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Three Essays on Portfolio and Investment Strategy Applications: A Higher Order Moment Approach

DOCTORAL THESIS

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To my parents, sister, and in memoriam to my other mom, Yolandita.

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ABSTRACT

Financial decisions constitute a key element to generate value in modern organizations. These decisions can be divided into three groups: first, financing decisions, concerning the best ways to finance or refinance debts and raising the necessary funds for companies' operations; second, investment decisions related to dividends, which are generally related to the policies that companies use for financing; and third, investment decisions that refer to the different alternatives in which the obtained money is employed. In turn, in this group it is worth considering those related to the acquisition of the assets in which it has decided to invest. Moreover, there are investments related to dividends, which are generally related to the policies that companies use for financing. Thirdly, there are investment decisions that refer to the way in which the money obtained is used so that it can be invested. These investments related to dividends are generally related to the policies that companies use for financing. The first category of investment decisions concerns the use of funds for financing purposes. The second category pertains to investment decisions that will determine how the funds obtained will be used to invest in resources that can be productive for the development of the companies. This category of investment decisions is focused on the study of the real assets (tangible or intangible) in which the company should invest.

The field of investments has been extensively studied by academics and implemented by practitioners over time. Prior to 1952, investment decisions were made based on individual securities, rather than in diversified portfolios. In 1952, Harry Markowitz introduced Modern Portfolio Theory (MPT), which placed an emphasis on the relationship between risk and return. One of the main aspects of his proposal is the ability for an investor to construct portfolios by either optimizing the return for a given level of risk or minimizing the risk for a given level of return. This can be achieved by diversifying the assets that present different correlations and

by utilizing only the first two moments of asset returns, namely the mean and variance. Later in 1970, Eugene Fama, widely known for his empirical and theoretical work in portfolio theory and asset pricing, made a fundamental contribution to the field of finance by presenting the Efficient Market Hypothesis (EMH), which postulates that asset prices reflect all available information. This hypothesis suggests that it is impossible to systematically outperform the market because prices already incorporate all relevant information, Nevertheless, it has been demonstrated that there is potential for abnormal profits to be generated through the contravention of technical trading rules, despite the theoretical fact that U.S. stock markets are subject to a random walk process. In the same line, researchers have conducted extensive examinations of various technical indicators and trading rules commonly utilized by professional traders. Many of these indicators and rules can be traced back to Charles H. Dow's editorials in the Wall Street Journal between 1900 and 1902. Momentum trading, a contemporary trading strategy for identifying trends and assessing their strength, is a topic of interest in technical analysis literature and is widely used by market traders. It employs a variety of indicators, with one based on moving averages, namely Bollinger bands, which within their architecture are characterized by the first two moments: the mean and the deviation.

On the other hand, over recent years, environmental, social, and corporate governance (ESG) criteria have become an increasingly important factor in the evaluation of companies. These are factors which do not relate to the company's financial position, but which can indirectly affect its performance and the portfolios of investors, this document outlines the three primary criteria, environmental, social, and governance (ESG), by which corporate sustainability and its potential impact on a company's financial performance are assessed.

It has been demonstrated that incorporating Environmental, Social, and Governance (ESG) criteria allows investors to incorporate their preferences to minimize the social and environmental impact of their investments. Furthermore, an optimal Mean-Variance, Skewness, Kurtosis- Environmental, Social and Governance (MVSK-ESG) portfolio has the capacity to generate superior indicators in comparison to any portfolio that solely pursues the risk-return ratio, while also outperforming its benchmark.

This document aims to incorporate into our study the higher moments: kurtosis and skewness in asset returns, which have been shown not to exhibit Gaussian behavior. To achieve the objective of incorporating the third and fourth stochastic moments in the distribution, we employ the Edgeworth expansion which allows us to obtain a probability distribution that incorporates the mentioned higher moments, in this way this approach allows us to take into account the biases present in the non-Gaussian probability distributions and thus obtain, from the transformation of the normal distribution, the probability distributions using this expansion. In the same line, the Gram-Charlier expansions are also utilized, which permit additional flexibility with respect to the normal density due to their inherent introduction of skewness and kurtosis as parameters. However, as polynomial approximations, they are subject to the disadvantage of yielding negative values for certain parameters. Furthermore, there appears to be no straightforward analytical characterization of those parameters for which the density will take positive values.

Therefore, three applications are presented. Two trading strategies using Edgeworth expansions with two alternative measures of risk using Gram Charlier distributions and Taylor expansions on an exponential utility function, and the optimization of a portfolio with ESG metrics with higher moments.

In the first chapter of the document, we propose two alternatives to the original specification of a technical analysis indicator widely used in the market: Bollinger bands. Our first proposal is to consider the confidence interval with Edgeworth expansions (Hall 1983) incorporating the third moment: skewness, and with the second proposal we design a confidence interval with

Edgeworth expansions (Hall and Jing 1995) incorporating the third and fourth moments: skewness and kurtosis, then we implement with the two previous proposals two trading strategies explained in detail within the paper: trend following and contrarian trading strategy, comparing their performance with traditional Bollinger bands using return, risk and risk-return indicators.

In the second chapter, higher moments are again analyzed: skewness and kurtosis with two proposed risk measures: behavioral variance and modified variance, incorporating different investor risk attitudes by implementing Taylor expansions and Gram-Charlier distributions for returns respectively with an exponential utility function, then two portfolios were optimized by variance minimization and Sharpe ratio maximization using Monte Carlo simulation, demonstrating the satisfactory performance of the proposal.

The third chapter implemented the optimization of a portfolio with ESG (environmental, social, and governance) metrics incorporating high moments through convex quadratic and convex difference of convex (DC) optimization algorithms under a multi-objective approach involving return, variance, skewness, and kurtosis. The proposed methodology was implemented with leading companies in ESG scores from the Dow Jones and Nasdaq indices. The optimized portfolio was tested using higher moments over traditional portfolio models with performance indicators such as Sharpe, Rachev, Delta and CVaR ratios. In addition, the effectiveness of the approach was tested through an in-sample and out-of-sample rolling window.

Finally, the conclusions derived from the mentioned chapters are presented at the end of the document as well as some recommendations for future research.

RESUMEN

El concepto de finanzas debe ser aplicado siempre que una empresa realice la toma de decisiones financieras, las cuales se divide en tres grupos diferentes: primero, las decisiones de financiación, en donde se buscan las mejores formas de financiar o refinanciar las deudas y los fondos necesarios para la operación de las empresas, estas decisiones, en concreto estudian la obtención de fondos provenientes de los inversores que adquieren los activos financieros emitidos por la empresa, para que pueda adquirir los activos en los que ha decidido invertir, segundo: las inversiones relacionadas con los dividendos, y que generalmente se encuentran relacionadas a las políticas que las empresas utilizan para el financiamiento y tercero: las decisiones de inversión que se van a referir al modo de empleo del dinero obtenido para que el mismo se invierta en recursos que pueden resultar productivos para el desarrollo de las empresas, es decir, se centran en el estudio de los activos reales (tangibles o intangibles) en los que la empresa debería invertir. El mundo de las inversiones es un campo de las finanzas bastante estudiado por académicos e implementado por los profesionales de industria a lo largo del tiempo. Anteriormente a 1952 las decisiones de inversión se tomaban sobre la base de títulos individuales y no en carteras diversificados, en 1952 Harry Markowitz introduce la teoría moderna de las carteras (MPT, por sus siglas en inglés), la cual enfatiza la relación entre riesgo y retorno y generando uno de los principales aspectos de su propuesta en la cual un inversionista puede construir carteras optimizando el rendimiento para un nivel de riesgo o minimizando el riesgo para un nivel de rendimiento, diversificando los activos los cuales presentan diferentes correlaciones y usando únicamente los dos primeros momentos de los rendimientos de los activos: media y varianza, posteriormente en 1970 Eugene Fama, ampliamente conocido por sus trabajos empíricos y teóricos en la teoría de carteras y valoración de activos, hace un aporte fundamental al campo de las finanzas presentando la hipótesis del

mercado eficiente (EMH, por sus siglas en inglés), la cual afirma que los precios de los activos reflejan toda la información disponible y sugiere que es imposible superar sistemáticamente al mercado porque los precios ya incorporan toda la información relevante, sin embargo ha sido demostrado que hay margen para obtener beneficios anormales mediante la aplicación contraria de reglas técnicas de negociación, a pesar del hecho teórico de que los mercados bursátiles estadounidenses siguen un proceso de caminata aleatorio. Además, los investigadores han examinado exhaustivamente diversos indicadores técnicos y reglas de negociación utilizados habitualmente por los operadores profesionales, muchos de los cuales remontan sus orígenes a los editoriales de Charles H. Dow en el Wall Street Journal de principios del siglo XX, entre 1900. La inversión basada en la estrategia de momentum, la cual es una estrategia de negociación contemporánea para la identificación de tendencias y la evaluación de su fuerza es un tema de interés en la literatura de análisis técnico y ampliamente usada por los operadores del mercado, utiliza diferentes indicadores entre los que sobresale uno basado en medias móviles: las bandas de Bollinger, las cuales dentro de su arquitectura están caracterizadas por los dos primeros momentos: la media y la desviación típica.

Por otro lado, en los últimos años han tomado relevancia, los criterios ambientales, sociales y de gobierno corporativo (ESG, por sus siglas en inglés), los cuales son factores no-financieros que afectan a las empresas de forma indirecta, al impactar el desempeño del portafolio de los inversionistas y de las finanzas de las empresas. Son estos tres principales criterios con los que se mide la sostenibilidad corporativa y el potencial asociado al desempeño financiero de una compañía y se resalta el hecho de que la incorporación de los criterios ESG les permite a los inversores incorporar sus preferencias para minimizar el impacto social y ambiental de sus inversiones, ya que la cartera óptima MV-ESG genera mejores indicadores que cualquier cartera que persiga sólo la relación óptima retorno-riesgo, además de superar el desempeño de un comparativo o modelo base.

Este documento tiene como objetivo incorporar al estudio del análisis técnico los momentos estadísticos de orden alto: curtosis y asimetría en la distribución de los rendimientos de los activos los cuales se ha demostrado no presentan comportamiento Gaussiano. Para el logro de este objetivo de incorporar el tercer y cuarto momento en la distribución: asimetría y curtosis se ha optado por emplear la expansión de Edgeworth, a partir de la cual se obtiene una distribución de probabilidad que incorpora los ya mencionados momentos estadísticos superiores. Esto permite tener en cuenta los sesgos presentes en las distribuciones probabilísticas no-gaussianas y obtener así, a partir de la transformación de la distribución normal, las distribuciones de probabilidad mediante esta expansión, por otro lado, también se utilizan las expansiones de Gram-Charlier, las cuales permiten una flexibilidad adicional con respecto a la densidad normal porque introducen de forma natural la asimetría y la curtosis de la distribución como parámetros. Sin embargo, al ser aproximaciones polinómicas, tienen el inconveniente de arrojar valores negativos para ciertos valores de sus parámetros. Además, no parece existir una caracterización sencilla y analítica de aquellos parámetros para los que la densidad tomará valores positivos.

En este orden de ideas, se presentan tres aplicaciones que corresponden a cada uno de los capítulos de este documento: en el primer capítulo se proponen dos alternativas a la arquitectura original de un indicador de análisis técnico ampliamente usado en el mercado: las bandas de Bollinger. Nuestra primera propuesta se centró en considerar el intervalo de confianza con Expansiones de Edgeworth (Hall, 1983) incorporando el tercer momento: la asimetría, y con la segunda propuesta diseñamos un intervalo de confianza con expansiones de Edgeworth (Hall y Jing, 1995) incorporando el tercer y cuarto momento: asimetría y curtosis, posteriormente implementamos con las dos propuestas anteriores dos estrategias de negociación explicadas en detalle al interior del documento: *trend following* y *contrarian trading strategy*, comparando su desempeño con las bandas de Bollinger tradicionales utilizando indicadores de rentabilidad, riesgo y riesgo-rentabilidad

En el segundo capítulo, se incorporan de nuevo los altos momentos: asimetría y curtosis con dos medidas de riesgo propuestas: *behavioral variance* y *modified variance*, incorporando diferentes actitudes de riesgo del inversionista implementando expansiones de Taylor y distribuciones Gram-Charlier para los rendimientos respectivamente con una función de utilidad exponencial, posteriormente se optimizaron dos carteras minimizando la varianza y maximizando el coeficiente de Sharpe utilizando simulación de Monte Carlo demostrando la bondad de la propuesta y, en el tercer capítulo se implementó la optimización de una cartera con métricas ESG (ambientales, sociales y de gobierno corporativo) incorporando momentos estadísticos de alto orden a través de algoritmos de optimización cuadrática convexa y de diferencias convexas (DC, por sus siglas en inglés) bajo un enfoque multiobjetivo que involucra el rendimiento, la varianza, la asimetría y la curtosis.

La metodología propuesta se implementó con empresas líderes en puntuaciones ESG de los índices Dow Jones y del Nasdaq, y se comprobó la bondad de desempeño de la cartera optimizada. Para ello, se compararon los resultados de la cartera propuesta usando momentos de orden alto de los rendimientos de los activos constituyentes y las puntuaciones ESG con los modelos tradicionales de cartera, usando los indicadores de desempeño como las ratios de Sharpe, Rachev, Delta y CVaR. Además, se comprobó la efectividad de la propuesta a través de una ventana móvil tanto dentro de la muestra como por fuera de la muestra.

Por último, al final del documento se presentan las conclusiones obtenidas de los capítulos mencionados, así como algunas recomendaciones para las futuras investigaciones.

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INTRODUCTION

The aim of this work is to highlight the novelty of incorporating high order moments of asset returns in financial applications. In the first chapter, high order moments are introduced to a technical analysis indicator: Bollinger bands with Edgeworth expansions and this proposal is implemented in two trading strategies to review their performance through profitability, risk, and risk-return indicators during calm and turbulence periods. In the second chapter, an innovative risk measure for portfolio selection is presented, and its performance is compared with two related measures: behavioral variance and Taylor-expanded variance, the methodology for the proposed measure considers investors' attitudes to risk in terms of skewness and kurtosis by assuming a Gram-Charlier distribution. In chapter three, we present the optimization of a portfolio with ESG criteria using the leading companies of the Dow Jones and Nasdaq indices, incorporating higher moments, and comparing their performance with metrics that capture the behavior of skewness and kurtosis.

Delving into the document, in the first chapter, presents two alternatives to the conventional specification of technical analysis indicators, which are frequently used by traders: Bollinger bands. The Bollinger bands are constructed using asset prices and a moving average with a value of N = 20, along with a confidence interval that encompasses two standard deviations above and below its average. The initial proposal entails modifying the confidence interval with Edgeworth expansions following to Hall (1983) by incorporating the third moment: skewness. The second proposal establishes a confidence interval with Edgeworth expansions following Hall and Jing (1995) incorporating the third and fourth moments: skewness and kurtosis. Subsequently, two trading strategies are implemented in detail: the first is a trend-following strategy with N = 20, 50, and 200 days, and the second is a contrarian trading strategy with N = 6, 10, 15 and 20 days. In addition, to measure the robustness of the two proposed confidence intervals to the traditional Bollinger bands in the two strategies, measures of return, risk and risk-return trading performance were used. It was concluded that the median cumulative returns and annual returns in the whole sample are positive for the contrarian trading strategy in both calm and crisis periods, in contrast to the trend following strategy where the results for returns are negative for both calm and crisis periods. The explanation for this phenomenon and the significant contribution of this chapter is that higher moments are more effectively captured for short periods, as evidenced using $N = 6$ and for $N = 10$ in the contrarian strategy.

The incorporation of higher moments to variance has been of great importance in finance. According to financial literature, one of the pioneers in addressing this issue was Scott and Horvath (1980), whose results reinforce that statement of preference for positive skew as an inherent result of risk aversion. One advantage of utility models is that they permit the modeling of different investor preferences, thereby enabling the construction of portfolios that are optimal with respect to those preferences. One utility function that is particularly favored by economists is the constant relative risk aversion (CRRA) as suggested by Kassimatis (2021). Furthermore, behavioral finance studies have demonstrated that agents with cumulative prospect theory preferences tend to select portfolios with a positive skewed distribution of returns (e.g., Barberis & Huang, 2008; Zhang, 2005). Conversely, they tend to avoid portfolios with a high kurtosis (Ågren, 2006). This is because risk-averse investors would prefer less excess kurtosis than risk-loving investors. In accordance with Nawrocki and Viole (2014), risk averse investors would prefer less excess kurtosis and positive skew than risk lover investors. In an effort to integrate such characteristics into a more comprehensive risk assessment tool, Davies and de Servigni (2012) developed a framework known as behavioral variance. In Chapter 2, a modified variance measure was proposed based on a Gram Charlier expansion on asset return density and a modified variance measure that were derived through Taylor expansions of the risk measures based on the investor's exponential utility. A case study of four stocks on the New York Stock Exchange (Case Study A) involved the implementation of three models: behavioral variance (Model 1), modified variance with Taylor's expansion (Model 2), and modified variance with Gram-Charlier returns (Model 3). The resulting portfolios consisted of 11 portfolios. A total of 11 portfolios were constructed: six with two assets, four with three assets, and one portfolio with all assets. The portfolios were optimized to maximize the Sharpe coefficient and minimize the variance, with the three risk aversion parameters (T = 0.5, T = 0.75, T = 1) applied to each model using Monte Carlo simulation. The results indicated that the proposed three-modified-variance with Gram-Charlier returns model is the optimal alternative for incorporating risk tolerance, as it also appears to provide more efficient portfolio than the corresponding alternative measures of behavioral variance.

Considering that the Gram-Charlier expansion allows us to capture the characteristics of an empirical distribution introducing higher moments, it has the disadvantage that for certain values of the moments, negative probability values are achieved, since it is a polynomial approximation, we chose the S&P 500 index, the 5-year US Treasury yield, corn futures and palladium futures (Case study B), where the skewness and kurtosis range over which the expansion is positive and obtain results that allow us to conclude again that the proposal (modified variance with Gram-Charlier returns) is better for different risk tolerance parameters with the criteria studied.

On the other hand, the classical mean-variance framework, which is the foundation of modern portfolio theory, is predicated on both risk and return measures. However, over the past two decades, the increasing prevalence of socially responsible investing (SRI) has demanded the integration of additional considerations into the investment decision-making process, particularly in light of the advent of the Sustainable Development Goals (SDGs) and the Principles for Responsible Investment (PRI) from the United Nations. At the same time, the empirical literature on investment strategies has been reoriented toward the integration of sustainability issues. For example, Hirschberger et al. (2013), Utz et al. (2014), and Gasser et al. (2017) expanded Markowitz's two-dimensional MV model into a three-dimensional portfolio model that incorporates environmental, social, and corporate governance (ESG) scores. To achieve this, they developed a model with multiple objective functions that extends the traditional MV model. In addition, Chen & Mussalli (2020) presented a portfolio approach integrating ESG investing based on the information ratio, with the objective of simultaneously maximizing alpha and ESG performance. It is important to note that portfolio approaches to portfolio optimization have not incorporated higher-order moments, which can lead to biased solutions, poor performance, or inferior results when compared to established benchmarks. As such, it is essential to develop a new approach that can effectively address and overcome these limitations.

In Chapter 3 of this document, a proposal is presented that concerns the optimization of a portfolio in the presence of higher-order moments and ESG metrics. This objective was achieved through a multi-objective approach that employed the use of a mean-variance-skewness-kurtosis (MVSK) model, through which a diversified portfolio with an ESG score was created, named as the MVSK-ESG portfolio model. The high-order optimization problem was solved using difference of convex (DC) algorithm. In addition, the analysis focuses on the optimal ESG portfolio comprising only leaders, that is, companies with the highest ESG scores, and demonstrates that MVSK-ESG portfolios achieve superior performance in terms of in-sample and out-of-sample measures compared to traditional models. The Sharpe ratio, the Rachev ratio, and the Delta ratio are employed for both United States stock markets, the Dow Jones and Nasdaq. The Rachev ratio is of particular significance, since it is the optimal performance measure when considering the high-order moments of the distribution of portfolio returns. The proposed approach represents a novel integration for both higher moments and

ESG metrics, resulting in a unified framework that can be efficiently implemented through Convex Quadratic Programming solvers.

Finally, the overall conclusions are summarized at the end of this document. Moreover, some future research and practical implications are discussed in that section.

CHAPTER 1: When Bollinger meets Edgeworth: An application to the Trend following and Contrarian trading strategies.

1.1. Introduction

In this chapter, we propose the Adjusted Bollinger Bands by using Edgeworth's expansions to adjust the confidence intervals used in the technical analysis and their implementation with the contrarian trading rule. To do that, we consider higher moments to incorporate the extensions into the confidence intervals that account for non-negligible characteristics of the underlying asset prices, such as skewness and kurtosis. As a result, we employ the Adjusted Bollinger Bands method as an alternative to the widely recognized Bollinger Bands technical analysis.

The fundamental premise of the Efficient Market Hypothesis (EMH) posits that if the stock market is working efficiently, the prices will follow a random walk, the prices will reflect the intrinsic values, and no one can benefit from trading. There has been a lot of research on testing the EMH for over eight decades (Brown, 2020; Chan et al., 1997; Degutis & Novickyte, 2014; Fama, 1970; Jain et al., 2010; Kelikume et al., 2020; Samsa, 2021; Shiller, 2003; Vidya, 2018). This extensive academic inquiry into EMH has established a consensus that no one can make money by trading securities when the markets are efficient. However, there is the possibility that investors may be reluctant to agree with this hypothesis (Elangovan et al., 2022), and many academic studies have empirically demonstrated the inefficiencies of financial markets by using technical strategies (see e.g., Mitra & Rohit, 2020; Agapova & Kaprielyan, 2020).

Likewise, Jain et al. (2020) analyzed the efficiency of the Indian stock market, from April 2010 to March 2019, based on Bombay Stock Exchange (BSE) and National Stock Exchange. Their findings suggest that the Indian market is weak-form inefficient, and therefore can be outperformed. Similarly, Patel et al. (2018) focused on the weak form of market efficiency three-year daily closing points were taken from the BSE (April 2015 to March 2018). Their results highlight that the market did not efficiently respond to news, creating opportunities for investors to outperform, particularly through informed decision-making. In contrast, Balsara et al. (2009) shows that there is room for making abnormal profits through the contrarian application of technical trading rules, despite the theoretical fact that US stock markets follow a random walk process. Furthermore, researchers have extensively examined various technical indicators and trading rules commonly used by professional traders, many of which trace their origins back to Charles H. Dow's early twentieth-century Wall Street Journal editorials between 1900 and 1902 (Taylor, 1997).

Momentum investing, a contemporary trading strategy for trend identification and strength assessment, is a focus in recent literature. Bettman et al. (2010) introduced the GH 52-week high momentum strategy, emphasizing purchasing stocks near their 52-week highs and short-selling those distant from this threshold (George & Hwang, 2005). This strategy outperforms the JT momentum and industry momentum strategies (Moskowitz & Grinblatt, 1999; George & Hwang, 2005). Momentum indicators, including moving averages and Bollinger Bands, are well-established (Pring, 1993).

The simple moving average (SMA) rule is extensively examined in trading literature (Leung & Chong, 2003), and one of its widely adopted techniques is Bollinger Bands, developed by John Bollinger in the early 1980s. This author proposes simple moving averages SMA20, SMA50, and SMA200, where 20, 50, and 200 indicate the number of days to assess the moving average (Kabasinskas & Macys, 2010), and two (upper and lower) standard deviations in order to construct the confidence interval around the SMA of the price. Similarly, Tse (2015) employed a cross-sectional momentum strategy for international iShares and US ETFs traded on the NYSE. The findings indicate poorer performance compared to a buy-and-hold strategy. Bley & Saad (2020) explored technical trading indicators in MENA equity markets, revealing variations in market efficiency, with Egypt, Jordan, Tunisia, and Turkey showing relative efficiency.

The nature of technical analysis calls into question the Efficient Market Hypothesis, even in its weakest form. This often leads scholars to dismiss its validity or collectively ignore it (Bley & Saad, 2020). However, Ebert & Hilpert (2019) studied the behavior of common stocks traded on major exchanges from 1962 to 2018 using Bollinger bands, revealing that investors' preference for positive skewness aligns with the popularity of technical analysis. Since prospect theory implies a strong skewness preference, this could explain why investors mostly resort to chart patterns that lack meaning, according to the efficient market hypothesis. Technicians often reference behavioral finance as a theoretical basis.

Contrary to this view, Ebert & Hilpert (2019) showed that ideas from behavioral finance explain why technical analysis is popular despite the lack of theoretical foundation and empirical success. In the same way, Jin (2022) studied the behavior of the SSE 180 index ETF in the Chinese market and found that an investor who applies technical trading rules on this ETF is inclined to do so because of the skewness feature of technical analysis can meet his skewness preference. Likewise, Cohen (2020) analyzed five ETFs: SPY, QQQ, IWM, XLF, and XLI that follow the S&P500, Nasdaq, Russell 2000, financial sector, and industrial sector, respectively for the years 1998-2018 using daily data with technical indicators such as Bollinger Bands, Relative Strength Index, Commodity Channel Index (CCI), and Chaikin Money Flow (CMF), showing that CCI and Bollinger Bands were found to be the best technical tools for trading the examined ETFs outperform the buy and hold strategy. Similarly, Vaidya (2021) applied a Bollinger Bands study to the NEPSE Index in Nepal from 1998 to 2020 proving that the results of trading signals based on the Bollinger Bands are seen as useful for an investor by providing a clear signal to "buy" or "sell". In addition, Chung et al. (2021) tested the performance of moving average strategies based on 20, 50, 100, and 200-day moving average prices with all companies listed on the Taiwan Stock Exchange (TWSE) between January 2000 and December 2018 and found that information asymmetry is another driving force affecting the effectiveness of technical analysis.

Likewise, Ni et al. (2020) studied the constituent stocks of the Taiwan 50 index and explored whether investors can beat the market by trading them as trading signals emitted by Bollinger Bands. The results revealed that investors might beat the market by taking long positions on stocks as share prices hit the lower Bollinger bands, as significantly shown in the positive abnormal returns. Similarly, for the Taiwan 50 index Chen et al. (2022) implemented a pairs trading strategy using genetic Bollinger Bands and a correlation-coefficient-based pairs trading strategy (GBCPT) using optimization technology to determine the parameters for correlation-based candidate pairs and discover Bollinger Bands-based trading signals showing the merits and effectiveness of this approach.

In an exploration of the Indian stock market, Tadas et al. (2023) focused on the Nifty 50 index from January to August 2022. They observed that the Bollinger Bands – Relative Strength Index strategy displayed superior performance, showcasing dynamic and active position management with the highest positive Sharpe and Sortino ratios. In times of crisis, Lento & Gradojevic (2022) analyzed the profitability of technical trading rules for the Comex gold spot, S&P 500 index, Bitcoin, crude oil WTI, and the VIX from January to May 2020 and found that trading rules can beat the buy-and-hold trading strategy. However, only the Bollinger Bands and trading range break-out rules become profitable after transaction costs during the market crash. Gerritsen et al. (2020) examined seven trend-following indicators in the Bitcoin market from July 2010 to January 2019. Their findings highlighted the significant forecasting power of specific technical analysis trading rules, especially trading range breakout, enabling the outperformance of the buy-and-hold strategy based on the Sharpe ratio.

Reviewing the contrarian approach to technical analysis (Lento et al., 2007) using daily closing prices for the Dow Jones Industrial Average, Nasdaq, and Toronto Stock Exchange Index from May 1995 to December 2004. Their findings indicated that Bollinger Bands, with 30-day and 20-day moving averages, after adjusting for transaction costs, consistently failed to outperform the buy-and-hold strategy. However, profitability improved with a contrarian approach, and robustness was determined by calculating the returns and the Sharpe Ratio. Similarly, Balsara et al. (2009) showed significant positive returns on trade signals generated by the contrarian version of the MA crossover rule, the channel breakout rule, and the Bollinger band trading rule, after accounting for transaction costs of 0.50 percent. In contrast, regular trend-following versions of these rules resulted in significant negative returns.

In this chapter, we propose Adjusted Bollinger Bands with Edgeworth-corrected confidence intervals, implemented alongside the Trend Following and contrarian trading rule. Within the statistics literature, a distinct focus emerges on examining confidence intervals beyond the conventional approach –Bollinger Bands may be regarded as a specific case of this approach– considering moments higher than the first and second order. Our contribution aims to incorporate these extensions by using higher moments for confidence intervals that account for non-negligible characteristics of underlying asset prices, such as skewness and kurtosis, in investment analysis for financial markets. The Adjusted Bollinger Bands method is recommended as an alternative to the commonly used Bollinger Bands in technical analysis.

The document is structured as follows: Section 1.2 Material and methods, describes the traditional Bollinger Bands and two confidence intervals built with Edgeworth expansions in order to construct the proposed Adjusted Bollinger Bands technique. The trading strategies with simulations are also presented. Section 1.3 shows the results of some performance indicators regarding return, risk, and risk-return with classical Bollinger Bands compared to our proposed Adjusted Bollinger Bands. Finally, Section 1.4 concludes.

1.2. Models

In the first part of this section, the traditional architecture of Bollinger Bands and its main applications in technical analysis − trend following, contrarian and squeeze frameworks − are reviewed. In the second part, two types of Adjusted Bollinger Bands confidence intervals with Edgeworth expansions will be presented: the Edgeworth-corrected confidence intervals of the type studied by Hall (1983) and a Berry-Esseen for Edgeworth expansions (Hall and Jing, 1995) in order to construct the Adjusted Bollinger Bands.

1.2.1. Model 1: The Bollinger Bands

This indicator was developed by John Bollinger in 1980's, and the construction of the Bollinger Bands starts with a 20-period simple moving average (middle band) as a measure of central tendency. In addition, the bands are constructed above and below that moving average, representing a 95% confidence interval, which is delineated by a measure of volatility. Usually, these upper and lower bands are two times away from the middle band underlying price, an *n*-day standard deviation (Bollinger, 2002). The traditional Bollinger Band calculation procedure is as follows (Chen and Chuang, 2014). In the first step of the procedure, the *n*-day moving average at time *t* is calculated:

$$MA_t = \frac{\sum_{i=0}^{n-1} P_{t-i}}{n}, \tag{1.1}$$

where P_t is the price of the underlying asset at time *t*. In the second step, the *n*-day standard deviation is computed:

$$SD_t = \left[\frac{\sum_{i=0}^{n-1}(P_{t-i}-MA_t)^2}{(n-1)}\right]^{0.5}. \tag{1.2}$$

Finally, the upper band (UB) and lower band (LB) for the confidence interval are assessed for each time *t*:

$$UB_t = MA_t + z_a * SD_t, \tag{1.3}$$

$$LB_t = MA_t - z_a * SD_t, \tag{1.4}$$

where $z_a = \Phi^{-1}(\alpha)$ is the α-level quantile of a standard normal distribution. These trading bands are lines plotted above and below the price behavior in the form of an envelope and are aimed to determining the probability of breaking through the bands. The latter serves as the basis for the decision-making process of buying or selling the analyzed asset. Therefore, it is expected that approximately 95% of the price activity occurs between the upper and lower Bollinger Bands and can be represented in the following confidence interval notation:

$$I_{2,a} = \left(\bar{X} - n^{-1/2}\hat{\sigma}z_\beta, \bar{X} + n^{-1/2}\hat{\sigma}z_\beta\right), \tag{1.5}$$

where $\beta = \frac{(1+\alpha)}{2}$, $\bar{X}$ represents the MA and $\hat{\sigma}$ stands for the SD given in equations (1.1) and (1.2), respectively. Treleaven et al. (2013) identifies opposing views of interpreting the price behavior associated with the Bollinger Bands, the so-called trend-following and contrarian frameworks.

First, the trend-following viewpoint assumes that prices will continue to move in the direction of the penetration. That is, a penetration of the upper band suggests that prices will continue to move higher, signaling a buy condition. Similarly, a penetration of the lower band suggests that prices will continue to move lower, and then, signaling a selling recommendation.

Second, the contrarian approach states that a penetration of the upper or lower band is reflective of an overreaction of the prices with a strong possibility of an impending trend reversal. As explained by Bollinger (2002), the closer a security price is to its lower (upper) level, the more oversold it is, signaling a buy (sell) condition. The idea is that the volatility tends to revert to its mean and after the price amount is located at the outer edge of a given band, this price tends to trade back toward the median level, represented by the mid-line moving average. Another consideration highlights the importance of including the change in the Bollinger bandwidth to capture volatility, which is referred to as the 'squeeze effect' (Bollinger, 2002). The latter means that periods of low volatility are followed by periods of high volatility. Relatively narrow bandwidth (also known as the 'squeeze') can foresee a significant advance or decline of the price.

Third, the squeeze Viewpoint argues that a new advance starts with a squeeze in the Bollinger bandwidth and subsequent break above the upper band. A new decline starts with a squeeze and subsequent break below the lower band (Gold, 2018).

1.2.2. Model 2: Adjusted Bollinger Bands (ABB1)

Our first proposal for the adjusted Bollinger Bands is based on the Edgeworth-corrected confidence intervals of the type examined by Pfanzagl (1979), Hall (1983), and Abramovitch and Singh (1985). Let $X_1, X_2, \dots, X_n$ denote the independent and identically distributed (i.i.d.) random variables, in our case the asset prices, with finite third moment. In addition, $\bar{X} = n^{-1}\sum_{i\le n} X_i$, $\hat{\sigma}^2 = n^{-1}\sum_{i\le n}(X_i - \bar{X})^2$ and $\hat{\gamma} = \hat{\sigma}^{-3} n^{-1}\sum_{i\le n}(X_i - \bar{X})^3$ are the estimators of $\mu = E(X_i), \sigma^2 = var(X_i)$ and $\gamma = \frac{E(X_i-\mu)^3}{\sigma^3}$, i.e., the mean, variance, and skewness, respectively. Accounting for skewness, Hall (1983) proposed the nominal α-level one-sided interval, which is given by:

$$J_{1,a} = \left(-\infty, \bar{X} + n^{-1/2}\hat{\sigma}\left\{z_a - n^{-1/2}\tfrac{1}{6}\hat{\gamma}(2z_a^2 + 1)\right\}\right), \qquad (1.6)$$

and

$$J_{1,a} = \left(\bar{X} - n^{-1/2}\hat{\sigma}\left\{z_a - n^{-1/2}\tfrac{1}{6}\hat{\gamma}(2z_a^2 + 1)\right\}, \infty\right). \qquad (1.7)$$

Whereas the two-sided interval is as follows:

$$J_{2,a} = \left(\bar{X} - n^{-\frac{1}{2}}\hat{\sigma}\left\{z_\beta + n^{-\frac{1}{2}}\tfrac{1}{6}\hat{\gamma}\left(2z_\beta^2 + 1\right)\right\}, \bar{X} + n^{-1/2}\hat{\sigma}\left\{z_\beta - n^{-\frac{1}{2}}\tfrac{1}{6}\hat{\gamma}\left(2z_\beta^2 + 1\right)\right\}\right). \qquad (1.8)$$

This confidence interval will be considered as an alternative to the usual confidence interval expressed in equation (5) used in the Bollinger Bands construction and it will be referred to as the ABB1 model. As observed, the Bollinger Bands is a special case of the ABB1 model when the skewness coefficient ($\hat{\gamma}$) is set zero.

