Taylor & Francis
Taylor & Francis Group

Linear and Multilinear Algebra

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/glma20

On the explicit computation of the set of 1-inverses of a square matrix

Fernando Pablos Romo & Diego Alba Alonso

To cite this article: Fernando Pablos Romo & Diego Alba Alonso (2023) On the explicit computation of the set of 1-inverses of a square matrix, Linear and Multilinear Algebra, 71:18, 2869-2876, DOI: 10.1080/03081087.2022.2122388

To link to this article: https://doi.org/10.1080/03081087.2022.2122388

Published online: 13 Sep 2022.

LINEAR AND MULTILINEAR ALGEBRA
2023, VOL. 71, NO. 18, 2869–2876
https://doi.org/10.1080/03081087.2022.2122388

Taylor & Francis
Taylor & Francis Group

On the explicit computation of the set of 1-inverses of a square matrix

Fernando Pablos Romo and Diego Alba Alonso

Departamento de Matemáticas, Universidad de Salamanca, Salamanca, España

ABSTRACT
The aim of this note is to offer a method for the explicit computation of all 1-inverses of an arbitrary square matrix. The main tool to obtain this algorithm is the characterization of the set of 1-inverses of a finite potent endomorphism.

ARTICLE HISTORY
Received 18 January 2022
Accepted 3 September 2022

COMMUNICATED BY
F. Poloni

KEYWORDS
1-inverse; finite potent endomorphism; square matrices

2020 MATHEMATICS SUBJECT CLASSIFICATIONS
15A09; 15A04

1. Introduction

If $A \in \mathrm{Mat}_{n\times m}(k)$ is a matrix with entries in an arbitrary field k, a matrix $A^- \in \mathrm{Mat}_{m\times n}(k)$ is a 1-inverse of A when $AA^-A = A$. Moreover, we say that a $A^+ \in \mathrm{Mat}_{m\times n}(k)$ is a reflexive generalized inverse of A when A^+ is a 1-inverse of A and A is a 1-inverse of A^+.

Similarly, given two k-vector spaces V and W and a linear map $f\colon V \to W$, we say that a morphism $f^-\colon W \to V$ is a 1-inverse of f when $f \circ f^- \circ f = f$ and that a linear map $f^+\colon W \to V$ is a reflexive generalized inverse of f when f^+ is a 1-inverse of f and f is a 1-inverse of f^+.

The notion of 1-inverse is basic in the theory of generalized inverses of matrices. Thus, if $A^\dagger$ denotes the Moore-Penrose inverse of A described by E. H. Moore in [1] and R. Penrose in [2], A^D is the Drazin inverse of A offered in [3], $A^{c\dagger}$ is the CMP inverse of A defined in [4] and $A^{d,\dagger}$ and $A^{\dagger,d}$ are the DMP inverses of A studied in [5], it is known that $A^\dagger$ is a reflexive generalized inverse of A and A is a 1-inverse of A^D, of $A^{c\dagger}$, of $A^{d,\dagger}$ and of $A^{\dagger,d}$

During the last years, the first-named author of this work has extended different notions of the theory of matrices to finite potent endomorphisms on arbitrary vector spaces. Thus, the existence and uniqueness of the Drazin inverse, the CMP inverse and the DMPs inverses of finite potent endomorphisms have been proved in [6–8], respectively. From

CONTACT Fernando Pablos Romo fpablos@usal.es Departamento de Matemáticas, Universidad de Salamanca, Plaza de la Merced 1-4, Salamanca 37008, España

these extensions, the main properties of these generalized inverses of matrices have been recovered and new relations between them have been given.

As far as we know an algorithm for the explicit computation of all 1-inverses of an arbitrary square matrix is not stated explicitly in the literature. This algorithm is offered in this work from the explicit characterization of the set of 1-inverses of a finite potent endomorphism on an arbitrary vector space.

A remaining question is to obtain an algorithm for the explicit computation of the set of reflexive generalized inverses of a square matrix.

The paper is organized as follows. In Section 2 we recall the basic definitions of the theory of finite potent endomorphisms and a summary of statements of the articles [6, 9, 10].

Section 3 deals with the characterization of the set of 1-inverses of a finite potent endomorphism on an arbitrary vector space and Section 4 contains the algorithm for computing the set of 1-inverses of a square matrix and an example with the explicit application of this algorithm.

Finally, we wish to point out that the offered method is compatible with the characterization of the $\{1\}$-inverses of a nilpotent matrix offered by M. I. Gareis, M. Lattanzi and N. Thome in [11, Theorem 2.1].

