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On G-Drazin inverses of finite potent endomorphisms and arbitrary square matrices

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ABSTRACT

The aim of this work is to extend to finite potent endomorphisms the notion of G-Drazin inverse of a finite square matrix. Accordingly, we determine the structure and the properties of a G-Drazin inverse of a finite potent endomorphism and, as an application, we offer an algorithm to compute the explicit expression of all G-Drazin inverses of a finite square matrix.

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1. Introduction

For an arbitrary $(n \times n)$ -matrix A with entries in the complex numbers, the index of A , $i(A) \geq 0$, is the smallest integer such that $\text{rk}(A^{i(A)}) = \text{rk}(A^{i(A)+1})$. Given $A \in \text{Mat}_{n \times n}(\mathbb{C})$ with $i(A) = r$, H. Wang and X. Liu introduced in [1] the notion of ‘G-Drazin inverse’ of A as a solution X of the system

$$\begin{aligned} AXA &= A; \\ XA^{r+1} &= A^r; \\ A^{r+1}X &= A^r, \end{aligned} \tag{1}$$

where X is a $(n \times n)$ -matrix with entries in $\mathbb{C}$.

Recently, C. Coll et al. have proved in [2] that a matrix $X \in \text{Mat}_{n \times n}(\mathbb{C})$ is a solution of the system (1) if and only if X satisfies that

$$\begin{aligned} AXA &= A; \\ XA^r &= A^rX. \end{aligned} \tag{2}$$

On the other hand, if k is a field, V is an arbitrary vector space over k and φ is an endomorphism of V , according to [3] we say that φ is ‘finite-potent’ if $\varphi^n V$ is finite dimensional for some n .

During recent years, the author has extended the notions of Drazin inverse, Core-Moore-Penrose inverse and Drazin-Moore-Penrose inverses of finite square matrices to finite potent endomorphisms, and has offered several properties of these extensions [4–6]. In particular, all the results obtained for finite potent endomorphisms are also valid for finite square matrices.

The aim of this work is to extend to finite potent endomorphisms the notion of G-Drazin inverse of a finite square matrix. Indeed, we determine the structure of a G-Drazin inverse of a finite potent endomorphism and, in particular, we offer the explicit expression of all G-Drazin inverses of a finite square matrix.

The paper is organized as follows. In Section 2 we briefly recall the basic definitions of this work: the definition of finite potent endomorphisms with the decomposition of the vector space given by Argerami et al. [7]; the Jordan bases of nilpotent endomorphisms of infinite-dimensional vector spaces; the Drazin inverse of an $(n \times n)$ -matrix and a finite potent endomorphism; the core-nilpotent decomposition of a finite potent endomorphism; and the basic properties of the G-Drazin inverses of an square matrix.

Section 3 is devoted to proving the existence of G-Drazin inverses of finite potent endomorphisms (Proposition 3.4) and to offering the explicit structure of these linear maps on arbitrary k -vector spaces (Lemma 3.10 and Corollary 3.11). Moreover, an explicit example of G-Drazin inverses of a finite potent endomorphism is given.

Finally, the goal of Section 4 is to apply the results of Section 3 to study the set $A\{GD\}$ of G-Drazin inverses of a square matrix $A \in \text{Mat}_{n \times n}(k)$, where k is an arbitrary field. Accordingly, if the index of A is r and $\text{rk}(A^i)$ is the rank of A^i , we can write $v_i(A) = n - \text{rk}(A^i)$ for all $i \in \{1, \dots, r\}$ and we check that there exists a bijection

$$k^{v_1(A) \cdot v_r(A)} \times k^{[v_r(A) - v_1(A)]} \times k^{[v_1(A) - 1][v_r(A) - v_1(A)]} \xrightarrow{\sim} A\{GD\},$$

from where we determine the explicit expression of all G-Drazin inverses of A .

2. Preliminaries

This section is added for the sake of completeness.

2.1. Finite potent endomorphisms

Let k be an arbitrary field, and let V be a k -vector space.

Let us now consider an endomorphism φ of V . We say that φ is ‘finite potent’ if $\varphi^n V$ is finite dimensional for some n . This definition was introduced by J. Tate in [3] as a basic tool for his elegant definition of Abstract Residues.

In 2007, M. Argerami et al. showed in [7] that an endomorphism φ is finite potent if and only if V admits a φ -invariant decomposition $V = U_\varphi \oplus W_\varphi$ such that $\varphi|_{U_\varphi}$ is nilpotent, W_φ is finite dimensional and $\varphi|_{W_\varphi} : W_\varphi \xrightarrow{\sim} W_\varphi$ is an isomorphism.

Indeed, if $k[x]$ is the algebra of polynomials in the variable x with coefficients in k , we may view V as an $k[x]$ -module via φ , and the explicit definition of the above φ -invariant subspaces of V is:

- $U_\varphi = \{v \in V \text{ such that } x^m v = 0 \text{ for some } m\};$
- $W_\varphi = \{v \in V \text{ such that } p(x)v = 0 \text{ for some } p(x) \in k[x] \text{ relative prime to } x\}.$

Note that if the annihilator polynomial of φ is $x^m \cdot p(x)$ with $(x, p(x)) = 1$, then $U_\varphi = \text{Ker } \varphi^m$ and $W_\varphi = \text{Ker } p(\varphi)$.

Hence, this decomposition is unique. In this paper we shall call this decomposition the φ -invariant AST-decomposition of V .

For a finite potent endomorphism φ , a trace $\text{tr}_V(\varphi) \in k$ may be defined as $\text{tr}_V(\varphi) = \text{tr}_{W_\varphi}(\varphi|_{W_\varphi})$

This trace has the following properties:

- (1) if V is finite dimensional, then $\text{tr}_V(\varphi)$ is the ordinary trace;
- (2) if W is a subspace of V such that $\varphi W \subset W$ then

$$\text{tr}_V(\varphi) = \text{tr}_W(\varphi) + \text{tr}_{V/W}(\varphi);$$

- (3) if φ is nilpotent, then $\text{tr}_V(\varphi) = 0$.

Usually, tr_V is named ‘Tate’s trace’.

It is known that in general tr_V is not linear; that is, it is possible to find finite potent endomorphisms $\theta_1, \theta_2 \in \text{End}_k(V)$ such that

$$\text{tr}_V(\theta_1 + \theta_2) \neq \text{tr}_V(\theta_1) + \text{tr}_V(\theta_2).$$

Moreover, with the previous notation, and using the AST-decomposition of V , D. Hernández Serrano and the author of this paper have offered in [8] a definition of a determinant for finite potent endomorphisms as follows:

$$\det_V^k(1 + \varphi) := \det_{W_\varphi}^k(1 + \varphi|_{W_\varphi}).$$

This determinant satisfies the following properties:

- if V is finite dimensional, then $\det_V^k(1 + \varphi)$ is the ordinary determinant;
- if W is a subspace of V such that $\varphi W \subset W$, then

$$\det_V^k(1 + \varphi) = \det_W^k(1 + \varphi) \cdot \det_{V/W}^k(1 + \varphi);$$

- if φ is nilpotent, then $\det_V^k(1 + \varphi) = 1$.

For details readers are referred to [3, 8–10].

2.2. Jordan bases of nilpotent endomorphisms of infinite-dimensional vector spaces

Let V be a vector space over an arbitrary field k and let $f \in \text{End}_k(V)$ be a nilpotent endomorphism.

If n is the nilpotency index of f , according to the statements [11], setting $W_i^f = \text{Ker} f^i / [\text{Ker} f^{i-1} + f(\text{Ker} f^{i+1})]$ with $i \in \{1, 2, \dots, n\}$, $\alpha_i(V, f) = \dim_k W_i^f$ and $S_{\alpha_i(V, f)}$ a set such that $\#S_{\alpha_i(V, f)} = \alpha_i(V, f)$ with $S_{\alpha_i(V, f)} \cap S_{\alpha_j(V, f)} = \emptyset$ for all $i \neq j$, one has that there exists a family of vectors $\{v_{s_i}\}$ that determines a Jordan basis of f :

$$B = \bigcup_{\substack{s_i \in S_{\alpha_i(V, f)} \\ 1 \leq i \leq n}} \{v_{s_i}, f(v_{s_i}), \dots, f^{i-1}(v_{s_i})\}. \quad (3)$$

Moreover, if we write $H_{s_i}^f = \langle v_{s_i}, f(v_{s_i}), \dots, f^{i-1}(v_{s_i}) \rangle$, the basis B induces a decomposition

$$V = \bigoplus_{\substack{s_i \in S_{\alpha_i(V, f)} \\ 1 \leq i \leq n}} H_{s_i}^f. \quad (4)$$

For a different method to construct Jordan bases of nilpotent endomorphisms of infinite-dimensional vector spaces readers can see [12].

