Linear and Multilinear Algebra

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/glma20

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To cite this article: Fernando Pablos Romo (2022) On the coincidence of the Drazin inverse and the Drazin-Moore-Penrose inverses, Linear and Multilinear Algebra, 70:19, 3617-3632, DOI: 10.1080/03081087.2020.1849004

To link to this article: https://doi.org/10.1080/03081087.2020.1849004

Published online: 18 Nov 2020.

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LINEAR AND MULTILINEAR ALGEBRA
2022, VOL. 70, NO. 19, 3617–3632
https://doi.org/10.1080/03081087.2020.1849004

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On the coincidence of the Drazin inverse and the Drazin-Moore-Penrose inverses

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ABSTRACT

The aim of this work is to offer necessary and sufficient conditions for the coincidence of the Drazin inverse and the DMP inverses of finite square complex matrices. This characterization is deduced from statements valid for finite potent endomorphisms and, in particular, several properties of matrices and generalized inverses are given.

ARTICLE HISTORY
Received 7 July 2020
Accepted 29 October 2020

COMMUNICATED BY
B. Kuzma

KEYWORDS
Drazin inverse; DMP inverse; finite potent endomorphism

2020 MATHEMATICS SUBJECT CLASSIFICATIONS
15A09; 15A03; 15A04

1. Introduction

For an arbitrary $n \times n$-matrix A with entries in the complex numbers, the index of A, $i(A) \geq 0$, is the smallest integer such that $\text{rk}(A^{i(A)}) = \text{rk}(A^{i(A)+1})$.

In 1958, M. P. Drazin in [1] showed the existence of a unique $n \times n$ complex matrix A^D, called the Drazin inverse, satisfying the equations:

- $A^{r+1}A^D = A^r$ for $r = i(A)$;
- $A^D A A^D = A^D$;
- $A^D A = A A^D$.

When $i(A) \leq 1$, it is known that the Drazin inverse A^D coincides with the group inverse $A^{\#}$.

If A is an $n \times m$ complex matrix, the Moore-Penrose inverse of A is an $m \times n$ complex matrix $A^{\dagger}$ such that:

- $AA^{\dagger}A = A$;
- $A^{\dagger}AA^{\dagger} = A^{\dagger}$;
- $(AA^{\dagger})^* = AA^{\dagger}$;
- $(A^{\dagger}A)^* = A^{\dagger}A$,

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A^* being the conjugate transpose of the matrix A.

The Moore-Penrose inverse of A always exists, it is unique, $[A^\dagger]^\dagger = A$, and, if $\tilde{A}$ is non-singular $n \times n$ complex matrix, then the Moore-Penrose inverse of $\tilde{A}$ coincides with the inverse matrix $\tilde{A}^{-1}$.

For more details about the Moore-Penrose inverse, readers are referred to [2].

Moreover, given again an $n \times n$ complex matrix A with $i(A) = r$, in 2014, S. B. Malik and N. Thome proved in [3] the existence of a unique $n \times n$ complex matrix $A^{d,\dagger}$ satisfying the equations:

- $A^r A^{d,\dagger} = A^r A^\dagger$;
- $A^{d,\dagger} A A^{d,\dagger} = A^{d,\dagger}$;
- $A^{d,\dagger} A = A^D A$.

The matrix $A^{d,\dagger}$ is called the left-DMP inverse of A (left-Drazin-Moore-Penrose inverse) and the right-DMP inverse $A^{\dagger,d}$ can be defined analogously as the unique matrix satisfying the conditions:

- $A^{\dagger,d} A^r = A^\dagger A^r$;
- $A^{\dagger,d} A A^{\dagger,d} = A^{\dagger,d}$;
- $A A^{d,\dagger} = A A^D$.

For every $n \times n$ complex matrix A with $i(A) \leq 1$, from the statements of [4] one has that:

- $A^\# = A^{d,\dagger}$ if and only if $N(A^\dagger) \subseteq N(A^D)$;
- $A^\# = A^{\dagger,d}$ if and only if $R(A) \subseteq R(A^*)$;

where $R(B)$ is the range and $N(B)$ is the nullspace of an arbitrary complex matrix B. The aim of this work is to prove that these conditions characterize matrices with arbitrary index such that $A^D = A^{d,\dagger}$ or $A^D = A^{\dagger,d}$, respectively.

As far as we know this characterization is not explicitly stated in the literature.

The main tool for proving these characterizations for finite matrices will be the theory of finite potent endomorphisms, although the properties involving the Drazin-Star or the Star-Drazin of a matrix (see [5] for details) will be proved with arguments exclusively concerned with finite square complex matrices. Indeed, the notions of Drazin inverse and DMP inverses have been recently extended by the author to finite potent endomorphisms in [6, 7]. From the structure of these linear maps, given an arbitrary inner product vector space V over $k = \mathbb{C}$ or $k = \mathbb{R}$ and a finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$, we shall prove that '$\varphi^{d,\dagger} = \varphi^D$ if and only if $\varphi \circ \varphi^{d,\dagger} = \varphi^{d,\dagger} \circ \varphi$' (Proposition 3.1) and '$\varphi^{\dagger,d} = \varphi^D$ if and only if $\varphi \circ \varphi^{\dagger,d} = \varphi^{\dagger,d} \circ \varphi$' (Proposition 3.6).

Accordingly, we shall show that '$A^D = A^{d,\dagger}$ if and only if $N(A^\dagger) \subseteq N(A^D)$' and '$A^D = A^{\dagger,d}$ if and only if $R(A_1) \subseteq R(A^*)$'. Moreover, several properties related to $A^{d,\dagger}$, $A^{\dagger,d}$ and A^D will be deduced from these characterizations.

The last part of this work provides explicit examples of matrices $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ such that $A^D = A^{d,\dagger} = A^{\dagger,d}$ (Example 4.1), $A^D = A^{d,\dagger}$ but $A^D \neq A^{\dagger,d}$ (Example 4.2) and $A^D = A^{\dagger,d}$ with $A^D \neq A^{d,\dagger}$ (Example 4.3).

2. Preliminaries

This section is added for the sake of completeness.

2.1. Finite potent endomorphisms

Let k be an arbitrary field, and let V be a k-vector space.

