PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 135, Number 7, July 2007, Pages 1977–1985
S 0002-9939(07)08690-X
Article electronically published on February 28, 2007

A NEW EXPLICIT EXPRESSION OF THE CONTOU-CARRÈRE SYMBOL

FERNANDO PABLOS ROMO

(Communicated by Wen-Ching Winnie Li)

ABSTRACT. The aim of this note is to offer a new explicit expression of the Contou-Carrère symbol that depends only on a product of a finite number of terms. As an application, we obtain an explicit formula for a Witt Residue.

1. INTRODUCTION

In 1994 C. Contou-Carrère [4] defined a natural transformation greatly generalizing the tame symbol. In the case of an artinian local base ring A with maximal ideal m, the natural transformation takes the following form. Let $f, g \in A((t))^\times$ be given, where t is a variable. (Here and below $R^\times$ denotes the multiplicative group of a ring R with unit.) It is possible in exactly one way to write

$$
\begin{aligned}
f &= a_0 \cdot t^{w(f)} \cdot \prod_{i=1}^{\infty}(1 - a_i t^i) \cdot \prod_{i=1}^{\infty}(1 - a_{-i} t^{-i}), \\
g &= b_0 \cdot t^{w(g)} \cdot \prod_{i=1}^{\infty}(1 - b_i t^i) \cdot \prod_{i=1}^{\infty}(1 - b_{-i} t^{-i}),
\end{aligned}
\tag{1.1}
$$

with $w(f), w(g) \in \mathbb{Z}$, $a_i, b_i \in A$ for $i > 0$, $a_0, b_0 \in A^\times$, $a_{-i}, b_{-i} \in m$ for $i > 0$, and $a_{-i} = b_{-i} = 0$ for $i \gg 0$. By definition, the value of the *Contou-Carrère symbol* is

$$
\langle f, g \rangle_A := (-1)^{w(f)w(g)} \frac{a_0^{w(g)} \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \big(1 - a_i^{j/(i,j)} b_{-j}^{i/(i,j)}\big)^{(i,j)}}{b_0^{w(f)} \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \big(1 - a_{-i}^{j/(i,j)} b_j^{i/(i,j)}\big)^{(i,j)}} \in A^\times .
$$

The definition makes sense because only finitely many of the terms appearing in the infinite products differ from 1. The symbol $\langle \cdot, \cdot \rangle_A$ is clearly antisymmetric and, although it is not immediately obvious from the definition, also bimultiplicative.

G. W. Anderson and the author [1] have interpreted the Contou-Carrère symbol $\langle f, g \rangle_A$—up to signs—as a commutator of liftings of f and g to a certain central extension of a group containing $A((t))^\times$, and they have exploited the commutator interpretation to prove, in the style of Tate [11], a reciprocity law for the Contou-Carrère symbol on a nonsingular complete curve defined over an algebraically closed field k, A being an artinian local k-algebra.

Received by the editors February 6, 2006 and, in revised form, February 23, 2006 and March 1, 2006.

2000 *Mathematics Subject Classification.* Primary 19F15, 19C20.

Key words and phrases. Contou-Carrère symbol, explicit expression, artinian local ring, Witt Residue.

This work was partially supported by the DGESYC research contract no. BFM2003-00078 and Castilla y León regional government contract SA071/04.

The author has also obtained a similar result for an algebraic curve over a perfect field [10], and A. Beilinson, S. Bloch and H. Esnault [3] have defined the Contou-Carrère symbol as the commutator pairing in a Heisenberg super extension. Moreover, recently, this symbol has played an important role in the work of M. Kapranov and E. Vasserot [6], and M. Asakura [2] has shown that the Contou-Carrère symbol coincides with the boundary map $\delta\colon K_{i+1}(A((t))) \longrightarrow K_i(A)$ described by K. Kato in [7].

In this context, the goal of this paper is to offer a new explicit expression of the Contou-Carrère symbol that depends only on a product of a finite number of terms (Proposition 3.4). As an application, we study a Witt Residue and we also obtain an explicit expression for it.