1.2.3. Model 3: Adjusted Bollinger Bands (ABB2)

In a posterior study of Hall (1983), Hall and Jing (1995) proposed the Berry-Esseen theorem for Edgeworth expansions. This will be used in our work as a second alternative for the Bollinger Bands. Let $\mu = 0$, $\sigma = 1$ and $T_0 = n^{1/2}\frac{\bar{X}}{\hat{\sigma}}$, which represents the 'studentized mean'. If X has a non-singular distribution, then the one-term Edgeworth expansion is valid. That is,

$$P(T_0 \leq x) = \Phi(x) + n^{-1/2}\frac{1}{6}\gamma(2x^2+1)\phi(x) + o(n^{-\frac{1}{2}}), \quad (1.9)$$

uniformly in x, as $n \to \infty$, where Φ and ϕ denote the standard normal distribution and density functions, respectively, and γ=$E(X^3)var(X)^{-3/2}$ is the skewness of the sampling distribution (see e.g., Chibishov, 1984, and Hall, 1987). Then, the equation (8) may be extended to a longer expansion:

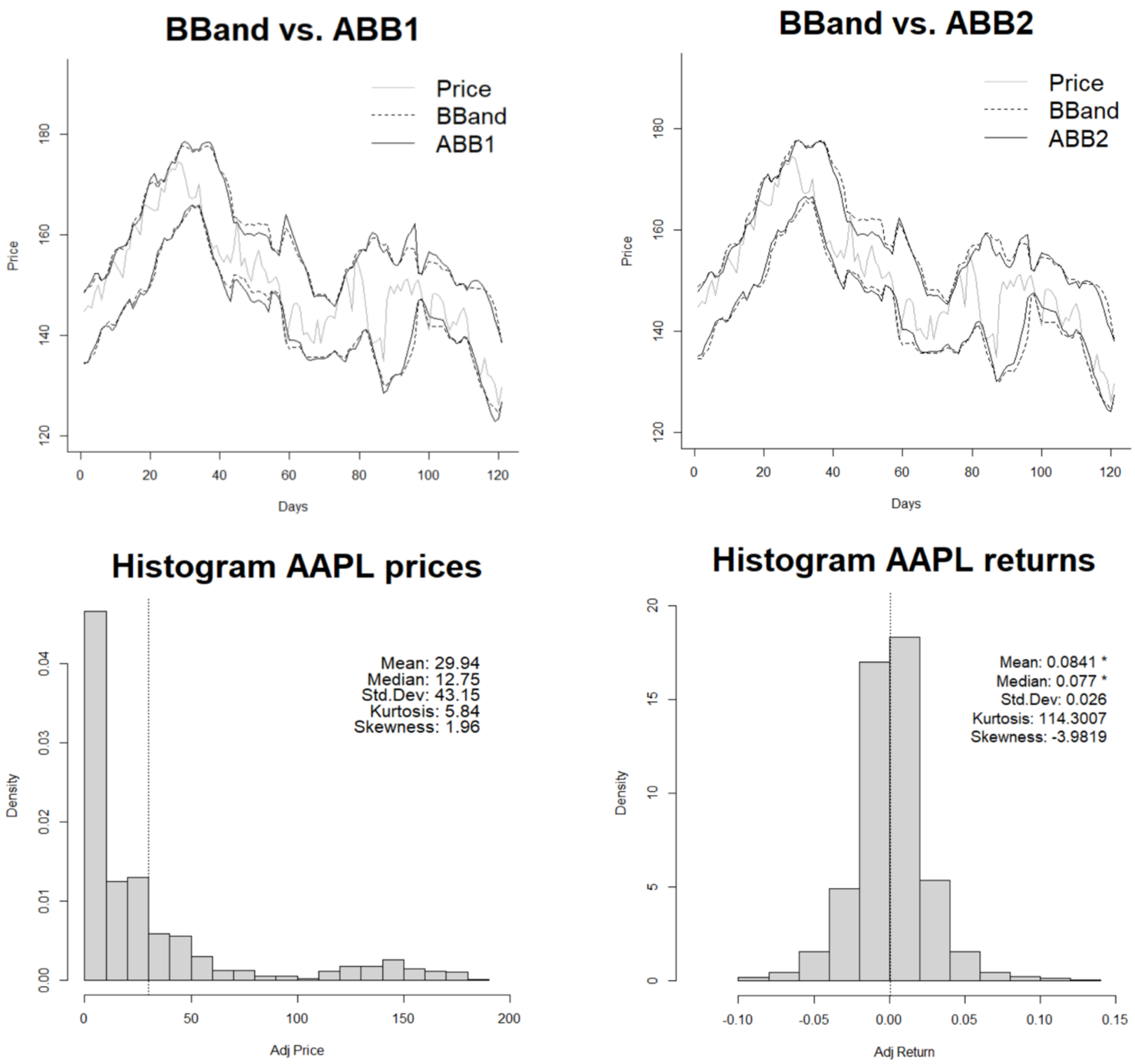
$$P(T_0 \leq x) = \Phi(x) + n^{-1/2}\frac{1}{6}\gamma(2x^2+1)\phi(x)$$
$$+n^{-1}\left\{\frac{1}{12}ek(x^2-1) - \frac{1}{18}\gamma^2(x^4+2x^2-3) - \frac{1}{4}(x^2+3)\right\}\phi(x) + o(n^{-1}), \quad (1.10)$$

uniformly in x, where ek= $E(X^4)var(X)^{-2} - 3$ denotes excess kurtosis. Therefore, a new confidence level can be constructed as:

$$K_{2,a}\text{=} \Big(\bar{X} - n^{-\frac{1}{2}}\hat{\sigma}\left\{z_\beta + n^{-\frac{1}{2}}\frac{1}{6}\hat{\gamma}\left(2z_\beta^2+1\right) + n^{-1}\left[\frac{1}{12}ek\left({z_\beta}^2-1\right) - \frac{1}{18}\gamma^2\left({z_\beta}^4 + 2{z_\beta}^2 - 3\right) - \frac{1}{4}\left({z_\beta}^2+3\right)\right]\right\}, \bar{X} + n^{-\frac{1}{2}}\hat{\sigma}\left\{z_\beta - n^{-\frac{1}{2}}\frac{1}{6}\hat{\gamma}\left(2z_\beta^2+1\right)\right\} + n^{-1}\left[\frac{1}{12}ek\left({z_\beta}^2-1\right) - \frac{1}{18}\gamma^2\left({z_\beta}^4 + 2{z_\beta}^2 - 3\right) - \frac{1}{4}\left({z_\beta}^2+3\right)\right]\Big). \quad (1.11)$$

This new confidence interval considers not only skewness but also excess kurtosis and will be denominated the ABB2 model. As can be noted, the Bollinger Bands and the ABB1 model are special cases of the ABB2 model. Figure 1.1. illustrates these bands for the APPLE stock.

Figure 1.1. Confidence bands comparison for the case of APPLE stock.



Note: AAPL stock price and returns (Upper Left and
Right Panels) from January 1, 2008, to April 30, 2008. *N*=20. * Mean and Median multiplied by 100.
Source: own elaboration.

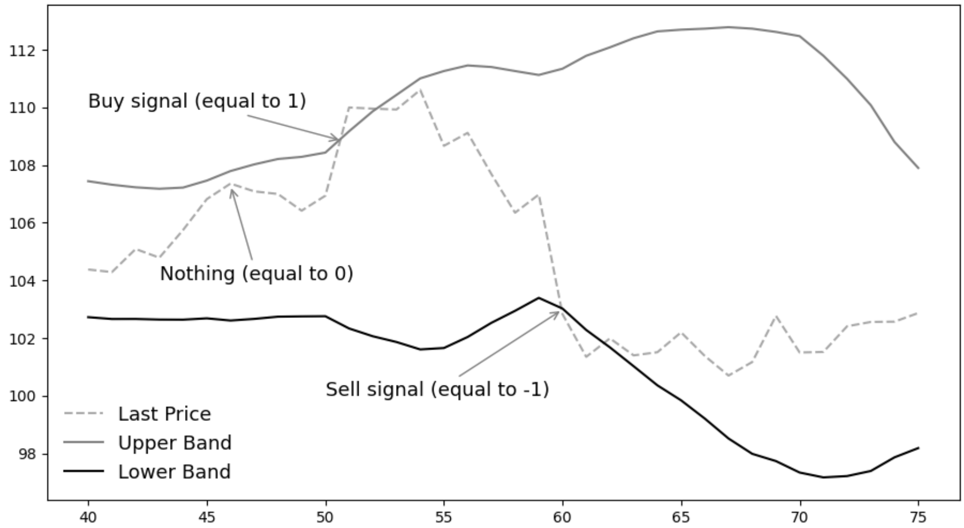
Next sections detail the trading strategy applied in our chapter and presents the results obtained using the three models presented above according to a battery of performance measures.

1.3. Trading Strategies

1.3.1. Trend Following Strategy

The Trend Following trading rule (Treleaven et al., 2013; Bollinger, 2002) is tested with (1) the (classical) Bollinger Bands; (2) ABB1 model, i.e., the confidence interval with Edgeworth expansion by Hall (1983), and (3) the ABB2 model, i.e., the confidence interval with the Berry-Esseen theorem for Edgeworth expansions proposed by Hall and Jing (1995). To this aim, the 30 stocks that form the Dow Jones Industrial Average Index (Jeet and Vats, 2017) are employed between January 1, 1998, and November 15, 2021. We use the Bollinger Bands with 20-day, 50-day and 250-day moving average to generate automated trading signals 1, −1, and zero as follows. A buy signal (equals to 1) rises when the last price of a stock is above the upper line of the indicator, while the middle line is considered as the support line. A sell signal (equals to −1) happens when the last price is down the lower line of the indicator, while the middle line will be considered the resistance level. Finally, out of market includes situations where the signal is zero. We did not include any transaction and slippage costs to calculate the performance measures, which might constitute an interesting refinement for further research. Figure 1.2 summarizes this strategy.

Figure 1.2. Trend following strategy explained



When the price is in an uptrend and touches the upper band, it is a buy signal (1). Meanwhile the price is in a downtrend and touches the lower band, it is a sell signal (-1), otherwise the signal is equal to zero.

Several measures of trading performance, commonly utilized by experienced traders and academics (Davey, 2014; Chande, 2008), are evaluated to test the success of the models with the trend following trading strategy. A brief description for each of these measures with their mathematical descriptions can be found in the Appendix of this chapter. Table 1.1 presents the performance measures grouped by return, risk, and risk-return indicators.

Table 1.1. Measures of trading performance: Trend Following Strategy

RETURN	RISK	RISK-RETURN
Cumulative Return	Annualized Standard Deviation	Sharpe Ratio
Annual Return	Maximum Drawdown (MDD)	Tracking Error
	Value at Risk (VaR)	Information Ratio
		Treynor Ratio

1.3.1.1. Simulations

Two simulations were performed for prices to contrast the results of the proposed models (ABB1 and ABB2) with the classical Bollinger Bands. The selected values for volatility, skewness and excess kurtosis are similar to the price statistics of the stocks in the DJIA index. That is, skewness value of -0.2005 and excess kurtosis of 4.67. Nevertheless, the first simulation accounts for a volatility of 30% (high volatility) and the second one with a volatility of 15% (low volatility) for N = 20, 50, and 200 days. To this aim, 500 prices were generated following a geometric Brownian motion (gBm) with a mean of 0.11 and a volatility of 0.30 (and 0.15) for the diffusion process is employed. In addition, a normal distribution with mean equal to −2.6 and standard deviation of 0.002 is also assumed for the shock process of the gBm. Furthermore, the cumulative return (return indicator), VaR (risk indicator) and tracking error (risk-return indicator) are calculated to compare the results for each methodology.

As observed in Table 1.2 (N=20), the two proposed models (ABB1 and ABB2) perform better than the classical Bollinger Bands for all indicators.

Table 1.2. Performance of trading strategies using 20-day moving averages: Trend Following Strategy

HIGH VOLATILTY			**LOW VOLATILITY**		
Cum Return	**VaR**	**Tracking Error**	**Cum Return**	**VaR**	**Tracking Error**
CLASSICAL BOLLINGER BANDS					
0.0500	0.2462	0.3177	−0.0478	0.0983	0.3197
ABB1					
0.2981	0.1071	0.3226	0.0762	0.0882	0.3371
ABB2					
0.2179	0.1000	2.1461	0.0621	0.0958	0.3468

The two new models (ABB1 and ABB2) are better than the classical Bollinger Bands for VaR and tracking error indicators as noticed in Table 1.3 (N=50).

Table 1.3. Performance of trading strategies using 50-day moving averages: Trend Following Strategy

HIGH VOLATILTY			**LOW VOLATILITY**		
Cum Return	**VaR**	**Tracking Error**	**Cum Return**	**VaR**	**Tracking Error**
CLASSICAL BOLLINGER BANDS					
−0.1954	0.1422	0.3489	0.0917	0.1056	0.3465
ABB1					
−0.0867	0.1341	0.3549	0.0565	0.1025	0.3571
ABB2					
−0.0652	0.1331	0.3568	0.0348	0.0993	0.3659

Table 1.4 shows that the VaR (and tracking error for the low volatility case) results for Bollinger Bands are outperformed by its adjusted versions (ABB1 and ABB2) when N=200.

Table 1.4. Performance of trading strategies using 200-day moving averages: Trend Following Strategy

HIGH VOLATILTY			LOW VOLATILITY		
Cum Return	**VaR**	**Tracking Error**	**Cum Return**	**VaR**	**Tracking Error**
CLASSICAL BOLLINGER BANDS					
−0.1056	0.1058	0.4298	0.0813	0.0814	0.3903
ABB1					
−0.0835	0.1043	0.4196	0.0621	0.0991	0.4111
ABB2					
−0.0831	0.1047	0.4195	0.0557	0.0989	0.3815

As a result of the simulations, the new proposals (ABB1 and ABB2) perform well in 'relatively' short periods for moving averages (i.e., N=20), which is usual in the financial industry, and N=200 days is considered for long-term trend analysis (Achelis, 2000). The disadvantage of using longer periods is that the strategy could not incorporate recent price changes and could yield exclusions of relevant trading signals. Furthermore, the averages (of data) tend to be normally distributed with larger N, due to the fact of the Central Limit Theorem. Therefore, the significance of high-order moments in the trading bands decreases as the period increases.

1.3.1.2. Results

1.3.1.2.1. Comparative Evaluation of Trend Following Strategy: Edgeworth vs. Bollinger Bands

The universe of this study corresponds to the Dow Jones Industrial Average Index components (i.e., 30 stocks) for the analyzed period ranged between January 1, 1998, and November 15, 2021, on a daily frequency basis, for a total of 186,217 observations. The series was obtained from the Bloomberg Platform. Table 1.5 shows the descriptive statistics of the returns, which have been calculated as $r_t = 100ln\left(\frac{P_t}{P_{T-1}}\right)$, where P_t is the price of the asset at time *t*.

Table 1.5. Descriptive Statistics for the analyzed stocks and Index: Dow Jones Industrial Average: Trend Following Strategy

Stock	Mean	Std Deviation	Skewness	Excess Kurtosis
IBM	0.0143	0.0173	-0.3060	8.7140
AAPL	0.1156	0.0272	-3.2962	90.2132
AXP	0.0325	0.0231	0.1251	10.6357
AMGN	0.0458	0.0208	0.2703	5.2998
BA	0.0251	0.0222	-0.4747	15.7946
CAT	0.0357	0.0209	-0.1840	4.5081
T	-0.0068	0.0169	-0.0323	6.1921
CSCO	0.0295	0.0242	0.0379	8.5575
CVX	0.0179	0.0176	-0.4014	19.4710
DIS	0.0264	0.0197	0.0013	8.9083
GE	-0.0094	0.0209	-0.0086	7.4470
XOM	0.0121	0.0166	-0.0173	8.9736
PFE	0.0122	0.0168	-0.1847	4.9823
HD	0.0491	0.0199	-0.9902	20.9344
WMT	0.0336	0.0158	0.1441	6.3373
INTC	0.0170	0.0241	-0.4973	8.8348
JNJ	0.0270	0.0128	-0.3824	11.9511
JPM	0.0251	0.0246	0.2093	12.5205
KO	0.0088	0.0139	-0.1939	8.3717
MCD	0.0392	0.0154	-0.0768	10.9985
MMM	0.0247	0.0153	-0.1862	6.1757
MRK	0.0082	0.0173	-1.1987	23.9640
MSFT	0.0503	0.0197	-0.1298	8.4985
NKE	0.0586	0.0200	-0.0320	10.1757
PG	0.0214	0.0143	-2.8257	73.4636
TRV	0.0224	0.0188	-0.1454	17.4083
UNH	0.0716	0.0214	-0.6947	25.5876
VZ	0.0041	0.0158	0.1375	5.9184
WBA	0.0194	0.0188	-0.2492	7.0662
HON	0.0296	0.0198	-0.2300	12.2893
DJI	0.0252	0.0119	-0.3818	12.4977

International Business Machine (IBM), Apple(AAPL), American Express (AXP), Amgen Inc. (AMGN), Boeing (BA), Caterpillar (CAT), American Telephone and Telegraph (T), Cisco (CSCO), Chevron (CVX), Disney (DIS), General Electric (GE), Exxon Mobil (XOM), Pfizer (PFE), Home Depot (HD), Walmart (WMT), Intel (INTC), Johnson & Johnson (JNJ), JP MORGAN (JPM), Coca Cola (KO), McDonald's (MCD), Minnesota Mining and Manufacturing (MMM), Merck (MRK), Microsoft (MSFT), Nike (NKE), Procter and Gamble (PG), Travelers (TRV), United Health (UNH), Verizon (VZ), Walgreens (WBA), Honeywell (HON), Dow Jones Industrial Average (DJI). The analyzed period ranged between January 1 ,1998 and November 15, 2021, observed on a daily basis (186.217 observations).

As observed, Apple (AAPL) is the stock with the highest daily mean, excess kurtosis, standard deviation, and negative skewness. The only assets that present positive skewness are Amgen Inc (AMGN), JP Morgan (JPM), Walmart (WMT), Verizon (VZ), American Express (AXP), Cisco (CSCO), and Disney (DIS). All assets and the index present leptokurtic behavior and fat tails. All these features support the non-Gaussianity property of daily return distributions (see e.g., Cont, 2001).

For the analyzed period, between January 1, 1998, and November 15, 2021, we calculated the performance indicators for return group: Cumulative Return and annual return; for risk group: Maximum Drawdown, annualized Standard Deviation and VaR; and for Risk-Return ratio group: Annualized Sharpe, Tracking-Error, Information Ratio and Treynor Ratio (see the Appendix in Section 1.5 for more details of these indicators). We measure the behavior of the Bollinger Bands and the Adjusted Bollinger Bands with Edgeworth expansions (ABB1 and ABB2) implemented with trading rule of trend following framework for 20, 50, and 200 days, as usual in the literature and in practice. In order to summarize the results, the minimum, maximum and median of the performance measures were calculated for all the 30 assets comprising the Dow Jones Industrial Average Index. Individual asset results are available upon request.

The performance of the proposed model is compared with respect to the behavior of classical Bollinger Bands as the benchmark model. Tables 1.6, 1.7 and 1.8 present the results for the performance measures with N=20 periods, N=50, and N=200 days respectively, in the analyzed period between January 1, 1998, and November 15, 2021.

Table 1.6. Performance of trading strategies on Dow Jones' stocks using 20-day moving averages ('Full Sample'): Trend Following Strategy

Metric	Return		Risk			Risk-Return			
	Cum Return	Annual Ret	Max Drawdown	Annualized Standard Deviation	VaR	Annualized Sharpe	Tracking Error	Information Ratio	Treynor Ratio
				CLASSICAL BOLLINGER BANDS					
min	-0.8991	-0.092	0.3632	0.0808	0.0868	-0.6535	0.2015	-0.6123	-0.2580
max	1.1472	0.0327	0.9099	0.1744	0.5341	0.2323	0.2563	-0.0610	3.1710
median	-0.5631	-0.034	0.6447	0.1176	0.1716	-0.3048	0.2152	-0.3818	0.3849
				ABB 1					
min	-0.9131	-0.098	0.4603	0.0907	0.0995	-0.7297	0.203	-0.6365	-0.238
max	2.2458	0.051	0.9250	0.1864	0.4726	0.2943	0.263	0.0148	1.6198
median	-0.6725	-0.046	0.7427	0.1332	0.1540	-0.3945	0.223	-0.4256	0.4359
				ABB 2					
min	-0.9288	-0.1053	0.5046	0.0960	0.0910	-0.7507	0.2051	-0.6817	-0.2675
max	2.8681	0.0586	0.9418	0.1966	0.4591	0.3246	0.2694	0.0453	1.0150
median	-0.6890	-0.0480	0.7574	0.1419	0.1475	-0.3494	0.2292	-0.4206	0.3438

Notes: Performance statistics on the 30 Dow Jones' stocks when the analyzed period is January 1, 1998, and November 15, 2021. ABB1 is based on Hall (1983), and ABB2 on Hall and Jing (1995). $N = 20$.

Table 1.7. Performance of trading strategies on Dow Jones' stocks using 50-day moving averages ('Full Sample'): Trend Following Strategy

Metric	Return		Risk			Risk-Return			
	Cum Return	Annual Ret	Max Drawdown	Annualized Standard Deviation	VaR	Annualized Sharpe	Tracking Error	Information Ratio	Treynor Ratio
				CLASSICAL BOLLINGER BANDS					
min	-0.9326	-0.1079	0.3507	0.0902	0.1074	-0.6879	0.1717	-0.6437	-0.2182
max	2.5452	0.0550	0.9359	0.1875	0.7756	0.2933	0.2576	0.0466	1.9573
median	-0.6414	-0.0425	0.7386	0.1336	0.1635	-0.3439	0.2240	-0.3921	0.2689
				ABB 1					
min	-0.9369	-0.1103	0.4211	0.0952	0.1033	-0.5868	0.2042	-0.5989	-0.1397
max	0.6373	0.0211	0.9549	0.1994	0.7272	0.1266	0.2667	-0.0825	0.7745
median	-0.6696	-0.0458	0.7377	0.1407	0.1599	-0.3234	0.2283	-0.3884	0.2595
				ABB 2					
min	-0.9092	-0.0965	0.4166	0.0969	0.1021	-0.5854	0.2046	-0.5393	-0.1961
max	0.8228	0.0257	0.9379	0.2027	0.6900	0.1534	0.2688	-0.0700	0.8592
median	-0.6946	-0.0489	0.7522	0.1458	0.1537	-0.3244	0.2311	-0.3890	0.2656

Notes: Performance statistics on the 30 Dow Jones' stocks when the analyzed period is January 1, 1998, and November 15, 2021. ABB1 is based on Hall (1983), and ABB2 on Hall and Jing (1995). $N = 50$.

Table 1.8. Performance of trading strategies on Dow Jones' stocks using 200-day moving averages ('Full Sample'): Trend Following Strategy

	Return		Risk			Risk-Return			
Metric	**Cum Return**	**Annual Ret**	**Max Drawdown**	**Annualized Standard Deviation**	**VaR**	**Annualized Sharpe**	**Tracking Error**	**Information Ratio**	**Treynor Ratio**
				CLASSICAL BOLLINGER BANDS					
min	-0.9543	-0.1253	0.3863	0.0877	0.0619	-0.6581	0.1934	-0.6199	-3.7637
max	1.3853	0.0384	0.9863	0.3533	0.4152	0.2020	0.4073	-0.0306	2.6081
median	-0.5745	-0.0365	0.6732	0.1458	0.1502	-0.2993	0.2296	-0.3831	0.1947
				ABB 1					
min	-0.8853	-0.0897	0.4534	0.0908	0.1012	-0.6374	0.1772	-0.5729	-74.3354
max	2.7289	0.0588	0.9018	0.2124	0.4166	0.2976	0.2767	0.0458	2.0560
median	-0.5442	-0.0335	0.6809	0.1412	0.1480	-0.2675	0.2233	-0.3781	0.2177
				ABB 2					
min	-0.8902	-0.0914	0.4641	0.0916	0.1003	-0.6353	0.2000	-0.5742	-11.7261
max	2.8544	0.0603	0.8980	0.2130	0.4136	0.3041	0.2774	0.0514	6.8089
median	-0.5431	-0.0334	0.6735	0.1424	0.1462	-0.2697	0.2272	-0.3594	0.2030

Notes: Performance statistics on the 30 Dow Jones' stocks when the analyzed period is January 1, 1998, and November 15, 2021. ABB1 is based on Hall (1983), and ABB2 on Hall and Jing (1995). N = 200.

It is observed that the medians of the VaR for the ABB1 and ABB2 models are lower than the median VaR for the classical Bollinger Bands for all N's (i.e., N=20, 50, and 200).

The median values of the Tracking Error for the Edgeworth-type intervals (ABB1 and ABB2) are greater than the median Tracking Error for the classical Bollinger Bands for N=20 and N=50. The Treynor Ratio median for the ABB1 model is greater than the Treynor Ratio median for the classical Bollinger Bands for N=20. For the Treynor Ratio case, median values for the ABB1 and ABB2 models are greater than those for the classical Bollinger Bands for N=200.

The medians of the annualized Standard deviation for the two new techniques (ABB1 and ABB2) are smaller than the median of the annualized Standard deviation for the classical Bollinger Bands for N=200.

The Maximum Drawdown median for the ABB1 model is less than the median Maximum Drawdown for classical Bollinger Bands for N=50. The maximum values of Cumulative Return and Annual Return for both the ABB1 and ABB2 models are greater than the maximum value of Cumulative Return and Annual Return for the classical Bollinger Bands for N=20.

The maximum values of VaR for Edgeworth's ABB1 and ABB2 models are lower than the maximum value of VaR for classical Bollinger Bands in periods N=20 and N=50. On the other hand, the maximum values of the Annualized Sharpe, Tracking Error, and Information Ratio for the proposed models (ABB1 and ABB2) are greater than the maximum value of the Annualized Sharpe, Tracking Error, and Information Ratio for the classical Bollinger Bands for N=20.

These results show that the confidence intervals based on the Edgeworth expansions generally outperform (return, risk, and risk-return measure groups) the classical Bollinger Bands in the trend following trading strategy for the thirty stocks that form the Down Jones Industrial Average Index in the analyzed period. To confirm this outperformance, in the following section the three models are applied to the same assets during the specific periods comprising the two last worldwide crises, namely, the subprime and COVID-19 crises.

1.3.1.3. Further results for trend following strategy.

As a robustness check, the performance indicators for return, risk and risk-return groups were analyzed for the subprime crisis between January 1 and December 31, 2008; and the COVID-19 crisis in a period ranged between January 1 to December 31, 2020, both in a daily frequency basis, for a total of 7,812 observations respectively.

Table 1.9. Performance of trading strategies on Dow Jones' stocks using 20-day moving averages ('Subprime crisis period'): Trend Following Strategy

	Return		Risk			Risk-Return			
Metric	**Cum Return**	**Annual Ret**	**Max Drawdown**	**Annualized Standard Deviation**	**VaR**	**Annualized Sharpe**	**Tracking Error**	**Information Ratio**	**Treynor Ratio**
				CLASSICAL BOLLINGER BANDS					
min	-0.4709	-0.4991	0.0554	0.0981	0.0061	-1.6177	0.3920	-0.2023	-4.7937
max	0.1904	0.2084	0.5250	0.4762	0.3437	1.4486	0.6260	1.4051	16.9934
median	-0.1221	-0.1319	0.1978	0.1812	0.1113	-0.7179	0.4322	0.5711	0.9466
				ABB 1					
min	-0.5219	-0.5513	0.0903	0.1191	0.0485	-1.5477	0.3947	-0.3589	-1.5678
max	0.1823	0.1994	0.5538	0.4771	0.3197	1.4031	0.6301	1.3654	17.7885
median	-0.1049	-0.1203	0.1980	0.2117	0.1245	-0.5511	0.4331	0.5554	0.6685
				ABB 2					
min	-0.5359	-0.5656	0.0903	0.1363	4.4018	-1.6595	0.3185	-0.3882	-1.3246
max	0.1554	0.1699	0.5417	0.4777	0.3091	1.1098	0.6361	1.2901	9.9656
median	-0.1673	-0.1803	0.2253	0.2179	0.1203	-0.7662	0.4378	0.4717	0.9079

Notes: Performance statistics on the 30 Dow Jones' stocks when the analyzed period is January 1, 2008, to December 31, 2008. ABB1 is based on Hall (1983), and ABB2 on Hall and Jing (1995). $N = 20$.

Table 1.10. Performance of trading strategies on Dow Jones' stocks using 50-day moving averages ('Subprime crisis period'): Trend Following Strategy

	Return		Risk			Risk-Return			
Metric	**Cum Return**	**Annual Ret**	**Max Drawdown**	**Annualized Standard Deviation**	**VaR**	**Annualized Sharpe**	**Tracking Error**	**Information Ratio**	**Treynor Ratio**
				CLASSICAL BOLLINGER BANDS					
min	-0.4319	-0.5061	0.0934	0.1631	0.0371	-2.1375	0.4208	-0.2370	-1.5746
max	0.2599	0.3340	0.4361	0.4619	0.2855	1.1228	0.6591	1.5005	15.0686
median	-0.1026	-0.1263	0.2473	0.2355	0.1233	-0.4045	0.4647	0.5219	0.3678
				ABB 1					
min	-0.4354	-0.5099	0.1070	0.1843	0.0496	-1.6668	0.4282	-0.2366	-1.4286
max	0.4701	0.6172	0.4560	0.4639	0.2726	1.8076	0.6535	1.9840	9.7021
median	-0.0993	-0.1222	0.2219	0.2608	0.1177	-0.4208	0.4662	0.5564	0.4114
				ABB 2					
min	-0.3896	-0.4598	0.0931	0.1847	0.0509	-1.4240	0.4385	-0.1027	-1.7802
max	0.3850	0.5013	0.4560	0.4789	0.2726	1.3386	0.6535	1.7366	9.7021
median	-0.1003	-0.1235	0.2108	0.2728	0.1178	-0.4362	0.4664	0.5812	0.4266

Notes: Performance statistics on the 30 Dow Jones' stocks when the analyzed period is January 1, 2008, to December 31, 2008. ABB1 is based on Hall (1983), and ABB2 on Hall and Jing (1995). $N = 50$.

Table 1.11. Performance of trading strategies on Dow Jones' stocks using 200-day moving averages ('Subprime crisis period'): Trend Following Strategy

	Return		Risk			Risk-Return			
Metric	**Cum Return**	**Annual Ret**	**Max Drawdown**	**Annualized Standard Deviation**	**VaR**	**Annualized Sharpe**	**Tracking Error**	**Information Ratio**	**Treynor Ratio**
					CLASSICAL BOLLINGER BANDS				
min	-0.4045	-0.9189	0.0433	0.0954	0.0009	-2.3009	0.5749	-1.2229	-0.0667
max	0.0113	0.0559	0.4872	0.7940	0.2079	0.1059	1.0827	0.2163	6.5928
median	-0.1367	-0.5086	0.1997	0.5240	0.1030	-1.2777	0.6862	-0.6360	1.3364
					ABB 1				
min	-0.4045	-0.9189	0.0433	0.0954	0.0009	-2.3349	0.5749	-1.2450	-0.1231
max	0.0241	0.1222	0.5165	0.7969	0.2054	0.1887	1.0806	0.3219	6.5928
median	-0.1367	-0.5086	0.1997	0.5240	0.1030	-1.2777	0.6862	-0.6360	1.3364
					ABB 2				
min	-0.4045	-0.9189	0.0433	0.0954	0.0009	-2.3349	0.5749	-1.2450	-0.1231
max	0.0241	0.1222	0.5165	0.8031	0.2054	0.1887	1.0806	0.3219	6.5928
median	-0.1491	-0.5428	0.2080	0.5197	0.1044	-1.3311	0.6800	-0.6783	1.3485

Notes: Performance statistics on the 30 Dow Jones' stocks when the analyzed period is January 1, 2008, to December 31, 2008. ABB1 is based on Hall (1983), and ABB2 on Hall and Jing (1995). $N = 200$.

Table 1.12. Performance of trading strategies on Dow Jones' stocks using 20-day moving averages ('COVID-19 crisis effect'): Trend Following Strategy

	Return		Risk			Risk-Return			
Metric	**Cum Return**	**Annual Ret**	**Max Drawdown**	**Annualized Standard Deviation**	**VaR**	**Annualized Sharpe**	**Tracking Error**	**Information Ratio**	**Treynor Ratio**
					CLASSICAL BOLLINGER BANDS				
min	-0.2820	-0.3023	0.1278	0.1183	0.0226	-2.2563	0.3469	-0.7389	-1.7827
max	0.3306	0.3638	0.3651	0.3471	0.2767	1.0481	0.5224	0.7313	3.1359
median	-0.1582	-0.1706	0.2279	0.1757	0.0990	-0.9879	0.3900	-0.3806	1.3199
					ABB 1				
min	-0.3845	-0.4097	0.1125	0.1220	0.0289	-2.1480	0.3576	-1.0289	-2.6333
max	1.5802	1.7999	0.4370	0.5384	0.2233	3.3428	0.5939	3.0615	3.8020
median	-0.1638	-0.1766	0.2550	0.2061	0.0917	-0.9145	0.3929	-0.4027	0.9338
					ABB 2				
min	-0.5036	-0.5327	0.1095	0.1241	0.0172	-2.3136	0.3473	-1.3942	-3.1219
max	1.8854	2.1614	0.5253	0.5485	0.3195	3.9402	0.5978	3.6459	3.7088
median	-0.1456	-0.1571	0.2560	0.2355	0.0944	-0.7372	0.4007	-0.3732	0.6933

Notes: Performance statistics on the 30 Dow Jones' stocks when the analyzed period is January 1, 2020, to December 31, 2020. ABB1 is based on Hall (1983), and ABB2 on Hall and Jing (1995). $N = 20$.

Table 1.13. Performance of trading strategies on Dow Jones' stocks using 50-day moving averages ('COVID-19 crisis effect'): Trend Following Strategy

	Return		Risk			Risk-Return			
Metric	**Cum Return**	**Annual Ret**	**Max Drawdown**	**Annualized Standard Deviation**	**VaR**	**Annualized Sharpe**	**Tracking Error**	**Information Ratio**	**Treynor Ratio**
				CLASSICAL BOLLINGER BANDS					
min	-0.2046	-0.2485	0.0937	0.1361	0.0518	-1.1404	0.3303	-1.5907	-0.7646
max	0.2784	0.3585	0.3507	0.5764	0.3000	0.9663	0.5855	0.0601	1.6218
median	0.0287	0.0360	0.1883	0.2503	0.1181	0.1489	0.3659	-0.7446	-0.0858
				ABB 1					
min	-0.2137	-0.2592	0.0937	0.1440	0.0605	-1.3190	0.3353	-1.7174	-0.9514
max	0.2666	0.3430	0.5016	0.6320	0.2965	1.1518	0.6812	0.0433	1.6257
median	0.0126	0.0157	0.1877	0.2520	0.1236	0.0632	0.3667	-0.8404	-0.0406
				ABB 2					
min	-0.2058	-0.2498	0.0937	0.1440	0.0666	-1.2690	0.3305	-1.6894	-0.9514
max	0.2666	0.3430	0.4835	0.6333	0.2942	1.1518	0.6859	0.0433	1.6257
median	-0.0140	-0.0174	0.2023	0.2521	0.1234	-0.0473	0.3712	-0.8760	0.0373

Notes: Performance statistics on the 30 Dow Jones' stocks when the analyzed period is January 1, 2020, to December 31, 2020. ABB1 is based on Hall (1983), and ABB2 on Hall and Jing (1995). $N = 50$.

