coordinates $\{u_\pm, v_\pm, x^A_\pm\}$ so that $\overline{\mathcal{N}}^\pm = \{u_\pm = 0\}$, $S^\pm = \{u_\pm = 0, v_\pm = 0\}$ and in which $\eta^\pm$ and the metrics $g^\pm$ are given by (8.23)-(8.24). Suppose that the matching of $(\mathcal{M}^\pm, g^\pm)$ across $\overline{\mathcal{N}}^\pm$ is feasible and that it identifies $\eta^\pm$, and let $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}^\pm\}$ be the corresponding null thin shell with metric part given by (8.28) and matching embeddings $\phi^\pm$ according to (8.27) and (8.29), where $\{\lambda, y^A\}$ are coordinates on $\mathcal{N}$ so that $\partial_\lambda \in \operatorname{Rad}\gamma$ and $H(\lambda, y^A) = \beta(y^A)\lambda$ for $0 < \beta(y^A) \in \mathcal{F}(\mathcal{N})$. Let $S \subset \mathcal{N}$ be the cross-section defined by $\phi^\pm(S) \stackrel{\text{def}}{=} S^\pm$ and $\mathrm{R}^\pm \stackrel{\text{def}}{=} (\phi^\pm)^*\mathbf{Ric}^\pm$ be the constraint tensors corresponding to the boundaries $\overline{\mathcal{N}}^\pm$. Denote by h, ∇^h and R^h the induced metric, Levi-Civita connection and Ricci tensor on the leaves $\{\lambda = const.\}$. Then, the gravitational and matter-energy content of the null shell $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}^\pm\}$ is given by*

$$\mathbf{Y}^\pm(\partial_\lambda, \partial_\lambda) = 0, \qquad \mathbf{Y}^\pm(\partial_\lambda, \partial_{y^A}) = r^\pm_A, \qquad \mathbf{Y}^\pm(\partial_{y^A}, \partial_{y^B}) = \Xi^\pm_{AB}\lambda, \tag{8.34}$$

$$\tau(d\lambda, d\lambda) = -\epsilon\gamma^{AB}[\Xi_{AB}]\lambda, \qquad \tau(d\lambda, dy^A) = \epsilon\gamma^{AB}[r_B], \qquad \tau(dy^A, dy^B) = 0, \tag{8.35}$$

where

$$r^-_A \stackrel{\text{def}}{=} \frac{m^-_A}{2}, \qquad r^+_A \stackrel{\text{def}}{=} \frac{b^B_A m^+_B}{2} - \frac{\nabla^h_A \beta}{\beta}, \qquad \Xi^\pm_{AB} \stackrel{\text{def}}{=} \frac{1}{2}\left(R^h_{AB} - \mathrm{R}^\pm_{AB} + 2\nabla^h_{(A} r^\pm_{B)} - 2r^\pm_A r^\pm_B \right) \tag{8.36}$$

and it holds that $r^\pm_A = r^\pm_A|_S$ and $\Xi^\pm_{AB} = \Xi^\pm_{AB}|_S$.

Remark 8.4.2. *The notation $r^\pm_A$ in Theorem 8.4.1 is consistent with (2.44).*

Proof. Recall that (8.26) forces entails $\mu_1^\pm = 1$, $\mu^\pm_I = 0$. We start by computing the quantity $\boldsymbol{\sigma}^\pm_L(v^\pm_I)$ (cf. (2.99)), for which we use (8.24)-(8.25) and the fact that $G^\pm$ is constant on $\mathcal{N}^\pm$:

$$\boldsymbol{\sigma}^\pm_L(v^\pm_I) \overset{\mathcal{N}^\pm}{=} \frac{1}{G^\pm} g^\pm\left(\nabla^\pm_{\partial_{x^I_\pm}} \partial_{v_\pm}, \partial_{u_\pm}\right) \overset{\mathcal{N}^\pm}{=} \frac{1}{G^\pm}\Gamma^{\pm\mu}{}_{vI}\, g^\pm_{\mu u} \overset{\mathcal{N}^\pm}{=} -\Gamma^{\pm v}{}_{vI} \overset{\mathcal{N}^\pm}{=} \frac{m^\pm_I}{2}. \tag{8.37}$$

The more direct way of proving the first and second results in (8.34) is by means of (7.88)-(7.89) and (7.91). In the present case $\mathbf{K}_{k^\pm} = 0$ (because $\mathcal{N}^\pm$ are Killing horizons), $\partial_\lambda\partial_\lambda H = 0$ (since $H(\lambda, y^A) = \beta(y^A)\lambda$) and $\kappa^\pm_k = 0$. Consequently, (7.88) and (7.91) give $\mathbf{Y}^\pm(\partial_\lambda, \partial_\lambda) = 0$, while (7.89) and (7.91) yield (recall (7.72))

$$\mathbf{Y}^+(\partial_\lambda, \partial_{y^A}) = b^B{}_A\, \boldsymbol{\sigma}^+_L(v^+_B) - \frac{\partial_\lambda\partial_{y^A} H}{\partial_\lambda H}, \qquad \mathbf{Y}^-(\partial_\lambda, \partial_{y^A}) = \boldsymbol{\sigma}^-_L(v^-_A). \tag{8.38}$$

Inserting (8.37) into (8.38) proves the second equations in (8.34).

To obtain an expression for $\mathbf{Y}^\pm(\partial_{y^A}, \partial_{y^B})$ we use an argument based on Proposition 6.4.5. First, from (8.28) we know that $\boldsymbol{\ell} = d\lambda$, which forces $n = \partial_\lambda$. Moreover, it holds that $\boldsymbol{\ell}_A \overset{\text{def}}{=} \boldsymbol{\ell}(\partial_{y^A}) = 0$ and that $s = 0$ (because $2s = \pounds_n\boldsymbol{\ell} = \iota_n(d^2\lambda) + d(\iota_n\boldsymbol{\ell}) = 0$, cf. (3.43)). The matching is assumed to be possible and such that $\eta^\pm$ are identified. Thus, there exists a vector field $\bar{\eta}$ verifying $\phi^\pm_\star\bar{\eta} \overset{\text{def}}{=} \eta^\pm$. As we know, the surface gravities $\kappa^\pm$ of $\eta^\pm$ are forced to be the same, so we write $\kappa \coloneqq \kappa^\pm$. From (8.27), it follows that $\bar{\eta} = \alpha n$ for $\alpha = \kappa\lambda$, and since the surface gravity κ of $\bar{\eta}$ coincides with κ (by Proposition 3.3.2), we get $\alpha = \kappa\lambda$. In particular, this yields $\nabla^h_A\alpha = 0$ and

$$\begin{aligned}(\partial_{y^A})^a(\partial_{y^B})^b\, \mathring{\nabla}_a\mathring{\nabla}_b\alpha &= \partial_{y^A}\partial_{y^B}\alpha - (\mathring{\nabla}_{\partial_{y^A}}\partial_{y^B})^b\,\mathring{\nabla}_b\alpha \\ &= \partial_{y^A}\partial_{y^B}\alpha - (\nabla^h_{\partial_{y^A}}\partial_{y^B})(\alpha) = 0, \end{aligned} \tag{8.39}$$

where in the next-to-last step we have used (3.88)-(3.89) for $\boldsymbol{\ell}_\parallel = 0$ and $\mathbf{U} = 0$. Taking into account the considerations above and noticing that here $\kappa^\pm_n = 0$ (because $\mathbf{Y}^\pm(\partial_\lambda, \partial_\lambda) = 0$), on both sides we can particularize Proposition 6.4.5 for the basis $\{n = \partial_\lambda, \partial_{y^A}\}$ and then the third equation in (8.34) follows at once. The expressions (8.35) are an immediate consequence of (7.58)-(7.60).

The fact that $r^\pm_A$ are constant along the null generators follows either from their definitions (8.36) (where nothing depends on λ or $v_\pm$) or from Proposition 6.4.5

(see (6.68)). To prove that $\Xi^{\pm}_{AB}$ are also λ-independent, we first particularize (6.21) for $s = 0$, $\kappa^{\pm}_{n} = 0$, $\mathbf{U} = 0$ and $\sqcap^{\eta}_{\pm} = 0$, obtaining

$$0 = \mathring{\nabla}_b\mathring{\nabla}_d\alpha - 2r^{\pm}_{(b}\mathring{\nabla}_{d)}\alpha - \alpha\pounds_n \mathrm{Y}^{\pm}_{bd} + \kappa\mathrm{Y}^{\pm}_{bd}. \tag{8.40}$$

Taking the Lie derivative along n and using (6.3) as well as $\kappa \stackrel{\mathrm{def}}{=} n(\alpha)$ (recall (3.102)), one gets

$$0 = \pounds_n\left(\mathring{\nabla}_b\mathring{\nabla}_d\alpha - 2r^{\pm}_{(b}\mathring{\nabla}_{d)}\alpha\right) - \alpha\pounds_n\pounds_n\mathrm{Y}^{\pm}_{bd}, \tag{8.41}$$

which upon contracting with $\{(\partial_{y^A})^b, (\partial_{y^B})^d\}$ and using that $\pounds_n\partial_{y^A} = 0$ and (8.39) gives

$$0 = \pounds_n\left(\nabla^h_A\nabla^h_B\alpha - 2r^{\pm}_{(A}\nabla^h_{B)}\alpha\right) - \alpha\pounds_n\pounds_n\mathrm{Y}^{\pm}_{AB} = -\alpha\pounds_n\pounds_n\mathrm{Y}^{\pm}_{AB} \quad \Longrightarrow \quad \pounds_n\pounds_n\mathrm{Y}^{\pm}_{AB} = 0,$$

where we used $\nabla^h_A\alpha = 0$ and the implication is a consequence of the fact that $\mathcal{N} \setminus \mathcal{S}$ is dense in $\mathcal{N}$. From this last result together with the last expression in (8.34) it is immediate that $\Xi^{\pm}_{AB}$ are independent of λ, i.e. constant along the null generators. □

Remark 8.4.3. *Note the intrinsic curvature term R^h drops out from the jump $[\Xi_{AB}]$. The underlying reason is the already mentioned isometry condition $(\phi^+)^\star R^+ = (\phi^-)^\star R^-$.*

The results (8.34)-(8.35) allow us to conclude that the matter-content of the shell, given by the tensor fields $\mathbf{Y}^{\pm}$ and τ, exclusively depends on the choice of $\beta(y^A)$, on the intrinsic and extrinsic geometry of the bifurcation surfaces $\mathcal{S}^{\pm}$ (we have just proved that $r^{\pm}_{A}$ and $\Xi^{\pm}_{AB}$ are constant along the null generators, so they are given by their values at $\mathcal{S}$), on the curvature tensor R^h_{AB} of $\mathcal{S}$ and on the pull-back to $\mathcal{S}$ of the constraint tensors $\mathcal{R}^{\pm}_{AB}$ of each side.

It is worth mentioning that the energy density of the shell (ruled by $\tau(d\lambda, d\lambda)$) is either identically zero or unavoidably changes its sign at the bifurcation surface (i.e. where $\lambda = 0$). Moreover, the energy current j^I is independent of λ, which means that the flux of energy is insensitive to the change of sign on the energy of the shell. This raises some questions concerning the physical interpretation of the quantities $\{\rho, j^A, p\}$ introduced in Remark 7.3.9. We include below some comments in this regard.

As we did in Section 7.3.3, let us call velocity the rate of change of the foliation functions $v_\pm$ along the null generators $e_1{}^\pm$ and acceleration to the rate of change of the velocity. In the present context we have

$$\begin{aligned} e_1{}^-(v_-) &= 1, & \nabla^-_{e_1{}^-} e_1{}^-(v_-) &= 0, \\ e_1^+(v_+) &= \partial_\lambda H = \beta(y^A), & \nabla^+_{e_1^+} e_1^+(v_+) &= \partial_\lambda \partial_\lambda H = 0. \end{aligned} \tag{8.42}$$

We have already discussed that the pressure p accounts for the effect of self-compression or self-stretching of points when crossing from $\mathcal{N}^-$ to $\mathcal{N}^+$. The trivial mapping between $\mathcal{N}^-$ and $\mathcal{N}$ always gives velocity equal to one on this side. For this reason, the effect of self-compression/self-stretching only appears when there exists non-constant acceleration along the generators of $\mathcal{N}^+$. As also shown in Section 7.3.3, the energy density of the shell increases when points are compressed and vice versa. Nevertheless, despite the fact that here the shell has vanishing pressure, some effect of compression or stretching of points is still taking place because the velocity along the null generators of $\mathcal{N}^+$, ruled by the function $\beta(y^A)$, is different for each generator.

We find the change of sign on the energy density of the shell ρ across the bifurcation surface really puzzling, and we do not know yet how to interpret this. The result suggests that the causality change of the Killing fields from future to past across the bifurcation surface somehow affects the energy density of the shell. We emphasize however, that this behaviour is fully compatible with the shell field equations obtained by Barrabés and Israel [63] for the case of null hypersurfaces. This of course had to be the case and we include an explicit proof in next section because this yields a non-trivial consistency check of our results.

8.4.1 *Surface layer equations*

As we have discussed in Chapter 2, the tensors $\mathbf{Y}^\pm$ and energy-momentum tensor τ of a shell satisfy the so-called Israel equations (also known as *shell equations* or *surface layer equations*). In the framework of hypersurface data, these equations are given by (2.160)-(2.161), where the bulk energy and momentum quantities $\rho_\ell{}^\pm$, $J^\pm_a$ are defined by (2.57)-(2.58). In the present case the unique normal vector field $\nu^\pm$ along $\mathcal{N}^\pm$ satisfying $g^\pm(\zeta^\pm, \nu^\pm) = 1$ is $\nu^\pm = e_1{}^\pm$. Moreover, $\mathbf{Ein}^\pm(e_1^\pm, e^\pm_a) = \mathbf{Ric}^\pm(e_1^\pm, e^\pm_a) = \mathrm{R}^\pm(n, \partial_{y^A}) = 0$, where we have used (4.17) first and then (4.19) (recall that $\mathbf{U} = 0$, $s = 0$, $\pounds_n r^\pm = 0$ and $\kappa_n^\pm = 0$). This immediately entails that $J^\pm = 0$. On the other hand, in terms of the basis $\{\zeta^\pm, e_1{}^\pm = \nu^\pm, e^\pm_A\}$ we can decompose the inverse metrics $g^{\pm}$ according to (2.27). By means of this decomposition, one obtains

$$
\begin{aligned}
\mathbf{Ein}^{\pm}(\zeta^{\pm}, e_1^{\pm}) &= \mathbf{Ric}^{\pm}(\zeta^{\pm}, e_1^{\pm}) - \frac{R^{\pm}}{2} \\
&= \mathbf{Ric}^{\pm}(\zeta^{\pm}, e_1^{\pm}) - \frac{1}{2}\Big(2\mathbf{Ric}^{\pm}(\zeta^{\pm}, e_1^{\pm}) + P^{cd}\mathbf{Ric}^{\pm}(e_c^{\pm}, e_d^{\pm}) \Big) \\
&= -\frac{1}{2} P^{cd} \mathrm{R}^{\pm}_{cd} = -\frac{1}{2}\gamma^{AB}\mathrm{R}^{\pm}_{AB},
\end{aligned}
\tag{8.43}
$$

where in the last two steps we have used (4.17), $\mathrm{R}^{\pm}(n,\cdot) = 0$ and the decomposition (3.80) for P^{cd}. Combining the definition (2.57) with equation (8.43) yields

$$
\rho_\ell^{\pm} = \frac{1}{2}\gamma^{AB}\mathrm{R}^{\pm}_{AB}. \tag{8.44}
$$

To prove that the shell equations hold for the tensor fields $\mathbf{Y}^{\pm}$ and τ of Theorem 8.4.1, we compute each term of the right hand side of (2.160)-(2.161) separately. We start with (2.160). The tensor $\boldsymbol{\mathcal{A}}$ (cf. (2.4)) is given in this case by

$$
\boldsymbol{\mathcal{A}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \gamma & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{8.45}
$$

because $\boldsymbol{\ell} = d\lambda$. Consequently, $|\det \gamma| = |\det \boldsymbol{\mathcal{A}}|$. On the other hand, the fact that $\mathcal{N}^{\pm}$ are totally geodesic entails $k^{\pm}(h^{\pm}_{IJ}) = 0$ (see (7.5)), from where it follows that $0 = e_1^{-}(h^{-}_{IJ}) = \partial_\lambda \gamma_{IJ}$. For spatial derivatives of $|\det\gamma|$, we use the well-known identity

$$
\frac{1}{2|\det\gamma|}\partial_{y^I}(|\det\gamma|) = \Gamma^{h\,A}_{\ AI}, \tag{8.46}
$$

where $\Gamma^{h\,A}_{\ BI}$ are the Christoffel symbols of ∇^h. The following expressions are immediate consequences of (8.34)-(8.35) together with $\ell_a = \delta^1_a$ and $\gamma_{1a} = 0$:

$$
\tau^{ab}\boldsymbol{\ell}_b = \tau^{1a} \tag{8.47}
$$

$$
\tau^{ab}\mathrm{Y}^{\pm}_{ab} = \tau^{11}\mathrm{Y}^{\pm}_{11} + 2\tau^{1I}\mathrm{Y}^{\pm}_{1I} + \tau^{IJ}\mathrm{Y}^{\pm}_{IJ} = 2\tau^{1I}\mathrm{Y}^{\pm}_{1I} \tag{8.48}
$$

$$
\tau^{ab}(\mathrm{Y}^{+}_{ab} + \mathrm{Y}^{-}_{ab}) = 2\epsilon\gamma^{IJ}[r_J](r^{+} + r^{-}) = 2\epsilon\gamma^{IJ}[r_I r_J]. \tag{8.49}
$$

$$
\tau^{bc}\gamma_{ca} = \tau^{bJ}\gamma_{aJ} = \delta^b_1\delta^I_a\tau^{1J}\gamma_{IJ} \tag{8.50}
$$

$$
\tau^{bd}\partial_{y^a}\gamma_{bd} = \tau^{11}\partial_{y^a}\gamma_{11} + 2\tau^{1I}\partial_{y^a}\gamma_{1I} + \tau^{IJ}\partial_{y^a}\gamma_{IJ} = 0. \tag{8.51}
$$

By (8.46) and (8.47), it follows

$$
\begin{aligned}
\frac{1}{\sqrt{|\det \boldsymbol{A}|}}\partial_{y^a}\left(\sqrt{|\det \boldsymbol{A}|}\,\tau^{ab}\ell_b\right) &= \frac{1}{\sqrt{|\det\gamma|}}\partial_{y^a}\left(\sqrt{|\det\gamma|}\,\tau^{1a}\right)\\
&= \partial_\lambda\tau^{11} + \frac{1}{\sqrt{|\det\gamma|}}\partial_{y^I}\left(\sqrt{|\det\gamma|}\,\tau^{1I}\right)\\
&= -\epsilon\gamma^{IJ}[\Xi_{IJ}] + \nabla^h_I\tau^{1I}\\
&= -\epsilon\gamma^{IJ}[\Xi_{IJ}] + \frac{\epsilon}{2}\gamma^{IJ}\left(\nabla^h_I[r_J] + \nabla^h_J[r_I]\right),\\
&= \frac{\epsilon}{2}\gamma^{IJ}[R_{IJ}] + \epsilon\gamma^{IJ}[r_I r_J],
\end{aligned}
\tag{8.52}
$$

where in the last equality we inserted (8.36) and (8.35). Combining (8.44), (8.49) and (8.52), the shell equation (2.160) follows immediately.

Checking the validity of equation (2.161) is almost direct. Since $J^{\pm}$ are zero, it suffices to substitute (8.50)-(8.51) into (2.161) to obtain

$$
0 = \frac{1}{\sqrt{|\det\gamma|}}\partial_{y^b}\left(\sqrt{|\det\gamma|}\,\delta^b_1\delta^I_a\tau^{1J}\gamma_{IJ}\right) = \frac{1}{\sqrt{|\det\gamma|}}\partial_\lambda\left(\sqrt{|\det\gamma|}\,\delta^I_a\tau^{1J}\gamma_{IJ}\right),
$$

which is automatically satisfied as nothing inside the parenthesis depends on λ.

8.5 spherical, plane or hyperbolic symmetric spacetimes

To conclude this chapter, we apply the formalism above to study particular matchings of interest. We start by determining the necessary and sufficient conditions that allow for the matching of two arbitrary spherical, plane or hyperbolic symmetric spacetimes admitting a Killing horizon with a bifurcation surface. We then particularize the results for the cases of two Schwarzschild spacetimes and two Schwarzschild-de Sitter spacetimes. We avoid ± notation until the actual matching is performed.

Let $(\mathcal{M}, g)$ be a spherical, plane or hyperbolic symmetric spacetime and Λ be its corresponding cosmological constant. Assume that it admits a Killing vector field η defining a bifurcation surface $S \subset \mathcal{M}$. Any spacetime of this kind is by definition a warped product of a 2-dimensional Lorentzian manifold $(\mathcal{N}, \bar{g})$ and an $(n-1)$-dimensional Riemannian space $(\mathcal{W}, h_k)$ of constant curvature $k \in \{1, 0, -1\}$ [130], [131]. We let $r \in \mathcal{F}(\mathcal{N})$ be the warping function and use Rácz-Wald coordinates $\{u, v, x^A\}$ constructed as in Section 8.4. We again scale a priori the Killing vectors defining the horizons of each spacetime so that they have the same surface gravity. In terms of $w \stackrel{\mathrm{def}}{=} uv$, the warped metric is

$$g = \overline{g} + r^2(w)\, \mathring{h}_k(x^A), \tag{8.53}$$

where $\overline{g} = -2G(w)\, du dv$, $G \in \mathcal{F}^*(\mathcal{M})$ (note that $G|_{\mathcal{N}}$ is constant).

The induced metric on the bifurcation surfaces $S^{\pm} = \{u_{\pm} = 0, v_{\pm} = 0\}$ is $g^{\pm}|_{S^{\pm}} = r^2_{\pm} \mathring{h}^{\pm}_k|_{S^{\pm}} = (r^{\pm}_0)^2 \mathring{h}^{\pm}_k$, where $r^{\pm}_0 \stackrel{\mathrm{def}}{=} r_{\pm}|_{\mathcal{N}^{\pm}} \neq 0$. The matching map $\Phi : S^- \longrightarrow S^+$ must be an isometry so the Ricci scalars of $g^{\pm}|_{S^{\pm}}$, which in this case are given by

$$(n-1)(n-2)k^{\pm}(r^{\pm}_0)^{-2}, \tag{8.54}$$

must be preserved by Φ. Therefore

$$\frac{k^-}{(r^-_0)^2} = \frac{k^+}{(r^+_0)^2}, \tag{8.55}$$

and we conclude that $k^{\pm}$ must coincide (recall that $k^{\pm} \in \{1, 0, -1\}$). From now on we simplify the notation and write k instead of $k^{\pm}$. An immediate consequence of (8.55) is that the jump $[r_0] \stackrel{\mathrm{def}}{=} r^+_0 - r^-_0$ is zero whenever $k \neq 0$, and can take whatever value in the plane case with $k = 0$.

Since $\mathcal{N}^{\pm}$ are totally geodesic, equation (7.31) constitutes an isometry condition between the leaves $\{v^{\pm} = \text{const.}\} \subset \mathcal{N}^{\pm}$. These sections are euclidean planes, spheres of radius $r^{\pm}_0$ and hyperbolic planes of curvature $-r_0^{-2}$ when $k = 0$, $k = 1$ and $k = -1$ respectively. The corresponding isometries are respectively euclidean motions, rotations and hyperbolic rotations. In each case they are also isometries of the ambient spacetimes, so the freedom in the matching, encoded in Φ, can be absorbed (with full generality) in the coordinates $\{u_+, v_+, x^A_+\}$ in such a way that the coefficients b^J_I take the simple form $b^J_I = \delta^J_I$. This will be assumed from now on. Thus (cf. (2.22))

$$\gamma_{IJ} \stackrel{\mathrm{def}}{=} g^{\pm}(e^{\pm}_I, e^{\pm}_J) = (r^{\pm}_0)^2 \mathring{h}^{\pm}_{k\,IJ}. \tag{8.56}$$

The metric (8.53) is of the form (8.24) with $m^{\pm}_A = 0$ and $\gamma^{\pm} = r^2_{\pm} \mathring{h}^{\pm}_k$. The tensor fields $\{\overset{\pm}{r}_A, \overset{\pm}{\Xi}_{AB}\}$ in this case read (cf. (8.36))

$$\begin{aligned} \overset{-}{r}_A &= 0, & \overset{-}{\Xi}_{AB} &= \frac{1}{2}\left(R^h_{AB} - \mathrm{R}^-_{AB}\right), \\ \overset{+}{r}_A &= -\frac{\nabla^h_A \beta}{\beta}, & \overset{+}{\Xi}_{AB} &= \frac{1}{2}\left(R^h_{AB} - \mathrm{R}^+_{AB} - \frac{2\nabla^h_A \nabla^h_B \beta}{\beta}\right). \end{aligned} \tag{8.57}$$

The fact that the metric (8.56) is of constant curvature kr_0^{-2} means that its Ricci tensor is (recall that h, also called γ_{IJ} when indices are used, is the induced metric at the bifurcation surface)

$$R^h_{AB} = \frac{(n-2)k}{(r_0^{\pm})^2}\gamma_{AB} \quad \Longrightarrow \quad \begin{aligned} \Xi^-_{AB} &= \frac{1}{2}\left(\frac{(n-2)k}{(r_0^-)^2}\gamma_{AB} - \mathrm{R}^-_{AB}\right), \\ \Xi^+_{AB} &= \frac{1}{2}\left(\frac{(n-2)k}{(r_0^+)^2}\gamma_{AB} - \mathrm{R}^+_{AB} - \frac{2\nabla^h_A\nabla^h_B\beta}{\beta}\right). \end{aligned} \tag{8.58}$$

Substituting this in the expressions of Theorem 8.4.1 and using (8.55), we obtain

$$\begin{aligned} \mathbf{Y}^-(\partial_\lambda, \partial_\lambda) &= 0, \qquad \mathbf{Y}^-(\partial_\lambda, \partial_{y^A}) = 0, \\ \mathbf{Y}^-(\partial_{y^A}, \partial_{y^B}) &= \frac{\lambda}{2}\left(\frac{(n-2)k}{(r_0^-)^2}\gamma_{AB} - \mathrm{R}^-_{AB}\right), \end{aligned} \tag{8.59}$$

$$\begin{aligned} \mathbf{Y}^+(\partial_\lambda, \partial_\lambda) &= 0, \qquad \mathbf{Y}^+(\partial_\lambda, \partial_{y^A}) = -\frac{\nabla^h_A\beta}{\beta}, \\ \mathbf{Y}^+(\partial_{y^A}, \partial_{y^B}) &= \frac{\lambda}{2}\left(\frac{(n-2)k}{(r_0^+)^2}\gamma_{AB} - \mathrm{R}^+_{AB} - \frac{2\nabla^h_A\nabla^h_B\beta}{\beta}\right), \end{aligned} \tag{8.60}$$

$$\begin{aligned} \tau(dy^A, dy^B) &= 0, \qquad \tau(d\lambda, dy^A) = -\epsilon\gamma^{AB}\frac{\nabla^h_B\beta}{\beta}, \\ \tau(d\lambda, d\lambda) &= \epsilon\frac{\lambda\gamma^{AB}}{2}\left([\mathrm{R}_{AB}] + \frac{2\nabla^h_A\nabla^h_B\beta}{\beta}\right). \end{aligned} \tag{8.61}$$

The resulting shells have therefore energy density ρ and energy flux j^J given by

$$\rho = \epsilon\frac{\lambda\gamma^{AB}}{2}\left([\mathrm{R}_{AB}] + \frac{2\nabla^h_A\nabla^h_B\beta}{\beta}\right), \qquad j^A = -\epsilon\gamma^{AB}\frac{\nabla^h_B\beta}{\beta}. \tag{8.62}$$

An interesting particular case occurs when the spacetimes $(\mathcal{M}^{\pm}, g^{\pm})$ to be matched are, in addition, solutions to the $\Lambda^{\pm}$-vacuum Einstein field equations

$$R^{\pm}_{\alpha\beta} = \frac{2\Lambda^{\pm}}{n-1}g^{\pm}_{\alpha\beta}. \tag{8.63}$$

In these circumstances, the constraint tensors $\mathrm{R}^{\pm}_{AB}$ are given by (cf. (4.17))

$$\mathrm{R}^{\pm}_{AB} = \frac{2\Lambda^{\pm}}{n-1}\gamma_{AB}. \tag{8.64}$$

Inserting this into (8.59)-(8.61) yields

$$
\begin{aligned}
\mathbf{Y}^-(\partial_\lambda,\partial_\lambda) &= 0, \qquad \mathbf{Y}^-(\partial_\lambda,\partial_{y^A}) = 0, \\
\mathbf{Y}^-(\partial_{y^A},\partial_{y^B}) &= \frac{\varepsilon^-\lambda}{2}\gamma_{AB}, \\
\mathbf{Y}^+(\partial_\lambda,\partial_\lambda) &= 0, \qquad \mathbf{Y}^+(\partial_\lambda,\partial_{y^A}) = -\frac{\nabla^h_A\beta}{\beta}, \\
\mathbf{Y}^+(\partial_{y^A},\partial_{y^B}) &= \frac{\lambda}{2}\left(\varepsilon^+\gamma_{AB} - \frac{2\nabla^h_A\nabla^h_B\beta}{\beta}\right), \\
\tau(dy^A,dy^B) &= 0, \qquad \tau(d\lambda,dy^A) = -\epsilon\gamma^{AB}\frac{\nabla^h_B\beta}{\beta}, \\
\tau(d\lambda,d\lambda) &= \epsilon\lambda\left([\Lambda] + \frac{\gamma^{AB}\nabla^h_A\nabla^h_B\beta}{\beta}\right).
\end{aligned}
\tag{8.65}
$$

where we have defined

$$
\varepsilon^\pm \stackrel{\mathrm{def}}{=} \frac{(n-2)k}{(r_0^\pm)^2} - \frac{2\Lambda^\pm}{n-1}. \tag{8.66}
$$

It is worth stressing that the constant curvature parameter k does not appear explicitly in (8.65). It however appears implicitly in the metric γ_{IJ} and in the corresponding covariant derivative ∇^h. In the next subsections we particularize further to (the maximally extended) Schwarzschild and Schwarzschild-de Sitter spacetimes.

8.5.1 *Schwarzschild spacetime*

If the metrics on both sides are Schwarzschild we have $\Lambda^\pm = 0$ and $k = 1$. By (8.55), the radii $r_0^\pm$ must coincide, so the Schwarzschild mass of both sides is necessarily the same. We write r_0 instead of $r_0^\pm$ from now on. Thus, (8.65) reduces to

$$
\begin{aligned}
\mathbf{Y}^-(\partial_\lambda,\partial_\lambda) &= 0, \qquad \mathbf{Y}^-(\partial_\lambda,\partial_{y^A}) = 0, \\
\mathbf{Y}^-(\partial_{y^A},\partial_{y^B}) &= \frac{(n-2)\lambda}{2r_0^2}\gamma_{AB}, \\
\mathbf{Y}^+(\partial_\lambda,\partial_\lambda) &= 0, \qquad \mathbf{Y}^+(\partial_\lambda,\partial_{y^A}) = -\frac{\nabla^h_A\beta}{\beta}, \\
\mathbf{Y}^+(\partial_{y^A},\partial_{y^B}) &= \left(\frac{(n-2)}{2r_0^2}\gamma_{AB} - \frac{\nabla^h_A\nabla^h_B\beta}{\beta}\right)\lambda, \\
\tau(dy^A,dy^B) &= 0, \qquad \tau(d\lambda,dy^A) = -\epsilon\gamma^{AB}\frac{\nabla^h_B\beta}{\beta}, \\
\tau(d\lambda,d\lambda) &= \epsilon\frac{\gamma^{AB}\nabla^h_A\nabla^h_B\beta}{\beta}\lambda.
\end{aligned}
\tag{8.67}
$$

The tensor γ is the round metric of radius r_0 so its Laplace-Beltrami operator is $r_0^{-2}\Delta_{\mathbb{S}^{n-1}}$ where $\Delta_{\mathbb{S}^{n-1}}$ is the Laplacian of the unit $(n-1)$-sphere.

For each natural number l we let $\{Y_{l,m}\}$, $m = 0, ..., N(n,l) - 1$ be a collection of linearly independent solutions of

$$\Delta_{\mathbb{S}^{n-1}} Y_{l,m} = -l(l+n-2) Y_{l,m} \tag{8.68}$$

which, as usual we call spherical harmonics. The number $N(n,l)$ is (see e.g. [132])

$$\begin{aligned} N(n,l) &= 1 && \text{if } l = 0, \\ N(n,l) &= \binom{l+n-2}{l} + \binom{l+n-3}{l-1} && \text{otherwise.} \end{aligned} \tag{8.69}$$

Since $\{Y_{l,m}\}$ form a complete set of functions over $\mathbb{S}^{n-1}$, any (sufficiently regular) function β can be decomposed in this basis. Observe that β can be ensured to be positive by selecting the coefficient of $Y_{0,0}$ suitably positive and large. By expressing β as

$$\beta = \sum_{l=0}^{\infty} \sum_{m=0}^{N(n,l)-1} a_{l,m} Y_{l,m}, \qquad \text{where} \quad a_{l,m} \in \mathbb{R}, \tag{8.70}$$

the energy density of the shell is given by (cf. (8.67), Remark 7.3.9)

$$\rho = \epsilon \frac{\Delta_{\mathbb{S}^{n-1}}\beta}{r_0^2 \beta} \lambda = -\epsilon \frac{\sum_{l=0}^{\infty} l(l+n-2) \sum_{m=0}^{N(n,l)-1} a_{l,m} Y_{l,m}}{r_0^2 \sum_{l=0}^{\infty} \sum_{m=0}^{N(n,l)-1} a_{l,m} Y_{l,m}} \lambda. \tag{8.71}$$

The simplest case occurs when β is a positive constant. Then $[\mathbf{Y}] = 0$ and we have complete absence of shell. The step function $H = \beta\lambda$ can be absorbed in the coordinates of the $(\mathcal{M}^+, g^+)$ side. This coordinate freedom is a consequence precisely of the fact that Schwarzschild admits a one-parameter isometry group leaving the Killing horizon, and its generators, invariant. This ensures that, in the absence of shell, we recover the global Schwarzschild spacetime, as we must.

We conclude with a simple but not trivial example in dimension four (i.e. $n = 3$). Take

$$\beta(\theta) = 3\sqrt{\pi} Y_{0,0} + \frac{2\sqrt{\pi}}{\sqrt{5}} Y_{2,0}(\theta) = 1 + \frac{3}{2}\cos^2\theta, \qquad \begin{aligned} Y_{0,0} &= \frac{1}{2\sqrt{\pi}}, \\ Y_{2,0} &= \frac{\sqrt{5}}{2\sqrt{\pi}} P_2(\cos\theta), \end{aligned} \tag{8.72}$$

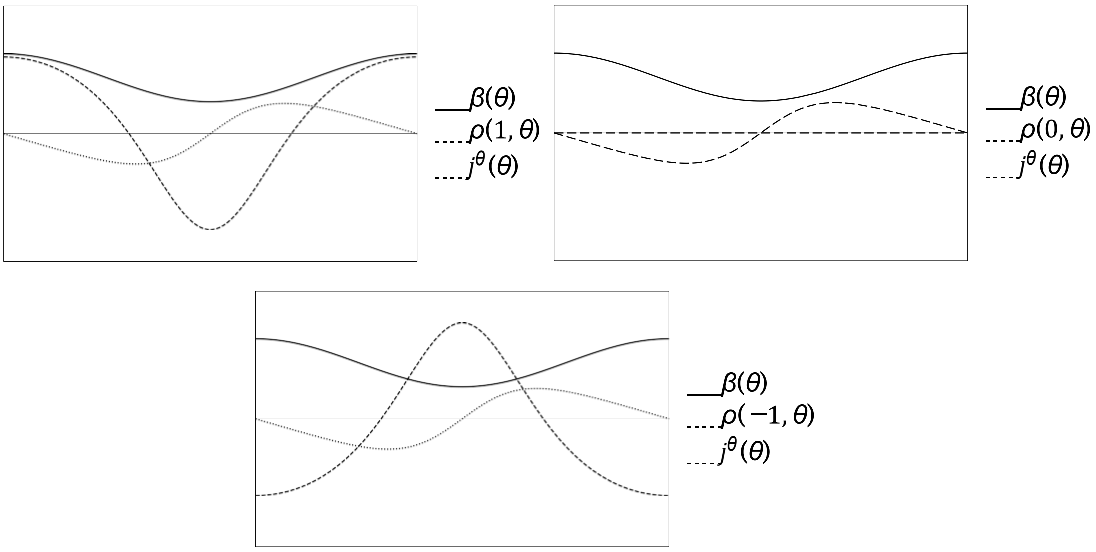


Figure 8.2: For $r_0 = 1$ and $\beta(\theta)$ given by (8.72), the up-left, up-right and bottom plots show $\beta(\theta)$, $j^\theta(\theta)$ and the energy density $\rho(\lambda, \theta)$ for $\lambda = 1$, $\lambda = 0$ and $\lambda = -1$ respectively. The figure corresponds to the case when $\mathbb{N}^-$ lies in the future of M^- so that the past rigging ζ^- points inwards, hence $\epsilon = -1$.

where $P_l(x)$ denote Legendre polynomials of degree l. This yields energy density and energy fluxes

$$\rho = -\epsilon \frac{3\ \ 3\cos^2\theta - 1}{1 + \frac{3}{2}\cos^2\theta} \frac{\lambda}{r_0^2} \qquad j^\theta = -\epsilon \frac{3\sin\theta\cos\theta}{r_0^2\ \ 1 + \frac{3}{2}\cos^2\theta}, \qquad j^\phi = 0. \tag{8.73}$$

In Figure 8.2 we plot the functions $\beta(\theta)$, $j^\theta(\theta)$ and the energy density $\rho(\lambda, \theta)$ for $\lambda = 1$, $\lambda = 0$ and $\lambda = -1$ in units where $r_0 = 1$. As already discussed, the energy density changes sign at the bifurcation surface, despite the fact that the energy flux is constant along each null generator, including at the crossing of the bifurcation surface. The figure shows clearly how the energy flux component j^θ is positive (resp. negative) whenever the function β decreases (resp. increases), i.e. the energy flows towards those null generators with higher values of β.