2.3. Drazin inverse of finite potent endomorphisms

2.3.1. Drazin inverse of $(n \times n)$ -matrices

Let $A \in \text{Mat}_{n \times n}(\mathbb{C})$.

Definition 2.1: The ‘index of A ’, $i(A) \geq 0$, is the smallest integer such that $\text{rk}(A^{i(A)}) = \text{rk}(A^{i(A)+1})$.

In 1958, given a matrix $A \in \text{Mat}_{n \times n}(\mathbb{C})$ with $i(A) = r$, M. P. Drazin [13] showed the existence of a unique $(n \times n)$ -matrix A^D satisfying the equations:

- $A^{r+1}A^D = A^r$ for $r = i(A)$;
- $A^D A A^D = A^D$;
- $A^D A = A A^D$.

The Drazin inverse A^D also verifies that

- $(A^D)^D = A$ if and only if $i(A) \leq 1$;
- if $A^2 = A$, then $A^D = A$.

2.3.2. Drazin inverse of finite potent endomorphisms

Let V be an arbitrary k -vector space and let $\varphi \in \text{End}_k(V)$ be a finite potent endomorphism of V . Let us consider the AST-decomposition $V = U_\varphi \oplus W_\varphi$ induced by φ (Subsection 2.1).

Definition 2.2: We shall call ‘index of φ ’, $i(\varphi)$, to the nilpotent order of $\varphi|_{U_\varphi}$.

In [6] (Lemma 3.2) is proved that for finite-dimensional vector spaces this definition of index coincides with Definition 2.1. Note that $i(\varphi) = 0$ if and only if V is a finite-dimensional vector space and φ is an automorphism.

For each finite potent endomorphism φ there exists a unique finite potent endomorphism φ^D that satisfies that:

- (1) $\varphi^{r+1} \circ \varphi^D = \varphi^r$;
- (2) $\varphi^D \circ \varphi \circ \varphi^D = \varphi^D$;
- (3) $\varphi^D \circ \varphi = \varphi \circ \varphi^D$,

where r is the index of φ .

The map φ^D is the Drazin inverse of φ and is the unique linear map such that:

$$\varphi^D(v) = \begin{cases} (\varphi|_{W_\varphi})^{-1} & \text{if } v \in W_\varphi \\ 0 & \text{if } v \in U_\varphi \end{cases}.$$

Moreover, φ^D satisfies the following properties:

- $(\varphi^D)^D = \varphi$ if and only if the $i(\varphi) \leq 1$;
- $\varphi = \varphi^D$ if and only if $\varphi|_{U_\varphi} = 0$ and $(\varphi|_{W_\varphi})^2 = \text{Id}_{|W_\varphi}$;
- $\text{tr}_V(\varphi + \varphi^D) = \text{tr}_V(\varphi) + \text{tr}_V(\varphi^D)$;
- if ψ is a projection finite potent endomorphism, then $\psi^D = \psi$.

2.4. CN decomposition of a finite potent endomorphism

Given a finite potent endomorphism $\varphi \in \text{End}_k(V)$, there exists a unique decomposition $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1, \varphi_2 \in \text{End}_k(V)$ are finite potent endomorphisms satisfying that:

- $i(\varphi_1) \leq 1$;
- φ_2 is nilpotent;
- $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1 = 0$.

According to [4, Theorem 3.2], one has that $\varphi_1 = \varphi \circ \varphi^D \circ \varphi$, which is the core part of φ . Also, φ_2 is named the nilpotent part of φ .

Moreover, one has that

$$\varphi = \varphi_1 \iff U_\varphi = \text{Ker} \varphi \iff W_\varphi = \text{Im} \varphi \iff (\varphi^D)^D = \varphi \iff i(\varphi) \leq 1. \quad (5)$$

2.5. G-Drazin inverses of a square matrix

Given $A \in \text{Mat}_{n \times n}(\mathbb{C})$ with $i(A) = r$, H. Wang and X. Liu introduced in [1] the notion of ‘G-Drazin inverse’ of A as a solution X of the system

$$\begin{aligned} AXA &= A; \\ XA^{r+1} &= A^r; \\ A^{r+1}X &= A^r, \end{aligned} \quad (6)$$

where X is a $(n \times n)$ -matrix with entries in $\mathbb{C}$.

Recently, C. Coll et al. have proved in [2] that a matrix $X \in \text{Mat}_{n \times n}(\mathbb{C})$ is a solution of the system (7) if and only if X satisfies that

$$\begin{aligned} AXA &= A; \\ XA^r &= A^r X. \end{aligned} \quad (7)$$

Usually, the set of G-Drazin inverses of a matrix A is denoted by $A\{GD\}$ and a G-matrix inverse of A is denoted by A^{GD} . We should note that G-Drazin inverses are considered for bounded linear operators.

If J is the Jordan matrix associated with $A \in \text{Mat}_{n \times n}(\mathbb{C})$, such that $A = B \cdot J \cdot B^{-1}$, with B being a non-singular matrix and

$$J = \begin{pmatrix} J_1 & 0 \\ 0 & J_0 \end{pmatrix},$$

J_0 and J_1 being the parts of J corresponding to zero and non-zero eigenvalues respectively, it is known that a G-Drazin inverse is

$$A^{GD} = B \cdot \begin{pmatrix} J_1^{-1} & 0 \\ 0 & J_0^- \end{pmatrix} \cdot B^{-1}, \quad (8)$$

where J_0^- is a generalized inverse of J_0 (1-inverse).

3. G-Drazin inverses of finite potent endomorphisms

The aim of this section is to generalize the definition and the main properties of the G-Drazin inverses of a matrix A to finite potent endomorphisms.

Let k be a field and let V be an arbitrary k -vector space.

Definition 3.1: Given a finite potent endomorphism $\varphi \in \text{End}_k(V)$, we say that an endomorphism $\varphi^{GD} \in \text{End}_k(V)$ is a G-Drazin inverse of φ when it satisfies that

$$\begin{aligned} \varphi \circ \varphi^{GD} \circ \varphi &= \varphi; \\ \varphi^{GD} \circ \varphi^r &= \varphi^r \circ \varphi^{GD}, \end{aligned} \quad (9)$$

where $i(\varphi) = r$.

Lemma 3.2: Let $\varphi \in \text{End}_k(V)$ be a finite potent endomorphism with $i(\varphi) = r$ and let $V = W_\varphi \oplus U_\varphi$ be the AST-decomposition determined by φ . If $f \in \text{End}_k(V)$ is an endomorphism such that $f \circ \varphi^r = \varphi^r \circ f$, then W_φ and U_φ are invariant under the action of f .

Proof: Let $\varphi = \varphi_1 + \varphi_2$ be the CN-decomposition of φ .

Since $i(\varphi) = r$, bearing in mind that $\varphi^r = \varphi_1^r$, $(\varphi_1^r)|_{W_\varphi} \in \text{Aut}_k(W_\varphi)$ and $\text{Im } \varphi_1^r = W_\varphi$, if $w \in W_\varphi$ and $\varphi_1^r(w') = w$, then

$$f(w) = (f \circ \varphi_1^r)(w') = (\varphi_1^r \circ f)(w') \in W_\varphi.$$

Accordingly, W_φ is f -invariant.

