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Has the interaction between skewness and kurtosis of asset returns information content for risk forecasting?

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ABSTRACT

This paper introduces the effect of the crossed products of Hermite polynomials on Gram-Charlier densities. This allows capturing the impact of the interaction between skewness and kurtosis and evaluating this new parameter as an additional source of information for risk management. We show that our modified Gram-Charlier density presents an improved accuracy, especially at distribution tails. Risk quantification is assessed for S&P500 losses with backtesting procedures for Value-at-Risk and Median Shortfall.

1. Introduction

The leptokurtosis and skewness pattern of asset returns is a well-known fact since the early work of Mandelbrot (1963). Financial crises and turbulences have evidenced that the shocks have a big impact on the asset return distribution, generating fat tails but also an asymmetric response on both tails. Despite both features have a joint impact on tail risk, in finance researchers have not paid attention to the interaction between both phenomena – e.g., in physics, Labit et al. (2007) found a parabolic relationship between the third and fourth normalized central moments. We address this gap by showing to what extent the moment interaction may contribute to financial risk forecasting.

For this purpose, we adopt a semi-nonparametric approach to the asset return density based on the Gram-Charlier (GC) series, which has been employed in the last decades to approximate frequency functions –see e.g. Del Brio et al. (2020), Jiménez et al. (2022), León and Ñíguez (2020), Mauleón (2010), Mauleón and Perote (2000), among others. This methodology allows a flexible fitting of the density by incorporating the effect of different moments with the addition of new parameters. The salient performance of this approach has been proved for derivative pricing and risk management, showing the importance of the consideration of skewness, kurtosis and even higher order moments. However, to the best of our knowledge, the impact of the interaction terms in the Hermite polynomial (HP) approximation has not been tested before. The rationale behind the consideration of the interaction terms is twofold: (i) From a theoretical viewpoint the expansion with crossed products of HPs is still a density in virtue of the orthogonality property of such

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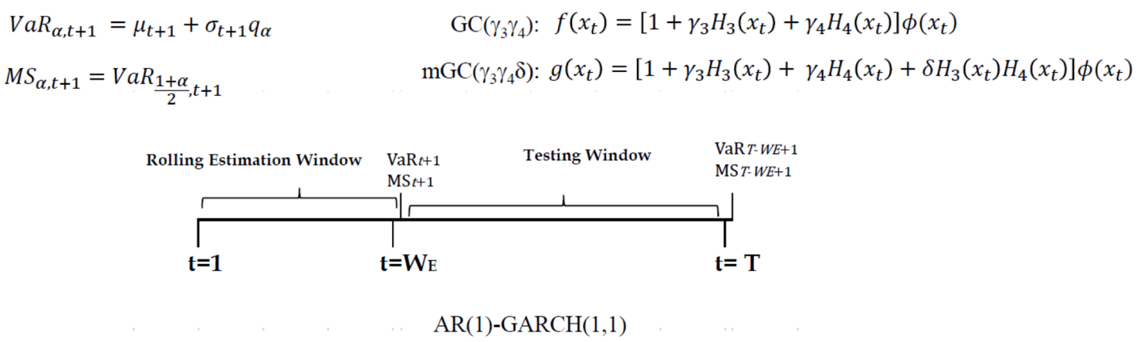

Fig. 1. Sketch on the backtesting procedure.

polynomials; (ii) The parameters associated to these 'crossed terms' capture the interaction between the moments, which might play an important role on the dynamic fitting. Therefore, this paper introduces a brand-new version of GC, named as Modified Gram Charlier (mGC), and shows the relevance of the interaction between even and odd moments (particularly between skewness and kurtosis) for the risk forecasting of a series of S&P500 index. Therefore, our contribution is valuable to both semi-nonparametric density fitting (a new density is provided) and risk management (risk assessment accuracy is proved) areas.

The next section presents the mGC distribution and the risk measures employed for assessing model performance. An empirical application is presented in Section 3 and, finally, Section 4 concludes.

