

GENERALIZED RECIPROCITY LAWS

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ABSTRACT. The aim of this paper is to give an abstract formulation of the classical reciprocity laws for function fields that could be generalized to the case of arbitrary (non-commutative) reductive groups as a first step to finding explicit non-commutative reciprocity laws. The main tool in this paper is the theory of determinant bundles over adelic Sato Grassmannians and the existence of a Krichever map for rank n vector bundles.

1. INTRODUCTION

In this paper we offer a unified approach to several geometric reciprocity laws, such as the Weil reciprocity law for the tame symbol and reciprocity laws for the Contou-Carrère symbol, in a way susceptible to being generalized to the non-abelian case.

The main tool used in our formulation of reciprocity laws is the algebraic theory of infinite determinants (as in [5]). The analytic counterpart of the theory has been developed by A. Pressley, G. Segal and G. Wilson ([12], [13]), and we follow the algebraic theory developed in [1] and [2]. An important difference with the paper of A. Beilinson, S. Bloch and H. Esnault ([5]) is the fact that we do not introduce a super-structure on the determinant bundles.

The determinant bundles over the Sato Grassmannian induce a central extension of the restricted linear group $\mathrm{Gl}(\infty)$. These central extensions induce central extensions of the loop groups, whose commutators allow us to recover the “classical symbols” up to a sign (§2).

When X is a complete, smooth and connected curve over a perfect field k with function field Σ_X , adèle ring \mathbb{A}_X , and $G \subset \mathrm{Gl}(n, k)$ a reductive algebraic group over k , we obtain a family of central extensions:

$$(1.1) \quad 1 \rightarrow \mathbb{G}_m \rightarrow \tilde{G}_x \rightarrow G_x \rightarrow 1,$$

$$(1.2) \quad 1 \rightarrow k^* \rightarrow \tilde{G}_{\mathbb{A}_X} \rightarrow G_{\mathbb{A}_X} \rightarrow 1,$$

$$(1.3) \quad 1 \rightarrow k^* \rightarrow \widetilde{G_{\Sigma_X}} \rightarrow G_{\Sigma_X} \rightarrow 1,$$

where G_x is the algebraic loop group over $k(x)$ (for each closed point $x \in X$); $G_{\mathbb{A}_X}$ is the adelic group, and G_{Σ_X} is the group defined over Σ_X . Our main result (Theorem 3.5) can be stated by saying that the extension (1.3) is trivial. In the case $G = \mathbb{G}_m$, this statement is equivalent to the Weil reciprocity law, as is proved

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in Remark 3.7. The proof of the main result is an easy consequence of the existence of a Krichever map for rank n vector bundles over X with formal trivialization at closed points.

Let $\{ \cdot, \cdot \}_{A_x}, \{ \cdot, \cdot \}_{\mathbb{A}_X}$ and $\{ \cdot, \cdot \}_{\Sigma_X}$ be the pairings defined by the commutators of the extensions (1.1), (1.2) and (1.3), respectively. The pairing $\{f, g\}_{\mathbb{A}_X}$ (for two commuting elements $f, g \in G_{\mathbb{A}_X}$) is not exactly the product $\prod_{x \in X} \{f, g\}_{A_x}$.

The precise relation (stated in §3 for $G = \mathbb{G}_m$, and in §4 for a non-commutative G) is that $\{f, g\}_{\mathbb{A}_X} = (\text{sign}) \cdot \prod_{x \in X} \{f, g\}_{A_x}$. This allows the recovery of the classical tame symbol in terms of the commutator without imposing an extra super-bundle structure on the determinant bundles.

The last section of the paper is devoted to exploring the reciprocity laws obtained in the non-commutative case $G = \text{Gl}(2, k)$. We make explicit computations in some examples and prove that the corresponding reciprocity laws are a consequence of the abelian one. Of course, it remains to study more general examples of commuting elements of $\text{Gl}(2, \Sigma_X)$, which could give new identities.

For contrast, here we do not impose the triviality of the central extension (1.3) (Theorem 3.5) in terms of the cocycles associated with some section of the map $\widetilde{G}_{\mathbb{A}_X} \rightarrow G_{\mathbb{A}_X}$.

We hope that further research in these directions could open a way to find explicit non-abelian reciprocity laws. To finish this Introduction, it is important to mention the very suggestive paper of E. Witten ([16]), in which the multiplicative Ward identities of a Quantum Field Theory are interpreted in terms of the triviality of the central extension (1.3).

2. PRELIMINARIES

2.A. Functor of points. Firstly, we shall recall the concept of S -valued points in the theory of schemes: if Y and S are k -schemes, an S -valued point of Y is a morphism of schemes from S to Y . The set of all such points is denoted

$$Y^\bullet(S) = \text{Hom}_{k\text{-sch.}}(S, Y).$$

The functor

$$Y^\bullet: \mathcal{C}_{k\text{-sch.}} \rightarrow \mathcal{C}_{sets}$$

is called the “functor of points” of the k -scheme Y , and the set $Y^\bullet(\text{Spec } k)$ consists of the “rational points of Y ”. We should remark that in this sense each k -scheme can be considered as a functor from the category of k -schemes to the category of sets (or to the category of abelian groups when Y is a group scheme).

In this work we shall write $Y(S)$ instead of $Y^\bullet(S)$.

2.B. Determinant bundles and central extensions of $\text{Gl}(\infty)$. Let V be a k -vector space endowed with a structure of a linear topological vector space and let us assume that V is complete with respect to this topology. We shall also assume that a basis, τ , of the topology of V is given by the family of vector subspaces A of V , which are commensurable (in the Tate sense [2], [4],[14]) with a fixed vector subspace $V_+ \subset V$. We use the symbol $A \sim V_+$ to denote commensurable vector subspaces.

In this paper we will be mainly interested in two examples:

- $V = k((z))$ and $V_+ = k[[z]]$.
- $V = \mathbb{A}_X$, the ring of adèles of a smooth curve X over k .