Moreover, if $u \in U_\varphi$, then

$$0 = (f \circ \varphi_1^r)(u) = (\varphi_1^r \circ f)(u)$$

and we deduce that $f(u) \in \text{Ker } \varphi_1^r = U_\varphi$. Hence, U_φ is also f -invariant. ■

Corollary 3.3: *If $\varphi^{GD} \in \text{End}_k(V)$ is a G-Drazin inverse of a finite potent endomorphism $\varphi \in \text{End}_k(V)$, with $i(\varphi) = r$ and AST-decomposition $V = W_\varphi \oplus U_\varphi$, then W_φ and U_φ are invariant under the action of φ^{GD} .*

The structure of a G-Drazin inverse of a finite potent endomorphism is given by the following proposition:

Proposition 3.4: *Given a finite potent endomorphism $\varphi \in \text{End}_k(V)$ with $i(\varphi) = r$ and AST-decomposition $V = W_\varphi \oplus U_\varphi$, one has that $\varphi^{GD} \in \text{End}_k(V)$ is a G-Drazin inverse of φ if and only if W_φ and U_φ are invariant under the action of φ^{GD} , $(\varphi^{GD})|_{W_\varphi} = (\varphi|_{W_\varphi})^{-1}$ and $(\varphi^{GD})|_{U_\varphi} = (\varphi|_{U_\varphi})^-$, where $(\varphi|_{U_\varphi})^-$ is a generalized inverse of $\varphi|_{U_\varphi}$.*

Proof: Let $\varphi^{GD} \in \text{End}_k(V)$ be a G-Drazin inverse of φ . It follows from Corollary 3.3 that W_φ and U_φ are invariant under the action of φ^{GD} . Also, since $\varphi \circ \varphi^{GD} \circ \varphi = \varphi$, it is clear that $(\varphi^{GD})|_{W_\varphi} = (\varphi|_{W_\varphi})^{-1}$ and $(\varphi^{GD})|_{U_\varphi} = (\varphi|_{U_\varphi})^-$, where $(\varphi|_{U_\varphi})^-$ is a generalized inverse of $\varphi|_{U_\varphi}$.

Conversely, if we consider an endomorphism $\psi \in \text{End}_k(V)$ satisfying that W_φ and U_φ are invariant under the action of ψ , $\psi|_{W_\varphi} = (\varphi|_{W_\varphi})^{-1}$ and $\psi|_{U_\varphi} = (\varphi|_{U_\varphi})^-$, with $(\varphi|_{U_\varphi})^-$ a generalized inverse of $\varphi|_{U_\varphi}$, it is easy to check that $\varphi \circ \psi \circ \varphi = \varphi$.

Moreover, since $(\varphi^r)|_{U_\varphi} = 0$, then it follows from the properties of ψ that

$$(\psi \circ \varphi^r)|_{W_\varphi} = (\varphi^{r-1})|_{W_\varphi} = (\varphi^r \circ \psi)|_{W_\varphi}$$

from where we deduce that

$$\psi \circ \varphi^r = \varphi^r \circ \psi$$

and the statement is proved. ■

Direct consequences of Proposition 3.4 are:

Corollary 3.5: *Given a finite potent endomorphism $\varphi \in \text{End}_k(V)$ with CN-decomposition $\varphi = \varphi_1 + \varphi_2$, if $\varphi^{GD} \in \text{End}_k(V)$ is a G-Drazin inverse of φ , one has that $\varphi^{GD} = \varphi^D + \varphi_2^{GD}$, where φ^D is the Drazin inverse of φ , and $\varphi_2^{GD} \in \text{End}_k(V)$ is the unique linear map satisfying that*

$$\varphi_2^{GD}(v) = \begin{cases} 0 & \text{if } v \in W_\varphi \\ (\varphi^{GD})|_{U_\varphi} & \text{if } v \in U_\varphi \end{cases},$$

which is a G-Drazin inverse of φ_2 .

Corollary 3.6: *If $\varphi = \varphi_1 + \varphi_2$ is the CN-decomposition of a finite potent endomorphism $\varphi \in \text{End}_k(V)$, then the Drazin inverse φ^D is a G-Drazin inverse of φ_1 .*

We shall now characterize all of the G-Drazin inverses of a finite potent endomorphism.

Lemma 3.7: *Let E be a k -vector space of dimension n and let $f \in \text{End}_k(E)$ be an endomorphism with annihilating polynomial $a_f(x) = x^n$. If $e \in E$ is a vector such that $f^{n-1}(e) \neq 0$, then every generalized inverse $f^- \in \text{End}_k(E)$ is determined by the expressions*

$$f^-(f^i(e)) = \begin{cases} f^{i-1}(e) + \lambda_i f^{n-1}(e) & \text{if } i \geq 1 \\ \tilde{e} & \text{if } i = 0 \end{cases},$$

with $i \in \{0, 1, \dots, n-1\}$, $\lambda_i \in k$ for every $i \in \{1, \dots, n-1\}$ and $\tilde{e} \in E$ being an arbitrary vector.

Proof: Since $f \circ f^- \circ f = f$, we have that f^- is a generalized inverse of f if and only if $(f \circ f^-)|_{\text{Im} f} = \text{Id}_{\text{Im} f}$. Hence, since $e \notin \text{Im} f$ we have that $f^-(e) = \tilde{e}$, where $\tilde{e} \in E$ is an arbitrary vector.

Moreover, bearing in mind that $(f \circ f^-)(f^i(e)) = f^i(e)$ for all $i \geq 1$, one has that

$$f^-(f^i(e)) \in f^{-1}(f^i(e)) + \text{Ker } f,$$

and we get that

$$f^-(f^i(e)) = f^{i-1}(e) + \lambda_i f^{n-1}(e)$$

for all $i \in \{1, \dots, n-1\}$.

Accordingly, since $\{e, f(e), \dots, f^{n-1}(e)\}$ is a Jordan basis of E induced by f , the claim is deduced. ■

We can reformulate the statement of Lemma 3.9 as follows:

Corollary 3.8: *Let E be a k -vector space of dimension n and let $f \in \text{End}_k(E)$ be an endomorphism with annihilating polynomial $a_f(x) = x^n$. If $e \in E$ is a vector such that $f^{n-1}(e) \neq 0$, then every generalized inverse $f^- \in \text{End}_k(E)$ is determined by the expressions*

$$f^-(f^i(e)) = \begin{cases} f^{i-1}(e) + \lambda_i f^{n-1}(e) & \text{if } i \geq 1 \\ \sum_{h=0}^{n-1} \alpha_h f^h(e) & \text{if } i = 0 \end{cases}, \quad (10)$$

with $i \in \{0, \dots, n-1\}$, $\lambda_i, \alpha_h \in k$ for every $i \in \{1, \dots, n-1\}$ and $h \in \{0, \dots, n-1\}$.

Furthermore, similar to Lemma 3.9 one can prove that

Lemma 3.9: If E is a finite-dimensional k -vector space, $f \in \text{End}_k(E)$ is an endomorphism with annihilating polynomial $a_f(x) = x^n$ and

$$\bigcup_{j=1}^r \{e_j, f(e_j), \dots, f^{n_j-1}(e_j)\}$$

is a Jordan basis of E induced by f , then every generalized inverse $f^- \in \text{End}_k(E)$ is determined by the expressions

$$f^-(f^i(e_j)) = \begin{cases} f^{i-1}(e_j) + \sum_{s=1}^r \lambda_{j,i}^s f^{n_s-1}(e_s) & \text{if } i \geq 1 \\ \sum_{s=1}^r \left[\sum_{h=0}^{n_s-1} \alpha_{j,h}^s f^h(e_s) \right] & \text{if } i = 0 \end{cases}, \quad (11)$$

with $\lambda_{j,i}^s, \alpha_{j,h}^s \in k$ for each $j, s \in \{1, \dots, r\}$, $i \in \{0, \dots, n_j - 1\}$ and $h \in \{0, \dots, n_s - 1\}$.

Let us again consider a finite potent endomorphism $\varphi \in \text{End}_k(V)$ of an arbitrary k -vector space V with $i(\varphi) = r$. If $V = W_\varphi \oplus U_\varphi$ is the AST-decomposition induced by φ , let

$$B_\varphi = \bigcup_{\substack{s_h \in S_{\alpha_h(V, \varphi|_{U_\varphi})} \\ 1 \leq h \leq r}} \{v_{s_h}, \varphi(v_{s_h}), \dots, \varphi^{h-1}(v_{s_h})\} \quad (12)$$

be a Jordan basis of U_φ induced by $\varphi|_{U_\varphi}$ (see Subsection 2.2).