2. Preliminaries

2.1. Finite potent endomorphisms

Let k be an arbitrary field, let V be a k-vector space and let us consider an endomorphism φ of V. We say that φ is 'finite potent' if $\varphi^n V$ is finite dimensional for some n, where the power means the composition $\varphi \circ \overset{n)}{\dots} \circ \varphi$. This definition was introduced by J. Tate in [12].

Basic examples of finite potent endomorphisms are endomorphisms on finite-dimensional vector spaces and finite rank or nilpotent endomorphisms on infinite-dimensional vector spaces.

To characterize finite potent endomorphisms, M. Argerami, F. Szechtman and R. Tifenbach proved in [9] that an endomorphism φ is finite potent if and only if V admits a φ-invariant decomposition $V = U_\varphi \oplus W_\varphi$ such that $\varphi_{|U_\varphi}$ is nilpotent, W_φ is finite dimensional and $\varphi_{|W_\varphi} \colon W_\varphi \overset{\sim}{\longrightarrow} W_\varphi$ is an isomorphism. In this paper we shall call this decomposition the φ-invariant AST-decomposition of V.

For details on this decomposition readers are referred to [9].

2.2. Jordan Basis of a nilpotent endomorphism

Let V be a vector space over an arbitrary field k and let $g \in \operatorname{End}_k(V)$ be a nilpotent endomorphism. If n is the nilpotency index of g, according to the statements of [10], setting

$$U_i^g = \operatorname{Ker} g^i / [\operatorname{Ker} g^{i-1} + f(\operatorname{Ker} g^{i+1})]$$

with $i \in \{1, 2, \dots, n\}$, $\mu_i(V, g) = \dim_k U_i^g$ and $S_{\mu_i(V,g)}$ a set such that $\# S_{\mu_i(V,g)} = \mu_i(V, g)$ with $S_{\mu_i(V,g)} \cap S_{\mu_j(V,g)} = \emptyset$ for all $i \neq j$, one has that there exists a family of vectors $\{v_{s_i}\}$

that determines a Jordan basis of g:

$$B = \bigcup_{\substack{s_i \in S_{\mu_i(V,g)} \\ 1 \le i \le n}} \{v_{s_i}, g(v_{s_i}), \dots, g^{i-1}(v_{s_i})\}. \tag{1}$$

Moreover, if we write $H^g_{s_i} = \langle v_{s_i}, g(v_{s_i}), \dots, g^{i-1}(v_{s_i})\rangle$, the basis B induces a decomposition

$$V = \bigoplus_{\substack{s_i \in S_{\mu_i(V,g)} \\ 1 \le i \le n}} H^g_{s_i}. \tag{2}$$

2.3. CN decomposition of a finite potent endomorphism

Let V be again an arbitrary k-vector space. Given a finite potent endomorphism $\varphi \in \operatorname{End}_k(V)$, from [6, Theorem 3.2] it is known that there exists a unique decomposition $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1, \varphi_2 \in \operatorname{End}_k(V)$ are finite potent endomorphisms satisfying that:

- $i(\varphi_1) \le 1$;
- φ_2 is nilpotent;
- $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1 = 0$.

If $V = W_\varphi \oplus U_\varphi$ is the AST-decomposition of V induced by φ, then φ_1 and φ_2 are the unique linear maps such that:

$$\varphi_1(v) = \begin{cases} \varphi(v) & \text{if } v \in W_\varphi \\ 0 & \text{if } v \in U_\varphi \end{cases} \quad \text{and} \quad \varphi_2(v) = \begin{cases} 0 & \text{if } v \in W_\varphi \\ \varphi(v) & \text{if } v \in U_\varphi \end{cases}. \tag{3}$$

3. {1}-inverses of finite potent endomorphisms

Let V be a k-vector space and let $f \in \operatorname{End}_k(V)$ be an arbitrary endomorphism. The goal of this section is to characterize the set of 1-inverse of a finite potent endomorphism.

Definition 3.1: We say that $f^- \in \operatorname{End}_k(V)$ is a $\{1\}$-inverse of $f \in \operatorname{End}_k(V)$ when

$$f \circ f^- \circ f = f.$$

Lemma 3.2: *Given a k-vector space V and an endomorphism $f \in \operatorname{End}_k(V)$, one has that $f^- \in \operatorname{End}_k(V)$ is a $\{1\}$-inverse of f if and only if for every $v \in V$ we have that*

$$f^-(f(v)) = v + u$$

with $u \in \operatorname{Ker} f$.