2. Preliminaries

2.1. **Contou-Carrère symbol.** Using the theory of groupoids, we can construct a central extension of groups

$$1 \to A^\times \to \widetilde{A((t))}^\times \xrightarrow{\pi} A((t))^\times \to 1\,, \tag{2.1}$$

and we have a commutator map

$$\{\cdot,\cdot\}^{A[[t]]}_{A((t))}\colon A((t))^\times \times A((t))^\times \longrightarrow A^\times\,.$$

That is, if τ and σ are two elements of $A((t))^\times$ and $\widetilde{\tau},\widetilde{\sigma} \in \widetilde{A((t))}^\times$ are elements such that $\pi(\widetilde{\tau}) = \tau$ and $\pi(\widetilde{\sigma}) = \sigma$, then the commutator map is

$$\{\tau,\sigma\}^{A[[t]]}_{A((t))} = \widetilde{\tau}\cdot\widetilde{\sigma}\cdot\widetilde{\tau}^{-1}\cdot\widetilde{\sigma}^{-1} \in A^\times\,.$$

With the notations of the preceding section, the Contou-Carrère symbol [4] is

$$\langle f,g\rangle_A = (-1)^{w(f)w(g)}\cdot\{f,g\}^{A[[t]]}_{A((t))}\,.$$

For details about the central extension (2.1) and the commutator $\{\cdot,\cdot\}^{A[[t]]}_{A((t))}$ readers are referred to [1].

For arbitrary elements $f,g,h \in A((t))^\times$, the following relations hold:

- $\langle f, g\cdot h\rangle_A = \langle f,g\rangle_A\cdot\langle f,h\rangle_A$.
- $\langle g,f\rangle_A = \langle f,g\rangle_A^{-1}$.
- $\langle f,-f\rangle_A = 1$.
- Given $\varphi \in A((t))^\times$ with positive winding number n, one has that

$$\langle f, g\circ\varphi\rangle_A = \langle \mathcal{N}_\varphi[f], g\rangle_A\,, \tag{2.2}$$

  where $\mathcal{N}_\varphi\colon A((t))^\times \to A((t))^\times$ denotes the corresponding norm mapping: viewing $A((t))$ via the homomorphism $h \mapsto h\circ\varphi$ as a free $A((t))$-module of rank n ([9], Proposition 3.6).

Remark 2.1. With the notation of the previous section, we should note that the original expression of the Contou-Carrère symbol [4] is

$$\langle f,g\rangle_A = (-1)^{w(f)w(g)}\frac{a_0^{w(g)}\cdot\exp\big(\sum_{i>0}[\delta_i(f)\cdot\delta_{-i}(g)/i]\big)}{b_0^{w(f)}\cdot\exp\big(\sum_{i>0}[\delta_{-i}(f)\cdot\delta_i(g)/i]\big)}\,, \tag{2.3}$$

where $\delta_m(f) = \operatorname{Res}(t^m\frac{df}{f})$.

Let $\xi_1, \xi_2, \dots;\ \eta_1, \eta_2, \dots$ now be indeterminates. Recall from [5] that we can define the sequences $\bar{x} = (\bar{x}_i)$ and $\bar{y} = (\bar{y}_j)$ by the equations

$$\prod_i (1 - \xi_i t) = 1 + \bar{x}_1 t + \bar{x}_2 t^2 + \dots,$$
$$\prod_i (1 - \eta_i t) = 1 + \bar{y}_1 t + \bar{y}_2 t^2 + \dots.$$

Hence, from these definitions we can construct a polynomial sequence $P_1, P_2, \dots$ satisfying the relations

$$\prod_{i,j} (1 - \xi_i \eta_j t) = 1 + P_1 t + P_2 t^2 + \dots. \tag{2.4}$$

Bearing in mind the fundamental theorem of symmetric functions, P_j can be written as $P_j(\bar{x}_1, \dots, \bar{x}_j; \bar{y}_1, \dots, \bar{y}_j)$.

Thus, if A is a commutative ring, we can consider on the multiplicative group

$$\bigwedge(A) = \{1 + a_1 t + a_2 t^2 + \dots, \quad a_i \in A\} \subseteq A[[t]]$$

a second operation by means of the formula

$$(1 + a_1 t + a_2 t^2 + \dots) * (1 + b_1 t + b_2 t^2 + \dots) = 1 + P_1(a, b) t + P_2(a, b) t^2 + \dots,$$

such that $(\bigwedge(A), \cdot, *)$ is a ring.

Moreover, $\mathbb{W}(A)$ being the ring of Witt vectors with coefficients in A, we can define a map

$$E_A \colon \mathbb{W}(A) \longrightarrow \bigwedge(A),$$
$$(a_1, a_2, \dots) \longrightarrow \prod_{i \geq 1} (1 - a_i t^i),$$

which is an isomorphism of rings because $E_A(a + b) E_A(a) \cdot E_A(b)$ and $E_A(a \cdot b) = E_A(a) * E_A(b)$.

Since $\mathbb{W}(A)$ is a commutative ring and with unit element ([5], page 117), we can denote by $\mathbb{W}_+(A)$ the abelian group induced by the ring structure.