Given the pair (V, V_+) we can construct a Grassmannian, $\text{Gr}(V)$, which is a k -scheme ([2], Th. 2.13). This scheme is the algebraic version of the Sato Grassmannian. The rational points of $\text{Gr}(V)$ are the discrete vector subspaces of V : that is, the vector subspaces $U \subseteq V$ such that $U \cap V_+$ and $V/U + V_+$ are finite-dimensional vector spaces over k . The index of a rational point $U \in \text{Gr}(V)(\text{Spec } k)$ is:

$$(2.1) \quad i_U = \dim_k(U \cap V_+) - \dim_k(V/U + V_+),$$

and the concept of index can be extended to arbitrary points of $\text{Gr}(V)$.

Denoting by $\text{Gr}_d(V)$ the subscheme over which the index takes values equal to $d \in \mathbb{Z}$, one can show that the $\text{Gr}_d(V)$ are open connected subschemes, and the decomposition of $\text{Gr}(V)$ in connected components is:

$$\text{Gr}(V) = \coprod_{d \in \mathbb{Z}} \text{Gr}_d(V).$$

Given a vector subspace $A \subset V$ commensurable with V_+ (i.e. such that $\dim_k(A + V_+/A \cap V_+) < \infty$), there exists a complex of modules over $\text{Gr}(V)$:

$$\mathcal{C}_A^\bullet \equiv 0 \rightarrow \mathbb{L} \oplus \widehat{A}_{\text{Gr}(V)} \xrightarrow{\delta} \widehat{V}_{\text{Gr}(V)} \rightarrow 0,$$

\mathbb{L} being the universal subspace of $\widehat{V}_{\text{Gr}(V)}$ and δ being the addition map (see [2], section 3 for details). \mathcal{C}_A^\bullet is a perfect complex and its determinant in the sense of Knudsen-Mumford ([9]) is denoted by Det_A and is a line bundle over $\text{Gr}(V)$. Given a rational point $U \in \text{Gr}(V)(k)$, the fibre of the bundle Det_A over U is:

$$\text{Det}_A(U) \simeq \Lambda(U \cap A) \otimes \Lambda(V/U + A)^*,$$

Λ being the maximal exterior power.

The bundle Det_{V_+} will be denoted simply by Det .

Remark 2.1. Kapranov has defined the notion of a determinantal theory over a locally linearly compact k -vector space ([8], Definition 1.2.2). The determinantal theories on V and their isomorphisms form a category (groupoid) denoted by $\text{Det}(V)$. In our setting, the groupoid $\text{Det}(V)$ can be identified with the set of line bundles:

$$\{\text{Det}_A, A \sim V_+\}.$$

Note that the line bundles Det_A are all isomorphic ([2], Theorem 3.5).

The restricted linear group, $\text{Gl}(V)$, of (V, V_+) is the contravariant functor over the category of k -schemes defined by:

$$S \rightsquigarrow \text{Gl}(V)(S) = \{g \in \text{Aut}_{\mathcal{O}_S}(\widehat{V}_S) \text{ such that } g \text{ is bicontinuous}\}$$

(see [2], section 4, for precise definitions). This group acts naturally on $\text{Gr}(V)$. In the case $V = k((z))$, the set of rational points of the restricted linear group is precisely the Japanese group $\text{Gl}(\infty)$.

Let us denote by $\text{Gl}_0(V)$ the subfunctor of $\text{Gl}(V)$ that preserves the connected components of $\text{Gr}(V)$. One can easily prove that $g^* \text{Det} \simeq \text{Det}$ for every $g \in \text{Gl}_0(V)$.

Let us define $\widetilde{\text{Gl}}_0(V)$ as the set of commutative diagrams:

$$\begin{array}{ccc} \text{Det} & \xrightarrow[\sim]{\bar{g}} & \text{Det} \\ \downarrow & & \downarrow \\ \text{Gr}(V) & \xrightarrow[\sim]{g} & \text{Gr}(V) \end{array}$$

for every $g \in \text{Gl}_0(V)$. $\widetilde{\text{Gl}}_0(V)$ is a functor of groups and there is a natural homomorphism $\pi: \widetilde{\text{Gl}}_0(V) \rightarrow \text{Gl}_0(V)$ defined by $\pi(\bar{g}) = g$.

In ([2], section 3) it is proved that $H^0(\text{Gr}_0(V), \mathcal{O}_{\text{Gr}_0(V)}) = k$. This implies that there exists a canonical central extension of the functor of groups:

$$1 \rightarrow \mathbb{G}_m \rightarrow \widetilde{\text{Gl}}_0(V) \rightarrow \text{Gl}_0(V) \rightarrow 1.$$

From the definitions, one has a natural action of $\widetilde{\text{Gl}}_0(V)$ on $\text{Det}|_{\text{Gr}_0(V)}$ and also on Det .

Let us now consider the case $V = k((z))$ and $V_+ = k[[z]]$, and let σ denote the product by z^{-1} in V (the “shift operator”). σ induces an automorphism of $\text{Gr}(V)$, also denoted by σ , and obviously σ is a rational point of $\text{Gl}(V)$. For each $d \in \mathbb{Z}$, σ is an isomorphism $\sigma: \text{Gr}_d(V) \xrightarrow{\sim} \text{Gr}_{d-1}(V)$. Let us denote by $\tilde{\sigma}$ a lifting of σ to Det :

$$\begin{array}{ccc} \text{Det}|_{\text{Gr}_1(V)} & \xrightarrow[\sim]{\tilde{\sigma}} & \text{Det}|_{\text{Gr}_0(V)} \\ \downarrow & & \downarrow \\ \text{Gr}_1(V) & \xrightarrow[\sim]{\sigma} & \text{Gr}_0(V) \end{array} .$$

Clearly, there exists such a $\tilde{\sigma}$, and it is unique up to multiplication by elements of \mathbb{G}_m .