8.5.2 *Schwarzschild-de Sitter spacetime*

Our final example in this chapter is the matching of two Schwarzschild-de Sitter spacetimes. As before, $k = 1$ and the horizon radii $r_0^\pm$ in both sides are forced to be the same so we can simply write r_0. A Schwarzschild-de Sitter spacetime of mass

M and cosmological constant $\Lambda > 0$ may have several, one or none Killing horizons located at r_i^H depending on the number of (positive) roots of the polynomial

$$0 = P(r_i^H) = (r_i^H)^{n-2} - \frac{2\Lambda(r_i^H)^n}{n(n-1)} - \frac{2M}{(n-1)(n-2)}. \tag{8.74}$$

Since we do the matching on a preselected horizon we change the point of view, namely we take any positive value r_0 and use this expression to determine M in terms of r_0 and Λ. The cosmological constants $\Lambda^{\pm}$ on both sides are allowed to be different. However, once they are selected, the corresponding masses $M^{\pm}$ must have jump

$$[M] = -\frac{(n-2)}{n}[\Lambda] r_0^n \tag{8.75}$$

Thus, a priori one can match across a horizon two Schwarzschild-de Sitter spacetimes with different masses and cosmological constants but *only* if the parameters are related by (8.75).

The matter content of the shell is in this case given by (8.65). As in the previous section we may decompose the function β in terms of spherical harmonics. The corresponding expression for the energy density is now

$$\rho = -\epsilon \left(\frac{\sum_{l=0}^{\infty} l(l+n-2) \sum_{m=0}^{N(n,l)-1} a_{l,m} Y_{l,m}}{r_0^2 \sum_{l=0}^{\infty} \sum_{m=0}^{N(n,l)-1} a_{l,m} Y_{l,m}} - [\Lambda] \right) \lambda. \tag{8.76}$$

Let us conclude with some interesting observations. The first is that it is impossible to construct a (non-trivial) shell with vanishing energy density. This is because in such case it must hold

$$\Delta_{S^{n-1}} \beta = -r_0^2 [\Lambda] \beta. \tag{8.77}$$

and all solutions of these equation must necessarily have zeroes, which is not allowed for the matching function β.

An interesting example is when the shell is composed on null dust, i.e. with identically zero energy-flux. By (8.65), this requires β to be a (positive) constant and then the energy density of the null dust is

$$\rho = \epsilon[\Lambda]\lambda. \tag{8.78}$$

The behaviour of this null dust is striking. Assume that $\mathcal{N}^-$ lies in the future of $(\mathcal{M}^-, g^-)$ so that ζ^- (which has been chosen past-directed) points inwards. Then we need to enforce $\epsilon = -1$. For definiteness suppose also that $[\Lambda] > 0$. In these circumstances, the energy density is $\rho = -[\Lambda]\lambda$, hence it is everywhere positive in the past of the bifurcation surface (i.e. for $\lambda < 0$) so the system starts being perfectly reasonable from a physical point of view. The shell then evolves on its own in a manner dictated by the field equations and ends up in a state in which the energy density is everywhere negative. This negative energy density cannot be considered as unphysical, since it has evolved from a physically reasonable initial state, the system itself is physically reasonable (a collection of incoherent massless particles) and its evolution is dictated by the gravitational shell equations that follow from the Einstein field equations. This is a rather surprising behaviour.

Furthermore, this behaviour is not exclusive of null dust. In fact, this occurs for more general functions β. Provided that we select β to be an everywhere positive function on $\mathbb{S}^{n-1}$, it holds that the energy density ρ is always positive for $\lambda < 0$ as long as the jump $[\Lambda]$ is suitably positive and large. Then, we have a shell with initial positive energy density and non-zero energy flux which unavoidably evolves into a state of negative energy density with no change along the evolution of the energy flux, which by (8.65) is independent of λ.

9 MATCHING FROM AN ABSTRACT VIEWPOINT

In Chapter 7, we have studied the matching of two general spacetimes with null boundaries that admit a foliation by diffeomorphic spacelike sections. The necessary and sufficient conditions that allow for the matching have been identified, and we have determined the geometrical objects upon which the matching depends. We have also provided explicit expressions for the gravitational and matter-energy content of the resulting shells (ruled by the tensors $\mathbf{Y}^{\pm}$ and the energy-momentum tensor), and particularized the corresponding results for the matching of two regions of Minkowski across a null hyperplane (see Section 7.3.3) and for the case when the boundaries are embedded AKH$_0$ (see Chapter 8). At this stage, at least two questions arise naturally. The first one is whether one can obtain analogous results without the topological assumptions on the boundaries and the second is whether there is a way of formulating the matching problem in a fully abstract manner, namely without making any reference to the actual spacetimes to be matched. Addressing these questions constitutes the aim of the last chapter of this thesis.

More concretely, we have seen that the matching of two given spacetimes is possible if the junction conditions are satisfied. These requirements are well understood from the point of view of the spacetimes, and even in the picture of embedded (metric) hypersurface data (recall Theorem 2.7.1), but it is not obvious how to write them in a purely abstract (in the sense of detached from the spacetimes to be matched) way. The specific purposes of this chapter are the following. First, we will provide a suitable abstract formulation of the junction conditions for the case of boundaries of any topology and any causal character. Then, we will study the actual problem of matching two spacetimes with null boundaries, analyzing (at the abstract level) the objects upon which the matching depends and including explicit expressions for the riggings identified in the matching process and the gravitational/matter-energy content of the (completely general) null shell. We will also address the problem of multiple matchings, relating the matter content and the energy-momentum tensor of two different shells arising from different matchings of the same two spacetimes. The particular case when one of the multiple matchings corresponds to having no shell will be studied in detail. These results apply to the context of the cut-and-paste procedure for the matching, and will allow us to describe this matching method in an abstract manner. Finally, assuming boundaries with product topology $S \times \mathbb{R}$ (where S is a spacelike section and the null generators are along $\mathbb{R}$) we will recover the results from Chapters 7 in the present setup, in particular the existence of a step function and the explicit form of the enegy-momentum tensor of the shell.

9.1 abstract matching without topological assumptions

Let us start with the abstract formulation of the junction conditions. As mentioned above, we first consider that the boundaries $\widetilde{\mathcal{N}}^{\pm}$ of the spacetimes $(\mathcal{M}^{\pm}, g^{\pm})$ to be matched have any topology and any causal character. Since $\widetilde{\mathcal{N}}^{-}$ is embedded, there exists an abstract manifold $\mathcal{N}$ and an embedding $\iota^{-} : \mathcal{N} \hookrightarrow \mathcal{M}^{-}$ such that $\iota^{-}(\mathcal{N}) = \widetilde{\mathcal{N}}^{-}$. From the embedding ι^{-}, one can construct an infinite number of embeddings simply by applying additional diffeomorphisms within $\mathcal{N}$. To elude this unavoidable redundancy, we henceforth let ι^{-} be one specific choice among all possible. This allows us to build embedded hypersurface data $\mathcal{D} \stackrel{\text{def}}{=} \{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ by requiring (2.22), (2.39), i.e. by defining

$$
\gamma \stackrel{\text{def}}{=} (\iota^{-})^{*}(g^{-}), \qquad \boldsymbol{\ell} \stackrel{\text{def}}{=} (\iota^{-})^{*}(g^{-}(\zeta^{-}, \cdot)), \qquad \ell^{(2)} \stackrel{\text{def}}{=} (\iota^{-})^{*}(g^{-}(\zeta^{-}, \zeta^{-})),
$$
$$
\mathbf{Y}^{-} \stackrel{\text{def}}{=} \frac{1}{2}(\iota^{-})^{*}(\pounds_{\zeta^{-}} g^{-}). \tag{9.1}
$$

As discussed in Chapter 7 and in Theorem 2.7.1, two spacetimes $(\mathcal{M}^{\pm}, g^{\pm})$ can be matched if there exists a pair of embeddings $\phi^{\pm} : \mathcal{N} \hookrightarrow \mathcal{M}^{\pm}$ related to a matching map Φ by $\phi^{+} = \Phi \circ \phi^{-}$. Moreover, the embedding and the rigging on one of the sides (say the minus side) can always be chosen freely. Suppose we enforce $\phi^{-} = \iota^{-}$ and take a specific rigging ζ^{-}. Then all the information about the matching (codified by Φ at the spacetime level) is encoded in ϕ^{+}, and the junction conditions can be written in terms of ϕ^{+} according to (7.23). The rigging ζ^{+} is uniquely determined by (7.23) from ϕ^{+} and the data $\{\gamma, \boldsymbol{\ell}, \ell^{(2)}\}$, and the gravitational/matter content of the shell is ruled by the tensor field $[\mathbf{Y}]$ (recall (7.25)).

The junction conditions (7.23), despite being of a more abstract nature than (7.22), still codify the matching information in the pair $\{\phi^+, \zeta^+\}$, which are not of abstract nature. In order to provide the matching information in terms of objects defined at the abstract level, we must take one step further. The following theorem, based on the existence of a diffeomorphism φ of the abstract manifold $\mathcal{N}$ onto itself, sets up the corresponding construction.

Theorem 9.1.1. *Consider two hypersurface data* $\mathcal{D} \stackrel{\text{def}}{=} \{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}^-\}$, $\widehat{\mathcal{D}} \stackrel{\text{def}}{=} \{\mathcal{N}, \hat{\gamma}, \hat{\boldsymbol{\ell}}, \hat{\ell}^{(2)}, \mathbf{Y}^+\}$ *embedded in two spacetimes* $(\mathcal{M}^-, g^-)$, $(\mathcal{M}^+, g^+)$ *with embeddings* Φ^-, Φ^+ *and riggings* L^-, L^+ *respectively. Assume that* $\Phi^{\pm}(\mathcal{N}) \stackrel{\text{def}}{=} \widetilde{\mathcal{N}}^{\pm}$ *are boundaries of* $(\mathcal{M}^{\pm}, g^{\pm})$ *and let* $\epsilon^+ = +1$ *(resp.* $\epsilon^+ = -1$*) if* L^+ *points outwards (resp. inwards) from* $\mathcal{M}^+$*. Define* ϵ^- *in the same way (i.e.* $\epsilon^- = +1$ *if* L^- *points outwards,* $\epsilon^- = -1$ *if inwards). The matching of* $(\mathcal{M}^{\pm}, g^{\pm})$ *across* $\widetilde{\mathcal{N}}^{\pm}$ *is possible if and only if*

(i) There exist a gauge group element $\mathcal{G}_{(z,V)}$ *and a diffeomorphism* φ *of* $\mathcal{N}$ *onto itself such that*

$$\mathcal{G}_{(z,V)}(\varphi^\star \hat{\gamma}) = \gamma, \qquad \mathcal{G}_{(z,V)}(\varphi^\star \hat{\boldsymbol{\ell}}) = \boldsymbol{\ell}, \qquad \mathcal{G}_{(z,V)}(\varphi^\star \hat{\ell}^{(2)}) = \ell^{(2)}; \tag{9.2}$$

(ii) $\operatorname{sign}(z) = -\operatorname{sign}(\epsilon^+)\operatorname{sign}(\epsilon^-)$.

Proof. The fact that $\mathcal{D}$, $\widehat{\mathcal{D}}$ are embedded on $(\mathcal{M}^{\pm}, g^{\pm})$ respectively means that

$$\begin{aligned}
&\gamma \stackrel{\text{def}}{=} (\Phi^-)^\star(g^-), && \boldsymbol{\ell} \stackrel{\text{def}}{=} (\Phi^-)^\star(g^-(L^-, \cdot)), && \ell^{(2)} \stackrel{\text{def}}{=} (\Phi^-)^\star(g^-(L^-, L^-)),\\
&\hat{\gamma} \stackrel{\text{def}}{=} (\Phi^+)^\star(g^+), && \hat{\boldsymbol{\ell}} \stackrel{\text{def}}{=} (\Phi^+)^\star(g^+(L^+, \cdot)), && \hat{\ell}^{(2)} \stackrel{\text{def}}{=} (\Phi^+)^\star(g^+(L^+, L^+)),\\
&\mathbf{Y}^- \stackrel{\text{def}}{=} \tfrac{1}{2}(\Phi^-)^\star(\pounds_{L^-} g^-), && \mathbf{Y}^+ \stackrel{\text{def}}{=} \tfrac{1}{2}(\Phi^+)^\star(\pounds_{L^+} g^+).
\end{aligned} \tag{9.3}$$

Since the spacetimes $(\mathcal{M}^{\pm}, g^{\pm})$, the embeddings $\Phi^{\pm}$ and the riggings $L^{\pm}$ are all given, the tensor fields in (9.3) are known. To prove the first part of the theorem, we start by assuming (*i*)-(*ii*). Thus, there exist a pair $\{z \in \mathcal{F}^\star(\mathcal{N}), V \in \Gamma(T\mathcal{N})\}$ and a diffeomorphism $\varphi : \mathcal{N} \longrightarrow \mathcal{N}$ so that (9.2) holds. These conditions can be rewritten as (cf. (2.30)-(2.32), Proposition 2.2.10)

$$\varphi^\star \hat{\gamma} = \mathcal{G}^{-1}_{(z,V)}(\gamma) = \mathcal{G}_{(z^{-1},-zV)}(\gamma) = \gamma, \tag{9.4}$$

$$\varphi^\star \hat{\boldsymbol{\ell}} = \mathcal{G}^{-1}_{(z,V)}(\boldsymbol{\ell}) = \mathcal{G}_{(z^{-1},-zV)}(\boldsymbol{\ell}) = \frac{\boldsymbol{\ell}}{z} - \gamma(V, \cdot), \tag{9.5}$$

$$\varphi^\star \hat{\ell}^{(2)} = \mathcal{G}^{-1}_{(z,V)}(\ell^{(2)}) = \mathcal{G}_{(z^{-1},-zV)}(\ell^{(2)}) = \frac{\ell^{(2)}}{z^2} - \frac{2\boldsymbol{\ell}(V)}{z} + \gamma(V, V). \tag{9.6}$$

Let us define the map $\phi^+ \stackrel{\text{def}}{=} \Phi^+ \circ \varphi$, the vector field $V' \stackrel{\text{def}}{=} \Phi^+_\star(\varphi_\star V)$, the function $z' \in \mathcal{F}^\star(\widetilde{\mathcal{N}}^+)$ given by $\varphi^\star((\Phi^+)^\star z') \stackrel{\text{def}}{=} z$ and the rigging $\zeta^+ \stackrel{\text{def}}{=} z'(L^+ + V')$ along $\widetilde{\mathcal{N}}^+$.

By definition of z', it holds that $\mathrm{sign}(z) = \mathrm{sign}(z')$. On the other hand, combining (9.4)-(9.6) with the fact that $\hat{\mathcal{D}}$ is embedded with embedding ι^+ and rigging L^+, it follows

$$
\begin{aligned}
\gamma &= \varphi^\star\hat{\gamma} = \varphi^\star\big((\iota^+)^\star(g^+)\big) = (\phi^+)^\star(g^+), && (9.7)\\
\boldsymbol{\ell} &= z\big(\varphi^\star\hat{\boldsymbol{\ell}} + (\varphi^\star\hat{\gamma})(V,\cdot)\big)\\
&= z\varphi^\star\big((\iota^+)^\star\big(g^+(L^+,\cdot) + g^+(V',\cdot)\big)\big) = (\phi^+)^\star(g^+(\zeta^+,\cdot)), && (9.8)\\
\ell^{(2)} &= z^2\big(\varphi^\star\hat{\ell}^{(2)} + 2(\varphi^\star\hat{\boldsymbol{\ell}})(V) + (\varphi^\star\hat{\gamma})(V,V)\big)\\
&= z^2\varphi^\star\big((\iota^+)^\star\big(g^+(L^+,L^+) + 2g^+(L^+,V') + g^+(V',V')\big)\big)\\
&= (\phi^+)^\star(g^+(\zeta^+,\zeta^+)). && (9.9)
\end{aligned}
$$

The data $\mathcal{D}$ is therefore embedded in $(\mathcal{M}^+, g^+)$ with embedding ϕ^+ and rigging ζ^+. Thus, conditions (7.24) are satisfied for $\phi^- = \iota^-$, $\phi^+ = \iota^+ \circ \varphi$ and for the riggings $\zeta^- = L^-$, ζ^+. Moreover, combining (ii) (which holds by assumption), the definition of ζ^+ and $\mathrm{sign}(z') = \mathrm{sign}(z)$, it follows

$$\zeta^+ = -\mathrm{sign}(\epsilon^+)\mathrm{sign}(\epsilon^-)|z'|\big(L^+ + V'\big). \qquad (9.10)$$

It is straightforward to check that (9.10) implies that whenever L^- points inwards (resp. outwards) then ζ^+ points outwards (resp. inwards) irrespectively of the orientation of L^+. Thus, $\mathcal{D}$ is embedded in $(\mathcal{M}^\pm, g^\pm)$ and L^-, ζ^+ are such that one points inwards and the other outwards, which means that the matching of $(\mathcal{M}^\pm, g^\pm)$ is possible.

To prove the converse, we assume that the matching is possible for two pairs $\{\phi^\pm, \zeta^\pm\}$. We have already discussed the flexibility of selecting at will the embedding and the rigging on one side (say the minus side). Let us therefore set $\phi^- = \iota^-$, $\zeta^- = L^-$. Since both L^+ and ζ^+ are riggings along $\widetilde{\mathcal{N}}^+$, there exists a pair $\{z' \in \mathcal{F}^\star(\widetilde{\mathcal{N}}^+), V' \in \Gamma(T\widetilde{\mathcal{N}}^+)\}$ such that $\zeta^+ = z'(L^+ + V')$. Moreover, one can define a diffeomorphism $\varphi : \mathcal{N} \longrightarrow \mathcal{N}$ by $\phi^+ \stackrel{\text{def}}{=} \iota^+ \circ \varphi$. But then one can follow the arguments of (9.7)-(9.9) backwards and prove (9.2) for a function $z \in \mathcal{F}^\star(\mathcal{N})$ defined by $z \stackrel{\text{def}}{=} \varphi^\star((\iota^+)^\star z')$. As before, $\mathrm{sign}(z) = \mathrm{sign}(z')$ so both $\zeta^+ = z'(L^+ + V')$ and $z'L^+ = \mathrm{sign}(z)|z'|L^+$ have the same orientation (because V' is tangent to $\widetilde{\mathcal{N}}^+$). By assumption the matching is possible, hence L^-, ζ^+ are such that one points inwards and the other outwards. If L^- points inwards (resp. outwards) then $\mathrm{sign}(z)L^+$ must point outwards (resp. inwards), so $\mathrm{sign}(z) = \mathrm{sign}(\epsilon^+)$ ($\mathrm{sign}(z) = -\mathrm{sign}(\epsilon^+)$) is forced. This means that (i)-(ii) are both fulfilled. □

Remark 9.1.2. *Theorem 9.1.1 does not impose any conditions on the topology of the abstract manifold* $\mathcal{N}$ *, except for the very mild one that hypersurface data sets can be defined on* $\mathcal{N}$.

Remark 9.1.3. *Observe that we have not restricted the gauges of the data sets* $\mathcal{D}, \widehat{\mathcal{D}}$ *(we let the two riggings* $L^{\pm}$ *be given, but no conditions have been imposed on them). Each specific choice of* $L^{\pm}$ *will fix a particular gauge on* $\mathcal{D}, \widehat{\mathcal{D}}$*. Moreover, Theorem 9.1.1 holds for data sets* $\mathcal{D}, \widehat{\mathcal{D}}$ *of any causal nature. In particular,* $\mathcal{D}, \widehat{\mathcal{D}}$ *are not required to contain non-null or null points exclusively.*

Remark 9.1.4. *When there are no null points on* $\mathcal{N}$ *, condition* (ii) *can always be fulfilled because there exist two gauge group elements which leave the hypersurface data invariant, namely* $\{\mathcal{G}_{(1,0)}, \mathcal{G}_{(-1,-2\ell)}\}$*, where* $\ell \stackrel{\mathrm{def}}{=} \gamma^{\sharp}(\boldsymbol{\ell},\cdot)$ *and* $\gamma^{\sharp}$ *is the inverse of* γ *(see Section 2.2.1.1). This means that when* (i) *is satisfied for a gauge group element* $\mathcal{G}_{(z,V)}$*, it also holds for* $\mathcal{G}_{(-1,-2\ell)} \circ \mathcal{G}_{(z,V)} = \mathcal{G}_{(-z,-2\ell-V)}$ *(recall Proposition 2.2.10). This ensures that there always exists a suitable choice of gauge parameter* z *for which* (i) *and* (ii) *hold.*

On the contrary, when $\mathcal{N}$ *contains null points only the gauge element* $\mathcal{G}_{(1,0)}$ *leaves the hypersurface data invariant, which means that* (i) *can be fulfilled for a gauge group element* $\mathcal{G}_{(z,V)}$ *but* z *may have the wrong sign. This is the underlying reason why the spacetime conditions* (7.22) *provide one unique solution for* ζ^{+} *for given* $\{\zeta^{-}, \Phi\}$ *(see the corresponding discussion in Section 7.2).*

Remark 9.1.5. *In Theorem 9.1.1, we have expressed the junction conditions as a restriction over two data sets and a requirement on the sign of a gauge parameter. Theorem 9.1.1 therefore constitutes an abstract formulation of the standard matching conditions. In particular, a remarkable advantage of Theorem 9.1.1 is that it allows us to study different matchings in two different levels. At the first level one takes whatever hypersurface data sets* $\mathcal{D}, \widehat{\mathcal{D}}$ *satisfying* (i) *and studies its properties from a fully detached point if view. At this level, the spacetimes need not even exist. The problem can then move on and study whether or not one can construct spacetimes in which such data can be embedded, and such that condition* (ii) *holds. In other words, by Theorem 9.1.1 one can produce a thin shell of any causality with full freedom to prescribe the gravitational and matter-energy content, and then study the problem of constructing the resulting spacetime* $(\mathcal{M}, g)$ *which contains it. This is of great use, as it provides a framework to build examples of spacetimes with thin shells of any type.*

In the setup of Theorem 9.1.1, the matching riggings are $\{L^{-}, \zeta^{+}\}$, where ζ^{+} is of the form (9.10). This means that the sign ϵ^{-} coincides with the sign ϵ introduced in the abstract notion of thin shell within Definitions 2.7.2 and 2.7.3. It is convenient not to fix the signs $\epsilon^{\pm}$ (or the riggings $L^{\pm}$) a priori because it may well occur that transverse vectors $L^{\pm}$ on each spacetime are already privileged or have been chosen for whatever other reason. The main point of the construction in Theorem 9.1.1 is firstly that it provides a fully abstract description of the matching and secondly that it keeps maximum flexibility so that one can adapt Theorem 9.1.1 to any particular scenario.

9.1.1 *Null boundaries*

Our main interest in this thesis is on null matching. Thus, for the remainder of the chapter, we focus on the case when both $\mathcal{D}$ and $\widehat{\mathcal{D}}$ are null hypersurface data. Under these circumstances, by Lemma 3.2.9 we know that there exists a pair $\{z, V\}$ ensuring the second and third equations in (9.2). It follows that the only restrictions are therefore condition (ii) in Theorem 9.1.1 and the first equality in (9.2), namely

$$\varphi^{\star}\widehat{\gamma} = \gamma. \tag{9.11}$$

Consequently, given two spacetimes $(\mathcal{M}^{\pm}, g^{\pm})$ with null boundaries $\widetilde{\mathcal{N}}^{\pm}$, either there exists (at least) one diffeomorphism φ satisfying (9.11) or not. In the former case the matching is possible (provided (ii) holds) and, as we shall see next, all information about the matching is codified by φ.

For the rest of the chapter and without loss of generality, we again make the harmless assumption that one of the boundaries lies in the future of its corresponding spacetime while the other lies in its spacetime past (see the discussion in Section 7.3). The following lemma provides the explicit form of the gauge parameters $\{z, V\}$ and of the matching rigging ζ^+ in terms of the diffeomorphism φ.

Lemma 9.1.6. *Assume that conditions* (i)-(ii) *in Theorem 9.1.1 hold for a pair of embedded null hypersurface data* $\mathcal{D}$, $\widehat{\mathcal{D}}$. *Then, the gauge parameters* $\{z, V\}$ *are given by*

$$z = \frac{1}{(\varphi^{\star}\widehat{\boldsymbol{\ell}})(n)}, \qquad V = -P(\varphi^{\star}\widehat{\boldsymbol{\ell}}, \cdot) + \frac{P(\varphi^{\star}\widehat{\boldsymbol{\ell}}, \varphi^{\star}\widehat{\boldsymbol{\ell}}) - \varphi^{\star}\widehat{\ell}^{(2)}}{2(\varphi^{\star}\widehat{\boldsymbol{\ell}})(n)}\, n. \tag{9.12}$$

Moreover, the matching identifies the rigging vector field L^- *in the minus side with the rigging in the plus side*

$$\zeta^+ = z'\left(L^+ - \iota^+_{\star}\varphi_{\star}\left(P(\varphi^{\star}\widehat{\boldsymbol{\ell}}, \cdot)\right) + \mu\,\iota^+_{\star}(\varphi_{\star}n)\right), \tag{9.13}$$

where $z' \in \mathcal{F}^\star(\mathcal{N}^+)$, $\mu \in \mathcal{F}(\mathcal{N}^+)$ *are scalar functions defined by*

$$\varphi^\star((\iota^+)^\star(z')) = \frac{1}{(\varphi^\star\hat{\boldsymbol{\ell}})(n)}, \qquad \varphi^\star((\iota^+)^\star(\mu)) = \frac{P(\varphi^\star\hat{\boldsymbol{\ell}},\varphi^\star\hat{\boldsymbol{\ell}}) - \varphi^\star\hat{\ell}^{(2)}}{2(\varphi^\star\hat{\boldsymbol{\ell}})(n)}. \tag{9.14}$$

Proof. The explicit form (9.12) for the function z follows from contracting (9.5) with n and using (2.7). The vector field V can be partially obtained also from (9.5) by particularizing Lemma 2.2.8 for $W = V$, $\boldsymbol{\varrho} = z^{-1}\boldsymbol{\ell} - \varphi^\star\hat{\boldsymbol{\ell}}$. This gives

$$V = P\left(\frac{\boldsymbol{\ell}}{z} - \varphi^\star\hat{\boldsymbol{\ell}},\cdot\right) + u_0\, n \overset{(2.8)}{=} -P(\varphi^\star\hat{\boldsymbol{\ell}},\cdot) + \left(u_0 - \frac{\ell^{(2)}}{z}\right) n, \tag{9.15}$$

where $u_0 \stackrel{\mathrm{def}}{=} \boldsymbol{\ell}(V)$ is a function yet to be determined. This is done by substituting (9.15) into (9.6). First, $\gamma(V,V) = \boldsymbol{\varrho}(V) = z^{-2}\ell^{(2)} + P(\varphi^\star\hat{\boldsymbol{\ell}},\varphi^\star\hat{\boldsymbol{\ell}})$ because of (2.7)-(2.8) and $z^{-1} = (\varphi^\star\hat{\boldsymbol{\ell}})(n)$. Thus,

$$\varphi^\star\hat{\ell}^{(2)} = \frac{2}{z}\left(\frac{\ell^{(2)}}{z} - u_0\right) + P(\varphi^\star\hat{\boldsymbol{\ell}},\varphi^\star\hat{\boldsymbol{\ell}})$$

$$\Longrightarrow \quad u_0 = \frac{\ell^{(2)}}{z} + \frac{z}{2}\left(P(\varphi^\star\hat{\boldsymbol{\ell}},\varphi^\star\hat{\boldsymbol{\ell}}) - \varphi^\star\hat{\ell}^{(2)}\right) \tag{9.16}$$

so that substituting this into (9.15) proves (9.12). Equation (9.13) is a direct consequence of (9.12) and the fact that $\zeta^+ = z'(L^+ + \iota^+_\star(\varphi_\star V))$. □

Whenever there exists a diffeomorphism φ solving (9.11) and given a basis $\{n, e_A\}$ of $\Gamma(T\mathcal{N})$, it is possible to obtain specific expressions for the pushforward vector fields $\{\varphi_\star n, \varphi_\star e_A\}$. This is done in the next corollary.

Corollary 9.1.7. *Assume that conditions* (*i*)-(*ii*) *in Theorem 9.1.1 hold for a pair of embedded null hypersurface data* $\mathcal{D}$, $\hat{\mathcal{D}}$ *and consider the tensor fields* $\{P, n\}$, $\{\hat{P}, \hat{n}\}$ *defined by particularizing* (2.5) *for these two data sets. Let* $\{n, e_A\}$ *be a basis of* $\Gamma(T\mathcal{N})$ *and define the covectors* $\{\boldsymbol{W}_A\}$ *and the functions* $\{\psi_A, \chi_{(A)}\}$ *along* $\mathcal{N}$ *by*

$$\varphi^\star \boldsymbol{W}_A \stackrel{\mathrm{def}}{=} \gamma(e_A,\cdot), \qquad \psi_A \stackrel{\mathrm{def}}{=} \boldsymbol{\ell}(e_A), \qquad \chi_{(A)} \stackrel{\mathrm{def}}{=} (\varphi^{-1})^\star(z^{-1}\psi_A) - \boldsymbol{W}_A(\varphi_\star V).$$

Then,

$$\varphi_\star n = \frac{\hat{n}}{(\varphi^{-1})^\star z}, \tag{9.17}$$

$$\varphi_\star e_A = \hat{P}(\boldsymbol{W}_A,\cdot) + \chi_{(A)}\hat{n}, \tag{9.18}$$

Moreover, it holds that $\hat{P}(\boldsymbol{W}_A,\hat{\boldsymbol{\ell}}) = 0$ *and* $\varphi^\star\chi_{(A)} = (\varphi^\star\hat{\boldsymbol{\ell}})(e_A)$.

Proof. Consider any point $p \in \mathcal{N}$. From (9.4) it follows that $\gamma(\varphi_\star n, \cdot)|_{\varphi(p)} = (\varphi^\star \gamma)(n, \cdot)|_p = \gamma(n,\cdot)|_p = 0$, so $\varphi_\star n = bn$ for some function $b \in \mathcal{F}(\mathcal{N})$. This, together with (9.12) and $\boldsymbol{\ell}(n) = 1$, entails that $z^{-1}|_p = (\varphi^\star \boldsymbol{\ell})(n)|_p = \boldsymbol{\ell}(\varphi_\star n)|_{\varphi(p)} = b|_{\varphi(p)} = \varphi^\star b|_p$, which proves (9.17). On the other hand, any vector field $X \in \Gamma(T\mathcal{N})$ satisfies

$$\gamma(\varphi_\star e_A, \varphi_\star X)|_{\varphi(p)} = (\varphi^\star \gamma)(e_A, X)|_p \overset{(9.4)}{=} \gamma(e_A, X)|_p = \varphi^\star W_A(X)|_p,$$

$$\boldsymbol{\ell}(\varphi_\star e_A)|_{\varphi(p)} = (\varphi^\star \boldsymbol{\ell})(e_A)|_p \overset{(9.5)}{=} \frac{\psi_A}{z} - \gamma(e_A, V)|_p = \frac{\psi_A}{z} - \varphi^\star W_A(V)|_p,$$

which means that

$$\gamma(\varphi_\star e_A, \cdot) = W_A, \qquad \boldsymbol{\ell}(\varphi_\star e_A) = (\varphi^{-1})^\star(z^{-1}\psi_A) - W_A(\varphi_\star V). \tag{9.19}$$

Particularizing Lemma 2.2.8 for the data $\mathcal{D}$ and for $W = \varphi_\star e_A$, $\varrho = W_A$ and $u_0 = (\varphi^{-1})^\star(z^{-1}\psi_A) - W_A(\varphi_\star V)$ yields (9.18). Finally, $P(W_A, \boldsymbol{\ell}) = 0$ because

$$\begin{aligned}
P(W_A, \boldsymbol{\ell})|_{\varphi(p)} &= -\ell^{(2)} W_A(n)|_{\varphi(p)} = -\ell^{(2)}((\varphi^{-1})^\star z) W_A(\varphi_\star n)|_{\varphi(p)} \\
&= -\ell^{(2)}((\varphi^{-1})^\star z)|_{\varphi(p)}(\varphi^\star W_A)(n)|_p \\
&= -\ell^{(2)}((\varphi^{-1})^\star z)|_{\varphi(p)}\gamma(e_A, n)|_p = 0,
\end{aligned}$$

while $\varphi^\star \chi_{(A)} = (\varphi^\star \boldsymbol{\ell})(e_A)$ follows from

$$\chi_{(A)}|_{\varphi(p)} = \frac{\psi_A}{z} - (\varphi^\star W_A)(V)|_p = \frac{\psi_A}{z} - \gamma(e_A, V)|_p \overset{(9.5)}{=} (\varphi^\star \boldsymbol{\ell})(e_A)|_p.$$

□

Remark 9.1.8. *From (9.17) it follows that φ is a diffeormorphism which sends null generators into null generators. Moreover, since the vector fields $\{W_A \overset{\text{def}}{=} P(W_A, \cdot)\}$ verify $\boldsymbol{\ell}(W_A) = 0$, it follows that $W_A \notin \operatorname{Rad}\gamma$. This, together with the fact that $\varphi_\star$ is necessarily of maximal rank, forces the vector fields $\{W_A\}$ to be everywhere non-zero on $\mathcal{N}$. In fact, $\{n, W_A\}$ constitutes a basis of $\Gamma(T\mathcal{N})$, since $\{W_A\}$ are all linearly independent. This can be proved by contradiction, by assuming that one can write one such vector field, e.g. W_2 as a linear combination of the remaining vector fields, i.e. $W_2 = \sum_{r=3}^n c_r W_r$. By (9.18), this would mean that $\varphi_\star(e_2 - \sum_{r=3}^n c_r e_r) = \chi_{(2)} - \sum_{r=3}^n c_r \chi_{(r)}\, n$, which we know it cannot occur (because only null generators can be mapped to null generators).*

The point of introducing the objects $\{W_A, \chi_{(A)}\}$ will become clear later when analyzing the particular case when the boundaries have product topology $S \times \mathbb{R}$, where S is a spacelike cross-section and the null generators are along $\mathbb{R}$. For the moment, let us simply anticipate that in such case the property $P(W_A, \boldsymbol{\ell}) = 0$ will allow us to conclude that the vector fields $\hat{P}(W_A, \cdot)$ are tangent to the leaves of a specific foliation of $\mathbb{N}^+$ while from $\varphi^\star \chi_{(A)} = (\varphi^\star \hat{\boldsymbol{\ell}})(e_A)$ we will conclude that the functions $\{\chi_{(A)}\}$ are actually spatial derivatives of the step function introduced in Chapter 7.

One of the most important results from Chapter 7 is the relation (7.54) between the second fundamental forms of each side. It turns out that in this abstract framework with no topological assumptions one can also recover an equation of this form. To obtain it, we start by noticing that $\mathbf{U} \stackrel{\text{def}}{=} \frac{1}{2}\pounds_n \gamma$, $\hat{\mathbf{U}} \stackrel{\text{def}}{=} \frac{1}{2}\pounds_{\hat{n}} \hat{\gamma}$ and that $\pounds_{f\hat{n}} \hat{\gamma} = f\pounds_{\hat{n}} \hat{\gamma}$ for any $f \in \mathcal{F}(\hat{\mathcal{N}})$ because $\hat{n} \in \mathrm{Rad}\,\hat{\gamma}$ (recall (5.19)). By direct computation one gets

$$\varphi^\star \hat{\mathbf{U}} \stackrel{\text{def}}{=} \frac{1}{2}\varphi^\star (\pounds_{\hat{n}} \hat{\gamma}) \stackrel{(9.17)}{=} \frac{z}{2}\varphi^\star \pounds_{\varphi_\star n} \hat{\gamma} \stackrel{(9.4)}{=} \frac{z}{2}\pounds_n \gamma = z\mathbf{U} \quad \Longrightarrow \quad \mathbf{U} = \frac{\varphi^\star \hat{\mathbf{U}}}{z}, \qquad (9.20)$$

which connects the second fundamental forms $\mathbf{U}$, $\hat{\mathbf{U}}$ corresponding to the hypersurface data sets $\mathcal{D}$, $\hat{\mathcal{D}}$. Equation (9.20) generalizes (7.54) to the case of boundaries with any topology, and has several implications that we discuss below.

In Theorem 9.1.1 we have seen that when the matching of two spacetimes $(\mathcal{M}^\pm, g^\pm)$ is possible there exists a diffeomorphism φ verifying (9.11). In such case, Lemma 9.1.6 and Corollary 9.1.7 provide explicit expressions for the gauge parameters $\{z, V\}$, the matching rigging ζ^+ and the pushforwards $\{\varphi_\star n, \varphi_\star e_A\}$ of any basis vector fields $\{n, e_A\}$ in terms of the map φ still to be determined.