Furthermore, we can define the abelian group

$$\hat{\mathbb{W}}_+(A) = \left\{ \begin{matrix} (b_1, b_2, \dots) \in \mathbb{W}_+(A) \text{ with } b_i \text{ nilpotent} \\ \text{for all } i \text{ and } b_i = 0 \text{ for almost all } i \end{matrix} \right\}.$$

If the artinian local ring A is a k-algebra with $\mathrm{ch}(k) = 0$, bearing in mind Proposition 2.4 ([10], p. 44) and denoting $E_A(\cdot) = E_A(\cdot, t)$, we have that

$$\exp\Big(\sum_{i>0} [\delta_i(f) \cdot \delta_{-i}(g)/i] \Big) = E_A(a \cdot b', 1),$$

where $a = (a_1, a_2, \dots) \in \mathbb{W}_+(A)$ and $b' = (b_{-1}, b_{-2}, \dots) \in \hat{\mathbb{W}}_+(A)$, with the notation of (1.1).

Remark 2.2. If A is again an artinian local k-algebra with $\mathrm{ch}(k) = 0$, let us now consider again $f, g \in A((t))^\times$ and let us assume that there exist decompositions

$$f \bar{a}_0 \cdot t^{w(f)} \cdot \prod_{i \geq 1} (1 - \bar{a}_i t) \cdot \prod_{i \geq 1} (1 - \bar{a}_{-i} t^{-1}),$$
$$g = \bar{b}_0 \cdot t^{w(g)} \cdot \prod_{j \geq 1} (1 - \bar{b}_j t) \cdot \prod_{j \geq 1} (1 - \bar{b}_{-j} t^{-1}),$$

with $\bar{a}_i, \bar{b}_j \in A$ and $\bar{a}_{-i}, \bar{b}_{-j} \in m$.

It follows from the above considerations, in particular from equality (2.4), and from the results of [8] (formulas (4.1) on p. 62 and (4.3) on p. 63), that

$$\exp\Big(\sum_{i>0}[\delta_i(f)\cdot\delta_{-i}(g)/i]\Big) = [\sum_{\lambda} z_\lambda^{-1} p_\lambda(\bar{a}) p_\lambda(\bar{b}')]^{-1},$$
$$\exp\Big(\sum_{i>0}[\delta_i(f)\cdot\delta_{-i}(g)/i]\Big) = [\sum_{\lambda} s_\lambda(\bar{a}) s_\lambda(\bar{b}')]^{-1},$$

summed over all partitions $\lambda = (\lambda_1, \lambda_2, \dots)$, with:

- $\bar{a} = (\bar{a}_1, \bar{a}_2, \dots)$;
- $\bar{b}' = (\bar{b}_{-1}, \bar{b}_{-2}, \dots)$;
- $z_\lambda = \prod_{i\geq 1} i^{m_i}\cdot m_i!$, where $m_i = m_i(\lambda)$ is the number of parts of λ equal to i;
- $p_\lambda = p_{\lambda_1}\cdot p_{\lambda_2}\dots$, where $p_{\lambda_i}(x)$ is the λ_i-th power sum $\sum x_j^{\lambda_i}$;
- s_λ is the Schur function associated with the partition λ.

Therefore, setting $\bar{b} = (\bar{b}_1, \bar{b}_2, \dots)$ and $\bar{a}' = (\bar{a}_{-1}, \bar{a}_{-2}, \dots)$, equivalent expressions to (2.3) are:

(1) $\langle f, g\rangle_A = (-1)^{w(f)w(g)} \frac{\bar{a}_0^{w(g)}\cdot \sum_\lambda z_\lambda^{-1} p_\lambda(\bar{b}) p_\lambda(\bar{a}')}{\bar{b}_0^{w(f)}\cdot \sum_\lambda z_\lambda^{-1} p_\lambda(\bar{a}) p_\lambda(\bar{b}')}$;

(2) $\langle f, g\rangle_A (-1)^{w(f)w(g)} \frac{\bar{a}_0^{w(g)}\cdot \sum_\lambda s_\lambda(\bar{b}) s_\lambda(\bar{a}')}{\bar{b}_0^{w(f)}\cdot \sum_\lambda s_\lambda(\bar{a}) s_\lambda(\bar{b}')}$.