The isomorphism $\tilde{\sigma}$ induces an automorphism, $\bar{\sigma}$, of the group $\widetilde{\text{Gl}}_0(V)$ by the following formula:

$$\bar{\sigma}(\tilde{A}) = \tilde{\sigma} \cdot \tilde{A} \cdot \tilde{\sigma}^{-1}.$$

The automorphism σ generates a group subscheme, $\langle \sigma \rangle$, of $\text{Gl}(V)$ isomorphic to $\mathbb{Z}_* := \coprod_{\alpha \in \mathbb{Z}} \text{Spec } k$, and there exists an isomorphism of group functors:

$$\text{Gl}(V) \simeq \text{Gl}_0(V) \rtimes \langle \sigma \rangle .$$

The automorphism $\bar{\sigma}$ allows us to define a semidirect product:

$$\widetilde{\text{Gl}}(V) := \widetilde{\text{Gl}}_0(V) \rtimes \mathbb{Z}_*$$

such that there exists an exact sequence of group functors:

$$(2.2) \quad 1 \rightarrow \mathbb{G}_m \rightarrow \widetilde{\text{Gl}}(V) \rightarrow \text{Gl}(V) \rightarrow 1 .$$

From the construction it is obvious that the central extension $\widetilde{\text{Gl}}(V)$ does not depend on the lifting $\tilde{\sigma}$ of σ . For each subgroup $G \subseteq \text{Gl}(V)$ the above sequence induces an exact sequence:

$$1 \rightarrow \mathbb{G}_m \rightarrow \tilde{G} \rightarrow G \rightarrow 1 .$$

Let Γ be the formal commutative group scheme representing the contravariant functor on the category of k -schemes:

$$S \rightsquigarrow k((z))^*(S) := H^0(S, \mathcal{O}_S)((z))^*$$

([2], section 4). It is clear that $\Gamma \subset \text{Gl}(V)$, and hence one has a central extension of group schemes:

$$1 \rightarrow \mathbb{G}_m \rightarrow \tilde{\Gamma} \rightarrow \Gamma \rightarrow 1.$$

Since Γ is commutative, the commutator of $\tilde{\Gamma}$ induces a pairing:

$$\begin{aligned} \{ \cdot, \cdot \}_{V_+}^V : \Gamma \times \Gamma &\longrightarrow \mathbb{G}_m \\ (a, b) &\longmapsto \{a, b\}_{V_+}^V = \tilde{a} \cdot \tilde{b} \cdot \tilde{a}^{-1} \cdot \tilde{b}^{-1}, \end{aligned}$$

\tilde{a} and \tilde{b} being liftings of a, b to $\tilde{\Gamma}$. The explicit computation of the pairing $\{ \cdot, \cdot \}_{V_+}^V$ can be found in [10].

Remark 2.2. Let X be a smooth and connected curve over a perfect field k and let $p \in X$ be a closed point on it. Let us consider Σ_X (the function field of X), $A_p = \hat{\mathcal{O}}_p$ (the completion of the local ring $\mathcal{O}_{X,p}$) and $K_p = (\hat{\mathcal{O}}_p)_0$ (the field of fractions of $\hat{\mathcal{O}}_p$, which coincides with the completion of Σ_X with respect to the valuation ring $\mathcal{O}_{X,p}$). Since k is a perfect field, one has that $A_p \simeq k(p)[[t]]$ and $K_p \simeq k(p)((t))$, $k(p)$ being the residue field of p . We set $\text{deg}(p) = \dim_k k(p)$.

Let us consider the k -vector spaces $V_+ = A_p$ and $V = K_p$, which define the Grassmannian $\text{Gr}(K_p)$ and the restricted linear group $\text{Gl}(K_p)$.

Let Γ_p be the formal commutative group scheme representing the contravariant functor on the category of k -schemes:

$$S \rightsquigarrow [H^0(S, \mathcal{O}_S) \otimes_k k(p)]((z))^* \quad ([10], \text{Section 3.B}).$$

If p is a rational point and Γ is the formal scheme defined above, then $\Gamma_p \simeq \Gamma$.

It is clear that $\Sigma_X^* \subseteq K_p^* \simeq \Gamma_p(\text{Spec } k) \subseteq \text{Gl}(K_p)(\text{Spec } k)$.

There exists a central extension of group schemes:

$$1 \rightarrow \mathbb{G}_m \rightarrow \tilde{\Gamma}_p \xrightarrow{\pi_p} \Gamma_p \rightarrow 1,$$

which, considering rational points and restricting to Σ_X^* , induces a central extension of groups:

$$1 \rightarrow k^* \rightarrow \widetilde{\Sigma_X^*} \xrightarrow{\pi_p} \Sigma_X^* \rightarrow 1,$$

whose commutator pairing is $\{ \cdot, \cdot \}_{A_p}^{K_p} : \Sigma_X^* \times \Sigma_X^* \rightarrow k^*$.

According to the statements of [11], if $f, g \in \Sigma_X^*$, then

$$\{f, g\}_{A_p}^{K_p} = N_{k(p)/k} \left(\frac{f^{v_p(g)}}{g^{v_p(f)}}(p) \right) \in k^*,$$

where $N_{k(p)/k}$ denotes the norm of the extension $k \hookrightarrow k(p)$, and v_p is the discrete valuation on Σ_X^* associated with the closed point p .

It is therefore possible to recover the tame symbol of an algebraic curve from this commutator, and its explicit expression is:

$$\begin{aligned} (f, g)_p &= (-1)^{\text{deg}(p) \cdot v_p(f) \cdot v_p(g)} \{f, g\}_{A_p}^{K_p} \\ &= (-1)^{\text{deg}(p) \cdot v_p(f) \cdot v_p(g)} N_{k(p)/k} \left(\frac{f^{v_p(g)}}{g^{v_p(f)}}(p) \right). \end{aligned}$$

When X is complete, the Weil reciprocity law shows that

$$\prod_{p \in X} (f, g)_p = 1.$$

Remark 2.3. With the notation of Remark 2.2, for each function $f \in \Sigma_X^*$ we can define the integer number:

$$i(f, A_p) = \dim_k(A_p/A_p \cap fA_p) - \dim_k(fA_p/A_p \cap fA_p) = \deg(p) \cdot v_p(f).$$

If U_p^f is a k -vector subspace of K_p such that $fA_p \oplus U_p^f = K_p$, then U_p^f is a rational point of the Grassmannian $\text{Gr}(K_p)$. One thus has that $i(f, A_p) = i_{U_p^f}$, where $i_{U_p^f}$ is the index defined by the formula (2.1).