Table 1.14. Performance of trading strategies on Dow Jones' stocks using 200-day moving averages ('COVID-19 crisis effect'): Trend Following Strategy

	Return		Risk			Risk-Return			
Metric	**Cum Return**	**Annual Ret**	**Max Drawdown**	**Annualized Standard Deviation**	**VaR**	**Annualized Sharpe**	**Tracking Error**	**Information Ratio**	**Treynor Ratio**
				CLASSICAL BOLLINGER BANDS					
min	-0.1108	-0.4340	0.0000	0.0000	0.0000	-3.0590	0.1719	-2.7056	-9.0326
max	0.2223	1.6456	0.1346	0.3313	0.0602	4.9677	0.3814	3.3953	19.7130
median	-0.0035	-0.0167	0.0099	0.0219	0.0025	-0.7040	0.1742	-2.0527	10.5245
				ABB 1					
min	-0.0915	-0.3720	0.0000	0.0000	0.0000	-4.1051	0.1719	-3.5099	-29.7359
max	0.2223	1.6456	0.1218	0.0209	0.0612	4.9677	0.3814	3.3953	19.7130
median	-0.0006	-0.0028	0.0099	0.0219	0.0028	-0.3824	0.1742	-2.0302	8.4519
				ABB 2					
min	-0.1028	-0.4089	0.0000	0.0009	0.0000	-4.1051	0.1719	-3.5099	-29.7359
max	0.2223	1.6456	0.1269	0.3313	0.0620	4.9677	0.3814	3.3953	19.7130
median	0.0004	0.0020	0.0100	0.0225	0.0031	0.0094	0.1742	-2.0186	10.5716

Notes: Performance statistics on the 30 Dow Jones' stocks when the analyzed period is January 1, 2020, to December 31, 2020. ABB1 is based on Hall (1983), and ABB2 on Hall and Jing (1995). $N = 200$.

Tables 1.9 - 1.14 present the results on the performance measures with *N*=20, 50, and 200 days. For the 2008 and 2020 crises, it is observed that the median Tracking Error for both the ABB1 and ABB2 models are higher than the median Tracking Error for the classical Bollinger Bands when *N*=20 and *N*=50 days.

In the 2008 crisis, the median Information Ratio for the new proposals (ABB1 and ABB2 models) are greater than the median Information Ratio for the classical Bollinger Bands when *N*=50. Moreover, in the same period of crisis (2008), the medians of the Treynor Ratio for the two new techniques (ABB1 and ABB2 models) are greater than the median Treynor Ratio for the classical Bollinger Bands in the for *N*=50. Furthermore, the median Treynor Ratio for the ABB2 model is greater than the median Treynor Ratio for classical Bollinger Bands when *N*=200. On the other hand, the median Max Drawdown Ratio for the ABB1 and ABB2 models is lower than that of the classical Bollinger Bands when *N*=50. The median VaR for both the ABB1 and ABB2 models are lower than the median Var for classical Bollinger Bands for *N*=50. The maximum VaR for the new two models (ABB1 and ABB2) is lower than the Var maximum for the classical Bollinger Bands for all *N*'s (i.e., *N*=20, *N*=50 and *N*=200).

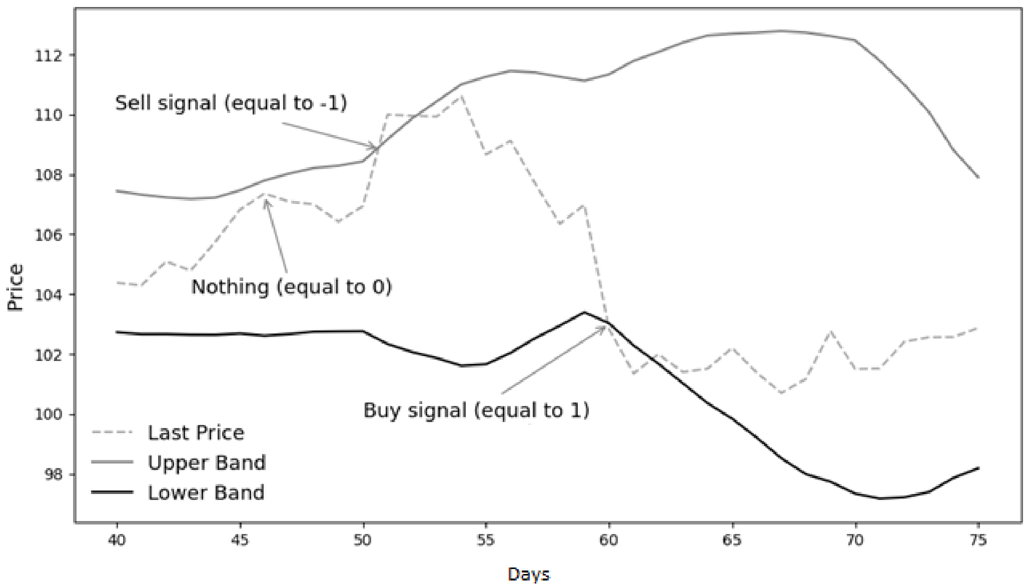
In the 2020 crisis period, the median Treynor Ratio for the ABB2 model is higher than the median Treynor Ratio for the classical Bollinger Bands in the periods *N*=50 and *N*=200. The median Max Drawdown for the ABB1 model is lower than the median Max Drawdown for classical Bollinger Bands with *N*=50. Finally, the maximum values of the ABB1 and ABB2 models are lower than the maximum value for the classical Bollinger Bands when *N*=200.

The findings for both crisis periods and full sample are in line with the simulation results, where the adjusted bands perform better for shorter periods of *N*, for the reasons explained in the Simulations section. Nevertheless, shorter periods for moving average calculations are employed in practice to capture short-term price fluctuations. Next section shows the model comparison for the contrarian trading strategy.

1.3.2. The Contrarian Trading Strategy

The Contrarian trading rule (Bollinger, 2002) is tested with (1) the (classical) Bollinger Bands; (2) ABB1 model, i.e., the confidence interval with Edgeworth expansion by Hall (1983), and (3) the ABB2 model, i.e., the confidence interval with the Berry-Esseen theorem for Edgeworth expansions proposed by Hall & Jing (1995). To this aim, the 30 stocks that form the Dow Jones Industrial Average Index (Jeet & Vats, 2017) are employed between January 1, 2000, and December 31, 2022. We use the Bollinger Bands with 6-day, 10-day, 15-day, and 20-day moving average to generate automated trading signals 1, −1, and zero as follows. A buy signal (equals to 1) arises when the last price of a stock is at the lower line of the indicator, while the middle line is considered the resistance line. A sell signal (equals to −1) occurs when the last price is at the upper line of the indicator, while the middle line is considered the support level. Finally, the out-of-market includes situations where the signal is zero. We did not include any transaction and slippage costs to calculate the performance measures, which might constitute an interesting refinement for further research. Figure 1.3 summarizes this strategy.

Figure 1.3. Contrarian trading strategy explained.



Note: When the price is in an uptrend and touches the lower band, it is a buy signal (1). Meanwhile the price is in a downtrend and touches the upper band, it is a sell signal (-1), otherwise the signal is equal to zero. Source: own elaboration.

Several measures of trading performance, commonly utilized by experienced traders and academics (Balsara et al., 2009), are evaluated to test the success of the models with the contrarian trading strategy. A brief description for each of these measures, along with their mathematical descriptions can be found in the Appendix of this chapter. Table 1.15 presents the performance measures grouped by return, risk, and risk-return indicators.

Table 1.15. Measures of trading performance: Contrarian Trading Strategy

RETURN	RISK	RISK-RETURN
Cumulative Return	Annualized Standard Deviation	Sharpe Ratio
Annual Return	Maximum Drawdown (MDD)	Omega Ratio
		Tracking Error
		Information Ratio Treynor Ratio

Source: own elaboration.

In addition, to statistically proof the validity of the best performance of our proposals on the median over the classical Bollinger bands in the different trading performance measures, we use the Non-Parametric Jonckheere Terpstra test (T_{JT}) (Ali et al., 2015) for ordered medians where the null hypothesis is: H_0: $\eta_1 = \eta_2 \ = \ldots = \eta_K$ with η_i the population median for the ith population and the alternative hypothesis is that the population medians have an *a priori* ordering e.g.: H_A: $\eta_1 \leq \eta_2 \leq \ldots \leq \eta_K$ (or $\eta_1 \geq \eta_2 \geq \ldots \geq \eta_k$) with at least one of the inequalities being strict.

The statistic T_{JT} is defined as $J = \sum U_{xy}$, where U_{xy} is the number of observations in group y that are greater than each observation in group x. Therefore, the standardized T_{JT} is computed as $Z = \frac{J-E(J)}{\sqrt{Var(J)}}$ with *E(J)=0.25**$\left(N^2 - \sum_{j=1}^{k} n_j^2\right)$ and $Var(J) = N^2(2n+3) - \sum_{j=1}^{k} n_j^2\,(2n_j + 3)$ where j is the number of group, N is the total number of observations in all groups, n_j is the observation in group j and k is the total number of groups.

1.3.2.1. Simulations

Two simulations were performed for prices to contrast the results of the proposed models (ABB1 and ABB2) with the classical Bollinger Bands (BB). We implemented a simulation of the price for 30 stocks, and the skewness value and excess kurtosis for each simulated stock are illustrated in Table 1.16. Nevertheless, the first simulation accounts for a volatility of 15% (low volatility) and the second one with a volatility of 30% (high volatility) for N = 6, 10, 15 and 20 days. For this purpose, 500 prices for each simulated stock were generated following a geometric Brownian motion (gBm) with a mean of 0.10 and a volatility of 0.30 (and 0.15) for the diffusion process. The gBm with a seed of 999 was employed. Furthermore, the Cumulative Return, Annual Return (return indicators), Sharpe, Omega, Information ratio and Tracking Error (risk-return indicators) are calculated to compare the results for each methodology.

Table 1.16. Simulated Stock for contrarian strategy

Simulated stock	Skewness	Excess Kurtosis
1	0.5463	2.4856
2	-0.5444	2.1377
3	-0.1240	1.9534
4	0.0043	2.1309
5	-0.0908	2.8561
6	-0.0357	2.2329
7	0.8220	3.3884
8	0.1985	2.6248
9	0.7381	2.9146
10	0.0348	2.5992
11	0.1067	2.1777
12	0.2954	1.5598
13	-0.5089	2.2453
14	-0.0302	2.3553
15	0.2602	1.8546
16	0.6015	1.8099
17	0.6658	2.1731
18	-0.4707	2.4541
19	0.7868	3.4506
20	0.5421	2.6652
21	-0.1350	2.1168
22	-0.0304	1.9973
23	0.7430	2.7144

24	-0.8243	3.0199
25	0.3716	3.1936
26	0.2241	2.0814
27	0.4510	1.9132
28	0.5265	2.3893
29	-0.0503	1.9992
30	0.2645	1.6171

Source: own elaboration.

Table 1.17. Performance of trading strategies using 6-day moving averages: Contrarian Trading Strategy

Metric	Low Volatility: 15%						High Volatility: 30%					
	Cum Return	Annual Return	Sharpe	Omega	Information Ratio	Tracking Error	Cum Return	Annual Return	Sharpe	Omega	Information Ratio	Tracking Error
						BB						
Median	0,0012	0,0006	0,2932	0,2320	0,3914	0,2110	0,0022	0,0011	0,2492	0,0410	0,3938	0,2112
						ABB1						
Median	0,0267	0,0134	0,7064	1,6639	0,4520	0,2123	0,0549	0,0273	0,6424	1,6266	0,5117	0,2154
P-Value JT (CBB-ABB1)	0,0002	0,0002	0,0042	0,0002	0,0002	0,0002	0,0002	0,0002	0,0052	0,0002	0,0002	0,0002
						ABB2						
Median	0,0604	0,0301	1,0487	1,9180	0,5292	0,2124	0,1067	0,0525	0,9243	1,8095	0,6142	0,2180
P-Value JT (CBB-ABB1)	0,0002*	0,0002*	0,0002*	0,0002*	0,0002*	0,0002*	0,0002*	0,0002*	0,0002*	0,0002*	0,0002*	0,0002*

Notes: CBB: Classical Bollinger Bands, ABB1 is based on Hall (1983), and ABB2 on Hall & Jing (1995). N = 6. JT: Test Jonckheere –Terpstra. * and ** significance at the 1% and 5% levels, respectively.
Source: own elaboration.

As observed in Table 1.17 (N=6), the two proposed models (ABB1 and ABB2) perform better than the classical Bollinger Bands for all indicators in low and high volatility.

Table 1.18. Performance of trading strategies using 10-day moving averages: Contrarian Trading Strategy

Metric	Low Volatility: 15%						High Volatility: 30%					
	Cum Return	Annual Return	Sharpe	Omega	Information Ratio	Tracking Error	Cum Return	Annual Return	Sharpe	Omega	Information Ratio	Tracking Error
						BB						
Median	0,0114	0,0057	0,2689	1,2549	0,4144	0,2115	0,0352	0,0176	0,5456	1,6850	0,4670	0,2136
						ABB1						
Median	0,0354	0,0177	0,7922	1,8817	0,4717	0,2119	0,0557	0,0278	0,6259	1,6593	0,5117	0,2153
P-Value JT (CBB-ABB1)	0,0008*	0,0004*	0,0118	0,0524	0,0014*	0,1930	0,0564	0,0508	0,1738	0,4232	0,0708	0,008*
						ABB2						
Median	0,0393	0,0197	0,7346	1,7173	0,4792	0,2122	0,0822	0,0407	0,7112	1,7314	0,5582	0,2160
P-Value JT (CBB-ABB1)	0,0002*	0,0002*	0,0062*	0,0670	0,0004*	0,0150	0,0042*	0,0046*	0,0780	0,3700	0,0086*	0,0002*

Notes: CBB: Classical Bollinger Bands, ABB1 is based on Hall (1983), and ABB2 on Hall & Jing (1995). N = 6. JT: Test Jonckheere –Terpstra. * and ** significance at the 1% and 5% levels, respectively.

Source: own elaboration.

The two new models (ABB1 and ABB2) outperform the classical Bollinger Bands in terms of cumulative return, annual return, information ratio in low volatility, and tracking error for high volatility in the adjusted versions (ABB1 and ABB2). Additionally, the ABB2 model surpasses BB for the Sharpe ratio in low volatility and excess in cumulative return and annual return in high volatility, as observed in Table 1.18 (*N*=10).

Table 1.19. Performance of trading strategies using 15-day moving averages: Contrarian Trading Strategy

Metric	Low Volatility: 15%						High Volatility: 30%					
	Cum Return	**Annual Return**	**Sharpe**	**Omega**	**Information Ratio**	**Tracking Error**	**Cum Return**	**Annual Return**	**Sharpe**	**Omega**	**Information Ratio**	**Tracking Error**
BB												
Median	0,0098	0,0049	0,3409	1,4115	0,4113	0,2116	0,0057	0,0029	0,0946	1,1028	0,3970	0,2134
ABB1												
Median	0,0077	0,0039	0,2280	1,2503	0,4058	0,2116	0,0243	0,0122	0,3131	1,3296	0,4354	0,2138
P-Value JT (CBB-ABB1)	0,492	0,482	0,4392	0,5036	0,4768	0,3274	0,1482	0,1364	0,1374	0,152	0,1294	0,2694
ABB2												
Median	0,0213	0,0107	0,6154	1,7321	0,4381	0,2118	0,0310	0,0155	0,4686	1,5131	0,4565	0,2139
P-Value JT (CBB-ABB1)	0,1446	0,1482	0,2056	0,2316	0,1464	0,4300	0,0756	0,0804	0,0874	0,0972	0,0718	0,2178

Notes: Classical Bollinger Bands, ABB1 is based on Hall (1983), and ABB2 on Hall & Jing (1995). N = 15. JT: Test Jonckheere –Terpstra. * and ** significance at the 1% and 5% levels, respectively.
Source: own elaboration.

Tables 1.18 ,1.19 and 1.20 reveal that neither for periods of high volatility nor for periods of low volatility does any performance indicator outperform the classical Bollinger Bands with the proposed models.

Table 1.20. Performance of trading strategies using 20-day moving averages

Metric	Low Volatility: 15%						High Volatility: 30%					
	Cum Return	Annual Return	Sharpe	Omega	Information Ratio	Tracking Error	Cum Return	Annual Return	Sharpe	Omega	Information Ratio	Tracking Error
						BB						
Median	0,0145	0,0073	0,4523	1,5596	0,4221	0,2116	0,0309	0,0155	0,4879	1,6389	0,4583	0,2135
						ABB1						
Median	0,0166	0,0084	0,5093	1,7509	0,4270	0,2116	0,0378	0,0189	0,6863	2,2571	0,4734	0,2133
P-Value JT (CBB-ABB1)	0,3326	0,3242	0,2784	0,2838	0,3332	0,4494	0,3082	0,3044	0,2476	0,2456	0,3006	0,6494
						ABB2						
Median	0,0194	0,0097	0,6295	1,8463	0,4331	0,2116	0,0326	0,0163	0,5064	1,7343	0,4579	0,2135
P-Value JT (CBB-ABB1)	0,2596	0,2536	0,2242	0,2636	0,2466	0,3172	0,4794	0,4888	0,4416	0,4698	0,5018	0,3668

Notes: Classical Bollinger Bands, ABB1 is based on Hall (1983), and ABB2 on Hall & Jing (1995). *N* = 20. JT: Test Jonckheere –Terpstra. * and ** significance at the 1% and 5% levels, respectively.
Source: own elaboration.

As a result of the simulations, the new proposals (ABB1 and ABB2) perform well in relatively short periods for moving averages (i.e., *N*=6, 10). The disadvantage of using longer periods is that the strategy may not incorporate recent price changes and could result in exclusions of relevant trading signals. Furthermore, the averages (of data) tend to be normally distributed with a larger *N*, due to the Central Limit Theorem. Therefore, the significance of high-order moments in the trading bands decreases as the period increases.

1.3.2.2. Results

1.3.2.2.1. Comparative Evaluation of Contrarian Trading Strategy: Edgeworth vs. Bollinger Bands

The universe of this study corresponds to the Dow Jones Industrial Average Index components (i.e., 30 stocks) for the analyzed period, which ranged between January 1, 2000, and December 31, 2022, on a daily frequency basis, for a total of 179,397 observations. The series was obtained from the Bloomberg Platform. Table 1.21 displays the descriptive statistics of the returns, calculated as $r_t = 100ln\left(\frac{P_t}{P_{T-1}}\right)$, where P_t is the price of the asset at time t.

Table 1.21. Descriptive Statistics for the analyzed stocks and Index: Dow Jones Industrial Average: Contrarian Trading Strategy

Stock	Mean	Std Deviation	Skewness	Excess Kurtosis
IBM	0.0041	0.0166	-0.2837	8.4178
AAPL	0.0841	0.0260	-3.9830	111.3980
AXP	0.0202	0.0228	0.1088	11.4298
AMGN	0.0247	0.0196	0.2407	6.1369
BA	0.0269	0.0224	-0.4224	14.9123
CAT	0.0395	0.0204	-0.1748	4.5122
CSCO	-0.0022	0.0236	0.0607	9.6280
CVX	0.0252	0.0177	-0.4420	19.9563
DIS	0.0187	0.0195	-0.0450	9.6317
HD	0.0273	0.0195	-1.1009	23.3084
INTC	-0.0086	0.0237	-0.5264	9.6980
JNJ	0.0232	0.0122	-0.4927	14.3232
JPM	0.0175	0.0241	0.2239	13.9299
KO	0.0141	0.0133	-0.1904	9.0681
MCD	0.0327	0.0147	-0.1827	12.4120
MMM	0.0161	0.0151	-0.2361	6.9338
MSFT	0.0244	0.0194	-0.1675	9.2898
NKE	0.0513	0.0194	-0.2455	11.3292
PG	0.0180	0.0137	-3.3023	89.6203
TRV	0.0300	0.0183	-0.2383	19.7552
UNH	0.0755	0.0197	0.1204	19.5338
VZ	-0.0054	0.0152	0.1189	6.7707
WBA	0.0046	0.0182	-0.3163	7.3637
WMT	0.0130	0.0150	0.0511	8.1617
HON	0.0238	0.0194	-0.2287	13.5289
MRK	0.0094	0.0170	-1.2866	26.9732
T	-0.0113	0.0164	-0.0371	7.1697
GE	-0.0263	0.0212	-0.0379	7.4038
XOM	0.0179	0.0168	-0.0578	8.8756
PFE	0.0091	0.0160	-0.1585	5.2446
DJI	0.0185	0.0119	-0.3692	12.5198

Note: International Business Machine (IBM), Apple(AAPL), American Express (AXP), Amgen Inc. (AMGN), Boeing (BA), Caterpillar (CAT), American Telephone and Telegraph (T), Cisco (CSCO), Chevron (CVX), Disney (DIS), General Electric (GE), Exxon Mobil (XOM), Pfizer (PFE), Home Depot (HD), Walmart (WMT), Intel (INTC), Johnson & Johnson (JNJ), JP MORGAN (JPM), Coca Cola (KO), McDonald's (MCD), Minnesota Mining and Manufacturing (MMM), Merck (MRK), Microsoft (MSFT), Nike (NKE), Procter and Gamble (PG), Travelers (TRV), United Health (UNH), Verizon (VZ), Walgreens (WBA), Honeywell (HON), Dow Jones Industrial Average (DJI). The analyzed period ranged between January 1, 2000, and December 31, 2022, observed on a daily basis (179.397 observations). Source: own elaboration.

As observed, Apple (AAPL) is the stock with the highest daily mean, excess kurtosis, standard deviation, and negative skewness. The only assets that present positive skewness are Amgen Inc (AMGN), JP Morgan (JPM), Walmart (WMT), Verizon (VZ), American Express (AXP), Cisco (CSCO), and United healthcare (UNH). All assets and the index present leptokurtic behavior and fat tails. All these features support the non-Gaussianity property of daily return distributions (Cont, 2001).

For the analyzed period (January 1, 2000, to December 31, 2022), we calculated the performance indicators for return group: Cumulative Return and annual return; for risk group: Maximum Drawdown and annualized Standard Deviation; and for Risk-Return ratio group: Annualized Sharpe, Omega ratio, Tracking-Error, Information Ratio and Treynor Ratio (refer to the Appendix of this chapter for additional indicator details). We measure the behavior of the Bollinger Bands and the Adjusted Bollinger Bands with Edgeworth expansions (ABB1 and ABB2) implemented with classical trading rule framework for 6, 10, 15 and 20 days. To summarize the results, the minimum, maximum and median of the performance measures were calculated for all the 30 assets comprising the Dow Jones Industrial Average Index. Individual asset results are available upon request.

The performance of the proposed model is compared with the behavior of the classical Bollinger Bands as the benchmark model. Tables 1.22, 1.23, 1.24, and 1.25 present the results for the performance measures with N=6 periods, N=10, N=15, and N=20 days, respectively, in the analyzed period between January 1, 2000, and December 31, 2022.

Table 1.22. Performance of trading strategies on Dow Jones' stocks using 6-day moving averages ('Full Sample'): Contrarian Trading Strategy

Metric	Return		Risk		Risk-Return				
	Cum Return	Annual Return	Max Drawdown	Annualized Standard Deviation	Sharpe	Omega	Tracking Error	Information Ratio	Treynor
					BB				
Min	-0.1289	-0.0060	0.0092	0.1066	-0.3622	0.2601	0.1897	-0.1841	-148.8155
Max	0.1697	0.0069	0.1760	0.4767	0.4647	6.1743	0.1924	-0.1158	6.9139
Mean	-0.0041	-0.0003	0.0693	0.2451	-0.0124	1.3617	0.1904	-0.1546	-9.7708
Median	-0.0220	-0.0010	0.0653	0.2245	-0.0674	0.8281	0.1902	-0.1582	-0.7683
					ABB1				
Min	0.1418	0.0058	0.0765	0.5376	0.0905	1.1244	0.1930	-0.1163	-41.7996
Max	3.4894	0.0676	0.3053	1.5005	1.1070	2.6236	0.2166	0.1864	109.6205
Mean	1.3556	0.0347	0.1457	0.9319	0.5944	1.7522	0.2002	0.0274	-0.5200
Median	1.2602	0.0362	0.1339	0.9478	0.6043	1.7286	0.1995	0.0355	-4.7333
P-Value JT (BB-ABB1)	0.0002*	0.0002*	1	1	0.0002*	0.0006*	0.0002*	0.0002*	0.9370
					ABB2				
Min	0.8937	0.0282	0.0747	0.7542	0.4788	1.3885	0.1953	-0.0046	-16.3114
Max	14.0002	0.1252	0.2689	1.8752	1.3301	2.6439	0.2259	0.4254	76.6387
Mean	5.1044	0.0763	0.1397	1.2217	0.9935	2.1310	0.2076	0.2245	0.8858
Median	5.0115	0.0813	0.1282	1.1992	1.0158	2.1391	0.2068	0.2476	-4.2991
P-Value JT (BB-ABB2)	0.0002*	0.0002*	1	1	0.0002*	0.0002*	0.0002*	0.0002*	0.9038

Notes: Performance statistics on the 30 Dow Jones' stocks when the analyzed period is January 1, 2000, and December 31, 2022. ABB1 is based on Hall (1983), and ABB2 on Hall & Jing (1995). $N = 6$. JT: Test Jonckheere-Terpstra.

* and ** significance at the 1% and 5% levels, respectively.
Source: own elaboration.

Table 1.23. Performance of trading strategies on Dow Jones' stocks using 10-day moving averages ('Full Sample'): Contrarian Trading Strategy

Metric	Return		Risk		Risk-Return				
	Cum Return	Annual Return	Max Drawdown	Annualized Standard Deviation	Sharpe	Omega	Tracking Error	Information Ratio	Treynor
					BB				
Min	-0.0553	-0.0025	0.0500	0.3754	-0.0638	0.9530	0.1913	-0.1635	-29.6409
Max	1.0031	0.0307	0.3140	1.0205	0.6341	2.0246	0.2017	0.0081	57.5950
Mean	0.3494	0.0122	0.1501	0.7247	0.2684	1.4208	0.1960	-0.0863	1.1531
Median	0.3595	0.0135	0.1532	0.7270	0.2721	1.4219	0.1959	-0.0792	-1.5703
					ABB1				

Min	-0.0276	-0.0012	0.0810	0.4976	-0.0345	0.9842	0.1924	-0.1570	-27.8029
Max	1.4947	0.0406	0.3715	1.1215	0.7299	2.1084	0.2040	0.0576	17.2651
Mean	0.6776	0.0216	0.1616	0.8564	0.4021	1.5411	0.1985	-0.0380	-2.9061
Median	0.6476	0.0220	0.1451	0.8956	0.4411	1.5442	0.1983	-0.0359	-3.0584
P-Value JT (BB-ABB1)	0.0008*	0.0004*	0.7298	0.999	0.0102**	0.0604	0.0014*	0.0004*	0.895
					ABB2				
Min	0.3062	0.0117	0.0693	0.6151	0.2079	1.2200	0.1929	-0.0898	-51.4973
Max	2.6485	0.0580	0.2550	1.4173	0.8689	2.3623	0.2153	0.1432	82.8284
Mean	1.3456	0.0361	0.1472	0.9849	0.5766	1.7402	0.2016	0.0335	-2.1689
Median	1.2608	0.0362	0.1398	1.0172	0.6415	1.7924	0.2009	0.0356	-3.8333
P-Value JT (BB-ABB2)	0.0002*	0.0002*	0.4974	1	0.0002*	0.0002*	0.0002*	0.0002*	0.9788

Notes: Performance statistics on the 30 Dow Jones' stocks when the analyzed period is January 1, 2000, and December 31 ,2022. ABB1 is based on Hall (1983), and ABB2 on Hall & Jing (1995). N = 10. JT: Test Jonckheere –Terpstra.
* and ** significance at the 1% and 5% levels, respectively.
Source: own elaboration.

Table 1.24. Performance of trading strategies on Dow Jones' stocks using 15-day moving averages ('Full Sample'): Contrarian Trading Strategy

	Return		**Risk**		**Risk-Return**				
Metric	**Cum Return**	**Annual Return**	**Max Drawdown**	**Annualized Standard Deviation**	**Sharpe**	**Omega**	**Tracking Error**	**Information Ratio**	**Treynor**
					BB				
Min	-0.0361	-0.0016	0.0533	0.4267	-0.0282	1.0005	0.1917	-0.1537	-60.5129
Max	1.1216	0.0333	0.3762	0.9519	0.7145	2.4032	0.2009	0.0211	32.9616
Mean	0.5166	0.0177	0.1244	0.6968	0.4069	1.6531	0.1959	-0.0586	-4.3269
Median	0.4906	0.0175	0.1136	0.7046	0.4272	1.6508	0.1960	-0.0592	-2.5649
					ABB1				
Min	0.0401	0.0017	0.0513	0.4855	0.0307	1.0677	0.1926	-0.1369	-28.3994
Max	1.2305	0.0356	0.2594	1.0163	0.6985	2.4622	0.2047	0.0325	38.4392
Mean	0.5405	0.0182	0.1288	0.7408	0.3918	1.6000	0.1967	-0.0558	-3.3719
Median	0.4833	0.0173	0.1178	0.7439	0.3796	1.5625	0.1965	-0.0600	-2.5853
P-Value JT (BB-ABB1)	0.5384	0.5294	0.7202	0.8586	0.7010	0.8384	0.1994	0.5174	0.4900
					ABB2				
Min	0.1284	0.0053	0.0564	0.4880	0.0898	1.1222	0.1926	-0.1181	-22.3317
Max	2.0573	0.0499	0.2180	1.2151	0.7838	2.5953	0.2056	0.1017	24.8897
Mean	0.8095	0.0250	0.1200	0.7959	0.4976	1.7531	0.1979	-0.0214	-3.3152
Median	0.7253	0.0240	0.1092	0.7844	0.5182	1.7006	0.1972	-0.0258	-2.5309
P-Value JT (BB-ABB2)	0.006*	0.0046*	0.5040	0.9824	0.0194**	0.1386	0.015**	0.004*	0.5806

Notes: Performance statistics on the 30 Dow Jones' stocks when the analyzed period is January 1, 200 and December 31, 2022. ABB1 is based on Hall (1983), and ABB2 on Hall & Jing (1995). *N* = 15. JT: Test Jonckheere –Terpstra.
* and ** significance at the 1% and 5% levels, respectively
Source: own elaboration.

Table 1.25. Performance of trading strategies on Dow Jones' stocks using 20-day moving averages ('Full Sample'): Contrarian Trading Strategy

	Return		Risk		Risk-Return				
Metric	**Cum Return**	**Annual Return**	**Max Drawdown**	**Annualized Standard Deviation**	**Sharpe**	**Omega**	**Tracking Error**	**Information Ratio**	**Treynor**
				BB					
Min	-0.0361	-0.0016	0.0533	0.4267	-0.0282	1.0005	0.1917	-0.1537	-60.5129
Max	1.1216	0.0333	0.3762	0.9519	0.7145	2.4032	0.2009	0.0211	32.9616
Mean	0.5166	0.0177	0.1244	0.6968	0.4069	1.6531	0.1959	-0.0586	-4.3269
Median	0.4906	0.0175	0.1136	0.7046	0.4272	1.6508	0.1960	-0.0592	-2.5649
				ABB1					
Min	-0.0062	-0.0003	0.0496	0.3895	-0.0050	1.0316	0.1921	-0.1489	-23.3571
Max	1.7169	0.0445	0.2273	1.2777	0.7515	2.7331	0.2097	0.0755	3.5693
Mean	0.4493	0.0154	0.1190	0.7037	0.3495	1.6334	0.1964	-0.0703	-3.0737
Median	0.3988	0.0147	0.1046	0.6606	0.3421	1.5441	0.1946	-0.0727	-1.7125
P-Value JT (BB-ABB1)	0.6268	0.6342	0.7656	0.5756	0.7682	0.8034	0.4336	0.6072	0.4412
				ABB2					
Min	0.0990	0.0041	0.0640	0.4075	0.0830	1.1273	0.1923	-0.1269	-16.1402
Max	1.9280	0.0479	0.2614	1.2687	0.8500	3.1009	0.2093	0.0928	52.8112
Mean	0.5820	0.0193	0.1133	0.7368	0.4208	1.7553	0.1969	-0.0504	-1.4852
Median	0.5365	0.0189	0.0968	0.7013	0.4183	1.7026	0.1957	-0.0516	-1.9811
P-Value JT (BB-ABB2)	0.0476**	0.0464**	0.4836	0.8748	0.2032	0.3098	0.1304	0.0416**	0.5260

Notes: Performance statistics on the 30 Dow Jones' stocks when the analyzed period is January 1, 2000, and December 31,2022. ABB1 is based on Hall (1983), and ABB2 on Hall & Jing (1995). N = 20. JT: Test Jonckheere -Terpstra
* and ** significance at the 1% and 5% levels, respectively.
Source: own elaboration.

The medians of Cumulative Return, Annual Return, Sharpe, Tracking Error, and Information Ratio for the ABB1 and ABB2 models are higher than the median values of Cumulative Return, Annual Return, Sharpe, Tracking Error, and Information Ratio for the classical Bollinger Bands for N=6 and N=10.

On the other hand, the median value of the Omega ratio for Edgeworth's ABB1 and ABB2 is higher than the Omega ratio for the classical Bollinger Bands for N=6.

The median value of the Omega ratio for ABB2 exceeds the Omega ratio for the classical Bollinger Bands for N=10. Similarly, the median values of Sharpe ratio and Tracking Error for ABB2 are higher than the Sharpe ratio and Tracking Error for the classical Bollinger Bands for

N=15. Finally, the median values of Cumulative Return, Annual Return, and Information Ratio for the proposed model ABB2 are higher than those for the classical Bollinger Bands for *N*=15 and 20.

These results indicate that the confidence intervals based on the Edgeworth expansions generally outperform the classical Bollinger Bands in the classical trading strategy for the thirty stocks that form the Dow Jones Industrial Average Index in the analyzed period. To confirm this outperformance, in the following section, the three models are applied to the same assets during the specific periods comprising the two recent worldwide crises, namely, the subprime and COVID-19 crises.

1.3.2.3. Further Results for contrarian strategy

As a robustness check, the performance indicators for return, risk and risk-return groups were analyzed for the subprime crisis between January 1 and December 31, 2008; and the COVID-19 crisis in a period ranged between January 1 to December 31, 2020, both in a daily frequency basis, for a total of 7,812 observations.

Tables 1.26 to 1.33 present the results on the performance measures with *N*=6, 10, 15 and 20 days.