Proof: Given $v \in V$, since $f(f^-(f(v)) - v) = 0$, then $f^-(f(v)) - v \in \operatorname{Ker} f$ and the claim is deduced. ■

Let us now consider a finite potent endomorphism $\varphi \in \operatorname{End}_k(V)$ with CN-decomposition $\varphi = \varphi_1 + \varphi_2$ and that induces the AST-decomposition $V = U_\varphi \oplus W_\varphi$.

Keeping the above notation, if n is the nilpotency order of φ_2, we fix a basis $B_V = B_{W_\varphi} \cup B_{U_\varphi}$ with

$$B_{W_\varphi} = \{w_1, \dots, w_r\}$$

where $r = \dim_k W_\varphi$ and

$$B_{U_\varphi} = \bigcup_{\substack{s_i \in S_{\mu_i(U_\varphi,\varphi)} \\ 1 \le i \le n}} \{v_{s_i}, \varphi(v_{s_i}), \dots, \varphi^{i-1}(v_{s_i})\}.$$

Moreover, it is clear that

$$B_{U_\varphi} = \bigcup_{\substack{s_i \in S_{\mu_i(U_\varphi,\varphi)} \\ 1 \le i \le n}} \{v_{s_i}, \varphi_2(v_{s_i}), \dots, \varphi_2^{i-1}(v_{s_i})\}$$

and

$$\operatorname{Ker} \varphi = \bigoplus_{\substack{s_i \in S_{\mu_i(U_\varphi,\varphi)} \\ 1 \le i \le n}} \langle \varphi^{i-1}(v_{s_i}) \rangle = \bigoplus_{\substack{s_i \in S_{\mu_i(U_\varphi,\varphi)} \\ 1 \le i \le n}} \langle \varphi_2^{i-1}(v_{s_i}) \rangle.$$

Proposition 3.3: *With the above notation, if $\varphi \in \operatorname{End}_k(V)$ is a finite potent endomorphism, then an endomorphism $\varphi^- \in \operatorname{End}_k(V)$ is a $\{1\}$-inverse of φ if an only if φ^- satisfies that*

- $\varphi^-(w_h) = (\varphi_{|W_\varphi})^{-1}(w_h) + u_h$ *for each* $h \in \{1, \dots, r\}$;
- $\varphi^-(\varphi^j(v_{s_i})) = \varphi^{j-1}(v_{s_i}) + u^j_{s_i}$ *for every* $s_i \in S_{\mu_i(U_\varphi,\varphi)}$ *and* $j \in \{1, \dots, i-1\}$;
- $\varphi^-(v_{s_i}) = \tilde{v}_{s_i}$ *for every* $s_i \in S_{\mu_i(U_\varphi,\varphi)}$;

where $\tilde{v}_{s_i} \in V$ *and* $u_h, u^j_{s_i} \in \operatorname{Ker} \varphi$ *for each* $h \in \{1, \dots, r\}$ *and for every* $s_i \in S_{\mu_i(U_\varphi,\varphi)}$ *and* $j \in \{1, \dots, i-1\}$.

Proof: The statement is a direct consequence of Lemma 3.2 and the structure of the Jordan basis B_V. ■

Let us now denote by $X_\varphi(1)$ the set of all $\{1\}$-inverses of a finite potent endomorphism $\varphi \in \operatorname{End}_k(V)$.

Theorem 3.4 (Structure of $X_\varphi(1)$): *If $\varphi \in \operatorname{End}_k(V)$ is a finite potent endomorphism, then there exists a bijection*

$$X_\varphi(1) \simeq \left[\prod_{h=1}^{r} \operatorname{Ker} \varphi\right] \times \left[\prod_{\substack{s_i \in S_{\mu_i(U_\varphi,\varphi)} \\ 1 \le i \le n}} \left[V \times \prod_{j=1}^{i-1} \operatorname{Ker} \varphi\right]\right]. \tag{4}$$