However, as the reader may have noticed, condition (9.11) does not fix φ completely, firstly because there can be more than one diffeomorphism φ satisfying (9.11) and secondly because the tensor fields γ and $\hat{\gamma}$ are both degenerate. As happened in Section 7.3.1.1, where the step function could not be fixed by the isometry condition (7.31), here one also needs an extra condition in order to fix φ fully. This second restriction is precisely (9.20) and, the same way as in Section 7.3.1.1, it provides useful information as long as both $\mathbf{U}$ and $\hat{\mathbf{U}}$ are non-zero. On the contrary, when $\mathbf{U}$ and $\hat{\mathbf{U}}$ vanish simultaneously then z (and hence part of φ, recall (9.12)) remains completely free. This means that one can find an infinite number of diffeomorphisms φ verifying (9.11), with which we recover the property that whenever the boundaries are totally geodesic then $(\mathcal{M}^\pm, g^\pm)$ can be matched in an infinite number of ways.

As anticipated before, one can obtain explicit expressions for the gravitational and matter-energy content of a null shell (i.e. for $[\mathbf{Y}]$ and τ) in terms of the diffeomorphism φ. This is done in the following theorem.

Theorem 9.1.9. *Assume that conditions* (i)-(ii) *in Theorem 9.1.1 hold for a pair of embedded null hypersurface data* $\mathcal{D}$, $\hat{\mathcal{D}}$ *and let* $\epsilon = \epsilon^-$. *Define*

$$\mathbf{Y}^- \stackrel{\text{def}}{=} \frac{1}{2}(\iota^-)^\star(\pounds_{L^-} g^-), \qquad \hat{\mathbf{Y}}^+ \stackrel{\text{def}}{=} \frac{1}{2}(\iota^+)^\star(\pounds_{L^+} g^+) \quad \text{and} \quad \mathbf{Y}^+ \stackrel{\text{def}}{=} \frac{1}{2}\varphi^\star\left((\iota^+)^\star(\pounds_{\zeta^+} g^+)\right),$$

where ζ^+ *is given by* (9.13). *Then, the tensor* $[\mathbf{Y}] \stackrel{\text{def}}{=} \mathbf{Y}^+ - \mathbf{Y}^-$ *reads*

$$[\mathrm{Y}_{ab}] = z\left((\varphi^\star\hat{\mathbf{Y}}^+)_{ab} + \frac{z}{2}\left(P(\varphi^\star\hat{\boldsymbol{\ell}}, \varphi^\star\hat{\boldsymbol{\ell}}) - \varphi^\star\hat{\ell}^{(2)}\right)\mathrm{U}_{ab} - \overset{\circ}{\nabla}_{(a}(\varphi^\star\hat{\boldsymbol{\ell}})_{b)}\right) - \mathrm{Y}^-_{ab}, \tag{9.21}$$

where z *is given by* (9.12). *The components of* $[\mathbf{Y}]$ *in any basis* $\{n, e_A\}$ *of* $\Gamma(T\mathcal{N})$ *are*

$$[\mathbf{Y}](n,n) = z(\varphi^\star\hat{\mathbf{Y}}^+)(n,n) - \mathbf{Y}^-(n,n) + \frac{n(z)}{z}, \tag{9.22}$$

$$\begin{aligned}[\mathbf{Y}](n,e_A) &= z(\varphi^\star\hat{\mathbf{Y}}^+)(n,e_A) - \mathbf{Y}^-(n,e_A) - \frac{z}{2}(\pounds_n\varphi^\star\hat{\boldsymbol{\ell}})(e_A) \\ &\quad + \frac{e_A(z)}{2z} + \boldsymbol{s}(e_A) + zP(\varphi^\star\hat{\boldsymbol{\ell}}, \mathrm{U}(e_A,\cdot)),\end{aligned} \tag{9.23}$$

$$\begin{aligned}[\mathbf{Y}](e_A,e_B) &= z(\varphi^\star\hat{\mathbf{Y}}^+)(e_A,e_B) - \mathbf{Y}^-(e_A,e_B) \\ &\quad + \frac{z^2}{2}\left(P(\varphi^\star\hat{\boldsymbol{\ell}}, \varphi^\star\hat{\boldsymbol{\ell}}) - \varphi^\star\hat{\ell}^{(2)}\right)\mathbf{U}(e_A,e_B) - z e_A^a e_B^b \overset{\circ}{\nabla}_{(a}(\varphi^\star\hat{\boldsymbol{\ell}})_{b)},\end{aligned} \tag{9.24}$$

while the components of the energy-momentum tensor of the shell in the dual basis $\{\mathbf{q}, \boldsymbol{\theta}^A\}$ *of* $\{n, e_A\}$ *are*

$$\begin{aligned}\tau(\mathbf{q},\mathbf{q}) &= -\epsilon h^{AB}\Big(z(\varphi^\star\hat{\mathbf{Y}}^+)(e_A,e_B) - \mathbf{Y}^-(e_A,e_B) \\ &\quad + \frac{z^2}{2}\left(P(\varphi^\star\hat{\boldsymbol{\ell}}, \varphi^\star\hat{\boldsymbol{\ell}}) - \varphi^\star\hat{\ell}^{(2)}\right)\mathbf{U}(e_A,e_B) - z e_A^a e_B^b \overset{\circ}{\nabla}_{(a}(\varphi^\star\hat{\boldsymbol{\ell}})_{b)}\Big),\end{aligned} \tag{9.25}$$

$$\begin{aligned}\tau(\mathbf{q},\boldsymbol{\theta}^A) &= \epsilon h^{AB}\Big(z(\varphi^\star\hat{\mathbf{Y}}^+)(n,e_B) - \mathbf{Y}^-(n,e_B) \\ &\quad - \frac{z}{2}(\pounds_n\varphi^\star\hat{\boldsymbol{\ell}})(e_B) + \frac{e_B(z)}{2z} + \boldsymbol{s}(e_B) + zP(\varphi^\star\hat{\boldsymbol{\ell}}, \mathbf{U}(e_B,\cdot))\Big),\end{aligned} \tag{9.26}$$

$$\tau(\boldsymbol{\theta}^A,\boldsymbol{\theta}^B) = -\epsilon h^{AB}\Big(z(\varphi^\star\hat{\mathbf{Y}}^+)(n,n) - \mathbf{Y}^-(n,n) + \frac{n(z)}{z}\Big), \tag{9.27}$$

where $h_{AB} \stackrel{\text{def}}{=} \gamma(e_A, e_B)$. *Finally, the purely gravitational content of the shell is ruled by*

$$\mathbf{Y}^G(e_A,e_B) \stackrel{\text{def}}{=} [\mathbf{Y}](e_A,e_B) + \epsilon\frac{\overline{\rho}}{n-1}\gamma(e_A,e_B), \tag{9.28}$$

$$\text{where} \quad \overline{\rho} \stackrel{\text{def}}{=} \rho + 2P(\mathbf{q},\boldsymbol{j}) + p\left(2\ell^{(2)} + P(\mathbf{q},\mathbf{q})\right)$$

and $\{\rho, p, \boldsymbol{j}\}$ *are defined as in Remark 7.3.9.*

Proof. First, we apply Lemma 3.1.4 for $V \stackrel{\text{def}}{=} \gamma(V,\cdot) = z^{-1}\boldsymbol{\ell} - \varphi^\star\boldsymbol{\ell}$ (recall (9.5)) and $u_0 \stackrel{\text{def}}{=} \boldsymbol{\ell}(V)$ (cf. (9.16)). This yields

$$\begin{aligned}\frac{z}{2}\pounds_V\gamma_{ab} &= \left(\ell^{(2)} + \frac{z^2}{2}\left(P(\varphi^\star\boldsymbol{\ell},\varphi^\star\boldsymbol{\ell}) - \varphi^\star\ell^{(2)}\right)\right)\mathrm{U}_{ab} + z\overset{\circ}{\nabla}_{(a}\left(\frac{\ell_{b)}}{z} - (\varphi^\star\boldsymbol{\ell})_{b)}\right)\\ &= \frac{z^2}{2}\left(P(\varphi^\star\boldsymbol{\ell},\varphi^\star\boldsymbol{\ell}) - \varphi^\star\ell^{(2)}\right)\mathrm{U}_{ab} - \frac{1}{z}(\overset{\circ}{\nabla}_{(a}z)\,\ell_{b)} - z\overset{\circ}{\nabla}_{(a}(\varphi^\star\boldsymbol{\ell})_{b)},\end{aligned} \tag{9.29}$$

where in the last step we used that $\overset{\circ}{\nabla}_{(a}\ell_{b)} = -\ell^{(2)}\mathrm{U}_{ab}$ (cf. (2.19)). By hypothesis the matching of $(\mathcal{M}^\pm, g^\pm)$ is possible, so the data sets $\{\mathcal{N}, \varphi^\star\gamma^+, \varphi^\star\boldsymbol{\ell}^+, \varphi^\star\ell^{(2)+}, \varphi^\star\mathbf{Y}^+\}$, $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}^+\}$ are embedded in $(\mathcal{M}^+, g^+)$ with embedding $\iota^+\circ\varphi$ and respective riggings L^+, ζ^+. This, together with (9.2), entails that the tensors $\varphi^\star\mathbf{Y}^+$, $\mathbf{Y}^+$ are related by $\mathbf{Y}^+ = \mathcal{G}_{(z,V)}(\varphi^\star\mathbf{Y}^+)$, where $\{z, V\}$ are given by (9.12). Thus (cf. (2.40), (9.11))

$$\mathbf{Y}^+ = z\varphi^\star\mathbf{Y}^+ + dz\otimes_s(\varphi^\star\boldsymbol{\ell} + \gamma(V,\cdot)) + \frac{z}{2}\pounds_V\gamma \stackrel{(9.5)}{=} z\varphi^\star\mathbf{Y}^+ + \frac{dz}{z}\otimes_s\boldsymbol{\ell} + \frac{z}{2}\pounds_V\gamma. \tag{9.30}$$

Inserting (9.29) into (9.30) yields the explicit form (9.21).

We now obtain the components of $[\mathbf{Y}]$ in the basis $\{n, e_A\}$, for which we recall that $\mathbf{U}(n,\cdot) = 0$ and $s(n) = 0$. Particularizing (3.11) for $\boldsymbol{\theta} = \varphi^\star\boldsymbol{\ell}$ and using (9.12) gives

$$\begin{aligned}n^a\overset{\circ}{\nabla}_{(a}(\varphi^\star\boldsymbol{\ell})_{b)} &= \frac{1}{2}\pounds_n(\varphi^\star\boldsymbol{\ell})_b + \frac{1}{2}\overset{\circ}{\nabla}_b((\varphi^\star\boldsymbol{\ell})(n)) - (\varphi^\star\boldsymbol{\ell})(n)s_b - P^{ac}\mathrm{U}_{bc}(\varphi^\star\boldsymbol{\ell})_a,\\ &= \frac{1}{2}\pounds_n(\varphi^\star\boldsymbol{\ell})_b - \frac{\overset{\circ}{\nabla}_b z}{2z^2} - \frac{s_b}{z} - P^{ac}\mathrm{U}_{bc}(\varphi^\star\boldsymbol{\ell})_a,\end{aligned} \tag{9.31}$$

$$n^a n^b\,\overset{\circ}{\nabla}_{(a}(\varphi^\star\boldsymbol{\ell})_{b)} = \frac{1}{2}\pounds_n\left((\varphi^\star\boldsymbol{\ell})(n)\right) - \frac{n(z)}{2z^2} = -\frac{n(z)}{z^2}. \tag{9.32}$$

Combining (9.31)-(9.32) with (9.21) yields (9.22)-(9.24). The results (9.25)-(9.27) for the components of the energy-momentum tensor of the shell are a direct consequence of (3.53)-(3.55).

Finally, we prove (9.28) as follows. First, we note that the one-forms $\boldsymbol{j}$ (see Remark 7.3.9) and $\boldsymbol{\ell}$ decompose in the basis $\{\mathbf{q}, \boldsymbol{\theta}^A\}$ as

$$\boldsymbol{j} = \boldsymbol{j}(e_A)\boldsymbol{\theta}^A, \qquad \boldsymbol{\ell} = \mathbf{q} + \boldsymbol{\ell}(e_A)\boldsymbol{\theta}^A \tag{9.33}$$

because $\boldsymbol{j}(n) = 0$ and $\boldsymbol{\ell}(n) = 1$. Also by Remark 7.3.9, we know that the one-form $\boldsymbol{j}$ verifies $[\mathbf{Y}](n, e_A) = \epsilon(\boldsymbol{j}(e_A) - p\boldsymbol{\ell}(e_A))$. Thus, a direct computation based on the decomposition (3.51) of the tensor field P yields

$$\operatorname{tr}_P[\mathbf{Y}] = P^{ab}[\mathrm{Y}_{ab}] = h^{AB}[\mathbf{Y}](e_A, e_B) + 2P(\mathbf{q}, \boldsymbol{\theta}^A)[\mathbf{Y}](n, e_A) + P(\mathbf{q},\mathbf{q})[\mathbf{Y}](n,n)$$

$$
\begin{aligned}
&= h^{AB}[\mathbf{Y}](e_A, e_B) + 2\epsilon P\big(\mathbf{q}, j(e_A)\boldsymbol{\theta}^A - p\boldsymbol{\ell}(e_A)\boldsymbol{\theta}^A\big) - \epsilon p P(\mathbf{q}, \mathbf{q}) \\
&= h^{AB}[\mathbf{Y}](e_A, e_B) + 2\epsilon P(\mathbf{q}, j) + \epsilon p \big(2\,\ell^{(2)} + P(\mathbf{q}, \mathbf{q})\big)
\end{aligned}
$$

where we used that $P(\boldsymbol{\theta}^A, \boldsymbol{\theta}^B) = h^{AB}$ (by Lemma 3.2.5), $P(\boldsymbol{\ell}, \cdot) = -\ell^{(2)} n$ (cf. (2.8)) and (9.33) in this order. Taking into account the definition of the energy density ρ (see (7.97)), one finds

$$
h^{AB}[\mathbf{Y}](e_A, e_B) = -\epsilon\Big(\rho + 2P(\mathbf{q}, j) + p\big(2\,\ell^{(2)} + P(\mathbf{q}, \mathbf{q})\big)\Big) \overset{\text{def}}{=} -\epsilon\bar{\rho}. \tag{9.34}
$$

Now, from (3.53)-(3.55) it is clear that the only part of $[\mathbf{Y}]$ that does not contribute to the energy-momentum tensor is the h-traceless part of $[\mathbf{Y}](e_A, e_B)$. By Lemma 3.2.5, we know that $h^{AB}\gamma(e_A, e_B) = n - 1$. Consequently, $[\mathbf{Y}](e_A, e_B)$ decomposes in a h-traceless and a h-trace part as

$$
[\mathbf{Y}](e_A, e_B) = \mathbf{Y}^G(e_A, e_B) + \frac{h^{IJ}[\mathbf{Y}](e_I, e_J)}{n-1}\gamma(e_A, e_B),
$$

from where (9.28) follows at once after inserting (9.34). □

Remark 9.1.10. *We emphasize that we have not made any assumption on the topology of the boundaries* $\mathcal{N}^{\pm}$ *in Theorems 9.1.1 and 9.1.9 or in Lemma 9.1.6. The results above therefore describe the most general matching of two spacetimes across null hypersurfaces and generalize the results in Chapters 7 and 8, where the existence of a foliation on the boundaries played an important role.*

The gravitational/matter-energy content of the resulting null shell is given by (9.22)-(9.28), and the associated energy density ρ*, energy flux* j *and pressure* p *are given by (7.97). The reason why we refer to* $\mathbf{Y}^G(e_A, e_B)$ *as the purely gravitational part of the shell is that only the components* $[\mathbf{Y}](n, n)$*,* $[\mathbf{Y}](n, e_A)$ *and the trace* $P(\boldsymbol{\theta}^A, \boldsymbol{\theta}^B)[\mathbf{Y}](e_A, e_B)$ *contribute to the energy-momentum tensor* τ *(cf. (3.53)-(3.55)). This means that even if* τ *vanishes identically* $\mathbf{Y}^G(e_A, e_B)$ *does not need to be zero. In fact, this case of* $\mathbf{Y}^G(e_A, e_B)$ *being the only non-zero contribution to the tensor* $[\mathbf{Y}]$ *corresponds to an impulsive gravitational wave propagating in the spacetime resulting from the matching.*

Remark 9.1.11. *By Lemma 3.2.5 we know that* $P(\mathbf{q}, \cdot) = 0$ *if and only if* $\boldsymbol{\ell}(e_A) = 0$ *and* $\ell^{(2)} = 0$*. In such case, the scalar* $\bar{\rho}$ *coincides with the energy density* ρ *of the shell. In the embedded picture, this restrictions amount to impose that the matching riggings* $\zeta^{\pm}$ *are null and orthogonal to the vector fields* $\phi^{\pm}_{\star} e_A$*. This holds, in particular, in Sections 7.3.3, 8.4 and 8.5, where we enforced that the rigging* ζ^{-} *is null and orthogonal to the leaves of the foliation on the minus side (recall the choice (7.26)).*

Remark 9.1.12. *In Theorems 9.1.1 and 9.1.9 and Lemma 9.1.6, all expressions are fully explicit in terms of the diffeomorphism φ. The two data sets* $\mathcal{D}$, $\widehat{\mathcal{D}}$ *are completely known (because the embeddings $\Phi^{\pm}$ and the spacetimes $(\mathcal{M}^{\pm}, g^{\pm})$ are given) and the rigging ζ^{+} is determined by the pair $\{z, V\}$ given by (9.12) in terms of φ. This also happened in Chapters 7 and 8, where the whole matching depended upon the step function H and the coefficients b^{J}_{I} which in turn determined the matching embedding ϕ^{+} (recall (7.43) and (7.51)) and the matching rigging ζ^{+} (according to (7.70)).*

Expressions (9.22)-(9.27) involve the pull-back $\varphi^{\star}\widehat{\mathbf{Y}}^{+}$, whose calculation can be cumbersome in general. It is more convenient to rewrite (9.22)-(9.27) in terms of pull-backs of scalar functions referred to the data $\widehat{\mathcal{D}}$ and objects defined with respect to $\mathcal{D}$. We provide the corresponding expressions in the following Lemma.

Lemma 9.1.13. *Assume that conditions (i)-(ii) in Theorem 9.1.1 hold for a pair of embedded null hypersurface data* $\mathcal{D}$, $\widehat{\mathcal{D}}$, *let $\epsilon = \epsilon^{-}$ and consider the tensor fields $\{P, n\}$, $\{\widehat{P}, \widehat{n}\}$ defined by particularizing (2.5) for these two data sets. Define the tensors $\{\mathbf{Y}^{-}, \widehat{\mathbf{Y}}^{+}, \mathbf{Y}^{+}\}$ as in Theorem 9.1.9, the covectors $\{\mathbf{W}_{A}\}$ and the functions $\{\chi_{(A)}, \psi_{A}\}$ along $\mathcal{N}$ according to Corollary 9.1.7 and the vector field $W_{A} \stackrel{\text{def}}{=} \widehat{P}(\mathbf{W}_{A}, \cdot)$. Let z be given by (9.12) and $\{n, e_{A}\}$ be a basis of $\Gamma(T\mathcal{N})$ with dual basis $\{\mathbf{q}, \boldsymbol{\theta}^{A}\}$. Then, equations (9.22)-(9.27) can be rewritten as*

$$[\mathbf{Y}](n,n) = \frac{1}{z}\varphi^{\star}\left(\widehat{\mathbf{Y}}^{+}(\widehat{n},\widehat{n})\right) - \mathbf{Y}^{-}(n,n) + \frac{n(z)}{z}, \tag{9.35}$$

$$\begin{aligned}[\mathbf{Y}](n,e_{A}) = {} & \varphi^{\star}\left(\widehat{\mathbf{Y}}^{+}(\widehat{n},W_{A}) + \chi_{(A)}\widehat{\mathbf{Y}}^{+}(\widehat{n},\widehat{n})\right) - \mathbf{Y}^{-}(n,e_{A}) \\ & - \frac{z}{2}\left(\pounds_{n}\varphi^{\star}\widehat{\boldsymbol{\ell}}\right)(e_{A}) + \frac{e_{A}(z)}{2z} + s(e_{A}) + zP(\varphi^{\star}\widehat{\boldsymbol{\ell}}, \mathbf{U}(e_{A},\cdot)),\end{aligned} \tag{9.36}$$

$$\begin{aligned}[\mathbf{Y}](e_{A},e_{B}) = {} & z\varphi^{\star}\Big(\widehat{\mathbf{Y}}^{+}(W_{A},W_{B}) + \chi_{(A)}\widehat{\mathbf{Y}}^{+}(\widehat{n},W_{B}) + \chi_{(B)}\widehat{\mathbf{Y}}^{+}(\widehat{n},W_{A}) \\ & + \chi_{(A)}\chi_{(B)}\widehat{\mathbf{Y}}^{+}(\widehat{n},\widehat{n})\Big) - \mathbf{Y}^{-}(e_{A},e_{B}) - ze^{a}_{A}e^{b}_{B}\overset{\circ}{\nabla}_{(a}(\varphi^{\star}\widehat{\boldsymbol{\ell}})_{b)} \\ & + \frac{z^{2}}{2}\left(P(\varphi^{\star}\widehat{\boldsymbol{\ell}},\varphi^{\star}\widehat{\boldsymbol{\ell}}) - \varphi^{\star}\widehat{\ell}^{(2)}\right)\mathbf{U}(e_{A},e_{B}).\end{aligned} \tag{9.37}$$

while the energy-momentum tensor of the shell reads

$$\begin{aligned}\tau(\mathbf{q},\mathbf{q}) = {} & -\epsilon h^{AB}\Big(z\varphi^{\star}\Big(\widehat{\mathbf{Y}}^{+}(W_{A},W_{B}) + \chi_{(A)}\widehat{\mathbf{Y}}^{+}(\widehat{n},W_{B}) + \chi_{(B)}\widehat{\mathbf{Y}}^{+}(\widehat{n},W_{A}) \\ & + \chi_{(A)}\chi_{(B)}\widehat{\mathbf{Y}}^{+}(\widehat{n},\widehat{n})\Big) - \mathbf{Y}^{-}(e_{A},e_{B}) - ze^{a}_{A}e^{b}_{B}\overset{\circ}{\nabla}_{(a}(\varphi^{\star}\widehat{\boldsymbol{\ell}})_{b)} \\ & + \frac{z^{2}}{2}\left(P(\varphi^{\star}\widehat{\boldsymbol{\ell}},\varphi^{\star}\widehat{\boldsymbol{\ell}}) - \varphi^{\star}\widehat{\ell}^{(2)}\right)\mathbf{U}(e_{A},e_{B})\Big)\end{aligned} \tag{9.38}$$

$$\tau(\mathbf{q},\boldsymbol{\theta}^{A}) = \epsilon h^{AB}\left(\varphi^{\star}\left(\widehat{\mathbf{Y}}^{+}(\widehat{n},W_{B}) + \chi_{(B)}\widehat{\mathbf{Y}}^{+}(\widehat{n},\widehat{n})\right) - \mathbf{Y}^{-}(n,e_{B})\right.$$

$$- \frac{z}{2}\left(\pounds_n \varphi^\star \hat{\boldsymbol{\ell}}\right)(e_B) + \frac{e_B(z)}{2z} + s(e_B) + zP\left(\varphi^\star \hat{\boldsymbol{\ell}}, \mathrm{U}(e_B,\cdot)\right)\Bigg], \tag{9.39}$$

$$\tau(\boldsymbol{\theta}^A,\boldsymbol{\theta}^B) = -\epsilon h^{AB}\left[\frac{1}{z}\varphi^\star\left(\widehat{\mathbf{Y}}^+(\hat{n},\hat{n})\right) - \mathbf{Y}^-(n,n) + \frac{n(z)}{z}\right], \tag{9.40}$$

where $h_{AB} \stackrel{\text{def}}{=} \gamma(e_A,e_B)$.

Proof. Inserting $(\varphi^\star \widehat{\mathbf{Y}}^+)(X,Y)|_p = \widehat{\mathbf{Y}}^+(\varphi_\star X, \varphi_\star Y)|_{\varphi(p)}$ into (9.22)-(9.27) and using (9.17)-(9.18), equations (9.38)-(9.40) follow at once. □

In Section 9.2, we shall recover the results of Proposition 7.3.7 by particularizing Lemma 9.1.13 to the case when the boundaries $\mathcal{N}^\pm$ have product topology. Lemma 9.1.13 therefore generalizes Proposition 7.3.7 to (null) boundaries with any topology, and states the matter-energy content of any null thin shell arising from the matching of two spacetimes.

9.1.1.1 *Pressure of the shell*

In Chapters 7 and 8, we have already discussed the effect and the importance of a non-zero pressure in a null shell. This, however, has been done in very specific contexts (namely in the matching of two regions of Minkowski across a null hyperplane or for matchings across embedded AKH$_0$s) and by following a non-fully geometric approach (i.e. by analyzing the effect of the pressure in some specific coordinates). Our aim in this section is to study the pressure of a completely general null shell at a fully abstract level, providing its explicit expression in terms of well-defined geometric quantities and reinforcing the geometric interpretation of Chapters 7 and 8.

In the following lemma we find explicit expressions for the pressure p in terms of the surface gravities of various null generators of $\mathcal{N}$.

Lemma 9.1.14. *Assume that conditions* (*i*)-(*ii*) *in Theorem 9.1.1 hold for a pair of embedded null hypersurface data* $\mathcal{D} = \{\mathcal{N},\gamma,\boldsymbol{\ell},\ell^{(2)},\mathbf{Y}^-\}$, $\widehat{\mathcal{D}} = \{\widehat{\mathcal{N}},\hat{\gamma},\hat{\boldsymbol{\ell}},\hat{\ell}^{(2)},\widehat{\mathbf{Y}}^+\}$, *a diffeomorphism* φ *and a gauge group element* $\mathcal{G}_{(z,V)}$ *(cf.* (9.12)*). Consider an arbitrary gauge parameter* $V \in \Gamma(T\mathcal{N})$. *Let* $\epsilon = \epsilon^-$ *and* $\{n,\hat{n}\}$ *be the null generators constructed from* $\{\mathcal{D},\widehat{\mathcal{D}}\}$ *respectively. Define*

$$\kappa_n \stackrel{\text{def}}{=} -\mathbf{Y}^-(n,n), \qquad \hat{\kappa}_{\hat{n}} \stackrel{\text{def}}{=} -\widehat{\mathbf{Y}}^+(\hat{n},\hat{n}), \qquad \varphi^\star\kappa_{\varphi_\star n} \stackrel{\text{def}}{=} \frac{1}{z}\left(\varphi^\star\hat{\kappa}_{\hat{n}} - n(z)\right). \tag{9.41}$$

Then, the pressure p of the corresponding null shell is given by

$$p = -\epsilon\left(\kappa_n - \mathcal{G}_{(z,\overline{V})}\left(\varphi^\star \widehat{\kappa}_n\right)\right) \qquad \textit{or, equivalently} \qquad p = -\epsilon\left(\kappa_n - \varphi^\star \kappa_{\varphi_\star n}\right) \tag{9.42}$$

for any vector field $\overline{V} \in \Gamma(T\mathcal{N})$. In particular, the pressure vanishes if and only if

$$\mathcal{G}_{(z,\overline{V})}\left(\varphi^\star \widehat{\kappa}_n\right) = \kappa_n \qquad \textit{or, equivalently} \qquad \varphi^\star \kappa_{\varphi_\star n} = \kappa_n. \tag{9.43}$$

Proof. Recall that $\mathcal{G}^{-1}_{(z,\overline{V})} = \mathcal{G}_{(z^{-1},-z\overline{V})}$. We start by noticing that (3.64) implies that $\mathcal{G}^{-1}_{(z,\overline{V})}(\kappa_n) = \mathcal{G}_{(z^{-1},-z\overline{V})}(\kappa_n) = z\left(\kappa_n + \frac{n(z)}{z}\right)$. On the other hand, combining (7.97) and (9.35), it follows

$$\begin{aligned} p &= -\epsilon\left(-\frac{1}{z}\varphi^\star \widehat{\kappa}_n + \kappa_n + \frac{n(z)}{z}\right) \\ &= -\frac{\epsilon}{z}\left(\mathcal{G}_{(z^{-1},-z\overline{V})}(\kappa_n) - \varphi^\star \widehat{\kappa}_n\right) = -\frac{\epsilon}{z}\left(\mathcal{G}^{-1}_{(z,\overline{V})}(\kappa_n) - \varphi^\star \widehat{\kappa}_n\right), \end{aligned} \tag{9.44}$$

from where we conclude that $-\mathcal{G}^{-1}_{(z,\overline{V})}(\epsilon p) = \mathcal{G}^{-1}_{(z,\overline{V})}(\kappa_n) - \varphi^\star \widehat{\kappa}_n$ (recall (2.154) and (7.99)). Applying $\mathcal{G}_{(z,\overline{V})}$ on both sides of the equation one obtains the left part of (9.42). The right part of (9.42) is an immediate consequence of inserting the definition of $\varphi^\star \kappa_{\varphi_\star n}$ into the first line of (9.44), while (9.43) is proven by setting $p = 0$ in (9.42). □

Remark 9.1.15. *The last expression in (9.41) defines a function $\kappa_{\varphi_\star n}$ on $\widehat{\mathcal{N}}$. However, we still need to justify this terminology. It turns out that $\kappa_{\varphi_\star n}$ coincides with the surface gravity of the vector field $\varphi_\star n$ with respect to the hypersurface connection $\overline{\widehat{\nabla}}$ constructed from the data $\widehat{\mathcal{D}}$. To prove this, we let $\widehat{z} \stackrel{\text{def}}{=} (\varphi^{-1})^\star z$, so that (cf. (9.41))*

$$\kappa_{\varphi_\star n} = \frac{1}{\widehat{z}}\left(\widehat{\kappa}_n - (\varphi^{-1})^\star(n(z))\right) \qquad \textit{and} \qquad (\varphi_\star n)(\widehat{z}) = (\varphi^{-1})^\star(n(z)),$$

where the right part follows from $(\varphi_\star n)(\widehat{z})|_{\varphi(p)} = (\varphi^\star d\widehat{z})(n)|_p = (d\varphi^\star \widehat{z})(n)|_p = n(z)|_p = (\varphi^{-1})^\star(n(z))|_{\varphi(p)}$. Then, the combination of (3.46) and (9.17) gives (recall (2.49))

$$\begin{aligned} \overline{\widehat{\nabla}}_{\varphi_\star n}\varphi_\star n &= \frac{1}{\widehat{z}}\overline{\widehat{\nabla}}_{\widehat{n}}\left(\frac{1}{\widehat{z}}\widehat{n}\right) = \frac{1}{\widehat{z}}\left(\frac{1}{\widehat{z}}\overline{\widehat{\nabla}}_{\widehat{n}}\widehat{n} - \frac{\widehat{n}(\widehat{z})}{\widehat{z}^2}\widehat{n}\right) = -\frac{1}{\widehat{z}}\left(\widehat{\mathbf{Y}}^{+}(\widehat{n},\widehat{n}) + \frac{\widehat{n}(\widehat{z})}{\widehat{z}}\right)\varphi_\star n \\ &= \frac{1}{\widehat{z}}\left(\widehat{\kappa}_n - (\varphi_\star n)(\widehat{z})\right)\varphi_\star n = \frac{1}{\widehat{z}}\left(\widehat{\kappa}_n - (\varphi^{-1})^\star(n(z))\right)\varphi_\star n = \kappa_{\varphi_\star n}\varphi_\star n. \end{aligned}$$

Remark 9.1.16. *The gauge parameter $\overline{V}$ is completely superfluous and plays no role in determining the pressure, which is only influenced by the function z given by (9.12). We keep $\overline{V}$ in the expression to emphasize this fact.*

Remark 9.1.17. *In Chapters 7 and 8, we have introduced the notion of self-compression and self-stretching on the boundaries of the spacetimes to be matched. We have seen that this effect is completely ruled by the pressure, and that it has to do with the differences in the acceleration along the null generators of both sides. With (9.42), we recover the same result but for the case of boundaries with any topology. Indeed, the surface gravities* κ_n *and* $\kappa_{\varphi_\star n}$ *verify*

$$\overline{\nabla}_n n = \kappa_n n, \qquad \widetilde{\overline{\nabla}}_{\varphi_\star n} \varphi_\star n = \kappa_{\varphi_\star n} \varphi_\star n, \tag{9.45}$$

so that, when the matching rigging ζ^- *points inwards (hence* $\epsilon = -1$*), the pressure is positive when* $\kappa_n > \varphi^\star \kappa_{\varphi_\star n}$ *(namely when the "acceleration" of* n *is greater than that of* $\varphi_\star n$*) and negative otherwise, just as happened in Section 7.3.3. The only scenario where there exists no pressure occurs when both surface gravities coincide, i.e. when the accelerations of* n *and* $\varphi_\star n$ *are the same.*

9.1.2 *Multiple matchings*

We have already seen that when two given spacetimes $(\mathcal{M}^\pm, g^\pm)$ can be matched, in general there exists at most one way of matching (i.e. only one matching map Φ or one single diffeomorphism φ). However, we are also aware of the fact that sometimes multiple (even infinite) matchings can be performed (e.g. when both second fundamental forms $\mathbf{U}$, $\widetilde{\mathbf{U}}$ vanish). In the language of (7.24), this means that given a choice of embedding ϕ^- and matching rigging ζ^- on the minus side, there exist several embeddings ϕ^+ for which the matching conditions hold, and each embedding gives rise to a unique solution for the rigging ζ^+ with suitable orientation.

In this section, our aim is to study the scenario of multiple matchings. The idea is to assume that *all* information about one of the matchings is known, in particular its corresponding diffeomorphism φ and hence the gravitational/matter-energy content. As we shall see, in these circumstances one only needs to consider a single hypersurface data set $\mathcal{D}$ (instead of two) and it is possible to provide explicit expressions for the jump $[\mathbf{Y}]$ and the energy-momentum tensor τ of any other shell in terms of their counterparts of the known matching. These results can be particularized to the case when the known matching gives rise to no-shell (i.e. when it is such that $[\mathbf{Y}] = 0$). This precisely happens in all cut-and-paste constructions, where $(\mathcal{M}^\pm, g^\pm)$ are two regions of the same spacetime.

Our setup will be the following. We make a choice $\{\phi^-, \zeta^-\}$ of embedding and rigging on the minus side and consider two matching embeddings ϕ^+, $\widetilde{\phi}^+$, each of them satisfying (7.24) for two riggings ζ^+, $\widetilde{\zeta}^+$ respectively. We also assume that the information about one of the matchings is completely known, namely we let $\{\hat{\phi}^+, \hat{\zeta}^+\}$ be given.