Table 1.26. Performance of trading strategies on Dow Jones' stocks using 6-day moving averages ('Subprime crisis period'): Contrarian Trading Strategy

Metric	Return		Risk		Risk-Return				
	Cum Return	Annual Return	Max Drawdown	Annualized Standard Deviation	Sharpe	Omega	Tracking Error	Information Ratio	Treynor
					BB				
Min	-0.0651	-0.0656	0.0000	0.0000	-1.1940	0.0000	0.3778	0.8665	-37.9130
Max	0.0734	0.0740	0.0651	0.9481	1.5037	36.5328	0.3858	1.2463	128.8565
Mean	0.0094	0.0095	0.0033	0.2115	0.4488	4.4864	0.3793	1.0711	11.3582
Median	0.0004	0.0004	0.0000	0.0764	1.0040	0.0000	0.3789	1.0478	7.1366
					ABB1				
Min	-0.2441	-0.2458	0.0006	0.2968	-2.0449	0.0000	0.3708	0.3680	-33.1620
Max	0.2702	0.2726	0.2802	3.3995	2.6552	243.2241	0.4310	1.5739	409.0257

Mean	0.0151	0.0153	0.0658	1.2745	0.2810	10.0953	0.3892	1.0584	13.1348
Median	0.0119	0.0120	0.0540	1.0731	0.2521	1.2564	0.3857	1.0659	-1.9426
P-Value JT (BB-ABB1)	0.2400	0.2396	1	1	0.7886	0.0130**	0.0002*	0.5164	0.9500
					ABB2				
Min	-0.2096	-0.2111	0.0073	0.3862	-1.6138	0.3526	0.3781	0.4505	-66.3144
Max	0.4321	0.4362	0.2332	3.8420	3.0135	15.3873	0.4374	1.9046	15.0058
Mean	0.0829	0.0836	0.0680	1.6423	0.8659	2.8521	0.3969	1.2100	-7.6829
Median	0.0391	0.0395	0.0504	1.4989	0.5406	1.4836	0.3933	1.1158	-1.8607
P-Value JT (BB-ABB2)	0.0028*	0.0024*	1	1	0.1556	0.0072*	0.0002*	0.0506	0.9954

Notes: Performance statistics on the 30 Dow Jones' stocks when the analyzed period is January 1, 2008, to December 31, 2008. ABB1 is based on Hall (1983), and ABB2 on Hall & Jing (1995). *N* = 6. JT: Test Jonckheere –Terpstra.
* and ** significance at the 1% and 5% levels, respectively.
Source: own elaboration.

Table 1.27. Performance of trading strategies on Dow Jones' stocks using 10-day moving averages ('Subprime crisis period'): Contrarian Trading Strategy

Metric	Return		Risk		Risk-Return				
	Cum Return	Annual Return	Max Drawdown	Annualized Standard Deviation	Sharpe	Omega	Tracking Error	Information Ratio	Treynor
					BB				
Min	-0.1343	-0.1353	0.0163	0.2638	-2.0662	0.0000	0.3753	0.6752	-36.3049
Max	0.2745	0.2769	0.1771	2.5235	1.8374	8.9392	0.4189	1.6291	107.1943
Mean	0.0081	0.0082	0.0541	1.0931	-0.0143	1.6991	0.3858	1.0500	1.1246
Median	0.0084	0.0084	0.0425	0.8879	0.1628	1.1930	0.3829	1.0603	1.0795
					ABB1				
Min	-0.1831	-0.1845	0.0093	0.3421	-2.0823	0.0122	0.3766	0.5415	-30.5877
Max	0.3080	0.3108	0.2168	2.7302	2.5191	11.9271	0.4176	1.7049	46.0303
Mean	0.0110	0.0111	0.0694	1.3352	0.0078	2.2051	0.3907	1.0431	-0.2404
Median	0.0190	0.0192	0.0589	1.1779	0.3151	1.3543	0.3879	1.0700	-0.2932
P-Value JT (BB-ABB1)	0.5228	0.5200	0.8038	0.9776	0.4580	0.4826	0.0334*	0.5156	0.3832
					ABB2				
Min	-0.1692	-0.1704	0.0093	0.4398	-2.0187	0.0079	0.3764	0.5720	-58.2772
Max	0.2427	0.2449	0.1806	2.7289	1.7218	5.8326	0.4195	1.5542	80.2897
Mean	0.0131	0.0132	0.0785	1.5113	0.0748	1.6418	0.3931	1.0424	1.4716
Median	0.0139	0.0141	0.0653	1.4541	0.2390	1.2459	0.3906	1.0651	1.6594
P-Value JT (BB-ABB2)	0.4622	0.4532	0.9756	0.9964	0.3400	0.3962	0.0050*	0.5276	0.1562

Notes: Performance statistics on the 30 Dow Jones' stocks when the analyzed period is January 1, 2008, to December 31, 2008. ABB1 is based on Hall (1983), and ABB2 on Hall & Jing (1995). *N* =10. JT: Test Jonckheere –Terpstra.
* and ** significance at the 1% and 5% levels, respectively.
Source: own elaboration.

Table 1.28. Performance of trading strategies on Dow Jones' stocks using 15-day moving averages ('Subprime crisis period'): Contrarian Trading Strategy

N=15

	Return		Risk		Risk-Return				
Metric	**Cum Return**	**Annual Return**	**Max Drawdown**	**Annualized Standard Deviation**	**Sharpe**	**Omega**	**Tracking Error**	**Information Ratio**	**Treynor**
				BB					
Min	-0.0819	-0.0825	0.0000	0.2109	-1.5107	0.0714	0.3775	0.7818	-91.9146
Max	0.1096	0.1105	0.0861	2.2981	1.9078	12.6174	0.4103	1.3106	27.3610
Mean	0.0315	0.0317	0.0392	1.0160	0.5274	2.9031	0.3879	1.1053	-2.4598
Median	0.0363	0.0366	0.0327	0.9794	0.6077	1.9743	0.3850	1.1219	1.2555
				ABB1					
Min	-0.1092	-0.1100	0.0000	0.3432	-1.5939	0.0482	0.3741	0.7303	-251.0622
Max	0.1881	0.1897	0.2094	3.0739	1.9078	490.3076	0.4622	1.4451	25.2540
Mean	0.0245	0.0247	0.0495	1.1523	0.3848	19.9567	0.3911	1.0802	-10.3112
Median	0.0131	0.0132	0.0327	1.1362	0.2526	1.2722	0.3866	1.0649	0.7284
P-Value JT (BB-ABB1)	0.7410	0.7430	0.7476	0.8068	0.7718	0.8081	0.2666	0.7662	0.6650
				ABB2					
Min	-0.0757	-0.0763	0.0000	0.3432	-0.9454	0.4011	0.3755	0.7936	-251.0622
Max	0.1881	0.1897	0.2094	3.0759	2.2859	12.4323	0.4593	1.4945	44.1864
Mean	0.0409	0.0412	0.0507	1.2562	0.5540	2.3210	0.3924	1.1182	-17.3363
Median	0.0353	0.0356	0.0365	1.1913	0.4919	1.6094	0.3895	1.1102	-0.5004
P-Value JT (BB-ABB2)	0.4244	0.4106	0.8146	0.9738	0.5432	0.6560	0.0932	0.5012	0.8812

Notes: Performance statistics on the 30 Dow Jones' stocks when the analyzed period is January 1, 2008, to December 31, 2008. ABB1 is based on Hall (1983), and ABB2 on Hall & Jing (1995). N =15. JT: Test Jonckheere –Terpstra.
* and ** significance at the 1% and 5% levels, respectively.
Source: own elaboration.

Table 1.29. Performance of trading strategies on Dow Jones' stocks using 20-day moving averages ('Subprime crisis period'): Contrarian Trading Strategy

	Return		Risk		Risk-Return				
Metric	**Cum Return**	**Annual Return**	**Max Drawdown**	**Annualized Standard Deviation**	**Sharpe**	**Omega**	**Tracking Error**	**Information Ratio**	**Treynor**
				BB					
Min	-0.0967	-0.0974	0.0000	0.2128	-1.5422	0.0000	0.3784	0.7514	-1070.1677
Max	0.1530	0.1544	0.1440	2.6319	2.1255	190.1128	0.3985	1.3856	117.8876
Mean	0.0282	0.0285	0.0351	0.9405	0.3676	13.2915	0.3863	1.1009	-32.6263
Median	0.0118	0.0119	0.0258	0.9870	0.2810	1.3633	0.3854	1.0709	0.7183
				ABB1					
Min	-0.0967	-0.0974	0.0000	0.2128	-1.5422	0.0000	0.3784	0.7514	-108.4493
Max	0.1663	0.1678	0.1440	2.6698	1.8374	19.8098	0.4012	1.4298	422.0984

Mean	0.0292	0.0295	0.0345	0.9354	0.4173	3.1973	0.3861	1.1041	15.6041
Median	0.0120	0.0121	0.0196	0.8496	0.4002	1.5447	0.3853	1.0766	0.7910
P-Value JT (BB-ABB1)	0.4522	0.4544	0.5240	0.4774	0.4192	0.4006	0.5588	0.4334	0.5852
					ABB2				
Min	-0.0967	-0.0974	0.0000	0.2128	-1.4281	0.0000	0.3760	0.7514	-108.4493
Max	0.2279	0.2300	0.1573	2.6507	2.1186	108.8287	0.4057	1.5448	422.0984
Mean	0.0327	0.0330	0.0403	1.0335	0.3637	8.9091	0.3871	1.1095	15.9524
Median	0.0195	0.0196	0.0373	0.9838	0.4856	1.6652	0.3855	1.0897	-0.4704
P-Value JT (BB-ABB2)	0.4866	0.4810	0.7762	0.7764	0.4844	0.4306	0.3894	0.4664	0.5624

Notes: Performance statistics on the 30 Dow Jones' stocks when the analyzed period is January 1, 2008, to December 31, 2008. ABB1 is based on Hall (1983), and ABB2 on Hall & Jing (1995). *N* =20. JT: Test Jonckheere -Terpstra
* and ** significance at the 1% and 5% levels, respectively
Source: own elaboration.

Table 1.30. Performance of trading strategies on Dow Jones' stocks using 6-day moving averages ('COVID-19 crisis effect'): Contrarian Trading Strategy

	Return		**Risk**		**Risk-Return**				
Metric	**Cum Return**	**Annual Return**	**Max Drawdown**	**Annualized Standard Deviation**	**Sharpe**	**Omega**	**Tracking Error**	**Information Ratio**	**Treynor**
					BB				
Min	-0.0525	-0.0529	0.0000	0.0000	-1.2829	0.0000	0.3720	-0.0873	-118.9459
Max	0.0596	0.0601	0.0525	0.8364	1.4737	5.2735	0.3788	0.2125	90.7146
Mean	-0.0016	-0.0017	0.0094	0.2463	0.0698	0.7776	0.3733	0.0492	-7.3343
Median	0.0009	0.0009	0.0041	0.1629	0.5058	0.0000	0.3725	0.0562	-8.0321
					ABB1				
Min	-0.0382	-0.0385	0.0000	0.5515	-0.7481	0.4816	0.3691	-0.0489	-49.1916
Max	0.2912	0.2938	0.0884	3.6461	2.7611	26.0501	0.5065	0.8091	252.3750
Mean	0.0681	0.0687	0.0408	1.2187	0.8334	2.9614	0.3855	0.2275	5.2168
Median	0.0480	0.0484	0.0346	1.0206	0.7144	1.8259	0.3802	0.1814	-0.6204
P-Value JT (BB-ABB1)	0.0006*	0.0006*	1	1	0.0092*	0.0002*	0.0002*	0.0008*	0.2872
					ABB2				
Min	-0.0675	-0.0680	0.0016	0.6034	-0.7114	0.5732	0.3737	-0.1245	-160.7665
Max	0.5120	0.5170	0.0874	4.7673	2.7194	73.5054	0.5526	1.1696	2785.2781
Mean	0.1498	0.1512	0.0456	1.8481	1.0593	5.8410	0.4049	0.4021	93.7282
Median	0.0974	0.0982	0.0442	1.3711	1.0505	2.4252	0.3889	0.3096	-1.2412
P-Value JT (BB-ABB2)	0.0002*	0.0002*	1	1	0.0006*	0.0002*	0.0002*	0.0002*	0.1460

Notes: Performance statistics on the 30 Dow Jones' stocks when the analyzed period is January 1, 2020, to December 31, 2020. ABB1 is based on Hall (1983), and ABB2 on Hall & Jing (1995). *N* = 6. JT: Test Jonckheere –Terpstra.
* and ** significance at the 1% and 5% levels, respectively.
Source: own elaboration.

Table 1.31. Performance of trading strategies on Dow Jones' stocks using 10-day moving averages ('COVID-19 crisis effect'): Contrarian Trading Strategy

	Return		Risk		Risk-Return				
Metric	**Cum Return**	**Annual Return**	**Max Drawdown**	**Annualized Standard Deviation**	**Sharpe**	**Omega**	**Tracking Error**	**Information Ratio**	**Treynor**
				BB					
Min	-0.1257	-0.1266	0.0046	0.2143	-1.5905	0.0000	0.3719	-0.2508	-53.1746
Max	0.2003	0.2021	0.1512	2.0315	1.7204	8.9506	0.4267	0.5662	17.4432
Mean	0.0026	0.0027	0.0485	0.9750	-0.0004	1.7159	0.3825	0.0589	-4.5172
Median	0.0050	0.0050	0.0411	0.9570	0.1295	1.1587	0.3762	0.0667	-2.4180
				ABB1					
Min	-0.0991	-0.0999	0.0046	0.2143	-1.3342	0.2759	0.3717	-0.2063	-107.7104
Max	0.1335	0.1347	0.1794	3.1556	1.4927	4.5134	0.4267	0.3999	27.8102
Mean	0.0029	0.0029	0.0565	1.1282	0.0607	1.3644	0.3840	0.0589	-2.6673
Median	0.0024	0.0024	0.0456	1.0736	0.0875	1.0906	0.3789	0.0599	-0.6702
P-Value JT (BB-ABB1)	0.4254	0.4128	0.7662	0.7740	0.4004	0.3988	0.2648	0.4178	0.1292
				ABB2					
Min	-0.1419	-0.1429	0.0073	0.3005	-1.1951	0.2683	0.3673	-0.2778	-16.0599
Max	0.5278	0.5330	0.1902	3.8947	2.1723	9.1458	0.4877	1.1337	409.6664
Mean	0.0648	0.0654	0.0520	1.4949	0.5913	2.5759	0.3954	0.2064	16.0846
Median	0.0366	0.0369	0.0428	1.4267	0.6228	1.8787	0.3883	0.1500	-0.7093
P-Value JT (BB-ABB2)	0.0072*	0.0084*	0.7512	0.9978	0.0184**	0.0238**	0.0098*	0.0060*	0.0568

Notes: Performance statistics on the 30 Dow Jones' stocks when the analyzed period is January 1, 2020, to December 31, 2020. ABB1 is based on Hall (1983), and ABB2 on Hall & Jing (1995). $N = 10$. JT: Test Jonckheere –Terpstra.
* and ** significance at the 1% and 5% levels, respectively.
Source: own elaboration.

Table 1.32. Performance of trading strategies on Dow Jones' stocks using 15-day moving averages ('COVID-19 crisis effect'): Contrarian Trading Strategy

	Return		Risk		Risk-Return				
Metric	**Cum Return**	**Annual Return**	**Max Drawdown**	**Annualized Standard Deviation**	**Sharpe**	**Omega**	**Tracking Error**	**Information Ratio**	**Treynor**
				BB					
Min	-0.1102	-0.1111	0.0000	0.1853	-1.6326	0.1437	0.3718	-0.2139	-45.6109
Max	0.0829	0.0836	0.1445	2.0189	2.1017	5.6862	0.4258	0.2750	128.8408
Mean	0.0000	0.0000	0.0346	0.6920	0.1132	1.4201	0.3776	0.0539	1.9155
Median	0.0040	0.0041	0.0296	0.6285	0.1263	1.1462	0.3739	0.0641	-0.7009
				ABB1					
Min	-0.1322	-0.1332	0.0000	0.3287	-1.6326	0.1437	0.3714	-0.2658	-9502.4682
Max	0.1075	0.1084	0.1484	2.0212	2.1502	6.8754	0.4257	0.3202	56.1200
Mean	0.0020	0.0020	0.0398	0.7941	0.1802	1.7293	0.3782	0.0590	-322.5991

Median	0.0075	0.0076	0.0324	0.6853	0.1802	1.1536	0.3745	0.0736	-1.7388
P-Value JT (BB-ABB1)	0.3704	0.3904	0.8102	0.9248	0.4076	0.4286	0.2138	0.3678	0.7110
					ABB2				
Min	-0.1717	-0.1730	0.0000	0.2622	-1.6326	0.1437	0.3724	-0.3593	-9502.4682
Max	0.1552	0.1565	0.1748	2.0906	2.1502	20.4306	0.4257	0.4236	91.4900
Mean	0.0121	0.0122	0.0451	0.9404	0.2839	3.1104	0.3807	0.0846	-311.1707
Median	0.0102	0.0103	0.0405	0.8635	0.2167	1.2158	0.3760	0.0812	-2.6969
P-Value JT (BB-ABB2)	0.2398	0.2446	0.8816	0.9984	0.2632	0.3474	0.0134*	0.2454	0.5258

Notes: Performance statistics on the 30 Dow Jones' stocks when the analyzed period is January 1, 2020, to December 31, 2020. ABB1 is based on Hall (1983), and ABB2 on Hall & Jing (1995). *N* = 15. JT: Test Jonckheere –Terpstra.
* and ** significance at the 1% and 5% levels, respectively.
Source: own elaboration.

Table 1.33. Performance of trading strategies on Dow Jones' stocks using 20-day moving averages ('COVID-19 crisis effect'): Contrarian Trading Strategy

	Return		**Risk**		**Risk-Return**				
Metric	**Cum Return**	**Annual Return**	**Max Drawdown**	**Annualized Standard Deviation**	**Sharpe**	**Omega**	**Tracking Error**	**Information Ratio**	**Treynor**
					BB				
Min	-0.1102	-0.1111	0.0000	0.1853	-1.6326	0.1437	0.3718	-0.2139	-45.6109
Max	0.0829	0.0836	0.1445	2.0189	2.1017	5.6862	0.4258	0.2750	128.8408
Mean	0.0000	0.0000	0.0346	0.6920	0.1132	1.4201	0.3776	0.0539	1.9155
Median	0.0040	0.0041	0.0296	0.6285	0.1263	1.1462	0.3739	0.0641	-0.7009
					ABB1				
Min	-0.1033	-0.1041	0.0022	0.1995	-1.8786	0.0000	0.3714	-0.2229	-145.7650
Max	0.1108	0.1117	0.1179	2.3151	2.3317	30.9674	0.4463	0.2951	95.2499
Mean	-0.0081	-0.0082	0.0483	0.8796	-0.1963	2.2778	0.3817	0.0284	-4.1171
Median	-0.0103	-0.0104	0.0354	0.7343	-0.3110	0.7606	0.3760	0.0259	-1.2730
P-Value JT (BB-ABB1)	0.6504	0.6466	0.7532	0.6964	0.6652	0.6360	0.3634	0.6512	0.2700
					ABB2				
Min	-0.1001	-0.1009	0.0022	0.3026	-1.7665	0.0000	0.3714	-0.2139	-1872.8702
Max	0.1455	0.1467	0.1068	2.5212	2.3317	30.9674	0.4463	0.4282	95.2499
Mean	0.0072	0.0073	0.0453	0.9866	-0.0214	2.7578	0.3841	0.0673	-70.2302
Median	-0.0152	-0.0153	0.0414	0.9396	-0.2668	0.7717	0.3769	0.0127	-1.3778
P-Value JT (BB-ABB2)	0.4132	0.3980	0.7576	0.8764	0.4296	0.3808	0.0958	0.3908	0.3948

Notes: Performance statistics on the 30 Dow Jones' stocks when the analyzed period is January 1, 2020, to December 31, 2020. ABB1 is based on Hall (1983), and ABB2 on Hall & Jing (1995). *N* = 20. JT: Test Jonckheere –Terpstra.
* and ** significance at the 1% and 5% levels, respectively
Source: own elaboration.

For the 2008 and 2020 crises, it is observed that the median Cumulative Return and Annual Return for ABB2 are higher than the median Cumulative Return and Annual Return for the classical Bollinger Bands when *N*=6. Moreover, in the same periods of crisis (2008 and 2020), the median Tracking Error and Omega ratio for both the new techniques (ABB1 and ABB2 models) are higher than the median Tracking Error for the classical Bollinger Bands when *N*=6.

In the 2008 crisis the medians of Tracking Error for ABB1 and ABB2 are higher than the median Tracking Error for the classical Bollinger Bands when *N*=10.

For the 2020 crises, the median Cumulative Return and Annual Return for ABB1 are higher than the median Cumulative Return and Annual Return for the classical Bollinger Bands when *N*=6. Furthermore, the medians of Information Ratio for ABB1 and ABB2 are higher than the median Information Ratio for the classical Bollinger Bands when *N*=6.

In the 2020 crisis, ABB2 demonstrates higher medians across Cumulative Return, Annual Return, Sharpe ratio, Omega ratio, Tracking Error, and Information Ratio compared to the classical Bollinger Bands when *N*=10. Moreover, the median Tracking Error for ABB2 surpasses that of classical Bollinger Bands when N=15 during the same crisis period.

The findings for both crisis periods and full sample are in line with the simulation results, where the adjusted bands perform better for shorter periods of *N*, for the reasons explained in the Simulations section (1.3.2.1).

1.4. Conclusions

In this chapter we compared three approaches for investment analysis in financial markets, the Bollinger Bands and the adjusted bands using the Edgeworth expansions methodologies (one proposed by Hall, 1983 and the other one by Hall and Jing, 1995) applied to the trend following trading strategy and contrarian trading strategy for the thirty stocks that form the Dow Jones Industrial Average Index. The performance of these methodologies is assessed by three groups of indicators, including return, risk, and risk-return measures in a period of 20 years. Moreover, the performance of the models is tested in two crisis periods to verify the results obtained for the complete sample.

Overall, the results show for trend following strategy that confidence intervals with Edgeworth expansions provide better median performance for indicators such as VaR (when N=20, 50, and 200 days), the median Tracking Error (for N=20 and 50 days), the Treynor Ratio median (with N=200 days) than the traditional Bollinger Bands approach in technical analysis for the whole sample period.

These results are also in line with the crisis periods studied in this chapter. Though the causes of the subprime crisis and COVID-19 pandemic are different, the effects of both crises result in higher volatility levels than the full period (i.e., the 20-year period) studied in our work. In these high uncertainty scenarios, the median Tracking Error with our proposal (for N=20 and 50 days), Information Ratio and Treynor Ratio (for N=50 and 200 days), and the median Max Drawdown (N=50 days) are better off.

However, we can see that the cumulative and annual returns of the trend-following strategy are negative in both calm and crisis periods. This leads us to conclude that this trading strategy is not attractive for the investor from the point of view of profitability.

In the same manner, the Bollinger Bands (BB) and the adjusted bands using Edgeworth expansions methodologies (Hall, 1983; Hall & Jing, 1995), named as ABB1 and ABB2, applied to the contrarian trading strategy for the thirty stocks of the Dow Jones Industrial Average Index. Again, the performance of these methodologies is assessed by three groups of indicators, including return, risk, and risk-return measures over a 20-year period. Moreover, the models' performance is tested in two crisis periods to verify the results obtained for the complete sample. Overall, the results show that confidence intervals with Edgeworth expansions for ABB1 (Hall, 1983) and ABB2 (Hall & Jing,1995) using the contrarian trading strategy provide better median performance for indicators such as Cumulative Return, Annual return, Sharpe Ratio, Tracking error, Omega, and Information ratio (when *N*=6 and 10 days) than traditional Bollinger Bands (BB) for the entire sample period. We support the statistical overperformance of our proposed strategy based on ABB1 and ABB2 over the BB using the Jonckheere -Terpstra Test.

These results are also in line with the crisis periods studied in this chapter Although the causes of the subprime crisis and COVID-19 pandemic are different, the effects of both crises result in higher volatility levels than the full period (i.e., 20 years) studied in our work. In these high uncertainty scenarios, the median Cumulative return, Annual return, Omega, and Tracking Error with our proposal (for *N*=6 and 10 days) perform better.

One contribution of this chapter is to incorporate confidence intervals that integrate skewness and kurtosis in the proposed trend following and contrarian trading strategy.

As the periods become shorter, as is the case with the contrarian trading strategy, both cumulative and annual returns are positive and perform better than those of the classical Bollinger bands at the median, as well as the risk and risk-return indicators. This indicates that, during shorter periods, these proposals can capture the nuances of excess kurtosis and asymmetry with greater efficacy.

1.5. Appendix

1.5.1. Appendix A

Trade return (tr) is calculated using the return of the stock price and the previous day signal.

Annual Return: is the profit or loss on an investment over a one-year period.

$$\text{Annual Return} = [(1 + tr_1) * (1 + tr_2) * ... * (1 + tr_{n-1}) * (1 + tr_n)]^{250/T} - 1 \qquad (1.A.1)$$

Cumulative return: is the aggregate amount that the investment has gained or lost over time, independent of the amount of time involved.

$$\text{Cumulative Return} = (1 + tr_1) * (1 + tr_2) \ldots (1 + tr_{n-1}) * (1 + tr_n) \qquad (1.A.2)$$

$$\text{Annualized Standard Deviation} \quad \sigma^A = \sqrt{252} * \sigma \qquad (1.A.3)$$

Maximum Drawdown is an indicator of downside risk over a specified period. It is the maximum observed loss from a peak to a trough of a portfolio before a new peak is attained.

$$\text{Maximum Drawdown} = \frac{(Trough\ Value - Peak\ Value)}{Peak\ Value} \qquad (1.A.4)$$

VaR is defined as the worst expected loss over a given time horizon at a given confidence level under normal market conditions:

$$\text{Value at Risk} \qquad VaR_\alpha(X) = F_Y^{-1}(1 - \alpha) \qquad (1.A.5)$$

Sharpe Ratio is defined as the difference between the returns of the investment and the risk-free return, divided by the standard deviation of the investment returns. It represents the additional amount of return that an investor receives per unit of increase in risk:

$$\text{Sharpe Ratio} = \frac{R_{TR} - R_f}{\sigma_{TR}} \qquad (1.A.6)$$

Tracking-Error is the divergence between the price behavior of a position or a portfolio and the price behavior of a benchmark. It is reported as a standard deviation percentage difference, which measures the difference between the return an investor receives and that of the benchmark they were attempting to imitate.

$$\text{Tracking Error} = \sigma_{TR} \sqrt{(1 - \rho_{TR,RETURN\ DOW\ JONES})^2} \qquad (1.A.7)$$

Information Ratio measures and compares the active return of an investment compared to a benchmark index relative to the volatility of the active return (also known as active risk or benchmark tracking risk).

$$\text{Information Ratio} = \frac{Annualized\ excess\ return}{Annualized\ tracking\ error} \qquad (1.A.8)$$

Treynor Ratio is a measurement of the returns earned in excess of that which could have been earned on an investment that has no diversifiable risk, per unit of market risk assumed.

$$\text{Treynor Ratio} = \frac{R_{TR} - R_f}{\beta_{TR}} \tag{1.A.9}$$

Omega is an indicator that measures the likelihood of achieving a target return in comparison to the potential downside risk. It also considers the third and fourth momentum effect, i.e., skewness & Kurtosis, which gives this an immense usefulness compared to other indicators.

$$\Omega(\theta) = \frac{\int_{\theta}^{\infty}[1 - F(r)]\, dr}{\int_{-\infty}^{\theta} F(r) dr} \tag{1.A.10}$$

CHAPTER 2: Technical Note: Modified variance incorporating high-order moments in risk measure with Gram-Charlier returns[1]

2.1. Introduction

In the finance literature, the optimal portfolio choice has been mainly studied under Markowitz' mean-variance scheme and its variations, thus measuring risk in terms of returns variance. As noted by Sarmas et al. (2020), new models based on the mean-variance methodology can be classified as: single index models, multi-index models, average correlation models, mixed models, and utility models. The single index model was pioneered by Sharpe (1963) and its advantages are: feasibility in calculating the optimal portfolio (Chauhan, 2014; Nalini, 2014), and less conservative capital charges (McAleer and Da Veiga, 2008). However, the main limitation of this type of model is that the risk factor only depends on the market, and other variables may affect the fluctuation of stock prices. On the other hand, in the multi-index models (Cohen and Pogue, 1967; Ross, 1978; Ingersoll, 1987), the return of each asset in the portfolio depends (in a linear way) of some benchmark or index and its main advantage rely "in identifying a more accurate set of efficient portfolios" (Aber, 1976). The drawback in this type of model is when the indexes are correlated, which affects the level of risk of the portfolio. According to Elton and Gruber (1973), Elton et al. (1978), and Aneja et al. (1989) a better future correlation matrix estimation is obtained with the average correlation models than the single-index and multi-index models. The initial idea with the average correlation models is to estimate the future correlation through the historical average correlation coefficient. Despite these good empirical results, the assumptions for these correlation models are not desirable from the theoretical point of view (Kempf et al., 2015). Mixed or combination of models may

[1] A version of this chapter was published as a technical note in The Engineering Economist Journal in 2022 (Volume 67, Issue 3, pp. 218-233).

also work well, for instance, combination of single-index model with historical correlation (Jagannathan and Ma, 2003), constant and historical correlation (Ledoit and Wolf, 2003). Finally, the same results of the mean – variance optimization can be obtained as an expected utility optimization. The advantage of utility models is that different investor preferences can be modelled and then, obtain different optimal portfolios. Though previous works argue that mean-variance portfolios are similar to portfolios from different utility functions, the constant relative risk aversion (CRRA) preferences seem to be the most preferred model among economists (Kassimatis, 2021).

Our work can be classified in the utility models since we depart from an exponential utility function and incorporate higher moments in the return distribution through the Gram-Charlier density. The analysis of moments higher than the variance has gained importance in the finance literature about portfolio selection since the publication of one of the most relevant works in this area, which is Scott and Horvath (1980). Therefore, our study contributes to the portfolio selection field by proposing a risk measure which considers higher moments than variance under the utility models framework. In this regard, Bell (1995) develops measures of risk compatible with families of utility functions, among these are the exponential utility functions. In addition, investor's risk attitudes seem to affect portfolio decisions under uncertainty (Feldstein, 1969) and therefore the skewness and kurtosis of the portfolio return distribution (linked to investor's 'prudence' or 'temperance' characteristics) should be incorporated in the risk measures. In this chapter, based on the work of Bell (1995), three methodologies to measure portfolio risk according to the investor's risk tolerance are discussed: the so-called 'behavioral' variance and modified variance with both Taylor and a Gram-Charlier expansions. These methodologies being straightforward procedures to incorporate the third and fourth moments of the return distribution in the portfolio analysis, recently studied in the literature –e.g., Ñíguez, Paya, Peel & Perote (2019) or León & Ñíguez (2020).

According to Nawrocki & Viole (2014) risk-averse investors would prefer less excess kurtosis and positive skew than risk-lover investors. That is, a few extreme events in the right tail of the asset return distribution represent profits to the investor (Rubinstein, 2006). In addition, behavioral finance studies confirm that agents, which have cumulative prospect theory preferences, select portfolios with positive skewed distributions of returns (e.g., Zhang, 2005; Barberis and Huang, 2008), and dislike kurtosis (Ågren, 2006). In this line, Davies & de Servigny (2012) attempt to incorporate such features in a more general risk measure, named behavioral variance. To the best of our knowledge, this behavioral risk measure has been only applied by Davies and Lim (2014) to obtain a desirability measure, which is defined as the difference between the expected excess return of the portfolio over the risk-free rate and the minimum compensation, Chereau (2014) in the context of supply chain management, and Rodríguez et al. (2021) utilize the behavioral variance (as risk measure) as one of the fuzzy goals to construct a diversified fuzzy-behavioral portfolio. In fact, fuzzy theory has been useful to model investor preferences (aka aspiration levels) in portfolio management framework, see e.g. Rodríguez et al. (2021) and the references therein. In this line, in the recent work of Yang et al. (2021), the authors examine a multi-period portfolio optimization problem considering the risk attitude of an investor in fuzzy environment.

Another strand of literature is focused on multi-criteria credibilistic framework. Based on the works of Jalota et al. (2017a, 2017b), there has been recent innovations in fuzzy portfolio selection context and credibility theory such as the salient work of García et al. (2019), who apply a multiobjective credibilistic model that besides risk and return, also considers the price-to-earnings ratio to measure portfolio performance. Moreover, García et al. (2020a, 2020b) incorporate liquidity to the usual measures like risk and return to quantify the performance of a portfolio under a fuzzy portfolio selection model.

Alternatively, in this chapter we propose a modified variance measure based on the consideration of a Gram-Charlier expansion on asset return density and a modified variance measure that have been derived through Taylor expansions of the risk measures based on investors' exponential utility. We believe that our proposal is a more natural way to incorporate behavioral aspects related to the skewness and excess kurtosis of the return distribution in the risk measure. Our case study shows that Monte Carlo simulations support this latter risk measure according to minimum variance and Sharpe's ratio criteria.

The chapter is divided as follows. Section 2.2 describes the three variance measures proposed to quantify the risk. Section 2.3 presents the results on random portfolio optimization comparing their relative performance. Finally, Section 2.4 concludes.

2.2. Model

Based on the work of Bell (1995), in this section the three measures of risk under an exponential utility function will be presented: behavioral variance and both modified variance applying Taylor's expansion and Gram-Charlier returns. Let *r* be the returns and *R* the risk measure, then Bell (1995) proposes a risk-return function of the form $f(r, R)$ and considers solutions to the equation:

$$E[U(r)] = f(r, R). \tag{2.1}$$

By considering an exponential utility function, the Expected Utility (EU) is as follows:

$$E[U(r)] = E\left[1 - e^{-\frac{2r}{T}}\right]$$
$$= 1 - E\left[e^{-\frac{2r}{T}}\right], \tag{2.2}$$

where T is the risk tolerance parameter (which is equal to 2/*c*, and *c* is the risk aversion parameter). Adding and subtracting by the expected return $\bar{r}$, and splitting the exponentials

$$E[U(r)] = 1 - E\left[e^{-\frac{2(r+\bar{r}-\bar{r})}{T}}\right]$$

$$= 1 - e^{-\frac{2\bar{r}}{T}} E\left[e^{\frac{2(\bar{r}-r)}{T}}\right]. \quad (2.3)$$

As stated in Bell (1995), a convenient choice of the risk measure *R* is:

$$e^{\frac{2R}{T}} = E\left[e^{\frac{2(\bar{r}-r)}{T}}\right]. \quad (2.4)$$

From equation (2.4), the three risk measures are derived and presented next.

2.2.1. Model 1: Behavioral variance

Davies & de Servigny (2012) propose 'behavioral variance' (σ_B^2) as an extension of return variance (σ_r^2) that admits skewness (sk) and excess kurtosis (ek) as relevant features reflecting investor's attitudes towards risk. Their proposal is stated in equation (2.5) –see Appendix A in Section 2.6.1 for the derivation of the behavioral variance expression, since the authors present a sketch of the proof.

$$\sigma_B^2 \approx \sigma_r^2\left[1 - \frac{2\sigma_r sk}{3T} + \frac{\sigma_r^2 ek}{3T^2}\right], \quad (2.5)$$

where *T* is the risk tolerance parameter of the investor, which moderates the impact of skewness and excess kurtosis, e.g., the impact of higher moments is smaller for investors with higher risk tolerance. Furthermore, when the returns are assumed to be normally distributed (sk = 0 and ek = 0), the behavioral variance collapses to the variance of the asset returns.