2.2. **Witt Residue symbol.** Similar to [1], let

$$\{\epsilon\}\coprod\{\mathbf{x}_i, \mathbf{y}_i\}_{i=1}^{\infty}$$

be a family of independent variables. Let us write

$$\prod_{i=1}^{\infty}((1-\mathbf{x}_i\epsilon^i)(1-\mathbf{y}_i\epsilon^i)) = \prod_{i=1}^{\infty}(1-\mathbf{A}_i\epsilon^i),$$
$$\prod_{i=1}^{\infty}\prod_{j=1}^{\infty}\Big(1-\mathbf{x}_i^{j/(i,j)}\mathbf{y}_j^{i/(i,j)}\epsilon^{ij/(i,j)}\Big)^{(i,j)}\prod_{i=1}^{\infty}(1-\mathbf{M}_i\epsilon^i),$$

thereby defining families of polynomials

$$\left\{\mathbf{A}_n, \mathbf{M}_n \in \mathbb{Z}\left[\{\mathbf{x}_i, \mathbf{y}_i\}_{i|n}\right]\right\}_{n=1}^{\infty}.$$

For any commutative ring B with unit and finite subset Δ of the set of positive integers closed under passage to divisors, let $\mathbb{W}_\Delta(B)$ denote the set of vectors with entries in B indexed by Δ. It can be shown that the $\mathbf{A}$'s and $\mathbf{M}$'s define addition and multiplication laws with respect to which $\mathbb{W}_\Delta(B)$ becomes a commutative ring with unit, functorially in commutative rings B with unit.

Let us fix a positive integer N. If $\mathbb{W}_{\leq N}(B)$ is the ring of Witt vectors associated with $\Delta = \{1, 2\dots, N\}$, one has that the map

$$x = (x_i)_{i=1}^N \mapsto \prod_{i=1}^{N}(1-x_i\epsilon^i) \bmod \epsilon^{N+1}$$

identifies the additive group underlying $\mathbb{W}_{\leq N}(B)$ with the group of units in the ring $B[\epsilon]/(\epsilon^{N+1})$ congruent to 1 modulo (ϵ) functorially in commutative rings B with unit.

If F is a field, and $A := F[\epsilon]/(\epsilon^{N+1})$, we can define a pairing

$$\mathrm{Res}_{\leq N}^{\mathbb{W}}(\cdot,\cdot) : F((t))^\times \times \mathbb{W}_{\leq N}(F((t))) \to \mathbb{W}_{\leq N}(F)$$

by the rule

$$\left\langle f, \prod_{i=1}^{N} \left(1 - x_i \epsilon^i\right) \right\rangle_{A((t))^\times} \equiv \prod_{i=1}^{N} \left(1 - \epsilon^i \left[\operatorname{Res}^{\mathbb{W}}_{\leq N}(f, x)\right]_i\right) \bmod (\epsilon^{N+1}),$$

where $\langle \cdot, \cdot \rangle_{A((t))^\times}$ is the Contou-Carrère symbol.

The pairing $\operatorname{Res}^{\mathbb{W}}_{\leq N}$ is essentially the pairing introduced in Witt's paper [12].

Hence, when $N = 1$ we have a map

$$\operatorname{Res}^{\mathbb{W}}(\cdot, \cdot) \colon F((t))^\times \times F((t)) \to F,$$

where

$$\langle f, 1 - \epsilon g \rangle_{A((t))^\times} \equiv 1 - \epsilon [\operatorname{Res}^{\mathbb{W}}(f, g)] \bmod (\epsilon^2),$$

for all $f \in F((t))^\times$ and $g \in F((t))$.

One has that:

- $\operatorname{Res}^{\mathbb{W}}(f \cdot f', g) = \operatorname{Res}^{\mathbb{W}}(f, g) + \operatorname{Res}^{\mathbb{W}}(f', g)$.
- $\operatorname{Res}^{\mathbb{W}}(f, g + g') = \operatorname{Res}^{\mathbb{W}}(f, g) + \operatorname{Res}^{\mathbb{W}}(f, g')$.

3. New explicit expression of the Contou-Carrère symbol

Given an element $f \in A((t))^\times$, it is possible in exactly one way to write

$$f = a_0 \cdot t^{w(f)} \cdot \prod_{i=1}^{\infty} \left(1 - a_{-i} t^{-i}\right) \cdot \prod_{i=1}^{\infty} \left(1 - a_i t^i\right),$$

where

$$w(f) \in \mathbb{Z}, \quad \begin{cases} a_i = 0 & \text{if } i \ll 0, \\ a_i \in m & \text{if } i < 0, \\ a_i \in A^\times & \text{if } i = 0, \\ a_i \in A & \text{if } i > 0. \end{cases}$$

The integer number $w(f)$ is the *winding number* of f, and the set $\{a_i\}_{i=-\infty}^{\infty}$ is the family of *Witt parameters* of f.