From the spacetimes $(\mathcal{M}^\pm, g^\pm)$, we can construct two data sets $\mathcal{D} = \{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}^-\}$, $\hat{\mathcal{D}} = \{\mathcal{N}, \hat{\gamma}, \hat{\boldsymbol{\ell}}, \hat{\ell}^{(2)}, \hat{\mathbf{Y}}^+\}$ and Theorem 9.1.1 ensures that we can find two diffeomorphisms $\varphi, \hat{\varphi}$ and two pairs $\{z, V\}$, $\{\hat{z}, \hat{V}\}$ for which (i)-(ii) hold. Even more, since the pair $\{\hat{\phi}^+, \hat{\zeta}^+\}$ is known, we can always make the choice $\{\iota^+ = \hat{\phi}^+, L^+ = \hat{\zeta}^+\}$ so that $\{\hat{\gamma}, \hat{\boldsymbol{\ell}}, \hat{\ell}^{(2)}\} = \{\gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ and $\hat{\varphi}$ is the identity map, i.e. $\hat{\varphi} = \mathbb{I}_{\mathcal{N}}$. In these circumstances, using (2.7)-(2.8) in (9.12) yields $\hat{z} = 1$ and $\hat{V} = 0$. Making the same choice of $\{\iota^+, L^+\}$ for the matching of φ transforms (9.2) into

$$\mathcal{G}_{(z,V)}(\varphi^\star\gamma) = \gamma, \qquad \mathcal{G}_{(z,V)}(\varphi^\star\boldsymbol{\ell}) = \boldsymbol{\ell}, \qquad \mathcal{G}_{(z,V)}(\varphi^\star \ell^{(2)}) = \ell^{(2)}, \tag{9.46}$$

and forces the embedding ϕ^+ to be given by

$$\phi^+ \circ \varphi \equiv \hat{\phi}^+. \tag{9.47}$$

Equations (9.4)-(9.6) now read

$$\varphi^\star\gamma = \gamma, \qquad \varphi^\star\boldsymbol{\ell} = \frac{\boldsymbol{\ell}}{z} - \gamma(V,\cdot), \qquad \varphi^\star \ell^{(2)} = \frac{\ell^{(2)}}{z^2} - \frac{2\boldsymbol{\ell}(V)}{z} + \gamma(V,V), \tag{9.48}$$

while the expressions (9.12) for the gauge parameters $\{z, V\}$ become

$$z = \frac{1}{(\varphi^\star\boldsymbol{\ell})(n)}, \qquad V = -P(\varphi^\star\boldsymbol{\ell},\cdot) + \frac{P(\varphi^\star\boldsymbol{\ell}, \varphi^\star\boldsymbol{\ell}) - \varphi^\star \ell^{(2)}}{2(\varphi^\star\boldsymbol{\ell})(n)}\, n. \tag{9.49}$$

It is important to emphasize that whereas $\hat{\varphi} = \mathbb{I}_{\mathcal{N}}$ forces the metric parts of $\mathcal{D}$, $\hat{\mathcal{D}}$ to be the same, the tensors $\mathbf{Y}^-$, $\hat{\mathbf{Y}}^+$ do not coincide in general. We let $[\hat{\mathbf{Y}}] \stackrel{\text{def}}{=} \hat{\mathbf{Y}}^+ - \mathbf{Y}^-$, $[\mathbf{Y}] \stackrel{\text{def}}{=} \mathbf{Y}^+ - \mathbf{Y}^-$ be the jumps codifying the gravitational/matter-energy content of the null shells associated to $\hat{\varphi}$ and φ respectively. Then, by (9.21) we know that $[\mathbf{Y}]$ must be given by

$$[Y_{ab}] = z\left((\varphi^\star\hat{\mathbf{Y}}^+)_{ab} + \frac{z}{2}\Big(P(\varphi^\star\boldsymbol{\ell}, \varphi^\star\boldsymbol{\ell}) - \varphi^\star \ell^{(2)} \Big) \mathrm{U}_{ab} - \mathring{\nabla}_{(a} (\varphi^\star\boldsymbol{\ell})_{b)} \right) - Y^-_{ab}. \tag{9.50}$$

The jumps $[\mathbf{Y}]$, $[\hat{\mathbf{Y}}]$ can actually be related, as we shall see next. Indeed, by defining the tensor

$$\boldsymbol{\Upsilon} \stackrel{\text{def}}{=} z\varphi^\star\hat{\mathbf{Y}}^+ - \hat{\mathbf{Y}}^+, \tag{9.51}$$

expression (9.50) can be rewritten as

$$[Y_{ab}] = \Upsilon_{ab} + \frac{z^2}{2}\Big(P(\varphi^\star \boldsymbol{\ell}, \varphi^\star\boldsymbol{\ell}) - \varphi^\star \ell^{(2)} \Big) \mathrm{U}_{ab} - z\, \mathring{\nabla}_{(a}(\varphi^\star \boldsymbol{\ell})_{b)} + [\hat{Y}_{ab}]. \tag{9.52}$$

Moreover, a direct calculation shows that the components (9.22)-(9.24) of $[\mathbf{Y}]$ in a basis $\{n, e_A\}$ of $\Gamma(T\mathcal{N})$ can be expressed in terms of $\boldsymbol{Y}$ as

$$[\mathbf{Y}](n,n) = \boldsymbol{Y}(n,n) + [\Upsilon](n,n) + \frac{n(z)}{z}, \tag{9.53}$$

$$\begin{aligned}[\mathbf{Y}](n,e_A) = {} & \boldsymbol{Y}(n,e_A) + [\Upsilon](n,e_A) - \frac{z}{2}(\pounds_n \varphi^\star \boldsymbol{\ell})(e_A) \\ & + \frac{e_A(z)}{2z} + s(e_A) + zP(\varphi^\star \boldsymbol{\ell}, \mathbf{U}(e_A,\cdot)),\end{aligned} \tag{9.54}$$

$$\begin{aligned}[\mathbf{Y}](e_A,e_B) = {} & \boldsymbol{Y}(e_A,e_B) + [\Upsilon](e_A,e_B) \\ & + \frac{z^2}{2}\left(P(\varphi^\star \boldsymbol{\ell}, \varphi^\star \boldsymbol{\ell}) - \varphi^\star \ell^{(2)}\right)\mathbf{U}(e_A,e_B) - z e^a_A e^b_B \mathring{\nabla}_{(a}(\varphi^\star \boldsymbol{\ell})_{b)}.\end{aligned} \tag{9.55}$$

Taking into account (9.25)-(9.27) allows us to connect the energy-momentum tensors $\tau, \bar{\tau}$ of the two shells. Specifically, for the dual basis $\{\mathbf{q}, \boldsymbol{\theta}^A\}$ of $\{n, e_A\}$ one finds (recall that $h_{AB} \stackrel{\text{def}}{=} \gamma(e_A, e_B)$)

$$\begin{aligned}\tau(\mathbf{q},\mathbf{q}) = {} & \bar{\tau}(\mathbf{q},\mathbf{q}) - \epsilon h^{AB}\Big(\boldsymbol{Y}(e_A,e_B) \\ & + \frac{z^2}{2}\left(P(\varphi^\star \boldsymbol{\ell}, \varphi^\star \boldsymbol{\ell}) - \varphi^\star \ell^{(2)}\right)\mathbf{U}(e_A,e_B) - z e^a_A e^b_B \mathring{\nabla}_{(a}(\varphi^\star \boldsymbol{\ell})_{b)}\Big),\end{aligned} \tag{9.56}$$

$$\begin{aligned}\tau(\mathbf{q},\boldsymbol{\theta}^A) = {} & \bar{\tau}(\mathbf{q},\boldsymbol{\theta}^A) + \epsilon h^{AB}\Big(\boldsymbol{Y}(n,e_B) \\ & - \frac{z}{2}(\pounds_n \varphi^\star \boldsymbol{\ell})(e_B) + \frac{e_B(z)}{2z} + s(e_B) + zP(\varphi^\star \boldsymbol{\ell}, \mathbf{U}(e_B,\cdot))\Big),\end{aligned} \tag{9.57}$$

$$\tau(\boldsymbol{\theta}^A,\boldsymbol{\theta}^B) = \bar{\tau}(\boldsymbol{\theta}^A,\boldsymbol{\theta}^B) - \epsilon h^{AB}\left(\boldsymbol{Y}(n,n) + \frac{n(z)}{z}\right). \tag{9.58}$$

The results (9.53)-(9.58) turn out to be of particular interest when one of the matchings of $(\mathcal{M}^\pm, g^\pm)$ gives no shell. In order to see this, let us assume that this is the case and take φ to be the diffeomorphism corresponding to the no-shell matching. Then, $[\Upsilon] = 0$ (i.e. $\hat{\mathbf{Y}}^+ = \mathbf{Y}^-$) holds and the tensor $\boldsymbol{Y}$ is given by (cf. (9.51))

$$\boldsymbol{Y} = z\varphi^\star \mathbf{Y}^- - \mathbf{Y}^-. \tag{9.59}$$

Setting $[\mathbf{Y}] = 0$ in equations (9.53)-(9.55) yields

$$[\mathbf{Y}](n,n) = \boldsymbol{Y}(n,n) + \frac{n(z)}{z}, \tag{9.60}$$

$$[\mathbf{Y}](n,e_A) = \boldsymbol{Y}(n,e_A) - \frac{z}{2}(\pounds_n \varphi^\star \boldsymbol{\ell})(e_A) + \frac{e_A(z)}{2z} + s(e_A) + zP(\varphi^\star \boldsymbol{\ell}, \mathbf{U}(e_A,\cdot)), \tag{9.61}$$

$$\begin{aligned}[\mathbf{Y}](e_A,e_B) = {} & \boldsymbol{Y}(e_A,e_B) + \frac{z^2}{2}\left(P(\varphi^\star \boldsymbol{\ell}, \varphi^\star \boldsymbol{\ell}) - \varphi^\star \ell^{(2)}\right)\mathbf{U}(e_A,e_B) \\ & - z e^a_A e^b_B \mathring{\nabla}_{(a}(\varphi^\star \boldsymbol{\ell})_{b)}.\end{aligned} \tag{9.62}$$

Consequently, when a no-shell matching is possible, the jump $[\mathbf{Y}]$ corresponding to any other possible matching is given by (9.60)-(9.62) in terms of the data fields $\{\gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}^-\}$ and the diffeomorphism φ. In other words, knowing the information about the no-shell matching automatically allows one to obtain the gravitational/matter-energy content of the remaining matchings by simply determining φ. In particular, there is no need to compute the new matching rigging ζ^+ or the tensor $\mathbf{Y}^+$ to determine the shell properties. One simple needs to compute the right-hand sides of (9.60)-(9.62) using (9.59).

We emphasize that (9.60)-(9.62) apply, in particular, when $(\mathcal{M}^\pm, g^\pm)$ are two regions of the same spacetime $(\mathcal{M}, g)$ and more than one matching can be performed. Then, the existence of a no-shell matching is always guaranteed, as one can always recover the full spacetime $(\mathcal{M}, g)$ from the matching of $(\mathcal{M}^\pm, g^\pm)$. This in fact occurs in all cut-and-paste constructions, which means that (9.60)-(9.62) *provide the matter content of a null shell generated by any cut-and-paste matching procedure, as long as the two regions* $(\mathcal{M}^\pm, g^\pm)$ *of* $(\mathcal{M}, g)$ *can be pasted in more than one way.*

We conclude this section by discussing a particular situation of interest, namely the case when a null hypersurface data $\mathcal{D} = \{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}^-\}$ can be embedded in two spacetimes $(\mathcal{M}^\pm, g^\pm)$ with embeddings $\iota^\pm$ (such that $\iota^\pm(\mathcal{N})$ are boundaries of $\mathcal{M}^\pm$) and riggings $L^\pm$ with the appropriate orientation. This means that $(\mathcal{M}^\pm, g^\pm)$ can be matched so that the resulting spacetime contains no shell (because $\mathbf{Y}^-$ is the same for both spacetimes). We assume, in addition, that $\mathcal{D}$ admits a vector field $\bar{\xi} \in \Gamma(T\mathcal{N})$ with the property $\pounds_{\bar{\xi}}\gamma = 0$. The vector $\bar{\xi}$ defines a (local) one-parameter group of transformations $\{\varphi_t\}$ of $\mathcal{N}$ satisfying

$$\varphi_t^\star \gamma = \gamma. \tag{9.63}$$

We now prove that, for each value of t, the diffeomorphism φ_t gives rise to a matching. First, we define gauge parameters $\{z, V\}$ according to (9.49) for $\varphi = \varphi_t$. Then, it is immediate to check that (9.46) holds for $\varphi = \varphi_t$ and that $z > 0$ (because φ_t depends continuously on t and $(\varphi_{t=0}^\star \boldsymbol{\ell})(n) = \boldsymbol{\ell}(n) = 1$). Therefore, conditions (*i*) and (*ii*) in Theorem 9.1.1 are both fulfilled (notice that, since $L^\pm$ are matching riggings, one points inwards and the other outwards, so (*ii*) is just $z > 0$) and indeed each φ_t corresponds to a different matching. The jump $[\mathbf{Y}] \stackrel{\text{def}}{=} \mathbf{Y}^+ - \mathbf{Y}^-$ where $\mathbf{Y}^+ \stackrel{\text{def}}{=} \frac{1}{2}\varphi_t^\star\ (\iota^+)^\star(\pounds_{\zeta^+} g^+)$ (and ζ^+ is given by (9.13)) rules the gravitational/matter-energy content of the resulting shell. The vector field $\bar{\xi}$ generates a multitude of new shells. The construction is further simplified when, in addtion to (9.63), it holds

$$\varphi_t^\star \mathbf{Y}^- = \mathbf{Y}^-. \tag{9.64}$$

Then (9.59) implies

$$\boldsymbol{Y} = (z-1)\,\mathbf{Y}^{-}, \tag{9.65}$$

which simplifies the expressions (9.60)-(9.62) considerably. One may wonder what is the final result when, in addition, $\overline{\xi}$ is the restriction to N of a Killing vector field ξ on $\mathcal{M}^{-}$ (i.e. $\imath_{\star}^{-}\overline{\xi} = \xi$) and $\pounds_{\xi} L^{-} = 0$ is fulfilled (so that (9.63) and (9.64) hold). It is straightforward to see that

$$\varphi_t^{\star}\boldsymbol{\ell} = \boldsymbol{\ell}, \qquad \varphi_t^{\star}\,\ell^{(2)} = \ell^{(2)}, \tag{9.66}$$

which combined with (9.49) means that $z = 1$, and $V = 0$, so $\boldsymbol{Y} = 0$ (cf. (9.65)). Moreover, one can easily check that the terms in the right-hand side of (9.60)-(9.62) cancel out. Thus, the procedure gives rise to another no-shell matching, as one would expect because the transformation induced by ξ does not affect in any geometric way the spacetime $(\mathcal{M}^{-}, g^{-})$. This constitutes a non-trivial consistency check of equations (9.60)-(9.62).

9.2 null boundaries with product topology $S \times \mathbb{R}$

In order to connect the results in this chapter with those from Chapters 7 and 8, we now consider the case when the boundaries of the spacetimes to be matched can be foliated by a family of spacelike cross-sections. In particular, we shall find a step function H and provide explicit expressions for the gauge parameters $\{z, V\}$ (cf. (9.12)) and the gravitational/matter-energy content of the shell. The results for the jump $[\mathbf{Y}]$ will be then compared with their counterparts from Chapter 7.

Our setup for the present section is the following. We consider two spacetimes $(\mathcal{M}^{\pm}, g^{\pm})$ with null boundaries $\widetilde{\mathcal{N}}^{\pm}$ and assume that $\widetilde{\mathcal{N}}^{\pm}$ have product topology $S^{\pm} \times \mathbb{R}$, where $S^{\pm}$ are spacelike cross-sections and the null generators are along $\mathbb{R}$. We select two future null generators $k^{\pm} \in \Gamma(T\mathcal{M}^{\pm})|_{\widetilde{\mathcal{N}}^{\pm}}$ of $\widetilde{\mathcal{N}}^{\pm}$ and two cross-sections $S_0^{\pm} \subset \widetilde{\mathcal{N}}^{\pm}$. We then construct foliation functions $v_{\pm} \in \mathcal{F}(\widetilde{\mathcal{N}}^{\pm})$ by solving $k^{\pm}(v_{\pm}) = 1$ with initial values $v_{\pm}|_{S_0^{\pm}} = 0$. Finally, the riggings $L^{\pm}$ are fixed by the conditions of being orthogonal to the respective leaves $\{v_{\pm} = \text{const}\}$, null and scaled to satisfy $\mu_1^{\pm} \stackrel{\text{def}}{=} g^{\pm}(L^{\pm}, k^{\pm}) = 1$.

We assume that $(\mathcal{M}^{\pm}, g^{\pm})$ can be matched, so that conditions (i)-(ii) in Theorem 9.1.1 are fulfilled for a diffeomorphism $\varphi : \mathcal{N} \longrightarrow \mathcal{N}$ verifying (9.11). This allows us to take two embeddings $\imath^{\pm} : \mathcal{N} \longhookrightarrow \mathcal{M}^{\pm}$ and construct the hypersurface data sets $\mathcal{D} = \{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}^-\}$, $\widehat{\mathcal{D}} = \{\mathcal{N}, \widehat{\gamma}, \widehat{\boldsymbol{\ell}}, \widehat{\ell}^{(2)}, \widehat{\mathbf{Y}}^+\}$ according to (9.3). We also introduce the functions

$$\lambda \stackrel{\text{def}}{=} (\iota^-)^\star(v_-), \qquad v \stackrel{\text{def}}{=} (\iota^+)^\star(v_+), \qquad \text{and} \qquad H \stackrel{\text{def}}{=} \varphi^\star v \tag{9.67}$$

on $\mathcal{N}$. Since by construction $\iota^-_\star(n) = k^-$ and $\iota^+_\star(\widehat{n}) = k^+$ (recall (2.25)), it is immediate to check that $\{\lambda, v\}$ are foliation functions of $\mathcal{N}$. Observe also that our choice for the riggings $L^\pm$ implies

$$\boldsymbol{\ell} = d\lambda, \qquad \ell^{(2)} = 0, \qquad \widehat{\boldsymbol{\ell}} = dv, \qquad \widehat{\ell}^{(2)} = 0, \tag{9.68}$$

which in turn means (recall (2.10)-(2.11))

$$n(\lambda) = 1, \quad \mathbf{F} = 0, \quad \boldsymbol{s} = 0, \qquad \widehat{n}(v) = 1, \quad \widehat{\mathbf{F}} = 0, \quad \widehat{\boldsymbol{s}} = 0. \tag{9.69}$$

We now select vector fields $\{e_A\}$ tangent to the leaves $\{\lambda = \text{const.}\}$ so that $\{n, e_A\}$ is a basis of $\Gamma(T\mathcal{N})$ satisfying $[n, e_A] = 0$. As before, we let h be induced metric on $\{\lambda = \text{const.}\}$ and ∇^h for its Levi-Civita derivative. In particular $h_{AB} \stackrel{\text{def}}{=} \gamma(e_A, e_B)$ and we note that, for any $f \in \mathcal{F}(\mathcal{N})$, we can write $e_A(f)$ also as $\nabla^h_A f$. Since in the present case $e_A(\lambda) = 0$, the pull-back of $\boldsymbol{\ell}$ to the leaves of constant λ is zero, i.e. $\ell_A = \psi_A = 0$. This, together with $\ell^{(2)} = 0$ and (3.50), means that $P = h^{AB} e_A \otimes e_B$. Observe also that

$$\varphi^\star \widehat{\boldsymbol{\ell}} = \varphi^\star dv = d(\varphi^\star v) = dH, \qquad \varphi^\star \widehat{\ell}^{(2)} = 0, \tag{9.70}$$

which in particular means that

$$P(\varphi^\star \widehat{\boldsymbol{\ell}}, \cdot) = P(dH, \cdot) = h^{AB}(\nabla^h_A H) e_B. \tag{9.71}$$

We can now particularize (9.12) to the case of boundaries with product topology. For that we insert (9.70)-(9.71) into (9.12) and get

$$z = \frac{1}{n(H)}, \qquad V = h^{AB} \nabla^h_A H \left(\frac{\nabla^h_B H}{2n(H)} n - e_B \right). \tag{9.72}$$

The push-forward vector fields $\{\varphi_\star n, \varphi_\star e_A\}$ can also be computed in terms of the function H and the vector fields $W_A \stackrel{\text{def}}{=} \widehat{\gamma}(\mathbf{W}_A, \cdot)$ (recall Corollary 9.1.7). In fact, combining (2.9) and (9.72) one finds

$$\gamma(e_A, V) = -\gamma(e_A, P(dH, \cdot)) = -e_A(H), \tag{9.73}$$

which in turn entails that

$$-W_A(\varphi_\star V)|_{\varphi(p)} = -(\varphi^\star W_A)(V)|_p = -\gamma(e_A, V)|_p = e_A(H)|_p \qquad \forall p \in \mathrm{N}. \tag{9.74}$$

Using $\psi_A = 0$, (9.72) and (9.74) in the expressions of Corollary 9.1.7 yields

$$\varphi_\star n = (\varphi^{-1})^\star(n(H))\,n, \tag{9.75}$$

$$\varphi_\star e_A = W_A + (\varphi^{-1})^\star(e_A(H))\,n. \tag{9.76}$$

Observe that $\{W_A\}$ are tangent to the leaves $\{v = \text{const.}\}$ (because by Corollary 9.1.7 we know that $0 = \gamma(W_A, n) = \ell(W_A) = W_A(v)$). Moreover, the vector fields $\{W_A\}$ commute with the null generator n, as we prove next.

Lemma 9.2.1. *The vector fields n and W_A satisfy $[n, W_A] = 0$.*

Proof. Define the functions $u \stackrel{\text{def}}{=} (\varphi^{-1})^\star(n(H))$ and $\chi_{(A)} \stackrel{\text{def}}{=} (\varphi^{-1})^\star(e_A(H))$, so that (9.75)-(9.76) can be written as $n = u^{-1}\varphi_\star n$ and $W_A = \varphi_\star e_A - \chi_{(A)} n$. Thus,

$$\begin{aligned}
[n, W_A] &= [u^{-1}\varphi_\star n, \varphi_\star e_A] - [n, \chi_{(A)} n] \\
&= u^{-1}\varphi_\star([n, e_A]) + u^{-2}\big(\varphi_\star e_A(u)\big)\varphi_\star n - n(\chi_{(A)})n \\
&= u^{-2}\big(\varphi_\star e_A(u) - \varphi_\star n(\chi_{(A)})\big)\varphi_\star n,
\end{aligned} \tag{9.77}$$

where in the last equality we used $[n, e_A] = 0$ and $n = u^{-1}\varphi_\star n$. To prove the claim we just need to show that the last parenthesis is zero. Indeed,

$$\begin{aligned}
\varphi_\star e_A(u) - \varphi_\star n(\chi_{(A)}) &= (du)(\varphi_\star e_A) - (d\chi_{(A)})(\varphi_\star n) = \varphi^\star(du)(e_A) - \varphi^\star(d\chi_{(A)})(n) \\
&= (d\varphi^\star u)(e_A) - (d\varphi^\star \chi_{(A)})(n) = e_A(n(H)) - n(e_A(H)) \\
&= [e_A, n](H) = 0.
\end{aligned}$$

□

By Remark 9.1.8 we also know that $\{n, W_A\}$ constitute a basis of $\Gamma(T\mathrm{N})$. Therefore, the vector fields

$$\{e^-_1 \stackrel{\text{def}}{=} \iota^-_\star n, \quad e^-_A \stackrel{\text{def}}{=} \iota^-_\star e_A\}, \qquad \{e^+_1 \stackrel{\text{def}}{=} \iota^+_\star(\varphi_\star n), \quad e^+_A \stackrel{\text{def}}{=} \iota^+_\star(\varphi_\star e_A)\} \tag{9.78}$$

form basis of $\Gamma(T\mathrm{N}^\pm)$ respectively. Inserting (9.75)-(9.76) into (9.78) and using again that $\iota^+_\star(n) = k^+$, one obtains

$$e^+_1 = n(H)k^+, \qquad e^+_A = e_A(H)k^+ + \iota^+_\star(W_A), \tag{9.79}$$

where for simplicity we have dropped pull-backs affecting functions. Given that $\{\iota^+_\star W_A\}$ are linearly independent and tangent to the leaves $\{v_+ = \text{const.}\} \subset \mathcal{N}^+$, they can be decomposed in a basis $\{L^+, k^+, v^+_A\}$ of $\Gamma(T\mathcal{M}^+)|_{\mathcal{N}^+}$ satisfying (7.2) as $\iota^+_\star W_A = b^B_A v^+_B$, with $\{b^B_A\}$ defining an invertible matrix. Moreover, $b_A{}^B$ are constant along the null generators as a consequence of Lemma 9.2.1:

$$0 = [\iota_\star(n), \iota_\star(W_A)] = [k^+, b^B_A v^+_A] = k^+(b^B_A) v^+_B \qquad \Longleftrightarrow \qquad k^+(b^B_A) = 0.$$

Thus, with expressions (9.79) we recover the form of the matching vector fields $\{e^+_a\}$ introduced in (7.30).

The matching rigging ζ^+ can be derived as well by inserting (9.72) into (9.13) and using (9.75)-(9.76), (9.78)-(9.79). Specifically, one obtains

$$\begin{aligned}\zeta^+ &= \frac{1}{n(H)}\left(L^+ - h^{AB}\nabla^h_A H\left(e^+_A - \frac{\nabla^h_B H}{2n(H)} e^+_1\right)\right)\\ &= \frac{1}{n(H)}\left(L^+ - h^{AB}\nabla^h_A H\left(\iota^+_\star(W_A) + \frac{\nabla^h_B H}{2} k^+\right)\right).\end{aligned} \tag{9.80}$$

In the language of Chapter 7, we have chosen $L^\pm$ so that $\mu_1{}^\pm = 1$, $\mu^\pm_A = 0$, firstly because we have identified the vector fields $\{v^-_A\}$ introduced in Section 7.3 with the push-forwards of $\{e_A\}$ (cf. (9.78)) and secondly because $\iota^+_\star W_A = b^B_A v^+_B$ means that $\mu^+_A = g^+(L^+, v^+_A) = (b^{-1})^B_A \boldsymbol{\ell}(W_B) = (b^{-1})^B_A W_B(v) = 0$. Moreover, the inverse metric h^{AB} is given by $h^{AB} = h^{IJ}_+(b^{-1})^A_I(b^{-1})^B_J$ (again due to $\iota^+_\star W_A = b^B_A v^+_B$). Thus, it is immediate to check that (9.80) is equivalent to (7.70).

Before studying the gravitational/matter-energy content of the shell, it is also worth mentioning that equation (7.54) can now be obtained by simply particularizing (9.20) for $z^{-1} = n(H)$. In the present case one gets

$$\mathbf{U} = n(H)\varphi^\star\mathbf{U}. \tag{9.81}$$

The expressions for $[\mathbf{Y}]$ are obtained as a particular case of Theorem 9.1.9.

Theorem 9.2.2. *In the setup and conditions of Theorem 9.1.9 suppose further that the boundaries $\mathcal{N}^\pm$ can be foliated by cross-sections and define λ, v, $H \in \mathcal{F}(\mathcal{N})$ as in (9.67). Let h be the induced metric and ∇^h the corresponding Levi-Civita covariant derivative on the leaves $\{\lambda = \text{const.}\} \subset \mathcal{N}$. Then,*

$$[\mathrm{Y}_{ab}] = \frac{1}{n(H)}\left((\varphi^\star\mathbf{Y}^+)_{ab} + \frac{h^{AB}(\nabla^h_A H)(\nabla^h_B H)}{2n(H)}\mathrm{U}_{ab} - \mathring{\nabla}_a\mathring{\nabla}_b H\right) - \mathrm{Y}^-_{ab}. \tag{9.82}$$

Let $\{e_A\}$ *be vector fields in* $\mathcal{N}$ *such that* $\{n, e_A\}$ *is a basis adapted to the foliation* $\{\lambda = \text{const.}\}$ *and define* W_A *by means of* (9.76). *Then the components the jump* $[\mathbf{Y}]$ *can be written as*

$$[\mathbf{Y}](n,n) = n(H)\varphi^\star\left(\mathbf{Y}^+(n,n)\right) - \mathbf{Y}^-(n,n) - \frac{n(n(H))}{n(H)}, \tag{9.83}$$

$$\begin{aligned}[\mathbf{Y}](n,e_A) = {} & \varphi^\star\left(\mathbf{Y}^+(n,W_A)\right) + (\nabla^h_A H)\varphi^\star\left(\mathbf{Y}^+(n,n)\right) - \mathbf{Y}^-(n,e_A) \\ & - \frac{\nabla^h_A(n(H))}{n(H)} + \frac{h^{IJ}\nabla^h_J H}{n(H)}\mathbf{U}_\parallel(e_A,e_J),\end{aligned} \tag{9.84}$$

$$\begin{aligned}[\mathbf{Y}](e_A,e_B) = {} & \frac{1}{n(H)}\Big(\varphi^\star\left(\mathbf{Y}^+(W_A,W_B)\right) + 2(\nabla^h_{(A}H)\varphi^\star\left(\mathbf{Y}^+(n,W_{B)})\right) \\ & + (\nabla^h_A H)(\nabla^h_B H)\varphi^\star\left(\mathbf{Y}^+(n,n)\right) - n(H)\mathbf{Y}^-(e_A,e_B) \\ & + \frac{h^{IJ}(\nabla^h_I H)(\nabla^h_J H)}{2n(H)}\mathbf{U}_\parallel(e_A,e_B) - \nabla^h_A\nabla^h_B H\Big).\end{aligned} \tag{9.85}$$

Proof. Equation (9.82) follows at once after inserting (9.70)-(9.72) into (9.21). To obtain (9.83)-(9.85), it suffices to particularize (9.35)-(9.37) for $z^{-1} = n(H)$, $\varphi^\star\boldsymbol{\ell} = dH$, $\chi_{(A)} = (\varphi^{-1})^\star(e_A(H))$, $\varphi^\star\ell^{(2)} = 0$, $s = 0$ and $P(\varphi^\star\boldsymbol{\ell},\cdot) = h^{AB}(\nabla^h_A H)e_B$ and notice that $\pounds_n(\varphi^\star\boldsymbol{\ell}) = \pounds_n dH = d(n(H))$, as well as $e^a_A e^b_B \nabla^h_a\nabla^h_b H = \nabla^h_A\nabla^h_B H$ (see (3.94)). □

Our aim now is to connect expressions (9.83)-(9.85) with those in Proposition 7.3.7. However, as a prior step we need to relate hypersurface data quantities with the tensors defined in (2.99). This is done in the following lemma.

Lemma 9.2.3. *Let* $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ *be hypersurface data embedded in a semi-Riemannian manifold* $(\mathcal{M}, g)$ *with embedding* ϕ *and rigging* ζ. *Define the null generator* $k \stackrel{\text{def}}{=} \phi_\star n$ *and denote by* κ_k *its surface gravity. Consider a transverse submanifold* $S \subset \mathcal{N}$ *and assume that the gauge is such that the rigging* ζ *is null and orthogonal to* $\phi(S)$. *Then, for any basis* $\{e_A\}$ *of* $\Gamma(TS)$ *it holds (we identify scalars and vectors with their images on* $\phi(\mathcal{N})$*)*

(a) $\kappa_k = -\mathbf{Y}(n,n)$,

(b) $\boldsymbol{\sigma}_\zeta(e_A) = \mathbf{Y}(e_A, n) + \mathbf{F}(e_A, n)$,

(c) $\overset{k}{\mathbf{K}}(e_A, e_B) = \mathbf{U}(e_A, e_B)$.

(d) $\boldsymbol{\Theta}^\zeta(e_{(A}, e_{B)}) = \mathbf{Y}(e_A, e_B)$,

where $\boldsymbol{\sigma}_\zeta$, $\overset{\zeta}{\boldsymbol{\Theta}}$ *are defined by* (2.99) *for* $L = \zeta$ *and* $\overset{k}{\mathbf{K}}$ *is the second fundamental form of* $\phi(\mathcal{N})$ *with respect to* k *(cf.* (2.84)*).*

Proof. Claim (*a*) follows at once from (2.44) and (3.47) (note that here $v = k$). To prove (*b*) we compute

$$\boldsymbol{\sigma}_{\zeta}(e_A) \overset{(2.99)}{=} -g(\nabla_{e_A} k, \zeta) = g(\nabla_{e_A} \zeta, k) = \mathbf{Y}(e_A, n) + \mathbf{F}(e_A, n),$$

where in the last step we used (5.5) for $y = k$ (so that $\beta = 0$ and $\bar{y} = n$). Item (*c*) has already been stated after definition (2.47) and (*d*) follows from

$$\mathbf{Y}(e_A, e_B) = \frac{1}{2}(\pounds_{\zeta} g)(e_A, e_B) = g\left(\nabla_{e_{(A}} \zeta, e_{B)}\right) \overset{(2.99)}{=} \boldsymbol{\Theta}^{\zeta}\left(e_{(A}, e_{B)}\right).$$

□

We are now in a position where the comparison can be made. Since in the present case $\mu_1^{\pm} = 1$, $\mu_A^{\pm} = 0$ and $n = \partial_\lambda$ (because $\boldsymbol{\ell}(\partial_\lambda) = 1$), the function A in equation (7.30) is $A = n(H)$ (recall (7.45)) and the one-form $\boldsymbol{\vartheta}$ of Lemma 7.3.2 verifies $\vartheta_A = -\nabla^h_A H$. This in turn forces the components $\{X^a\}$ of (7.66)-(7.67) to be given by

$$X^1 = \frac{h^{AB}\nabla^h_A H \nabla^h_B H}{2n(H)}, \qquad X^A = -h^{AB}\nabla^h_B H. \tag{9.86}$$

Thus, expressions (7.88)-(7.90) become

$$[\mathbf{Y}](n, n) = -n(H)\left(\kappa^{+}_{k+} + \kappa^{-}_{k-}\right) - \frac{n(n(H))}{n(H)}, \tag{9.87}$$

$$\begin{aligned}[\mathbf{Y}](n, e_J) = {} & \boldsymbol{\sigma}^{+}_{L}(W^{k+}_{J}) - \boldsymbol{\sigma}^{k-}_{L}(v^{-}_{J}) - (\nabla^h_J H)\kappa^{+}_{k+} \\ & - \frac{\nabla^h_J(n(H))}{n(H)} + \frac{h^{LB}\nabla^h_B H}{n(H)} \mathbf{K}^{k}_{-}(v^{-}_{J}, v^{-}_{L}),\end{aligned} \tag{9.88}$$

$$\begin{aligned}[\mathbf{Y}](e_I, e_J) = {} & \frac{1}{n(H)}\Big(2(\nabla^h_{(I} H)\boldsymbol{\sigma}^{+}_{L}(W_{J)}) - \kappa^{+}_{k+}(\nabla^h_I H)(\nabla^h_J H) + \boldsymbol{\Theta}^{L}_{+}(W_{(I}, W_{J)}) \\ & - n(H)\boldsymbol{\Theta}^{L}_{-}(v^{-}_{(I}, v^{-}_{J)}) + \frac{h^{AB}\nabla^h_A H \nabla^h_B H}{2n(H)} \mathbf{K}^{k}_{-}(v^{-}_{I}, v^{-}_{J}) \Big) - \nabla^h_I \nabla^h_J H.\end{aligned} \tag{9.89}$$

Particularizing Lemma 9.2.3 to the sections $\{\lambda = \text{const}\}$ of D (with basis e_A) and the sections $\{v = \text{const}\}$ of D (with basis W_A), and recalling that $\mathbf{F} = \mathbf{F} = 0$ (see (9.69)), it is straightforward to check that (9.87)-(9.89) coincide with (9.83)-(9.85).

9.3 cut-and-paste construction: (anti-)de sitter spacetime

We have already mentioned that (9.60)-(9.62) hold for the specific case when the two spacetimes to be matched are actually two regions of the same spacetime (with the additional requirement that more than one matching is allowed). In this section, our aim is to provide an example of a cut-and-paste construction, namely the matching of two regions of a constant-curvature spacetime across a totally geodesic null hypersurface. For previous works on the cut-and-paste construction describing non-expanding impulsive gravitational waves in constant curvature backgrounds we refer e.g. to [88], [91], [92] [5], [6] and references therein. In these publications, the discontinuity in the coordinates when crossing the shell is given by the Penrose's jump (2.170). Moreover, as we shall see next, this jump can only be recovered when the shell has neither pressure nor energy-flux, just as happened in the case of Minkowski described in Section 7.3.3.

It is well-known that in any constant curvature spacetime $(\mathcal{M}, g)$ there exists only one totally geodesic null hypersurface up to isometries (see e.g. [133], [134]). Let us denote one such hypersurface by $\widetilde{\mathcal{N}}$. Then, one can always construct coordinates $\{U, V, x^A\}$ adapted to $\widetilde{\mathcal{N}}$ so that the metric is conformally flat and $\widetilde{\mathcal{N}} \stackrel{\text{def}}{=} \{U = 0\}$, namely

$$g = \frac{g_{Mk}}{\mu^2} \qquad \text{where} \quad g_{Mk} = -2dU\,dV + \delta_{AB}dx^A dx^B, \quad \mu \stackrel{\text{def}}{=} 1 + \frac{\Lambda}{12}\left(\delta_{AB}x^A x^B - 2UV\right). \tag{9.90}$$

Here Λ stands for the cosmological constant, so $\Lambda = 0, \Lambda > 0, \Lambda < 0$ correspond to Minkowski, de Sitter and anti-de Sitter spacetimes respectively. When $\Lambda \leq 0$, the coordinates $\{U, V, x^A\}$ cover a whole neighbourhood of $\widetilde{\mathcal{N}}$. However, for the de Sitter case one needs to remove one generator of $\widetilde{\mathcal{N}}$. This is because the topology of $\widetilde{\mathcal{N}}$ is $\mathbb{S}^n \times \mathbb{R}$ while stereographic coordinates only cover the sphere minus one point. In this section, we will analyze the three cases $\Lambda = 0$, $\Lambda < 0$ and $\Lambda > 0$ at once with the matching formalism introduced before.

The induced metric on $\widetilde{\mathcal{N}}$ reads

$$ds^2 \stackrel{\widetilde{\mathcal{N}}}{=} \frac{\delta_{AB}dx^A dx^B}{\left(1 + \frac{\Lambda}{12}\delta_{AB}x^A x^B\right)^2}, \tag{9.91}$$

and obviously the topology of $\widetilde{\mathcal{N}}$ is $S \times \mathbb{R}$, S being a spacelike section and the null generators being along $\mathbb{R}$. Therefore, all results from Section 9.2 apply in the present context.

Let us construct hypersurface data associated to $\widetilde{\mathcal{N}}$. Since $\widetilde{\mathcal{N}}$ is embedded on $(\mathcal{M}, g)$, there exists an abstract manifold $\mathcal{N}$ and an embedding ι such that

$\iota(\mathcal{N}) = \widetilde{\mathcal{N}}$. We can select ι to be as trivial as possible by constructing coordinates $\{\lambda, y^A\}$ on $\mathcal{N}$ so that

$$\begin{aligned} \iota: \mathcal{N} &\longhookrightarrow \widetilde{\mathcal{N}} \\ (\lambda, y^A) &\longmapsto \iota(\lambda, y^A) \equiv (U = 0, V = \lambda, x^A = y^A). \end{aligned} \tag{9.92}$$

We also need a choice of rigging vector field ζ along $\widetilde{\mathcal{N}}$. For convenience, we set $\zeta = -\mu^2 \partial_U$ (observe that $\mu^2|_{\widetilde{\mathcal{N}}} \neq 0$). The corresponding null metric hypersurface data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ defined by (2.22) is

$$\gamma = \frac{\delta_{AB}}{\mu_{\mathcal{N}}^2} dy^A \otimes dy^B, \qquad \boldsymbol{\ell} = d\lambda, \qquad \ell^{(2)} = 0, \tag{9.93}$$

where $\mu_{\mathcal{N}} \stackrel{\text{def}}{=} \iota^\star \mu = 1 + \frac{\Lambda}{12} \delta_{AB} y^A y^B$. Observe that $\partial_\lambda \in \operatorname{Rad}\gamma$ and that $n = \partial_\lambda$ because $\boldsymbol{\ell}(\partial_\lambda) = 1$. Moreover, $\mathbf{F} = 0$ and $s = 0$ (cf. (2.10)-(2.11)) and $\mathbf{U} = 0$ as a consequence of (2.12) and (9.93).