2.2.2. Model 2. Modified variance with Taylor's expansion

The effect of sk and ek on the risk measure may be incorporated by means of a Taylor's expansion, which, for an exponential utility function, yields to the following modified variance (σ_M^2) –see Appendix B in Section 2.6.2. for the proof:

$$\sigma_M^2 \approx \frac{T}{2}\ln\left(1 + \frac{2\sigma_r^2}{T^2}\left[1 - \frac{2\sigma_r sk}{3T} + \frac{\sigma_r^2 ek}{3T^2}\right]\right). \quad (2.6)$$

In this model, when sk = 0 and ek = 0, the modified variance is a function of the return's variance and the tolerance risk parameter.

2.2.3. Model 3. Modified variance with Gram-Charlier returns

In addition to the previous model, we present a new proposal of the modified variance when the risk measure is modelled as in equation (2.4) and the asset returns follow a Gram-Charlier distribution, which is an expansion of a standard normal distribution that has been used in varied financial applications to account for the impact of skewness, kurtosis and even higher order moments –e.g., Mauleón & Perote (2000), León et al., (2009), Del Brio, Mora-Valencia & Perote (2020), Jiménez, I., Mora-Valencia, A., Perote, J. (2022), among others. Following Kendall's (1943) notation, a random variable x is Gram-Charlier distributed if:

$$f(x) = \sum_{j=0}^{\infty} c_j H_j(x)\phi(x), \tag{2.7}$$

where $H_j(x)$ are the Chebyshev-Hermite polynomials defined as

$$(-D)^j \phi(x) = H_j(x)\phi(x), \tag{2.8}$$

where D is the differential operator, $\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}x^2\right\}$, $H_0(x) = 1$, and the first four polynomials are $H_1(x) = x,\ H_2(x) = x^2 - 1, H_3(x) = x^3 - x,$ and $H_4(x) = x^4 - 6x^2 + 3$. In addition,

$$c_j = \frac{1}{j!}\int_{-\infty}^{\infty} f(x)\, H_j(x)dx, \tag{2.9}$$

in particular, these parameters are linked to the Gaussian distribution central moments (μ_i), e.g. $c_0 = 1, c_1 = 0,\ c_2 = \frac{1}{2}(\mu_2 - 1),\ c_3 = \frac{1}{6}\mu_3,\ c_4 = \frac{1}{24}(\mu_4 - 6\mu_3 + 3)$.

In financial applications, the density is usually truncated up to the fourth term:

$$f(x) = \phi(x)[1 + c_3 H_3(x) + c_4 H_4(x)], \tag{2.10}$$

where $\phi(x)$ is the standard normal density, c_3 is related to skewness ($c_3 = sk/6$), and c_4 to excess kurtosis ($c_4 = ek/24$).

By following the same idea of the previous modified variance model, we obtain the modified variance when the returns are Gram-Charlier distributed (σ_{MGC}^2) –see Appendix C in Section 6.2.3:

$$\sigma_{MGC}^2 = \frac{\sigma_r^2}{T} + \frac{T}{2}\ln\{1 + d_2\sigma_X^2 + d_3\sigma_X^3 + d_4\sigma_X^4\}, \tag{2.11}$$

where:

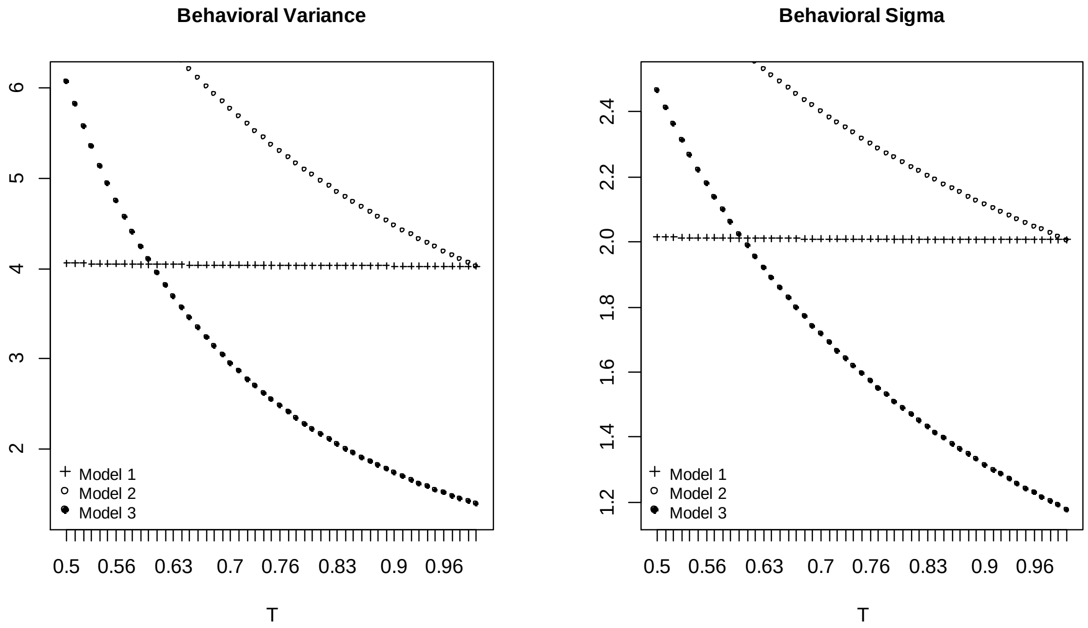
$$\begin{aligned} d_2 &= \frac{1}{2}(\sigma_X^2 - 1), & \sigma_X^2 &= \frac{4\sigma_r^2}{T^2}, \\ d_3 &= \frac{sk_x}{6}, & sk_x &= \frac{\bar{r}^3}{\sigma_r^3} + \frac{3\bar{r}}{\sigma_r} - sk_r, \\ d_4 &= \frac{ek_x}{24}, & ek_x &= \frac{3\bar{r}^4}{\sigma_r^4} + \frac{6\bar{r}^2}{\sigma_r^2} - \frac{4s_r}{\sigma_r} + ek_r, \end{aligned} \tag{2.12}$$

and $\bar{r}$, σ_r, sk_r, and ek_r is the mean, standard deviation, skewness, and excess kurtosis of the returns, respectively.

Besides the nice properties and its broad application in finance, we argue that considering the impact of skewness and kurtosis on the return distribution through Gram-Charlier series, and thus incorporating these moments on the risk measure, is a more consistent and appropriate manner to measure risk than assuming normally distributed returns and then applying a Taylor expansion to 'ad hoc' incorporating the effect of the third and fourth moments.

As a matter of fact, salient differences derive from the three models. Figure 2.1 illustrates how risk tolerance parameter (*T*) affects the three risk measures. As observed, behavioral variance (Model 1) is not very sensitive with variations of risk tolerance parameter, as the other two models.

Figure 2.1. Modified variance models according to different risk tolerance (*T*)



Model 1 represents the behavioral risk, Model 2 is the modified risk employing Taylor's expansion, and Model 3 stands for the modified risk when returns are Gram-Charlier distributed.

2.3. Case study A

This section analyzes the performance of the models for four stocks: Apple (AAPL), American International Group (AIG), Mosaic Company (MOS), and Marathon Oil Corporation (MRO). The selected assets are part of the "financial, exploration and production, agricultural chemicals, communications equipment" sectors according to the Bloomberg Industry Classification Standard (BICS). In addition, the beta coefficient of the assets is 1.1 higher than the S&P500 index for the analyzed period. Since stocks exhibit heavy tails, it is possible to obtain corner solutions – that is, a solution that may be concentrated in one single asset – in the classical portfolio selection due to the fatter tails of some stocks (see e.g., Hyung and de Vries, 2007, and the references therein). Thus, a methodology that incorporates the third and fourth moment of the return distribution may mitigate this type of issue. The analyzed stocks in our work, in particular, present heavier tails than traditional stock indices and we compare the performance of the three methodologies described in Section 2.2.

The sample period ranged between May 30, 2000, and November 30, 2020, on a daily frequency basis, for a total of 20,636 observations. Table 2.1 shows the descriptive statistics of the returns, which have been calculated as $r_t = 100\ln(P_t/P_{t-1})$, where P_t is the price of the asset at time *t*.

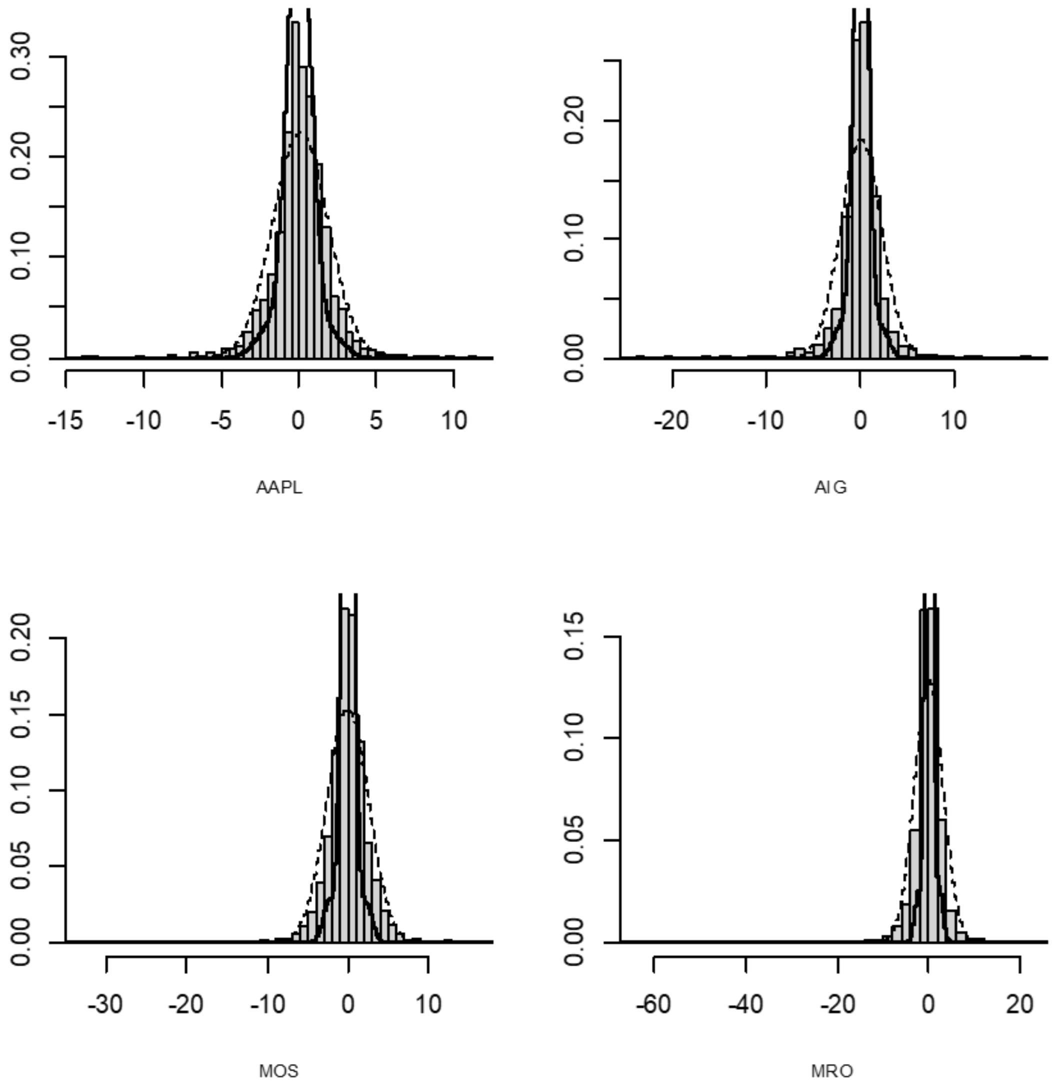
Table 2.1. Descriptive Statistics for the analyzed stocks

	AAPL	AIG	MOS	MRO
Mean	0.099	- 0.062	0.011	0.005
Std deviation	2.606	3.799	3.045	2.778
Skewness	-4.471	-3.254	-1.566	-2.211
Excess Kurtosis	123.867	111.732	25.170	57.492

Apple (AAPL), American International Group (AIG), Mosaic Company (MOS), and Marathon Oil Corporation (MRO). The analyzed period ranged between May 30, 2000, and November 30, 2020, observed on a daily basis (20,636 observations).

As observed, AIG is the only stock with negative daily average returns and it is also the riskiest asset, with the highest standard deviation. On the other hand, AAPL is the asset with the highest average daily return, and it presents the lowest standard deviation. Moreover, the returns of the four stocks exhibit relatively high negative values of skewness and fat tails. All asset returns present leptokurtic behavior; however, APPL and AIG, exhibit higher excess kurtosis than MRO and MOS. All these features support the non-Gaussianity property of daily return distributions and the use of Gram-Charlier type of densities for fitting non-Gaussian data, as proposed in the early work of Samuelson (1943). In the last two decades, there are many applications to financial assets (Jondeau and Rockinger, 2001), portfolio of stock indices (Del Brio et al., 2009), option pricing (Schlögl, 2013), and, more recently, for a regime-switching Lévy process (Asmussen and Bladt, 2021), and financial risk applications (León and Ñíguez, 2021; Molina-Muñoz et al., 2021). Figure 2.2 presents the fitting of the normal and GC densities to the analyzed stock returns in the present study.

Figure 2.2. Comparison of Gram-Charlier and Normal distributions to the analyzed data



The histogram represents the empirical distribution of the analyzed returns, whereas the dotted line depicts the density of the normal distribution and the solid line the GC density. The estimated parameters for the GC density are $d3 = -0.0087$, $d4 = 0.0667$ (AAPL), $d3 = -0.0031$, $d4 = 0.0795$ (AIG), $d3 = -0.0080$, $d4 = 0.0934$ (MOS), and $d3 = 0.0054$, $d4 = 0.0976$ (MRO).

As observed in Figure 2.2, the GC distribution performs well for the fitting of the stock returns, particularly the fat tails of the empirical distributions.

Furthermore, all the assets pairwise-correlations are below 50%, as presented in Table 2.2. The most correlated stock returns are MRO and MOS, while the pair with the lowest correlation are AIG and AAPL, which means that portfolios formed on these assets seem to be diversified.

Table 2.2. Correlation matrix of returns for the analyzed stocks

Stock	AAPL	AIG	MOS	MRO
AAPL	1.000	0.223	0.275	0.249
AIG	0.223	1.000	0.258	0.296
MOS	0.275	0.258	1.000	0.457
MRO	0.249	0.296	0.457	1.000

Apple (AAPL), American International Group (AIG), Mosaic Company (MOS), and Marathon Oil Corporation (MRO). The analyzed period ranged between May 30, 2000, and November 30, 2020, observed on a daily basis (20,636 observations).

Table 2.3 provides the comparison of the standard deviation of returns with the three risk measures obtained with the methodologies described in Section 2.2. We select different risk tolerance parameter values as in Fabozzi, Kolm, Pachamanova & Focardi (2007). The authors argue that risk aversion parameter (c) is between 2 and 4. That is, the risk tolerance parameter (T) values are between 0.5 and 1, provided that $T = 2/c$, as previously stated. The behavioral risk (Model 1) and modified risk (Model 2) values are higher than standard deviation for all cases. On the contrary, modified risk of Model 3 is only higher than the volatility of the assets in two cases (AAPL and AIG) when $T = 0.5$. As expected, the values of the three risk measures decrease when the risk tolerance parameter increases for each model. As particular case, at certain level of risk tolerance parameter, AIG exhibits the highest risk values. This can be explained by the combination of high values of standard deviation and excess kurtosis for this stock. On the other hand, MOS presents the lowest risk at a certain level of T for Model 3, mainly explained by the lowest excess kurtosis value and lower standard deviation compared with AIG. However, AAPL is the asset with the lowest risk figures for Model 2, and mixed results are presented for Model 1.

Table 2.3. Standard deviation versus different risk measures

Stock	Standard deviation of returns	Behavioral standard deviation (Model 1)			Modified standard deviation (Model 2)			Modified standard deviation (Model 3)		
		T=0.5	T=0.75	T=1.0	T=0.5	T=0.75	T=1.0	T=0.5	T=0.75	T=1.0
AAPL	2.606	3.062	2.859	2.775	4.143	3.230	2.740	3.505	1.989	1.343
AIG	3.799	4.798	4.333	4.144	6.286	4.809	4.046	5.532	3.122	2.098
MOS	3.045	3.231	3.150	3.117	4.497	3.611	3.103	2.595	1.471	0.993
MRO	2.778	3.044	2.924	2.875	4.190	3.333	2.854	2.724	1.543	1.041

Model 1 represents the behavioral risk, Model 2 is the modified risk employing Taylor's expansion, and Model 3 stands for the modified risk when returns are Gram-Charlier distributed. T represents the risk tolerance parameter. The analyzed period ranged between May 30, 2000, and November 30, 2020, observed on a daily basis (20,636 observations).

To compare the performance of the three models in this chapter, we employ random portfolio optimization, which is a hybrid of mean-variance portfolio analysis and Monte Carlo simulation. To this end, we employed R software and PortfolioAnalytics package (Peterson et al., 2018). A recent application of the hybrid Monte Carlo simulation and Markowitz model on stocks of the Shanghai stock is performed by Shadabfar and Cheng (2020). Random sample portfolios are also employed by Mehlawat et al. (2020), and the authors show how to obtain efficient portfolios rebalancing the inefficient random sample portfolios. Other works that have also employed random portfolios are Meade et al. (2021), Gempesaw et al. (2021), Mehralizade et al. (2021), López-García et al. (2021), Bollerslev et al. (2021), among others.

First, we analyze the portfolio risk minimization problem, that is min σ_p^2, subject to $\sum_{i=1}^{n} w_i = 1, w_i \geq 0$ where w_i is the asset *i* weight. The portfolio variance is given by $\sigma_p^2 = \boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}$, where $\boldsymbol{w}$ is the (column) weight vector and $\boldsymbol{\Sigma}$ is the asset variance-covariance matrix, where the diagonal of this matrix is replaced by the risk measure of each model and according to the risk tolerance parameter (T = 0.5, 0.75, and 1.0) for each simulation. A simulation is performed to generate 5,000 scenarios with the weights of each portfolio with a uniform random variable between 0 and 1, according to the restriction of the optimization. Therefore, we obtain the minimum variance (MV) measure to compare the performance of the models.

Second, we want to maximize the Sharpe ratio, which is given by the following optimization program: max Θ_p, subject to $\sum_{i=1}^{n} w_i = 1$, $w_i \geq 0$. The Sharpe coefficient is given by $\Theta_p = \frac{r_p - r_f}{\sigma_p}$, where $r_p = \sum_{i=1}^{n} w_i \, r_i$, is the portfolio return; r_f is the risk-free rate, which is equal to zero for our study, without loss of generality. Again, we employ the Monte Carlo simulation to generate 5,000 scenarios with the weights of each portfolio in a random manner. Thus, we get a second performance measure based on the maximum Sharpe ratio (MSR).

We opt for Monte Carlo simulation since this technique is preferable for the treatment of pricing and risk management problems, even though analytical solutions are available for a given model –e.g., Giesecke, Kim & Zhu (2011), moreover, "[m]any real-world portfolio optimization problems are global optimization problems, and therefore are not suitable for linear or quadratic programming routines" (Peterson et al., 2018). In addition, historical and Monte Carlo simulation techniques have been employed by practitioners in portfolio optimization problems (Marchioro, 2011).

Table 2.4 shows the performance of two models: behavioral sigma (Model 1) and our proposal based on Gram-Charlier returns (Model 3) according to the minimum variance (MV) and the maximum Sharpe ratio (MSR) of all combinations of portfolios formed with two, three, and four assets. Thus, we have a total of 11 combinations of portfolios: six of two-asset portfolios, four of three-asset portfolios (Table 2.5 summarizes the asset combinations in the two- and three-asset portfolios), and one for the portfolio formed of all the individual assets. Totalizing the combinations of portfolios with the three different risk tolerance parameters (*T* = 0.5, 0.75, and 1.0), we have 33 performance measures assessed with MV and the other 33 calculated using MSR. For all cases, Model 3 is better off than Model 1, according to the MV indicator, except for combination C1 when *T* = 0.5. For the 33 MSR measures, Model 3 is outperformed by Model 1 in only six cases (when *T* = 0.5). As particular cases, for two-asset portfolios, combinations C3 (AAPL and MRO) and C6 (AAPL and MOS) present the best performance measures for Model 3. Therefore, the combination (C9) of these three assets (AAPL, MRO, and MOS) provides the best MSR values (for Model 3) and all the risk tolerance parameters in the three-asset portfolio case.

Table 2.4. Comparison of metrics for Model 1 and Model 3

Model	Model 1						Model 3					
T	T=0.5		T=0.75		T=1.0		T=0.5		T= 0.75		T=1.0	
Metric	**MV**	**MSR**	**MV**	**MSR**	**MV**	**MSR**	**MV**	**MSR**	**MV**	**MSR**	**MV**	**MSR**
						2 asset portfolios						
C1	0.027	9.339	0.025	6.598	0.024	10.572	0.031	5.169	0.019	9.157	0.013	13.629
C2	0.026	0.904	0.025	0.925	0.025	0.942	0.023	1.061	0.014	1.872	0.009	2.775
C3	0.024	9.430	0.022	9.781	0.022	10.574	0.023	10.496	0.015	18.523	0.010	27.463
C4	0.029	0.893	0.027	0.916	0.027	0.935	0.025	1.112	0.014	1.962	0.009	2.905
C5	0.028	0.478	0.029	0.498	0.026	0.513	0.026	0.534	0.015	0.943	0.010	1.398
C6	0.025	9.340	0.023	9.077	0.023	10.574	0.023	8.158	0.014	14.373	0.009	21.286
						3 asset portfolios						
C7	0.023	8.678	0.022	8.896	0.022	9.072	0.022	7.998	0.014	13.790	0.010	19.705
C8	0.025	0.949	0.023	0.725	0.024	1.014	0.022	1.091	0.015	1.782	0.010	2.361
C9	0.022	8.744	0.021	8.964	0.021	9.142	0.021	8.076	0.014	13.969	0.010	20.061
C10	0.024	5.874	0.022	6.497	0.022	7.049	0.022	8.004	0.015	13.843	0.010	19.875
4asset	0.022	8.646	0.020	9.187	0.020	9.645	0.021	7.679	0.015	12.575	0.012	16.894

MV stands for minimum variance and MSR is maximum Sharpe ratio. Model 1 represents the behavioral risk, and Model 3 stands for the modified risk when returns are Gram-Charlier distributed. Results obtained for 5,000 Monte Carlo simulations. See Table 6 for the combinations of two and three assets in the portfolio (C1 to C10).

Table 2.5. Asset combination in every portfolio

Combination	Assets	Combination	Assets
C1	AAPL, AIG	C6	AAPL, MOS
C2	MOS, MRO	C7	AAPL, AIG, MOS
C3	AAPL, MRO	C8	AIG, MRO, MOS
C4	AIG, MOS	C9	AAPL, MRO, MOS
C5	AIG, MRO	C10	AAPL, AIG, MRO

Apple (AAPL), American International Group (AIG), Mosaic Company (MOS), and Marathon Oil Corporation (MRO).

On the other hand, Table 2.6 presents the results of MV and MSR comparing Model 2 (modified variance with Taylor's expansion) and Model 3 (modified variance according to Gram-Charlier returns). As observed, only for the 2-asset portfolio in combination C1 (AAPL-

AIG), when the investor exhibits low risk tolerance (*T*=0.5), the Model 2 outperforms Model 3 considering the Sharpe ratio performance. For the other 32 performance measures Model 3 is better off than Model 2 in terms of both MV and MSR criteria. In Model 2, the lowest MV was obtained with the 4-asset portfolio for all levels of risk tolerance and the higher MSR for the combination C6 (AAPL-MOS). In Model 3, the higher MSR was observed for the 2-asset combination C3 (AAPL-MRO) and high-risk tolerance (*T*=1), and the MV corresponded to combinations C2 (MOS-MRO), C4 (AIG-MOS), and C6 (AAPL-MOS).

Table 2.6. Comparison of metrics for Model 2 and Model 3

Model	Model 2						Model 3					
T	T=0.5		T=0.75		T=1.0		T=0.5		T=0.75		T=1.0	
Metric	**MV**	**MSR**	**MV**	**MSR**	**MV**	**MSR**	**MV**	**MSR**	**MV**	**MSR**	**MV**	**MSR**
						2 asset portfolios						
C1	0.036	6.901	0.032	8.852	0.029	7.385	0.031	5.169	0.019	9.157	0.013	13.629
C2	0.033	0.714	0.030	0.877	0.025	1.013	0.023	1.061	0.014	1.872	0.009	2.775
C3	0.031	6.824	0.025	8.879	0.022	10.020	0.023	10.496	0.015	18.523	0.010	27.463
C4	0.038	0.642	0.031	0.799	0.027	0.930	0.025	1.112	0.014	1.962	0.009	2.905
C5	0.036	0.231	0.029	0.303	0.026	0.360	0.026	0.534	0.015	0.943	0.010	1.398
C6	0.032	6.903	0.026	8.853	0.023	10.438	0.023	8.158	0.014	14.373	0.009	21.286
						3 asset portfolios						
C7	0.030	6.799	0.025	8.677	0.022	10.181	0.022	7.998	0.014	13.790	0.010	19.705
C8	0.03	0.699	0.026	0.871	0.024	1.008	0.022	1.091	0.015	1.782	0.010	2.361
C9	0.027	6.847	0.023	8.746	0.021	10.270	0.021	8.076	0.014	13.969	0.010	20.061
C10	0.029	6.785	0.024	8.667	0.021	10.176	0.022	8.004	0.015	13.843	0.010	19.875
4asset	0.026	6.528	0.022	8.226	0.020	9.537	0.021	7.679	0.015	12.575	0.012	16.894

MV stands for minimum variance and MSR is maximum Sharpe ratio. Model 2 is the modified risk employing Taylor's expansion, and Model 3 stands for the modified risk when returns are Gram-Charlier distributed. Results obtained for 5,000 Monte Carlo simulations. See Table 5 for the combinations of two and three assets in the portfolio (C1 to C10).

2.4. Case Study B

Although the Gram-Charlier expansion allows the variations on the normal standard introduced by moments of order greater than two to be taken into account, it has the disadvantage that, for certain values of moments of order greater than two, negative probability values are reached because it is a polynomial approximation, so in addition to the stocks initially used, we chose the S&P 500 index (GSPC), the 5-year US Treasury yield (FVX), corn futures (ZC) and palladium futures (PA) where the skewness and kurtosis domain on which the expansion is positive.

Descriptive statistics for analyzed assets, correlation matrix of returns for the analyzed assets, standard deviation versus different risk measures, comparison of metrics for Model 1 and Model 3 and asset combination in every portfolio are shown in the tables below, obtaining results equivalent to those of case study A.

Table 2.7. Descriptive Statistics for the analyzed assets

	FVX	GSPC	ZC	PA
Mean	-0.004	0.386	0.062	-0.006
Std deviation	2.578	0.826	1.652	1.412
Skewness	-0.056	-0.517	-0.420	-0.008
Excess Kurtosis	5.039	6.720	5.155	5.130

5-year US treasury yield bond (FVX), S&P500 (GSPC), Corn (ZC), and Palladium (PA). The analyzed period ranged between January 1, 2014, and December 31, 2019, observed on a daily basis (1499 observations).

Table 2.8. Correlation matrix of returns for the analyzed assets

Stock	FVX	GSPC	ZC	PA
FVX	1.000	0.347	0.048	0.062
GSPC	0.347	1.000	0.207	0.068
ZC	0.048	0.207	1.000	0.065
PA	0.062	0.068	0.065	1.000

5-year US treasury yield bond (FVX), S&P500 (GSPC), Corn (ZC), and Palladium (PA). The analyzed period ranged between January 1, 2014, and December 31, 2019, observed on a daily basis (1499 observations).

Table 2.9. Standard deviation versus different risk measures

Stock	Standard deviation of returns	Behavioral standard deviation (Model 1)			Modified standard deviation (Model 2)			Modified standard deviation (Model 3)		
		T=0.5	T=0.75	T=1.0	T=0.5	T=0.75	T=1.0	T=0.5	T=0.75	T=1.0
FVX	2.578	2.586	2.582	2.581	3.649	2.979	2.579	0.507	0.281	0.186
GSPC	0.826	0.829	0.828	0.828	1.172	0.956	0.828	0.206	0.118	0.081
ZC	1.652	1.661	1.658	1.656	2.347	1.913	1.656	0.538	0.308	0.210
PA	1.412	1.413	1.413	1.412	1.997	1.631	1.412	0.117	0.063	0.041

Model 1 represents the behavioral risk, Model 2 is the modified risk employing Taylor's expansion, and Model 3 stands for the modified risk when returns are Gram-Charlier distributed. T represents the risk tolerance parameter. The analyzed period ranged between Jan 1, 2014, and Dec 31, 2019, observed on a daily basis (1499 observations).

Table 2.10. Comparison of metrics for Model 1 and Model 3

Model	Model 1						Model 3					
T	T=0.5		T=0.75		T=1.0		T=0.5		T= 0.75		T=1.0	
Metric	**MV**	**MSR**	**MV**	**MSR**	**MV**	**MSR**	**MV**	**MSR**	**MV**	**MSR**	**MV**	**MSR**
2 asset portfolios												
C1	0.830	11.846	0.829	11.860	0.828	11.866	0.207	47.408	0.120	81.905	0.082	118.649
C2	1.110	10.199	1.109	10.219	1.108	10.229	0.118	31.511	0.064	54.939	0.041	80.634
C3	1.426	10.381	1.424	10.401	1.423	10.411	0.484	32.013	0.278	55.798	0.184	81.876
C4	0.735	12.348	0.734	12.362	0.734	12.369	0.115	49.689	0.062	86.350	0.040	126.366
C5	0.795	14.298	0.794	14.316	0.793	14.324	0.207	47.331	0.119	82.130	0.082	119.841
C6	1.270	0.042	1.270	0.042	1.269	0.042	0.115	0.211	0.062	0.379	0.041	0.570
3 asset portfolios												
C7	0.736	11.963	0.736	11.976	0.735	11.982	0.138	42.852	0.101	61.431	0.090	72.315
C8	0.715	14.526	0.714	14.544	0.714	14.552	0.130	43.424	0.086	62.403	0.071	73.595
C9	1.046	10.068	1.045	10.087	1.045	10.097	0.144	30.264	0.106	49.780	0.094	66.998
C10	0.795	14.251	0.794	14.269	0.794	14.277	0.276	37.372	0.224	53.509	0.208	75.150
4asset	0.715	13.905	0.715	13.920	0.715	13.927	0.970	10.387	0.812	12.382	0.715	13.929

MV stands for minimum variance and MSR is maximum Sharpe ratio. Model 1 represents the behavioral risk, and Model 3 stands for the modified risk when returns are Gram-Charlier distributed. Results obtained for 5,000 Monte Carlo simulations. See Table 6 for the combinations of two and three assets in the portfolio (C1 to C10).

Table 2.11. Asset combination in every portfolio

Combination	Assets	Combination	Assets
C1	FVX, GSPC	C6	FVX, ZC
C2	ZC, PA	C7	FVX, GSPC, ZC
C3	FVX, PA	C8	GSPC, PA, ZC
C4	GSPC, ZC	C9	FVX, PA, ZC
C5	GSPC, PA	C10	FVX, GSPC, PA

5-year US treasury yield bond (FVX), S&P500 (GSPC), Corn (ZC), and Palladium (PA).

Table 2.12. Comparison of metrics for Model 2 and Model 3

Model	Model 2						Model 3					
T	T=0.5		T=0.75		T=1.0		T=0.5		T=0.75		T=1.0	
Metric	**MV**	**MSR**	**MV**	**MSR**	**MV**	**MSR**	**MV**	**MSR**	**MV**	**MSR**	**MV**	**MSR**
2 asset portfolios												
C1	1.160	8.380	0.955	10.273	0.828	11.867	0.207	47.408	0.120	81.905	0.082	118.649
C2	1.545	7.219	1.270	8.854	1.108	10.231	0.118	31.511	0.064	54.939	0.041	80.634
C3	1.994	7.347	1.635	9.011	1.422	10.413	0.484	32.013	0.278	55.798	0.184	81.876
C4	1.025	8.734	0.842	10.707	0.734	12.370	0.115	49.689	0.062	86.350	0.040	126.366
C5	1.090	10.567	0.903	12.668	0.793	14.326	0.207	47.331	0.119	82.130	0.082	119.841
C6	1.774	0.029	1.457	0.036	1.269	0.042	0.115	0.211	0.062	0.379	0.041	0.570

3 asset portfolios												
C7	1.018	8.500	0.841	10.397	0.735	11.983	0.138	42.852	0.101	61.431	0.090	72.315
C8	0.973	10.741	0.810	12.874	0.714	14.554	0.130	43.424	0.086	62.403	0.071	73.595
C9	1.442	7.138	1.192	8.747	1.045	10.099	0.144	30.264	0.106	49.780	0.094	66.998
C10	1.080	10.546	0.902	12.635	0.794	14.279	0.276	37.372	0.224	53.509	0.208	75.150
4asset	0.240	32.378	0.222	42.316	0.217	50.356	0.970	10.387	0.812	12.382	0.715	13.929

MV stands for minimum variance and MSR is maximum Sharpe ratio. Model 2 is the modified risk employing Taylor's expansion, and Model 3 stands for the modified risk when returns are Gram-Charlier distributed. Results obtained for 5,000 Monte Carlo simulations. See Table 5 for the combinations of two and three assets in the portfolio (C1 to C10).

Table 2.7 shows the descriptive statistics of the returns, which have been calculated as $r_t = 100\ln(P_t/P_{t-1})$, where P_t is the price of the asset at time *t*.

As observed, FVX and PA presents negative daily average returns and the riskiest asset is FVX, with the highest standard deviation. On the other hand, GSPC is the asset with the highest average daily return, and it presents the lowest standard deviation. Moreover, the returns of the four assets exhibit negative values of skewness and fat tails.

Furthermore, all the assets pairwise-correlations are below 50%, as presented in Table 2.8, the most correlated stock returns are GSPC and FVX, while the pair with the lowest correlation are ZC and FVX, which means that portfolios formed on these assets seem to be diversified.

Table 2.9 provides the comparison of the standard deviation of returns with the three risk measures obtained with the methodologies described in Section 2.2. The behavioral risk (Model 1) and modified risk (Model 2) values are higher than standard deviation for all cases. As expected, the values of the three risk measures decrease when the risk tolerance parameter increases for each model. As particular case, PA presents the lowest risk at certain level of T for Model 3, mainly explained by the lowest excess kurtosis value and lower standard deviation compared with FVX. However, GSPC is the asset with the lowest risk figures for Models 1 and 2.