Thus, we can consider the morphism of groups

$$\begin{aligned} \phi \colon A((t))^\times &\longrightarrow A((t))^\times, \\ f &\longmapsto a_0 . \end{aligned}$$

Remark 3.1 (Characterization of the Contou-Carrère symbol). It follows from the properties described in Subsection 2.1 that the commutator map $\{\cdot, \cdot\}^{A[[t]]}_{A((t))}$ is the only bimultiplicative map

$$\{\cdot, \cdot\}^{A[[t]]}_{A((t))} \colon A((t))^\times \times A((t))^\times \longrightarrow A^\times$$

that satisfies the conditions:

- $\{\cdot, \cdot\}^{A[[t]]}_{A((t))}\big|_{A[[t]]^\times \times A[[t]]^\times} = 1$.
- $\{f, g\}^{A[[t]]}_{A((t))} = \phi(\mathcal{N}_g[f])$ for all $g \in A((t))^\times$ with $w(g) > 0$.

Hence, the Contou-Carrère symbol is the only map

$$\langle \cdot, \cdot \rangle_A \colon A((t))^\times \times A((t))^\times \longrightarrow A^\times$$

that satisfies the properties:

- $\langle \cdot, \cdot \rangle_A$ is bimultiplicative.
- $\langle \cdot, \cdot \rangle_A\big|_{A[[t]]^\times \times A[[t]]^\times} = 1$.

- $\langle f, g\rangle_A = (-1)^{w(f)w(g)} \cdot \phi(\mathcal{N}_g[f])$ for all $g \in A((t))^\times$ with $w(g) > 0$.

As far as we know a characterization relating the Contou-Carrere symbol with a norm map is not stated explicitly in the literature.

Let us now consider a series $s(t) \in A[[t]]^\times$ and an element $a \in m$. For each positive integer n, setting

$$s(t) = s_0^n(t^n) + t \cdot s_1^n(t^n) + \cdots + t^{n-1} \cdot s_{n-1}^n(t^n)\,, \tag{3.1}$$

we can construct a matrix $C^{n,a}_{s(t)} \in \mathrm{Gl}(n, A)$ where the coefficients are

$$[C^{n,a}_{s(t)}]_{ij} = \begin{cases} s^n_{i-j}(a) \text{ if } i \geq j, \\ a \cdot s^n_{j-i}(a) \text{ if } i < j\,. \end{cases}$$

That is, the expression of the matrix is

$$C^{n,a}_{s(t)} \begin{pmatrix} s_0^n(a) & a \cdot s^n_{n-1}(a) & \dots & a \cdot s_2^n(a) & a \cdot s_1^n(a) \\ s_1^n(a) & s_0^n(a) & \dots & a \cdot s_3^n(a) & a \cdot s_2^n(a) \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ s^n_{n-2}(a) & \dots & s_1^n(a) & s_0^n(a) & a \cdot s^n_{n-1}(a) \\ s^n_{n-1}(a) & \dots & s_2^n(a) & s_1^n(a) & s_0^n(a) \end{pmatrix}.$$

Note that in (3.1) the series $s_0^n(t) \in A[[t]]^\times$. Moreover, for all $s(t) \in A[[t]]^\times$ and $a \in m$, one has that

$$\det C^{1,a}_{s(t)} = \phi(s(t)) = s(a)\,.$$

We shall now give an explicit expression of the Contou-Carrere symbol by using the determinants of the matrices $C^{n,a}_{s(t)}$.

Lemma 3.2. *If $n > 0$, $a \in m$ and $s(t) \in A[[t]]^\times$, one has that*

$$\langle s(t), t^n - a\rangle_A = \det C^{n,a}_{s(t)}\,.$$

Proof. Viewing $A((t))$ via the homomorphism $h \mapsto h \circ (t^n - a)$ as a free $A((t))$-module of rank n with basis $\{1, \dots, t^{n-1}\}$, from the relations

$$s_i^n(t^n) = s_i^n(a) + (t^n - a) \cdot \widetilde{s}_i^n(t^n - a),$$
$$t^n \cdot s_j^n(t^n) = a \cdot s_j^n(a) + (t^n - a) \cdot \widetilde{s}_j^n(t^n - a),$$

we have that the matrix of the homothety $h_{s(t)}$ is obtained from the equality