In order to compute the explicit form of the tensor $\mathbf{Y}$, we first obtain the derivative $\pounds_\zeta g$. Using that ∂_U is a Killing vector of g_{Mk}, one gets

$$\begin{aligned} \pounds_\zeta g &= \frac{2}{\mu} (\partial_U \mu) g_{Mk} - 2 d\mu \otimes_s g_{Mk}(\partial_U, \cdot) = -\frac{2}{\mu} \left(\frac{\Lambda V}{6} g_{Mk} + 2 d\mu \otimes_s dV \right) \\ &= -\frac{\Lambda}{3\mu} \left(V g_{Mk} - 2 \left(\delta_{AB} x^B dx^A - V dU - U dV \right) \otimes_s dV \right), \end{aligned} \tag{9.94}$$

from where it follows (cf. (2.39))

$$\mathbf{Y} = -\frac{\Lambda \delta_{AB}}{6\mu_{\mathcal{N}}} \left(\lambda dy^A \otimes dy^B - 2 y^B dy^A \otimes_s d\lambda \right). \tag{9.95}$$

When $\Lambda = 0$ we recover $\mathbf{Y} = 0$, which is in accordance with the first equation of (7.121). The explicit expressions for the components of $\mathbf{Y}$ are

$$Y_{\lambda\lambda} = 0, \qquad Y_{\lambda y^B} = \frac{\Lambda \delta_{BJ} y^J}{6\mu_{\mathcal{N}}}, \qquad Y_{y^I y^J} = -\frac{\Lambda\lambda}{6\mu_{\mathcal{N}}} \delta_{IJ}. \tag{9.96}$$

Cutting the spacetime across the hypersurface $\{U = 0\}$ leaves two spacetimes $(\mathcal{M}^\pm, g^\pm)$ defined to be the regions $U \gtreqless 0$ endowed with the metrics

$$g^\pm = \frac{g^\pm_{Mk}}{\mu_\pm^2}, \quad \text{where} \quad \begin{cases} g^\pm_{Mk} \stackrel{\text{def}}{=} -2 dU_\pm dV_\pm + \delta_{AB} dx^A_\pm dx^B_\pm, \\ \mu_\pm \stackrel{\text{def}}{=} 1 + \frac{\Lambda}{12} \left(\delta_{AB} x^A_\pm x^B_\pm - 2 U_\pm V_\pm \right). \end{cases} \tag{9.97}$$

Obviously, the boundaries are $\widetilde{\mathcal{N}}^\pm \equiv \{U_\pm = 0\}$.

It is clear that one can always perform a matching of $(\mathcal{M}^{\pm}, g^{\pm})$ and give rise to a resulting spacetime with no matter/gravitational content on the matching hypersurface. It suffices to select the same two riggings along the boundaries and paste $(\mathcal{M}^{\pm}, g^{\pm})$ across $\widetilde{\mathcal{N}}^{\pm}$ with the identity matching map. With this procedure we simply recover the global (anti-)de Sitter spacetime. Moreover, since $\widetilde{\mathcal{N}}^{\pm}$ are totally geodesic, by (7.54) (or (9.20)) we know that multiple matchings can be performed.

We therefore proceed as in Section 9.1.2, i.e. we let the two embeddings $\iota^{\pm}$ be given by $\iota^{\pm} = \iota$ and take $\zeta^{-} = -\mu_{-}^{2}\partial_{U_-}$, $\zeta^{+} = -\mu_{+}^{2}\partial_{U_+}$ as the riggings defining the no-shell matching, namely the matching for which $[\mathbf{Y}] = 0$. Any other possible matching will be ruled by a diffeomorphism φ of $\mathcal{N}$ onto itself and it will correspond to a different rigging ζ^{+} along $\widetilde{\mathcal{N}}^{+}$. Specifically, the hypersurface data corresponding to the no-shell matching is $\mathcal{D} = \{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$, where $\{\gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ and $\mathbf{Y}$ are respectively given by (9.93) and (9.95), while the matter/gravitational content of the shell of any other possible matching (ruled by φ) is given by the the jump $[\mathbf{Y}] \stackrel{\mathrm{def}}{=} \mathbf{Y}^{+} - \mathbf{Y}$ with

$$\mathbf{Y}^{+} \stackrel{\mathrm{def}}{=} \frac{1}{2}\varphi^{\star}\left(\iota^{\star}(\pounds_{\zeta^{+}} g^{+})\right). \tag{9.98}$$

From Section 9.1.2, we know that there is no need to compute the new rigging ζ^{+} or its corresponding $\mathbf{Y}^{+}$, but the jump $[\mathbf{Y}]$ is explicitly given by (9.60)-(9.62). Consequently, we only need to worry about the diffeomorphism φ. The only restriction that φ must satisfy is $\varphi^{\star}\gamma = \gamma$, which in coordinates reads

$$\frac{(\partial_{y^a}\varphi^{A})(\partial_{y^b}\varphi^{B})\delta_{AB}}{\left(1+\frac{\Lambda}{12}\delta_{IJ}\varphi^{I}\varphi^{J}\right)^{2}} = \frac{\delta^{A}_{a}\delta^{B}_{b}\delta_{AB}}{\left(1+\frac{\Lambda}{12}\delta_{IJ}y^{I}y^{J}\right)^{2}}. \tag{9.99}$$

It follows that the components $\{\varphi^{A}\}$ cannot depend on the coordinate λ. In particular, if we let $\{h^{A}(y^{B})\}$ be a set of functions such that (a) the Jacobian matrix $\frac{\partial(h^{2},\dots,h^{n+1})}{\partial(y^{2},\dots,y^{n+1})}$ has non-zero determinant and (b) $\{h^{A}(y^{B})\}$ satisfy $\left(1+\frac{\Lambda}{12}\delta_{IJ}y^{I}y^{J}\right)^{-2}\delta_{CD} = \left(1+\frac{\Lambda}{12}\delta_{IJ}h^{I}h^{J}\right)^{-2}(\partial_{y^C}h^{A})(\partial_{y^D}h^{B})\delta_{AB}$, any diffeomorphism $\varphi : \mathcal{N} \longrightarrow \mathcal{N}$ of the form

$$\begin{aligned} \varphi : \quad \mathcal{N} \quad &\longrightarrow \quad \mathcal{N} \\ (\lambda, y^{B}) \quad &\longmapsto \quad \varphi(\lambda, y^{B}) \equiv (H(\lambda, y^{B}), h^{A}(y^{B})). \end{aligned} \tag{9.100}$$

with $\partial_{\lambda}H \neq 0$ fulfils $\varphi^{\star}\gamma = \gamma$. A particular simple example is $\{h^{A} = y^{A}\}$, but many more exist. In fact since the metric on any section of $\widetilde{\mathcal{N}}$ is of constant curvature, it is also maximally symmetric (and of dimension $n-1$) so $h^{A}(y^{B})$ can depend on $n(n-1)/2$ arbitrary parameters. Once we find one such set $\{h^{A}(y^{B})\}$, the gauge parameters z and V are given by (9.72) for $\{n = \partial_\lambda, e_A = \partial_{y^A}\}$ and for an arbitrary step function $H(\lambda, y^A)$.

In the present case the tensor $\boldsymbol{Y}$ is given by $\boldsymbol{Y} = \frac{1}{n(H)}\varphi^\star \mathbf{Y} - \mathbf{Y}$ (cf. (9.59)), so we now compute the pull-back $\varphi^\star \mathbf{Y}$. Defining $\overline{\mu}_N \stackrel{\text{def}}{=} 1 + \frac{\Lambda}{12}\delta_{AB}h^A h^B$, from (9.95) and (9.100) it is straightforward to get

$$(\varphi^\star \mathbf{Y})_{\lambda\lambda} = 0, \qquad (\varphi^\star \mathbf{Y})_{\lambda y^B} = \frac{\Lambda\delta_{IJ}h^J}{6\overline{\mu}_N}\frac{\partial h^I}{\partial y^B}\frac{\partial H}{\partial \lambda}, \tag{9.101}$$

$$(\varphi^\star \mathbf{Y})_{y^A y^B} = \frac{\Lambda\delta_{IJ}}{6\overline{\mu}_N}\left(h^J\left(\frac{\partial H}{\partial y^A}\frac{\partial h^I}{\partial y^B} + \frac{\partial h^I}{\partial y^A}\frac{\partial H}{\partial y^B}\right) - H\frac{\partial h^I}{\partial y^A}\frac{\partial h^J}{\partial y^B}\right) \tag{9.102}$$

so that, multiplying (9.101)-(9.102) by $\frac{1}{n(H)}$ and subtracting $\mathbf{Y}$ (cf. (9.95)) yields

$$\begin{aligned} Y_{\lambda\lambda} &= 0, \qquad Y_{\lambda y^B} = \frac{\Lambda\delta_{IJ}}{6}\left(\frac{h^J}{\overline{\mu}_N}\frac{\partial h^I}{\partial y^B} - \frac{\delta^I_B y^J}{\mu_N}\right), \\ Y_{y^A y^B} &= \frac{\Lambda\delta_{IJ}}{6n(H)}\left(\frac{h^J}{\overline{\mu}_N}\left(\frac{\partial H}{\partial y^A}\frac{\partial h^I}{\partial y^B} + \frac{\partial h^I}{\partial y^A}\frac{\partial H}{\partial y^B}\right) - \frac{H}{\overline{\mu}_N}\frac{\partial h^I}{\partial y^A}\frac{\partial h^J}{\partial y^B} + \frac{\delta^I_A\delta^J_B\lambda}{\mu_N}n(H)\right). \end{aligned} \tag{9.103}$$

Inserting these expressions into (9.60)-(9.62) and using $n = \partial_\lambda$, $\boldsymbol{s} = 0$, $\mathbf{U} = 0$ together with the identity $(\pounds_n \varphi^\star \boldsymbol{\ell})(e_A) = (\pounds_n dH)(e_A) = d(n(H))(e_A) = e_A(n(H))$ (here $\varphi^\star \boldsymbol{\ell} = dH$ by (9.70) and $\boldsymbol{\ell} = \boldsymbol{\ell}$), one finds

$$\begin{aligned} [Y_{\lambda\lambda}] &= -\frac{n(n(H))}{n(H)}, \\ [Y_{\lambda y^A}] &= Y_{\lambda y^A} - \frac{\nabla^h_A(n(H))}{n(H)}, \\ [Y_{y^A y^B}] &= Y_{y^A y^B} - \frac{\nabla^h_A\nabla^h_B H}{n(H)}, \end{aligned} \tag{9.104}$$

which can be interpreted as the sum of the jump corresponding to the matching of two regions of Minkowski across a null hyperplane (see Section 7.3.3, equation (7.121)) plus the contribution of the tensor $\boldsymbol{Y}$. Observe that $\Lambda = 0$ entails $\boldsymbol{Y} = 0$, so in this way we recover expressions (7.121) for the most general shell in the spacetime of Minkowski.

A direct computation that combines the definitions (7.97), (9.93) and (9.104) yields energy-density, energy flux and pressure (note that here we need to take $\epsilon = -1$)

$$\rho = \mu_N^2\,\delta^{AB}\left(Y_{y^A y^B} - \frac{\nabla^h_A\nabla^h_B H}{n(H)}\right),$$

$$j = \mu_{\mathrm{N}}^2 \delta^{AB} \left(\frac{\nabla^h_B(n(H))}{n(H)} - \mathrm{Y}_{\lambda y^B} \right) \partial_{y^A}, \qquad (9.105)$$
$$p = -\frac{n(n(H))}{n(H)}.$$

Observe that only the pressure is independent of the value of the cosmological constant Λ (ρ and j depend on the conformal factor μ_{N} and on $\boldsymbol{Y}$). The pressure p takes the same value for the matchings of two regions of (anti-)de Sitter or Minkowski (in fact, p coincides with the pressure obtained in Section 7.3.3). In particular, in the case $h^A = y^A$ (i.e. when the mapping between null generators of both sides is trivial), then $\mathrm{Y}_{\lambda y^B} = 0$ (cf. (9.103)) and (9.105) simplifies to

$$\rho = \mu_{\mathrm{N}} \overset{2}{\delta}{}^{AB} \left(\mathrm{Y}_{y^A y^B} - \frac{\nabla^h_A \nabla^h_B H}{n(H)} \right),$$
$$j = \mu_{\mathrm{N}}^2 \delta^{AB} \frac{\nabla^h_B(n(H))}{n(H)} \partial_{y^A}, \qquad (9.106)$$
$$p = -\frac{n(n(H))}{n(H)}.$$

In the cut-and-paste constructions corresponding to constant-curvature spacetimes, the so-called *Penrose's junction conditions* impose the jump (2.170) in the coordinates across the shell. In the present case the matching embeddings $\phi^- = \imath$ and $\phi^+ = \imath \circ \varphi$ are given by

$$\phi^-(\lambda, y^B) = \left(\mathrm{U}_- = 0, \mathrm{V}_- = \lambda, x^A_- = y^A \right),$$
$$\phi^+(\lambda, y^B) = \left(\mathrm{U}_+ = 0, \mathrm{V}_+ = H(\lambda, y^B), x^A_+ = h^A(y^B) \right),$$

so the step function corresponding to Penrose's jump is $H(\lambda, y^A) = \lambda + \mathrm{H}(y^A)$, $\mathrm{H} \in \mathcal{F}(N)$. In order to recover such an H, one needs that there is no energy flux and no pressure on the shell. Indeed, imposing this in (9.106) and integrating for H yields $H(\lambda, y^A) = a\lambda + \mathrm{H}(y^A)$ with $\mathrm{H} \in \mathcal{F}(N)$ and $a \in \mathbb{R}$ being positive[1]. Thus, in this more general context with arbitrary cosmological constant, the Penrose's jump (2.170) still describes either purely gravitational waves (when ρ, j and p are all zero) or shells of null dust (when j and p vanish but $\rho \neq 0$), analogously to what happened in Section 7.3.3 for the Minkowski spacetime.

[1] In the present case, ζ^- point inwards with respect to $(\mathcal{M}^-, g^-)$, so ζ^+ points outwards (because the matching is possible). In these circumstances, condition (*ii*) in Theorem 9.1.1 imposes $\mathrm{sign}(z) = -\mathrm{sign}(\epsilon^+)\mathrm{sign}(\epsilon) = -(+1)(-1) = +1$. This, together with (9.72), means that $n(H) > 0$ necessarily, which in turn forces the constant a to be strictly positive.

10 CONCLUSIONS

This thesis consists of two different parts. In Chapters 3, 4, 5, 6, we have studied in deep detail the geometry of abstract null hypersurfaces by means of the formalism of hypersurface data introduced in Chapter 2. The second part of the thesis, corresponding to Chapters 7, 8, 9, is devoted to addressing the problem of matching two completely general spacetimes across a null hypersurface. In view of the structure of the thesis, we split the conclusions in two sections, one for each part.

10.1 formalism of hypersurface data

Let us start by summarizing the main results within Chapter 3. As we have seen, the tensor field "Lie derivative of a connection D along a vector Z", denoted by $\Sigma_Z \stackrel{\text{def}}{=} \pounds_Z D$, plays a fundamental role in the context of hypersurfaces equipped with a privileged vector field. In Section 3.1, we have obtained explicit expressions for Σ_Z whenever D is torsion-free and the manifold is endowed with a symmetric 2-covariant tensor (Lemma 3.1.6 and Corollary 3.1.7). Precisely these last results allow us to relate Σ_Z with the deformation tensor K^Z of Z in Chapter 5. As a particular case, we have considered hypersurface data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ and introduced the tensor "Lie derivative of the metric hypersurface connection", namely $\mathring{\Sigma} \stackrel{\text{def}}{=} \pounds_n \mathring{\nabla}$. We have computed $\mathring{\Sigma}$ explicitly in terms of the metric part of the data (Lemma 3.1.8) and we have connected $\mathring{\Sigma}$ with the tensor "Lie derivative of the hypersurface connection" $\overline{\Sigma} \stackrel{\text{def}}{=} \pounds_n \overline{\nabla}$ (Lemma 3.1.9).

Section 3.2 concentrates on hypersurface data in the null case. In Definitions 3.2.1 and 3.2.2 as well as in Lemma 3.2.3, we set up the notion of null (metric) hypersurface data. These concepts are based upon the fact that the tensor γ has one degenerate direction given by the data vector n, which must necessarily be everywhere non-zero. The data one-form $\boldsymbol{\ell}$ is also restricted to verify $\boldsymbol{\ell}(n) \neq 0$.

In the spirit of recovering the standard notions of null hypersurfaces that are well-known in the embedded context, at the abstract level we have introduced a notion of surface gravity κ_n associated to n. When a null hypersurface data set $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ happens to be embedded with embedding ϕ and rigging ζ, the scalar function κ_n defined in (2.44) coincides with the surface gravity of the unique null normal ν along $\phi(\mathcal{N})$ satisfying $g(\nu, \zeta) = 1$ (see (3.47)).

Since the tensor γ is degenerate, another key point is how to codify abstractly the intrinsic geometry of a null hypersurface (i.e. the information about the full metric tensor of the would-be ambient space). Two important results in this regard are that one can always select the metric tensors $\{\boldsymbol{\ell}, \ell^{(2)}\}$ at will as long as $\boldsymbol{\ell}(n) \neq 0$ everywhere on $\mathcal{N}$ (Lemma 3.2.9) and that two null metric hypersurface data sets $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$, $\{\mathcal{N}, \underline{\gamma}, \underline{\boldsymbol{\ell}}, \underline{\ell}^{(2)}\}$ are related by a gauge transformation if and only if $\gamma = \underline{\gamma}$ (Corollary 3.2.11). This implies that in the null case one can codify all the intrinsic geometric information of the hypersurface in the tensor γ up to gauge freedom.

We have also studied the geometry of any transverse submanifold S within $\mathcal{N}$. The most important result in this context is Lemma 3.2.19, where we obtain the relation of the Levi-Civita covariant derivative ∇^h of S with $\overset{\circ}{\nabla}$. Finally, we analyze the case when a null hypersurface data admits a cross-section S. In this context we prove that the one-form s, the scalar $\ell^{(2)}$ and the pull-back $\boldsymbol{\ell}_{\parallel}$ can be selected freely while κ_n can always be set to zero (Proposition 3.2.23, Lemma 3.2.24, Lemma 3.2.25).

The last part of Chapter 3 is Section 3.3, where we consider null hypersurface data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ equipped with an additional gauge-invariant vector field $\overline{\eta} \in \operatorname{Rad}\gamma$. We introduce a new gauge-invariant scalar function $\kappa \in \mathcal{F}(\mathcal{N})$ (Lemma 3.3.1) which in the embedded case happens to coincide with the surface gravity of $\eta \stackrel{\text{def}}{=} \phi_\star \overline{\eta}$ (at points where η is non-zero). This makes κ a smooth extension of the surface gravity of η to the points of $\phi(\mathcal{N})$ where η vanishes (see Proposition 3.3.2).

Prior to this thesis, the formalism of hypersurface data had already succeeded in determining various components of the ambient Riemann tensor at the abstract level. Following in this direction, in this work we have been able to codify information about the ambient Ricci tensor by introducing the constraint tensor $\mathcal{R}$, to which we devote Chapter 4. In Section 4.1, we provide the abstract definition of $\mathcal{R}$ purely in terms of (general) hypersurface data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ (cf. (4.15)). Such definition is built so that the tensor $\mathbf{Y}$ appears explicitly. This turns out to be advantageous in many situations, e.g. this fact allows us to obtain the generalized master equation in Chapter 6 (see below). The definition of $\mathcal{R}$ does not require any global topological assumption on $\mathcal{N}$. Moreover, it is fully covariant on $\mathcal{N}$ despite the fact that $\mathcal{N}$ is not equipped with a metric tensor. In the embedded case, the constraint tensor codifies a certain combination of components of the ambient Riemann and Ricci tensors (see Proposition 4.1.4) and, at null points, it coincides with the pull-back to $\mathcal{N}$ of the ambient Ricci tensor (cf. (4.17)). The rest of Chapter 4 concentrates on the null case. In particular, in Section 4.2 we compute the contractions $\mathcal{R}_{ab}n^a$, $\mathcal{R}_{ab}n^a n^b$ (Theorem 4.2.2). With the latter we obtain the equivalent to the Raychaudhuri equation (2.103) at the abstract level. We then study the pull-back of $\mathcal{R}$ to a transverse submanifold $S \subset \mathcal{N}$, obtaining the explicit relation between $\mathcal{R}_{AB}$ and the Ricci tensor of S (Theorem 4.2.3).

Chapter 4 concludes with Section 4.3, where we find several $\mathcal{G}_1$-invariant quantities on any transverse submanifold S of $\mathcal{N}$. From equation (4.30) and using the fact that the constraint tensor is gauge-invariant, we identify the tensors $\boldsymbol{\omega}_\parallel$, $\mathbf{P}_\parallel$ and $\mathbf{S}_\parallel$, which exhibit a simple gauge behaviour (see Lemma 4.3.1 and Corollary 4.3.3) and in fact are invariant under the action of the subgroup $\mathcal{G}_1$. This in turn allows us to write the pull-back of $\mathcal{R}$ to S in terms of $\mathcal{G}_1$-invariant objects (Proposition 4.3.2). The tensor $\mathbf{S}_\parallel$ codifies information on the first order variation of the tensor field $\mathbf{Y}$ along n and its features are worth further consideration. It is intrinsic to S and it codifies information on the curvature. Moreover, it plays a core role in the study of abstract Killing horizons of order one because it is related to the pull-back to S of the tensor field $\mathring{\Sigma} - n \otimes \pounds_n \mathbf{Y}$, which happens to vanish in this sort of horizons in the gauge where the symmetry generator coincides with the null generator n (which of course requires that the horizon does not contain fixed points).

Chapter 5 constitutes one of the main parts of the thesis. It is divided in four sections. In Section 5.1 we consider completely general hypersurface data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ embedded in a semi-Riemannian manifold $(\mathcal{M}, g)$ with embedding ϕ and rigging ζ. We let $(\mathcal{M}, g)$ be equipped with an additional vector field $y \in \Gamma(T\mathcal{M})$ (non-necessarily tangent to $\phi(\mathcal{N})$ at any of its points). In this context, we compute the Lie bracket $[y, \zeta]$ (for any extension of the rigging off $\phi(\mathcal{N})$) in terms of the deformation tensor $\mathcal{K}^y \stackrel{\text{def}}{=} \pounds_y g$ (Lemma 5.1.2). We then concentrate on the case when y is tangent to $\phi(\mathcal{N})$ everywhere therein, and find the explicit expression for the Lie derivative $\pounds_{\bar{y}}\mathbf{Y}$, $y \stackrel{\text{def}}{=} \phi_\star \bar{y}$ in terms of the data, the components of $\mathcal{K}^y$ on $\phi(\mathcal{N})$ and the pull-back $\phi^\star(\pounds_\zeta \mathcal{K}^y)$ (Proposition 5.1.3).

In Section 5.2 we focus on the case when $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ defines null embedded hypersurface data. We also restrict $y, \bar{y}$ to be null and tangent to the hypersurface, and denote them as η, $\bar{\eta}$ respectively. In such circumstances, we prove that the components $\mathcal{K}^\eta(\zeta, \nu)$ (where, as usual, $\nu \in \Gamma(T\mathcal{M})|_{\phi(\mathcal{N})}$ is the unique null normal verifying $g(\zeta, \nu) = 1$) are gauge-invariant (Lemma 5.2.1).

Section 5.3 is devoted to the geometric properties of the tensor field $\Sigma_\eta \stackrel{\text{def}}{=} \pounds_\eta \nabla$. Its relation with the deformation tensor K^η of η is given by (5.30). We start with Lemma 5.3.1, where we obtain the pull-back $\phi^\star(W(\Sigma_\eta))$ in terms of {ק, ש, i, ד} (cf. (5.15)-(5.16)), the null data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ and the function α defined by $\overline{\eta} = \alpha n$. The analysis of $\phi^\star(W(\Sigma_\eta))$ reveals the new tensor ח$^\eta$, given by a certain combination of {ק, ש, i, ד} (cf. (5.39)). It turns out that ח$^\eta$, ח$^\eta(n, \cdot)$ exhibit a simple gauge behaviour (Lemma 5.3.2). In particular, when $\mathbf{U} = 0$ the tensor ח$^\eta$ simply rescales under gauge transformations, while ח$^\eta(n, \cdot)$ is gauge-invariant. Another important result within Section 5.3 is Lemma 5.3.5, where we find explicit expressions for the vector field $\Sigma_\eta(\phi_\star Y, \phi_\star Z)$, $\forall Y, Z \in \Gamma(T\mathcal{N})$. This becomes essential in the study of horizons at a purely abstract level, as we shall see next.

Chapter 5 concludes with Section 5.4, where we present the notions of abstract Killing horizons of order zero and one (see Definitions 5.4.1 and 5.4.5). The main advantages of AKH0/1s are firstly that they allow us to study horizons without the necessity of them being embedded on any ambient space and secondly that they do not require any global topological assumption whatsoever. The notions of AKH0/1s generalize those of non-expanding, isolated and Killing horizons (of order zero and one). At the embedded level, we have also introduced the concepts of Killing horizon of order zero and ½. The former corresponds to an embedded null hypersurface admitting a null tangent vector field η such that *all* components of the deformation tensor vanish on the hypersurface. In the latter, in addition, the tangent-tangent components of $\pounds_\zeta \mathrm{K}^\eta$ are zero. As anticipated above, the results of Section 5.3 are essential in this context because it occurs that an embedded AKH1 verifies that $\Sigma_\eta(\phi_\star X, \phi_\star W) = 0$, $\forall X, W \in \Gamma(T\mathcal{N})$ and ח$^\eta = 0$ (Lemma 5.4.7). The tensors Σ_η and ח$^\eta$ therefore play a crucial role in the understanding these sort of abstract horizons.

Chapter 6 is another key part of the thesis. For any null hypersurface data $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}\}$ embedded in a semi-Riemannian manifold with embedding ϕ and rigging ζ, and assuming that the data admits a gauge-invariant vector field $\overline{\eta} \in \mathrm{Rad}\gamma$, we derive the generalized master equation as well as its contractions with the data vector field n (see Theorem 6.1.1). The generalized master equation (6.1) is a fully covariant identity involving hypesurface data, derivatives of the proportionality function α between $\overline{\eta}$ and n, the curvature tensors R and $\overset{\circ}{R}$, the surface gravity κ of $\overline{\eta}$ and the ambient objects {ק, ח$^\eta$}, where η is *any* extension of $\phi_\star\overline{\eta}$ off $\phi(\mathcal{N})$ (in particular, the deformation tensor K^η has not been restricted in any sense). This identity holds everywhere on the hypersurface (even at points where $\overline{\eta}$ vanishes), is valid in any gauge and it does not require any topological assumption besides the existence of an everywhere non-zero, smooth vector field

n. The contractions of (6.1) with n (namely (6.2)-(6.3)) are of interest as well. For a completely general null hypersurface admitting an additional vector field, these identities allow us to identify the necessary and sufficient conditions for the surface gravity κ to remain constant along the null generators of N (Corollary 6.1.3) and everywhere on N (Corollary 6.1.4). The behaviour of the surface gravity κ turns out to be ruled by the tensor Σ_η.

In Section 6.2 we study the generalized master equation in the case when the deformation tensor K^η is proportional to the metric, i.e. $K^\eta = 2\chi g$ for a suitable function χ. The fact that the function $\chi\alpha^{-1}$ must be everywhere regular on N imposes that χ has a zero of higher order than α at least at those points where α vanishes (Proposition 6.2.1), otherwise there exist singularities in the manifold where the data is embedded (Remark 6.2.3). In particular, this fact allows us to prove that a smooth homothetic Killing horizon cannot admit fixed points (Remark 6.2.4). Particularizing (6.1)-(6.3) to the case $K^\eta = 2\chi g$, we obtain the identity (6.48) for the surface gravity κ, in which the components $R(n, n)$ of the constraint tensor, the proportionality function α and the function χ are involved (Proposition 6.2.7). Specifically, when χ vanishes no-where on N , κ is given by (6.49) in terms of the quantities χ, $\overline{\eta}(\chi)$ and $R(\overline{\eta}, \overline{\eta})$. We also conclude that the surface gravity of a homothetic vector field is everywhere constant if and only if $R(\overline{\eta}, \overline{\eta})$ is constant therein.

In Section 6.3 we address the case of abstract Killing horizons of order zero and one. In such context, we have obtained the necessary and sufficient conditions under which κ is everywhere constant on N (Proposition 6.3.1). We have also proven that when N admits a cross-section, if $d\kappa$ is non-zero at any point of N then N cannot be geodesically complete (Proposition 6.3.2).

Section 6.4 is devoted to computing the pull-back of the generalized master equation to any transverse submanifold S within N (see Lemma 6.4.1 and Theorem 6.4.2). The main advantage of our approach is that S does not need to be a cross-section (in fact such section does not even need to exist). From this identity (namely (6.61)), we recover the near horizon equation for isolated horizons (cf. (2.129)) as well as the master equation for multiple Killing horizons (cf. (2.153)) as particular cases (see Remarks 6.4.3 and 6.4.4).

Chapter 6 concludes with a section on vacuum degenerate Killing horizons of order one. By identifying points along the same null generator, we introduce a quotient structure and obtain a near horizon equation in the quotient space as long as this quotient has a manifold structure (Proposition 6.5.1). The fact that we are able to find the near horizon equation in a more general context opens up the possibility to exploit all the results concerning the possible solutions of these equation available the literature (see e.g. [100]). These results should allow one to address the problem of classifying near horizon geometries without assuming that the manifold has a product structure. We intend to look into this problem in the future.

10.2 matching of spacetimes

We start addressing the problem of matching of spacetimes in Chapter 7. We let $(\mathcal{M}^{\pm}, g^{\pm})$ be two spacetimes with boundaries $\mathcal{N}^{\pm}$. Our approach in this chapter is based on considering $\mathcal{N}^{\pm}$ as embedded in $(\mathcal{M}^{\pm}, g^{\pm})$ rather than at a purely abstract level (this is actually done in Chapter 9, see below). The main results in this chapter can be found in Section 7.2 and 7.3. In the former, we discuss briefly the matching problem for boundaries of arbitrary causal character. In particular, we show that the matching requires the existence of a metric hypersurface data set which is embeddable in both spacetimes, and that the gravitational/matter-energy content of the shell is ruled by the jump in the extrinsic part of the data, namely the tensors $\mathbf{Y}^{\pm}$. We also prove that all the information about the matching is encoded in a diffeomorphism between the boundaries, the so-called matching map Φ.

Section 7.3 constitutes one core part of the thesis. It is entirely devoted to study the matching problem across null hypersurfaces. Our analysis is completely general except by the fact that the boundaries are assumed to admit a foliation by a family of spacelike cross-sections. The corresponding conclusions and results are as follows.

We begin by rewriting the Standard Junction Conditions in terms of (the components of) two basis of vector fields that are to be identified in the matching process. The outcome is that the necessary and sufficient conditions for the matching to be possible are (7.31)-(7.34) together with the requirement that one matching rigging points inwards whereas the other points outwards with respect to their corresponding spacetimes.

The solvability of (7.31) constitutes the key problem for the existence of the matching. This condition forces any section on the minus side and its image through Φ to be isometric. Moreover, given a point $p^{-} \in \mathcal{N}^{-}$, its image point $p^{+} = \Phi(p^{-}) \in \mathcal{N}^{+}$ and the null generators $\sigma^{\pm} \subset \mathcal{N}^{\pm}$ containing $p^{\pm}$, such isometry must be universal in the sense of being the same for all points along $\sigma^{\pm}$. Condition (7.32), on the other hand, forces null generators on one side to be mapped to null generators on the other side. This, together with the fact that there must exist a one-to-one correspondence between points of the boundaries, means that there must exist a diffeomorphism Ψ between the sets of null generators of both sides. Finally, conditions (7.33)-(7.34) determine, once we have selected one of the riggings (say ζ^-), the tangent part of the matching rigging ζ^+.

Another important result is that the whole information about the matching is encoded in a scalar function H and in the diffeomorphism Ψ. The function H, called step function, must necessarily be monotonic along the generators of the plus side. The name step function is justified by the fact that H accounts for the jump along the tangent null direction of the matching hypersurface when crossing from one side to the other. It is precisely the step function that connects the matching formalism with the cut-and-paste constructions, where there exists a jump in the null coordinate when crossing the shell (recall (2.170)).

Of course, in general it will be impossible to find a pair $\{H, \Psi\}$ verifying the junction conditions, and hence the matching will be infeasible. However, we have seen that sometimes more than one matching can be performed. This occurs, for instance, when the second fundamental forms $\mathbf{K}^{k}_{\pm}$ with respect to any two null generators $k^{\pm}$ of $\mathcal{N}^{\pm}$ vanish simultaneously. In these circumstances, the matching is not only feasible but it even allows for an infinite number of possibilities, since the step function cannot be fixed or restricted in any way (cf. (7.54)).

Chapter 7 concludes with Sections 7.3.2 and 7.3.3. In the former, we obtain explicit expressions for the matching rigging ζ^+ (Corollary 7.3.4), the tensor fields $\mathbf{Y}^{\pm}$ and the energy-momentum tensor $\boldsymbol{\tau}$ in terms of known tensor fields codifying the geometry of the boundaries and the pair $\{H, \Psi\}$ (Proposition 7.3.7). We emphasize that throughout this process the only assumption we have made is that the boundaries have product topologies. Apart from this, all the results are valid for the matching of any two given spacetimes with null boundaries. We also provide the first geometric notions of energy density, pressure and energy flux of any null thin shell (Remark 7.3.9). These definitions are valid for any topology of the boundaries. The definitions of energy density, pressure and energy flux are normally presented in the literature (see e.g. [128]) in a specific gauge and in a concrete local basis. Instead, our definitions are fully covariant and valid in any gauge. In Section 7.3.3 we study the particular case of the matching of two regions of the spacetime of Minkowski across a null hyperplane. We find that Penrose's jump (2.170) corresponds to either shells of null dust or to purely gravitational waves propagating in the spacetime of Minkowski. We also obtain the step function corresponding to the most general shell that can be generated by a matching of this type. This, in turn, allows us to analyze the effect of the pressure, and find that a positive (resp. negative) pressure is responsible for an effect of self-compression (resp. self-stretching) of points on one of the boundaries. The last part of the chapter is devoted to building a coordinate system in which the metric of the spacetime resulting from the most general matching of two regions of Minkowski takes a C^0 form (Lemma 7.3.10). In fact, as a particular case we recover the Lipschitz-continuous metric (2.169) corresponding to the four-dimensional Penrose's cut-and-paste construction (see e.g. [5]). This result is part of an ongoing project with Argam Ohanyan and Roland Steinbauer at the University of Vienna. The purpose of this research collaboration is to find the distributional form of the metric of the spacetime resulting from the most general matching of two regions of Minkowski across a null hyperplane (see Section 7.3.3).

In Chapter 8 we particularize the results above to the case when the boundaries $\mathcal{N}^{\pm}$ are embedded abstract Killing horizons of order zero with (spacetime) symmetry generators $\eta^{\pm}$. As we did in Chapter 7, here we also assume that $\mathcal{N}^{\pm}$ have product topology. We suppose further that the surface gravities $\kappa^{\pm}$ of $\eta^{\pm}$ are constant everywhere on $\mathcal{N}^{\pm}$.

In these circumstances the boundaries are totally geodesic, which in principle would mean that the step function H cannot be determined. However, we are interested in the case when the spacetime resulting from the matching preserves the symmetry associated to $\eta^{\pm}$, which happens whenever the matching identifies the vectors $\eta^{\pm}$. This introduces new conditions on the matching, namely that the fixed points sets $S^{\pm}$ of $\eta^{\pm}$ are forced to be mapped to each other and that both $\eta^{\pm}$ must be either future or past. Moreover, the identification of $\eta^{\pm}$ restricts the step function and by extension the set of feasible matchings.

In Sections 8.1 to 8.3 we obtain all possible matchings of this type, providing the explicit expression of the step function for each of them and analyzing the nature of the remaining matching freedom. The corresponding results are collected in Theorem 8.3.1. A particularly interesting result is that a geodesically complete degenerate boundary can be matched with a geodesically incomplete non-degenerate boundary provided that that neither of them contain fixed points. This sort of matching is characterized by having a non-zero pressure that is responsible for the necessary self-compression of points in the non-degenerate boundary.

The condition that $S^{\pm}$ must be mapped to each other may seem superfluous but in fact it is not. This is so because, according to Lemma 5.4.11, the causal character of $S^{\pm}$ depends on the properties of $\eta^{\pm}$. Since throughout the matching process null generators from one side are sent to null generators on the other side, even if all matching conditions are fulfilled it may occur that there is no possible matching in which $\eta^{\pm}$ are identified.

In Section 8.4 we analyze the particular case when the boundaries are non-degenerate (full) Killing horizons containing a bifurcation surface. We again consider matchings in which the Killing vector fields $\eta^{\pm}$ are identified. As before, the surface gravities $\kappa^{\pm}$ of $\eta^{\pm}$ are assumed to be constant. These types of matchings are of physical interest because they cover all possible cases of spacetimes obtained from the matching of two stationary black holes glued across their event horizons provided they have non-zero constant temperature.