Proof: With the notation of Proposition 3.3, we have that the map

$$X_\varphi(1) \longrightarrow \left[\prod_{h=1}^{r} \operatorname{Ker} \varphi\right] \times \left[\prod_{\substack{s_i \in S_{\mu_i(U_\varphi,\varphi)} \\ 1 \le i \le n}} \left[V \times \prod_{j=1}^{i-1} \operatorname{Ker} \varphi\right]\right]$$

$$\varphi^- \longmapsto (u_h, (\tilde{v}_{s_i}, u_{s_i}^j)_{j\in\{1,\dots,i-1\}})_{h\in\{1,\dots,r\};s_i\in S_{\mu_i(U_\varphi,\varphi)}}$$

is clearly a bijection. ∎

Example 3.1: Let V be a vector space of countable dimension over an arbitrary ground field k. Let $\{v_1, v_2, v_3, \dots\}$ be a basis of V indexed by the natural numbers and let us consider the finite potent endomorphism $\varphi \in \operatorname{End}_k(V)$ defined as:

$$\varphi(v_i) = \begin{cases} v_1 - v_2 + v_4 & \text{if } i = 1 \\ -2v_1 - v_5 & \text{if } i = 2 \\ v_2 + v_4 + v_5 & \text{if } i = 3 \\ v_{i+1} & \text{if } i = 2r \\ 0 & \text{if } i = 2r+1 \end{cases}$$

for all $r \geq 2$. We shall characterize the set $X_\varphi(1)$.

A computation shows that the AST-decomposition of φ is $V = W_\varphi \oplus U_\varphi$ with

$$\begin{aligned} W_\varphi &= \langle v_1 + v_2 - v_4 + 2v_5, -2v_1 + v_2 - v_4 - v_5 \rangle \quad \text{and} \\ U_\varphi &= \langle 2v_1 + v_2 + 2v_3, v_4, v_5, \dots \rangle, \end{aligned}$$

that the matrix associated with $\varphi_{|W_f}$ in this basis is

$$\varphi_{|W_\varphi} \equiv \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$$

and that $\operatorname{Ker}\varphi = \langle v_{2r+1} \rangle_{r\geq 2}$.

Given an endomorphism $\varphi^- \in X_\varphi(1)$, since $\varphi(2v_1 + v_2 + 2v_3) = 4v_4 + v_5$, if we consider the basis $B_V = B_{W_\varphi} \cup B_{U_\varphi}$ with

$$B_{W_\varphi} = \{v_1 + v_2 - v_4 + 2v_5, -2v_1 + v_2 - v_4 - v_5\}$$

and

$$B_{U_\varphi} = \{2v_1 + v_2 + 2v_3, 4v_4 + v_5, 4v_5\} \cup \left[\bigcup_{r\geq 3} \{v_{2r}, v_{2r+1}\} \right],$$

it follows from Proposition 3.3 that:

- $\varphi^-(v_1 + v_2 - v_4 + 2v_5) = -v_1 - v_2 + v_4 - 2v_5 + u_1$;
- $\varphi^-(-2v_1 + v_2 - v_4 - v_5) = -v_1 + \frac{1}{2}v_2 - \frac{1}{2}v_4 - \frac{1}{2}v_5 + u_2$;
- $\varphi^-(2v_1 + v_2 + 2v_3) = \tilde{v}_3$;
- $\varphi^-(4v_4 + v_5) = 2v_1 + v_2 + 2v_3 + u_4$;
- $\varphi^-(4v_5) = 4v_4 + v_5 + u_5$;
- $\varphi^-(v_{2r}) = \tilde{v}_{2r}$;
- $\varphi^-(v_{2r+1}) = v_{2r} + u_{2r+1}$;

with $u_1, u_2, u_4, u_5, u_{2r+1} \in \operatorname{Ker}\varphi$ and $\tilde{v}_3, \tilde{v}_{2r} \in V$ for every $r \geq 3$.

Accordingly, from a non-difficult computation we obtain that the above equalities are equivalent to the following:

- $\varphi^-(v_1) = -\frac{1}{2}v_2 - \frac{1}{2}v_4 - \frac{3}{4}v_5 + \frac{1}{3}u_1 - \frac{1}{3}u_2 - \frac{1}{4}u_5$;
- $\varphi^-(v_2) = -\frac{1}{2}v_1 - \frac{1}{4}v_2 + \frac{1}{2}v_3 - \frac{3}{4}v_4 - \frac{29}{16}v_5 - \frac{2}{3}u_1 + \frac{1}{3}u_2 + \frac{1}{4}u_4 - \frac{5}{16}u_5$;
- $\varphi^-(v_3) = \frac{1}{2}v_1 + \frac{5}{4}v_2 - \frac{1}{2}v_3 + \frac{5}{4}v_4 + \frac{53}{16}v_5 + \frac{1}{3}u_2 - \frac{1}{4}u_4 + \frac{13}{16}u_5 + \tilde{v}_3$;
- $\varphi^-(v_4) = \frac{1}{2}v_1 + \frac{1}{4}v_2 + \frac{1}{2}v_3 - \frac{1}{4}v_4 - \frac{1}{16}v_5 + \frac{1}{4}u_4 - \frac{1}{16}u_5$;
- $\varphi^-(v_5) = v_4 + \frac{1}{4}v_5 + \frac{1}{4}u_5$;
- $\varphi^-(v_{2r}) = \tilde{v}_{2r}$;
- $\varphi^-(v_{2r+1}) = v_{2r} + u_{2r+1}$;