Table 2.10 shows the performance of two models: behavioral sigma (Model 1) and our proposal based on Gram-Charlier returns (Model 3) according to the minimum variance (MV) and the maximum Sharpe ratio (MSR) of all combinations of portfolios formed with two, three, and four assets. Thus, we have a total of 11 combinations of portfolios: six of two-asset portfolios, four of three-asset portfolios (Table 2.11 summarizes the asset combinations in the two- and three-asset portfolios), and one for the portfolio formed of all the individual assets. Totalizing the combinations of portfolios with the three different risk tolerance parameters ($T = 0.5$, 0.75, and 1.0), we have 33 performance measures assessed with MV and the other 33 calculated using MSR. For all cases, Model 3 is better off than Model 1, according to the MV and MSR indicators and all risk tolerance parameters. As particular cases, for two-asset portfolios, combination C4 (GSPC and ZC) present the best performance measures for Model 3. Therefore, the combination (C8) of these three assets (GSPC, PA, and ZC) provides the best MSR and MV values (for Model 3), and all the risk tolerance parameters in the three-asset portfolio case, except for combination C10 when $T = 1$ for MSR value.

On the other hand, Table 2.12 presents the results of MV and MSR comparing Model 2 (modified variance with Taylor's expansion) and Model 3 (modified variance according to Gram-Charlier returns). As observed, for the 33 performance measures Model 3 is better off than Model 2 in terms of both MV and MSR criteria. In Model 2, the lowest MV and the highest MSR was obtained with the 4-asset portfolio for all levels of risk tolerance. In Model 3, the higher MSR and lower MV was observed for the 2-asset combination C4 (GSPC-ZC) for all risk tolerance parameters.

2.5. Conclusion

Several extensions and variations have been proposed to the mean-variance analysis in portfolio theory. Our work is framed in the utility models since our proposed methodology is based on the exponential utility function, which implies a constant absolute risk aversion model, the most preferred utility amongst economists (Kassimatis, 2021). In addition, we assume that asset returns, as in several financial applications, follow a Gram-Charlier distribution to incorporate higher moments than variance in the risk expression. Therefore, the financial theory has explored various methodologies to quantify measures of risk when asset returns exhibit non-Gaussian distributions, and the utility function is not quadratic. In this context, investor's risk attitudes linked to skewness and kurtosis have a significant impact on the investment portfolio optimization process. In this chapter we propose a new risk measure corresponding to a modified variance compatible to Gram-Charlier returns (Model 3) and compare it with the so-called behavioral sigma (Model 1) and a variance modified through Taylor's expansion (Model 2). The performance of the models is evaluated by minimizing the portfolio risk (MV) and maximizing the Sharpe ratio (MSR) criteria. According to the case studies A and B, where simulations of 11 portfolio combinations of three assets and with three different values for risk tolerance are performed, our proposal is outperformed by Model 1 (Model 2) in just 9 (10) cases out of total of 33 cases in each comparison.

All in all, the modified variance according to the Gram-Charlier return distribution seems not only to be a natural and straightforward procedure to incorporate risk tolerance features to investor decisions, but also seems to provide more efficient decisions on portfolio choice than those corresponding to alternative behavioral variance measures. Future research may be focused on the analysis of our modified variance with other types of financial assets and different performance metrics.

2.6. Appendix

2.6.1. Appendix A. Proof of behavioral variance

By solving R in equation (2.4), we obtain the same expression derived by Bell (1995) for an exponential utility function (where $T = 2/c$)

$$R = \frac{T}{2}\ln\left(E\left[e^{\frac{2(\bar{r}-r)}{T}}\right]\right). \quad (2.A.1)$$

Multiplying both sides of the equation by T, we have:

$$RT = \frac{T^2}{2}\ln\left(E\left[e^{\frac{2(\bar{r}-r)}{T}}\right]\right). \quad (2.A.2)$$

Now we show that if the returns are assumed to be normally distributed ($r{\sim}N(\bar{r},\sigma_r^2)$), then the risk measure, R is equal to ${\sigma_r^2}/{T}$.

If a random variable $y{\sim}N(\mu,\sigma^2)$, then $E[\exp\{y\}]$ is lognormally distributed and its mean is given by $\exp\left\{\mu+\frac{\sigma^2}{2}\right\}$.

Let $x = \frac{2(\bar{r}-r)}{T}$, as $r{\sim}N(\bar{r},\sigma_r^2)$ then,

$$x{\sim}N\left(0,\frac{4\sigma_r^2}{T^2}\right). \quad (2.A.3)$$

Therefore,

$$E[e^{\mathrm{x}}] = e^{\frac{2\sigma_r^2}{T^2}}. \quad (2.A.4)$$

By replacing expression (A.4) in equation (A.1), we get

$$R = \frac{T}{2}\ln\left(e^{\frac{2\sigma_r^2}{T^2}}\right). \quad (2.A.5)$$

Then,

$$R = \frac{\sigma_r^2}{T}. \quad (2.A.6)$$

The obtained variance from the last equation $\sigma_r^2 = RT$, is named the behavioral variance σ_B^2, and replacing the latter expression in equation (2.A.2), we obtain the general expression for the behavioral variance when returns follow a normal distribution.

$$\sigma_B^2 = \frac{T^2}{2}\ln\left(E\left[e^{\frac{2(\bar{r}-r)}{T}}\right]\right). \tag{2.A.7}$$

To obtain the behavioral variance of Davies and de Servigny (2012), recall that the cumulant generating function is the natural logarithm of the moment generating function, that is:

$$K(t) = \ln(\mathrm{E}[e^{tX}]). \tag{2.A.8}$$

Moreover,

$$K(t) = \sum_{n=1}^{\infty} \kappa_n \frac{t^n}{n!}, \tag{2.A.9}$$

where κ_n are the cumulants, with $\kappa_1 = \mu_1$, $\kappa_2 = \mu_2$, $\kappa_3 = \mu_3$, and $\kappa_1 = \mu_4 - 3\mu_2^2$, and μ_n are the central moments. By setting $t = {}^{2}/_{T}$ and $X = \bar{r} - r$ we get (and assuming $\mu = 0$, without loss of generality)

$$\ln\left(E\left[e^{\frac{2(\bar{r}-r)}{T}}\right]\right) = \frac{2}{T^2}\sigma_r^2\left[1 - \frac{2\sigma^3 sk}{3T} + \frac{\sigma^2 ek}{3T^2}\right]. \tag{2.A.10}$$

Then,

$$\sigma_B^2 \approx \sigma_r^2\left[1 - \frac{2\sigma^3 sk}{3T} + \frac{\sigma^2 ek}{3T^2}\right]. \tag{2.A.11}$$

2.6.2. Appendix B. Proof of modified variance with Taylor expansion

Skewness and kurtosis can be introduced in the risk measure in equation (2.A.1) by considering a four-order Taylor expansion of the term $\exp(x)$ where $x = \frac{2(\bar{r}-r)}{T}$. Applying the expectation operator and, finally, the logarithm transformation, we obtain the approximation for the risk measure R, which is called modified variance:

$$\sigma_M^2 \approx \frac{T}{2}\ln\left(1 + \frac{2\sigma_r^2}{T^2}\left[1 - \frac{2\sigma_r sk}{3T} + \frac{\sigma_r^2 ek}{3T^2}\right]\right). \tag{2.B.1}$$

2.6.3. Appendix C. Proof of modified variance when returns are Gram-Charlier distributed.

Let us assume the risk measure in equation (2.A.1), as per in Bell (1995), and that returns follow a Gram-Charlier distribution, that is $r \sim GC(\bar{r}, \sigma_r^2, sk_r, ek_r)$. If a random variable y is Gram-Charlier distributed, then $E[\exp\{y\}]$ is log-SNP distributed and its mean is given by $\exp\left\{\mu + \frac{\sigma^2}{2}\right\}[1 + d_1\sigma + d_2\sigma^2 + d_3\sigma^3 + d_4\sigma^4]$ (Ñíguez et al., 2013). A recent application of the log-SNP in finance maybe found in Cortés, Mora-Valencia and Perote (2020).

If $r \sim GC(\bar{r}, \sigma_r^2, sk, ek)$ and $x = \frac{2(\bar{r}-r)}{T}$, then $x \sim GC(\mu_x, \sigma_x^2, sk_x, ek_x)$, where:

$$\mu_x = 0, \tag{2.C.1}$$

$$\sigma_x^2 = \frac{4\sigma_r^2}{T^2}, \tag{2.C.2}$$

$$sk_x = \frac{\bar{r}^3}{\sigma_r^3} + \frac{3\bar{r}}{\sigma_r} - sk_r, \tag{2.C.3}$$

$$ek_x = \frac{3\bar{r}^4}{\sigma_r^4} + \frac{6\bar{r}^2}{\sigma_r^2} - \frac{4s_r}{\sigma_r} + ek_r. \tag{2.C.4}$$

By replacing these values in equation (A.1), we obtain our proposal for modified variance:

$$\sigma_{MGC}^2 = \frac{\sigma_r^2}{T} + \frac{T}{2}\ln\{1 + d_2\sigma_X^2 + d_3\sigma_X^3 + d_4\sigma_X^4\}, \tag{2.C.5}$$

Where:

$$\begin{aligned} d_2 &= \frac{1}{2}(\sigma_X^2 - 1), & \sigma_X^2 &= \frac{4\sigma_r^2}{T^2}, \\ d_3 &= \frac{sk_x}{6}, & sk_x &= \frac{\bar{r}^3}{\sigma_r^3} + \frac{3\bar{r}}{\sigma_r} - sk_r, \\ d_4 &= \frac{ek_x}{24}, & ek_x &= \frac{3\bar{r}^4}{\sigma_r^4} + \frac{6\bar{r}^2}{\sigma_r^2} - \frac{4s_r}{\sigma_r} + ek_r. \end{aligned} \tag{2.C.6}$$

CHAPTER 3: ESG Portfolio Optimization: The Relevance of Higher Order Moments

3.1. Introduction

In 1952, Harry Markowitz developed the Mean-Variance Portfolio Theory, also known as the Modern Portfolio Theory (MPT). Under the assumption that stock returns are normally distributed as random variables, the MV model considers that investors are rational, and markets are efficient. Based on the MV model, Markowitz tries to minimize the risk of portfolio for a given expected return or to maximize the expected return for a given risk. The Markowitz' MV model establishes the foundation for modern portfolio theory (MPT) and the Capital Asset Pricing Model (CAPM) developed shortly thereafter by Sharpe (1964), Lintner (1965), and Mossin (1966).

The MV approach to portfolio selection has been heavily criticized for implicitly assuming that portfolio measures such as expected return and variance are appropriate for constructing optimal portfolios. Jondeau and Rockinger (2006) and Weide (2010) stated that expected return and variance are sufficient for optimal portfolio only if: (1) the utility function is quadratic; and (2) the probability distribution of returns is normal. However, neither of these assumptions is strictly met. Empirical evidence has shown that stock returns do not exhibit Gaussian behavior and, therefore, the highest moments of the distribution should be considered in portfolio construction and optimization, like Chen and Zhou (2018) and Zhou and Palomar (2021) suggested. These authors, like Cvitanic et al. (2007) and Jondeau and Rockinger (2006), found that if the distribution of returns has a significantly different skewness and kurtosis from a normal distribution, the variance as a measure of risk will not be appropriate and, therefore, the Classic MV framework will not produce efficient results.

In this sense, an asymmetric and heavy-tailed return distribution requires the integration of higher-order moments such as skewness and kurtosis in portfolio optimization, in addition to mean and variance. Nevertheless, incorporating higher-order moments turns portfolio selection into a non-convex optimization problem considered by multiple objective functions, such as minimizing variance and kurtosis and maximizing mean and skewness, respectively (Zhou and Palomar, 2021). Therefore, solving the optimization problem can be much more complex than the traditional formulation of the MV model.

On the other hand, the classical MV framework is based on both risk and return measures (or mean and variance). However, in the last two decades, the increasing popularity of socially responsible investing (SRI) has imposed additional requirements on investment decision-making (Hartzmark and Sussman, 2019; Coqueret, 2022), especially since the adoption of the Sustainable Development Goals (SDG) and the Principles for Responsible Investment (PRI) from the United Nations. This has given rise to the broad field of investing, also known as sustainable investing or environmental, social, and governance (ESG) investing, as noted by Steuer and Utz (2023). While ESG investing is rising worldwide, its incorporation into investment decision-making is becoming increasingly relevant and necessary. Henriksson et al. (2019) argued that SRI/ESG investing includes new preferences driven by sustainability criteria and concerns about climate change, thus adding a new 'fiduciary duty' to society. However, there are concerns about how its incorporation is carried out within the process of creating portfolios and investment strategies.

Utz et al. (2014), Henriksson et al. (2019), Steuer and Utz (2023), among others, suggested that SRI/ESG investing adds a third dimension to investment strategies, as these should not only consider purely financial criteria such as return and risk, but also ESG criteria. They agreed that its incorporation into the portfolio model requires that the efficient portfolio frontier (EPF) can be theorized as a three-dimensional surface (or 3D-space) and that it requires optimizing risk, return, and ESG metrics[2]. Although the financial literature on portfolio optimization and the design of investment strategies considering ESG criteria or metrics has grown significantly in recent decades, there is still no model that incorporates high order moments in the construction of optimal portfolios considering ESG metrics.

In this chapter, we propose a portfolio optimization model in the presence of higher order moments and ESG metrics. To this end, we use a multi-objective approach that employs a mean-variance-skewness-kurtosis (MVSK) model to construct a diversified portfolio with an ESG score named as the MVSK-ESG portfolio. We use a difference of convex (DC) algorithm to solve high-order optimal portfolios. Furthermore, we focus on the optimal ESG portfolio with only leaders –i.e. companies with the highest ESG scores and show that the MVSK-ESG portfolios achieve better performance measures (in-sample and out-of-sample) compared to the traditional models. We use measures such as the Sharpe ratio, the Rachev ratio, and the Delta ratio for the United States stock markets. Particularly, we use the Rachev ratio, as it is the suitable performance measure when high-order moments in the distribution of portfolio returns are considered. Our proposed approach improves previous models by combining both higher moments and ESG metrics in a unified framework and by using Quadratic Programming (QP) solvers, which can be easily applied. Furthermore, the proposed model is highly computationally efficient.

This chapter is organized as follows. Section 3.2 presents a literature review of the MV framework and its extensions considering high-order moments and ESG metrics. Section 3.3 presents the theoretical considerations and models for portfolio construction and optimization.

[2] A similar approach considers measures of climate change or climate risk like carbon intensity or CO2 emissions (For more details see Bender et al., 2020; Coqueret, 2022; Roncalli et al., 2021).

In Section 3.4, we implement the proposed MVSK-ESG approach to build optimal portfolios for the United States stock markets. Finally, Section 3.5 concludes the chapter.

3.2. Literature review

The rise of SRI and ESG investing has attracted the attention of both investment and academic communities, as suggested by Chen and Mussalli (2020), Chen et al. (2021), and Coqueret (2022). Investors are adopting strategies that combine these ESG criteria to promote sustainable investment opportunities. Numerous academic studies have focused on SRI/ESG investing from numerous perspectives and methodologies, including portfolio construction and optimization, financial performance, abnormal returns, as well as the mitigation of ESG risks. Particularly, some studies have concentrated on portfolio optimization techniques. For instance, Hirschberger et al. (2013), Utz et al. (2014), and Gasser et al. (2017) extended Markowitz's two-dimensional MV model to a tri-criterion portfolio model involving ESG rating scores. To do so, they used a model with multiple objective functions to expand the traditional MV model.

In the same line, we also found studies that extended this approach to find optimal portfolios for different portfolio measures and strategies. For example, Alessandrini and Jondeau (2020) optimized ESG metrics using some constraints such as turnover, tracking error, factor exposure, and others. Chen et al. (2023) combined ESG scores with financial measures like risk and return based on a cross-efficiency method and they applied a data envelopment analysis (DEA) model. Pedersen et al. (2021) proposed a model to find the ESG-efficient frontier and the highest Sharpe ratio feasible for all ESG levels, and based on that model, they showed advantages of responsible investing. Geczy et al. (2018) and Geczy et al. (2021) applied the APT-MV tracking-error model, combining mutual funds to create portfolios with maximum Sharpe ratios and by using prior beliefs with historical data, they tested pricing models. Steuer and Utz (2023) extended the tri-criterion approach that accounts for ESG factors, creating an efficient surface and special non-contour curves. Abate et al. (2021) implemented different techniques of ESG-efficient portfolio optimization models, identifying the effect of constraints incorporating ESG scores, and extending traditional models using downside risk measures.

Additionally, Chen and Mussalli (2020) proposed a portfolio approach combining ESG investing based on factors where they used the information ratio to solve the dual objectives of maximizing alpha and ESG performance. Xidonas and Essner (2022) proposed an optimization model with multi objective functions, where they maximized the risk measure for ESG investment objectives. Finally, Lauria et al. (2022) proposed a unified framework for incorporating ESG scores into portfolio construction. They introduced an ESG-valued return and found that the model preserves the traditional risk aversion parameter.

On the other hand, the empirical literature has also considered climate risk measures into portfolio optimization models, such as carbon intensity (CI) and carbon risk (CR). For example, Chan et al. (2020) introduced a factor model under the traditional quadratic utility function with ESG scores and CI constraints. Bender et al. (2020) introduced CI metrics into portfolio optimization with objectives focused on carbon reduction and volatility. Finally, Blitz et al. (2023) implemented a 3D investing approach for optimal portfolios. They established conditions under which a 3D approach outperforms a 2D approach with ESG constraints.

In summary, the empirical literature on investment strategies has been reoriented to integrate sustainability issues. To this end, ESG factors have been included into optimal portfolio construction, in addition to financial measures such as expected return and risk. However, portfolio approaches have not incorporated higher-order moments in portfolio optimization, which can lead to biased solutions, showing poor performance or being inferior to their benchmarks. It is therefore necessary to develop an approach that allows dealing with these shortcomings.

3.3. The Models

3.3.1 Mean-Variance Model

Markowitz (1952) proposed the Mean-Variance (MV) model, which considers the first two moments of the return distribution, where the mean and the variance are in fact the first two central moments of the portfolio return distribution. The main assumption of the model is that the distribution of returns is normal, which is equivalent to considering a quadratic utility function dependent on the mean and variance. In this context, MV model involves the mathematical expectation of asset returns ($\mu \in \mathbb{R}^{n\times 1}$) and the covariance matrix ($\Sigma \in \mathbb{R}^{n\times n}$) given the returns' distribution.

The Markowitz' optimization problem is solved by minimizing the variance, $\sigma_P^2 = w'\Sigma w$, which is the risk measure of the portfolio, for a given level of expected return, $\mu_p = w'\mu$.

$$\min_{\{w\}} \{w'\Sigma w\} \qquad \begin{array}{l} s.t.\ \ \mu' w = \mu_p, \\ \qquad w'\mathbf{1} = 1, \\ \qquad w \in \mathcal{W} \end{array} \tag{3.1}$$

Where, μ_p is the expected return of the lower risk portfolio, $w \in \mathbb{R}^{n\times 1}$ is the vector of weights of the assets, $\mathbf{1} \in \mathbb{R}^{n\times 1}$ is a vector of ones, and $\mathcal{W}$ is a set of feasible solutions, including any portfolio constraints. Similarly, a further formulation of the optimization problem may incorporate restrictions on the sign of the weights. For example, if short selling is not allowed, the asset weights must be non-negative, i.e. $w \geq 0$. With all these constraints, the optimal MV portfolio is solved using quadratic programming (QP).

On the other hand, the portfolio selection can also be represented as a problem of optimizing the relationship between the expected return of the portfolio and its variance (Levy and Markowitz, 1979) and it depends on the investor´s risk aversion parameter, ζ. This assumption implies that the utility function for portfolio problems is quadratic and only depends on the first two moments. The optimization problem is given by:

$$\min_{\{w\}} \{-\mu' w + \zeta w' \Sigma w\} \qquad \begin{aligned} s.t.\ w'\mathbf{1} &= 1, \\ w &\in \mathcal{W} \end{aligned} \tag{3.2}$$

This quadratic utility function exhibits both increasing absolute and relative risk aversion, as well as satiation, i.e., based on that utility function any investor prefers less wealth rather than more, which is problematic from the utility theory point of view , since the utility function must exhibit non-satiation, risk aversion, decreasing absolute risk aversion (DARA), and constant relative risk aversion (CRRA).[3] Furthermore, assuming that asset returns are normally distributed is also problematic. Therefore, investor decisions based on the mean and variance can generate inconsistent results. This is because investors can never lose more than their entire wealth.

Markowitz (1959), Levy and Markowitz (1979) also recognized that investor utility functions are unlikely to be quadratic and that the empirical distribution of returns does not follow a Gaussian distribution. However, they prove the MV approach based on the belief that at least one of the following statements is true:

1) Utility functions are approximately quadratic

2) Probability distributions can be approximated by their first two moments.

[3] Mossin (1968) and Levy and Markowitz (1979) showed that the logarithmic function and the power function are both utility functions that satisfy all the above desirable features.

3) The MV model provides relevant information to investors and helps them make investment decisions.

However, the classical Markowitz's Mean-Variance (MV) approach ignores heavy tails and skewness, as shown above. Consequently, the MV formulation does not yield optimal portfolios in the presence of skewness and kurtosis (Zhou and Palomar, 2021). In the next section, we describe some proposed models for high-order portfolios considering the mean, variance, skewness, and kurtosis.

3.3.2 Portfolio Construction under Higher-Order Moments

Following the methodology of Jondeau and Rockinger (2006) and Chen and Zhou (2018), we incorporated higher order moments (third and fourth moments) into the portfolio selection process. To do so, we define the statistical measures of skewness as $s_i = E(r_i - \mu_i)^3$, and kurtosis, $k_i = E(r_i - \mu_i)^4$, for a given asset return's distribution function. In that sense, for a list of *n* assets, co-skewness, s_{ijk}, and co-kurtosis, k_{ijkl}, between asset returns are:

$$s_{ijk} = E\left[(r_i - \mu_i)(r_j - \mu_j)(r_k - \mu_k)\right]^{1/3} \tag{3.3a}$$

$$k_{ijkl} = E\left[(r_i - \mu_i)(r_j - \mu_j)(r_k - \mu_k)(r_l - \mu_l)\right]^{1/4} \tag{3.3b}$$

We adjusted co-skewness and co-kurtosis in a convenient way. As suggested by Athayde and Flôres (2004), we transform the co-skewness (n, n, n) matrix into the (n, n^2) matrix, and the co-kurtosis (n, n, n, n) matrix into the (n, n^3) matrix. For example, in the case of $n = 2$ assets, the resulting co-skewness matrix and co-kurtosis matrix are given by:

$$S = \begin{vmatrix} s_{111} & s_{112} \\ s_{121} & s_{122} \end{vmatrix}\begin{vmatrix} s_{211} & s_{212} \\ s_{221} & s_{222} \end{vmatrix} = \left| s_{1jk} \; s_{2jk} \right| \tag{3.4}$$

$$K = \begin{vmatrix} k_{1111} & k_{1112} \\ k_{1121} & k_{1122} \end{vmatrix} \begin{vmatrix} k_{1211} & k_{1212} \\ k_{1221} & k_{1222} \end{vmatrix} \begin{vmatrix} k_{2111} & k_{2112} \\ k_{2121} & k_{2122} \end{vmatrix} \begin{vmatrix} k_{2211} & k_{2212} \\ k_{2221} & k_{2222} \end{vmatrix} = \tag{3.5}$$

$$|k_{11kl}\ k_{12kl}|k_{21kl}\ k_{22kl}|$$

For a given portfolio weight vector w, skewness $s_p \in \mathbb{R}^{n\times n^2}$, and kurtosis $k_p \in \mathbb{R}^{n\times n^3}$ of the portfolio, respectively, are given by:

$$s_p = w'S(w \otimes w) \tag{3.6}$$

$$k_p = w'K(w \otimes w \otimes w) \tag{3.7}$$

where ⊗ is the Kronecker product. Even though some authors suggested a numerical solution to this problem, others proposed quasi-analytical solutions using Polynomial Goal Programming (PGP) (Lai et al., 2006; Mhiri and Prigent, 2010; Škrinjarić, 2013; Proelss and Schweizer, 2014) and employing multi-objective portfolio optimization (Chen y Zhou, 2018; Mukesh et al., 2021; Noraveshcand Kerstens, 2022; Xidonas and Essner, 2022). Following Jondeau and Rockinger (2006) and Chen and Zhou (2018), the portfolio optimization problem based on the MVSK framework can be expressed by:

$$\min_{\{w\}} \{-\lambda_1\mu'w + \lambda_2 w'\Sigma w - \lambda_3 w'S(w \otimes w) + \lambda_4 w'K(w \otimes w \otimes w)\} \qquad s.t.\ w'\mathbf{1} = 1,\ w \in \mathcal{W} \tag{3.8}$$

Where, λ_1, λ_2, λ_3, $\lambda_4 \geq 0$ are the parameters for combining the four moments of the portfolio return. Additionally, as Chen y Zhou (2018) and Abid et al (2023) argued, this optimization problem implies that the investor’s decision is to maximize expected returns and skewness as well as minimize the variance and kurtosis. In line with Jondeau and Rockinger (2003, 2005, 2006) , Martellini and Ziemann (2010) and Boudt, Lu & Peeters (2015) the investor´s preference implies maximizing the expected value of the fourth-order expansion of the

(CRRA)[4] utility function with parameters $\lambda_{1,2,3,4}$, which, according to Zhou and Palomar (2021) are given by $\lambda_1 = 1$, $\lambda_2 = \frac{\zeta}{2}, \lambda_3 = \frac{\zeta(\zeta+1)}{6}$, and $\lambda_4 = \frac{\zeta(\zeta+1)(\zeta+2)}{24}$. However, the solution is highly nonlinear in the vector w, but it can be easily solved using nonlinear optimization routines by applying numerical methods. In this sense, we have adopted the algorithm and routines proposed by Zhou and Palomar (2021).

Zhou and Palomar (2021) suggested successive convex approximation (SCA) algorithms given the difficulty due to the non-convexity and high computational cost.[5] Also, the authors proposed a method based on the difference of convex (DC) algorithm to approximate non-convex functions, and they used solvers to obtain the solutions to the original high-order portfolio optimization problem. They transformed the problem described in Equation (3.8) into a convex QP problem by introducing a variable $u \in \mathbb{R}^{n\times 1}$.

$$\min_{\{w\}} \left\{\frac{\tau}{2} w'w - w'\left(\tau w^i - \nabla f(w^i)\right)\right\} \qquad \begin{aligned} \text{s.t. } & -u \le w \le u, \\ & u'\mathbf{1} \le l \\ & w'\mathbf{1} = 1, \end{aligned} \tag{3.9}$$

$\nabla f(w^i) = -\lambda_1 \nabla\psi_1(w^i) + \lambda_2 \nabla\psi_2(w^i) - \lambda_3 \nabla\psi_3(w^i) + \lambda_4 \nabla\psi_4(w^i)$, $\nabla\psi_j(w^i)$ with $i = 1,2,3,4$, which refer to the gradient of the four moments and w^i denotes the current iterate of w at i-th iteration and l is the leverage constraint. Additionally, if ψ_i: $D \rightarrow \mathbb{R}$ is a convex function of class C^2 defined on a convex set D of $\mathbb{R}^n$, then the spectral radius of the Hessian matrix denoted by $\rho(\nabla^2\psi(x))$ is bounded by D to guarantee convex approximation. We set the decomposition for ψ_i as $\frac{\tau}{2} w'w - w'\left(\tau w^i - \nabla f(w^i)\right)$, where $\frac{\tau}{2} w'w$ *and* $w'\left(\tau w^i -\right.$

[4] In Appendix D (Section 3.7.4), we described in detail the CRRA utility function and their properties.

[5] According to Zhou and Palomar (2021), the SCA algorithm is a general framework for solving non-convex optimization problems as a sequence of inner convex approximating problems.

$\nabla f(w^i))$ are convex on D, and τ is a constant verifying $\tau \geq \max_{\{w \in D\}} \rho(\nabla^2 \psi(x))$ to guarantee the strong convexity of the approximating function. This algorithm can be very efficiently solved using a QP solver. Following this approach, Zhou and Palomar (2021) implemented the MVSK portfolio.

3.3.3 MV-ESG Portfolio

It is straightforward to extend the MV framework in order to incorporate ESG metrics into the portfolio optimization problem. Following to Utz et al. (2014) and Gasser et al. (2017), the MV-ESG optimal portfolio with a given ESG metric $\mathcal{E}_{\mathrm{p}} = w'e$, for the quadratic utility function is given by:

$$\min_{\{\mathrm{w}\}} \{-w'\mu + \zeta\, w'\Sigma\, w - \gamma\, w'e\} \qquad \begin{aligned} s.t.\ & w'\mathbf{1} = 1, \\ & w \in \mathcal{W}. \end{aligned} \tag{3.10}$$

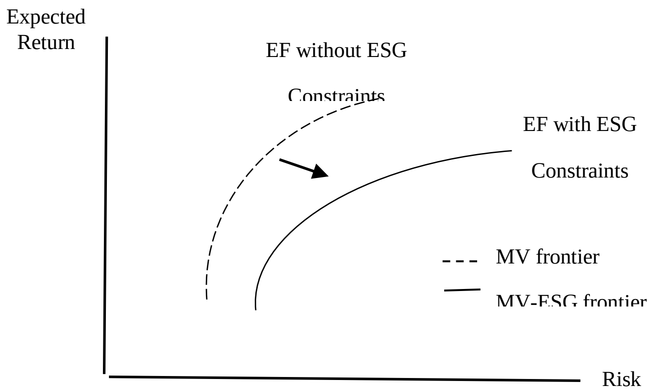
Applying the classical MV formulation with ESG metrics from assets, the optimization is given by:

$$\min_{\{w\}} \{w'\Sigma\, w\} \qquad \begin{aligned} s.t.\ & w'\mu = \mu_p, \\ & w'e = \mathcal{E}_p, \\ & w'\mathbf{1} = 1, \\ & w \in \mathcal{W} \end{aligned} \tag{3.11}$$

where, e is a vector of ESG scores from assets, and $\mathcal{E}_{\mathrm{p}}$ is the ESG target from the optimal portfolio. According to Blitz et al. (2023) and Steuer and Utz (2023) equations 3.10 and 3.11 construct portfolios on an efficient frontier (EF) surface in three dimensions. This implies that the EF is a function of the ESG indicator objective; however, the EF with and without ESG constraints must be differentiated. They demonstrated the effects of incorporating objective

ESG scores into the optimization problem. Similarly, Méndez-Rodríguez et al. (2013) found a rightward shift in the EF, which implies a lower degree of diversification or a higher level of risk, as shown in figure 3.1a.

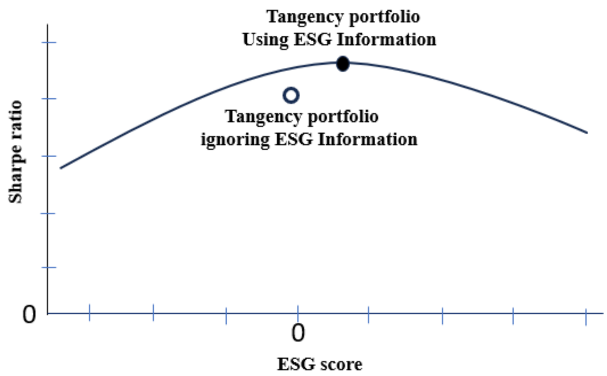
Figure 3.1.a. MV-ESG efficient portfolios.



Source: Based on Méndez-Rodríguez et al (2013) and Bell and Van Vuuren (2022).

Pedersen et al. (2021) provided an economic explanation for the shift in the efficiency frontier. They argued that the higher the ESG score of a security, the higher the demand from ESG investors, the lower the expected return, the different the expected future profits, which can increase the expected return if the market underreacts to this predictability of fundamentals; and the stronger the flows from investors, which can increase the price in the short run. In addition, a downward shift in the ESG SR frontier is observed when realistic constraints are imposed on the portfolio. This result is expected because the imposition of constraints reduces the maximum Sharpe ratio that can be achieved for a given ESG score. Along the same lines, Bell and Van Vuuren (2022) argue that as the ESG score of the portfolio increases, returns deteriorate and risk or volatility increases.

Figure 3.1.b. ESG-Sharpe Frontier



Source: Pedersen et al. (2021)

Figure 3.1b shows the ESG Sharpe frontier, which is the maximum Sharpe ratio that can be achieved for all portfolios with a given ESG score. The peak of the ESG Sharpe frontier is the Sharpe ratio (SR) of the standard tangent portfolio. Investors concerned with both SR and ESG should select a frontier portfolio to the right of this portfolio, on the ESG efficient frontier. The investor calculates the maximum Sharpe ratio as the slope of the line from the risk-free security to the tangent portfolio, based only on portfolios with that ESG level.

3.3.4. MV-ESG Portfolio with Higher-Order Moments

Based on the portfolio optimization using the utility function, we reformulated the original problem of the MVSK portfolio, as described in equation 3.8, to incorporate the ESG scores of the assets and the optimal portfolio. To achieve this, we reformulated the original non-convex function to include a five-objective into the optimization algorithm.

$$\min_{\{w\}} \{-\lambda_1 \mu' w + \lambda_2 w' \Sigma w - \lambda_3 w' S (w \otimes w) + \lambda_4 w' K (w \otimes w \otimes w) - \lambda_5 w' e\} \quad s.t.\ w'e = \mathcal{E}_p,\ w'\mathbf{1} = 1,\ w \in \mathcal{W} \qquad (3.12)$$

To solve the problem in equation 3.12, we use the same DC algorithm proposed by Zhou and Palomar (2021). In that sense, we combine multi-objectives into the non-convex optimization problem defined in equation 3.12. The optimization problem can be written as follows:

$$\min_{\{w\}} \left\{\frac{\tau}{2} w'w - w'\left(\tau w^{i} - \nabla f(w^{i})\right)\right\} \qquad \begin{aligned} s.t. \quad & w'e = \mathcal{E}_p, \\ & -u \leq w \leq u, \\ & w'\mathbf{1} = 1, \\ & u'\mathbf{1} \leq l \end{aligned} \qquad (3.13)$$

where $\nabla f(\mathrm{w}^{i})$ also includes ESG risk preferences given by $-\lambda_5 \nabla \psi_5(\mathrm{w}^{i})$. In that sense, the MVSK-ESG portfolio can be implemented like the MVSK portfolio where the ESG constraint is included into the DC algorithm.

3.4. Numerical example and results

3.4.1. Data

The proposed model is implemented for the United States stock market. We selected the DJIA and NASDAQ100 indices for empirical testing and validation. The DJIA includes the 30 most important and representative industrial companies listed in the United States, and the NASDAQ100 comprises the 100 most important and representative technology companies listed on the Nasdaq Stock Market. We used monthly returns on assets and indices from January 2010 to December 2023. All data was downloaded from the Refinitiv database. Additionally, we adopted the ESG scores[6] from the ESG Refinitiv database, which is one of the most comprehensive ESG databases in the global market.

[6] On the other hand, for the model implementation, we used two different types of normalization methods to adjust ESG scores: i) the min-max feature scaling measure defined as $\frac{ESG_i - min(ESG)}{Max(ESG) - min(ESG)}$, and; ii) the standard ESG score, given by $\frac{ESG_i - mean(ESG)}{Stad.\ Dev.\ (ESG)}$. The portfolio model results remain the same under both the original ESG scores and the normalized ESG scores.