Let us now consider an endomorphism φ of V. We say that φ is 'finite potent' if $\varphi^n V$ is finite dimensional for some n. This definition was introduced by J. Tate in [8] as a basic tool for his elegant definition of Abstract Residues.

In 2007 M. Argerami, F. Szechtman and R. Tifenbach showed in [9] that an endomorphism φ is finite potent if and only if V admits a φ-invariant decomposition $V = U_\varphi \oplus W_\varphi$ such that $\varphi_{|U_\varphi}$ is nilpotent, W_φ is finite dimensional and $\varphi_{|W_\varphi}: W_\varphi \xrightarrow{\sim} W_\varphi$ is an isomorphism.

Indeed, if $k[x]$ is the algebra of polynomials in the variable x with coefficients in k, we may view V as an $k[x]$-module via φ, and the explicit definition of the above φ-invariant subspaces of V is:

- $U_\varphi = \{v \in V \text{ such that } \varphi^m(v) = 0 \text{ for some m}\}$;
- $W_\varphi = \{v \in V \text{ such that } p(\varphi)v = 0 \text{ for some } p(x) \in k[x] \text{ relatively prime to } x\}$.

Note that if the annihilator polynomial of φ is $x^m \cdot p(x)$ with $(x, p(x)) = 1$, then $U_\varphi = \operatorname{Ker} \varphi^m$ and $W_\varphi = \operatorname{Ker} p(\varphi)$.

Hence, this decomposition is unique. We shall call this decomposition the φ-invariant AST-decomposition of V.

Moreover, we shall call 'index of φ', $i(\varphi)$, the nilpotent order of $\varphi_{|U_\varphi}$, which coincides with the smallest $n \in \mathbb{N}$ such that $\dim_k \varphi^n V < \infty$. One has that $i(\varphi) = 0$ if and only if V is a finite-dimensional vector space and φ is an automorphism.

2.2. Drazin inverse of finite potent endomorphisms

Let V be an arbitrary k-vector space, and let $\varphi \in \operatorname{End}_k(V)$ be a finite potent endomorphism of V. Let us consider the AST-decomposition $V = U_\varphi \oplus W_\varphi$ induced by φ.

According to [7, Theorem 3.4], for each finite potent endomorphism φ there exists a unique finite potent endomorphism φ^D that satisfies:

(1) $\varphi^{r+1} \circ \varphi^D = \varphi^r$;
(2) $\varphi^D \circ \varphi \circ \varphi^D = \varphi^D$;
(3) $\varphi^D \circ \varphi = \varphi \circ \varphi^D$,

where r is the index of φ.

The map φ^D is the Drazin inverse of φ and is the unique linear map such that:

$$\varphi^D(v) = \begin{cases} (\varphi_{|W_\varphi})^{-1}(v) & \text{if } v \in W_\varphi \\ 0 & \text{if } v \in U_\varphi \end{cases}.$$

Moreover, φ^D satisfies the following properties:

- $(\varphi^D)^D = \varphi$ if and only if the $i(\varphi) \leq 1$;
- $\varphi = \varphi^D$ if and only if $\varphi_{|U_\varphi} = 0$ and $(\varphi_{|W_\varphi})^2 = \mathrm{Id}_{|W_\varphi}$;
- $\mathrm{Tr}_V(\varphi + \varphi^D) = \mathrm{Tr}_V(\varphi) + \mathrm{Tr}_V(\varphi^D)$;
- If ψ is a projection finite potent endomorphism, then $\psi^D = \psi$.

Readers can see [7] for more details on the Drazin inverse of a finite potent endomorphism.

2.3. Moore-Penrose inverse of a linear map over arbitrary vector spaces

Let (V, g) and $(W, \bar{g})$ be inner product vector spaces over $k = \mathbb{C}$ or $k = \mathbb{R}$.

Given a linear map $f\colon V \to W$, a linear map $f^+\colon W \to V$ is a reflexive generalized inverse of f when

- $f \circ f^+ \circ f = f$;
- $f^+ \circ f \circ f^+ = f^+$.

Definition 2.1: Given a linear map $f\colon V \to W$, we say that f is admissible for the Moore-Penrose inverse when $V = \mathrm{Ker} f \oplus [\mathrm{Ker} f]^\perp$ and $W = \mathrm{Im} f \oplus [\mathrm{Im} f]^\perp$.

If (V, g) and $(W, \bar{g})$ are inner product spaces over k and $f\colon V \to W$ is a linear map admissible for the Moore-Penrose inverse, according to [10, Theorem 3.11] there exists a unique linear map $f^\dagger\colon W \to V$ such that:

(1) $f^\dagger$ is a reflexive generalized inverse of f;
(2) $f^\dagger \circ f$ and $f \circ f^\dagger$ are self-adjoint, that is,
   - $g([f^\dagger \circ f](v), v') = g(v, [f^\dagger \circ f](v'))$;
   - $\bar{g}([f \circ f^\dagger](w), w') = \bar{g}(w, [f \circ f^\dagger](w'))$;

for all $v, v' \in V$ and $w, w' \in W$. The operator $f^\dagger$ is called the Moore-Penrose inverse of f and it is the unique linear map satisfying

$$f^\dagger(w) = \begin{cases} (f_{|_{[\mathrm{Ker} f]^\perp}})^{-1}(w) & \text{if } w \in \mathrm{Im} f \\ 0 & \text{if } w \in [\mathrm{Im} f]^\perp \end{cases}.$$

The Moore-Penrose inverse $f^\dagger\colon W \to V$ also satisfies the following properties:

- $f^\dagger$ is admissible for the Moore-Penrose inverse and $(f^\dagger)^\dagger = f$;
- If $f \in \mathrm{End}_k(V)$ and f is an isomorphism, then $f^\dagger = f^{-1}$;
- $f^\dagger \circ f = P_{[\ker f]^\perp}$;
- $f \circ f^\dagger = P_{\mathrm{Im} f}$,

where $P_{[\ker f]^\perp}$ and $P_{\mathrm{Im} f}$ are the projections induced by the decompositions $V = \mathrm{Ker} f \oplus [\mathrm{Ker} f]^\perp$ and $W = \mathrm{Im} f \oplus [\mathrm{Im} f]^\perp$ respectively.