2. Methodology

2.1. The semi-nonparametric approach

Traditionally, the semi-nonparametric estimation of a frequency function is characterized by a series of derivatives of a standard normal density $\varphi(x_t)$, which, considering n terms, can be expressed as a function $f(x_t)$ satisfying

$$f(x_t) = \left[1 + \sum_{s=1}^{n} \gamma_s H_s(x_t)\right] \phi(x_t), \tag{1}$$

where $H_s(x_t)$ is the orthogonal series of HPs, which are recursively obtained as $(-D)^s \varphi(x_t) = H_s(x_t)\varphi(x_t)$, $D = \frac{d}{dx_t}$. As $n \to \infty$ the Eq. (1) is called the GC series of Type A (Kendall and Stuart, 1977), which asymptotically represents any "regular" frequency function. This expansion integrates one but, for n finite, positivity is only satisfied in a restricted range −see e.g. Jondeau and Rockinger (2001). For solving this problem there have been proposed different positive transformations (León et al., 2009; Ñíguez and Perote, 2012) but also it has been recently argued that the positivity region enlarges as n increases (Lin and Zhang, 2022). Furthermore, the γ_s parameters are directly related to the first s moments, which allows measuring their relative importance in the fitting. We argue that by including the crossed HP terms as in Eq. (2) −modified GC, hereafter mGC− the interaction between the moments can be explicitly captured by δ_{ij} parameters. Even more, the inclusion of the crossed terms might contribute to avoiding negative values by means of a more flexible parametrization.

$$g(x_t) = \left[1 + \sum_{s=1}^{n} \gamma_s H_s(x_t) + \sum_{i=1}^{n} \gamma_s \sum_{j=1, j\neq i}^{n} \delta_{ij} H_i(x_t) H_j(x_t)\right] \phi(x_t). \tag{2}$$

Most applications in finance −e.g. Jurczenko et al. (2004), León et al. (2009), Schlögl (2013)− only consider $H_3(x_t) = x_t^3 - 3x_t$ and $H_4(x_t) = x_t^4 - 6x_t^2 + 3$, so as γ_3 and γ_4 are related to skewness and excess kurtosis, respectively. We additionally consider the interaction between $H_3(x_t)$ and $H_4(x_t)$ −captured with parameter δ− to better accommodate the fitting at the tails:

$$g(x_t) = [1 + \gamma_3 H_3(x_t) + \gamma_4 H_4(x_t) + \delta H_3(x_t) H_4(x_t)]\phi(x_t) \tag{3}$$

Note that $\int g(x_t)dx_t = 1$ since $\int H_i(x_t) H_j(x_t) \phi(x_t) dx_t = 0, \forall i \neq j$.

2.2. Risk measures

Model validation in risk management is usually tested in terms Value-at-Risk (VaR). Alternatively, Kou and Peng (2014) suggest the use of Median Shortfall (MS), which outperforms expected shortfall (ES) in terms of tail risk, robustness, elicitability, backtesting, and surplus invariance −He et al. (2021). For a return with conditional mean μ_t, variance σ_t and mGC distribution, and given the confidence level α and the time horizon $t+1$, VaR and MS may be obtained as

$$VaR_{\alpha,t+1} = \mu_{t+1} + \sigma_{t+1} q_\alpha, \tag{4}$$

Table 1
Descriptive statistics for S&P500 losses.

Obs.	Min.	Median	Mean	Max.	Standard Deviation	Variance	Excess Kurtosis	Skewness
2790	−8.968	−0.070	−0.048	12.765	1.078	1.162	17.844	0.927

S&P 500 index percentage losses from December 7th, 2010 to January 6th, 2022.

Table 2
Fitted GC densities for S&P500.

Parameter	GC3(γ_3)	GC4(γ_4)	GC($\gamma_3\gamma_4$)	mGC($\gamma_3\gamma_4\delta$)
φ_0	−0.088	−0.105	−0.082	−0.112
	(0.000)	(0.000)	(0.000)	(0.000)
φ_1	−0.040	−0.045	−0.057	−0.052
	(0.337)	(0.328)	(0.319)	(0.313)
ω	0.041	0.035	0.034	0.037
	(0.000)	(0.000)	(0.000)	(0.000)
α	0.140	0.149	0.147	0.138
	(0.000)	(0.000)	(0.000)	(0.000)
β	0.810	0.810	0.811	0.813
	(0.000)	(0.000)	(0.000)	(0.000)
γ_3	0.033	–	0.065	0.033
	(0.000)	–	(0.000)	(0.058)
γ_4	–	0.050	0.045	0.049
	–	(0.000)	(0.000)	(0.000)
δ	–	–	–	0.002
	–	–	–	(0.000)
AIC	591.068	567.568	554.500	550.929
LogLik	−289.534	−277.784	−270.250	−268.464