$$\begin{aligned} h_{s(t)}(1) &= s_0^n(t^n) + t \cdot s_1^n(t^n) + \cdots + t^{n-1} \cdot s^n_{n-1}(t^n) \\ &= \big[[s_0^n(a) + t \cdot \widetilde{s}_0^n(t)] \circ (t^n - a)\big] + \cdots + t^{n-1} \cdot \big[[s^n_{n-1}(a) + t \cdot \widetilde{s}^n_{n-1}(t)] \circ (t^n - a)\big] \\ &\equiv (s_0^n(a) + t \cdot \widetilde{s}_0^n(t), \dots, s^n_{n-1}(a) + t \cdot \widetilde{s}^n_{n-1}(t)) \end{aligned}$$

and from the expressions

$$\begin{aligned} h_{s(t)}(t^i) &= t^i \cdot s_0^n(t^n) + \cdots + t^{n-1} \\ &\quad \cdot s^n_{n-1-i}(t^n) + t^n \cdot s^n_{n-i}(t^n) + \cdots + t^{i-1} \cdot t^n \cdot s^n_{n-1}(t^n) \\ &= t^i \cdot \big[[s_0^n(a) + t \cdot \widetilde{s}_0^n(t)] \circ (t^n - a)\big] + \cdots + t^{i-1} \\ &\quad \cdot \big[[a \cdot s^n_{n-1}(a) + t \cdot \widetilde{s}^n_{n-1}(t)] \circ (t^n - a)\big] \\ &\equiv (a \cdot s^n_{n-i}(a) + t \cdot \widetilde{s}^n_{n-i}(t), \dots, s^n_{n-1-i}(a) + t \cdot \widetilde{s}^n_{n-1-i}(t)) \end{aligned}$$

when $i \geq 1$.

Thus, bearing in mind the definition of $C^{n,a}_{s(t)}$, since $\mathcal{N}_{t^n-a}[s(t)] \in A[[t]]^\times$ and

$$\langle s(t), t^n - a\rangle_A = \phi(\mathcal{N}_{t^n-a}[s(t)])\,,$$

the claim is deduced. □

A direct consequence of Lemma 3.2 is the following result that appeared in [2].

Corollary 3.3. *For every $s(t) \in A[[t]]^\times$ and $a \in m$, one has that*

$$\langle s(t), t - a\rangle_A = s(a)\,.$$

Given two elements $f, g \in A((t))^\times$, let us now write

$$f = t^{-N} \cdot s(t) \cdot \prod_{i=1}^{k} \left(t^i - a_{-i}\right), \qquad g = t^{-M} \cdot s'(t) \cdot \prod_{j=1}^{h} \left(t^j - b_{-j}\right),$$

with $N, M \in \mathbb{Z}^+$; $a_{-i}, b_{-j} \in m$; $s(t), s'(t) \in A[[t]]^\times$; $w(f) = \frac{k\cdot(k+1)}{2} - N$ and $w(g) = \frac{h\cdot(h+1)}{2} - M$.

Proposition 3.4 (Explicit expression of the Contou-Carrère symbol)**.** *With the previous notation, if $\phi(f) = a_0$ and $\phi(g) = b_0$, the Contou-Carrère symbol is*

$$\langle f, g\rangle_A = (-1)^{w(f)w(g)} \cdot \frac{b_0^M \cdot \prod_{j=1}^h \det C^{j,b_{-j}}_{s(t)}}{a_0^N \cdot \prod_{i=1}^k \det C^{i,a_{-i}}_{s'(t)}}\,.$$

Proof. The expression follows from Lemma 3.2, bearing in mind that

- $\langle t^{-N}, t^{-M}\rangle_A = (-1)^{N\cdot M}$;
- $\langle t^{-N}, t^j - b_{-j}\rangle_A = (-1)^{j\cdot N}$;
- $\langle t^i - a_{-i}, t^{-M}\rangle_A = (-1)^{i\cdot M}$;
- $\langle t^{-N}, s'(t)\rangle_A = b_0^{-M}$;
- $\langle s(t), t^{-M}\rangle_A = a_0^{-N}$;
- $\langle t^i - a_{-i}, t^j - b_{-j}\rangle_A = (-1)^{i\cdot j}$. □

Corollary 3.5. *Given two elements $f, g \in A((t))^\times$, such that*

$$f = t^{-N} \cdot a_0 \cdot \bar{s}(t) \cdot \prod_{i=1}^{k} \left(t^i - a_{-i}\right), \qquad g = t^{-M} \cdot b_0 \cdot \bar{s}'(t) \cdot \prod_{j=1}^{h} \left(t^j - b_{-j}\right),$$

with $N, M \in \mathbb{Z}^+$; $a_0, b_0 \in A^\times$; $a_{-i}, b_{-j} \in m$; $\bar{s}(t), \bar{s}'(t) \in A[[t]]^\times$; $w(f) = \frac{k\cdot(k+1)}{2} - N$ and $w(g) = \frac{h\cdot(h+1)}{2} - M$, the Contou-Carrère symbol is

$$\langle f, g\rangle_A = (-1)^{w(f)w(g)} \cdot \frac{a_0^{w(g)} \cdot \prod_{j=1}^h \det C^{j,b_{-j}}_{\bar{s}(t)}}{b_0^{w(f)} \cdot \prod_{i=1}^k \det C^{i,a_{-i}}_{\bar{s}'(t)}}\,.$$