Let us now denote

$$\bar{S}_{\alpha, \varphi} = S_{\alpha_1(V, \varphi|_{U_\varphi})} \cup \dots \cup S_{\alpha_r(V, \varphi|_{U_\varphi})}$$

and $\beta(V, \varphi) = \#\bar{S}_{\alpha, \varphi}$.

Accordingly, similar to Lemma 3.9, it is easy to check that

Lemma 3.10: Given an arbitrary k -vector space V and a finite potent endomorphism $\varphi \in \text{End}_k(V)$ with $i(\varphi) = r$ and AST-decomposition $V = W_\varphi \oplus U_\varphi$, fixing a Jordan basis B_φ of U_φ as in (12), then every generalized inverse $(\varphi|_{U_\varphi})^- \in \text{End}_k(U_\varphi)$ is determined by the expressions

$$(\varphi|_{U_\varphi})^-(\varphi^i(v_{s_h})) = \begin{cases} \varphi^{i-1}(v_{s_h}) + \sum_{s_t \in \bar{S}_{\alpha, \varphi}} \lambda_{s_h, s_t}^i \varphi^{t-1}(v_{s_t}) & \text{if } 1 \leq i \leq h-1 \\ v & \text{if } i = 0 \end{cases},$$

with $v \in U_\varphi$, $1 \leq h \leq r$, $i \in \{0, 1, \dots, h-1\}$, $\lambda_{s_h, s_t}^i \in k$ and $\lambda_{s_h, s_t}^i = 0$ for almost all $s_t \in \bar{S}_{\alpha, \varphi}$ (for each $s_h \in S_{\alpha_h(V, \varphi|_{U_\varphi})}$).

Writing

$$(\varphi|_{U_\varphi})^-(v_{s_h}) = \sum_{\substack{s_l \in \bar{S}_{\alpha, \varphi} \\ 0 \leq l \leq h-1}} \alpha_{s_h, s_l}^j \varphi^j(v_{s_l}) \in U_\varphi,$$

one has that

Corollary 3.11: *Given an arbitrary k -vector space V and a finite potent endomorphism $\varphi \in \text{End}_k(V)$ with $i(\varphi) = r$ and AST-decomposition $V = W_\varphi \oplus U_\varphi$, fixing a Jordan basis B_φ of U_φ as in (12), then every generalized inverse $(\varphi|_{U_\varphi})^- \in \text{End}_k(U_\varphi)$ is determined by the expressions*

$$(\varphi|_{U_\varphi})^-(\varphi^i(v_{s_h})) = \begin{cases} \varphi^{i-1}(v_{s_h}) + \sum_{\substack{s_t \in \bar{S}_{\alpha, \varphi} \\ 0 \leq j \leq l-1}} \lambda_{s_h, s_t}^i \varphi^{t-1}(v_{s_t}) & \text{if } 1 \leq i \leq h-1 \\ \sum_{\substack{s_l \in \bar{S}_{\alpha, \varphi} \\ 0 \leq j \leq l-1}} \alpha_{s_h, s_l}^j \varphi^j(v_{s_l}) & \text{if } i = 0 \end{cases},$$

with $1 \leq h \leq r$, $i \in \{0, 1, \dots, h-1\}$, $\lambda_{s_h, s_t}^i, \alpha_{s_h, s_l}^j \in k$, and $\lambda_{s_h, s_t}^i = 0 = \alpha_{s_h, s_l}^j$ for almost all $s_t, s_l \in \bar{S}_{\alpha, \varphi}$ and $0 \leq j \leq l-1$.

Corollary 3.12: *With the notation of Corollary 3.5, $\varphi_2^{GD} = 0$ if and only if $i(\varphi) \leq 1$.*

Proof: It follows from Lemma 3.10 that $\varphi_2^{GD} = 0$ if and only if $\text{Ker} \varphi = U_\varphi$, from where the claim is proved. ■

Corollary 3.13: *Given a finite potent endomorphism $\varphi \in \text{End}_k(V)$, one has that the Drazin inverse $\varphi^D \in \text{End}_k(V)$ is a G-Drazin inverse of φ if and only if $i(\varphi) \leq 1$.*

Proof: If $\varphi = \varphi_1 + \varphi_2$ is the CN-decomposition, then according to Corollary 3.5 one has that $\varphi^D \in \text{End}_k(V)$ is a G-Drazin inverse of φ if and only if 0 is a G-Drazin inverse of φ_2 and, bearing in mind Corollary 3.12, the statement is deduced. ■

Accordingly, from Proposition 3.4 and Lemma 3.10, we have characterized all the G-Drazin inverses of a finite potent endomorphism $\varphi \in \text{End}_k(V)$. Note that, in general, a G-Drazin inverse $\varphi^{GD} \in \text{End}_k(V)$ is not a finite potent endomorphism.

If we denote by $X_\varphi^{\{GD\}}$ the set of all G-Drazin inverses of a finite potent endomorphism $\varphi \in \text{End}_k(V)$, fixing a Jordan basis B_φ of U_φ as in (12), with the notation of Subsection 2.2, $i(\varphi) = r$ and we have a bijection

$$X_\varphi^{\{GD\}} \xrightarrow{\sim} \prod_{h=1}^r \left(\prod_{\alpha_h(V, \varphi|_{U_\varphi})} \left[U_\varphi \times \prod_{i=1}^{h-1} \left(\bigoplus_{\beta(V, \varphi)} k \right) \right] \right) \quad (13)$$

$$\varphi^{GD} \mapsto ((\varphi^{GD}(s_h), ((\lambda_{s_h, s_t}^i)_{s_t \in \bar{S}_{\alpha, \varphi}})_{1 \leq i \leq h-1})_{s_h \in S_{\alpha_h}(V, \varphi|_{U_\varphi})})_{h \in \{1, \dots, r\}}$$

Example 3.1: Let k be an arbitrary ground field, let V be a k -vector space of countable dimension over k and let $\{v_1, v_2, v_3, \dots\}$ be a basis of V indexed by the natural numbers.

Let $\varphi \in \text{End}_k(V)$ the finite potent endomorphism defined as follows:

$$\varphi(v_i) = \begin{cases} v_2 + v_5 + v_7 & \text{if } i = 1 \\ v_1 + 3v_2 & \text{if } i = 2 \\ v_4 & \text{if } i = 3 \\ v_1 - v_3 & \text{if } i = 4 \\ -v_3 + 2v_5 + 2v_7 & \text{if } i = 5 \\ 3v_{i+1} & \text{if } i = 5h + 1 \\ 0 & \text{if } i = 5h + 2 \\ -v_{i-2} + 2v_{i+1} & \text{if } i = 5h + 3 \\ v_{i-2} + v_{i+1} & \text{if } i = 5h + 4 \\ -v_{i-4} + 5v_{i-3} & \text{if } i = 5h + 5 \end{cases}$$

for all $h \geq 1$.

We have that the AST-decomposition $V = U_\varphi \oplus W_\varphi$ is determined by the subspaces

$$W_\varphi = \langle v_1, v_2, v_3, v_4, v_5 + v_7 \rangle \quad \text{and} \quad U_\varphi = \langle v_j \rangle_{j \geq 6}.$$

In this basis of W_φ one has that

$$\varphi|_{W_\varphi} \equiv A_{W_\varphi} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 \end{pmatrix},$$

and, computing $A_{W_\varphi}^{-1}$, we obtain that the Drazin inverse of φ is

$$\varphi^D(v_i) = \begin{cases} 6v_1 - 2v_2 + 3v_4 - 3v_5 & \text{if } i = 1 \\ -2v_1 + v_2 - v_4 + v_5 & \text{if } i = 2 \\ 6v_1 - 2v_2 + 2v_4 - 3v_5 & \text{if } i = 3 \\ v_3 & \text{if } i = 4 \\ 3v_1 - v_2 + v_4 - v_5 & \text{if } i = 5 \\ 0 & \text{if } i \geq 6 \end{cases}.$$

Moreover, we can write $U_\varphi = \bigoplus_{i \geq 2} H_i$ with $H_i = \langle v_{5i-4}, v_{5i-3}, v_{5i-2}, v_{5i-1}, v_{5i} \rangle$ for all $i \geq 2$, and in the same bases we have that

$$\varphi|_{H_i} \equiv A_{H_i} = \begin{pmatrix} 0 & 0 & -1 & 0 & -1 \\ 3 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

for all $i \geq 2$.