In these circumstances, the matching freedom is encoded in a positive function β defined at the bifurcation surface and extended as a constant along the generators. In Theorem 8.4.1 we find explicit expressions for the tensor fields $\mathbf{Y}^{\pm}$, τ of any null thin shell of this type. We show that the gravitational/matter-energy content of the shell depends only on the function β, on the intrinsic and extrinsic geometric properties of the bifurcation surfaces $S^{\pm}$, on the Ricci tensor of $S^{\pm}$ and on the pull-back of the constraint tensor to $S^{\pm}$.

In this sort of shells, the pressure is identically zero. Moreover, the energy density either vanishes everywhere or unavoidably changes its sign at the bifurcation surface. This behaviour is striking and suggests that the change in the causality of the Killing fields that takes place at the bifurcation surfaces affects the energy density. It is even more puzzling that the energy current j is constant along the generators, which in particular makes it insensitive to the change of sign on the energy density. We emphasize, however, that the behaviour of the pressure, energy density and energy flux is fully compatible with the shell field equations, as we proved in Section 8.4.1.

The last part of Chapter 8 is Section 8.5, where we particularize the results from Section 8.4 to the case of spherical, plane or hyperbolic symmetric spacetimes. We derive explicit expressions for the gravitational/matter-energy content of the most general shell that can be constructed by matching two spacetimes of this type. We then include two examples, namely the matchings of two regions of the Schwarzschild and Schwarzschild-de Sitter spacetimes.

The thesis concludes with Chapter 9, devoted to the problem of matching as well. There are two main differences with respect to our approach in Chapters 7 and 8. First, the matching problem is addressed from a completely abstract viewpoint, i.e. we want to provide a fully abstract formulation of the matching. The final aim is to be able to describe matchings in a detached way from the actual two spacetimes involved. The second difference is that in Chapter 9 we make no assumptions whatsoever on the boundaries (in particular, there are no topological restrictions). In this sense, the results within Chapter 9 are completely general.

The chapter consists of three parts. In Section 9.1, we start by setting up an abstract formulation of the matching conditions for boundaries of arbitrary causal character (Theorem 9.1.1). Given two hypersurface data sets $\mathcal{D} \stackrel{\text{def}}{=} \{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathbf{Y}^-\}$, $\widehat{\mathcal{D}} \stackrel{\text{def}}{=} \{\widehat{\mathcal{N}}, \widehat{\gamma}, \widehat{\boldsymbol{\ell}}, \widehat{\ell}^{(2)}, \widehat{\mathbf{Y}}^+\}$, we find that the matching of $\mathcal{D}$, $\widehat{\mathcal{D}}$ is possible provided that there exists a diffeomorphism $\varphi : \mathcal{N} \longrightarrow \widehat{\mathcal{N}}$ and a gauge group element $\mathcal{G}_{(z,V)}$ such that the metric data $\{\mathcal{N}, \varphi^\star\widehat{\gamma}, \varphi^\star\widehat{\boldsymbol{\ell}}, \varphi^\star\widehat{\ell}^{(2)}\}$ transforms into $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ under the action of $\mathcal{G}_{(z,V)}$ (see (9.2)). The condition on the orientations of the riggings translates, at the abstract level, into a restriction upon the sign of the gauge parameter z.

Theorem 9.1.1 is of interest because it applies for *any possible thin shell*. It is fully abstract in nature given that the matching conditions are written solely as a restriction over two data sets and a requirement on the sign of a gauge parameter. The main advantage of Theorem 9.1.1 is that it allows us to split the matching problem into two different levels. In the first one, we can consider a thin shell abstractly and prescribe the gravitational and matter-energy content at will. Then, on a second stage, we can address the problem of building a spacetime $(\mathcal{M}, g)$ containing such shell. This can be of use e.g. to find examples of spacetimes containing a certain type of shell.

In Section 9.1.1, we particularize the construction of Theorem 9.1.1 to the null case. As we already know, the intrinsic geometry $\{\mathcal{N}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ of an abstract null hypersurface is entirely codified by the tensor γ (recall that $\{\boldsymbol{\ell}, \ell^{(2)}\}$ can be selected at will). In the null case the matching conditions therefore reduce to $\gamma = \varphi^\star\widehat{\gamma}$ (cf. (9.11)) (together with the restriction on the sign of z). Assuming that they are satisfied for a diffeomorphism φ, we find the gauge parameters $\{z, V\}$ explicitly in terms of φ, $\mathcal{D}$ and $\widehat{\mathcal{D}}$ (cf. (9.12)). Moreover, by embedding $\mathcal{D}$, $\widehat{\mathcal{D}}$ and making a choice of rigging on the minus side, we obtain the matching rigging ζ^+ also in terms of φ, $\mathcal{D}$ and $\widehat{\mathcal{D}}$ (Lemma 9.1.6). The restriction (7.54) on the second fundamental forms can be generalized in this context (cf. (9.20)). In Theorem 9.1.9 and Lemma 9.1.13 we derive explicit expressions for the gravitational and matter-energy content of any null thin shell. Since all these results depend exclusively on $\mathcal{D}$, $\widehat{\mathcal{D}}$ and φ (and the data sets $\mathcal{D}$, $\widehat{\mathcal{D}}$ are known a priori), it follows that all the information of the matching is encoded in φ.

In Section 9.1 we also study the pressure of any null shell. In Lemma 9.1.14 we show that the pressure is given by the difference of the surface gravities of the null generators n and $\varphi_\star n$. This, in particular, allows us to confirm that also in this more general context the pressure accounts for an effect of self-compression/self-stretching of points. Section 9.1 concludes with a discussion on multiple matchings. Assuming that $\mathcal{D}$, $\widehat{\mathcal{D}}$ can be matched in more than one way and that the information about one of the matchings is fully known (concretely the diffeomorphism and the contents of the shell), in (9.52)-(9.58) we find the matter-energy content of any other possible matching in terms of its corresponding diffeomorphism.

The last two parts of Chapter 9 are Sections 9.2 and 9.3. In the former, we examine the case when the null boundaries can be foliated by a family of diffeomorphic spacelike cross-sections. We recover the results from Chapter 7, in particular the existence of the step function H and the expressions for the jump $[\mathbf{Y}^{\pm}]$ of the shell (Theorem 9.2.2). In Section 9.3 we apply the abstract formalism to study the matching of two regions of the spacetime of (anti-)de Sitter across a totally geodesic null hypersurface. In particular, we compute the matter-energy content of the most general null thin shell that can be generated in a matching of this type. We prove that the pressure takes the same form as in the case of Minkowski (i.e. that it is given by second derivatives of the step function). Moreover, we find that Penrose's jump (2.170) describes either shells of null dust or purely gravitational waves, with which we connect our results with the cut-and-paste constructions of [88], [91], [92] [5], [6]. In the limit of vanishing cosmological constant, we recover the results of Section 7.3.3.

A

GENERAL IDENTITIES FOR CURVATURE TENSORS

We start with the proof of two identities for the curvature tensor of any completely general torsion-free connection.

Lemma A.0.1. *Let* V *be a smooth manifold endowed with a torsion-free connection* D*,* Ψ_{ab} *a symmetric two-covariant tensor field and* $R^d{}_{abc}$ *the curvature tensor of* D*. Define* $\Delta_{abcd} := -(D_c D_d \Psi_{ab} - D_d D_c \Psi_{ab})$*. Then,*

$$\Psi_{af} R^f{}_{bcd} + \Psi_{bf} R^f{}_{acd} = \Delta_{abcd}, \tag{A.1}$$

$$\Psi_{af} R^f{}_{bcd} - \Psi_{cf} R^f{}_{dab} = \frac{1}{2}\left(\Delta_{abcd} + \Delta_{acbd} + \Delta_{adcb} + \Delta_{bcda} + \Delta_{bdac} + \Delta_{cdba}\right). \tag{A.2}$$

Proof. The Ricci identity for Ψ_{ab}, i.e. $\Psi_{af} R^f{}_{bcd} + \Psi_{bf} R^f{}_{acd} = D_d D_c \Psi_{ab} - D_c D_d \Psi_{ab}$, immediately yields (A.1). To prove (A.2), we use indices $\{a_1, a_2, a_3, a_4\}$ and write the Bianchi identity four times:

$$\Psi_{a_4 f}\left(R^f{}_{a_1 a_2 a_3} + R^f{}_{a_2 a_3 a_1} + R^f{}_{a_3 a_1 a_2}\right) = 0, \qquad \Psi_{a_1 f}\left(R^f{}_{a_2 a_3 a_4} + R^f{}_{a_3 a_4 a_2} + R^f{}_{a_4 a_2 a_3}\right) = 0,$$

$$\Psi_{a_2 f}\left(R^f{}_{a_3 a_4 a_1} + R^f{}_{a_4 a_1 a_3} + R^f{}_{a_1 a_3 a_4}\right) = 0, \qquad \Psi_{a_3 f}\left(R^f{}_{a_4 a_1 a_2} + R^f{}_{a_1 a_2 a_4} + R^f{}_{a_2 a_4 a_1}\right) = 0.$$

Adding the first two and subtracting the second two gives

$$\begin{aligned} 0 = {} & 2\Psi_{a_1 f} R^f{}_{a_2 a_3 a_4} - 2\Psi_{a_3 f} R^f{}_{a_4 a_1 a_2} - \Delta_{a_1 a_2 a_3 a_4} + \Delta_{a_3 a_4 a_1 a_2} \\ & - \Delta_{a_1 a_3 a_2 a_4} + \Delta_{a_1 a_4 a_2 a_3} + \Delta_{a_4 a_2 a_3 a_1} - \Delta_{a_3 a_2 a_4 a_1} \end{aligned}$$

where (A.1) has been used to swap indices several times. Since by construction Δ_{abcd} is symmetric in the first two indices and antisymmetric in the last two, (A.2) follows immediately after renaming $a_1 = a$, $a_2 = b$, $a_3 = c$, $a_4 = d$. □

B

A GENERALIZED GAUSS IDENTITY

In this section, we obtain a generalized form of the well-known Gauss identity (see e.g. [106]). On any semi-Riemannian manifold, the Gauss identity relates the curvature tensor of the Levi-Civita connection along tangential directions of a non-degenerate hypersurface with the curvature tensor of the induced metric and the second fundamental form. It has been generalized in a number of directions, e.g. when dealing with induced connections associated to a transversal (rigging) vector [64]. Here we find an identity where the connection of the space and of the hypersurface are completely general, except for the condition that they are both torsion-free.

Our primary interest will be in applying this identity when the space defines null hypersurface data and the codimension one submanifold is non-degenerate. However, the identity is far more general and may be of independent value. We remark that the tensor γ in the statement of the lemma is completely arbitrary, so neither γ nor its pullback h to the submanifold are assumed to be non-degenerate.

Theorem B.0.1. *Consider a smooth manifold* $\mathcal{N}$ *endowed with a symmetric* 2*-covariant tensor field* γ *and a torsion-free connection* ∇*. Let* S *be an embedded hypersurface in* $\mathcal{N}$ *and assume that* S *is equipped with another torsion-free connection* D*. Define* $h \stackrel{\text{def}}{=} \psi^{\star}\gamma$ *(where* $\psi : S \longrightarrow \mathcal{N}$ *is the corresponding embedding) and the tensor* P *by means of*

$$\nabla_X Y = D_X Y + \mathrm{P}(X,Y) \qquad \forall X, Y \in \Gamma(TS),$$

and assume that there exists a transversal vector field n *along* S *satisfying* $\gamma(n, X) = 0$ *for all* X *tangent to* S*. Define the* 2*-covariant tensor* Ω *and the* 1*-contravariant,* 2*-covariant tensor* A *on* S *by decomposing* $\mathrm{P}(X,Y)$ *in tangential and transverse parts as follows:*

$$\mathrm{P}(X,Y) = A(X,Y) + \Omega(X,Y)n. \tag{B.1}$$

Then, for all $X, Y, Z, W \in \Gamma(TS)$ *it holds*

$$
\begin{aligned}
\gamma(W, R^{\nabla}(X,Y)Z) = {} & h(W, R^{D}(X,Y)Z) + (D_X A_h)(W,Y,Z) - (D_Y A_h)(W,X,Z) \\
& + h(A(Y,W), A(X,Z)) - h(A(X,W), A(Y,Z)) \\
& - (\nabla_X \gamma)(W, \mathrm{P}(Y,Z)) + (\nabla_Y \gamma)(W, \mathrm{P}(X,Z)) \\
& + \gamma(n,n)\left(\Omega(Y,W)\Omega(X,Z) - \Omega(X,W)\Omega(Y,Z)\right), \qquad \text{(B.2)}
\end{aligned}
$$

where $A_h(W,X,Z) \stackrel{\text{def}}{=} h(W, A(X,Z))$.

Proof. Since the connections are torsion-free, the tensors $\mathrm{P}(X,Y)$, $A(X,Y)$ and $\Omega(X,Y)$ are all symmetric in X, Y. First, we find

$$
\nabla_X \nabla_Y Z = \nabla_X (D_Y Z + \mathrm{P}(Y,Z)) = D_X D_Y Z + \mathrm{P}(X, D_Y Z) + \nabla_X(\mathrm{P}(Y,Z)), \qquad \text{(B.3)}
$$

$$
\nabla_{[X,Y]} Z = D_{[X,Y]} Z + \mathrm{P}(D_X Y, Z) - \mathrm{P}(D_X Y, Z). \qquad \text{(B.4)}
$$

The quantity $(D_X\mathrm{P})(Y,Z) \stackrel{\text{def}}{=} \nabla_X(\mathrm{P}(Y,Z)) - \mathrm{P}(D_X Y, Z) - \mathrm{P}(Y, D_X Z)$ is tensorial in X, Y, Z, and takes values in the space of vector fields (not necessarily tangent) along $\psi(S)$. Inserting (B.3)-(B.4) into the definition of the curvature tensor (2.3) yields

$$
R^{\nabla}(X,Y)Z = R^{D}(X,Y)Z + (D_X\mathrm{P})(Y,Z) - (D_Y\mathrm{P})(X,Z).
$$

We now insert the decomposition (B.1). Using that $\gamma(n, W) = 0$ and $h(X,Y) \stackrel{\text{def}}{=} \gamma(X,Y)$ gives

$$
\begin{aligned}
\gamma(\nabla_X W, \mathrm{P}(Y,Z)) &= \gamma(\nabla_X W, A(Y,Z) + \Omega(Y,Z)n) \\
&= h(D_X W, A(Y,Z)) + h(A(X,W), A(Y,Z)) \\
&\quad + \gamma(n,n)\Omega(X,W)\Omega(Y,Z), \qquad \text{(B.5)}
\end{aligned}
$$

from where it follows

$$
\begin{aligned}
\gamma(W, (D_X\mathrm{P})(Y,Z)) &= \gamma(W, \nabla_X(\mathrm{P}(Y,Z))) - \gamma(W, \mathrm{P}(D_X Y, Z)) - \gamma(W, \mathrm{P}(Y, D_X Z)) \\
&= \nabla_X\left(\gamma(W, \mathrm{P}(Y,Z))\right) - (\nabla_X \gamma)(W, \mathrm{P}(Y,Z)) - \gamma(\nabla_X W, \mathrm{P}(Y,Z)) \\
&\quad - \gamma\left(W, \mathrm{P}(D_X Y, Z)\right) - \gamma(W, \mathrm{P}(Y, D_X Z)) \\
&\stackrel{\text{(B.5)}}{=} D_X\, h(W, A(Y,Z)) - (\nabla_X \gamma)(W, \mathrm{P}(Y,Z)) - h(D_X W, A(Y,Z)) \\
&\quad - h(A(X,W), A(Y,Z)) - \gamma(n,n)\Omega(X,W)\Omega(Y,Z) \\
&\quad - h(W, A(D_X Y, Z)) - h(W, A(Y, D_X Z)) \\
&= (D_X h)(W, A(Y,Z)) - (\nabla_X \gamma)(W, \mathrm{P}(Y,Z)) + h(W, (D_X A)(Y,Z))) \\
&\quad - h(A(X,W), A(Y,Z)) - \gamma(n,n)\Omega(X,W)\Omega(Y,Z).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\gamma(W, R^{\nabla}(X, Y)Z) = {} & \hat{h}(W, R^{\hat{D}}(X, Y)Z) + (\hat{D}_X \hat{h})(W, A(Y, Z)) - (\nabla_X \gamma)(W, \mathrm{P}(Y, Z)) \\
& + \hat{h}(W, (\hat{D}_X A)(Y, Z))) - \hat{h}(A(X, W), A(Y, Z)) - (\hat{D}_Y \hat{h})(W, A(X, Z)) \\
& + (\nabla_Y \gamma)(W, \mathrm{P}(X, Z)) - \hat{h}(W, (\hat{D}_Y A)(X, Z))) + \hat{h}(A(Y, W), A(X, Z)) \\
& + \gamma(n, n)\,(\Omega(Y, W)\Omega(X, Z) - \Omega(X, W)\Omega(Y, Z))\,. \qquad \text{(B.6)}
\end{aligned}
$$

By virtue of the definition of $A_{\hat{h}}$, it holds

$$
(\hat{D}_Y A_{\hat{h}})(W, X, Z) = (\hat{D}_Y \hat{h})(W, A(X, Z)) + \hat{h}(W, \hat{D}_Y A(X, Z)).
$$

This allows us to rewrite (B.6) as (B.2). □

In abstract index notation the generalized Gauss identity (B.2) takes the form

$$
\begin{aligned}
v^a_A \gamma_{af} (R^{\nabla})^f{}_{bcd} v^b_B v^c_C v^d_D = {} & \hat{h}_{FA} (R^{\hat{D}})^F{}_{BCD} + \hat{D}_C A_{\hat{h}ABD} - \hat{D}_D A_{\hat{h}ABC} \\
& + \hat{h}_{FL} A^L{}_{AD} A^F{}_{BC} - \hat{h}_{FL} A^L{}_{AC} A^F{}_{BD} + v^d_D (\nabla_d \gamma_{af}) v^a_A \mathrm{P}^f{}_{BC} \\
& - v^c_C (\nabla_c \gamma_{af}) v^a_A \mathrm{P}^f{}_{BD} + \gamma(n, n)\,(\Omega_{AD}\Omega_{BC} - \Omega_{AC}\Omega_{BD})\,, \qquad \text{(B.7)}
\end{aligned}
$$

where the vectors v^a_A are the push forward with ψ of any basis vectors $\{\hat{v}_A\}$ in S.

C

GAUGE BEHAVIOUR OF $\boldsymbol{\mathcal{S}}_{\parallel}$

In this appendix we provide an alternative proof of Corollary 4.3.3. This serves as a highly non-trivial test of the validity of the expressions obtained in Section 4.3.

Since for any gauge group element $\mathcal{G}_{(z,V)}$ it holds (cf. Proposition 2.2.10)

$$\mathcal{G}_{(z,V)} = \mathcal{G}_{(1,V)} \circ \mathcal{G}_{(z,0)}, \tag{C.1}$$

it suffices to prove that $\boldsymbol{\mathcal{S}}_{\parallel}$ behaves as claimed in Corollary 4.3.3 for the gauge parameters $(z, 0)$ and that it is gauge invariant under the subgroup $\mathcal{G}_1$. We establish these two facts consecutively

Lemma C.0.1. *Assume Setup 3.2.15 and let $z \in \mathcal{F}^{\star}(\mathcal{N})$, $\hat{z} \stackrel{\text{def}}{=} z|_S$ and $\hat{z}_n \stackrel{\text{def}}{=} n(z)|_S$. Then*

$$\mathcal{G}_{(z,0)}(\boldsymbol{\mathcal{S}}_{\parallel})_{AB} = S_{AB} + \frac{1}{\hat{z}}\nabla^h_A\nabla^h_B\hat{z} - \frac{2}{\hat{z}^2}\nabla^h_A \hat{z}\nabla^h_B\hat{z} + \frac{2}{\hat{z}}\omega_{(A}\nabla^h_{B)}\hat{z} + \frac{\hat{z}_n}{\hat{z}}P_{AB}. \tag{C.2}$$

Proof. As usual, we denote gauge-transformed quantities with a prime. We start by computing the Lie derivative $\pounds_{n'}\mathbf{Y}'$. Using (2.34) and (2.44) we obtain

$$\pounds_{n'}\mathbf{Y}' = \pounds_{\frac{1}{z}n}\mathbf{Y}' = \frac{1}{z}\pounds_n\mathbf{Y}' - \frac{2dz}{z^2}\otimes_s \mathbf{Y}'(n,\cdot) = \frac{1}{z}\pounds_n\mathbf{Y}' - \frac{2dz}{z}\otimes_s \boldsymbol{r}'.$$

By Lemma 3.2.7, we know that $\boldsymbol{r}'$ reads

$$\boldsymbol{r}' = \boldsymbol{r} + \frac{1}{2z}dz + \frac{n(z)}{2z}\boldsymbol{\ell}.$$

Since under the action of $\mathcal{G}_{(z,0)}$ the tensor $\mathbf{Y}$ transforms as $\mathbf{Y}' = z\mathbf{Y} + \boldsymbol{\ell}\otimes_s dz$ (cf. (2.40)) and $\pounds_n\boldsymbol{\ell} = 2\boldsymbol{s}$ (cf. (3.43)), we conclude that

$$\pounds_{n'}\mathbf{Y}' = \pounds_n\mathbf{Y} + \frac{n(z)}{z}\mathbf{Y} + \frac{1}{z}\boldsymbol{\ell}\otimes_s d\left(n(z)\right) + \frac{1}{z}dz\otimes_s\left(2\boldsymbol{s} - 2\boldsymbol{r} - \frac{dz}{z} - \frac{n(z)}{z}\boldsymbol{\ell}\right).$$

A similar, but much simpler calculation gives

$$\pounds_{n'}\mathbf{U}' = \frac{1}{z^2}\left(\pounds_n\mathbf{U} - \frac{n(z)}{z}\mathbf{U}\right),$$

while the quantities $\ell^{(2)}$, $\ell^{(2)}_\parallel$, $\boldsymbol{\ell}_\parallel$ and $\mathbf{U}_\parallel$ simply scale as (cf. (4.33)-(4.34))

$$\ell^{(2)\prime} = z^2\ell^{(2)}, \qquad \ell^{(2)\prime}_\parallel = \hat{z}^2\ell^{(2)}_\parallel, \qquad \boldsymbol{\ell}'_\parallel = \hat{z}\boldsymbol{\ell}_\parallel, \qquad \mathbf{U}'_\parallel = \frac{1}{\hat{z}}\mathbf{U}_\parallel.$$

It only remains to determine the gauge behaviour of $\nabla^h_{(A}s_{B)}$. Firstly we pull-back (3.62) to S and get $s'_\parallel = s_\parallel + \frac{\hat{z}_n}{2\hat{z}}\boldsymbol{\ell}_\parallel - \frac{1}{2\hat{z}}d\hat{z}$. Thus, the symmetrized covariant derivative satisfies

$$\begin{aligned}\left(\nabla^h_{(A}s_{B)}\right)' &= \nabla^h_{(A}s_{B)} + \frac{\hat{z}_n}{2\hat{z}}\nabla^h_{(A}\ell_{B)}\\ &+ \frac{1}{2\hat{z}}\left(\nabla^h_{(A}\hat{z}_n - \frac{\hat{z}_n}{\hat{z}}\nabla^h_{(A}\hat{z}\,\ell_{B)}\right) - \frac{1}{2\hat{z}}\nabla^h_A\nabla^h_B\hat{z} + \frac{1}{2\hat{z}^2}\nabla^h_A\hat{z}\nabla^h_B\hat{z}\end{aligned}$$

Note also that (recall that $\ell^\sharp_\parallel \stackrel{\text{def}}{=} h^\sharp(\boldsymbol{\ell}_\parallel,\cdot)$)

$$\left(-\frac{1}{2}n(\ell^{(2)}) + 2s_{\not\parallel}(\ell^\sharp_\parallel)\right)' \stackrel{S}{=} \hat{z}\left(-\frac{1}{2}n(\ell^{(2)}) + 2s_{\not\parallel}(\ell^\sharp_\parallel)\right) + \hat{z}_n\left(\ell^{(2)}_\parallel - \ell^{(2)}|_S\right) - \ell^C\nabla^h_C\hat{z}.$$

and $(\mathbf{U}_\parallel(\ell^\sharp_\parallel,\ell^\sharp_\parallel))' = \hat{z}\mathbf{U}_\parallel(\ell^\sharp_\parallel,\ell^\sharp_\parallel)$. Inserting all these expressions into the definition (4.36) of $\mathbf{S}_\parallel$ yields, after a direct computation,

$$\begin{aligned}\mathcal{G}_{(z,0)}(\mathbf{S}_\parallel)_{AB} &= S_{AB} + \frac{1}{\hat{z}}\nabla^h_A\nabla^h_B\hat{z} - \frac{1}{\hat{z}^2}\nabla^h_A\hat{z}\nabla^h_B\hat{z}\\ &+ \frac{1}{\hat{z}}\nabla^h_{(A}\hat{z}\left(2s_{B)} - 2r_{B)} - 2U_{BC}\ell^C - \nabla^h_{B)}\hat{z}\right)\\ &+ \frac{\hat{z}_n}{\hat{z}}\left(Y_{AB} + \frac{1}{2}(\ell^{(2)}_\parallel - \ell^{(2)}|_S)U_{AB} - \nabla^h_{(A}\ell_{B)}\right)\end{aligned}$$

which upon substituting the definitions of $\boldsymbol{\omega}_\parallel$ and $\mathbf{P}_\parallel$ in Lemma 4.3.1 yields the transformation law (C.2). □

Lemma C.0.2. *Assume Setup 3.2.15. The tensor $\mathbf{S}_\parallel$ defined in (4.36) is gauge invariant under the action of the subgroup* $\mathcal{G}_1$.

Proof. Consider the gauge parameters $(0, V)$ and decompose V as in (3.57). We shall need the commutator $[n, V]$. Since $\gamma_{ab}V^b = w_a$ and $\ell_a V^a = f$ we can write

$$\gamma_{ab}[n, V]^b = \pounds_n w_a - (\pounds_n\gamma_{ab})V^b = \pounds_n w_a - 2\mathrm{U}_{ab}P^{bc}w_c$$

$$\ell_a[n,V]^a = £_n(\ell_a V^a) - (£_n \ell_a)V^a \stackrel{(3.43)}{=} n(f) - 2s_a P^{ab} w_b.$$

This implies (by Lemma 2.2.8)

$$[n,V]^a = \Big(n(f) - 2P^{bc} s_b w_c \Big) n^a + P^{ab} \Big(£_n w_b - 2\mathrm{U}_{bc} P^{cd} w_d \Big).$$

We define the covector $p \stackrel{\mathrm{def}}{=} £_n w$ and note that $p(n) = 0$. To compute the gauge transformation law of $\psi^\star(£_n \mathbf{Y})$, we first need to determine $£_n £_V \gamma$. For that purpose, we use

$$£_n £_V \gamma = £_V £_n \gamma + £_{[n,V]}\gamma = 2£_V \mathbf{U} + £_{[n,V]}\gamma$$

and compute its pull-back to S by applying Lemma 3.2.22 twice, namely to $t = V, T = \mathbf{U}$ and $t = [n,V], T = \gamma$, and cancelling terms. The result is

$$\begin{aligned}\psi^\star\left(£_n £_V \gamma\right)_{AB} =& 2\Big(f|_S - \ell^C w_C \Big)(£_n \mathrm{U})_{AB} + 2w^C \Big(\nabla^h_C \mathrm{U}_{AB} - \nabla^h_A \mathrm{U}_{BC} + \nabla_B \mathrm{U}_{AC} \Big) \\ &+ 2\Big(n(f)|_S - 2s^C w_C - \ell^C p_C + 2\ell^C \mathrm{U}_{CD} w^D \Big)\mathrm{U}_{AB} + 2\nabla^h_{(A} p_{B)}.\end{aligned}$$

Given that $\mathbf{Y}' = \mathbf{Y} + \frac{1}{2}£_V \gamma$ we conclude

$$\begin{aligned}(£_n \mathbf{Y}')_{AB} =& (£_n \mathbf{Y})_{AB} + \Big(f|_S - \ell^C w_C \Big)(£_n \mathrm{U})_{AB} + w^C \Big(\nabla^h_C \mathrm{U}_{AB} - \nabla^h_A \mathrm{U}_{BC} + \nabla_B \mathrm{U}_{AC} \Big) \\ &+ \Big(n(f)|_S - 2s^C w_C - \ell^C p_C + 2\ell^C \mathrm{U}_{CD} w^D \Big)\mathrm{U}_{AB} + \nabla^h_{(A} p_{B)}.\end{aligned} \tag{C.3}$$

The transformation law of $\ell^{(2)}$, $\ell^{(2)}_\parallel$, $\mathbf{U}_\parallel$ and $\boldsymbol{\ell}_\parallel$ has already been obtained in (4.33)-(4.34). Setting $z = 1, \hat{z} = 1$ therein gives

$$\Big(\ell^{(2)} - \ell^{(2)}_\parallel \Big)' = \ell^{(2)} - \ell^{(2)}_\parallel + 2\Big(f|_S - \ell^C w_C \Big). \tag{C.4}$$

$$\mathbf{U}'_\parallel = \mathbf{U}_\parallel, \qquad \boldsymbol{\ell}'_\parallel = \boldsymbol{\ell}_\parallel + w_\parallel. \tag{C.5}$$

On the other hand, pulling-back (3.62) to S yields

$$s'_\parallel = s_\parallel + \frac{1}{2}\mathbf{p}. \tag{C.6}$$

It only remains to determine how $n(\ell^{(2)})$ transforms. Given the fact that $\ell^{(2)\prime}$ involves P^{ab} (see (3.59)) we need to know $£_n P^{ab}$. Thus, we compute

$$\begin{aligned}£_n P^{ab} &= n^c \overset{\circ}{\nabla}_c P^{ab} - 2P^{c(a} \overset{\circ}{\nabla}_c n^{b)} \stackrel{(2.21)}{=} -2P^{c(a}\Big(n^f \mathrm{F}_{fc} n^{b)} + \overset{\circ}{\nabla}_c n^{b)} \Big) - n^c n^a n^b \overset{\circ}{\nabla}_c \ell^{(2)} \\ &\stackrel{(3.44)}{=} -2P^{c(a}\Big(2s_c n^{b)} + P^{b)f}\mathrm{U}_{cf} \Big) - n^a n^b n(\ell^{(2)}),\end{aligned}$$

which inserted in the $\pounds_n$ derivative of (3.59) yields

$$n(\ell^{(2)})' = n(\ell^{(2)}) + 2n(f) - 2P^{ca}P^{bf}\mathrm{U}_{cf}w_a w_b + 2P^{ab}w_a p_b$$

and hence

$$n(\ell^{(2)})'|_S = n(\ell^{(2)})|_S + 2n(f)|_S - 2w^A w^B \mathrm{U}_{AB} + 2w^A p_B. \tag{C.7}$$

Applying $\mathrm{G}_{(0,V)}$ to the right-hand side of (4.36) and inserting (C.3)-(C.7), the gauge invariance of $\mathbf{s}_{\parallel}$ follows after a somewhat long but straightforward calculation. □

D

COORDINATES NEAR A NULL HYPERSURFACE

In the main body of this thesis we have made use of so-called *Gaussian null coordinates* and *Rácz-Wald coordinates*. The former ones can be built in a neighbourhood of any null hypersurface, whereas the latter exist near a bifurcation surface. The standard procedures that raise these two coordinate sets can be found e.g. in [100] and [101] respectively. However, for this thesis to be self-contained, we next provide their construction. This derivation is original and relies on a more geometric approach compared with those in [100], [101]. This is advantageous for at least two reasons. Firstly, it allows us to construct both coordinate systems as particular cases of a single, more general setup (the general framework is interesting in itself, and it is likely that other useful particular cases can be extracted). Secondly, and perhaps more important, it allows us to prove that Rácz-Wald-type coordinates exist not only near bifurcation surfaces of Killing horizons but also near *any* null hypersurface.

This appendix has two parts. In the first one we find a very flexible coordinate system near a general null hypersurface, while in the second one we particularize the results for non-degenerate Killing horizons with constant surface gravity.

d.1 coordinates near a null hypersurface

The idea of the following construction is to introduce a function ρ from which we build a pair of Lie-commuting vector fields $\{L, k\}$. Then we set up coordinates adapted to $\{L, k\}$ so that different choices of ρ give rise to different coordinate systems. The following lemma constitutes the keystone for the construction.

Lemma D.1.1. *Consider a spacetime* $(\mathcal{M}, g)$ *of dimension at least two, an embedded smooth connected null hypersurface* $\mathcal{N} \subset \mathcal{M}$ *and a null generator* k *of* $\mathcal{N}$. *Select a vector field* ξ *along* $\mathcal{N}$ *with the properties of being null, everywhere transversal to* $\mathcal{N}$ *and* *such that* $g(\xi, k)|_{\mathcal{N}} = -1$. *Extend* ξ *uniquely to a neighbourhood* $\mathcal{O}$ *of* $\mathcal{N}$ *as an affinely parametrized geodesic vector field, i.e. enforcing* $\nabla_\xi \xi = 0$. *Let* $G \in \mathcal{F}^*(\mathcal{O})$ *be a non-zero function and* k *be any extension of* $k|_{\mathcal{N}}$ *to* $\mathcal{O}$. *If* $\{G, k\}$ *satisfy*

$$[\xi, k] = \frac{1}{2}\Big(\big(\pounds_\xi g\big)(k,k) - \rho\Big)\xi, \tag{D.1}$$

$$k(G) = \frac{1}{2}\Big(\big(\pounds_\xi g\big)(k,k) - \rho\Big)G, \tag{D.2}$$

for some function $\rho \in \mathcal{F}(\mathcal{O})$ *then* $\{L \stackrel{\mathrm{def}}{=} G\xi, k\}$ *verify* $[k, L] = 0$ *and* $\xi(g(k,k)) = \rho$. *Conversely, given a function* $\rho \in \mathcal{F}(\mathcal{O})$ *that is allowed to depend on* k *but not on its derivatives, then* (D.1) *yields a unique solution* k *given initial data* $k|_{\mathcal{N}}$, *while* (D.2) *gives a unique solution* G *for each value* $G|_{\mathcal{H}_t}$ *on a hypersurface* $\mathcal{H}_t \subset \mathcal{O}$ *to which* k *is everywhere transverse. If the initial data for* G *is nowhere zero then the solution* G *is also nowhere zero.*

Proof. First, we note that ξ is null everywhere on $\mathcal{O}$ because $\nabla_\xi(g(\xi,\xi)) = 2g(\nabla_\xi\xi, \xi) = 0$. To prove that $[L,k] = 0$, it suffices to substitute (D.2) into (D.1), which yields

$$[\xi, k] = \frac{k(G)}{G}\xi \quad \Longrightarrow \quad 0 = G[\xi,k] - k(G)\xi = [G\xi, k] = [L,k].$$

This entails, in turn, that $g(\xi,k) = -1$ everywhere on $\mathcal{O}$. Indeed, combining $[k,L] = 0$ with ξ being geodesic and null and $L = G\xi$ being also null gives

$$\begin{aligned}\nabla_\xi(g(\xi,k)) &= g\big(\nabla_\xi\xi, k\big) + g\big(\xi, \nabla_\xi k\big) = \frac{1}{G^2}g(L, \nabla_L k)\\ &= \frac{1}{G^2}g(L, \nabla_k L) = \frac{1}{2G^2}\nabla_k(g(L,L)) = 0,\end{aligned}$$

which means that $g(\xi,k) = g(\xi,k)|_{\mathcal{N}} = -1$. Using (D.1) together with $g(\xi,k) = -1$ one also gets

$$\begin{aligned}\xi(g(k,k)) = \pounds_\xi(g(k,k)) &= \big(\pounds_\xi g\big)(k,k) + 2g([\xi,k], k)\\ &= \big(\pounds_\xi g\big)(k,k) - \big(\pounds_\xi g\big)(k,k) + \rho = \rho,\end{aligned}$$

as claimed. Since ξ is transverse to $\mathcal{N}$, (D.1) constitutes a system of ordinary differential equations along the integral curves of ξ (here we use the fact that ρ may depend on k, but not on its derivatives), hence admitting a unique solution for given initial data $k|_{\mathcal{N}}$. Once the unique extension k is known, (D.2) is simply an ordinary differential equation along the integral curves of k. This yields a unique solution provided the initial data $G|_{\mathcal{H}_t}$ is given on a hypersurface $\mathcal{H}_t$ transverse to

k everywhere. The claim about the solution being nowhere zero is immediate from the structure of the ODE (D.2). □

Remark D.1.2. *Observe that in Lemma D.1.1 the vector field ξ is null everywhere on $\mathcal{O}$, and that $\mathcal{H}_t$ is not assumed to be null. For later purposes, we define $\psi \stackrel{\text{def}}{=} \frac{1}{2}\left(\pounds_\xi g\,(k,k) - \rho\right)$. Although ψ depends on ρ, on ξ and on k (but not on its derivatives), for simplicity we do not reflect this dependence in the notation.*

From now on, we make the extra assumption that $\overline{\mathcal{N}}$ can be foliated by a family of diffeomorphic spacelike cross-sections. This can always be fulfilled by restricting $\overline{\mathcal{N}}$ if necessary. In these circumstances, one can introduce a foliation function $v \in \mathcal{F}(\overline{\mathcal{N}})$ as the solution of $k(v)|_{\overline{\mathcal{N}}} = 1$, $v|_S = 0$, where S is one such cross-section of $\overline{\mathcal{N}}$. In the following we also restrict $\xi|_{\overline{\mathcal{N}}}$ to be orthogonal to the leaves $S_v \stackrel{\text{def}}{=} \{v = \text{const.}\}$, which makes $\xi|_{\overline{\mathcal{N}}}$ unique (recall that $g(\xi,\xi) = 0$, $g(\xi,k) = -1$ also hold in $\overline{\mathcal{N}}$).