Proof. Since $\langle a_0, t^j - b_{-j}\rangle_A = a_0^j$ and $\langle t^i - a_{-i}, b_0\rangle_A = b_0^i$, this formula follows immediately from the explicit expression given in Proposition 3.4. □

Finally, as an application of this explicit formula of the Contou-Carrère symbol, we shall study the map $\operatorname{Res}^{\mathbb{W}}(\cdot,\cdot)\colon F((t))^\times \times F((t)) \to F$ defined in Subsection 2.2 by the rule

$$\langle f, 1 - \epsilon g\rangle_{A((t))^\times} \equiv 1 - \epsilon[\operatorname{Res}^{\mathbb{W}}(f, g)] \mod (\epsilon^2)\,.$$

Note that if $g = \sum_{j\geq -h} b_j t^j$, then

$$1 - \epsilon g = t^{-\frac{h(h+1)}{2}} \cdot (1-\epsilon b_0) \cdot \bar{s}'(t) \cdot \prod_{j=1}^{h} (t^j - \epsilon b_{-j}) \,,$$

where $\bar{s}'(t) \in A[[t]]^\times$ with $A = F[\epsilon]/(\epsilon^2)$.

Thus, if $f = t^{w(f)} \cdot a_0 \cdot \bar{s}(t)$, we have that

$$\langle f, 1-\epsilon g\rangle_{A((t))^\times} = \frac{\prod_{j=1}^h \det C^{j,\epsilon b_{-j}}_{\bar{s}(t)}}{[1-\epsilon b_0]^{w(f)}} = \frac{\prod_{j=1}^h \det C^{j,\epsilon b_{-j}}_{\bar{s}(t)}}{[1-\epsilon w(f) b_0]} \,.$$

Accordingly, if $\mathrm{Tr}\, A$ denotes the trace of a square matrix A, bearing in mind that

$$C^{j,\epsilon b_{-j}}_{\bar{s}(t)} = A^j_{\bar{s}(t)} \cdot \big(\mathrm{Id} \ + \epsilon \cdot b_{-j} \cdot [A^j_{\bar{s}(t)}]^{-1} \cdot B^j_{\bar{s}(t)}\big) \,,$$

with

$$A^j_{\bar{s}(t)} = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ a_1 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \dots & \vdots \\ a_{j-2} & \dots & a_1 & 1 & 0 \\ a_{j-1} & \dots & a_2 & a_1 & 1 \end{pmatrix}$$

and

$$B^j_{\bar{s}(t)} = \begin{pmatrix} a_j & a_{j-1} & \dots & \dots & a_1 \\ a_{j+1} & a_j & a_{j-1} & \dots & a_2 \\ \vdots & \ddots & \ddots & \dots & \vdots \\ a_{2j-2} & \dots & a_{j+1} & a_j & a_{j-1} \\ a_{2j-1} & \dots & a_{j+2} & a_{j+1} & a_j \end{pmatrix} \,,$$

where $\bar{s}(t) = 1 + \sum_{i\geq 1} a_i \cdot t^i$, we have that

$$\det C^{j,\epsilon b_{-j}}_{\bar{s}(t)} = 1 + \epsilon \cdot b_{-j} \cdot \mathrm{Tr}([A^j_{\bar{s}(t)}]^{-1} \cdot B^j_{\bar{s}(t)}) \,.$$

Hence, since

$$\big([A^j_{\bar{s}(t)}]^{-1}\big)_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha < \beta, \\ \displaystyle\sum_{\substack{i_1+\dots+i_t = \alpha-\beta \\ 1\leq i_1,\dots,i_t\leq j-1}} (-1)^t a_{i_1}\cdots a_{i_t} & \text{if } \alpha > \beta \,, \end{cases}$$

then

$$\mathrm{Tr}([A^j_{\bar{s}(t)}]^{-1} \cdot B^j_{\bar{s}(t)}) = j \cdot a_j + \sum_{r=1}^{j-1} \big[\sum_{\substack{i_1+\dots+i_t = r \\ 1\leq i_1,\dots,i_t\leq j-1}} (-1)^t a_{i_1}\cdots a_{i_t}\big] a_{j-r} \,,$$