For every $i \geq 2$, one has that

$$\{v_{5i-2}, -v_{5i-4} + 2v_{5i-1}, -v_{5i-3} + 2v_{5i}, -2v_{5i-4} + 10v_{5i-3}, -6v_{5i-3}\}$$

is a Jordan basis of H_i induced by $\varphi|_{H_i}$ and, therefore,

$$\varphi|_{H_i} \equiv P \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \cdot P^{-1}$$

with

$$P = \begin{pmatrix} 0 & -1 & 0 & -2 & 0 \\ 0 & 0 & -1 & 10 & -6 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix}.$$

Accordingly, it follows from Corollaries 3.5 and 3.11, that $\varphi_2^{GD} \in \text{End}_k(V)$ is the unique linear map such that:

- $\varphi_2^{GD}(v_i) = 0$ for $i \in \{1, 2, 3, 4\}$;
- $\varphi_2^{GD}(v_5 + v_7) = 0$;
- $\varphi_2^{GD}(v_{5i-2}) = \sum_{j \geq 6} \alpha_{ij} v_j$
- $\varphi_2^{GD}(-v_{5i-4} + 2v_{5i-1}) = v_{5i-2} + \sum_{h \geq 2} \lambda_{i,5h-3}^1 v_{5h-3}$;
- $\varphi_2^{GD}(-v_{5i-3} + 2v_{5i}) = -v_{5i-4} + 2v_{5i-1} + \sum_{h \geq 2} \lambda_{i,5h-3}^2 v_{5h-3}$;
- $\varphi_2^{GD}(-2v_{5i-4} + 10v_{5i-3}) = -v_{5i-3} + 2v_{5i} + \sum_{h \geq 2} \lambda_{i,5h-3}^3 v_{5h-3}$;
- $\varphi_2^{GD}(-6v_{5i-3}) = -2v_{5i-4} + 10v_{5i-3} + \sum_{h \geq 2} \lambda_{i,5h-3}^4 v_{5h-3}$;

for every $i \geq 2$, and with $\alpha_{ij} = 0$ for almost all $j \geq 6$ (for each $i \geq 2$) and $\lambda_{5h-3,s}^i = 0$ for almost all $h \geq 2$ (for every $i \geq 2$ and for every $s \in \{2, 3, 4, 5\}$).

Thus, a non-difficult computation shows that $\varphi_2^{GD} \in \text{End}_k(V)$ is the unique linear map such that:

- $\varphi_2^{GD}(v_i) = 0$ for $i \in \{1, 2, 3, 4\}$;
- $\varphi_2^{GD}(v_5) = -\frac{1}{3}v_6 + \frac{5}{3}v_7 + \frac{1}{6} \sum_{h \geq 2} \lambda_{5h-3,5}^2 v_{5h-3}$;
- $\varphi_2^{GD}(v_{5i-4}) = -\frac{5}{3}v_{5i-4} - \frac{47}{6}v_{5i-3} - v_{5i} - \sum_{h \geq 2} (\frac{1}{2}\lambda_{i,5h-3}^3 + \frac{5}{6}\lambda_{i,5h-3}^4) v_{5h-3}$;
- $\varphi_2^{GD}(v_{5i-3}) = \frac{1}{3}v_{5i-4} - \frac{5}{3}v_{5i-3} - \frac{1}{6} \sum_{h \geq 2} \lambda_{i,5h-3}^4 v_{5h-3}$;
- $\varphi_2^{GD}(v_{5i-2}) = \sum_{j \geq 6} \alpha_{ij} v_j$;
- $\varphi_2^{GD}(v_{5i-1}) = \frac{5}{6}v_{5i-4} - \frac{47}{12}v_{5i-3} + \frac{1}{2}v_{5i-2} - \frac{1}{2}v_{5i} + \sum_{h \geq 2} (\frac{1}{2}\lambda_{i,5h-3}^1 - \frac{1}{4}\lambda_{i,5h-3}^3) v_{5h-3} - \frac{5}{12}\lambda_{i,5h-3}^4$;
- $\varphi_2^{GD}(v_{5i}) = -\frac{1}{3}v_{5i-4} - \frac{5}{6}v_{5i-3} + v_{5i-1} - \frac{5}{6}v_{5i} + \sum_{h \geq 2} (\frac{1}{2}\lambda_{i,5h-3}^2 - \frac{1}{12}\lambda_{i,5h-3}^4) v_{5h-3}$;

for every $i \geq 2$, and with $\alpha_{ij} = 0$ for almost all $j \geq 6$ (for each $i \geq 2$) and $\lambda_{5h-3,s}^i = 0$ for almost all $h \geq 2$ (for every $i \geq 2$ and for every $s \in \{2, 3, 4, 5\}$).

Hence, bearing in mind that $\varphi^{GD} = \varphi^D + \varphi_2^{GD}$ (Corollary 3.5), one has that a G-Drazin inverse φ^{GD} is determined by:

- $\varphi^{GD}(v_1) = 6v_1 - 2v_2 + 3v_4 - 3v_5$;
- $\varphi^{GD}(v_2) = -2v_1 + v_2 - v_4 + v_5$;
- $\varphi^{GD}(v_3) = 6v_1 - 2v_2 + 2v_4 - 3v_5$;
- $\varphi^{GD}(v_4) = v_3$;
- $\varphi^{GD}(v_5) = 3v_1 - v_2 + v_4 - v_5 - \frac{1}{3}v_6 + \frac{5}{3}v_7 + \frac{1}{6} \sum_{h \geq 2} \lambda_{5h-3,5}^2 v_{5h-3}$;
- $\varphi^{GD}(v_{5i-4}) = -\frac{5}{3}v_{5i-4} - \frac{47}{6}v_{5i-3} - v_{5i} - \sum_{h \geq 2} (\frac{1}{2}\lambda_{i,5h-3}^3 + \frac{5}{6}\lambda_{i,5h-3}^4) v_{5h-3}$;
- $\varphi^{GD}(v_{5i-3}) = \frac{1}{3}v_{5i-4} - \frac{5}{3}v_{5i-3} - \frac{1}{6} \sum_{h \geq 2} \lambda_{i,5h-3}^4 v_{5h-3}$;
- $\varphi^{GD}(v_{5i-2}) = \sum_{j \geq 6} \alpha_{i,j} v_j$;
- $\varphi^{GD}(v_{5i-1}) = \frac{5}{6}v_{5i-4} - \frac{47}{12}v_{5i-3} + \frac{1}{2}v_{5i-2} - \frac{1}{2}v_{5i} + \sum_{h \geq 2} (\frac{1}{2}\lambda_{i,5h-3}^1 - \frac{1}{4}\lambda_{i,5h-3}^3) v_{5h-3} - \frac{5}{12}\lambda_{i,5h-3}^4$;
- $\varphi^{GD}(v_{5i}) = -\frac{1}{3}v_{5i-4} - \frac{5}{6}v_{5i-3} + v_{5i-1} - \frac{5}{6}v_{5i} + \sum_{h \geq 2} (\frac{1}{2}\lambda_{i,5h-3}^2 - \frac{1}{12}\lambda_{i,5h-3}^4) v_{5h-3}$;

for every $i \geq 2$, and with $\alpha_{i,j} = 0$ for almost all $j \geq 6$ (for each $i \geq 2$) and $\lambda_{5h-3,s}^i = 0$ for almost all $h \geq 2$ (for every $i \geq 2$ and for every $s \in \{2, 3, 4, 5\}$).

Remark 3.14: Note that the explicit expression of the bijection (13) for the finite potent endomorphism φ of Example 3.1 is

$$\begin{aligned} X_\varphi^{\{GD\}} &\xrightarrow{\sim} \prod_{i \in \mathbb{N}} \left(U_\varphi \times \prod_{j=1}^4 \left(\bigoplus_{h \in \mathbb{N}} k \right) \right) \\ \varphi^{GD} &\longmapsto (\varphi^{GD}(v_{5i-2}), ((\lambda_{i,5h-3}^j)_{h \in \mathbb{N}})_{1 \leq j \leq 4})_{i \in \mathbb{N}} \end{aligned}$$

To finish this section, we shall briefly study the G-Drazin inverses of a finite potent endomorphism that also are finite potent.