Table 3.1 summarizes the descriptive statistics for the monthly data (mean or expected return, standard deviation or volatility, skewness, and kurtosis) of the selected companies with the highest ESG scores. For the sake of comparison, we only consider ESG scores that are equal to or greater than the threshold 80 (ESG score ≥ 80) for the DJIA index and equal to or greater than the threshold 74 (ESG score ≥ 74) for the NASDAQ100 index. We have found the following leading companies.

Table 3.1. Descriptive statistics of assets and ESG scores.

DJIA market					
	Mean	**Volatility**	**Skewness**	**Kurtosis**	**ESG Score**
CAT	0.012	0.0845	-0.0312	3.461	82.95
CSCO	0.0067	0.0734	-0.1809	3.092	82.81
HON	0.0121	0.0583	-0.0092	4.4739	81.64
INTC	0.0079	0.0759	-0.1775	3.9454	85.44
JNJ	0.0077	0.044	-0.1194	3.2361	87.18
KO	0.0069	0.0456	-0.7877	4.5518	82.26
MMM	0.0042	0.0592	-0.5083	3.0381	90.26
MSFT	0.0166	0.0619	-0.1078	3.1687	88.46
UNH	0.0182	0.0547	0.0423	2.9876	80.06
WBA	0.0005	0.083	0.436	3.738	90.15
DJIA	**0.0076**	**0.0419**	**-0.3539**	**4.022**	**-**

Note: Caterpillar (CAT), Cisco (CSCO), Honeywell (HON), Intel (INTC), Johnson & Johnson (JNJ), Coca Cola (KO), Minnesota Mining and Manufacturing, Microsoft (MSFT), United Health (UNH), Walgreens (WBA), Dow Jones Industrial Average (DJIA). The analyzed period ranged between January 1, 2010, and December 31, 2023, observed monthly (4394 observations). Source: own elaboration.

NASDAQ market					
	Mean	**Volatility**	**Skewness**	**Kurtosis**	**ESG Score**
AAPL	0.0203	0.0775	-0.257	2.6714	79.64
ADP	0.0129	0.054	-0.2605	3.6971	75.76
ADSK	0.0135	0.0993	-0.1461	2.7301	80.72
AMAT	0.016	0.0944	-0.076	3.0164	77.82
AMGN	0.0116	0.0632	-0.037	3.1069	76.02
AMZN	0.0186	0.0856	-0.0391	3.5485	75.38
AVGO	0.0264	0.0812	-0.0726	3.3641	74.01
AZN	0.0097	0.0599	0.14	3.2443	95.51
BIIB	0.0094	0.0912	0.1397	4.6702	77.80
BKR	0.0028	0.1188	-0.5091	4.2881	87.60
CCEP	0.0131	0.0804	-0.0293	8.222	84.44
CDNS	0.0227	0.0718	0.0657	3.2378	75.07
CSCO	0.0067	0.0734	-0.1809	3.092	82.82
MDLZ	0.0102	0.0667	-0.7382	18.5532	79.96
MSFT	0.0166	0.0619	-0.1078	3.1687	88.43
NVDA	0.0283	0.1272	-0.1923	3.8398	75.00
PEP	0.0086	0.041	-0.0993	2.9736	85.92
TXN	0.0133	0.0643	-0.2071	2.7743	77.64
WBA	0.0005	0.083	0.436	3.738	90.16
XEL	0.0092	0.0464	-0.4627	3.2216	81.88
NDX	**0.0112**	**0.0504**	**-0.3801**	**3.2189**	**-**

Note: Apple (AAPL), Automatic Data Processing (ADP), Autodesk (ADSK), Applied Materials (AMAT), Amgen (AMGN), Amazon (AMZN), Broadcom (AVGO), AstraZeneca (AZN), Biogen (BIIB), Baker Hughes Company (BKR), Coca-Cola Euro pacific Partners PLC (CCEP), Cadence Design Systems, Inc. (CDNS), Cisco Systems (CSCO), Mondelez International (MDLZ), Microsoft (MSFT), NVIDIA Corporation (NVDA), PepsiCo (PEP), Texas Instruments Incorporated (TXN), Walgreens Boots Alliance (WBA), Xcel Energy Inc. (XEL), Nasdaq 100 Index (NDX) .The analyzed period ranged between January 1, 2010 and December 31, 2023, observed at monthly basis (8619 observations). Source: own elaboration.

To select leading companies, or companies with the highest ESG scores, we define the 60th percentile as a threshold. Although we use the 60^{th} percentile of ESG scores as the initial threshold, we provide comparisons for different thresholds ranging from 30% to 80%. In this way, those companies with an ESG score above this threshold for both the DJIA and the NASDAQ100 markets were selected. Furthermore, this selection excluded those companies that did not have complete information for the period analyzed.

On the other hand, as shown in Table 3.1, the skewness and kurtosis statistics confirm the presence of bias and heavy tails in the distribution of the returns of the selected assets. Failure to comply with the assumption of normality in the distribution of returns can generate strong limitations in optimal portfolios. In this way, an optimization approach that incorporates these high moments of the distributions is necessary.

3.4.2. Empirical implementation and Comparisons

To test the proposed model, we used two different procedures, considering in-sample and out-of-sample evaluation methods, where optimal portfolios taking only long positions ($w \geq 0$) are constructed for all approaches (MV, MV-ESG, MVSK, and MVSK-ESG).[7] In the first procedure, we used the full sample period (2010-2023), while in the second one, we implemented a rolling sample approach proposed by DeMiguel and Nogales (2009). Additionally, we set a target ESG score for the optimization algorithm equal to the average ESG score from all considered assets. For each optimal portfolio, we report its composition (optimal assets weights) as well as portfolio performance measures such as expected return (mean), risk (volatility), and risk-reward measures such as the Sharpe ratio and the Rachev ratio.

[7] All models are implemented using R software version 4.3. We used packages like: quadprog, moments and highOrderPortfolios.

Although the Sharpe ratio is the most used risk-reward ratio in portfolio performance evaluation, we also used the Rachev ratio (RR), which is consistent with the proposed approach based on high-order moments, as suggested by Rachev et al. (2008) and Campbell et al. (2010). The Rachev ratio (RR) is a risk-reward ratio introduced by Biglova et al. (2004), and it is calculated based on the conditional value at risk (CVaR). The RR is defined as the average of the quantiles of the portfolio return distribution that are above a threshold (or a target quantile), and the risk measure is CVaR at a given tail probability, i.e., RR is the ratio of the expected tail return (ETR) divided by the expected tail loss (ETL).

$$RR_{\alpha,\beta} = \frac{CVaR_{\alpha}(r_p)}{CVaR_{\beta}(r_p)} \tag{3.14}$$

where the tail probability α defines the quantile level and β is the tail probability of CVaR[8]. Additionally, the ETR or the numerator represents a measure of reward which is given by:

$$CVaR_{\alpha} = \alpha^{-1} \int_{0}^{\alpha} F^{-1}(p)dp \tag{3.15}$$

Additionally, following Gasser et al. (2017) and Chen et al. (2021), we implemented another important metric to assess ESG performance, known as Delta ratio (DR), as a risk adjusted ESG score.

$$DR = \frac{ESG\ score}{\sigma_p} \tag{3.16}$$

According to Gasser et al. (2017), the DR relates the ESG measure and risk in the same way that the Sharpe Ratio relates return and risk. In that sense, a portfolio with a high Delta ratio has a higher ESG value. Finally, following Zhou and Palomar (2021), we settled the same

[8] The probability α is often called lower tail probability and β is known as upper tail probability.

parameters and initial values for portfolio weights, and we used the DC algorithm to guarantee the strong convexity to approximate the objective function and its convergence; on the order hand, we calculated the risk aversion parameter[9] as ζ.

The DC algorithm to solve the optimization problem for MVSK and MVSK-ESG portfolios is implemented in the following way:[10]

DC algorithm for the MVSK and the MVSK-ESG portfolios

1: Calculate the high order moments using generalized hyperbolic multivariate skew-t distribution (ghMST)[10]

2: Initialize $w^{o} \in \mathcal{W}$ and compute τ

3: For iteration $i = 0,1,2,3, \ldots do$

4: Calculate $\nabla f(w^{i})$

5: Solve the problem described in equations 3.9 and 3.13, and update w^{i+1}

6: Find the optimal portfolio if the DC algorithm converges (Terminate loop)

7: End for

Algorithm 1: DC method following Zhou and Palomar (2021).

For both the MV-ESG and MVSK-ESG portfolios, we got an ESG target score of 85.12 for DJIA, and 81.08 for NASDQ100. ESG scores are equal to the average score of the assets in each stock market. Table 3.2 reports the results for all the optimal portfolios and benchmarks, including return (mean), risk (volatility), and risk-reward measures, considering all sample period. Even though the MVSK portfolios show the higher expected return and risk, respectively; the MVSK-ESG portfolio achieves the best results for CVaR and risk-reward

[9] The ζ parameter was calculated based on Boudt, Lu, and Peeters (2015). In that sense, we set this parameter to 7.5.

[10] In Appendix D (Section 3.7.4), we described all the pseudo-codes for more details.

[10] In Appendix E (Section 3.7.5), we described in detail the ghMST distribution.

ratios like the Sharpe Ratio, and the Rachev Ratio for both markets, where the MVSK-ESG portfolio is ranked as the best over all the sample periods. Furthermore, for the Delta ratio, the MVSK-ESG portfolio achieves the best results.

Table 3.2. Results for the MV, the MVSK and the MVSK-ESG portfolios

DJIA market | ESG score: 85.12					
	MV	**MV-ESG**	**MVSK**	**MVSK-ESG**	**DJIA**
Mean	0.0104	0.0101	**0.0175**	0.0159	0.0076
Volatility	**0.0359**	0.0363	0.0452	0.0363	0.0419
CVaR	-0.0636	-0.0648	-0.0758	**-0.0591**	-0.0788
Sharpe Ratio	0.2889	0.2792	0.3864	**0.4364**	0.1825
Rachev Ratio	0.1017	0.0956	0.1657	**0.2016**	0.0393
Delta Ratio	23.41	**23.43**	18.43	**23.43**	**-**
NASDAQ market | ESG score: 81.08					
	MV	**MV-ESG**	**MVSK**	**MVSK-ESG**	**Nasdaq**
Mean	0.0107	0.011	**0.016**	0.0156	0.0112
Volatility	**0.0328**	0.033	0.0385	0.0331	0.0504
CVaR	-0.057	-0.0571	-0.0635	**-0.0528**	-0.0927
Sharpe Ratio	0.3248	0.3333	0.4148	**0.4704**	0.2231
Rachev Ratio	0.1244	0.13	0.1858	**0.2273**	0.0622
Delta Ratio	**25.2304**	24.5367	20.585	24.4702	-

Source: own elaboration.

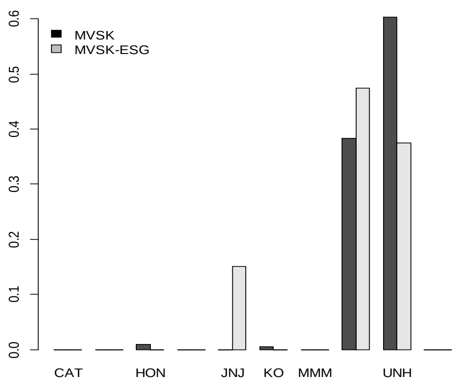
Furthermore, it is important to highlight that the MVSK-ESG portfolio achieves better diversification than its MVSK counterpart for both the DJIA and the NASDAQ100 markets, as shown in Figure 3.2. The improvements in diversification can be verified in the lower volatility and the CVaR risk measures for the MVSK-ESG portfolio, which is likely to result in a more balanced allocation across assets. However, the result is below that of the traditional MV portfolio (Figure 3.2 b), which is consistent with previous work and MPT literature, although it should be noted that the MVSK-ESG portfolio reaches a lower degree of concentration compared to the MV-ESG portfolio.

As both portfolios, the MV-ESG and the MVSK-ESG were built with the average ESG scores of the assets, we also show the EF of each market with different ESG targets (EF-ESG), to show the effect that each EF would have, especially the lowest diversification obtained when a higher ESG score is imposed in the optimization process. Figure 3.3 shows the ESG frontiers under an ESG-risk relation.

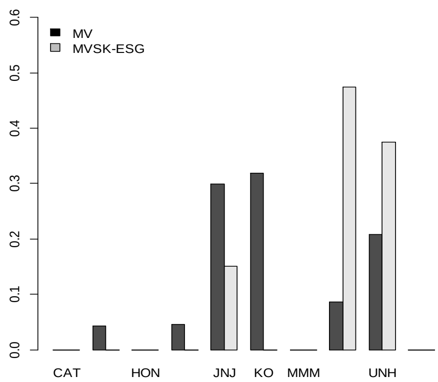
Figure 3.2. Optimal weights for MV, MVSK and MVSK-ESG portfolios

DJIA market

a. MVSK vs. MVSK-ESG

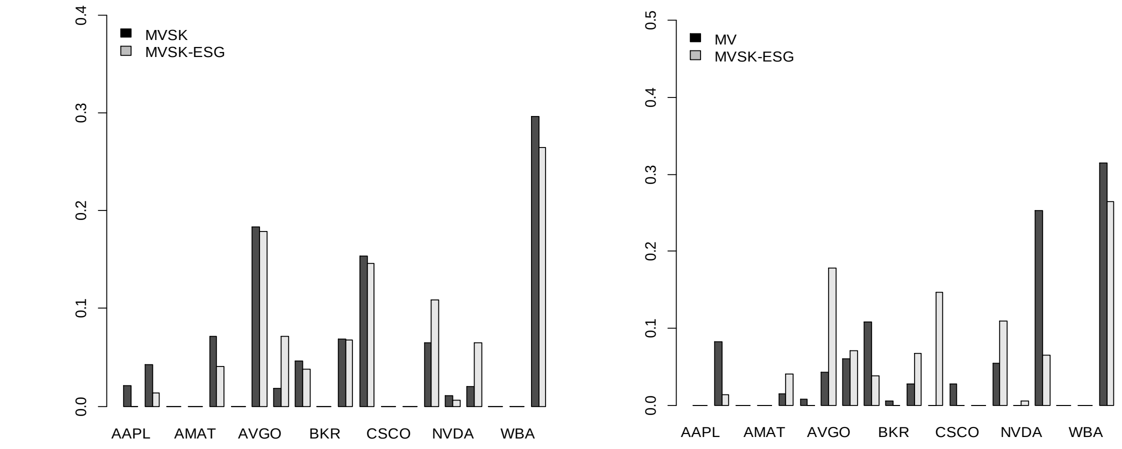


b. MV vs. MVSK-ESG



NASDAQ100 market

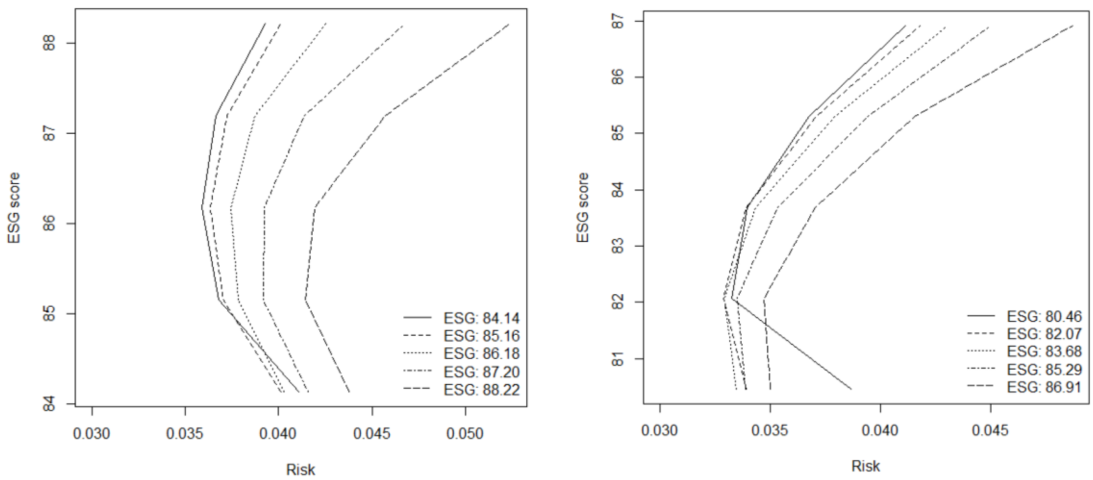
a. MVSK vs. MVSK-ESG b. MV vs. MVSK-ESG



Source: own elaboration.

Figure 3.3. EF-ESG for DJIA and NASDAQ markets

a. FE-ESG DJIA b. FE-ESG Nasdaq

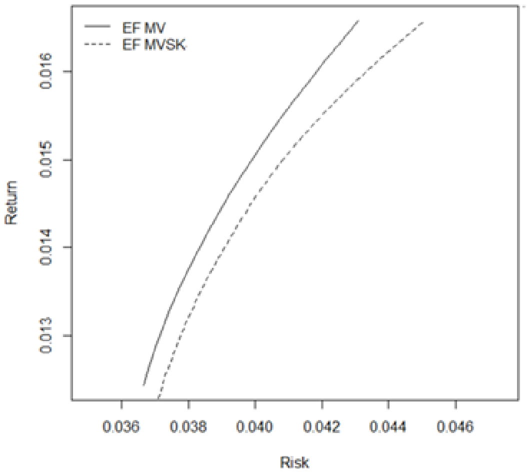


Source: own elaboration.

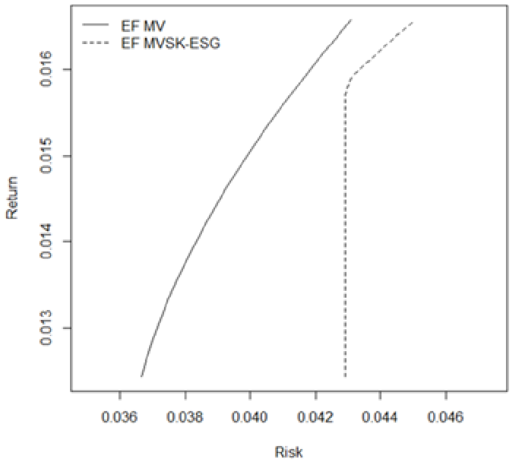
Figure 3.4. Risk-Return EF for DJIA and NASDAQ markets

DJIA market

a. EF for MV and MVSK

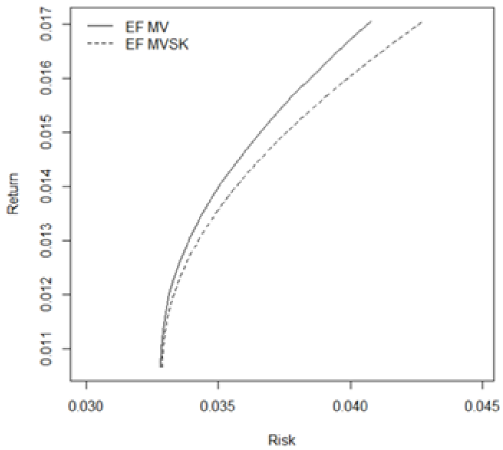


b. EF for MV and MVSK-ESG

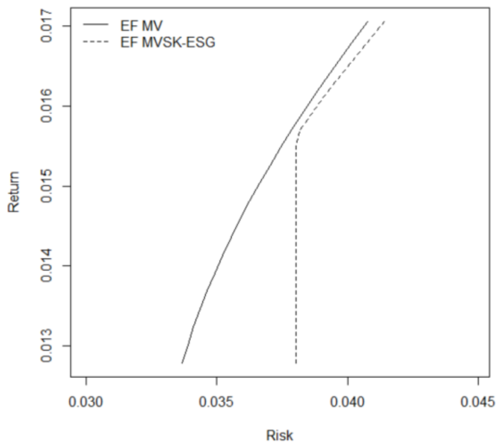


NASDAQ100 market

a. EF for MV and MVSK



b. EF for MV and MVSK-ESG



Source: own elaboration.

On the other hand, Figures 3.4a and 3.4b for the Dow and Nasdaq markets confirm the correct EF shift effect when ESG scores are introduced into the optimal portfolio. In both cases, the MVSK and the MVSK-ESG portfolios are less diversified than their MV counterparts, respectively. Furthermore, the lower part of the EF MVSK-ESG shows a lower degree of adjustment of the risk measure of these portfolios compared to the EF MV.

Finally, we implemented a rolling sample approach proposed by DeMiguel and Nogales (2009) to compute the performance evaluation measures, where each window has a length of 60 months for the in-sample period and 12 months for the out-of-sample period, respectively. The moving window is one year in each iteration. For example, we used data from 2010 to 2014 to compute the statistics for all assets and then we optimized their weights. The optimal portfolio is evaluated in 2015 using different performance measures. This procedure is repeated until 2023. For each rolling sample period we calculated all parameters as means, covariances and ESG scores for all assets. Table 3.3 shows the results for the average calculations of all the measures used in the performance evaluation of the portfolios (Appendices A and B in Section 3.7. provide detailed calculations for each year).

Table 3.3. Average measures calculated in the rolling window method.

DJIA market					
	MV	**MV-ESG**	**MVSK**	**MVSK-ESG**	**DJIA**
Mean	0.0092	0.009	**0.0141**	0.0140	0.0064
Volatility	0.0401	**0.0397**	0.047	0.0468	0.0468
CVaR	-0.0736	**-0.0729**	-0.0829	-0.0825	-0.0825
Sharpe Ratio	0.301	0.2947	0.3439	**0.3463**	0.2416
Rachev Ratio	0.1733	0.1658	0.1724	**0.1754**	0.1511
Delta Ratio	22.1267	**22.8021**	17.2664	17.5459	-

NASDAQ market					
	MV	**MV-ESG**	**MVSK**	**MVSK-ESG**	**Nasdaq**
Mean	0.0089	0.0092	0.0109	**0.0115**	0.0089
Volatility	0.038	**0.0379**	0.0429	0.0423	0.0423
CVaR	-0.0694	**-0.0689**	-0.0775	-0.0757	-0.0757
Sharpe Ratio	0.2568	0.2677	0.2917	**0.3079**	0.2981
Rachev Ratio	0.1008	0.108	0.119	0.1285	**0.246**
Delta Ratio	20.399	**20.7117**	16.7804	17.024	-

Source: own elaboration.

For the DJIA market, the MVSK-ESG portfolio has the best risk-return ratios (Sharpe and Rachev ratios) and the lowest volatility. For the NASDAQ100 market, the MVSK-ESG portfolio has the best Sharpe ratio and the highest average return. However, it is worth noting that during the Covid-19 pandemic period (2020-2021), the MVSK-ESG portfolio achieved the best results in all the measures evaluated for the DJIA market, although it achieved the lowest risk measures, such as volatility and CVaR, for the NASDAQ100 market (see Appendix A, Section 3.7.1).

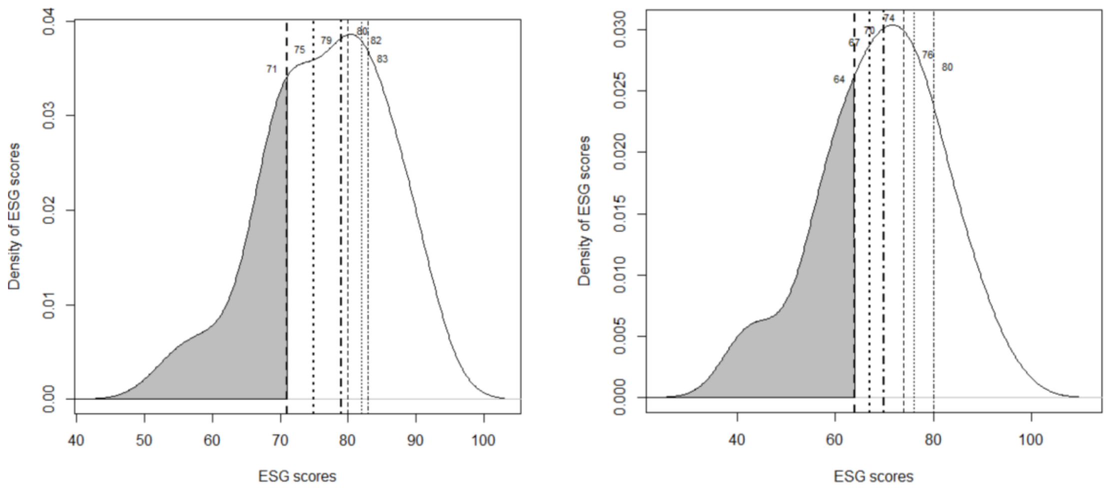
Based on the risk-rewards ratios (Sharpe and Rachev ratios), the MVSK and the MVSK-ESG portfolios are more robust and suitable, especially compared to their traditional counterparts (both the MV model and the MV-ESG model), given that the risk-reward ratios are better than those of the MV and the MV-ESG portfolios. In the next section, we provide additional tests to check the consistency of the proposed approach and compare the results for different thresholds that we used to select the set of leading companies based on their ESG scores.

3.4.3. Comparisons for different thresholds

In this section, we performed additional tests to check the consistency and robustness of the proposed approach for the MVSK-ESG portfolio construction and compare the results by taking different thresholds. In this way, we define thresholds ranging from 30% to 80%, and select the set of leading companies for both the DJIA and the NASDAQ100 markets, according to their ESG scores, and considering the whole period. On this basis, we demonstrated that the results for the MVSK-ESG portfolio are maintained. Figure 3.5 shows the set of companies that are excluded as the threshold increases.

Figure 3.5. Exclusions of companies under different thresholds.

a. DJIA market b. NASDAQ100 market



Source: own elaboration.

By defining each threshold, for example 30%, the set of companies that are below the minimum score are excluded (grey area in Figure 3.4), while the companies that exceed this threshold are selected to carry out the optimization and construction of optimal portfolios. As the threshold is increased, the resulting number of companies becomes increasingly lower. As a result, we calculate all the portfolio measures (expected returns, volatility, CVaR, and Sharpe, Rachev and Delta ratios) used to compare their performance. Tables 3.4 and 3.5 show the results for both markets.

Table 3.4. Comparison for different thresholds in the DJIA Index

	MV	MV-ESG	MVSK	MVSK-ESG	DJIA
Threshold:	***0.3***	***# Assets:***	***17***	***Min ESG:***	***71***
Mean	0.0093	0.0097	**0.0168**	0.0162	0.0076
Volatility	**0.0323**	0.033	0.0412	0.0331	0.0419
CVaR	-0.0572	-0.0583	-0.0682	**-0.0521**	-0.0788
Sharpe	0.2881	0.2954	0.4074	**0.4894**	0.1825
Rachev	0.1011	0.1057	0.1805	**0.2421**	0.0393
Delta	24.3703	24.7206	19.6878	**24.7791**	-
Threshold:	***0.4***	***# Assets:***	***15***	***Min ESG:***	***75***
Mean	0.0093	0.0097	**0.0168**	0.0162	0.0076
Volatility	**0.0323**	0.033	0.0412	0.0331	0.0419
CVaR	-0.0572	-0.0583	-0.0682	**-0.0521**	-0.0788
Sharpe	0.2881	0.2954	0.4074	**0.4894**	0.1825
Rachev	0.1011	0.1057	0.1805	**0.2421**	0.0393
Delta	24.3703	24.7206	19.6878	**24.7791**	-
Threshold:	***0.5***	***# Assets:***	***12***	***Min ESG:***	***79***
Mean	0.01	0.01	**0.0175**	0.0166	0.0076
Volatility	**0.0345**	0.035	0.0436	0.035	0.0419
CVaR	-0.0612	-0.0621	-0.0725	**-0.0555**	-0.0788
Sharpe	0.2905	0.2855	0.4012	**0.4741**	0.1825
Rachev	0.1026	0.0995	0.1762	**0.2301**	0.0393
Delta	24.0559	**24.0928**	18.8294	**24.0928**	-
Threshold:	***0.6***	***# Assets:***	***10***	***Min ESG:***	***80***
Mean	0.0104	0.0101	**0.0175**	0.0159	0.0076
Volatility	**0.0359**	0.0363	0.0452	0.0363	0.0419
CVaR	-0.0636	-0.0648	-0.0758	**-0.0591**	-0.0788

Sharpe Ratio	0.2889	0.2792	0.3864	**0.4364**	0.1825
Rachev Ratio	0.1017	0.0956	0.1657	**0.2016**	0.0393
Delta Ratio	23.41	**23.43**	18.43	**23.43**	-
Threshold:	***0.7***	***# Assets:***	***8***	***Min ESG:***	***82***
Mean	0.0083	0.0085	**0.013**	**0.013**	0.0076
Volatility	**0.0374**	0.0378	0.0457	0.0383	0.0419
CVaR	-0.0689	-0.0695	-0.0812	**-0.0661**	-0.0788
Sharpe	0.2209	0.2256	0.2851	**0.3388**	0.1825
Rachev	0.061	0.0637	0.0993	**0.1335**	0.0393
Delta	22.7507	22.7749	19.0932	**22.813**	-
Threshold:	***0.8***	***# Assets:***	***5***	***Min ESG:***	***83***
Mean	0.0083	0.0087	0.0133	**0.0146**	0.0076
Volatility	**0.0406**	0.0414	0.047	0.0414	0.0419
CVaR	-0.0755	-0.0767	-0.0837	**-0.0708**	-0.0788
Sharpe	0.2034	0.211	0.2831	**0.3533**	0.1825
Rachev	0.051	0.0553	0.0981	**0.1431**	0.0393
Delta	**21.5892**	21.3052	18.7108	21.3052	-

Source: own elaboration.

Table 3.5. Comparison for different thresholds in the NASDAQ Index

	MV	**MV-ESG**	**MVSK**	**MVSK-ESG**	**Nasdaq**
Threshold:	***0.3***	***# Assets:***	***32***	***Min ESG:***	***64***
Mean	0.0107	0.011	**0.016**	0.0158	0.0112
Volatility	**0.0328**	0.0332	0.0386	0.0333	0.0504
CVaR	-0.057	-0.0575	-0.0636	**-0.0529**	-0.0927
Sharpe	0.3248	0.3324	0.4152	**0.4745**	0.2231
Rachev	0.1244	0.1293	0.1861	**0.2304**	0.0622
Delta	**25.2304**	24.1985	20.5682	24.1434	-
Threshold:	***0.4***	***# Assets:***	***31***	***Min ESG:***	***67***

Mean	0.0113	0.0119	**0.0175**	0.0174	0.0112
Volatility	**0.0324**	0.0333	0.0391	0.0334	0.0504
CVaR	-0.0555	-0.0567	-0.0631	**-0.0515**	-0.0927
Sharpe	0.3498	0.3567	0.4474	**0.5208**	0.2231
Rachev	0.1408	0.1454	0.2097	**0.2674**	0.0622
Delta	**25.2313**	23.179	19.6569	23.0852	-
Threshold:	***0.5***	***# Assets:***	***24***	***Min ESG:***	***70***
Mean	0.0112	0.0116	**0.017**	0.0167	0.0112
Volatility	**0.0324**	0.0334	0.0386	0.033	0.0504
CVaR	-0.0558	-0.0564	-0.0626	**-0.0514**	-0.0927
Sharpe	0.344	0.3532	0.4402	**0.5069**	0.2231
Rachev	0.137	0.1431	0.2044	**0.256**	0.0622
Delta	**25.3691**	24.1407	20.446	24.086	-
Threshold:	***0.6***	***# Assets:***	***20***	***Min ESG:***	***74***
Mean	0.0107	0.011	**0.016**	0.0156	0.0112
Volatility	**0.0328**	0.0335	0.0385	0.0331	0.0504
CVaR	-0.057	-0.0571	-0.0635	**-0.0528**	-0.0927
Sharpe Ratio	0.3248	0.3333	0.4148	**0.4704**	0.2231
Rachev Ratio	0.1244	0.13	0.1858	**0.2273**	0.0622
Delta Ratio	**25.2304**	24.5367	20.585	24.4702	-
Threshold:	***0.7***	***# Assets:***	***15***	***Min ESG:***	***76***
Mean	0.0099	0.0099	**0.0131**	0.0123	0.0112
Volatility	**0.0332**	**0.0332**	0.0367	**0.0332**	0.0504
CVaR	-0.0585	-0.0585	-0.0626	**-0.0528**	-0.0927
Sharpe	0.2982	0.2994	0.3561	**0.3696**	0.2231

Rachev	0.1075	0.1082	0.1451	**0.1643**	0.0622
Delta	**25.2439**	25.0289	22.6175	25.0953	-
Threshold:	***0.8***	***# Assets:***	***9***	***Min ESG:***	***80***
Mean	0.0097	0.0098	**0.0127**	**0.0127**	0.0112
Volatility	**0.035**	**0.035**	0.0388	**0.035**	0.0504
CVaR	-0.0624	-0.0625	-0.0675	**-0.0596**	-0.0927
Sharpe	0.2774	0.2783	0.3257	**0.3612**	0.2231
Rachev	0.0946	0.0951	0.125	**0.1485**	0.0622
Delta	24.546	24.6567	22.2232	**24.6567**	-

Source: own elaboration.

Results for both markets are once again satisfactory, since the MVSK-ESG portfolio continues to have the best Sharpe and Rachev ratios and the lowest CVaR risk measure. Furthermore, for the DJIA market, the MVSK-ESG portfolio also has the best Delta ratio for almost all thresholds.

3.5. Discussion

Incorporating ESG considerations into portfolio selection is a complex issue, with academics and practitioners disagreeing on its impact on performance. Some argue that ESG considerations inevitably lead to lower expected returns, as suggested by Hartzmark and Sussman (2019), Henriksson et al. (2019), and Pedersen et al. (2021), while others, such as Abate et al. (2021), Chen et al. (2021), and Blitz et al. (2023), argue that the outperformance of ESG strategies is undeniable, as ESG scores can have a significant impact on performance even after controlling for stock fundamentals, and investors can improve their decision-making by using both financial and ESG analysis.

The results for MV-ESG and MVSK-ESG in this chapter are consistent with the first set of empirical applications and confirm that introducing higher order moments such as skewness and kurtosis into the portfolio optimization problem leads to lower expected returns. However, we find important implications for diversification, as MVSK-ESG portfolios are more concentrated compared to their traditional counterparts, especially when ESG scores are introduced into the optimization problem by excluding assets with scores below the specified threshold. In both cases, a rightward shift of the EF is observed, suggesting that a socially responsible investor concerned about environmental and climate change issues should be willing to sacrifice some of his expected return to invest in the companies with the best ESG scores, considering skewness and kurtosis. While this effect is present in the MVSK-ESG portfolio, we find that this portfolio still maintains good risk-return ratios, such as Sharpe ratio, Rachev ratio and Delta ratio, and even outperforms the MV and MV-ESG portfolios. Thus, an investor can fulfil this new 'fiduciary duty', as stated by Hartzmark and Sussman (2019) and Coqueret (2022), by prioritizing the allocation of his capital to companies with better ESG scores.

Although diversification is an important issue and has implications for portfolio managers to consider, it should not be as important in creating optimal investments with companies with better ESG scores. Even the MV model tends to create concentrated optimal portfolios by significantly overweighting assets with high expected returns and low or negative correlations.