This construction of the central extension $\widetilde{\Gamma}$ can be generalized to more general loop groups.

Let us consider $V^n = V \oplus \dots \oplus V$ and $V_+^n = V_+ \oplus \dots \oplus V_+$, with $V = k((z))$ and $V_+ = k[[z]]$. Following the same steps described above, we construct the vector Grassmannian $\text{Gr}(V^n)$, the determinant bundle Det_n , and the groups $\text{Gl}(V^n)$ and $\widetilde{\text{Gl}}(V^n)$. The loop group of $\text{Gl}(n, k)$ is defined as the group scheme $\text{LGl}(n)$, representing the functor of groups:

$$S \rightsquigarrow \underline{\text{LGl}}(n)(S) = \left\{ \begin{array}{l} \text{automorphisms of } V^n \widehat{\otimes}_k H^0(S, \mathcal{O}_S) \\ \text{as } H^0(S, \mathcal{O}_S)\text{-modules} \end{array} \right\}.$$

$\text{LGl}(n)$ is a subgroup of $\text{Gl}(V^n)$, and therefore there exists a central extension of group functors:

$$1 \rightarrow \mathbb{G}_m \rightarrow \widetilde{\text{LGl}}(n) \rightarrow \text{LGl}(n) \rightarrow 1.$$

3. ADELIC CENTRAL EXTENSIONS AND GENERALIZED RECIPROCITY LAWS

Let X be a complete, smooth and connected curve over a perfect field k . Let Σ_X be its function field, \mathbb{A}_X the adèle ring of X , and \mathbb{A}_X^+ the subring

$$\prod_{\substack{x \in X \\ x \text{ closed}}} \widehat{\mathcal{O}}_x$$

of \mathbb{A}_X .

To the pair of k -vector spaces $(\mathbb{A}_X, \mathbb{A}_X^+)$ we can associate a Grassmannian $\text{Gr}_X = \text{Gr}(\mathbb{A}_X, \mathbb{A}_X^+)$, as described in the previous section. Let us denote by \mathbb{D} the determinant bundle over Gr_X . The group of ideles of X , \mathbb{I}_X , is a subgroup of $\text{Gl}(\mathbb{A}_X)(\text{Spec } k)$, the rational points of the restricted linear group, and then, applying the results of Subsection 2.B, there exists a canonical central extension of groups:

$$(3.1) \quad 1 \rightarrow k^* \rightarrow \widetilde{\mathbb{I}}_X \rightarrow \mathbb{I}_X \rightarrow 1.$$

In general, for each positive integer n , let us consider the k -vector spaces:

$$\begin{aligned} \mathbb{A}_X^n &= \mathbb{A}_X \oplus \dots \oplus \mathbb{A}_X, \\ (\mathbb{A}_X^+)^n &= \mathbb{A}_X^+ \oplus \dots \oplus \mathbb{A}_X^+, \end{aligned}$$

and let us denote by Gr_X^n the vector adelic Grassmannian $\text{Gr}(\mathbb{A}_X^n, (\mathbb{A}_X^+)^n)$ and by \mathbb{D}_n its determinant bundle. The rank- n restricted linear group $\text{Gl}(\mathbb{A}_X^n)$ acts on Gr_X^n , and we denote by $\text{Gl}(n, \mathbb{A}_X)$ the adelic linear group of rank n (that is, the group of invertible $n \times n$ matrices with entries in \mathbb{A}_X). Obviously, $\text{Gl}(n, \mathbb{A}_X) \subset \text{Gl}(\mathbb{A}_X^n)(\text{Spec } k)$, and one has a canonical central extension of groups:

$$(3.2) \quad 1 \rightarrow k^* \rightarrow \widetilde{\text{Gl}}(n, \mathbb{A}_X) \xrightarrow{\pi} \text{Gl}(n, \mathbb{A}_X) \rightarrow 1.$$

If g and h are two commuting elements of $\mathrm{Gl}(n, \mathbb{A}_X)$ and $\widetilde{g}, \widetilde{h} \in \widetilde{\mathrm{Gl}(n, \mathbb{A}_X)}$ are elements such that $\pi(\widetilde{g}) = g$ and $\pi(\widetilde{h}) = h$, then one has a commutator pairing:

$$\{g, h\}_{(\mathbb{A}_X^+)^n}^{\mathbb{A}_X^+} = \widetilde{g} \cdot \widetilde{h} \cdot \widetilde{g}^{-1} \cdot \widetilde{h}^{-1} \in k^* .$$

Definition 3.1. The moduli space of rank n vector bundles over X with an infinite-level trivialization is the set:

$$\mathbb{M}_n^\infty(X)_k = \{(\mathcal{E}, \Phi^\infty)\} / \text{Isomorphisms} ,$$

where \mathcal{E} is a locally-free sheaf of rank n over X and Φ^∞ an infinite-level trivialization of \mathcal{E} :

$$\Phi^\infty : \varprojlim_D \widehat{\mathcal{E}}_D \xrightarrow{\sim} (\mathbb{A}_X^+)^n = \mathbb{A}_X^+ \oplus \dots \oplus \mathbb{A}_X^+$$

with $\widehat{\mathcal{E}}_D = \mathcal{E} \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_D$ ($\widehat{\mathcal{O}}_D = \varprojlim_n \mathcal{O}_X / \mathcal{O}_X(-nD)$), D being effective divisors on X .

One can easily define the moduli functor $\underline{\mathbb{M}}_n^\infty(X)$ of rank n vector bundles over X with infinite-level trivialization in such a way that $\mathbb{M}_n^\infty(X)_k$ are the points of $\underline{\mathbb{M}}_n^\infty(X)$ with values in $\mathrm{Spec} k$. See [1] for details in the case of trivialization at a point.