and the explicit expression of the Witt Residue is:

$$\mathrm{Res}^{\mathbb{W}}(f,g) = -w(f)\cdot b_0 - \sum_{j=1}^{h} b_{-j} \cdot \big(j\cdot a_j + \sum_{r=1}^{j-1} \big[\sum_{\substack{i_1+\dots+i_t = r \\ 1\leq i_1,\dots,i_t\leq j-1}} (-1)^t a_{i_1}\cdots a_{i_t}\big] a_{j-r}\big)$$

for $f = t^{w(f)} \cdot a_0 \cdot [1 + \sum_{i\geq 1} a_i \cdot t^i]$ and $g = \sum_{j\geq -h} b_j t^j$.

Remark 3.6 (Residue Theorem). Let X be a nonsingular and irreducible curve over the field F and let Σ_X be its function field. Fixing a closed point $x \in X$, we have an immersion of rings

$$i_x \colon \Sigma_X \hookrightarrow (\widehat{\mathcal{O}_x})_0 \simeq F((t)) ,$$

such that, by restriction of $\operatorname{Res}^{\mathbb{W}}(\cdot, \cdot)$, we can consider the map

$$\operatorname{Res}_x^{\mathbb{W}}(\cdot, \cdot) \colon \Sigma_X^{\times} \times \Sigma_X \to F ,$$

which is the Witt Residue associated with the closed point x.

When X is a complete curve, a direct consequence of the reciprocity law for the Contou-Carrère symbol [1] is the following Residue Theorem:

$$\sum_{x \in X} \operatorname{Res}_x^{\mathbb{W}}(f, g) = 0$$

for every $f \in \Sigma_X^{\times}$ and $g \in \Sigma_X$.

Acknowledgment

The author thanks the anonymous referee for her/his constructive comments aimed at improving the preliminary version of this paper.

References

[1] Anderson, G. W.; Pablos Romo, F. Simple Proofs of Classical Explicit Reciprocity Laws on Curves using Determinant Grupoids over an Artinian Local Ring. Comm. Algebra. **2004**, *32(1)*, 79-102. MR2036223 (2005d:11099)

[2] Asakura, M. On *dlog* image of K_2 of elliptic surface minus singular fibers. Preprint, 2005, math. AG./0511190.

[3] Beilinson, A.; Bloch S.; Esnault H. ϵ-factors for Gauss-Manin determinants. Moscow Math. J. **2002**, *2(3)*, 477-532. MR1988970 (2004m:14011)

[4] Contou-Carrere, C. Jacobienne Locale, Groupe de Bivecteurs de Witt Universel et Symbole Modéré. C.R. Acad. Sci. Paris. **1994**, *318, Série I*, 743-746. MR1272340 (95c:14059)

[5] Hazewinkel, M., *Formal Groups and Applications*, Academic Press, New York, San Francisco, London, 1978. MR0506881 (82a:14020)

[6] Kapranov, M.; Vasserot, E. Formal loops groups II: A local Riemann-Roch Theorem for determinantal gerbes. Preprint, 2005, math. AG./0509646.

[7] Kato, K. A Generalization of local class field theory by using K-groups II. J. Fac. Sci. Univ. Tokyo Sect. IA **1980**, *27(3)*, 603-683. MR0603953 (83g:12020a)

[8] Macdonald, I. G., *Symmetric Functions and Hall Polynomials*, Oxford Science Publications, second edition, Oxford, 1995. MR1354144 (96h:05207)

[9] Pablos Romo, F. A Contou-Carrère symbol on $\mathrm{Gl}(n, A((t)))$ and a Witt Residue Theorem on $\mathrm{Mat}(n, \Sigma_C)$. Int. Math. Res. Not. **2006**, Article ID 56824, 1-21. MR2211145 (2006m:11092)

[10] Pablos Romo, F. A Generalization of the Contou-Carrère symbol. Israel J. Math. **2004**, *141*, 39-60. MR2063024 (2005g:11115)

[11] Tate, J. Residues of Differentials on Curves. Ann. Scient. Éc. Norm. Sup. **1968**, 4a série, *1*, 149-159. MR0227171 (37:2756)

[12] Witt, E. Zyklische Körper und Algebren der Charakteristik p vom Grad p^n. J. Reine Angew. Math. **1937**, *176*, 126-140.

Departamento de Matemáticas, Universidad de Salamanca, Plaza de la Merced 1-4, 37008 Salamanca, España

E-mail address: fpablos@usal.es