The construction of the coordinates is as follows. In the setup of Lemma D.1.1, we let k be the unique extension of $k|_{\overline{\mathcal{N}}}$ given by (D.1) for some function ρ that may depend on k but not on its derivatives. Given a choice $G|_{\mathcal{H}_t}$, we also let G be the only solution of (D.2) constructed from k. As before, we introduce the vector field $L \stackrel{\text{def}}{=} G\xi$ which by Lemma D.1.1 verifies $[L,k] = 0$. Then, we take coordinates $\{x^A\}$ on S and transfer them to all leaves $\{S_v\}$ by enforcing $k(x^A)|_{\overline{\mathcal{N}}} = 0$. It follows that $\{v, x^A\}$ are coordinates on $\overline{\mathcal{N}}$ and that $k|_{\overline{\mathcal{N}}} = \partial_v$. We complete the construction by defining the functions $u, \hat{v}, \hat{x}^A \in \mathcal{F}(\mathcal{O})$ as the unique solutions of

$$\begin{cases} L(u) = 1 \\ u \stackrel{\overline{\mathcal{N}}}{=} 0 \end{cases}, \qquad \begin{cases} L(\hat{v}) = 0 \\ \hat{v} \stackrel{\overline{\mathcal{N}}}{=} v \end{cases}, \qquad \begin{cases} L\left(\hat{x}^A\right) = 0 \\ \hat{x}^A \stackrel{\overline{\mathcal{N}}}{=} x^A \end{cases}.$$

This allows us to drop the hat in $\{\hat{v}, \hat{x}^A\}$ and let the context determine if we are referring to $v, x^A \in \mathcal{F}(\mathcal{O})$ or $v, x^A \in \mathcal{F}(\overline{\mathcal{N}})$. The set $\{u, v, x^A\}$ constitutes a local coordinate system in $\mathcal{O}$. Let us show that $L = \partial_u$ and $k = \partial_v$ everywhere in $\mathcal{O}$. The former is obvious from the construction and the latter holds because $[L, k - \partial_v] = 0$ (which is true because both k and the coordinate vector ∂_v commute with L). Since $k|_{\overline{\mathcal{N}}} = \partial_v$, it follows from uniqueness of this ordinary differential equation that $k = \partial_v$ everywhere on $\mathcal{O}$.

We can now write the metric g in the coordinates $\{u, v, x^A\}$. For that we notice that $g(\partial_u, \partial_u) = 0$ (because ξ, L are null) and that $g(\partial_u, \partial_{x^A}) = 0$ since

$$\nabla_\xi\left(g(\xi, \partial_{x^A})\right) = g(\xi, \nabla_\xi \partial_{x^A}) = \frac{1}{G^2}\, g(L, \nabla_L \partial_{x^A})$$

$$= \frac{1}{G^2} g(L, \nabla_{\partial_{x^A}} L) = \frac{1}{2G^2} \nabla_{\partial_{x^A}} (g(L, L)) = 0,$$

which means that $g(\xi, \partial_{x^A}) = g(\xi, \partial_{x^A})|_{\mathcal{N}} = 0$. The metric g therefore reads

$$g = -2G dv \left(du + q_A dx^A + \hat{g}_{vv} dv \right) + \gamma_{AB} dx^A dx^B, \tag{D.3}$$

where we have defined $\hat{g}_{vv} \stackrel{\text{def}}{=} -\frac{g(\partial_v, \partial_v)}{2G}$ and $q_A, \gamma_{AB} \in \mathcal{F}(\mathcal{O})$. The fact that $k|_{\mathcal{N}}$ is a null generator of $\mathcal{N}$ implies that $g(k, k)|_{\mathcal{N}} = 0, g(k, \partial_{x^A})|_{\mathcal{N}} = 0$. Consequently, there must exists functions $H, h_A \in \mathcal{F}(\mathcal{O})$ such that $q_A = u h_A(u, v, x^B)$, $\hat{g}_{vv} = uH(u, v, x^B)$. Thus,

$$g = -2G dv \left(du + u h_A dx^A + uH dv \right) + \gamma_{AB} dx^A dx^B, \tag{D.4}$$

with G, h_A, H and γ_{AB} depending on all variables. The components of the inverse metric $g^{\sharp}$ are given by

$$g^{uv} = -\frac{1}{G}, \qquad g^{uu} = u^2 \gamma^{AB} h_A h_B + \frac{2uH}{G}, \qquad g^{uA} = -u\gamma^{AB} h_B, \tag{D.5}$$

$$g^{vv} = 0, \qquad g^{vA} = 0, \qquad \delta^A_C = \gamma^{AB} \gamma_{BC}. \tag{D.6}$$

The freedom in the construction above is the choice of (i) a function $\rho \in \mathcal{F}(\mathcal{O})$, (ii) a null generator $k|_{\mathcal{N}}$, (iii) a cross-section S and (iv) a nowhere zero function G on a hypersurface $\mathcal{H}_t$ transversal to $\mathcal{N}$ (in fact it must be transversal to the vector field k constructed by solving (D.1), but after restriction of $\mathcal{O}$ if necessary this will always be true if $\mathcal{H}_t$ is transversal to $\mathcal{N}$).

d.1.1 *Generalized Rácz-Wald coordinates*

The Rácz-Wald form of the metric is characterized by the property that the vector field k is null everywhere on $\mathcal{O}$. This can be achieved by enforcing $\rho = 0$. By Lemma D.1.1 this entails that $\xi(g(k, k)) = 0$, i.e. $g(k, k) = g(k, k)|_{\mathcal{N}} = 0$. Therefore, in this case $H = 0$ and (D.4) becomes

$$g = -2G dv \left(du + u h_A dx^A \right) + \gamma_{AB} dx^A dx^B. \tag{D.7}$$

We call **generalized Rácz-Wald coordinates** to the coordinates constructed with the above setup and in which the metric takes the form (D.7). They exist in some neighbourhood of any point p of a null hypersurface $\mathcal{N}$ (corresponding to $\{u =$

0} in this coordinate system). The generalized Rácz-Wald coordinates admit the freedom of items (*ii*)-(*iv*) above.

d.1.2 *Gaussian null coordinates*

The Gaussian null coordinates are characterized by G being everywhere constant (usually equal to one) and L being null and affinely geodesic. By enforcing $\rho = £_\xi g\ (k, k)$ in the general construction above, equations (D.1)-(D.2) become

$$[\xi, k] = 0, \qquad k(G) = 0. \tag{D.8}$$

As before, the former provides a unique solution for k for given initial data $k|_{\overline{N}}$ while the second entails that $G = G|_{H^t}$. By simply taking $G|_{H^t} = 1$, one obtains the metric form for Gaussian null coordinates, namely

$$g = -2dv\left(du + uh_A dx^A + uHdv\right) + \gamma_{AB}dx^A dx^B, \tag{D.9}$$

and it holds that $G \in \mathbb{R} - \{0\}$ and hence that $L = G\xi$ is affinely geodesic. Observe that in this specific construction we recover the well-known freedom associated to Gaussian null coordinates, namely a choice of a null generator k of $\overline{N}$ and a choice of a spacelike section S on $\overline{N}$.

d.2 coordinates near a killing horizon

We now study the case when $\overline{N}$ is a null hypersurface constructed by taking null geodesics starting orthogonally from a bifurcation surface S. The hypersurface $\overline{N}$ is therefore the closure of a non-degenerate Killing horizon H with respect to a Killing vector field η. Moreover, it holds that the surface gravity κ of η, defined according to (2.81), is constant everywhere on $\overline{N}$. Since by construction $\overline{N}$ admits a cross-section (namely S), we can take $k|_{\overline{N}}$ affine, i.e. satisfying $\nabla_k k = 0$ on $\overline{N}$. Then, $k|_{\overline{N}}$ is given by its value at S and by Remark 2.4.8 it holds that $\eta|_{\overline{N}} = (f + \kappa v)k$, where $f \in \mathcal{F}(\overline{N})$ fulfils $k(f)|_{\overline{N}} = 0$. If, in addition, one enforces $S =$ S in the construction above, then $\eta|_{\overline{N}} = \kappa v k$.

In order to study the properties of the Killing vector field η off $\overline{\mathcal{N}}$, we decompose it as $\eta = \eta^u L + \eta^v k + \eta^A \partial_{x^A}$. Since ξ is geodesic, we know that $\nabla_\xi (g(\eta, \xi)) = 0$, so that $g(\eta, \xi) = g(\eta, \xi)|_{\overline{\mathcal{N}}}$. This, together with $\xi = G^{-1} L = G^{-1} \partial_u$ and (D.4), means

$$\eta^v = -g(\eta, \xi) \overset{\overline{\mathcal{N}}}{=} -g(\eta, \xi) \overset{\overline{\mathcal{N}}}{=} \eta^v \qquad \Longrightarrow \qquad \eta^v = \eta^v|_{\overline{\mathcal{N}}}, \tag{D.10}$$

Thus $\eta = \eta^u L + \bar{\kappa} v k + \eta^A \partial_{x^A}$, where we have extended κ to $\mathcal{O}$ as the same constant. We can now write down the Killing equations $\pounds_\eta g = 0$ for the metric (D.7). Considering that $\partial_u \eta^v = 0$ and $g_{vv} = 0$, the non-trivial Killing equations are

$$0 = \eta(G) + (\partial_u \eta^u + \underline{\kappa}) G - g_{vA} \partial_u \eta^A, \tag{D.11}$$

$$0 = \gamma_{AB} \partial_u \eta^A, \tag{D.12}$$

$$0 = -G \partial_v \eta^u + g_{vA} \partial_v \eta^A, \tag{D.13}$$

$$0 = \eta(g_{vB}) + \bar{\kappa} g_{vB} + \gamma_{AB} \partial_v \eta^A - G \partial_{x^B} \eta^u + g_{vA} \partial_{x^B} \eta^A, \tag{D.14}$$

$$0 = \eta(\gamma_{AB}) + \gamma_{BC} \partial_{x^A} \eta^C + \gamma_{AC} \partial_{x^B} \eta^C. \tag{D.15}$$

Equation (D.12) entails that $\partial_u \eta^A = 0$ because $\gamma_{AB} \overset{\text{def}}{=} g(\partial_{x^A}, \partial_{x^B})$ is necessarily positive definite. Consequently, $\eta^A = \eta^A|_{\overline{\mathcal{N}}} = 0$ and hence (D.13) implies $0 = \partial_v \eta^u$. Taking this into account and deriving (D.11) along k, one obtains

$$0 \overset{(D.2)}{=} k(\eta(G)) + (\partial_u \eta^u + \underline{\kappa}) \psi G \overset{(D.11)}{=} k(\eta(G)) - \psi \eta(G)$$

$$\Longrightarrow \qquad k(\eta(G)) = \psi \eta(G). \tag{D.16}$$

The null hypersurface $\overline{\mathcal{N}}'$ constructed by taking null geodesics starting from S with tangent vector field $\xi|_S$ constitutes a branch of the bifurcate Killing horizon. In particular, this means that $\overline{\mathcal{N}}'$ is a hypersurface everywhere transverse to k and that there exists a function $\beta \in \mathcal{F}(\overline{\mathcal{N}}')$ such that $\eta = \beta \xi$ on $\overline{\mathcal{N}}'$. If we select $H_t = \overline{\mathcal{N}}'$ and $G|_{H_t}$ so that $\xi(G)|_{H_t} = 0$, then $\eta(G)|_{H_t} = 0$. This, together with (D.16), entails that $\eta(G) = 0$ everywhere on $\mathcal{O}$. The explicit form of η on $\mathcal{O}$ can be derived now by enforcing $\eta(G) = 0$ into (D.11). This turns (D.11) into $\partial_u \eta^u = -\bar{\kappa}$, from where it follows that $\eta^u = -\bar{\kappa} u + \eta^u_0(x^A)$. Since $\eta^u|_{\overline{\mathcal{N}}} = \eta^u|_{\{u=0\}} = 0$, $\eta^u_0(x^A)$ must vanish and the Killing vector field η reads

$$\eta = \bar{\kappa}(-u \partial_u + v \partial_v), \tag{D.17}$$

which is the standard form for the Killing vector field in Rácz-Wald coordinates (see e.g. [101]). Particularizing (D.11), (D.14) and (D.15) for (D.17) yields the ODEs

$$0 = \eta(G), \qquad 0 = \eta(g_{vB}) + \bar{\kappa} g_{vB}, \qquad 0 = \eta(\gamma_{AB}), \tag{D.18}$$

whose respective solutions are (recall that $g_{vB} = uGh_B$)

$$G = G(uv, x^C), \qquad g_{vB} = uG(uv, x^C)h_B(uv, x^C), \qquad \gamma_{AB} = \gamma_{AB}(uv, x^C). \tag{D.19}$$

Substituting (D.19) in (D.7) yields the well-known form of the metric g in Rácz-Wald coordinates, namely

$$g = -2G(uv, x^C)dv \left(du + uh_A(uv, x^C)dx^A \right) + \gamma_{AB}(uv, x^C)dx^A dx^B. \tag{D.20}$$

Observe that $\eta(G)|_{\mathcal{H}_t} = 0$ means that $G|_{\mathcal{H}_t}$ is given by its value at $\mathcal{S}$. Therefore, the remaining freedom in this coordinates is the choice of $G|_{\mathcal{S}}$ and $k|_{\mathcal{S}}$ (recall that k has been selected affine, so it verifies $\nabla_k k = 0$ on $\widetilde{\mathcal{N}}$). In particular, by enforcing $G|_{\mathcal{S}} = \text{const.}$ one obtains $G = G(uv)$ on $\mathcal{O}$, and hence G is constant everywhere on $\overline{\mathcal{N}}$ and $\mathcal{H}_t$.

BIBLIOGRAPHY

[1] A. Ashtekar, C. Beetle and J. Lewandowski, 'Geometry of generic isolated horizons', *Classical and Quantum Gravity*, vol. **19**, no. 6, p. 1195, 2002.

[2] M. Mars, T. Paetz and J. M. M. Senovilla, 'Multiple Killing horizons', *Classical and Quantum Gravity*, vol. **35**, no. 15, p. 155 015, 2018.

[3] R. Penrose, 'A remarkable property of plane waves in General Relativity', *Reviews of Modern Physics*, vol. **37**, p. 215, 1965.

[4] J. Podolskỳ, R. Steinbauer and R. Švarc, 'Gyratonic pp-waves and their impulsive limit', *Physical Review D*, vol. **90**, p. 044 050, 2014.

[5] J. Podolskỳ, R. Švarc, R. Steinbauer and C. Sämann, 'Penrose junction conditions extended: Impulsive waves with gyratons', *Physical Review D*, vol. **96**, p. 064 043, 2017.

[6] J. Podolskỳ, C. Sämann, R. Steinbauer and R. Švarc, 'Cut-and-paste for impulsive gravitational waves with Λ: The geometric picture', *Physical Review D*, vol. **100**, p. 024 040, 2019.

[7] J. Podolsky` and R. Steinbauer, 'Penrose junction conditions with Λ: Geometric insights into low-regularity metrics for impulsive gravitational waves', *General Relativity and Gravitation*, vol. **54**, no. 9, p. 96, 2022.

[8] A. Einstein, 'Die Feldgleichungen der Gravitation', *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften*, pp. 844–847, 1915.

[9] A. Einstein, 'Die Grundlage der allgemeinen Relativitätstheorie', *Annalen der Physik*, vol. **354**, no. 7, pp. 769–822, 1916. doi: https://doi.org/10.1002/andp.19163540702.

[10] A. Einstein, 'Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie', *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, pp. 831–839, 1915.

[11] F. W. Dyson, A. S. Eddington and C. Davidson, 'A determination of the deflection of light by the Sun's gravitational field, from observations made at the total eclipse of may 29, 1919', *Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character*, vol. **220**, no. 571-581, pp. 291–333, 1920.

[12] W. S. Adams, 'The relativity displacement of the spectral lines in the companion of Sirius', *Proceedings of the National Academy of Sciences*, vol. **11**, no. 7, pp. 382–387, 1925.

[13] D. M. Popper, 'Red Shift in the Spectrum of 40 Eridani B', *Astrophysical Journal, vol. 120, p. 316*, vol. **120**, p. 316, 1954.

[14] A. Einstein, 'Näherungsweise Integration der Feldgleichungen der Gravitation', *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften*, pp. 688–696, 1916.

[15] A. Einstein, 'Über Gravitationswellen', *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, pp. 154–167, 1918.

[16] R. Penrose, 'Gravitational collapse and space-time singularities', *Physical Review Letters*, vol. **14**, no. 3, p. 57, 1965.

[17] S. W. Hawking and R. Penrose, 'The singularities of gravitational collapse and cosmology', *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, vol. **314**, no. 1519, pp. 529–548, 1970.

[18] G. F. R. Ellis, R. Maartens and S. D. Nel, 'The expansion of the Universe', *Monthly Notices of the Royal Astronomical Society*, vol. **184**, no. 3, pp. 439–465, 1978.

[19] S. W. Hawking and G. F. R. Ellis, 'The cosmic black-body radiation and the existence of singularities in our universe', *Astrophysical Journal*, vol. **152**, p. 25, 1968.

[20] A. Einstein, 'Zur Elektrodynamik bewegter Körper', *Annalen der physik*, vol. **4**, 1905.

[21] L. Corry, 'David Hilbert between Mechanical and Electromagnetic Reductionism (1910-1915)', *Archive for History of Exact Sciences*, vol. **53**, no. 6, pp. 489–527, 1999.

[22] H. Bondi, M. G. J. Van der Burg and A. W. K. Metzner, 'Gravitational Waves in General Relativity, VII. Waves from Axi-Symmetric Isolated System', *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, vol. **269**, no. 1336, pp. 21–52, 1962. doi: 10.1098/rspa.1962.0161.

[23] R. K. Sachs, 'Gravitational Waves in General Relativity VIII. Waves in Asymptotically Flat Space-time', *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, vol. **270**, no. 1340, pp. 103–126, 1962. doi: 10.1098/rspa.1962.0206.

[24] E. Newman and R. Penrose, 'An approach to gravitational radiation by a method of spin coefficients', *Journal of Mathematical Physics*, vol. **3**, no. 3, pp. 566–578, 1962.

[25] B. P. Abbott *et al.*, 'Binary Black Hole Mergers in the First Advanced LIGO Observing Run', *Phys. Rev. X*, vol. **6**, p. 041 015, 4 2016. doi: 10.1103/PhysRevX.6.041015.

[26] K. Schwarzschild, 'Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie', *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, pp. 189–196, 1916.

[27] R. Penrose, 'Gravitational collapse: The role of General Relativity', *Nuovo Cimento Rivista Serie*, vol. **1**, p. 252, 1969.

[28] R. Penrose, 'Singularities and time-asymmetry', in *General Relativity, an Einstein Centenary Survey. Eds. S. W. Hawking and W. Israel*, Cambridge University Press, Cambridge, UK, 1979.

[29] M. Oda, P. Gorenstein, H. Gursky, E. Kellogg, E. Schreier, H. Tananbaum and R. Giacconi, 'X-ray pulsations from Cygnus X-1 observed from Uhuru', *The Astrophysical Journal*, vol. **166**, p. L1, 1971.

[30] T. E. H. T. Collaboration *et al.*, 'First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole', *The Astrophysical Journal Letters*, vol. **875**, no. 1, p. L1, 2019. doi: 10.3847/2041-8213/ab0ec7.

[31] K. Akiyama *et al.*, 'First M87 Event Horizon Telescope Results. II. Array and Instrumentation', *The Astrophysical Journal Letters*, vol. **875**, no. 1, p. L2, 2019.

[32] K. Akiyama *et al.*, 'First M87 Event Horizon Telescope Results. III. Data Processing and Calibration', *The Astrophysical Journal Letters*, vol. **875**, no. 1, p. L3, 2019.

[33] K. Akiyama *et al.*, 'First M87 Event Horizon Telescope Results. IV. Imaging the Central Supermassive Black Hole', *The Astrophysical Journal Letters*, vol. **875**, no. 1, p. L4, 2019.

[34] K. Akiyama *et al.*, 'First M87 Event Horizon Telescope Results. V. Physical Origin of the Asymmetric Ring', *The Astrophysical Journal Letters*, vol. **875**, no. 1, p. L5, 2019.

[35] K. Akiyama *et al.*, 'First M87 Event Horizon Telescope Results. VI. The Shadow and Mass of the Central Black Hole', *The Astrophysical Journal Letters*, vol. **875**, no. 1, p. L6, 2019.

[36] K. Akiyama *et al.*, 'First M87 Event Horizon Telescope Results. VII. Polarization of the Ring', *The Astrophysical Journal Letters*, vol. **910**, no. 1, p. L12, 2021.

[37] F. Schwabl, *Quantum mechanics*. Springer Science & Business Media, 2007.

[38] J. F. Donoghue, E. Golowich and B. R. Holstein, *Dynamics of the standard model*. Cambridge university press, 2014.

[39] C. Rovelli, 'Loop quantum gravity', *Living reviews in relativity*, vol. **11**, pp. 1–69, 2008.

[40] K. R. Dienes, 'String theory and the path to unification: A review of recent developments', *Physics Reports*, vol. **287**, no. 6, pp. 447–525, 1997.

[41] S. Shankaranarayanan and J. P. Johnson, 'Modified theories of Gravity: Why, How and What?', *General Relativity and Gravitation*, vol. **54**, no. 5, p. 44, 2022.

[42] L. Lehner, 'Numerical relativity: A review', *Classical and Quantum Gravity*, vol. **18**, no. 17, R25, 2001.

[43] I. B. Zeldovich and I. D. Novikov, *Relativistic Astrophysics, Vol. 2: The Structure and Evolution of the Universe*. University of Chicago press, 1971.

[44] S. Dodelson and F. Schmidt, *Modern cosmology*. Academic press, 2020.

[45] P. Chrus´ciel, G. Galloway and D. Pollack, 'Mathematical General Relativity: A sampler', *Bulletin of the American Mathematical Society*, vol. **47**, no. 4, pp. 567–638, 2010.

[46] D. Christodoulou, 'On the global initial value problem and the issue of singularities', *Classical and Quantum Gravity*, vol. **16**, no. 12A, A23, 1999.

[47] K. Landsman, 'Singularities, black holes, and cosmic censorship: A tribute to Roger Penrose', *Foundations of Physics*, vol. **51**, pp. 1–38, 2021.

[48] S. Yau, *Seminar on differential geometry*. Princeton University Press, 1982.

[49] Y. Foures-Bruhat, 'Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires', *Acta mathematica*, vol. **88**, pp. 141–225, 1952.

[50] S. Klainerman *et al.*, *The evolution problem in General Relativity*. Springer Science & Business Media, 2002, vol. **25**.

[51] H. Ringström, *The Cauchy problem in General Relativity*. European Mathematical Society, 2009, vol. **6**.

[52] Y. Choquet-Bruhat, P. T. Chrus´ciel and J. M. Martín-García, 'The Cauchy problem on a characteristic cone for the Einstein equations in arbitrary dimensions', in *Annales Henri Poincaré*, Springer, vol. **12**, 2011, pp. 419–482.

[53] A. D. Rendall, 'Reduction of the characteristic initial value problem to the Cauchy problem and its applications to the Einstein equations', *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, vol. **427**, no. 1872, pp. 221–239, 1990.

[54] G. Caciotta and F. Nicolò, 'Global characteristic problem for Einstein vacuum equations with small initial data: (I) The initial data constraints', *Journal of Hyperbolic Differential Equations*, vol. **2**, pp. 201–277, 2005.

[55] P. T. Chrus´ciel and T.-T Paetz, 'The many ways of the characteristic Cauchy problem', *Classical and Quantum Gravity*, vol. **29**, 2012.

[56] A. Cabet, P. T. Chrus´ciel and R. T. Wafo, 'On the characteristic initial value problem for nonlinear symmetric hyperbolic systems, including Einstein equations', *Dissertationes Mathematicae*, vol. **515**, pp. 1–67, 2016.

[57] D. Hilditch, J. A Valiente-Kroon and P. Zhao, 'Revisiting the characteristic initial value problem for the vacuum Einstein field equations', *General Relativity and Gravitation*, vol. **52**, pp. 1–76, 2020.

[58] M. Mars, 'Constraint equations for general hypersurfaces and applications to shells', *General Relativity and Gravitation*, vol. **45**, pp. 2175–2221, 2013.

[59] M. Mars, 'Hypersurface data: General properties and Birkhoff theorem in spherical symmetry', *Mediterranean Journal of Mathematics*, vol. **17**, pp. 1–45, 2020.

[60] M. Mars and G. Sánchez-Pérez, 'Double Null Data and the Characteristic Problem in General Relativity', *Journal of Physics A: Mathematical and Theoretical*, vol. **56**, no. 3, p. 035 203, 2023.

[61] M. Mars and G. Sánchez-Pérez, 'Covariant definition of Double Null Data and geometric uniqueness of the characteristic initial value problem', *Journal of Physics A: Mathematical and Theoretical*, vol. **56**, no. 25, p. 255 203, 2023.

[62] J. A. Schouten, *Ricci Calculus: An introduction to tensor analysis and its geometrical applications*, 2nd edition. Springer-Verlag, 1954.

[63] C. Barrabés and W. Israel, 'Thin shells in general relativity and cosmology: The lightlike limit', *Physical Review D*, vol. **43**, pp. 1129–1142, 1991.

[64] M. Mars and J. M. M. Senovilla, 'Geometry of general hypersurfaces in spacetime: Junction conditions', *Classical and Quantum Gravity*, vol. **10**, p. 1865, 1993.

[65] M. Mars, J. Senovilla and R. Vera, 'Lorentzian and signature changing branes', *Physical Review D*, vol. **76**, p. 044 029, 2007.

[66] M. W. Hirsch, *Differential Topology*. Springer, 1997, vol. **33**.

[67] C. J. S. Clarke and T. Dray, 'Junction conditions for null hypersurfaces', *Classical and Quantum Gravity*, vol. **4**, p. 265, 1987.

[68] G. Darmois, 'Les équations de la gravitation einsteinienne', *Mémorial des Sciences Mathématiques, Fascicule XXV (Paris: Gauthier-Villars)*, vol. **44**, 1927.

[69] S. O'Brien, 'Jump conditions at discontinuities in General Relativity', *Communications of the Dublin Institute for Advanced Studies*, vol. **9**, 1952.

[70] P. Le Corbeiller, 'Theories Relativistes de la Gravitation et de l'Electromagnetisme. Relativite Generale et Theories Unitaires. A. Lichnerowicz. Masson, Paris, 1955. xii+ 298 pp. Illus. F. 2800.', *Science*, vol. **122**, no. 3166, pp. 424–424, 1955.

[71] W. B. Bonnor and P. A. Vickers, 'Junction conditions in General Relativity', *General Relativity and Gravitation*, vol. **13**, no. 1, pp. 29–36, 1981.

[72] L. Bel and A. Hamoui, 'Les conditions de raccordement en Relativité Générale', in *Annales de l'IHP Physique théorique*, vol. **7**, 1967, pp. 229–244.

[73] W. Israel, 'Singular hypersurfaces and thin shells in General Relativity', *Il Nuovo Cimento B*, vol. **44**, pp. 1–14, 1966.

[74] A. H. Taub, 'Spacetimes with distribution valued curvature tensors', *Journal of Mathematical Physics*, vol. **21**, no. 6, pp. 1423–1431, 1980.

[75] R. Geroch and J. Traschen, 'Strings and other distributional sources in General Relativity', *Physical Review D*, vol. **36**, no. 4, p. 1017, 1987.

[76] P. G. LeFloch and C. Mardare, 'Definition and weak stability of spacetimes with distributional curvature', *Portugaliae Mathematica*, vol. **64**, pp. 535–573, 2007.

[77] M. Kunzinger and R. Steinbauer, 'Foundations of a nonlinear distributional geometry', *Acta Applicandae Mathematica*, vol. **71**, pp. 179–206, 2002.

[78] M. Grosser, M. Kunzinger, R. Steinbauer and J. A. Vickers, 'A global theory of algebras of generalized functions', *Advances in Mathematics*, vol. **166**, no. 1, pp. 50–72, 2002.

[79] R. Steinbauer and J. A. Vickers, 'The use of generalized functions and distributions in General Relativity', *Classical and Quantum Gravity*, vol. **23**, no. 10, R91, 2006.

[80] M. Kunzinger, R. Steinbauer and J. Vickers, 'Sheaves of nonlinear generalized functions and manifold-valued distributions', *Transactions of the American Mathematical Society*, vol. **361**, no. 10, pp. 5177–5192, 2009.

[81] M. Grosser, M. Kunzinger, R. Steinbauer and J. Vickers, 'A global theory of algebras of generalized functions II: Tensor distributions', *New York Journal of Mathematics*, vol. **18**, pp. 139–199, 2012.

[82] K. Lanczos, 'Bemerkung zur de sitterschen welt', *Physikalische Zeitschrift*, vol. **23**, no. 539-543, p. 15, 1922.

[83] K. Lanczos, 'Flächenhafte verteilung der materie in der einsteinschen gravitationstheorie', *Annalen der Physik*, vol. **379**, no. 14, pp. 518–540, 1924.

[84] J. M. M. Senovilla, 'Equations for general shells', *Journal of High Energy Physics*, vol. **2018**, p. 134, 2018.

[85] R. Penrose, 'Twistor quantisation and curved space-time', *International Journal of Theoretical Physics*, vol. **1**, pp. 61–99, 1968.

[86] *Battelle Rencontres, 1967 Lectures in mathematics and physics*.

[87] R. Penrose, 'The geometry of impulsive gravitational waves', in *General Relativity: Papers in honour of J.L. Synge*, L. O'Raifeartaigh, Ed., 1972, pp. 101–115.

[88] J. Podolsky` and J. B. Griffiths, 'Nonexpanding impulsive gravitational waves with an arbitrary cosmological constant', *Physics Letters A*, vol. **261**, pp. 1–4, 1999.

[89] J. Podolsky` and J. B. Griffiths, 'Expanding impulsive gravitational waves', *Classical and Quantum Gravity*, vol. **16**, no. 9, p. 2937, 1999.

[90] J Podolsky` and M Ortaggio, 'Symmetries and geodesics in (anti-) de Sitter spacetimes with non-expanding impulsive waves', *Classical and Quantum Gravity*, vol. **18**, no. 14, p. 2689, 2001.

[91] J. Podolskỳ, 'Exact impulsive gravitational waves in spacetimes of constant curvature', in *Gravitation: Following the Prague Inspiration: A Volume in Celebration of the 60th Birthday of Jirí Bicˇák*, World Scientific, 2002, pp. 205–246.

[92] J. B. Griffiths and J. Podolskỳ, *Exact space-times in Einstein's General Relativity*. Cambridge University Press, 2009.

[93] M. Mars, T. Paetz and J. M. M. Senovilla, 'Multiple Killing horizons and near horizon geometries', *Classical and Quantum Gravity*, vol. **35**, no. 24, p. 245 007, 2018.

[94] M. Mars, T. Paetz and J. M. M. Senovilla, 'Multiple Killing horizons: The initial value formulation for Λ-vacuum', *Classical and Quantum Gravity*, vol. **37**, no. 2, p. 025 010, 2019.

[95] A. Ashtekar, S. Fairhurst and B. Krishnan, 'Isolated horizons: Hamiltonian evolution and the first law', *Physical Review D*, vol. **62**, no. 10, p. 104 025, 2000.

[96] A. Ashtekar, C. Beetle, O. Dreyer, S. Fairhurst, B. Krishnan, J. Lewandowski and J. Wis´niewski, 'Generic isolated horizons and their applications', *Physical Review Letters*, vol. **85**, no. 17, p. 3564, 2000.

[97] B. Krishnan, *Isolated horizons in numerical relativity*. The Pennsylvania State University, 2002.

[98] E. Gourgoulhon and J. L. Jaramillo, 'A 3 + 1 perspective on null hypersurfaces and isolated horizons', *Physics Reports*, vol. **423**, pp. 159–294, 2006.

[99] J. L. Jaramillo, 'Isolated horizon structures in quasiequilibrium black hole initial data', *Physical Review D*, vol. **79**, no. 8, p. 087 506, 2009.

[100] H. K. Kunduri and J. Lucietti, 'Classification of near-horizon geometries of extremal black holes', *Living Reviews in Relativity*, vol. **16**, pp. 1–71, 2013.

[101] I. Rácz and R. M. Wald, 'Extensions of spacetimes with Killing horizons', *Classical and Quantum Gravity*, vol. **9**, no. 12, p. 2643, 1992.

[102] B. C. Nolan, B. Reina and K. Sousa, 'First order perturbations of hypersurfaces of arbitrary causal character', *Classical and Quantum Gravity*, vol. **36**, no. 18, p. 185 016, 2019.

[103] M. Manzano and M. Mars, 'Null shells: General matching across null boundaries and connection with cut-and-paste formalism', *Classical and Quantum Gravity*, 2021.

[104] M. Manzano and M. Mars, 'General matching across Killing horizons of zero order', *Physical Review D*, vol. **106**, no. 4, p. 044 019, 2022.

[105] R. Bartnik and J. Isenberg, 'The Constraint Equations', in *The Einstein Equations and the Large Scale Behavior of Gravitational Fields: 50 Years of the Cauchy Problem in General Relativity*, Springer, 2004, pp. 1–38.

[106] K. Nomizu and S. Kobayashi, *Foundations of differential geometry, Vol. II*. New York: Interscience Publisher, 1969.

[107] W. P. A. Klingenberg, *Riemannian geometry*. Walter de Gruyter, 2011, vol. **1**.

[108] J. M. Lee, 'Introduction to smooth manifolds', *Graduate Texts in Mathematics*, vol. **218**, pp. 191–194, 2003.

[109] G. J. Galloway, 'Null geometry and the Einstein equations', in *The Einstein equations and the large scale behavior of gravitational fields*, Springer, 2004, pp. 379–400.

[110] K. L. Duggal and H. J. Dae, *Null curves and hypersurfaces of semi-Riemannian manifolds*. World Scientific Publishing Company, 2007.

[111] B. Carter and J. B. Hartle, *Gravitation in astrophysics: Cargèse 1986*. Springer Science & Business Media, 2012, vol. **156**.

[112] V. Frolov and I. Novikov, *Black hole physics: basic concepts and new developments*. Springer Science & Business Media, 2012, vol. **96**.

[113] R. M. Wald, 'General Relativity', *Chicago, University of Chicago Press*, 1984.

[114] S. Kobayashi, 'Fixed points of isometries', *Nagoya Mathematical Journal*, vol. **13**, pp. 63–68, 1958.

[115] J. M. Bardeen, B. Carter and S. W. Hawking, 'The four laws of black hole mechanics', *Communications in mathematical physics*, vol. **31**, no. 2, pp. 161–170, 1973.

[116] B. S. Kay and R. M. Wald, 'Theorems on the uniqueness and thermal properties of stationary, nonsingular, quasifree states on spacetimes with a bifurcate Killing horizon', *Physics Reports*, vol. **207**, no. 2, pp. 49–136, 1991.

[117] M. Mars, 'Stability of MOTS in totally geodesic null horizons', *Classical and Quantum Gravity*, vol. **29**, no. 14, p. 145 019, 2012.

[118] H. W. Brinkmann, 'On Riemann spaces conformal to Einstein spaces', *Proceedings of the National Academy of Sciences of the United States of America*, vol. **9**, p. 172, 1923.

[119] L. Witten, *Gravitation: an introduction to current research*. Wiley, New York, 1962.

[120] K. Yano, *The Theory of Lie Derivatives and Its Applications*. North-Holland Publishing CO, 1957.

[121] P. Chrusciel and J. Isenberg, 'Compact Cauchy horizons and Cauchy surfaces', *Directions in General Relativity: Papers in Honor of Dieter Bril*, vol. **2**, p. 97, 1993.

[122] E. Minguzzi, 'Area theorem and smoothness of compact Cauchy horizons', *Communications in Mathematical Physics*, vol. **339**, pp. 57–98, 2015.

[123] I. Bustamante and M. Reiris, 'A classification theorem for compact Cauchy horizons in vacuum spacetimes', *General Relativity and Gravitation*, vol. **53**, pp. 1–10, 2021.

[124] S. W. Hawking and G. F. R. Ellis, *The large scale structure of space-time*. Cambridge university press, 2023.

[125] H. P. Roesch, 'Proof of a null Penrose conjecture using a new quasi-local mass', *Communications in Analysis and Geometry*, vol. **29**, no. 8, pp. 1847–1915, 2021.

[126] M. Blau and M. O'Loughlin, 'Horizon shells and BMS-like soldering transformations', *Journal of High Energy Physics*, vol. **2016**, p. 29, 2016.

[127] S. Bhattacharjee and A. Bhattacharyya, 'Soldering freedom and BMS-like transformations', *arXiv:1707.01112*, 2017.

[128] E. Poisson, *A relativist's toolkit: the mathematics of black-hole mechanics*. Cambridge university press, 2004, pp. 98–107.