Proposition 3.2. *There exists a Krichever map*

$$\mathrm{Kr} : \mathbb{M}_n^\infty(X)_k \longrightarrow \mathrm{Gr}_X^n(\mathrm{Spec} k)$$

defined by: $\mathrm{Kr}(\mathcal{E}, \Phi^\infty) = \Phi^\infty[\varinjlim_D H^0(X - D, \mathcal{E})]$, D being the set of effective divisors on X .

Proof. This is the same as in ([1], Theorem 3.9). □

Remark 3.3. $\mathrm{Kr}[\mathbb{M}_n^\infty(X)_k]$, as a subset of $\mathrm{Gr}_X^n(\mathrm{Spec} k)$, is stable under the action of $\mathrm{Gl}(n, \mathbb{A}_X)$. This follows from the well-known description of the moduli space of vector bundles in terms of $\mathrm{Gl}(n, \mathbb{A}_X)$ ([6], [15]).

Let us consider the diagonal immersion:

$$\delta : \mathrm{Gl}(n, \Sigma_X) \hookrightarrow \mathrm{Gl}(n, \mathbb{A}_X)$$

induced by the diagonal immersion $\Sigma_X^* \hookrightarrow \mathbb{I}_X \subset \mathbb{A}_X$. This immersion induces an action of $\mathrm{Gl}(n, \Sigma_X)$ on $\mathrm{Gr}_X^n(\mathrm{Spec} k)$.

Proposition 3.4. *The points of $\mathrm{Kr}[\mathbb{M}_n^\infty(X)_k]$ are invariant under the action of $\mathrm{Gl}(n, \Sigma_X)$.*

Proof. This statement is a reinterpretation of the usual description of the moduli stack of bundles over X as a quotient of the loop group (see for example [6], 4.1.9).

This can be proved directly by observing that given an element $g \in \mathrm{Gl}(n, \Sigma_X)$, the image of the vector subspace $U = H^0(X - D, \mathcal{E}) \subset k((z))^n$ (\mathcal{E} again being a locally free sheaf of rank n over X) under the action of g is a subspace of $H^0(X - (D + D_g), \mathcal{E})$, D_g being the set of poles of the matrix g . Of course, this statement is only true if X is a complete curve. □

Theorem 3.5 (Abstract reciprocity law). *The central extension*

$$1 \rightarrow k^* \rightarrow \widetilde{\mathrm{Gl}(n, \Sigma_X)} \xrightarrow{\pi} \mathrm{Gl}(n, \Sigma_X) \rightarrow 1$$

induced, through $\delta: \text{Gl}(n, \Sigma_X) \hookrightarrow \text{Gl}(n, \mathbb{A}_X)$, by the central extension (3.2), is trivial.

Proof. Let $[X_n] = (\mathcal{O}_X^n, Id)$ be the canonical point of $\mathbb{M}_n^\infty(X)_k$ and let $\mathbb{D}[X_n]$ be the fibre of the determinant bundle over the point $[X_n]$. From the invariance of $[X_n]$ under the action of $\text{Gl}(n, \Sigma_X)$ it follows that $\widetilde{\text{Gl}(n, \Sigma_X)}$ acts on the fibre $\mathbb{D}[X_n]$, which is a k -vector space of dimension one. Thus, the character of this representation is a homomorphism of groups:

$$\mu: \widetilde{\text{Gl}(n, \Sigma_X)} \rightarrow k^* .$$

Let $\sigma: \text{Gl}(n, \Sigma_X) \rightarrow \widetilde{\text{Gl}(n, \Sigma_X)}$ be a section (in general not a group homomorphism) of the map $\pi: \widetilde{\text{Gl}(n, \Sigma_X)} \rightarrow \text{Gl}(n, \Sigma_X)$. The cocycle defined by σ is:

$$a: \text{Gl}(n, \Sigma_X) \times \text{Gl}(n, \Sigma_X) \rightarrow k^* \\ (g, h) \mapsto a(g, h) := \sigma(g)\sigma(h)\sigma(gh)^{-1} .$$

Let us define $\bar{\mu}: \text{Gl}(n, \Sigma_X) \rightarrow k^*$ as follows:

$$\bar{\mu}(g) = \mu(\sigma(g)) .$$

Applying μ to the identity defining the cocycle a , one has

$$(3.3) \quad \bar{\mu}(g) \cdot \bar{\mu}(h) = a(g, h) \cdot \bar{\mu}(gh) .$$

Let us now define another section $\tilde{\sigma}$:

$$\tilde{\sigma}: \text{Gl}(n, \Sigma_X) \rightarrow \widetilde{\text{Gl}(n, \Sigma_X)} \\ g \mapsto \tilde{\sigma}(g) = \bar{\mu}(g)^{-1}\sigma(g) .$$

Bearing in mind the identity (3.3), it can be easily proved that:

$$\tilde{a}(g, h) := \tilde{\sigma}(g)\tilde{\sigma}(h)\tilde{\sigma}(gh)^{-1} = 1 ,$$

and therefore $\tilde{\sigma}$ is a homomorphism of groups. □

Corollary 3.6. *Let g and h be two commuting elements of $\text{Gl}(n, \Sigma_X)$ and let $\tilde{g}, \tilde{h} \in \widetilde{\text{Gl}(n, \Sigma_X)}$ be elements such that $\pi(\tilde{g}) = g$ and $\pi(\tilde{h}) = h$. One has*

$$\{g, h\}_{(\mathbb{A}_X^\pm)^n}^{\mathbb{A}_X^\pm} = \tilde{g} \cdot \tilde{h} \cdot \tilde{g}^{-1} \cdot \tilde{h}^{-1} = 1 .$$

Proof. This is a trivial consequence of Theorem 3.5. □

Remark 3.7 (Reciprocity law for the tame symbol). Let us show that the usual reciprocity law for the tame symbol can be recovered from Theorem 3.5.

Let X again be a complete, smooth and connected curve over a perfect field k , and let $p \in X$ be a closed point on it. Let us denote again by $A_p = \hat{\mathcal{O}}_p$ the completion of the local ring $\mathcal{O}_{X,p}$ and by $K_p = (\hat{\mathcal{O}}_p)_0$ its field of fractions.