Let V be again an arbitrary k -vector space and let $\varphi \in \text{End}_k(V)$ be a finite potent endomorphism.

With the above notation, if we denote by $(\varphi^{GD})_{\{\lambda_{s_h}^i, \alpha_{s_h}^j\}}$ to the unique linear map of V such that

$$(\varphi^{GD})_{\{\lambda_{s_h}^i, \alpha_{s_h}^j\}}(v) = \begin{cases} (\varphi|_{W_\varphi})^{-1}(v) & \text{if } v \in W_\varphi \\ (\varphi|_{U_\varphi})^-(v) & \text{if } v \in U_\varphi \end{cases},$$

where $(\varphi|_{U_\varphi})^-$ is the generalized inverse of $\varphi|_{U_\varphi} \in \text{End}_k(U_\varphi)$ characterized in Corollary 3.11, it is clear that $(\varphi^{GD})_{\{\lambda_{s_h}^i, \alpha_{s_h}^j\}}$ is finite potent when

$$\lambda_{s_h}^i = 0 = \alpha_{s_h}^j \quad \text{for almost all } i, j \text{ and } s_h. \quad (14)$$

It is clear that the condition (14) is sufficient for determining that $(\varphi^{GD})_{\{\lambda_{s_h}^i, \alpha_{s_h}^j\}}$ is a finite potent endomorphism, but this condition is not necessary for this fact as it is immediately deduced from the following counter-example: given a countable k -vector space V

with a basis $\{v_1, v_2, v_3, \dots\}$ indexed by the natural numbers, if we consider the finite potent endomorphism $\varphi \in \text{End}_k(V)$ defined as

$$\varphi(v_i) = \begin{cases} v_1 + v_2 & \text{if } i = 1 \\ v_1 + 2v_2 & \text{if } i = 2 \\ v_{i+1} & \text{if } i = 2j + 1 \\ 0 & \text{if } i = 2j + 2 \end{cases}$$

for every $j \geq 1$, then it is clear that

$$\varphi^{GD}(v_i) = \begin{cases} 2v_1 - v_2 & \text{if } i = 1 \\ -v_1 + v_2 & \text{if } i = 2 \\ -v_i - v_{i+1} & \text{if } i = 2j + 1 \\ v_{i-1} + v_i & \text{if } i = 2j + 2 \end{cases}$$

for every $j \geq 1$, is a G-Drazin inverse of φ that does not satisfy the condition (14).

Remark 3.15: A remaining problem is obtaining a computable method for determining when a G-Drazin inverse φ^{GD} of a finite potent endomorphism φ is also finite potent.

Remark 3.16: If φ^{GD} is a finite potent G-Drazin inverse of a finite potent endomorphism φ such that $W_{\varphi^{GD}} = W_{\varphi}$, then

$$\text{tr}_V \varphi^{GD} = \text{tr}_V \varphi^D \quad \text{and} \quad \det_V^k \varphi^{GD} = \det_V^k \varphi^D.$$

Indeed, in this case, if $\{\lambda_1, \dots, \lambda_n\}$ are the eigenvalues of $\varphi|_{W_{\varphi}}$ in the algebraic closure of k (with their multiplicity), one has that:

- $\text{tr}_V(\varphi^{GD}) = \lambda_1^{-1} + \dots + \lambda_n^{-1}$;
- $\det_V^k(1 + \varphi^{GD}) = \prod_{i=1}^n (1 + \lambda_i^{-1})$.

4. Explicit computation of the G-Drazin inverses of a square matrix

The final section of this work is devoted to offer a method for computing explicitly all the G-Drazin inverses of a square matrix.

If k is an arbitrary ground field, let us consider a square matrix $A \in \text{Mat}_{n \times n}(k)$ with $i(A) = r$.

Fixing a k -vector space E with dimension n , a basis $B = \{e_1, \dots, e_n\}$ of E and an endomorphism $\varphi \in \text{End}_k(E)$ associated with A in the basis B , from the AST-decomposition $E = W_{\varphi} \oplus U_{\varphi}$ one has that

$$A = P \cdot \begin{pmatrix} A_W & 0 \\ 0 & J_U \end{pmatrix} \cdot P^{-1}, \quad (15)$$

where $\varphi|_{W_{\varphi}} \equiv A_W$, J_U is the Jordan matrix determined by $\varphi|_{U_{\varphi}}$ and P is the corresponding basis change matrix.

If $A \in \text{Mat}_{n \times n}(k)$ is again a square matrix with $i(A) = r$ and $\text{rk}(A^i)$ is the rank of A^i , we can write $v_i(A) = n - \text{rk}(A^i)$ for all $i \in \{1, \dots, r\}$ and we can consider the non-negative integers $\{\delta_1(A), \dots, \delta_r(A)\}$ defined from the equations:

$$\begin{aligned} \delta_r(A) &= v_r(A) - v_{r-1}(A) \\ 2\delta_r(A) + \delta_{r-1}(A) &= v_r(A) - v_{r-2}(A) \\ &\vdots \\ (r-1)\delta_r(A) + \dots + 2\delta_3(A) + \delta_2(A) &= v_r(A) - v_1(A) \\ r\delta_r(A) + (r-1)\delta_{r-1}(A) + \dots + 2\delta_2(A) + \delta_1(A) &= v_r(A) \end{aligned}$$

From these relations it is clear that

$$v_1(A) = \sum_{i=1}^r \delta_i(A) \quad \text{and} \quad \sum_{j=1}^r \delta_j(A)(v_r(A) - j) = [v_1(A) - 1]v_r(A).$$

Accordingly, the explicit expression of the matrix J_U is

$$J_U = \begin{pmatrix} A_1^1 & 0 & \dots & \dots & \dots & 0 & 0 \\ 0 & \ddots & \ddots & \dots & \dots & \vdots & 0 \\ \vdots & \ddots & A_1^{\delta_1(A)} & \ddots & \dots & \vdots & \vdots \\ \vdots & \dots & \ddots & \ddots & \ddots & \dots & \vdots \\ \vdots & \dots & \dots & \ddots & A_r^1 & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & \dots & \dots & 0 & A_r^{\delta_r(A)} \end{pmatrix} \in \text{Mat}_{v_r \times v_r}(k),$$

where

$$A_j^s = \begin{pmatrix} 0 & 0 & \dots & \dots & \dots & 0 \\ 1 & 0 & \ddots & \dots & \dots & 0 \\ 0 & 1 & 0 & \ddots & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix} \in \text{Mat}_{j \times j}(k)$$

for every $j \in \{1, \dots, r\}$ and $1 \leq s \leq \delta_j(A)$.

With the above notation, if $A\{GD\}$ is the set of the G-Drazin inverses of A , it follows from Lemma 3.9 that there exists a bijection

$$\begin{aligned} k^{v_1(A) \cdot v_r(A)} \times k^{[v_r(A) - v_1(A)]} \times k^{[v_1(A) - 1][v_r(A) - v_1(A)]} &\xrightarrow{\sim} A\{GD\} \\ ((\alpha_{j,h}^s), (\alpha_{j,j',z}^s), (\lambda_{j,t}^s), (\lambda_{j,j',x}^s)) &\longmapsto (A^{GD})_{((\alpha_{j,h}^s), (\lambda_{j,t}^s))}^{((\alpha_{j,j',z}^s), (\lambda_{j,j',x}^s))}, \end{aligned} \quad (16)$$

where $j, j' \in \{1, \dots, r\}$; $j \neq j'$; $1 \leq h \leq j$; $z \in \{1, \dots, j\}$; $t \in \{2, \dots, j\}$; $x \in \{2, \dots, j'\}$; $s \in \{1, \dots, \delta_j(A)\}$ and

$$(A^{GD})_{((\alpha_{j,h}^s), (\lambda_{j,t}^s))}^{((\alpha_{j,j',z}^s), (\lambda_{j,j',x}^s))} = P \cdot \begin{pmatrix} (A_W)^{-1} & 0 \\ 0 & (J_U^-)_{((\alpha_{j,h}^s), (\lambda_{j,t}^s))}^{((\alpha_{j,j',z}^s), (\lambda_{j,j',x}^s))} \end{pmatrix} \cdot P^{-1}$$