For details on this Moore-Penrose inverse readers are referred to [10].

2.4. Core-Nilpotent decomposition of a finite potent endomorphism

Given a finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$, there exists a unique decomposition $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1, \varphi_2 \in \mathrm{End}_k(V)$ are finite potent endomorphisms satisfying that:

- $i(\varphi_1) \leq 1$;
- φ_2 is nilpotent;
- $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1 = 0$.

According to [11, Theorem 3.2], one has that $\varphi_1 = \varphi \circ \varphi^D \circ \varphi$, which is called the core part of φ. Also, φ_2 is called the nilpotent part of φ. This expression is the CN-decomposition of φ.

Moreover, one has that

$$\varphi = \varphi_1 \Longleftrightarrow U_\varphi = \mathrm{Ker}\,\varphi \Longleftrightarrow W_\varphi = \mathrm{Im}\varphi \Longleftrightarrow (\varphi^D)^D = \varphi \Longleftrightarrow i(\varphi) \leq 1. \quad (1)$$

2.5. Core-Moore-Penrose inverse of a finite potent endomorphism

Let $\varphi \in \mathrm{End}_k(V)$ be a finite potent endomorphism admissible for the Moore-Penrose inverse of an arbitrary k-vector space V.

According to [11, Theorem 4.5], there exists a unique finite-potent endomorphism $\varphi^{c\dagger}$ verifying that:

(1) $\varphi^{c\dagger} \circ \varphi \circ \varphi^{c\dagger} = \varphi^{c\dagger}$;
(2) $\varphi \circ \varphi^{c\dagger} \circ \varphi = \varphi_1$;
(3) $\varphi \circ \varphi^{c\dagger} = \varphi_1 \circ \varphi^{\dagger}$;
(4) $\varphi^{c\dagger} \circ \varphi = \varphi^{\dagger} \circ \varphi_1$,

where φ_1 is the core part of its CN-decomposition.

The finite potent endomorphism $\varphi^{c\dagger}$ is the CMP inverse of φ.

2.6. Drazin-Moore-Penrose inverses

Given an $n \times n$ complex matrix A with $i(A) = r$, in 2014 S. B. Malik and N. Thome showed in [3] the existence of a unique $n \times n$ complex matrix $A^{d,\dagger}$ satisfying the equations:

- $A^r A^{d,\dagger} = A^r A^{\dagger}$;
- $A^{d,\dagger} A A^{d,\dagger} = A^{d,\dagger}$;
- $A^{d,\dagger} A = A^D A$.

$A^{d,\dagger}$ is called the left-Drazin-Moore-Penrose (left-DMP) inverse of A and its explicit expression is $A^{d,\dagger} = A^D A A^{\dagger}$. Moreover, if A has index 1, the generalized inverse $A^{d,\dagger}$ coincides with the core inverse of A offered by O. M. Baksalary and G. Trenkler in [12].

Readers can see [3, Proposition 2.14] for several properties satisfied by the DMP inverse.

Similarly, the right-DMP inverse $A^{\dagger,d}$ can be defined from the equations referred to in Section 1 and its explicit expression is $A^{\dagger,d} = A^{\dagger} A A^D$.

The notions of left-DMP inverse and right-DMP inverse have been extended to finite potent endomorphisms in [6]. In particular, again given an $n \times n$ complex matrix A, readers can check the properties of the right-DMP inverse $A^{\dagger,d}$ in [6, p.16].

2.7. Drazin-star and star-drazin inverses

Recently, D. Mosić has introduced in [5] the notions of Drazin-Star and Star-Drazin inverses.

The Drazin-Star inverse of a square complex matrix A is $A^{D,*} = A^D AA^*$ and it is defined as the unique solution of the system of equations:

$$\begin{aligned} X(A^\dagger)^* X &= X \\ A^r X &= A^r A^*, \\ X(A^\dagger)^* &= A^D A \end{aligned}$$

where $i(A) = r$.

Analogously, the Star-Drazin of A is the unique solution of

$$\begin{aligned} X(A^\dagger)^* X &= X \\ XA^r &= A^* A^r \\ (A^\dagger)^* X &= A^D A \end{aligned}$$

and its explicit expression is $A^{*,D} = A^* AA^D$.

3. Coincidence of the Drazin inverse and the DMP inverses

In this main section we shall characterize the conditions under which the Drazin inverse coincides with the left-DMP inverse or with the right-DMP inverse of a finite potent endomorphism. These conditions are valid for finite matrices and, as an application, several properties of matrices and generalized inverses are obtained.

3.1. Coincidence of the Drazin inverse and the left-DMP inverse

Let V be an inner product vector space over $k = \mathbb{C}$ or $k = \mathbb{R}$.

According to [6, Theorem 3.2], given a finite potent endomorphism $\varphi \in \operatorname{End}_k(V)$ admissible for the Moore-Penrose inverse, the left-DMP inverse $\varphi^{d,\dagger}$ is the unique finite potent endomorphism satisfying the following conditions:

(1) $\varphi^{d,\dagger} \circ \varphi \circ \varphi^{d,\dagger} = \varphi^{d,\dagger}$;
(2) $\varphi^r \circ \varphi^{d,\dagger} = \varphi^r \circ \varphi^\dagger$ with $r = i(\varphi)$;
(3) $\varphi^{d,\dagger} \circ \varphi = \varphi^D \circ \varphi$,

where φ^D is the Drazin inverse and $\varphi^\dagger$ is the Moore-Penrose inverse of φ. In fact, $\varphi^{d,\dagger} = \varphi^D \circ \varphi \circ \varphi^\dagger$.