Let Γ_p be the formal commutative scheme referred to in Remark 2.2.

One has the following central extensions:

$$(3.4) \quad 1 \rightarrow k^* \rightarrow \tilde{\Gamma}_p(\text{Spec } k) \xrightarrow{\pi_p} \Gamma_p(\text{Spec } k) \rightarrow 1 ,$$

$$(3.5) \quad 1 \rightarrow k^* \rightarrow \tilde{\mathbb{I}}_X \xrightarrow{\pi} \mathbb{I}_X \rightarrow 1 ,$$

$$(3.6) \quad 1 \rightarrow k^* \rightarrow \widetilde{\Sigma}_X^* \xrightarrow{\pi} \Sigma_X^* \rightarrow 1 ,$$

and there exists the commutative diagram:

$$\begin{array}{ccccccc}
 1 & \longrightarrow & k^* & \longrightarrow & \tilde{\mathbb{I}}_X & \xrightarrow{\pi} & \mathbb{I}_X \longrightarrow 1 \\
 & & \parallel & & \uparrow & & \uparrow \\
 1 & \longrightarrow & k^* & \longrightarrow & \Sigma_X^* & \xrightarrow{\pi} & \Sigma_X^* \longrightarrow 1
 \end{array}$$

relating the central extensions (3.5) and (3.6).

By Theorem 3.5, the extension (3.6) is trivial. Moreover, the groups $\Gamma_p(\text{Spec } k)$ and \mathbb{I}_X define pairings:

$$\begin{aligned}
 \{ \ , \ }_{A_p}^{K_p} : \Gamma_p(\text{Spec } k) \times \Gamma_p(\text{Spec } k) &\longrightarrow k^* \\
 (a, b) &\longmapsto \{a, b\}_{A_p}^{K_p} = \tilde{a} \cdot \tilde{b} \cdot \tilde{a}^{-1} \cdot \tilde{b}^{-1},
 \end{aligned}$$

with $\pi_p(\tilde{a}) = a$, $\pi_p(\tilde{b}) = b$, and

$$\begin{aligned}
 \{ \ , \ }_{\mathbb{A}_X^+}^{\mathbb{A}_X} : \mathbb{I}_X \times \mathbb{I}_X &\longrightarrow k^* \\
 (f, g) &\longmapsto \{f, g\}_{\mathbb{A}_X^+}^{\mathbb{A}_X} = \tilde{f} \cdot \tilde{g} \cdot \tilde{f}^{-1} \cdot \tilde{g}^{-1},
 \end{aligned}$$

where $\pi(\tilde{f}) = f$ and $\pi(\tilde{g}) = g$.

Via the natural immersion $\Sigma_X^* \hookrightarrow K_p^* = \Gamma_p(\text{Spec } k)$, we have a pairing $\{ \ , \ }_{A_p}^{K_p} : \Sigma_X^* \times \Sigma_X^* \rightarrow k^*$ for each closed point $p \in X$.

The triviality of (3.6) implies that given $f, g \in \Sigma_X^*$ one has

$$(3.7) \quad \{f, g\}_{\mathbb{A}_X^+}^{\mathbb{A}_X} = 1.$$

We shall show that the identity (3.7) is precisely the reciprocity law. Let us compute the restriction of the pairing $\{ \ , \ }_{\mathbb{A}_X^+}^{\mathbb{A}_X}$ to $\Sigma_X^* \times \Sigma_X^*$ in terms of the pairings $\{ \ , \ }_{A_p}^{K_p}$.

One can describe $\tilde{\mathbb{I}}_X$ as in [11]:

$$\tilde{\mathbb{I}}_X = \left\{ (f, s) \text{ such that } f \in \mathbb{I}_X \text{ and } 0 \neq s \in \Lambda(\mathbb{A}_X^+ / [\mathbb{A}_X^+ \cap f\mathbb{A}_X^+]) \otimes_k \Lambda(f\mathbb{A}_X^+ / [\mathbb{A}_X^+ \cap f\mathbb{A}_X^+])^* \right\},$$

with $f = (f_x)_{x \in X}$, and $f_{x_i} \notin A_{x_i}^*$ for a finite number of points $\{x_i\}$.

It is clear that

$$\begin{aligned}
 \mathbb{A}_X^+ / [\mathbb{A}_X^+ \cap f\mathbb{A}_X^+] &= \bigoplus_{x_i \in X} A_{x_i} / [A_{x_i} \cap f_{x_i} A_{x_i}], \\
 f\mathbb{A}_X^+ / [\mathbb{A}_X^+ \cap f\mathbb{A}_X^+] &= \bigoplus_{x_i \in X} f_{x_i} A_{x_i} / [A_{x_i} \cap f_{x_i} A_{x_i}].
 \end{aligned}$$

Given two elements $f = (f_x)_{x \in X}$ and $g = (g_x)_{x \in X}$ of $\Sigma_X^* \subseteq \mathbb{I}_X$, let $\{x_i\}$ be the finite set of points such that $f_{x_i} \notin A_{x_i}^*$, and let $\{y_j\}$ be the finite set of points such that $g_{y_j} \notin A_{y_j}^*$. Let us denote $D_{fg} = \{x_i\} \cup \{y_j\}$: this is a finite set of closed points

of X and it follows from the statements of [11] that

$$\begin{aligned} \{f, g\}_{\mathbb{A}_X^+}^{\mathbb{A}_X} &= \{f, g\}_{\bigoplus_{x \in D_{fg}} \mathbb{A}_x}^{\bigoplus_{x \in D_{fg}} K_x} \\ &= (-1)^{\sum_{x_i, x_h \in D_{fg}} i(f_{x_i}, A_{x_i})i(f_{x_h}, A_{x_h})} \prod_{x \in D_{fg}} \{f_x, g_x\}_{A_x}^{K_x} \\ &= (-1)^{\sum_{x_i, x_h \in X} i(f_{x_i}, A_{x_i})i(f_{x_h}, A_{x_h})} \prod_{x \in X} \{f_x, g_x\}_{A_x}^{K_x}, \end{aligned}$$

where $\deg(x) = \dim_k k(x)$, and $i(f_x, A_x) = \deg(x)v_x(f_x)$. Applying this computation to the case $f, g \in \Sigma_X^*$ one concludes the proof of the reciprocity law.