and, therefore, one has that an endomorphism $\varphi^- \in \mathrm{End}_k(V)$ is a 1-inverse of φ if and only if

$$\varphi^-(v_i) = \begin{cases} -\frac{1}{2}v_2 - \frac{1}{2}v_4 - \frac{3}{4}v_5 + \frac{1}{3}u_1 - \frac{1}{3}u_2 - \frac{1}{4}u_5 & \text{if } i = 1 \\ -\frac{1}{2}v_1 - \frac{1}{4}v_2 + \frac{1}{2}v_3 - \frac{3}{4}v_4 - \frac{29}{16}v_5 - \frac{2}{3}u_1 & \\ \qquad +\frac{1}{3}u_2 + \frac{1}{4}u_4 - \frac{5}{16}u_5 & \text{if } i = 2 \\ \tilde{v}_3 & \text{if } i = 3 \\ \frac{1}{2}v_1 + \frac{1}{4}v_2 + \frac{1}{2}v_3 - \frac{1}{4}v_4 - \frac{1}{16}v_5 + \frac{1}{4}u_4 - \frac{1}{16}u_5 & \text{if } i = 4 \\ v_4 + \frac{1}{4}v_5 + \frac{1}{4}u_5 & \text{if } i = 5 \\ \tilde{v}_i & \text{if } i = 2r \\ v_{i-1} + u_i & \text{if } i = 2r+1 \end{cases}$$

with $u_1, u_2, u_4, u_5, u_{2r+1} \in \operatorname{Ker}\varphi$ and $\tilde{v}_3, \tilde{v}_{2r} \in V$ for every $r \geq 3$.

4. Computation of {1}-inverses of a square matrix

This final section is devoted to apply the previous results for the characterization of the set $A(1)$ of $\{1\}$-inverses of a finite square matrix A with entries in an arbitrary field k and to offer a method for the explicit computation of $A(1)$.

Theorem 4.1 (Structure of $A(1)$): *Let $A \in \mathrm{Mat}_{m\times m}(k)$ be a square matrix with entries in an arbitrary field k and let $A = A_1 + A_2$ be its core-nilpotent decomposition. Then, the structure of $A(1)$ is determined from the following bijection:*

$$A(1) \simeq \left[\prod_{i=1}^{\mathrm{rk}\, A_1} N_u(A)\right] \times \left[\prod_{i=1}^{\mathrm{rk}\, A_2} N_u(A)\right] \times \left[\prod_{i=1}^{\dim N_u(A)} k^m\right] \simeq k^{[\dim N_u(A)](m+\mathrm{rk}\, A)}, \quad (5)$$

where $N_u(A)$ denotes the nullspace of A.

Proof: Bearing in mind the well-known relationship between finite square matrices and endomorphisms of finite-dimensional vector spaces, the statement is immediately deduced from Theorem 3.4. ■

Accordingly, an algorithm for computing $A(1)$ is the following:

(1) Write A in its Jordan canonical form: $A = PJP^{-1}$.
(2) If $J = \left(\begin{smallmatrix} C & 0 \\ 0 & N \end{smallmatrix}\right)$ with C invertible and N nilpotent, compute the inverse C^{-1} and the transpose N^t.
(3) Calculate the nullspace $N_u(J)$ of J.
(4) Setting $J' = \begin{pmatrix} C^{-1} & 0 \\ 0 & N^t \end{pmatrix}$, complete with parameters all the zero columns of J'.
(5) To conclude, add a general element of $N_u(J)$ in the rest of columns of J' to obtain the expression of J^- depending of $[\dim N_u(A)](m + \operatorname{rk} A)$ parameters so that every $\{1\}$-inverse of A is of the form

$$A^- = PJ^-P^{-1}.$$

Remark 4.2: We wish to point out that this algorithm is compatible with the characterization of the $\{1\}$-inverses of a nilpotent matrix offered by M. I. Gareis, M. Lattanzi and N. Thome in [11, Theorem 2.1].