Proposition 3.1: *Given an inner product vector space V over $\mathbb{R}$ or $\mathbb{C}$, and a finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$ admissible for the Moore-Penrose inverse, one has that $\varphi^{d,\dagger} = \varphi^D$ if and only if*

$$\varphi \circ \varphi^{d,\dagger} = \varphi^{d,\dagger} \circ \varphi.$$

Proof: Let us first assume that $\varphi^{d,\dagger} = \varphi^D$. In this case, it follows from the properties of the Drazin inverse that

$$\varphi \circ \varphi^{d,\dagger} = \varphi \circ \varphi^D = \varphi^D \circ \varphi = \varphi^{d,\dagger} \circ \varphi.$$

Conversely, when $\varphi \circ \varphi^{d,\dagger} = \varphi^{d,\dagger} \circ \varphi$ we shall check that $\varphi^{d,\dagger} = \varphi^D$. To do this we only need to see that $\varphi^{r+1} \circ \varphi^{d,\dagger} = \varphi^r$, where r is the index of φ.

Thus, bearing in mind that

$$\varphi^{r+1} \circ \varphi^{d,\dagger} = \varphi^r \circ (\varphi^{d,\dagger} \circ \varphi) = \varphi^r \circ (\varphi^D \circ \varphi) = \varphi^{r+1} \circ \varphi^D = \varphi^r,$$

the statement is proved. ∎

Corollary 3.2: *With the previous notation, one has that $\varphi^{d,\dagger} = \varphi^D$ if and only if* $\mathrm{Ker}\,\varphi^\dagger \subseteq \mathrm{Ker}\,\varphi^D$.

Proof: The claim is immediately deduced from Proposition 3.1 because it is known from [6, Proposition 3.9] that $\varphi \circ \varphi^{d,\dagger} = \varphi^{d,\dagger} \circ \varphi$ if and only if $\mathrm{Ker}\,\varphi^\dagger \subseteq \mathrm{Ker}\,\varphi^D$. ∎

Corollary 3.3: *Given an inner product vector space V over $k = \mathbb{R}$ or $k = \mathbb{C}$, and a finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$ admissible for the Moore-Penrose inverse, if* $\mathrm{Ker}\,\varphi^\dagger \subseteq \mathrm{Ker}\,\varphi^D$, *then*

$$(\varphi^D)^n = \begin{cases} (\varphi^D \circ \varphi^\dagger)^{\frac{n}{2}} & \text{if } n \text{ is even} \\ \varphi \circ (\varphi^D \circ \varphi^\dagger)^{\frac{n+1}{2}} & \text{if } n \text{ is odd} \end{cases}.$$

Proof: The result is a direct consequence of Corollary 3.2 and [6, Proposition 3.3]. ∎

Lemma 3.4: *Given an inner product vector space V over $k = \mathbb{R}$ or $k = \mathbb{C}$, and a finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$ admissible for the Moore-Penrose inverse, if $\varphi^{d,\dagger} = \varphi$, then $\varphi = \varphi^\dagger = \varphi^D$.*

Proof: If $\varphi^{d,\dagger} = \varphi$, it is clear that $\mathrm{Ker}\,\varphi^\dagger \subseteq \mathrm{Ker}\,\varphi$ and, since $\mathrm{Ker}\,\varphi \subseteq \mathrm{Ker}\,\varphi^D$, it follows from Corollary 3.2 that $\varphi^{d,\dagger} = \varphi^D$.

Then, bearing in mind that from [6, Lemma 3.12] we know that $\varphi^\dagger = \varphi^D$ when $\varphi^{d,\dagger} = \varphi$, the claim is proved. ∎

Corollary 3.5: *With the hypothesis of Lemma* 3.4, *one has that $\varphi^{d,\dagger}$ is the group inverse $\varphi^\#$.*

Proof: Since $i(\varphi) \leq 1$, the assertion is immediately deduced from Lemma 3.4 and [4, Theorem 3.5]. ∎

3.2. Coincidence of the Drazin inverse and the right-DMP inverse

Let V be again an inner product vector space over $k = \mathbb{C}$ or $k = \mathbb{R}$.

With the previous notation, if $\varphi \in \operatorname{End}_k(V)$ a finite potent endomorphism admissible for the Moore-Penrose inverse, it follows from [6, Theorem 3.17] that there exists a unique finite potent endomorphism $\varphi^{\dagger,d}$ satisfying the following conditions:

(1) $\varphi^{\dagger,d} \circ \varphi \circ \varphi^{\dagger,d} = \varphi^{\dagger,d}$;
(2) $\varphi^{\dagger,d} \circ \varphi^r = \varphi^{\dagger} \circ \varphi^r$ with $r = i(\varphi)$;
(3) $\varphi \circ \varphi^{\dagger,d} = \varphi \circ \varphi^D$.

The linear map $\varphi^{\dagger,d}$ is the right-Drazin Moore-Penrose (right-DMP) inverse of φ.

Proposition 3.6: *Given an inner product vector space V over $k = \mathbb{R}$ or $k = \mathbb{C}$, and a finite potent endomorphism $\varphi \in \operatorname{End}_k(V)$ admissible for the Moore-Penrose inverse, one has that $\varphi^{\dagger,d} = \varphi^D$ if and only if*

$$\varphi \circ \varphi^{\dagger,d} = \varphi^{\dagger,d} \circ \varphi.$$

Proof: Assuming that $\varphi^{\dagger,d} = \varphi^D$, from the properties of the Drazin inverse one has that

$$\varphi \circ \varphi^{\dagger,d} = \varphi \circ \varphi^D = \varphi^D \circ \varphi = \varphi^{\dagger,d} \circ \varphi.$$

Hence, similar to above, if $\varphi \circ \varphi^{\dagger,d} = \varphi^{\dagger,d} \circ \varphi$ and r is the index of φ, then we only need to check that $\varphi^{r+1} \circ \varphi^{\dagger,d} = \varphi^r$ for proving that $\varphi^{d,\dagger} = \varphi^D$.