φ_0 and φ_1 are the parameters of the AR(1) model; ω, α and β are the parameters of the GARCH (1,1) model; γ_3, γ_4 and δ are the parameters of the GC/mGC distributions. P-values in parentheses. AIC and LogLik correspond to Akaike Information Criterion and Log-Likelihood, respectively.

$$MS_{\alpha,t+1} = VaR_{\frac{1+\alpha}{2},\ t+1} \tag{5}$$

where q_α is the α-quantile of the GC distribution computed for the mGC in Eq. (3) according to

$$\int_{-\infty}^{q_\alpha} g(x_t)dx_t = \int_{-\infty}^{q_\alpha} \phi(x_t)dx_t - \phi(q_\alpha)[\gamma_3 H_2(q_\alpha) + \gamma_4 H_3(q_\alpha) + \delta H_4(q_\alpha)H_2(q_\alpha) + 4\delta H_3(q_\alpha)H_1(q_\alpha) + 12\delta H_2(q_\alpha) + 24\delta]. \tag{6}$$

Following Basel Committee's recommendations, we select 99%-VaR and 99%-MS. In the next section, backtesting techniques (with a rolling window of size W_E) are considered to examine mGC's performance to forecast ex-post real losses – see e.g., Jiménez et al. (2020, 2022). Fig. 1 illustrates the procedure.

3. Empirical application

To show the risk assessment of incorporating the mGC density, we select an arbitrary series of S&P500, comprising 12 years of daily prices (P_t), spanning from December 7th, 2010 to January 6th, 2022 (2790 observations).[1] Focusing on the right tail, we compute daily percentage losses as

$$L_t = -100[\ln(P_t) - \ln(P_{t-1})]. \tag{7}$$

Descriptive statistics in Table 1 feature the characteristics of a heavy-tailed and skewed distribution.

Table 2 shows the Maximum likelihood (ML) estimates of GC (with parameters γ_3 or/and γ_4) and mGC with an AR(1)-GARCH(1,1) for the mean-variance modeling.

The ML estimates are computed for the first in-sample window of the backtesting with a size of 1000 observations, jointly estimated with an AR-GARCH structure and the four different density specifications. The odd and even parameters (γ_3 and γ_4) are significant, showing positive skewness and leptokurtosis. Importantly, the "crossed-parameter" (δ) is also significant and the mGC is better off in terms of Log-likelihood and AIC criteria, reflecting the information content of the interaction between skewness and kurtosis.[2] The relevance of this parameter in the fitting is illustrated in Fig. 2, where the right tails for the S&P500 losses reveal that the mGC density seems to outperform the other alternatives at the extremes.

[1] Results are robust to different time periods and the use of other leptokurtic and skewed financial return series.
[2] The parameter (δ) is significant for the full backtesting (1,790 times) with only 17 exceptions at 10% confidence.