Remark 3.8 (Reciprocity laws associated with finite coverings). Let us consider a finite covering $\pi: Y \rightarrow X$ of a complete, smooth and connected curve over a perfect field k . Let n be the degree of π . $\pi_*\mathcal{O}_Y$ is a locally free sheaf of \mathcal{O}_X -modules of rank n . Let Φ^∞ be an infinite-level trivialization of $\pi_*\mathcal{O}_Y$, that is,

$$\Phi^\infty: \varprojlim_D (\widehat{\pi_*\mathcal{O}_Y})_D \xrightarrow{\sim} \mathbb{A}_X^n.$$

Note that an infinite-level trivialization of the sheaf $\pi_*\mathcal{O}_Y$ is equivalent to giving a structure of an \mathbb{A}_X -algebra on \mathbb{A}_X^n with certain conditions. The trivialization Φ^∞ induces an inclusion of groups:

$$\Sigma_Y^* \subset \mathrm{Gl}(n, \mathbb{A}_X),$$

where Σ_Y is the function field of Y . Therefore, the extension (3.2) induces a central extension:

$$(3.8) \quad 1 \rightarrow k^* \rightarrow \widetilde{\Sigma}_Y^* \xrightarrow{\pi} \Sigma_Y^* \rightarrow 1.$$

Let $U = \mathrm{Kr}(\pi_*\mathcal{O}_Y, \Phi^\infty) \in \mathrm{Gr}(\mathbb{A}_X^n)$ be the point of the adelic Grassmannian defined by U . Obviously U is invariant under the action of Σ_Y^* , and hence the same arguments used in the proof of Theorem 3.5 prove that the extension (3.8) is trivial. Since Σ_Y^* is commutative, one has that

$$(3.9) \quad \{g, h\}_{(\mathbb{A}_X^+)^n}^{\mathbb{A}_X^n} = \widetilde{g}\widetilde{h}\widetilde{g}^{-1}\widetilde{h}^{-1} = 1$$

for all $g, h \in \Sigma_Y^*$.

In some sense, the reciprocity law (3.9) together with some maximality condition on $\mathrm{Gl}(n, \mathbb{A}_X)$ could be used to characterize finite coverings $Y \xrightarrow{\pi} X$. One problem directly related to a hypothetical non-abelian class field theory should be the characterization of Galois coverings $Y \xrightarrow{\pi} X$ and the construction, in terms of $\mathrm{Gl}(n, \mathbb{A}_X)$, of its Galois group.

Remark 3.9 (The non-abelian case). Keeping the assumptions about X of Remark 3.7, let G be a reductive algebraic group over k and let us assume we have an immersion $G \subset \mathrm{Gl}(n, k)$. For each closed point $x \in X$, let us denote by G_x the loop group over $k(x)$, $G_x \subset \mathrm{LGl}(n)_{k(x)}$, where $\mathrm{LGl}(n)_{k(x)}$ is the group scheme representing the functor of groups:

$$S \rightsquigarrow \underline{\mathrm{LGl}}(n)_{k(x)}(S) = \underline{\mathrm{LGl}}(n)(S \times_{\mathrm{Spec} k} \mathrm{Spec} k(x)).$$

Let $G_{\mathbb{A}_X}$ be the adelic group ($G_{\mathbb{A}_X} \subset \text{Gl}(n, \mathbb{A}_X)$), and let G_{Σ_X} be the group over Σ_X ($G_{\Sigma_X} \subset \text{Gl}(n, \Sigma_X)$). The above results imply the existence of the following central extensions:

$$(3.10) \quad 1 \rightarrow \mathbb{G}_m \rightarrow \tilde{G}_x \rightarrow G_x \rightarrow 1,$$

$$(3.11) \quad 1 \rightarrow k^* \rightarrow \tilde{G}_{\mathbb{A}_X} \rightarrow G_{\mathbb{A}_X} \rightarrow 1,$$

$$(3.12) \quad 1 \rightarrow k^* \rightarrow \widetilde{G_{\Sigma_X}} \rightarrow G_{\Sigma_X} \rightarrow 1,$$

where the extension (3.10) is induced by the morphism of the functor of groups

$$N_{k(x)/k} : \mathbb{G}_{m, k(x)}(S) = [H^0(S, \mathcal{O}_S) \otimes_k k(x)]^* \rightarrow \mathbb{G}_m(S) = H^0(S, \mathcal{O}_S)^*,$$

and there again exists a commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & k^* & \longrightarrow & \tilde{G}_{\mathbb{A}_X} & \longrightarrow & G_{\mathbb{A}_X} \longrightarrow 1 \\ & & \parallel & & \uparrow & & \uparrow \\ 1 & \longrightarrow & k^* & \longrightarrow & \widetilde{G_{\Sigma_X}} & \longrightarrow & G_{\Sigma_X} \longrightarrow 1 \end{array}$$

relating the central extensions (3.11) and (3.12).

From Theorem 3.5 it follows that (3.12) is a trivial central extension.

We have two ways to translate the triviality of the extension (3.12) into explicit identities that can be understood as non-abelian analogues of the reciprocity law. The first consists in applying Corollary 3.6 and explicitly computing the identity of that corollary in terms of the commutators of the groups \tilde{G}_x . This approach will be explored in the next section in the case $n = 2$.

The second approach consists in imposing the condition of triviality of the sequence (3.12) in terms of cocycles associated with some section of $\tilde{G}_{\mathbb{A}_X} \rightarrow G_{\mathbb{A}_X}$ and computing these cocycles in terms of the cocycles of the local extensions (3.10). This approach will be developed in another paper.