Example 4.1: Let $\mathbb{F}_7$ be the field with seven elements. We shall now characterize the set of $\{1\}$-inverses of the matrix

$$A = \begin{pmatrix} 4 & 1 & 3 & 2 \\ 6 & 4 & 1 & 3 \\ 1 & 6 & 2 & 6 \\ 3 & 0 & 3 & 2 \end{pmatrix} \in \operatorname{Mat}_{4\times 4}(\mathbb{F}_7)\,.$$

Since the Jordan canonical form of A is

$$A = \begin{pmatrix} 1 & 1 & 6 & 0 \\ 1 & 2 & 0 & 0 \\ 6 & 6 & 2 & 4 \\ 0 & 0 & 6 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 6 & 6 & 4 \\ 3 & 1 & 4 & 5 \\ 3 & 0 & 3 & 2 \\ 3 & 0 & 3 & 3 \end{pmatrix},$$

keeping the above notation, we have that $N_u(J) = \{(0,0,0,\lambda)\}_{\lambda\in\mathbb{F}_7}$ and

$$J' = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

because $\frac{1}{2} = 4$ and $\frac{1}{3} = 5$ in $\mathbb{F}_7$.

Applying the algorithm we obtain that A^- is a $\{1\}$-inverse of A if and only if

$$A^- = \begin{pmatrix} 1 & 1 & 6 & 0 \\ 1 & 2 & 0 & 0 \\ 6 & 6 & 2 & 4 \\ 0 & 0 & 6 & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 & a & 0 \\ 0 & 5 & b & 0 \\ 0 & 0 & c & 1 \\ d & e & f & g \end{pmatrix} \cdot \begin{pmatrix} 1 & 6 & 6 & 4 \\ 3 & 1 & 4 & 5 \\ 3 & 0 & 3 & 2 \\ 3 & 0 & 3 & 3 \end{pmatrix},$$

with $a, b, c, d, e, f, g \in \mathbb{F}_7$.

Bearing in mind that $\dim N_u(A) = 1$, $\operatorname{rk}(A) = 3$ and $m = 4$, it is known from (5) that $A(1) \simeq (\mathbb{F}_7)^7$, which is coherent with the computations made in this example.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

The first-named author of this work is partially supported by the Spanish Government research projects no. PGC2018-099599-B-I00 and the Regional Government of Castile and Leon research project no. J416/463AC03.

ORCID

Fernando Pablos Romo http://orcid.org/0000-0003-3444-1800
Diego Alba Alonso http://orcid.org/0000-0001-7147-8368

References

[1] Moore EH. On the reciprocal of the general algebraic matrix. Bull Am Math Soc. 1920;26(9):394–95.
[2] Penrose R. A generalized inverse for matrices. Proc Camb Philos Soc. 1955;51(3):406–413.
[3] Drazin MP. Pseudo-inverses in associative rings and semigroups. Am Math Monthly. 1958;65(7):506–514.
[4] Mehdipour M, Salemi M. On a new generalized inverse of matrices. Linear Multilinear Algebra. 2018;66(5):1046–1053. DOI:10.1080/03081087.2017.1336200
[5] Malik SB, Thome N. On a new generalized inverse for matrices of an arbitrary index. Appl Math Comput. 2014;226(1):575–580.
[6] Pablos Romo F. Core-nilpotent decomposition and new generalized inverses of finite potent endomorphisms. Linear Multilinear Algebra. 2020;68(11):2254–2275.
[7] Pablos Romo F. On Drazin-Moore-Penrose inverses of finite potent endomorphisms. Linear Multilinear Algebra. 2021;69(4):627–647.
[8] Pablos Romo F. On the drazin inverse of finite potent endomorphisms. Linear Multilinear Algebra. 2019;67(10):2135–2146.
[9] Argerami M, Szechtman F, Tifenbach R. On Tate's trace. Linear Multilinear Algebra. 2007;55(6):515–520.
[10] Pablos Romo F. Classification of finite potent endomorphisms. Linear Algebra Appl. 2014;440:266–277.
[11] Gareis MI, Lattanzi M, Thome N. Nilpotent matrices and the minus partial order. Quaest Math. 2017;40(4):519–525. DOI:10.2989/16073606.2017.1300612
[12] Tate J. Residues of differentials on curves. Ann Scient Éc Norm Sup. 1968;1:149–159. 4a série.