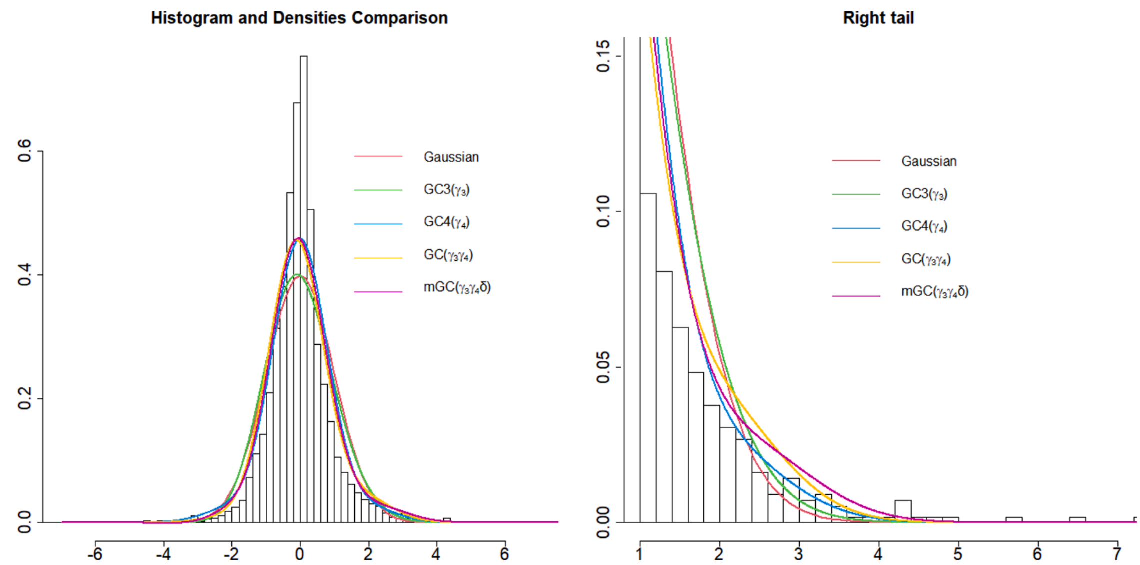

Fig. 2. Fitted densities for standardized S&P500 losses.

Table 3
Backtesting 99%-VaR and 99%-MS.

Model	Exc.	Bern. Test		Indep. Test		CC Test		DQ Test		QL	QL Ratio
Panel A: VaR 99% E.E. 18											
γ_3	39	18.796	(0.000)	1.193	(0.275)	19.989	(0.000)	48.356	(0.000)	0.038	–
γ_4	23	1.347	(0.246)	1.073	(0.300)	2.419	(0.298)	6.173	(0.520)	0.036	0.943
$\gamma_3\gamma_4$	24	1.897	(0.168)	4.195	(0.041)	6.092	(0.048)	16.487	(0.021)	0.036	0.947
$\gamma_3\gamma_4\delta$	19	0.067	(0.796)	1.674	(0.196)	1.741	(0.419)	6.035	(0.536)	0.036	0.945
Panel B: MS 99% E.E. 9											
Model	Exc.	Bern. Test		Indep. Test		CC Test		DQ Test		QL	QL Ratio
γ_3	18	21.519	(0.000)	3.633	(0.057)	25.151	(0.000)	62.322	(0.000)	0.024	–
γ_4	16	4.518	(0.034)	0.289	(0.591)	4.807	(0.090)	8.263	(0.310)	0.021	0.875
$\gamma_3\gamma_4$	20	10.132	(0.001)	1.505	(0.220)	11.637	(0.003)	24.621	(0.001)	0.021	0.889
$\gamma_3\gamma_4\delta$	15	3.412	(0.065)	0.254	(0.615)	3.666	(0.160)	7.643	(0.365)	0.020	0.855

E.E. stands for the expected exceptions and Exc. is the number of observed. Conditional Coverage (CC) tests the null hypothesis of correct model specification, where exceptions satisfy the Unconditional Coverage and Independence test. Dynamic Quantile (DQ) (with 4 lags) tests the null hypothesis of correct model specification. P-values in parentheses (significance level is 5%). QL (Quantile Loss) ratio compares every model with GC (γ_3), QL<1 means that the model outperforms the first model. Furthermore, QL ratio for GC ($\gamma_3\gamma_4$) in comparison to mGC ($\gamma_3\gamma_4\ \delta$) provides values of 0.997 and 0.962 for 99%-VaR and 99%-MS, respectively, being GC ($\gamma_3\gamma_4$) the first model for this case.

The in-sample results are corroborated by the out-of-sample performance in terms of both VaR and MS criteria. Table 3 presents backtesting results in panels A and B, for 99%-VaR and 99%-MS, respectively.[3] Model performance is assessed in terms of the Bernoulli test, Independence test, Conditional Coverage Test (CC), Dynamic Quantile (DQ) test, and QL ratio, which exhibit robust results. Expansions that only account for skewness (γ_3) are rejected and the model with kurtosis parameter alone (γ_4), although significant, presents a poorer performance than the mGC model. In addition, mGC outperforms GC($\gamma_3\gamma_4$), according to the QL ratio. Furthermore, in both risk measures the Bernoulli test shows the best performance for mGC considering the number of observed exceptions which are the closest to the expected exceptions. Thus, the mGC model exhibits a remarkable performance for backtesting 99%-VaR and 99%-MS.