However, for $\varphi^{\dagger,d}$ this property is immediate because

$$\varphi^{r+1} \circ \varphi^{\dagger,d} = \varphi^{r+1} \circ \varphi^D = \varphi^r,$$

and the proof is concluded. ∎

Moreover, bearing in mind the properties of the right-DMP inverse $\varphi^{\dagger,d}$ studied in [6], a direct consequence of Proposition 3.6 is:

Corollary 3.7: *With the above notation, one has that $\varphi^{d,\dagger} = \varphi^D$ if and only if $W_\varphi \subseteq [\operatorname{Ker} \varphi]^\perp$.*

Similar to Corollary 3.3, one can check that

Corollary 3.8: *Given an inner product vector space V over $k = \mathbb{R}$ or $k = \mathbb{C}$, and a finite potent endomorphism $\varphi \in \operatorname{End}_k(V)$ admissible for the Moore-Penrose inverse, if $W_\varphi \subseteq [\operatorname{Ker} \varphi]^\perp$, then*

$$(\varphi^D)^n = \begin{cases} (\varphi^{\dagger} \circ \varphi^D)^{\frac{n}{2}} & \text{if } n \text{ is even} \\ (\varphi^{\dagger} \circ \varphi^D)^{\frac{n+1}{2}} \circ \varphi & \text{if } n \text{ is odd} \end{cases}.$$

Lemma 3.9: *Given an inner product vector space V over $\mathbb{R}$ or $\mathbb{C}$, and a finite potent endomorphism $\varphi \in \operatorname{End}_k(V)$ admissible for the Moore-Penrose inverse, if $\varphi^{\dagger,d} = \varphi$, then $\varphi = \varphi^{\dagger} = \varphi^D$.*

Proof: If $\varphi^{\dagger,d} = \varphi$, one has that

$$W_\varphi \subseteq \mathrm{Im}\varphi = \mathrm{Im}\varphi^{\dagger,d} \subseteq \mathrm{Im}\varphi^{\dagger} = [\mathrm{Ker}\,\varphi]^{\perp},$$

and it follows from Corollary 3.7 that $\varphi^{d,\dagger} = \varphi^D$.

Now, according to the properties of $\varphi^{\dagger,d}$ offered in [6], it is known that $\varphi^{\dagger} = \varphi^D$ when $\varphi^{\dagger,d} = \varphi$ and the statement is deduced. ■

With the same arguments of Corollary 3.5, one can prove that:

Corollary 3.10: *With the hypothesis of Lemma* 3.9, *one has that* $\varphi^{\dagger,d}$ *is the group inverse* $\varphi^{\#}$.

3.3. Coincidence of the left-DMP and the right-DMP inverses

Keeping the previous notation, we shall now consider finite potent endomorphisms where the DMP inverses are equal, that is: we shall study a finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$ satisfying that $\varphi^{d,\dagger} = \varphi^{\dagger,d}$, where V is an inner product vector space over $k = \mathbb{R}$ or $k = \mathbb{C}$.

Proposition 3.11: *Let V be an inner product vector space over $k = \mathbb{R}$ or $k = \mathbb{C}$ and let $\varphi \in \mathrm{End}_k(V)$ be a finite potent endomorphism admissible for the Moore-Penrose inverse. If $\varphi^{d,\dagger} = \varphi^{\dagger,d}$, then one has that*

$$\varphi^{d,\dagger} = \varphi^D = \varphi^{\dagger,d}.$$

Proof: Let us assume that $\varphi^{d,\dagger} = \varphi^{\dagger,d}$. Then, it follows from the conditions that characterize $\varphi^{d,\dagger}$ and $\varphi^{\dagger,d}$ that

$$\varphi \circ \varphi^{\dagger,d} = \varphi \circ \varphi^D = \varphi^D \circ \varphi = \varphi^{d,\dagger} \circ \varphi = \varphi^{\dagger,d} \circ \varphi$$

and, similarly, one can check that $\varphi \circ \varphi^{d,\dagger} = \varphi^{d,\dagger} \circ \varphi$.

Thus, we deduce that $\varphi^{\dagger,d} = \varphi^D$ (Proposition 3.1) and $\varphi^{d,\dagger} = \varphi^D$ (Proposition 3.6), from where the statement is proved. ■

Corollary 3.12: *Given an inner product vector space V over $k = \mathbb{R}$ or $k = \mathbb{C}$, if $\varphi \in \mathrm{End}_k(V)$ is a finite potent endomorphism admissible for the Moore-Penrose inverse, then $\varphi^{d,\dagger} = \varphi^{\dagger,d}$ if and only if* $\mathrm{Ker}\,\varphi^{\dagger} \subseteq \mathrm{Ker}\,\varphi^D$ *and* $W_\varphi \subseteq [\mathrm{Ker}\,\varphi]^{\perp}$.

Proof: The assertion is immediately deduced from Proposition 3.11, Corollaries 3.2 and 3.7. ■

Given an inner product vector space V over $k = \mathbb{R}$ or $k = \mathbb{C}$, recall now from [11, Definition 4.12] that a finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$ admissible for the Moore-Penrose inverse is called 'Core-EP' when $\varphi_1 \circ \varphi^{\dagger} = \varphi^{\dagger} \circ \varphi_1$, where φ_1 is the core part of its CN-decomposition.

Moreover, if V is an inner product vector space over $k = \mathbb{R}$ or $k = \mathbb{C}$, according to [11, Definition 4.14] we say that a finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$ admissible for the Moore-Penrose inverse with $i(\varphi) = r$ is 'r-EP' when $\varphi^r \circ \varphi^{\dagger} = \varphi^{\dagger} \circ \varphi^r$.

Lemma 3.13: *If V is an inner product vector space over $k = \mathbb{R}$ or $k = \mathbb{C}$, a finite potent endomorphism $\varphi \in \operatorname{End}_k(V)$ admissible for the Moore-Penrose inverse is Core-EP if and only if* $\operatorname{Ker} \varphi^{\dagger} \subseteq \operatorname{Ker} \varphi^D$ *and* $W_{\varphi} \subseteq [\operatorname{Ker} \varphi]^{\perp}$.

Proof: It is known that a finite potent endomorphism $\varphi \in \operatorname{End}_k(V)$ admissible for the Moore-Penrose inverse is Core-EP if and only if $\varphi^{d,\dagger} = \varphi^{\dagger,d} = \varphi^{c\dagger}$ (see [6, page 16]). Hence, the statement is a direct consequence of Corollary 3.12. ■

Corollary 3.14: *Given an inner product vector space V over $k = \mathbb{R}$ or $k = \mathbb{C}$, a finite potent endomorphism $\varphi \in \operatorname{End}_k(V)$ admissible for the Moore-Penrose inverse satisfies that $\varphi^{c\dagger} = \varphi^D$ if and only if* $\operatorname{Ker} \varphi^{\dagger} \subseteq \operatorname{Ker} \varphi^D$ *and* $W_{\varphi} \subseteq [\operatorname{Ker} \varphi]^{\perp}$.