Finally, we are concerned about the limitations of the proposed approach, particularly the use of ESG scores from a single data provider, given the advantages that one scoring methodology may have over another. Therefore, we are not exempt from the problem of greenwashing, as some companies in the sample may overestimate their environmental impact. However, given the scope of the work, we do not address this issue, and this will be the focus of future research.

3.6. Conclusions

In this chapter, we have proposed a portfolio selection model that considers ESG scores under high-order moments. The proposed approach consists of a multi-objective optimization model focused on ESG scores and high-order moments of asset returns. Considering leader companies, –i.e. companies with the highest ESG scores for different thresholds, for the DJIA and the NASDAQ100 of the United States stock market, as well as performance measures such as the Sharpe ratio, the Rachev ratio, and the Delta ratio, we implemented the MV, MVSK, MV-ESG, and MVSK-ESG portfolios using a difference of convex (DC) algorithm to solve high-order optimal portfolios and QP optimization solvers, which are easily and computationally efficient.

The results for both the DJIA and the NASDAQ100 markets are satisfactory and confirm the benefits of the MVSK-ESG approach to build optimal portfolios. In general, the MVSK-ESG portfolios achieved the best risk-reward ratios (Sharpe and Rachev ratios) and the lowest volatility and CVaR risk measures for the entire 2014-2023 period, as well as for several years within the out-of-sample analysis period based on the rolling sample approach.

We provided evidence on the effect of asset exclusion considering the minimum ESG score defined as threshold. This exercise allows us to build sets of leading companies for different markets, considering the availability of companies´ ESG information. This application can be extended to different stock markets, which helps to address the issue of selection bias, as well as using different risk measures such as CVaR, in order to implement a Mean-CVaR-ESG approach. Furthermore, much more efficient optimization methods and algorithms can be used, without forgetting their quantitative and computational complexity. In addition, the proposed methodology can be extended to include different ESG data providers such as Bloomberg, S&P

Global Ratings, Sustainalytics, MSCI ESG Ratings, among others, to consider the heterogeneity and controversies between their methodologies.

3.7. Appendix

3.7.1 Appendix A. Portfolio Metrics for the DJIA Index

Returns

	MV	MV-ESG	MVSK	MVSK-ESG	Index
2015	0.0039	0.0035	**0.0101**	**0.0101**	-0.0018
2016	0.0124	0.0121	**0.0206**	**0.0206**	0.0084
2017	0.0198	0.0206	**0.0263**	**0.0263**	0.0197
2018	0.0041	0.0035	**0.0053**	**0.0053**	-0.0031
2019	0.0075	0.0075	**0.0091**	**0.0091**	0.0085
2020	0.0087	0.0115	**0.0204**	**0.0204**	0.0067
2021	0.0198	0.0194	0.0289	**0.0296**	0.0157
2022	**0.0045**	0.0012	-0.0009	-0.0048	-0.003
2023	0.0019	0.002	0.0067	**0.0096**	0.0066

Volatility

	MV	MV-ESG	MVSK	MVSK-ESG	Index
2015	**0.0331**	0.0343	0.0344	0.0344	0.0344
2016	**0.0242**	0.0244	0.0321	0.0321	0.0321
2017	**0.0173**	0.0184	0.0292	0.0292	0.0292
2018	0.0435	**0.043**	0.0557	0.0557	0.0557
2019	**0.0393**	0.0443	0.0655	0.0655	0.0655
2020	0.0622	**0.058**	0.0586	0.0586	0.0586
2021	0.0538	**0.0464**	0.0558	0.0554	0.0554
2022	0.0504	**0.0501**	0.0534	0.0526	0.0526
2023	**0.0374**	0.0386	0.0385	0.0377	0.0377

CVaR					
	MV	**MV-ESG**	**MVSK**	**MVSK-ESG**	**Index**
2015	**-0.0644**	-0.0672	-0.0609	-0.0609	-0.0609
2016	**-0.0374**	-0.0383	-0.0456	-0.0456	-0.0456
2017	**-0.0159**	-0.0174	-0.0338	-0.0338	-0.0338
2018	-0.0856	**-0.0851**	-0.1097	-0.1097	-0.1097
2019	**-0.0735**	-0.084	-0.1259	-0.1259	-0.1259
2020	-0.1196	-0.1081	**-0.1005**	**-0.1005**	**-0.1005**
2021	-0.0912	**-0.0764**	-0.0862	-0.0846	-0.0846
2022	**-0.0995**	-0.1021	-0.1111	-0.1134	-0.1134
2023	-0.0752	-0.0776	-0.0727	**-0.0683**	**-0.0683**

Sharpe Ratio					
	MV	**MV-ESG**	**MVSK**	**MVSK-ESG**	**Index**
2015	0.1188	0.1008	**0.2937**	**0.2937**	-0.0466
2016	0.5139	0.494	**0.6412**	**0.6412**	0.2646
2017	1.143	1.1173	0.9025	0.9025	**1.1869**
2018	0.0947	0.0808	**0.0949**	**0.0949**	-0.073
2019	**0.1904**	0.1683	0.1395	0.1395	0.1835
2020	0.1397	0.1986	**0.3477**	**0.3477**	0.0889
2021	0.3685	0.4177	0.5185	**0.534**	0.4579
2022	**0.0893**	0.0238	-0.016	-0.0903	-0.0474
2023	0.0505	0.0509	0.173	**0.2532**	0.1599

Rachev Ratio					
	MV	**MV-ESG**	**MVSK**	**MVSK-ESG**	**Index**
2015	0.0053	-0.0039	**0.1046**	**0.1046**	-0.0736
2016	0.2617	0.2457	**0.3747**	**0.3747**	0.0868
2017	1.1247	1.0669	0.6844	0.6844	**1.2313**
2018	**-0.007**	-0.014	**-0.007**	**-0.007**	-0.085
2019	**0.0437**	0.0315	0.0161	0.0161	0.0399

2020	0.0162	0.0483	**0.1394**	**0.1394**	-0.0099
2021	0.1534	0.1879	0.2655	**0.2783**	0.2177
2022	**-0.0098**	-0.0416	-0.0599	-0.0924	-0.0739
2023	-0.0288	-0.0286	0.0341	**0.0799**	0.027

Delta Ratio

	MV	MV-ESG	MVSK	MVSK-ESG
2015	21.946	**22.0355**	17.8045	17.8044
2016	30.1754	**31.9623**	20.8176	20.8177
2017	**45.4635**	43.5897	27.0525	27.0525
2018	17.454	**18.3719**	13.7292	13.7293
2019	**19.5175**	18.2704	12.3613	12.3613
2020	11.7883	**14.0783**	13.1181	13.1181
2021	14.5395	**17.9106**	13.9786	14.2118
2022	16.3579	**16.9233**	15.2246	15.9948
2023	21.8981	22.0767	21.3108	**22.8234**

3.7.2. Appendix B. Portfolio Metrics for the NASDAQ Index

Returns

	MV	MV-ESG	MVSK	MVSK-ESG	Index
2015	0.0071	0.0075	0.0082	**0.0115**	0.0034
2016	0.0085	**0.0094**	0.0085	0.0092	0.004
2017	0.0188	0.0186	0.023	**0.0227**	0.02
2018	0.0007	0.0006	0.0012	**0.0016**	-0.0027
2019	**0.0159**	**0.0159**	**0.0159**	0.0156	0.0155
2020	0.0094	0.0097	**0.0181**	0.0167	0.0305
2021	0.0136	0.014	0.0178	0.0179	**0.0191**
2022	0.007	**0.0078**	-0.0059	-0.0044	-0.0304
2023	-0.0007	-0.0005	0.0116	**0.0125**	0.0207

Volatility

	MV	MV-ESG	MVSK	MVSK-ESG	Index
2015	0.0316	**0.0315**	0.0387	0.0399	0.0399
2016	0.0319	**0.0307**	0.0331	0.033	0.033
2017	0.0252	**0.0245**	0.0348	0.0344	0.0344
2018	**0.038**	0.0369	0.0383	0.0391	0.0391
2019	**0.0352**	**0.0352**	0.0394	0.0392	0.0392
2020	**0.0495**	0.0501	0.0537	0.052	0.052
2021	0.0444	0.0441	0.0393	**0.0387**	**0.0387**
2022	**0.0524**	0.0544	0.0754	0.0714	0.0714
2023	0.0333	0.0334	0.033	**0.0328**	**0.0328**

CVaR

	MV	MV-ESG	MVSK	MVSK-ESG	Index
2015	-0.0582	**-0.0574**	-0.0717	-0.0708	-0.0708
2016	-0.0574	**-0.0539**	-0.0598	-0.0589	-0.0589
2017	-0.0332	**-0.0321**	-0.0488	-0.0482	-0.0482
2018	-0.0776	**-0.0756**	-0.0778	-0.079	-0.079
2019	**-0.0567**	**-0.0567**	-0.0654	-0.0653	-0.0653
2020	-0.0929	-0.0937	-0.0927	**-0.0905**	**-0.0905**
2021	-0.078	-0.0769	-0.0634	**-0.0619**	**-0.0619**
2022	**-0.1011**	-0.1044	-0.1615	-0.1516	-0.1516
2023	-0.0694	-0.0694	-0.0565	**-0.0551**	**-0.0551**

Sharpe Ratio

	MV	MV-ESG	MVSK	MVSK-ESG	Index
2015	0.224	0.2384	0.2123	**0.2886**	0.0795
2016	0.265	**0.3069**	0.2576	0.2798	0.1002
2017	0.7447	0.7566	0.6612	0.6607	**1.3649**
2018	0.0193	0.0153	0.0316	**0.0407**	-0.0528
2019	**0.4513**	0.4511	0.4029	0.397	0.2802

2020	0.1887	0.1927	0.3365	0.322	**0.4147**
2021	0.3067	0.3187	0.4519	0.4623	**0.5593**
2022	0.1332	**0.1435**	-0.0788	-0.0612	-0.4038
2023	-0.0221	-0.0139	0.3501	**0.3809**	0.3409

Rachev Ratio

	MV	MV-ESG	MVSK	MVSK-ESG	Index
2015	0.0628	0.0712	0.056	**0.1015**	-0.0147
2016	0.087	**0.1129**	0.0826	0.0961	-0.0043
2017	0.4826	0.4962	0.3943	0.3938	**1.8004**
2018	-0.0437	-0.0455	-0.0379	**-0.0336**	-0.0763
2019	**0.2127**	0.2125	0.1773	0.1732	0.0963
2020	0.0428	0.045	0.132	0.1226	**0.1858**
2021	0.1128	0.1205	0.2131	0.2211	**0.2998**
2022	0.0128	**0.0182**	-0.0875	-0.0799	-0.2077
2023	-0.0627	-0.059	0.141	**0.1619**	0.1349

Delta Ratio

	MV	MV-ESG	MVSK	MVSK-ESG
2015	**21.8959**	21.1835	14.7996	14.1752
2016	**21.6645**	21.3914	17.2378	17.2735
2017	25.7689	**29.4889**	17.9149	18.1094
2018	18.9902	**19.614**	16.4456	16.0387
2019	20.5632	**21.1035**	16.5317	16.6043
2020	15.2017	**16.0185**	13.8464	14.2969
2021	18.512	18.2494	19.9707	**20.4725**
2022	**15.9757**	14.8059	10.1313	11.2733
2023	**25.0188**	24.5507	24.1456	24.9723

3.7.3. Appendix C. Utility function details

The utility function CRRA has a constant relative risk aversion function equal to $\boldsymbol{\zeta}$.

$$U(R) = \frac{1}{1-\zeta} R^{1-\zeta}$$

This function is translation invariant; we can write it as follows:

$$U(W) = \frac{1}{1-\zeta} R^{1-\zeta} = exp\left\{\frac{\log(R)(1-\zeta)}{1-\zeta}\right\}$$

That tends to Log(R) when $\zeta \rightarrow 1$. As for the negative exponential utility function and before determining the corresponding risk measure of the power utility function, we must check if this function satisfies the four following properties:

$$\begin{cases} U^{(1)} = R^{-\zeta} > 0 \\ U^{(2)} = -\zeta R^{-\zeta-1} < 0 \\ U^{(3)} = \zeta(1+\zeta) R^{-\zeta-2} > 0 \\ U^{(4)} = -\zeta(1+\zeta)(2+\zeta) R^{-\zeta-3} > 0 \end{cases}$$

Similarly, if $\zeta = 1$, we have $U^{(1)} > 0, U^{(2)} < 0, U^{(3)} > 0, U^{(4)} < 0$

The Taylor approximation truncated to the fourth order applied to the expected utility of this function implies:

$$\begin{cases} \mathbb{E}\left[\frac{1}{1-\zeta} R^{1-\zeta}\right] \simeq (1-\zeta)^{-1} \mathbb{E}(R)^{(1-\zeta)} - \frac{\gamma}{2} \mathbb{E}(R)^{-(1+\zeta)} \sigma^2 \\ \qquad + \frac{\zeta(1+\zeta)}{3!} \mathbb{E}(R)^{-(\zeta+2)} \lambda_3 - \frac{\zeta(1+\zeta)(2+\zeta)}{4!} \mathbb{E}(R)^{-(\zeta+3)} \lambda_4 \\ \mathbb{E}[\log(R)] \simeq \log(\mathbb{E}(R)) - \frac{1}{2} \sigma^2 \mathbb{E}(R)^{-2} + \frac{2}{3!} \lambda_3 \mathbb{E}(R)^{-3} - \frac{6}{4!} \lambda_4 \mathbb{E}(R)^{-4} \end{cases}$$

Maximizing this quantity is equivalent to minimizing the risk defined as follows:

$$\begin{cases} \mathcal{R}_{iso(\zeta\neq 1)} = \sigma^2 - \frac{(1+\zeta)}{3}\mathbb{E}(R)^{-1}\left[\lambda_3 - \frac{(2+\zeta)}{4}\lambda_4\mathbb{E}(R)^{-1}\right] \\ \mathcal{R}_{iso(\zeta=1)} = \sigma^2 - \frac{2}{3}\lambda_3\mathbb{E}(R)^{-1} + \frac{1}{2}\lambda_4\mathbb{E}(R)^{-2} \end{cases}$$

As for the risk measure of the previous function, this measure is also a linear combination of variance, skewness, and kurtosis. We also note that the weights of the moments depend, this time, on the relative risk aversion (ζ) and the final wealth mean.

3.7.4. Appendix D. Pseudo-codes for Optimal Portfolios

Algorithm for MV portfolios

0: Identify the objective function: $\min_{\{w\}} \{-\mu' w + \zeta w' \Sigma w\}$ and constrain: $s.t.\ w'\mathbf{1} = 1$

1: Calculate the first two sample moments of stock returns: mean (μ) and the covariance matrix (Σ).

2. Calculate the portfolio return μ_p: $\mu' w$ and the portfolio variance: σ_P^2: $w'\Sigma\, w$

3: Found the optimal portfolio using MV procedure to obtain optimal w_i using "quadprog" with one equality constraint.

Algorithm for MV-ESG portfolios

0: Identify the objective function: $\min_{\{w\}} \{w'\Sigma\, w\}$ and

constrains: $s.t. w'\mu = \mu_p,\ \ w'e = \mathcal{E}_p,\ \ w'\mathbf{1} = 1,$

1: Calculate the first two sample moments of stock returns: mean (μ) and the covariance matrix (Σ), as well as the average ESG scores (e) ≥ 80 for DJIA and ≥ 74 for NASDAQ100.

2. Calculate the portfolio return μ_p: $\mu' w,$ the portfolio variance: σ_P^2: $w'\Sigma\, w,$ and the Portfolio ESG score given by $\mathcal{E}_{p,mean} = w'e.$

3: Found the optimal portfolio using MV-ESG procedure in order to obtain optimal w_i using “quadprog” with constraints.

DC algorithm for MVSK portfolios

0: Identify the objective function: $\min_{\{w\}} \{-\lambda_1 \mu' w + \lambda_2 w' \Sigma w - \lambda_3 w' S(w \otimes w) + \lambda_4 w' K(w \otimes w \otimes w)\}$ and constrain: $s.t.\ w'\mathbf{1} = 1$

1: Calculate the high order moments using generalized hyperbolic multivariate skew-t distribution.

2: Initialize $w^o \in \mathcal{W}$ and compute τ

3: For iteration $i = 0,1,2,3, \dots do$

4: Calculate $\nabla f(w^i)$

5: Solve the problem described in equations 3.12 and 3.13, and update w^{i+1}

6: Found the optimal portfolio if the DC algorithm converges (Terminate loop)

7: End for

DC algorithm for MVSK-ESG portfolios

0: Identify the objective function: $\min_{\{w\}} \{-\lambda_1 \mu' w + \lambda_2 w' \Sigma w - \lambda_3 w' S(w \otimes w) + \lambda_4 w' K(w \otimes w \otimes w) - \lambda_5 w' e\}$ and constrain: $s.t.\ w'\mathbf{1} = 1, w'e = \mathcal{E}_p$.

1: Calculate the four sample moments of stock returns moments such as mean, variance, skewness, and kurtosis, as well as the average ESG scores (e) ≥ 80 for DJIA and ≥ 74 for NASDAQ100.

2: Initialize $w^o \in \mathcal{W}$ *and compute* τ

3: For iteration $i = 0,1,2,3, \dots do$

Calculate $\nabla f(w^i)$

Solve the problem described in equations 16 and 17, and update w^{i+1}

Found the optimal portfolio if the DC algorithm converges (Terminate loop)

End for

3.7.5. Appendix E. ghMST Distribution

According to Wang, Zhou and Palomar (2023), the generalized hyperbolic multivariate skew-*t* (ghMST) distribution, is a sub-class of the generalized hyperbolic (GH) distribution, which is often used in economics to model the data with skewness and heavy tails. They argued that a variable X follows a *d*-dimensional skewed-*t* distribution, i.e. $\mathrm{X} \sim \mathrm{ghMST}(\mu, \Omega, \gamma, \upsilon)$, the probability density function (pdf) is given by:

$$f_{\mathrm{ghMST}}\ (\mathrm{x}|\mu, \Omega, \gamma, \upsilon) = c_{ST} \in \frac{K_{\frac{\upsilon+d}{2}}\left(\sqrt{(\upsilon+Q(x))\gamma^T\Omega^{-1}\gamma}\right)e^{(X-\mu)^T\ \Omega^{-1}\gamma}}{\sqrt{(\upsilon+Q(x))\ \gamma^T\ \Omega^{-1}\gamma}^{-\frac{\upsilon+d}{2}}\ (1+Q(x)/\upsilon)^{\frac{\upsilon+d}{2}}},$$

where $Q(\mathbf{x})=(x-\mu)^T\Omega^{-1}(x-\mu)$, and the normalizing constant C_{ST} is

$$C_{ST} = \frac{2^{1-\frac{\upsilon+d}{2}}}{\Gamma\left(\frac{\upsilon}{2}\right)(\pi\upsilon)^{\frac{d}{2}}|\Omega|^{\frac{1}{2}}},$$

where $\upsilon \in \mathbb{R}_{++}$ is the degree of freedom, μ $\mathbb{R}^N$ is the location vector, $\gamma \in \mathbb{R}^N$ is the skewness vector, $\Omega \in \mathbb{R}^{NXN}$ is the scatter matrix, Γ is the gamma function, and K_λ is the modified Bessel function of the second kind with index λ. Additionally, by assuming a random vector $\mathbf{r} \sim \mathrm{ghMST}(\mu, \Omega, \gamma, \upsilon)$, then the mean and covariance of **r**are given as follows:

$$\mathbb{E}[\mathbf{r}] = \mu + a_1\gamma,$$

$$Cov[\mathbf{r}] = a_{21}\Omega + a_{22}\gamma\gamma^{\mathrm{T}}$$

Where, $a_1 = \frac{\nu}{\nu-2}, a_{21} = \frac{\nu}{\nu-2}$ $, and\ a_{22} = \frac{2\nu^2}{(\upsilon-2)^2(\upsilon-4)}$ are scalar coefficients decided by ν. The third moment co-skewness matrix **Φ** is expressed as:

$$\Phi_{i,(j-1)xN+k} = a_{31}\gamma_i\gamma_j\gamma_k + \frac{a_{32}}{3}(\gamma_i\Omega_{jk} + \gamma_j\Omega_{ik} + \gamma_k\Omega_{ij}),$$

The fourth moment co-kurtosis matrix **Ψ** is expressed as:

$$\Psi_{i,(j-1)N^2+(K-1)N+l}$$

$$=a_{41}\gamma_i\gamma_j\gamma_k\gamma_l + \frac{a_{42}}{6}\underbrace{(\Omega_{ij}\gamma_k\gamma_l + \cdots + \Omega_{kl}\gamma_i\gamma_j)}_{\text{6 items}}+\frac{a_{43}}{3}(\Omega_{ij}\Omega_{kl} + \Omega_{ik}\Omega_{jl} + \Omega_{il}\Omega_{jk})$$

Here $a_{31} = \frac{16\upsilon^3}{(\upsilon-2)^3(\upsilon-4)(\upsilon-6)}$ $a_{32} = \frac{6\upsilon^2}{(\upsilon-2)^2(\upsilon-4)}$ $a_{41} = \frac{(12\upsilon+120)\upsilon^4}{(\upsilon-2)^4(\upsilon-4)(\upsilon-6)(\upsilon-8)}$

$a_{42} = \frac{6(2\upsilon+4)\upsilon^3}{(\upsilon-2)^3(\upsilon-4)(\upsilon-6)}$, $a_{43} = \frac{3\upsilon^2}{(\upsilon-2)(\upsilon-4)}$ are coefficients determined by υ.

CONCLUSIONS

This document makes a significant contribution by incorporating higher moments into the analysis of financial assets. This is demonstrated in the first chapter, where higher moments are applied to trading strategies. In the second chapter, a modified variance measure was proposed based on a Gram Charlier expansion on asset return density and a modified variance measure that were derived through Taylor expansions of the risk measures based on the investor's exponential utility. Finally, in the third chapter, a portfolio model is proposed under higher moments and ESG scores.

Despite the theoretical fact that stock markets follow a random walk process, numerous empirical studies have demonstrated the inefficiencies of financial markets by employing technical strategies. These studies include Balsara et al. (2009), Mitra & Rohit (2020), Agapova & Kaprielyan (2020), and Jain et al. (2020). As previously demonstrated in the first chapter, the integration of two confidence intervals with Edgeworth expansions can be achieved as follows: Hall (1983) incorporated skewness and Edgeworth expansions, while Hall & Jing (1995) incorporated skewness and kurtosis. Both studies implemented two trading strategies: a trend-following strategy with N = 20, 50, and 200 days, and a contrarian trading strategy with N = 6, 10, 15, and 20 days. This strategy demonstrated superior performance in the median of risk and risk-return indicators over the classical Bollinger bands. The Edgeworth expansions demonstrate superior median performance for a range of indicators, including VaR (for N = 20, 50 and 200 days), median tracking error (for N = 20 and 50 days), and median Treynor ratio (for N = 200 days), when compared to the traditional Bollinger Bands approach in technical analysis across the entire sample period. It was identified that the annual and cumulative returns of the trend-following strategy exhibited losses in both the entire sample and the crisis periods, as well as for the traditional Bollinger bands.

However, it should be noted that the second trading strategy: contrarian trading strategy provides better median performance for indicators such as cumulative return, annual return, Sharpe ratio, tracking error, omega, and information ratio (when $N = 6$ and 10 days) were found to outperform traditional Bollinger bands (BB) for the entire sample period. The statistical overperformance of our proposed strategy based on ABB1 and ABB2 is supported when compared to the BB. In the same manner, in high uncertainty scenarios (crisis studied in this chapter), the median cumulative return, annual return, Omega, and tracking error with our proposal (for $N = 6$ and 10 days) perform better.

It is therefore concluded that a disadvantage of using longer periods is that the strategy may not incorporate recent price changes and could result in exclusions of relevant trading signals in the case of the trend following strategy, which was implemented with long periods ($N = 20$, $N = 50$, $N = 200$). Therefore, the importance of high order moments in the trading bands decreases as the period increases. As the periods become shorter, as in the case of the contrarian trading strategy, both cumulative and annual returns are positive and outperform the classical Bollinger bands at the median, as well as risk and risk-return indicators. This indicates that in shorter periods, these proposals effectively capture the excess of kurtosis and skewness. Hence, the recommendation is to consider their application in intraday trading schemes or even high-frequency trading, incorporating higher moments as there is no trading scheme with high moments in the practitioners' world until today and our contribution may be novel for the investment industry.

Future research could be focused on implementing our proposal of contrarian trading strategy reviewing the performance measures with other trading methodologies for shorter periods such as squeeze trading strategy and trend following and including transaction costs. In contrast with the traditional portfolio theory, which is based on the mean-variance approach, contemporary approaches have been developed with the aim of measuring risk when the utility function is not quadratic and asset returns do not have a Gaussian distribution. In order to achieve this goal, higher moments of asset returns have to be considered. In this line, the second chapter, incorporates the higher moments (asymmetry and kurtosis) into the investor's risk attitudes to review their effect on portfolio optimization. The proposal consists of two risk measures framed in an exponential utility function: modified variance applying Taylor's expansion and with Gram Charlier returns. These will be compared in their performance with the proposal of behavioral finance proposed by Davies & Servigny (2012) in its architecture with higher moments explained in this second chapter. The optimal results were achieved with the proposed modified variance when returns were distributed according to the Gram-Charlier model and for the three different risk tolerance parameters. Consequently, for an investor, a practitioner, or a portfolio manager, the proposed modified variance, according to the Gram-Charlier return distribution, represents the optimal choice among the behavioral variance measures.

Further research may be conducted to analyze the modified variance with other types of financial assets and distinct performance metrics. This may also involve investigating the impact of positive transformations of Gram-Charlier expansions, as proposed by Ñíguez and Perote (2012).

In Chapter three the results confirm the benefits of the MVSK-ESG approach to build optimal portfolios. In general, the MVSK-ESG portfolios achieved the best risk-reward ratios (Sharpe and Rachev ratios) and the lowest volatility and CVaR risk measures for the entire 2014-2023 period, as well as for several years within the out-of-sample analysis period based on the rolling sample approach. We provided evidence on the effect of asset exclusion considering the minimum ESG score defined as threshold. This exercise allows us to build sets of leading companies for different markets, considering the availability of companies' ESG information.

Future research could be directed along four lines: first, the use of alternative risk measures such as CVaR to implement a Mean-CVaR-ESG approach; second, the implementation of the methodology proposed in Chapter three with other more efficient nonlinear optimization methods; third, the detailed study of the phenomenon of greenwashing, the incorporation of ESG variables in the asset selection process to design a portfolio; MVSK-ESG becomes more difficult for investors when companies greenwash, i.e. disclose misleading ESG information; and fourth, given the lack of uniformity in the methodologies used by ESG data providers, the proposal to construct and optimize ESG portfolios incorporating higher moments can be extended to each ESG data provider, the efficiency of our proposal with each provider's approach can then be measured.

CONCLUSIONES

Este documento aporta una contribución significativa al incorporar los momentos superiores al análisis de los activos financieros. Esto se demuestra en el primer capítulo, donde los momentos superiores se aplican a dos estrategias de negociación utilizando un indicador ampliamente usado en el mundo del análisis técnico: las bandas de Bollinger. En el segundo capítulo, se propone una medida de varianza modificada basada en una expansión de Gram Charlier sobre la rentabilidad de los activos y una medida de varianza modificada que se deriva mediante expansiones de Taylor de las medidas de riesgo basadas en una función de utilidad exponencial para el inversionista. Por último, en el tercer capítulo, se propone un modelo de optimización de cartera incorporando momentos superiores y puntuaciones ESG.

A pesar del hecho teórico de que los mercados bursátiles siguen un proceso de paseo aleatorio, numerosos estudios empíricos han demostrado las ineficiencias de los mercados financieros mediante el empleo de estrategias de Análisis técnico. Entre estos estudios se encuentran los de Balsara et al. (2009), Mitra & Rohit (2020), Agapova & Kaprielyan (2020), y Jain et al. (2020) . Como se demostró anteriormente en el primer capítulo, la integración de dos intervalos de confianza con expansiones de Edgeworth puede lograrse del siguiente modo: Hall (1983) incorporó la asimetría (ABB1), mientras que Hall & Jing (1995) incorporaron la asimetría y la curtosis (ABB2). En ambos estudios se aplicaron dos estrategias de negociación utilizando bandas de Bollinger: una estrategia de seguimiento de la tendencia *trend following* con $N = 20$, 50 y 200 días, y una estrategia de negociación contraria *contrarian strategy* con $N = 6$, 10, 15 y 20 días. Esta última estrategia (contraria) demostró un rendimiento superior en la mediana de los indicadores de riesgo y rentabilidad con respecto a las bandas de Bollinger clásicas.

La propuesta con expansiones de Edgeworth (ABB1 y ABB2) presentaron un comportamiento medio superior para una serie de indicadores, incluido el VaR (para N = 20, 50 y 200 días), la mediana del error de seguimiento (para N = 20 y 50 días) y la mediana del ratio de Treynor (para N = 200 días), en comparación con el enfoque tradicional de las bandas de Bollinger (BB) en todo el periodo de la muestra, sin embargo, se observó que los rendimientos anuales y acumulados con la estrategia de seguimiento de la tendencia presentaban pérdidas tanto en la totalidad de la muestra como en los periodos de crisis, e igual comportamiento de pérdidas con las bandas de Bollinger tradicionales.

Analizando el comportamiento de la segunda estrategia de negociación: contraria, los resultados mostraron un mejor rendimiento medio para indicadores como la rentabilidad acumulada, la rentabilidad anual, la ratio de Sharpe, el error de seguimiento, el omega y la ratio de información (con N = 6 y 10 días) superando a las bandas de Bollinger (BB) tradicionales durante todo el periodo de muestra en estas ratios de desempeño. El mejor rendimiento de nuestra estrategia propuesta basada en ABB1 y ABB2 se confirma cuando se compara con el enfoque tradicional de las bandas de Bollinger (BB). De la misma manera, con esta estrategia de negociación contraria, en escenarios de alta incertidumbre (crisis estudiadas en este capítulo), tanto la mediana de la rentabilidad acumulada como la rentabilidad anual, la ratio omega y ratio de error de seguimiento son mejores con nuestra propuesta (para N = 6 y 10 días).

Por lo tanto, se concluye que una desventaja de utilizar periodos más largos es que la estrategia puede no incorporar los cambios recientes en los precios y podría dar lugar a la exclusión de señales de negociación relevantes en el caso de la estrategia de seguimiento de la tendencia en la cual se usaron periodos largos (N = 20, 50, 200), por lo tanto, la importancia de los momentos de alto orden en las bandas de negociación disminuye a medida que aumenta el periodo.

A medida que los periodos se acortan, como en el caso de la estrategia de negociación contraria, tanto la rentabilidad acumulada como la anual son positivas y superan a las bandas de Bollinger clásicas en la mediana, así como a los indicadores de riesgo y de riesgo-rentabilidad, esto indica que, como se anotaba anteriormente, en periodos más cortos, nuestras propuestas, ABB1 y ABB2, captan eficazmente el exceso de curtosis y asimetría, al obtener mejor rentabilidad mejores indicadores de desempeño sobre las bandas de Bollinger tradicionales. Por lo tanto, la recomendación es considerar su aplicación en esquemas de negociación intradía o incluso de alta frecuencia, incorporando los momentos más altos ya que hasta hoy no existe ningún esquema de negociación que incorpore los momentos altos en el mundo de los profesionales de industria, además nuestra contribución puede ser novedosa para la industria de la inversión. La investigación futura podría centrarse en aplicar nuestra propuesta de estrategia de negociación contraria revisando las medidas de desempeño para el rendimiento, el riesgo y el rendimiento-riesgo con otras metodologías de negociación para periodos más cortos, como la estrategia de negociación: compresión *squeeze* y la estrategia de seguimiento de la tendencia (*trend following*), e incluyendo los costes de transacción.

A diferencia de la teoría tradicional de carteras, que se basa en el enfoque media-varianza, se han desarrollado enfoques contemporáneos con el objetivo de medir el riesgo cuando la función de utilidad no es cuadrática y los rendimientos de los activos no tienen una distribución gaussiana. Para lograr este objetivo, hay que considerar los momentos superiores de los rendimientos de los activos. En esta línea, el segundo capítulo, incorpora los momentos superiores (asimetría y curtosis) a las actitudes de riesgo del inversor para medir su efecto en la optimización de la cartera. La propuesta consiste en dos medidas de riesgo enmarcadas en una función de utilidad exponencial: varianza modificada aplicando expansiones de Taylor y varianza modificada a los rendimientos utilizando una distribución Gram-Charlier. Estas dos medidas son comparadas en su desempeño con la propuesta de finanzas conductuales propuesta por Davies & Servigny (2012) con momentos superiores explicada en el segundo capítulo de este documento. Los resultados óptimos se obtuvieron con la varianza modificada propuesta cuando los rendimientos se distribuían según el modelo de Gram-Charlier y para los tres parámetros diferentes de tolerancia al riesgo. Por consiguiente, para un inversor, un profesional o un gestor de carteras, la varianza modificada propuesta, según la distribución de rentabilidad de Gram-Charlier, representa la elección óptima entre las medidas de varianza conductual.

Pueden realizarse más investigaciones para analizar la varianza modificada con otros tipos de activos financieros y distintas métricas de rendimiento. Esto también puede implicar investigar el impacto de las transformaciones positivas de las expansiones de Gram-Charlier, como proponen Ñíguez y Perote (2012).

En el capítulo tres, los resultados confirman las ventajas del enfoque "Mean, Variance, Skewness, Kutosis - Envioronmental, Social and Governance" (MVSK-ESG, por sus siglas en inglés) para construir carteras óptimas. En general, de acuerdo con los resultados obtenidos en nuestro estudio, las carteras MVSK-ESG lograron las mejores ratios de riesgo-rendimiento (ratios de Sharpe y Rachev) y las medidas de riesgo de volatilidad y CVaR más bajas para todo el periodo 2014-2023, así como para varios años dentro del periodo de análisis y fuera de muestra basado en el enfoque de ventana móvil.

Aportamos pruebas sobre el efecto de la exclusión de activos considerando la puntuación ESG mínima definida como umbral. Este ejercicio nos permite construir conjuntos de empresas líderes para diferentes mercados, teniendo en cuenta la disponibilidad de información ESG de las empresas.

La investigación futura podría orientarse en torno a cuatro líneas: en primer lugar, el uso de medidas de riesgo alternativas, como el CVaR, para aplicar un enfoque Mean-CVaR-ESG, en segundo lugar, la aplicación de la metodología propuesta en el capítulo tres con otros métodos de optimización no lineal más eficientes, en tercer lugar, el estudio detallado del fenómeno del *greenwashing*, (práctica de marketing verde destinada a crear una imagen ilusoria de responsabilidad ecológica) en la incorporación de variables ESG ya que el proceso de selección de activos para construir y optimizar una cartera MVSK-ESG se hace más difícil para los inversores cuando las empresas divulgan información ESG engañosa y en cuarto lugar, dada la falta de uniformidad en las metodologías utilizadas por los proveedores de datos ESG, la propuesta de construir y optimizar carteras ESG incorporando momentos superiores puede extenderse a cada proveedor de datos ESG, pudiendo medirse entonces la eficiencia de nuestra propuesta con el enfoque de cada proveedor.

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