with

$$(J_U^-)_{((\alpha_{j,h}^s), (\lambda_{j,t}^s))}^{((\alpha_{j,j',z}^s), (\lambda_{j,j',x}^s))} = ((J_U^-)_{((\alpha_{j,h}^s), (\lambda_{j,t}^s))}^{((\alpha_{j,j',z}^s), (\lambda_{j,j',x}^s))})_{lm} \Big|_{1 \leq l, m \leq v_1} \in \text{Mat}_{v_r \times v_r}(k)$$

such that

- if $j \in \{1, \dots, r\}$ and $1 \leq s \leq \delta_j(A)$ are such that $l = (\sum_{i=1}^{j-1} \delta_i) + s$, then $(J_U^-)_{ll} \in \text{Mat}_{j \times j}(k)$ with

$$\begin{aligned} ((J_U^-)_{((\alpha_{j,h}^s), (\lambda_{j,t}^s))}^{((\alpha_{j,j',z}^s), (\lambda_{j,j',x}^s))})_{ll} &= (A_{\{\alpha_{j,1}^s, \dots, \alpha_{j,j-1}^s\}}^{\{\lambda_{j,1}^s, \dots, \lambda_{j,j}^s\}})^{-} \\ &= \begin{pmatrix} \alpha_{j,1}^s & 1 & 0 & \dots & \dots & 0 \\ \alpha_{j,2}^s & 0 & 1 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \alpha_{j,j-2}^s & 0 & \dots & \ddots & 1 & 0 \\ \alpha_{j,j-1}^s & 0 & \dots & \dots & 0 & 1 \\ \alpha_{j,j}^s & \lambda_{j,2}^s & 0 & \dots & \lambda_{j,j-1}^s & \lambda_{j,j}^s \end{pmatrix}, \end{aligned}$$

for all $\alpha_{j,h}^s, \lambda_{j,t}^s \in k$, $h \in \{1, \dots, j\}$, $t \in \{2, \dots, j\}$, $j \in \{1, \dots, r\}$ and $s \in \{1, \dots, \delta_j(A)\}$;

- if $j, j' \in \{1, \dots, r\}$, $1 \leq s \leq \delta_j(A)$ and $1 \leq s' \leq \delta_{j'}(A)$ are such that

$$l = \left(\sum_{i=1}^{j-1} \delta_i \right) + s \quad \text{and} \quad m = \left(\sum_{i=1}^{j'-1} \delta_i \right) + s',$$

with $l \neq m$, then $(J_U^-)_{lm} \in \text{Mat}_{j \times j'}(k)$ where

$$\begin{aligned} ((J_U^-)_{((\alpha_{j,h}^s), (\lambda_{j,t}^s))}^{((\alpha_{j,j',z}^s), (\lambda_{j,j',x}^s))})_{lm} &= (A_{\{\alpha_{j,j',1}^s, \dots, \alpha_{j,j',j}^s\}}^{\{\lambda_{j,j',2}^s, \dots, \lambda_{j,j',j'}^s\}})^{-} \\ &= \begin{pmatrix} \alpha_{j,j',1}^s & 0 & \dots & \dots & 0 \\ \alpha_{j,j',2}^s & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ \alpha_{j,j',j-1}^s & 0 & \dots & \dots & 0 \\ \alpha_{j,j',j}^s & \lambda_{j,j',2}^s & \dots & \dots & \lambda_{j,j',j'}^s \end{pmatrix} \end{aligned}$$

for every $\alpha_{j,j',z}^s, \lambda_{j,j',x}^s \in k$, $j \neq j'$, $1 \leq j, j' \leq r$, $z \in \{1, \dots, j\}$, $x \in \{2, \dots, j'\}$ and $s \in \{1, \dots, \delta_j(A)\}$.

If $\tilde{A} \in \text{Mat}_{n \times n}(k)$ with $i(\tilde{A}) = 1$, it follows from (16) that there exists a bijection $\tilde{A}\{GD\} \xrightarrow{\sim} k^{[v_1(\tilde{A})^2]}$.

From the results of this work, we can finally offer the following algorithm for computing the G-Drazin inverses of $A \in \text{Mat}_{n \times n}(k)$.

- (1) Fix a k -vector space E with dimension n , a basis $B = \{e_1, \dots, e_n\}$ of E and an endomorphism $\varphi \in \text{End}_k(E)$ associated with A in the basis B , to facilitate the computations.
- (2) Compute the AST-decomposition $E = W_\varphi \oplus U_\varphi$ and the matrix expression (15) for A .
- (3) Calculate the non-negative integer numbers $\{v_1(A), \dots, v_r(A)\}$ and $\{\delta_1(A), \dots, \delta_r(A)\}$.
- (4) Construct the matrices $(J_U^-)^{((\alpha_{jj',z}^s), (\lambda_{jj',x}^s))_{((\alpha_{j,h}^s), (\lambda_{j,t}^s))}}$ and compute $(A_W)^{-1}$.
- (5) Get all the G-Drazin inverses $(A^{GD})^{((\alpha_{jj',z}^s), (\lambda_{jj',x}^s))_{((\alpha_{j,h}^s), (\lambda_{j,t}^s))}}$ of A .

Remark 4.1: We wish remark that is not necessary to compute the characteristic polynomial $c_A(x)$ in the method offered in this paper for calculate all the G-Drazin inverses of a square matrix A with $i(A) = r$, because we can obtain the matrices A_W and A_U by computing $R(A^r)$ and $N(A^r)$, where $R(B)$ and $N(B)$ are the range and the nullspace of a matrix B respectively.

Example 4.1: Let us consider an arbitrary field k and the matrix

$$A = \begin{pmatrix} -9 & -7 & 11 & -3 & -6 & -4 & -2 \\ -3 & 1 & 2 & 1 & 1 & 0 & 1 \\ -13 & -8 & 15 & -3 & -7 & -5 & -2 \\ -4 & -3 & 5 & -1 & -3 & -2 & -1 \\ -13 & -12 & 17 & -6 & -11 & -7 & -4 \\ 11 & 10 & -14 & 5 & 9 & 6 & 3 \\ 8 & 6 & -10 & 3 & 6 & 4 & 2 \end{pmatrix} \in \text{Mat}_{7 \times 7}(k).$$

We shall compute all the G-Drazin inverses A^{GD} of A .

Let us now fix a k vector space E with basis $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ and an endomorphism $\varphi \in \text{End}_k(E)$ such that $\varphi \equiv A$ in this basis.

It is easy to check that $i(A) = 3$, $\text{rk}(A) = 5$, $\text{rk}(A^2) = 3$ and $\text{rk}(A^3) = 2$. Accordingly, $v_1(A) = 2$, $v_2(A) = 4$, $v_3(A) = 5$, $\delta_1(A) = 0$, $\delta_2(A) = 1$ and $\delta_3(A) = 1$.

Now, a non-difficult computation shows that $W_\varphi = \langle e_1 + e_2 + 2e_3 + e_5 - e_7, e_4 - e_6 \rangle$, $U_\varphi = \langle e_1 + e_3, e_2 - e_5, -e_5 + e_6, e_4 - e_7, e_1 + e_3 - e_7 \rangle$,

$$A_W = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

and

$$A_U = \begin{pmatrix} 3 & -1 & 3 & -2 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ -3 & 1 & -3 & 2 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 \end{pmatrix}.$$

Moreover, from the Jordan basis

$$\{-e_1 + e_2 - e_3 - e_5 - e_7, -e_1 - e_3 - e_5 + e_6, e_1 + e_2 + e_3 + e_4 - e_5 - e_7, \\ -e_2 + e_4 + e_5 - e_7, -e_5 + e_6 + e_7\}$$

of U_φ induced by $\varphi|_{U_\varphi}$, one gets that

$$A = P \cdot \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \cdot P^{-1},$$

with

$$P = \begin{pmatrix} 1 & 0 & -1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & -1 & 0 \\ 2 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & -1 & -1 & 1 & -1 \\ 0 & -1 & 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & -1 & -1 & 1 \end{pmatrix}$$

and

$$P^{-1} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & -1 & 2 & 0 & -1 & -1 & 0 \\ 5 & 6 & -7 & 3 & 5 & 3 & 2 \\ -8 & -8 & 10 & -4 & -7 & -4 & -3 \\ -1 & -2 & 2 & -1 & -2 & -1 & -1 \\ 3 & 3 & -4 & 2 & 3 & 2 & 1 \\ 6 & 7 & -8 & 4 & 6 & 4 & 3 \end{pmatrix}.$$