Proof: Bearing in mind that from [11, Proposition 4.13] one has that φ is Core-EP if and only if $\varphi^{c\dagger} = \varphi^D$, the claim is immediately deduced from Lemma 3.13. ■

Moreover, with the above notation, it follows from [11, Proposition 4.16] that a finite potent endomorphism $\varphi \in \operatorname{End}_k(V)$ admissible for the Moore-Penrose inverse with $i(\varphi) = r$ is Core-EP if and only if it is r-EP, and, therefore, φ is also r-EP if and only if $\operatorname{Ker} \varphi^{\dagger} \subseteq \operatorname{Ker} \varphi^D$ and $W_{\varphi} \subseteq [\operatorname{Ker} \varphi]^{\perp}$.

3.4. Coincidence of the Drazin and the DMP inverses on finite matrices

Given an $n \times n$ complex matrix A, we denote by $\operatorname{rk}(A)$ the rank of A. From the well-known equivalence between endomorphisms of finite-dimensional vector spaces and square matrices, we shall now offer properties relating the Drazin inverse to the DMP inverses of A.

If $i(A) = r$, it is known that $R(A^r) = R(A_1)$, where A_1 is the core part of the CN-decomposition $A = A_1 + A_2$ offered in [13, Theorem 2.2.21] with

- $\operatorname{rk}(A_1) = \operatorname{rk}(A_1^2)$, i.e $i(A_1) \leq 1$;
- A_2 is nilpotent;
- $A_1A_2 = A_2A_1 = 0$.

Moreover, if we consider the standard complex inner product

$$\begin{aligned} \langle \cdot, \cdot \rangle \colon \mathbb{C}^n \times \mathbb{C}^n &\longrightarrow \mathbb{C} \\ (x, y) &\longmapsto \overline{x_1} \cdot y_1 + \overline{x_2} \cdot y_2 + \cdots + \overline{x_n} \cdot y_n \end{aligned},$$

where $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{C}^n$ and $\overline{x_i}$ is the conjugate of x_i, one has that $N(A)^{\perp} = R(A^*)$ with A^* being the conjugate transpose of A.

Accordingly, from the preceding results we obtain the following properties for finite complex square matrices:

(1) $A^D = A^{d,\dagger}$ if and only if $N(A^{\dagger}) \subseteq N(A^D)$;
(2) $A^D = A^{\dagger,d}$ if and only if $R(A_1) \subseteq R(A^*)$;
(3) If $N(A^{\dagger}) \subseteq N(A^D)$, then $(A^D)^n = \begin{cases} (A^DA^{\dagger})^{\frac{n}{2}} & \text{if } n \text{ is even} \\ A(A^DA^{\dagger})^{\frac{n+1}{2}} & \text{if } n \text{ is odd} \end{cases}$;

(4) If $R(A_1) \subseteq R(A^*)$, then $(A^D)^n = \begin{cases} (A^\dagger A^D)^{\frac{n}{2}} & \text{if } n \text{ is even} \\ (A^\dagger A^D)^{\frac{n+1}{2}} A & \text{if } n \text{ is odd} \end{cases}$;
(5) If $A^{d,\dagger} = A$ or $A^{\dagger,d} = A$, then $A = A^D = A^\dagger$;
(6) If $A^{d,\dagger} = A^{\dagger,d}$, then $A^{d,\dagger} = A^{\dagger,d} = A^D$;
(7) If $i(A) = r$, then the following conditions are equivalent:
   (a) A is Core-EP;
   (b) A is r-EP;
   (c) $A^D = A^{c\dagger}$;
   (d) $A^{d,\dagger} = A^{\dagger,d}$;
   (e) $N(A^\dagger) \subseteq N(A^D)$ and $R(A_1) \subseteq R(A^*)$,

where $A^{c\dagger}$ is the CMP-inverse of A (see [14, Theorem 2.1]).

Lemma 3.15: *Given an $n \times n$ complex matrix A with $i(A) = r$, the following conditions are equivalent:*

- $A^r A^{d,\dagger} = A^{d,\dagger} A^r$;
- $A^{c\dagger} = A^{\dagger,d}$;
- $N(A^\dagger) \subseteq N(A^D)$.

Proof: It is known from [15, Theorem 3.14] that the following conditions are equivalent:

- $A^r A^{d,\dagger} = A^{d,\dagger} A^r$;
- $A^{c\dagger} = A^{\dagger,d}$;
- $A^{d,\dagger} = A^D$.

Hence, the statement is immediately deduced from Corollary 3.2. ∎

Lemma 3.16: *If A is an $n \times n$ complex matrix, then the following conditions are equivalent:*

- $A^r A^{\dagger,d} = A^{\dagger,d} A^r$;
- $A^{c\dagger} = A^{d,\dagger}$;
- $R(A_1) \subseteq R(A^*)$.

Proof: The claim follows from Corollary 3.7 bearing in mind that [15, Theorem 3.17] states that the following conditions are equivalent:

- $A^r A^{\dagger,d} = A^{\dagger,d} A^r$;
- $A^{c\dagger} = A^{d,\dagger}$;
- $A^{\dagger,d} = A^D$.

∎

Lemma 3.17: *Given an $n \times n$ complex matrix A, one has that*

$$A^{d,\dagger} = A^{\dagger,d} = A \iff A^{D,*} = A^{*,D} = A^* \quad \textit{and } A \textit{ is tripotent}.$$

Proof: According to [5, Lemma 2.2] one has that

- $A^{D,*} = A^* \iff A^{d,\dagger} = A^\dagger$;
- $A^{*,D} = A^* \iff A^{\dagger,d} = A^\dagger$.