Our analysis corroborates that the interaction term between skewness and kurtosis seems to be meaningful for risk forecasting. However, Fig. 3 shows that the correlation dynamics of the interaction term (δ) with γ_3 (γ_4) is high (mild). In particular, excess kurtosis seems to be quite stable, but skewness and the interaction term seem to be significantly affected by turmoil periods. This means that the interaction term is particularly useful to capture the asymmetric impact of shocks at the distribution tails in times of high instability. In these scenarios, considering the interaction terms of skewness with the parameters related to heavy tails seems to be a relevant source of information to provide accurate risk measures.

[3] Expected Shortfall (ES) was also backtested with similar results.

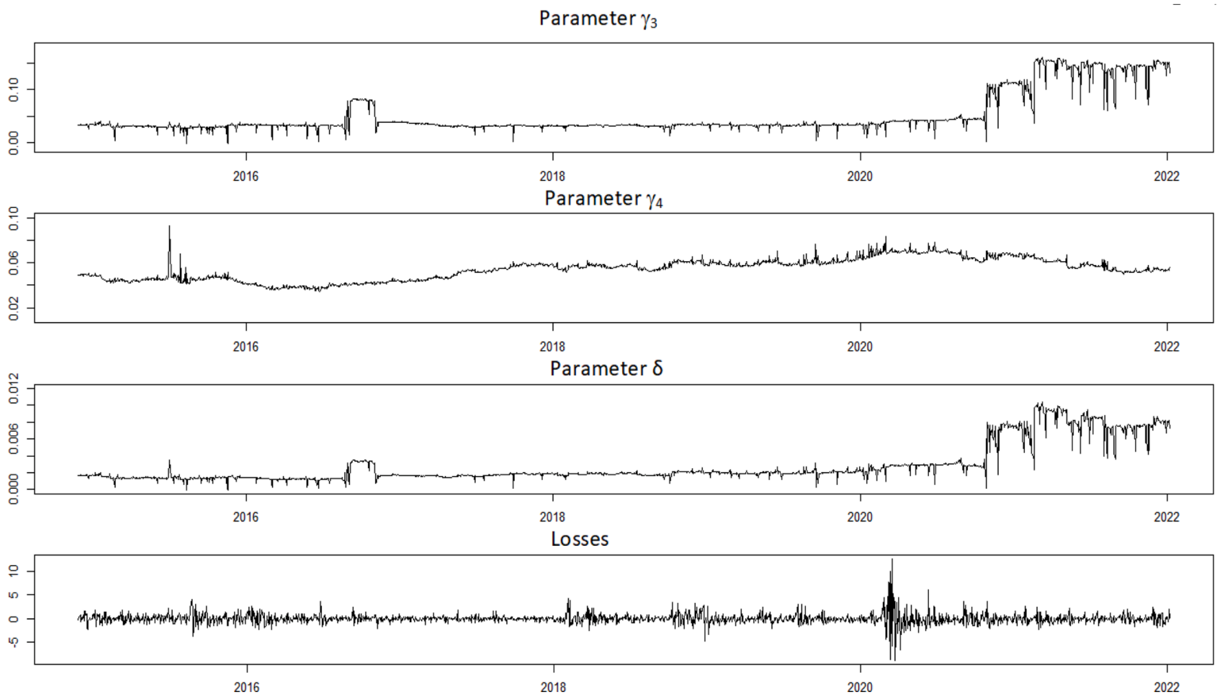

Fig. 3. Parameter dynamics along the backtesting of the mGC model.

4. Conclusions

The empirical analysis in this work points out the significant information content of the "interaction" terms in the GC expansions. From a theoretical viewpoint, these new parameters do not impose problems on the up-to-one integration of the GC expansion and might help to solve the positivity problems of the short CG expansions. From an empirical perspective, we illustrate that they contribute to improving the fitting at the tails and thus the risk measures. This finding is not only good news for risk management but also for the understanding of how distribution tails evolve in response to shocks that induce higher leptokurtosis and skewness at the same time.

A more exhaustive analysis of the GC expansion with all possible interaction terms would be an outstanding contribution to future research. In particular, the contribution of these terms on parameter identification and/or positivity conditions of GC densities is still to be determined. The multiple applications of this new density (e.g. to tail dependence or derivative pricing) and the multivariate extensions for portfolio choice and hedging are also interesting avenues for research.

Declaration of Competing Interest

None.

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