[129] V. Moncrief and J. Isenberg, 'Symmetries of cosmological Cauchy horizons', *Communications in Mathematical Physics*, vol. **89**, pp. 387–413, 1983.

[130] B. Chen, *Pseudo-Riemannian geometry, δ-invariants and applications*. World Scientific, 2011.

[131] B. O'Neill, *Semi-Riemannian geometry with applications to relativity*. Academic press, 1983.

[132] C. Efthimiou and C. Frye, *Spherical harmonics in p dimensions*. World Scientific, 2014.

[133] Á. Ferrández, Á. Giménez and P. Lucas, 'Geometry of lightlike submanifolds in Lorentzian space forms', in *Proc. del Congreso Geometria de Lorentz, Benalmadena*, 2001.

[134] M. Navarro, O. Palmas and D. A. Solis, 'Null hypersurfaces in generalized Robertson–Walker spacetimes', *Journal of Geometry and Physics*, vol. **106**, pp. 256–267, 2016.

GEOMETRÍA DE HIPERSUPERFICIES NULAS ABSTRACTAS Y ENLACE DE ESPACIOTIEMPOS

Autor:
Miguel Manzano

Supervisor:
Prof. Marc Mars

Resumen en castellano de la tesis doctoral
presentada para obtener el título de
DOCTOR EN FÍSICA FUNDAMENTAL Y MATEMÁTICAS

Departamento de Física Fundamental
Facultad de Ciencias
Instituto Universitario de Física Fundamental y Matemáticas

Universidad de Salamanca
Salamanca, España

Octubre 2023

DECLARACIÓN DEL SUPERVISOR

Dr. D. Marc Mars Lloret, Catedrático de Física Teórica en el Departamento de Física Fundamental de la Universidad de Salamanca,

CERTIFICA:

Que el trabajo de investigación que se recoge en la siguiente memoria titulada *Geometría de hipersuperficies nulas abstractas y enlace de espaciotiempos*, presentada por D. Miguel Manzano Rodríguez para optar al Título de Doctor por la Universidad de Salamanca con la Mención de Doctorado Internacional, ha sido realizada en su totalidad bajo su dirección y autoriza su presentación.

En Salamanca, a 2 de octubre de 2023.

Dr. D. Marc Mars Lloret,
Catedrático de Física Teórica,
Departamento de Física Fundamental,
Universidad de Salamanca.

PUBLICACIONES

1. M. Manzano and M. Mars, "Null shells: General matching across null boundaries and connection with cut-and-paste formalism", *Classical and Quantum Gravity*, 2021. DOI: 10.1088/1361-6382/abfd91.
2. M. Manzano and M. Mars, "General matching across Killing horizons of zero order", *Physical Review D*, vol. 106, no. 4, p. 044 019, 2022. DOI: 10.1103/PhysRevD.106.044019.
3. M. Manzano and M. Mars, "The Constraint Tensor: General Definition and Properties", 2023. ArXiv:2309.14813 [gr-qc].
4. M. Manzano and M. Mars, "Abstract Formulation of the Spacetime Matching Problem and Null Thin Shells", 2023. ArXiv:2309.14874 [gr-qc].
5. M. Manzano and M. Mars, "Embedded Hypersurface Data and Ambient Vector Fields: Abstract Killing Horizons of order zero/one", en preparación.
6. M. Manzano and M. Mars, "Master Equation for General Null Hypersurfaces", en preparación.

ESTANCIAS DE INVESTIGACIÓN

1. **Lugar:** Universität Wien, Fakultät für Mathematik, Viena, Austria

   Fechas: 8 al 13 de noviembre de 2021

2. **Lugar:** Universität Wien, Fakultät für Mathematik, Viena, Austria

   Fechas: 7 de junio al 6 de septiembre de 2022

3. **Lugar:** Erwin Schrödinger International Institute, Viena, Austria

   Fechas: 13 al 24 de marzo de 2023

ÍNDICE DE LA TESIS DOCTORAL

1 Introducción 1
1.1 Contexto y motivación 1
1.2 Objetivo de la tesis doctoral 13
1.3 Contenidos 13

2 Preliminares 19
2.1 Notación y convenciones 19
2.2 Formalismo de dato de hipersuperficie 20
2.2.1 Dato métrico de hypersuperficie 20
2.2.2 Dato de hypersuperficie 28
2.2.3 Dato de materia-hypersuperficie y ecuaciones de ligadura . . 32
2.3 Geometría de subvariedades 34
2.4 Geometría de hipersuperficies nulas 39
2.4.1 Espacio cociente en hipersuperficies nulas 43
2.4.2 Ecuación de Raychaudhuri en hipersuperficies nulas 45
2.4.3 Hipersuperficies nulas totálmente geodésicas 47
2.5 Horizontes no expansivos y (débilmente) aislados 49
2.6 Horizontes de Killing 55
2.6.1 Horizontes de Killing múltiples 58
2.6.2 Constancia de la gravedad superficial 63
2.7 Enlace de espaciotiempos 63
2.7.1 Capas delgadas: formalismo de dato de hipersuperficie 65
2.7.2 Método de *corte* y *pegado* 68

3 Nuevos resultados en el formalismo de dato de hipersuperficie 73
3.1 Dato de hipersuperficie general 74
3.1.1 Derivada de Lie de la conexión $\overset{\circ}{\nabla}$ a lo largo de n 78
3.1.2 Curvatura de la conexión métrica de hipersuperficie $\overset{\circ}{\nabla}$ 82
3.2 Dato nulo de hipersuperficie 85
3.2.1 Resultados de fijado de gauge 88
3.2.2 Curvatura de la conexión métrica de hipersuperficie $\overset{\circ}{\nabla}$ 92
3.2.3 Subvariedades transversas 94
3.2.4 Dato nulo de hipersuperficie que admite una sección 100
3.3 Vector invariante gauge a lo largo de la dirección degenerada 103

4 El tensor de ligadura 107
4.1 Definición y primeras propiedades 108
4.2 Tensor de ligadura: caso nulo 113
4.2.1 Tensor de ligadura a lo largo de la dirección nula n 114
4.2.2 Tensor de ligadura en una subvariedad transversa S 115
4.3 Cantidades gauge invariantes en una subvariedad transversa 117

5 Dato de hipersuperficie embebido y campos vectoriales en el ambiente . . 123
5.1 Derivada de Lie del tensor $\mathbf{Y}$ 124
5.2 Campo vectorial a lo largo de la dirección degenerada 128
5.3 El tensor $\Sigma_\eta \overset{\text{def}}{=} \pounds_\eta \nabla$ 131
5.4 Horizontes de Killing abstractos de orden cero y uno 138
5.4.1 Aspectos de los horizontes de Killing abstractos de orden cero 144
5.4.2 Aspectos de los horizontes de Killing abstractos de orden uno 145
5.4.3 Conexión con horizontes no expansivos y aislados 146

6 Ecuación maestra generalizada 149
6.1 Ecuación maestra generalizada en una hipersuperficie nula 150
6.2 Tensor deformación $\mathcal{K}^\eta$ proporcional a la métrica 155
6.3 Horizontes de Killing abstractos de orden cero y uno 161

6.4 Ecuación maestra generalizada en una subvariedad transversa 163
6.5 Horizontes degenerados de orden uno en vacío 167

7 Problema de enlace: enfoque espaciotemporal 171
7.1 Consideraciones previas y setup 172
7.2 Enlace de espaciotiempos con bordes de cualquier carácter causal . . 177
7.3 Enlace de espaciotiempos con bordes nulos 181
7.3.1 La función salto 185
7.3.2 Tensor de energía-momento de la capa delgada 189
7.3.3 Enlace de dos regiones del espaciotiempo de Minkoski a lo largo de un hiperplano nulo 201

8 Enlace a través de horizontes de Killing abstractos de orden cero 217
8.1 Caso de $\eta^{\pm}$ degenerados 221
8.2 Caso de $\eta^{\pm}$ no degenerados 222
8.3 Caso de η^{-} degenerado, η^{+} no degenerado 225
8.4 Horizontes de Killing con superficies de bifurcación 227
8.4.1 Ecuaciones de la capa delgada 233
8.5 Espaciotiempos con simetría esférica, plana o hiperbólica 235
8.5.1 Espaciotiempo de Schwarzschild 238
8.5.2 Espaciotiempo de Schwarzschild-de Sitter 240

9 Problema de enlace: enfoque abstracto 243
9.1 Enlace abstracto sin restricciones topológicas 244
9.1.1 Bordes nulos 248
9.1.2 Enlaces múltiples 258
9.2 Bordes nulos con topología producto $S \times \mathbb{R}$ 262
9.3 Construcción de corte y pegado: espaciotiempo de (anti-)de Sitter . . 267

10 Conclusiones 273
10.1 Formalismo de dato de hipersuperficie 273
10.2 Enlace de espaciotiempos 278

A Identidades generales para tensores de curvatura 285

B Identidad de Gauss generalizada 287

C Comportamiento gauge de $\mathfrak{S}_{\parallel}$ 291

D Coordenadas cerca de una hipersuperficie nula 295

D.1 Construcción de coordenadas cerca de una hipersuperficie nula . . . 295

D.1.1 Coordenadas de Rácz-Wald generalizadas 298

D.1.2 Coordenadas nulas gaussianas 299

D.2 Coordenadas cerca de un horizonte de Killing 299

Bibliografía 301

1 INTRODUCCIÓN

La teoría de la Relatividad General, formulada por Albert Einstein de manera completa en [1] por primera vez, ha demostrado ser la teoría fundamental más precisa para describir efectos gravitacionales a grandes escalas. Desde sus primeras predicciones (la precesión del perihelio de Mercurio [2], la curvatura de los rayos de luz [3], el efecto redshift gravitacional [4], [5] y la emisión de ondas gravitacionales [6], [7]) hasta otras más recientes (por ejemplo la existencia de agujeros negros [8], [9] o el efecto de lente gravitatoria [10]), la teoría de la Relatividad General ha anticipado con precisión muchos de los fenómenos naturales apoyados tiempo después por las observaciones empíricas. Ya desde su nacimiento, la teoría de la Relatividad General se ha mostrado inquebrantable y completamente consistente con las observaciones experimentales, sin importar el creciente nivel de precisión de los resultados observacionales. Su solidez la convierte hoy en la teoría de la gravedad más aceptada por la comunidad científica.

Dependiendo del enfoque y del tipo de problemas que se estudian, la teoría de Relatividad General se divide en varias áreas de investigación. A saber, *Relatividad Numérica* [11], basada en métodos numéricos y códigos de programación; *Astrofísica Relativista* [12] y *Cosmología* [13], cuyo objetivo es proveer modelos teóricos y computacionales así como estudiar aspectos experimentales de la teoría; o *Relatividad Matemática* [14], que aborda cuestiones fundamentales de la fśica gravitatoria con máximo rigor matemático. La tesis doctoral que aquí se describe se encuadra precisamente dentro de ésta última área.

A pesar de ser una teoría centenaria, existen multitud de problemas abiertos en Relatividad General Matemática. Por ejemplo, (las versiones débil y fuerte de) la Conjetura de Censura Cósmica [15], [16], [17], [18] y la Conjetura del Estado Final [19] (y problemas relacionados como la unicidad de agujeros negros y la estabilidad de agujeros negros de Kerr-Newman). Esto convierte a la Relatividad General Matemática en un campo de investigación muy activo en la actualidad.

La principal disciplina matemática en la que se apoya la Relatividad General Matemática es la geometría, cuyos objetos matemáticos principales son las variedades y los tensores. Uno de los pilares fundamentales de la geometría es el estudio de las hipersuperficies. En particular, éstas pueden ser temporales, espaciales, nulas o mixtas. Precisamente las de tipo nulo desempeñan un papel fundamental en Relatividad General, y constituyen el objeto central de estudio de esta tesis.

El ejemplo por excelencia de hipersuperficie nula es el cono de luz de un punto en un espaciotiempo. Sin embargo, existen incontables escenarios en los que las hipersuperficies nulas están involucradas. Por ejemplo, juegan un papel fundamental en el estudio de causalidad, en el contexto de emisión de ondas gravitaciones, en el análisis de la geometría del infinito nulo, en el problema característico, en el estudio de cualquier horizontes (de Cauchy, de eventos, de Killing...). Es por ello que entender la geometría de hipersuperficies nulas es clave para la comprensión de los aspectos físicos y matemáticos de la teoría de la Relatividad General. De hecho, las hipersuperficies nulas son esenciales porque describen, localmente, las trayectorias de rayos de luz que se emiten perpendicularmente a una superficie espacial de codimension dos.

La tesis doctoral que aquí se resume consta de dos partes diferenciadas. En la primera, se estudia la geometría de hipersuperficies nulas abstractas (esto es, considerando a la hipersuperficie como variedad en sí misma, sin necesidad de entenderla como embebida en un espacio ambiente). Este análisis se lleva a cabo por medio del llamado *formalismo de dato de hipersuperficie*, que permite codificar la geometría intrínseca y extrínseca de una hipersuperficie de cualquier carácter causal de manera abstracta.

La segunda parte de la tesis se centra en el problema de enlace de dos espaciotiempos (véase, por ejemplo, [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30]). Determinar las condiciones bajo las cuales dos espaciotiempos generales pueden enlazarse y dar lugar a un espaciotiempo nuevo es un problema fundamental en cualquier teoría métrica de la gravedad. Es importante analizar las propiedades del espaciotiempo resultante (en particular de la hipersuperficie de enlace). Un ejemplo paradigmático ocurre cuando se estudian campos gravitatorios generados por un objeto autogravitante, por ejemplo una estrella de neutrones. En ese contexto, el contenido material en la región interna de la estrella es distinto de cero, por tanto el campo gravitatorio debe satisfacer las ecuaciones de Einstein (o cualesquiera ecuaciones de campo que uno quiera imponer) con un término de fuente no cero. Por otro lado, en la región exterior no existe materia y por tanto el campo gravitatorio debe ser solución de las ecuaciones de campo de vacío. Las ecuaciones para el campo gravitatorio son distitas en las diferentes regiones, y por tanto las soluciones también han de serlo. Sin embargo, el espaciotiempo no está separado en dos partes, lo que hace esencial que se puedan enlazar las regiones externa e interna, dando lugar a una única solución. En el contexto de esta tesis doctoral, se consideraran capas de matería-energía de grosor tan fino que se puede asumir que se localizan sobre una hipersuperficie de carácter nulo.

2

CONTENIDOS DE LA TESIS DOCTORAL

La tesis doctoral consta de tres partes. En la primera, correspondiente al Capítulo 2, se discuten las definiciones matemáticas, herramientas y resultados de la literatura que se requieren más adelante a lo largo de la tesis. Se establecen nuestras convenciones de notación y se presenta el formalismo de dato de hipersuperficie [29], [31] (definiciones de dato (métrico) de hipersuperficie, construcciones de las conexiones abstractas $\overset{\circ}{\nabla}$, $\overline{\nabla}$, etcétera). Además, se repasan algunos aspectos clave de geometría de subvariedades, en particular de hipersuperficies nulas embebidas. También se revisan las definiciones y propiedades geométricas de varios tipos de hipersuperficies nulas que juegan un papel esencial más adelante en la tesis. A saber, horizontes no expansivos, (débilmente) aislados y de Kiling. Finalmente, se incluyen algunas consideraciones previas sobre enlace de espaciotiempos a través de una hipersuperficie.

El resto de la disertación presenta resultados originales obtenidos en el transcurso de la tesis doctoral. En particular, la segunda parte está dedicada al desarrollo del formalismo de dato de hipersuperficie. Esta tarea se lleva a cabo en los Capítulos 3, 4, 5 y 6, cuyo contenido se describe a continuación.

En el Capítulo 3, se proporcionan varios resultados nuevos en el marco del formalismo de dato de hipersuperficie. En particular, se estudia el tensor "derivada de Lie de una conexión" a lo largo de un vector privilegiado Z. Se obtienen varias identidades que involucran a Σ_Z y se analiza el caso particular del tensor "derivada de Lie de $\overset{\circ}{\nabla}$" a lo largo de un campo vectorial n que puede definirse a partir de cualquier dato métrico. También dentro del Capítulo 3 se estudia el caso de un dato nulo (esto es, el dato que se corresponde con la abstracción de una hipersuperficie nula). Se demuestran varios resultados de fijado de gauge, así como varias identidades nuevas que involucran a la conexión $\overset{\circ}{\nabla}$ y a los tensores de curvatura y de Ricci asociados a ella. Además, se incluye un análisis detallado de la geometría de una hipersuperficie nula abstracta que admite una subvariedad no degenerada de codimension uno. Finalmente, se estudia el caso en que una hipersuperficie nula abstracta admite un campo vectorial extra que es nulo e invariante bajo transformaciones gauge.

El Capítulo 4 se dedica al denominado *tensor de ligadura* $\mathcal{R}$. Dicho tensor se define para cualquier hipersuperficie abstracta de manera que, cuando el dato está embebido en una variedad semi-Riemanniana, éste captura una cierta combinación de componentes del tensor de Riemann del espacio ambiente. En primer lugar, se motiva su definición abstracta y se derivan algunas de sus propiedades. Posteriormente, se particulariza el análisis al caso nulo, encontrando las contracciones de $\mathcal{R}$ con un generador nulo y proporcionando su *pull-back* a cualquier subvariedad no degenerada dentro de la hipersuperficie abstracta. En particular, se calcula su relación con el tensor de Ricci de la métrica inducida en dicha subvariedad Riemanniana. Finalmente, se presentan varias cantidades que son invariantes bajo transformaciones gauge o tienen un comportamiento gauge simple. Los resultados de este capítulo son de utilidad en otras partes de la tesis.

El Capítulo 5 constituye una de las partes fundamentales de la tesis doctoral. En primer lugar, se considera un dato de hipersuperficie completamente general embebido en una variedad semi-Riemanniana equipada con un campo vectorial privilegiado $\mathfrak{y}$. Inicialmente, se permite que $\mathfrak{y}$ sea completamente arbitrario (en particular, no necesariamente tangente a la hipersuperficie). En este contexto, se derivan expresiones explícitas para el corchete de Lie de $\mathfrak{y}$ con cualquier extensión de un campo vectorial de *rigging* (esto es, cualquier campo transverso a la hipersuperficie en todos sus puntos). Luego, se examina el caso en el que $\mathfrak{y}$ es tangente y se obtiene la derivada de Lie del tensor de dato $\mathbf{Y}$, que codifica la geometría extrínseca de una hipersuperficie, a lo largo de $\mathfrak{y}$. Estos resultados involucran al tensor de deformación de $\mathfrak{y}$. Posteriormente, el análisis se centra en el caso en que la hipersuperficie es nula y $\mathfrak{y}$ es nulo y tangente a la hipersuperficie. En este contexto, se estudia el tensor "Derivada de Lie a lo largo de $\mathfrak{y}$ de la conexión de Levi-Civita", es decir, $\Sigma_{\mathfrak{y}}$. Se calcula la forma explícita de $\Sigma_{\mathfrak{y}}$ en términos del dato más un campo tensorial adicional $\text{ה}^{\mathfrak{y}}$ que resulta desempeñar un papel crucial en la descripción abstracta de horizontes de Killing de orden cero y uno. Éstos son nuevos conceptos abstractos de horizontes que motivamos y presentamos también en el Capítulo 5.

El Capítulo 6 es el último dedicado al desarrollo del formalismo de dato de hipersuperficie. En él, se obtiene la denominada *ecuación maestra generalizada*. Dicha ecuación se cumple para toda hipersuperficie nula que admita un vector nulo tangente privilegiado $\mathfrak{y}$. La ecuación maestra generalizada involucra la función de proporcionalidad entre $\mathfrak{y}$ y un generador nulo de la hipersuperficie, el *tensor de ligadura* $\mathcal{R}$, el tensor $\text{ה}^{\mathfrak{y}}$ mencionado anteriormente y varios tensores abstractos. En este capítulo, se obtienen también las contracciones de la ecuación maestra generalizada con un generador nulo. Posteriormente, se particulariza el análisis al caso en el que el tensor de deformación de η es proporcional a la métrica. En este contexto, se obtienen varios resultados interesantes relacionados con el conjunto de puntos fijos de η, la regularidad del tensor de Ricci del espacio ambiente y la constancia de la gravedad superficial $\widetilde{\kappa}$ de η. También se particularizan los resultados para horizontes de Killing abstractos de orden cero y uno. Esto permite identificar algunas consecuencias de que $\widetilde{\kappa}$ no sea constante. Otro resultado clave de este capítulo es la restricción de la ecuación maestra generalizada a cualquier subvariedad no degenerada dentro de una hipersuperficie nula. Como caso particular, recuperamos la *ecuación maestra* de horizontes de Killing múltiples [32], [33], [34], así como la *near-horizon equation* para un horizonte aislado [35], [36], [37], [38], [39], [40]. Finalmente, se aplican los resultados previos al caso de un horizonte de Killing degenerado en el vacío.

La tercera parte de esta tesis, correspondiente a los Capítulos 7, 8 y 9, se dedica al problema de enlace de dos espaciotiempos completamente generales a través de una hipersuperficie nula.

En el Capítulo 7 se aborda el problema de enlace desde un punto de vista espaciotiemporal, es decir, sin considerar los bordes de los espaciotiempos a enlazar de manera independiente. Asumiendo que los bordes pueden ser foliados por una familia de secciones espaciales, se determinan las condiciones necesarias y suficientes para que el enlace sea posible, y éstas se escriben en términos de una base de campos vectoriales. Además, se demuestra que toda la información de enlace se puede codificar en la llamada *función salto* y en un difeomorfismo entre el conjunto de generadores nulos de cada borde. Resulta que, cuando los bordes son totalmente geodésicos y los espaciotiempos pueden enlazarse de una manera, entonces infinitos enlaces son posibles. Se obtienen expresiones explícitas para el contenido de energía-materia de la capa delgada nula más general posible resultante de un enlace de este tipo. Finalmente, se aplican los resultados al caso del enlace de dos regiones del espaciotiempo de Minkowski a través de un hiperplano nulo. Esto nos permite relacionar nuestros resultados con los de las construcciones de *corte y pegado* de la literatura.

En el Capítulo 8, se estudia un caso particular de lo anterior. A saber, el escenario en que los bordes de los espaciotiempos a enlazar son horizontes de Killing abstractos de orden cero. La idea es analizar la situación en la que el enlace identifica los campos vectoriales de "Killing" de orden cero. Se abordan los casos en que (a) ambos bordes son no degenerados, (b) ambos son degenerados y (c) un borde es degenerado y el otro no degenerado. También se particularizan los resultados para el caso de horizontes de Killing con superficies de bifurcación. El capítulo concluye con un análisis detallado del caso en que se enlazan dos espaciotiempos con simetría esférica, plana o hiperbólica.

El Capítulo 9 es el último dedicado al problema de enlace, y constituye otra parte fundamental de la tesis por varias razones. Primero, porque se aborda el problema de enlace desde un punto de vista puramente abstracto (es decir, sin requerir que las hipersuperficies de enlace estén embebidas) y, en segundo lugar, porque los resultados son completamente generales (en el sentido de que no imponemos restricciones topológicas ni ninguna otra condición en las hipersuperficies nulas y los espaciotiempos). Primero, se establece una formulación abstracta del problema de enlace. Posteriormente, se analiza el caso nulo, para el que se obtienen expresiones explícitas del contenido gravitatorio/material de la capa delgada nula resultante. También se analiza el escenario de múltiples enlaces, se recuperan los resultados del Capítulo 7 para el caso con bordes con topología de producto y se incluye un ejemplo de enlace a través de una hipersuperficie nula totalmente geodésica en los espaciotiempos de (anti-)de Sitter y Minkowski.

Finalmente, en el Capítulo 10 de la tesis doctoral se recogen las conclusiones de nuestro trabajo, así como algunas perspectivas de trabajo futuras.

La tesis doctoral incluye cuatro apéndices. En el Apéndice A, demostramos varias identidades generales relacionadas con el tensor de curvatura de una conexión libre de torsión. El Apéndice B está dedicado a la derivación de una forma generalizada de la identidad de Gauss. En el Apéndice C, ofrecemos una comprobación de la consistencia del comportamiento de calibre de un campo tensorial introducido en el Capítulo 4. La tesis concluye con el Apéndice D, donde presentamos una nueva construcción geométrica de coordenadas cerca de cualquier hipersuperficie nula. El punto esencial de dicha construcción es que permite recuperar las llamadas coordenadas nulas gaussianas (ver, por ejemplo, [41]) y coordenadas de Rácz-Wald [42] en un entorno de una hipersuperficie nula y una superficie de bifurcación, respectivamente.

3

CONCLUSIONES

Dada la estructura de la tesis doctoral, es conveniente separar las conclusiones en dos partes. En la primera, expondremos los resultados correspondientes al estudio de hipersuperficies nulas abstractas, mientras que en la segunda presentaremos los avances relacionados con el problema de enlace de espaciotiempos.

HIPERSUPERFICIES NULAS ABSTRACTAS

Hemos demostrado que, en el caso de hipersuperficies nulas abstractas, toda la información acerca de la geometría intrínseca de la hipersuperficie puede ser codificada en un único tensor γ, que juega el papel de primera forma fundamental en el contexto embebido. Además, cuando la hipersuperficie admite un campo privilegiado que es invariante gauge y nulo, se puede definir una función invariante gauge que, en el caso embebido, coincide con la gravedad superficial de dicho vector privilegiado. Esta función constituye, por tanto, una extensión (a nivel abstracto) de la gravedad superficial ambiente a los puntos donde el vector privilegiado se anula.

Para una hipersuperficie abstracta $\mathcal{N}$ de cualquier carácter causal, hemos construido un tensor abstracto, el tensor de ligadura $\mathcal{R}$, que, en el caso embebido, codifica una cierta combinación de componentes de los tensores de Riemann y Ricci del espacio ambiente y que, en puntos nulos, coincide con el pull-back a $\mathcal{N}$ del tensor de Ricci ambiente. Esto se ha conseguido sin requerir ninguna suposición topológica global sobre $\mathcal{N}$. Además, la definición de $\mathcal{R}$ es completamente covariante en $\mathcal{N}$ a pesar de que $\mathcal{N}$ no esté equipado con un tensor métrico. Cuando $\mathcal{N}$ es nula, hemos obtenido una versión abstracta de la ecuación de Raychaudhuri. Además, hemos identificado varias cantidades con un comportamiento gauge simple. En particular, una de ellas, el tensor $\mathfrak{S}_{\parallel}$ (que puede definirse en una subvariedad no degenerada de codimension uno en $\mathcal{N}$), codifica información sobre la curvatura y juega un papel clave en la geomertría de horizontes de Killing abstractos de orden uno.

Otro punto fundamental de la tesis es el análisis de las propiedades del tensor $\Sigma_{\mathfrak{y}}$ para un vector $\mathfrak{y}$ invariante gauge, nulo y tangente a una hipersuperficie nula. Esto nos ha permitido relacionar $\Sigma_{\mathfrak{y}}$ con el tensor deformación de $\mathfrak{y}$, así como encontrar varios tensores que desempeñan un papel clave en la descripción abstracta de horizontes de Killing abstractos de orden cero y uno, definidos por primera vez en el Capítulo 5.

Precisamente el estudio del tensor $\Sigma_{\mathfrak{y}}$, junto con el cálculo de la derivada de Lie del tensor de dato $\mathbf{Y}$ (que, recordemos, codifica la geometría extrínseca de $\mathcal{N}$), nos ha permitido obtener la ecuación maestra generalizada para cualquier hipersuperficie nula. Esta ecuación (y sus contracciones con un generador nulo) permiten identificar bajo qué condiciones la gravedad superficial κ de $\mathfrak{y}$ permanece constante. Además, para un horizonte de Killing homotético, esta identidad permite demostrar que no pueden existir puntos fijos, y que κ es constante si y solo si $\mathcal{R}(\mathfrak{y}, \mathfrak{y})$ es también constante.

Las condiciones necesarias y suficientes para que la gravedad superficial de $\mathfrak{y}$ sea constante en todas partes en un horizonte de Killing de orden cero o uno han sido obtenidas. Esto nos ha permitido demostrar que, si el horizonte es tal que la gravedad superficial no es constante en un punto, entonces éste no puede esr geodésicamente completo.

ENLACE DE ESPACIOTIEMPOS

En lo que se refiere al problema de enlace de espaciotiempos, las principales conclusiones son las siguientes. En primer lugar, se ha conseguido formular el problema de enlace de manera completamente abstracta, lo que permite incluso estudiar capas delgadas de manera desligada de cualquier espacio ambiente, para luego analizar si dicha capa es embebible o no. Tanto para bordes nulos con topología arbitraria como para bordes nulos con topología producto, se ha conseguido determinar el contenido material de la capa delgada de manera explícita, así como demostrar que toda la información del enlace está codificada por un difeomorfismo φ a nivel abstracto (que contiene exactamente la misma información que la función salto y el difeomorfismo de enlace entre los conjuntos de generadores nulos de ambos bordes).

Se ha estudiado ampliamente el caso de enlaces múltiples, que en particular ocurre cuando los bordes nulos son totálmente geodésicos. En este contexto, se ha demostrado que, dado el contenido material de uno de los enlaces, se puede obtener el de cualquier otro enlace simplemente determinando su difeomorfismo φ asociado. Esto, en particular, permite conocer de manera automática el contenido material de cualquier capa delgada nula generada mediante el procedimiento de corte y pegado.

Hemos analizado en detalle el caso del enlace de dos regiones del espaciotiempo de Minkowski a lo largo de un hiperplano nulo, obteniendo la forma explícita de la métrica C^0 del espaciotiempo resultante y expresiones explícitas para el contenido material. El enlace de dos regiones del espaciotiempo de (anti-)de Sitter a lo largo de una hipersuperfice nula totálmente geodésica se ha estudiado también. En concreto, se ha obtenido que la densidad de energía y el flujo de energía son los correspondientes al enlace de Minkowski más un término adicional. La presión, por otra parte, es la misma en los casos de (anti-)de Sitter y Minkowski. Estos ejemplos han permitido conectar las construcciones de corte y pegado con el formalismo de enlace, tanto espaciotemporal como abstracto.

BIBLIOGRAPHY

[1] A. Einstein, 'Die Grundlage der allgemeinen Relativitätstheorie', *Annalen der Physik*, vol. **354**, no. 7, pp. 769–822, 1916. DOI: https://doi.org/10.1002/andp.19163540702.

[2] A. Einstein, 'Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie', *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, pp. 831–839, 1915.

[3] F. W. Dyson, A. S. Eddington and C. Davidson, 'A determination of the deflection of light by the Sun's gravitational field, from observations made at the total eclipse of may 29, 1919', *Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character*, vol. **220**, no. 571-581, pp. 291–333, 1920.

[4] W. S. Adams, 'The relativity displacement of the spectral lines in the companion of Sirius', *Proceedings of the National Academy of Sciences*, vol. **11**, no. 7, pp. 382–387, 1925.

[5] D. M. Popper, 'Red Shift in the Spectrum of 40 Eridani B', *Astrophysical Journal, vol. 120, p. 316*, vol. **120**, p. 316, 1954.

[6] A. Einstein, 'Näherungsweise Integration der Feldgleichungen der Gravitation', *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften*, pp. 688–696, 1916.

[7] A. Einstein, 'Über Gravitationswellen', *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, pp. 154–167, 1918.

[8] R. Penrose, 'Gravitational collapse and space-time singularities', *Physical Review Letters*, vol. **14**, no. 3, p. 57, 1965.

[9] S. W. Hawking and R. Penrose, 'The singularities of gravitational collapse and cosmology', *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, vol. **314**, no. 1519, pp. 529–548, 1970.

[10] D. Walsh, R. F. Carswell and R. J. Weymann, '0957+561 a, b: Twin quasistellar objects or gravitational lens?', *Nature*, vol. 279, p. 381, 1979.

[11] L. Lehner, 'Numerical relativity: A review', *Classical and Quantum Gravity*, vol. **18**, no. 17, R25, 2001.

[12] I. B. Zeldovich and I. D. Novikov, *Relativistic Astrophysics, Vol. 2: The Structure and Evolution of the Universe*. University of Chicago press, 1971.

[13] S. Dodelson and F. Schmidt, *Modern cosmology*. Academic press, 2020.

[14] P. Chruściel, G. Galloway and D. Pollack, 'Mathematical General Relativity: A sampler', *Bulletin of the American Mathematical Society*, vol. **47**, no. 4, pp. 567–638, 2010.

[15] R. Penrose, 'Gravitational collapse: The role of General Relativity', *Nuovo Cimento Rivista Serie*, vol. **1**, p. 252, 1969.

[16] R. Penrose, 'Singularities and time-asymmetry', in *General Relativity, an Einstein Centenary Survey. Eds. S. W. Hawking and W. Israel*, Cambridge University Press, Cambridge, UK, 1979.

[17] D. Christodoulou, 'On the global initial value problem and the issue of singularities', *Classical and Quantum Gravity*, vol. **16**, no. 12A, A23, 1999.

[18] K. Landsman, 'Singularities, black holes, and cosmic censorship: A tribute to Roger Penrose', *Foundations of Physics*, vol. **51**, pp. 1–38, 2021.

[19] S. Yau, *Seminar on differential geometry*. Princeton University Press, 1982.

[20] G. Darmois, 'Les équations de la gravitation einsteinienne', *Mémorial des Sciences Mathématiques, Fascicule XXV (Paris: Gauthier-Villars)*, vol. **44**, 1927.

[21] S. O'Brien, 'Jump conditions at discontinuities in General Relativity', *Communications of the Dublin Institute for Advanced Studies*, vol. **9**, 1952.

[22] P. Le Corbeiller, 'Theories Relativistes de la Gravitation et de l'Electromagnetisme. Relativite Generale et Theories Unitaires. A. Lichnerowicz. Masson, Paris, 1955. xii+ 298 pp. Illus. F. 2800.', *Science*, vol. **122**, no. 3166, pp. 424–424, 1955.

[23] W. Israel, 'Singular hypersurfaces and thin shells in General Relativity', *Il Nuovo Cimento B*, vol. **44**, pp. 1–14, 1966.

[24] W. B. Bonnor and P. A. Vickers, 'Junction conditions in General Relativity', *General Relativity and Gravitation*, vol. **13**, no. 1, pp. 29–36, 1981.

[25] C. J. S. Clarke and T. Dray, 'Junction conditions for null hypersurfaces', *Classical and Quantum Gravity*, vol. **4**, p. 265, 1987.

[26] C. Barrabés and W. Israel, 'Thin shells in general relativity and cosmology: The lightlike limit', *Physical Review D*, vol. **43**, pp. 1129–1142, 1991.

[27] M. Mars and J. M. M. Senovilla, 'Geometry of general hypersurfaces in spacetime: Junction conditions', *Classical and Quantum Gravity*, vol. **10**, p. 1865, 1993.

[28] M. Mars, J. Senovilla and R. Vera, 'Lorentzian and signature changing branes', *Physical Review D*, vol. **76**, p. 044 029, 2007.

[29] M. Mars, 'Constraint equations for general hypersurfaces and applications to shells', *General Relativity and Gravitation*, vol. **45**, pp. 2175–2221, 2013.

[30] J. M. M. Senovilla, 'Equations for general shells', *Journal of High Energy Physics*, vol. **2018**, p. 134, 2018.

[31] M. Mars, 'Hypersurface data: General properties and Birkhoff theorem in spherical symmetry', *Mediterranean Journal of Mathematics*, vol. **17**, pp. 1–45, 2020.

[32] M. Mars, T. Paetz and J. M. M. Senovilla, 'Multiple Killing horizons', *Classical and Quantum Gravity*, vol. **35**, no. 15, p. 155 015, 2018.

[33] M. Mars, T. Paetz and J. M. M. Senovilla, 'Multiple Killing horizons and near horizon geometries', *Classical and Quantum Gravity*, vol. **35**, no. 24, p. 245 007, 2018.

[34] M. Mars, T. Paetz and J. M. M. Senovilla, 'Multiple Killing horizons: The initial value formulation for Λ-vacuum', *Classical and Quantum Gravity*, vol. **37**, no. 2, p. 025 010, 2019.

[35] A. Ashtekar, S. Fairhurst and B. Krishnan, 'Isolated horizons: Hamiltonian evolution and the first law', *Physical Review D*, vol. **62**, no. 10, p. 104 025, 2000.

[36] A. Ashtekar, C. Beetle, O. Dreyer, S. Fairhurst, B. Krishnan, J. Lewandowski and J. Wiśniewski, 'Generic isolated horizons and their applications', *Physical Review Letters*, vol. **85**, no. 17, p. 3564, 2000.

[37] B. Krishnan, *Isolated horizons in numerical relativity*. The Pennsylvania State University, 2002.

[38] A. Ashtekar, C. Beetle and J. Lewandowski, 'Geometry of generic isolated horizons', *Classical and Quantum Gravity*, vol. **19**, no. 6, p. 1195, 2002.

[39] E. Gourgoulhon and J. L. Jaramillo, 'A $3+1$ perspective on null hypersurfaces and isolated horizons', *Physics Reports*, vol. **423**, pp. 159–294, 2006.

[40] J. L. Jaramillo, 'Isolated horizon structures in quasiequilibrium black hole initial data', *Physical Review D*, vol. **79**, no. 8, p. 087 506, 2009.

[41] H. K. Kunduri and J. Lucietti, 'Classification of near-horizon geometries of extremal black holes', *Living Reviews in Relativity*, vol. **16**, pp. 1–71, 2013.

[42] I. Rácz and R. M. Wald, 'Extensions of spacetimes with Killing horizons', *Classical and Quantum Gravity*, vol. **9**, no. 12, p. 2643, 1992.