Hence, if $A^{d,\dagger} = A^{\dagger,d} = A$, it follows from Lemmas 3.4 and 3.9 that $A^{d,\dagger} = A^{\dagger,d} = A^\dagger$, and we deduce that $A^{D,*} = A^{*,D} = A^*$. Moreover, in this case A is tripotent because it is known from [3, Proposition 2.14 (h)] and [6, p.16] that

$$A \text{ is EP and tripotent} \iff A^{d,\dagger} = A^{\dagger,d} = A.$$

Conversely, if $A^{D,*} = A^{*,D} = A^*$ and A is tripotent, then

$$A^D = A^{d,\dagger} = A^{\dagger,d} = A^\dagger$$

and, therefore, one deduces that A is EP and the statement is proved. ■

Lemma 3.18: *If A is an $n \times n$ complex matrix such that $A^{D,*} = A^D$, then $A^{D,*} = A^{d,\dagger}$.*

Proof: The claim follows immediately from the following properties of the Drazin-Star inverse offered in [5, Lemma 2.2]:

- $A^{D,*}A = AA^D \iff A^{d,\dagger} = A^{D,*}$;
- $AA^{D,*} = AA^D \iff A^D = A^{D,*}$.

■

A direct consequence of this lemma and Corollary 3.2 is:

Corollary 3.19: *If A is an $n \times n$ complex matrix such that $A^{D,*} = A^D$, then $N(A^\dagger) \subseteq N(A^D)$.*

Lemma 3.20: *If A is an $n \times n$ complex matrix such that $A^{*,D} = A^D$, then $A^{*,D} = A^{\dagger,d}$.*

Proof: This statement is directly deduced for the following properties of the Star-Drazin inverse given in [5, Lemma 2.2]:

- $A^{*,D}A = AA^D \iff A^{\dagger,d} = A^{*,D}$;
- $AA^{*,D} = AA^D \iff A^D = A^{*,D}$.

■

It follows immediately from this lemma and Corollary 3.7 that

Corollary 3.21: *If A is an $n \times n$ complex matrix such that $A^{*,D} = A^D$, then $R(A_1) \subseteq R(A^*)$.*

Furthermore, from Proposition 3.11, Lemmas 3.17 and 3.20 one deduces that

Corollary 3.22: *Given an $n \times n$ complex matrix A, one has that*

$$A^{D,*} = A^{*,D} = A^D \iff A^{D,*} = A^{d,\dagger} = A^{\dagger,d} = A^{*,D}.$$

Remark 3.1: Given an $n \times n$ complex matrix A with $i(A) = r$, we wish to recall that $N(A^\dagger) = R(A)^\perp$, $N(A^D) = N(A^r)$, $R(A_1) = R(A^r)$ and $R(A^*) = N(A)^\perp$.

4. Illustrative examples

To finish this work, we shall provide examples of matrices $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ such that $A^D = A^{d,\dagger} = A^{\dagger,d}$ in the first case, $A^D = A^{d,\dagger}$ but $A^D \neq A^{\dagger,d}$ in the second and $A^D = A^{\dagger,d}$ with $A^D \neq A^{d,\dagger}$ in the last case.

Example 4.1: We shall now study from the point of view of this work the matrix

$$A = \begin{pmatrix} 1 & \frac{3}{2} & -\frac{3}{2} & -1 \\ -1 & -1 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ -1 & -\frac{3}{2} & \frac{3}{2} & 1 \end{pmatrix},$$

whose core part is

$$A_1 = \begin{pmatrix} 1 & \frac{3}{2} & -\frac{3}{2} & -1 \\ -\frac{1}{2} & -1 & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & -1 & -\frac{1}{2} \\ -1 & -\frac{3}{2} & \frac{3}{2} & 1 \end{pmatrix}.$$

Since $N(A) = \langle(0,1,1,0)\rangle,$, $R(A) = \langle(-1,0,0,1), (0,-1,1,0), (0,1,1,0)\rangle$, $i(A) = 2$ and

$$A^2 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix},$$

one obtains that

$$N(A^\dagger) \subseteq N(A^D) \quad \text{and} \quad R(A_1) \subseteq R(A^*)$$

because:

- $N(A^\dagger) = R(A)^\perp = \langle(1,0,0,1)\rangle$;
- $N(A^D) = N(A^2) = \langle(1,0,0,1), (0,1,1,0)\rangle$;
- $R(A_1) = R(A^2) = \langle(1,0,0,-1), (0,-1,1,0)\rangle$;
- $R(A^*) = N(A)^\perp = \langle(1,0,0,0), (0,-1,1,0), (0,0,0,1)\rangle$.

Thus, readers can easily check that

$$A^\dagger = \begin{pmatrix} 1 & 1 & -2 & -1 \\ -\frac{1}{2} & -1 & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & -1 & -\frac{1}{2} \\ -1 & -2 & 1 & 1 \end{pmatrix}, \quad A^D = A^{d,\dagger} = A^{\dagger,d} = \begin{pmatrix} 1 & \frac{3}{2} & -\frac{3}{2} & -1 \\ -\frac{1}{2} & -1 & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & -1 & -\frac{1}{2} \\ -1 & -\frac{3}{2} & \frac{3}{2} & 1 \end{pmatrix}$$

and that A is Core-EP and 2-EP.

Example 4.2: Let us now consider the 4×4 complex matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & -i & 1+i \\ 1 & -1 & 1+i & -2-i \\ 1 & 0 & -1 & 1 \end{pmatrix},$$

with core part

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -i & 1+i \\ 0 & -1 & 1+i & -2-i \\ 0 & 0 & -1 & 1 \end{pmatrix},$$

$N(A) = \langle(0,-1,1,1)\rangle$, $R(A) = \langle(0,-1,1,0),(0,1,0,-1),(0,-1,1,1)\rangle$, $i(A) = 2$ and

$$A^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1+i & -3i & 1+4i \\ 0 & -2-i & 2+4i & -4-5i \\ 0 & 1 & -2-i & 3+i \end{pmatrix}.$$

Therefore, one obtains the following facts:

- $N(A^\dagger) = R(A)^\perp = \langle(1,0,0,0)\rangle$;
- $N(A^D) = N(A^2) = \langle(1,0,0,0),(0,-1,1,1)\rangle$;
- $R(A_1) = R(A^2) = \langle(0,-1,1,0),(0,1,0,-1)\rangle$;
- $R(A^*) = N(A)^\perp = \langle(1,0,0,0),(0,1,1,0),(0,1,0,1)\rangle$.