Thus, it follows from the method described above that a G-Drazin inverse of A has the explicit expression

$$(A^{GD})_{((\alpha_{j,h}^1), (\lambda_{j,t}^1))}^{(\gamma_{j,j',z}^1)} = P \cdot \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_{2,1}^1 & 1 & \alpha_{2,3,1}^1 & 0 & 0 \\ 0 & 0 & \alpha_{2,2}^1 & \lambda_{2,2}^1 & \alpha_{2,3,2}^1 & \lambda_{2,3,2}^1 & \lambda_{2,3,3}^1 \\ 0 & 0 & \alpha_{3,2,1}^1 & 0 & \alpha_{3,1}^1 & 1 & 0 \\ 0 & 0 & \alpha_{3,2,2}^1 & 0 & \alpha_{3,2}^1 & 0 & 1 \\ 0 & 0 & \alpha_{3,2,3}^1 & \lambda_{3,2,2}^1 & \alpha_{3,3}^1 & \lambda_{3,2}^1 & \lambda_{3,3}^1 \end{pmatrix} \cdot P^{-1},$$

with $\alpha_{j,h}^1, \lambda_{j,t}^1, \alpha_{j,j',z}^1, \lambda_{j,j',x}^1 \in k$ for every $j \neq j', 2 \leq j, j' \leq 3, h \in \{1, \dots, j\}, t \in \{2, \dots, j\}, z \in \{1, \dots, j\}$ and $x \in \{2, \dots, j'\}$.

Hence, with the data of this example, we have the following bijection that determines all the G-Drazin inverses of A :

$$k^{10} \times k^3 \times k^3 \xrightarrow{\sim} A\{GD\}$$

$$((\alpha_{j,h}^1), (\alpha_{j,j',z}^1)_{\{j,j',h,z\}}, (\lambda_{j,t}^1)_{\{j,t\}}, (\lambda_{j,j',x}^1)_{\{j,j',x\}}) \mapsto (A^{GD})_{((\alpha_{j,h}^1), (\lambda_{j,t}^1))}^{(\gamma_{j,j',z}^1)},$$

with $j \neq j', 2 \leq j, j' \leq 3, h \in \{1, \dots, j\}, t \in \{2, \dots, j\}, z \in \{1, \dots, j\}$ and $x \in \{2, \dots, j'\}$.

Finally, we shall study the relationships that there exist between G-Drazin inverses and the core-nilpotent decomposition of a matrix A .

Remark 4.2: If k is an arbitrary ground field, $A \in \text{Mat}_{n \times n}(k)$, $A = A_1 + A_2$ is its core-nilpotent decomposition and A^{GD} is a G-Drazin inverse of A with core-nilpotent decomposition $A^{GD} = (A^{GD})_1 + (A^{GD})_2$, Example 4.1 shows that, in general, $(A^{GD})_1$ is not a G-Drazin inverse of A_1 and $(A^{GD})_2$ is not a G-Drazin inverse of A_2 .

As a counterexample of this fact we offer the following: if A is the matrix studied in Example 4.1, an easy computation shows that its core-nilpotent decomposition is $A = A_1 + A_2$ with

$$A_1 = \begin{pmatrix} -4 & -1 & 4 & 0 & -1 & -1 & 0 \\ -4 & -1 & 4 & 0 & -1 & -1 & 0 \\ -8 & -2 & 8 & 0 & -2 & -2 & 0 \\ -3 & -1 & 3 & 0 & -1 & -1 & 0 \\ -4 & -1 & 4 & 0 & -1 & -1 & 0 \\ 3 & 1 & -3 & 0 & 1 & 1 & 0 \\ 4 & 1 & -4 & 0 & 1 & 1 & 0 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} -5 & -6 & 7 & -3 & -5 & -3 & -2 \\ 1 & 2 & -2 & 1 & 2 & 1 & 1 \\ -5 & -6 & 7 & -3 & -5 & -3 & -2 \\ -1 & -2 & 2 & -1 & -2 & -1 & -1 \\ -9 & -11 & 13 & -6 & -10 & -6 & -4 \\ 8 & 9 & -11 & 5 & 8 & 5 & 3 \\ 4 & 5 & -6 & 3 & 5 & 3 & 2 \end{pmatrix}.$$

If we now consider

$$A^{GD} = \begin{pmatrix} 7 & 6 & -8 & 3 & 6 & 4 & 2 \\ -10 & -11 & 13 & -6 & -9 & -5 & -5 \\ 8 & 7 & -9 & 3 & 7 & 5 & 2 \\ 6 & 8 & -9 & 6 & 7 & 4 & 4 \\ 8 & 9 & -10 & 4 & 8 & 5 & 4 \\ 7 & 6 & -8 & 2 & 5 & 4 & 1 \\ -3 & -5 & 5 & -3 & -5 & -4 & 2 \end{pmatrix},$$

which is the G-Drazin inverse of A determined by $\lambda_{2,1}^1 = \lambda_{3,1}^1 = 1$ and otherwise $\alpha_{j,h}^1 = \lambda_{j,t}^1 = \gamma_{j,j',z}^1 = 0$, one has that $(A^{GD})_1 = (A^{GD})^{-1}$ and $(A^{GD})_2 = 0$, and we can immediately check that $(A^{GD})_1$ is not a G-Drazin inverse of A_1 and $(A^{GD})_2$ is not a G-Drazin inverse of A_2 .

Furthermore, if $(A_1)^{GD}$ is a G-Drazin inverse of A_1 and $(A_2)^{GD}$ is a G-Drazin inverse of A_2 , in general, one has that $\bar{A}^{GD} = (A_1)^{GD} + (A_2)^{GD}$ is not a G-Drazin inverse of A , as can be deduced from this counterexample: keeping again the data of Example 4.1, if we consider

$$(A_1)^{GD} = P \cdot \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot P^{-1}$$

and

$$(A_2)^{GD} = P \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot P^{-1},$$

then it is clear that

$$\bar{A}^{GD} = P \cdot \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot P^{-1}$$

is not a G-Drazin inverse of A .

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References

- [1] Wang X, Liu H. Partial orders based on core-nilpotent decomposition. *Linear Algebra Appl.* 2016;488:235–248.
- [2] Coll C, Lattanzi M, Thome N. Weighted G-Drazin inverses and a new pre-order on rectangular matrices. *Appl Math Comput.* 2018;317:12–24.
- [3] Tate J. Residues of differentials on curves. *Ann Sci École Norm Sup 4.* 1968;1:149–159.
- [4] Pablos Romo F. Core-Nilpotent decomposition and new generalized inverses of finite potent endomorphisms. *Linear Multilinear Algebra.* 2019. doi:10.1080/03081087.2019.1578332
- [5] Pablos Romo F. On Drazin-Moore-Penrose inverses of finite potent endomorphisms. *Linear Multilinear Algebra.* 2019. doi:10.1080/03081087.2019.1612834
- [6] Pablos Romo F. On the Drazin inverse of finite potent endomorphisms. *Linear Multilinear Algebra.* 2019;67(10):2135–2146.
- [7] Argerami M, Szechtman F, Tifenbach R. On Tate's trace. *Linear Multilinear Algebra.* 2007;55(6):515–520.
- [8] Hernández Serrano D, Pablos Romo F. Determinants of finite potent endomorphisms, symbols and reciprocity laws. *Linear Algebra Appl.* 2013;439:239–261.
- [9] Pablos Romo F. On the linearity property of Tate's trace. *Linear Multilinear Algebra.* 2007;55(4):323–326.
- [10] Ramos González J, Pablos Romo F. A negative answer to the question of the linearity of Tate's trace for the sum of two endomorphisms. *Linear Multilinear Algebra.* 2014;62(4):548–552.
- [11] Pablos Romo F. Classification of finite potent endomorphisms. *Linear Algebra Appl.* 2014;440:266–277.
- [12] López-Pellicer M, Bru R. Jordan basis for an infinite-dimensional space. *Port Math.* 1985/86;43(1):153–156.
- [13] Drazin MP. Pseudo-inverses in associative rings and semigroups. *Amer Math Monthly.* 1958;65(7):506–514.