Accordingly, since $N(A^\dagger) \subseteq N(A^D)$ and $R(A_1) \nsubseteq R(A^*)$, bearing in mind that a computation shows that

$$A^\dagger = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & \frac{5}{3}+\frac{2}{3}i & 1+\frac{2}{3}i & \frac{2}{3} \\ 0 & \frac{4}{3}+\frac{1}{3}i & 1+\frac{1}{3}i & \frac{1}{3} \\ 0 & \frac{1}{3}+\frac{1}{3}i & \frac{1}{3}i & \frac{1}{3} \end{pmatrix}, \quad A^D = A^{d,\dagger} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 5+2i & 3+2i & 2 \\ 0 & -2-i & -1-i & -1 \\ 0 & -3-i & -2-i & -1 \end{pmatrix}$$

and

$$A^{\dagger,d} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{5}{3}+\frac{2}{3}i & 1+\frac{2}{3}i & \frac{2}{3} \\ 0 & \frac{4}{3}+\frac{1}{3}i & 1+\frac{1}{3}i & \frac{1}{3} \\ 0 & \frac{1}{3}+\frac{1}{3}i & \frac{1}{3}i & \frac{1}{3} \end{pmatrix},$$

one can check that this example also agrees with the statements of this work that characterize matrices such that $A^{d,\dagger} = A^D \neq A^{\dagger,d}$

Example 4.3: Finally, if we consider the 5×5 complex matrix

$$A = \begin{pmatrix} -\frac{7}{3} & -2 & \frac{7}{3} & -\frac{14}{3} & 3 \\ -2 & -2 & 1 & -3 & 2 \\ -4 & -5 & 4 & -8 & 4 \\ \frac{2}{3} & 0 & -\frac{2}{3} & \frac{4}{3} & -1 \\ -\frac{11}{3} & -4 & \frac{8}{3} & -\frac{19}{3} & 4 \end{pmatrix},$$

where

$$A_1 = \begin{pmatrix} -\frac{8}{3} & -3 & \frac{8}{3} & -\frac{16}{3} & 3 \\ -1 & -1 & 1 & -2 & 1 \\ -\frac{11}{3} & -4 & \frac{11}{3} & -\frac{22}{3} & 4 \\ 1 & 1 & -1 & 2 & -1 \\ -\frac{8}{3} & -3 & \frac{8}{3} & -\frac{16}{3} & 3 \end{pmatrix},$$

$R(A) = \langle (1,0,1,0,1), (0,1,1,-1,0), (-1,0,1,1,0), (0,-1,0,0,-1) \rangle$, $N(A) = \langle (-1,0,1,1,0) \rangle$, $i(A) = 3$ and

$$A^3 = \begin{pmatrix} -\frac{187}{3} & -67 & \frac{187}{3} & -\frac{374}{3} & 67 \\ -\frac{67}{3} & -24 & \frac{67}{3} & -\frac{134}{3} & 24 \\ -\frac{254}{3} & -91 & \frac{254}{3} & -\frac{508}{3} & 91 \\ \frac{67}{3} & 24 & -\frac{67}{3} & \frac{134}{3} & -24 \\ -\frac{187}{3} & -67 & \frac{187}{3} & -\frac{374}{3} & 67 \end{pmatrix},$$

one has that:

- $N(A^\dagger) = R(A)^\perp = \langle (1,1,0,1,-1) \rangle$;
- $N(A^D) = N(A^3) = \langle (-1,0,1,1,0), (0,-1,0,0,-1), (1,0,1,0,0) \rangle$;
- $R(A_1) = R(A^3) = \langle (1,0,1,0,1), (0,1,1,-1,0) \rangle$;;
- $R(A^*) = N(A)^\perp = \langle (1,0,1,0,0), (0,1,0,0,0), (1,0,0,1,0), (0,0,0,0,1) \rangle$.

Now, readers can check that

$$A^\dagger = \begin{pmatrix} \frac{17}{12} & -\frac{9}{4} & -\frac{2}{3} & \frac{25}{12} & \frac{5}{4} \\ -\frac{1}{3} & 1 & \frac{1}{3} & -\frac{5}{3} & -1 \\ \frac{3}{4} & -\frac{5}{4} & 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{2}{3} & -1 & -\frac{2}{3} & \frac{4}{3} & 1 \\ \frac{19}{12} & -\frac{7}{4} & -\frac{4}{3} & \frac{23}{12} & \frac{7}{4} \end{pmatrix}, \quad A^D = A^{\dagger,d} = \begin{pmatrix} 1 & -2 & -1 & 2 & 2 \\ -\frac{2}{3} & 1 & \frac{2}{3} & -\frac{4}{3} & -1 \\ \frac{1}{3} & -1 & -\frac{1}{3} & \frac{2}{3} & 1 \\ \frac{2}{3} & -1 & -\frac{2}{3} & \frac{4}{3} & 1 \\ 1 & -2 & -1 & 2 & 2 \end{pmatrix}$$

and

$$A^{d,\dagger} = \begin{pmatrix} \frac{5}{4} & -\frac{7}{4} & -1 & \frac{9}{4} & \frac{7}{4} \\ -\frac{2}{3} & 1 & \frac{2}{3} & -\frac{4}{3} & -1 \\ \frac{7}{12} & -\frac{3}{4} & -\frac{1}{3} & \frac{11}{12} & \frac{3}{4} \\ \frac{2}{3} & -1 & -\frac{2}{3} & \frac{4}{3} & 1 \\ \frac{5}{4} & -\frac{7}{4} & -1 & \frac{9}{4} & \frac{7}{4} \end{pmatrix}$$

from which we observe that $A^{\dagger,d} = A^D \neq A^{d,\dagger}$. Note that $N(A^\dagger) \nsubseteq N(A^D)$ but $R(A_1) \subseteq R(A^*)$.

Acknowledgments

The author would like to thank the anonymous reviewer for his/her valuable comments to improve the quality of the paper.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

This work is partially supported by the Spanish Government research projects MTM2015-66760-P and PGC2018-099599-B-I00 and the Regional Government of Castile and Leon research project J416/463AC03.

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