Remark 3.10. Here we have used the theory of the Sato Grassmannian and determinant bundles to construct the central extensions. Of course, one can construct these extensions using a more elementary language, as is done for example in [3]. The reason for using these more sophisticated techniques is that in this way the results can be generalized to the case of arithmetic curves and, in general, curves defined over a ring R . We hope to develop this theory over arbitrary Noetherian rings elsewhere.

4. RECIPROCITY LAWS ON $\text{Gl}(2, \Sigma_X)$

This final section is devoted to applying the above theory to study reciprocity laws on the group $\text{Gl}(2, \Sigma_X)$, where Σ_X is again the function field of a smooth and connected curve over a perfect field k . Keeping the previous notation, we set $A_p = \hat{\mathcal{O}}_p$ and $K_p = (\hat{\mathcal{O}}_p)_0$.

Let us now consider $V_2^+ = A_p \oplus A_p = A_p^2$ and $V_2 = K_p \oplus K_p = K_p^2$. It is clear that $\text{Gl}(2, \Sigma_X) \subseteq \text{Gl}(K_p^2)(\text{Spec } k)$, and hence the central extension of the functor of groups (2.2) induces a central extension of groups:

$$1 \rightarrow k^* \rightarrow \widetilde{\text{Gl}}(2, \Sigma_X) \rightarrow \text{Gl}(2, \Sigma_X) \rightarrow 1,$$

whose commutator is denoted by $\{\varphi, \phi\}_{A_p^2}^{K_p^2}$ for all $\varphi, \phi \in \text{Gl}(2, \Sigma_X)$, such that $\varphi \cdot \tau = \tau \cdot \varphi$. Similar to Remark 2.3, for each matrix $\varphi \in \text{Gl}(2, \Sigma_X)$, one has the index

$$i(\varphi, A_p^2) = \dim_k(A_p^2/A_p^2 \cap \varphi A_p^2) - \dim_k(\varphi A_p^2/A_p^2 \cap \varphi A_p^2).$$

Lemma 4.1. *If $\varphi \in \text{Gl}(2, \Sigma_X)$, then*

$$i(\varphi, A_p^2) = \deg(p) \cdot v_p(\det \varphi).$$

Proof. Bearing in mind the properties of the index, from the decomposition

$$\varphi = \begin{pmatrix} \det \varphi & 0 \\ 0 & 1 \end{pmatrix} \cdot \tilde{\varphi},$$

where $\det \tilde{\varphi} = 1$, one has that $i(\varphi, A_p^2) = i\left(\begin{pmatrix} \det \varphi & 0 \\ 0 & 1 \end{pmatrix}, A_p^2\right) + i(\tilde{\varphi}, A_p^2)$, with

$$i\left(\begin{pmatrix} \det \varphi & 0 \\ 0 & 1 \end{pmatrix}, A_p^2\right) = i(\det \varphi, A_p) = \deg(p) \cdot v_p(\det \varphi).$$

Thus, to prove the claim of the lemma it is sufficient to see that $i(\tilde{\varphi}, A_p^2) = 0$.

Moreover, it also follows from the additivity of the index that $i(\tilde{\varphi}, A_p^2) = i(J_{\tilde{\varphi}}, A_p^2)$, where $J_{\tilde{\varphi}}$ is the Jordan matrix associated with $\tilde{\varphi}$.

Since $J_{\tilde{\varphi}} \in \left\{ \begin{pmatrix} \pm 1 & 0 \\ 1 & \pm 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & \text{tr } \tilde{\varphi} \end{pmatrix}, \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \right\}$, one has that

$$\dim_k(A_p^2/[A_p^2 \cap J_{\tilde{\varphi}} A_p^2]) = \dim_k(A_p^2/[A_p^2 \cap J_{\tilde{\varphi}}^{-1} A_p^2]),$$

and we conclude that $i(\tilde{\varphi}, A_p^2) = i(J_{\tilde{\varphi}}, A_p^2) = 0$. □

Theorem 4.2 (Reciprocity Law on $\text{Gl}(2, \Sigma_X)$). *If X is a complete, smooth and connected curve, for all commuting elements $\tau, \varphi \in \text{Gl}(2, \Sigma_X)$ one has that*

$$\prod_{p \in X} (-1)^{\deg(p)v_p(\det \tau)v_p(\det \varphi)} \{\tau, \varphi\}_{A_p^2}^{K_p^2} = 1.$$

Proof. If $\{\tau, \varphi\}_{(\mathbb{A}_X^\pm)^2}^{\mathbb{A}_X^2}$ is the commutator of the adelic central extension induced by $\delta: \text{Gl}(2, \Sigma_X) \hookrightarrow \text{Gl}(2, \mathbb{A}_X)$, it follows from Theorem 3.5 and Corollary 3.6 that $\{\tau, \varphi\}_{(\mathbb{A}_X^\pm)^2}^{\mathbb{A}_X^2} = 1$ for all commuting elements $\tau, \varphi \in \text{Gl}(2, \Sigma_X)$.

With similar arguments to Remark 3.7 one can see that

$$\{\tau, \varphi\}_{(\mathbb{A}_X^\pm)^2}^{\mathbb{A}_X^2} = (-1)^{\sum_{p_i, p_h \in X} i(\tau, A_{p_i}^2) \cdot i(\varphi, A_{p_h}^2)} \prod_{p \in X} \{\tau, \varphi\}_{A_p^2}^{K_p^2},$$

and the claim is a direct consequence of Lemma 4.1 because

$$\sum_{p \in X} i(\tau, A_p^2) = \sum_{p \in X} i(\varphi, A_p^2) = 0$$

when X is complete. □

Remark 4.3 (Generalized tame symbol on $\text{Gl}(n, \Sigma_X)$). For two commuting elements $\tau, \varphi \in \text{Gl}(2, \Sigma_X)$, we can define a generalization of the tame symbol of an algebraic curve by the expression:

$$(\tau, \varphi)_{2,p} = (-1)^{i(\tau, A_p^2) \cdot i(\varphi, A_p^2)} \{\tau, \varphi\}_{A_p^2}^{K_p^2} \in k^*,$$

and Theorem 4.2 is a generalization of the Weil reciprocity law because it shows that

$$\prod_{p \in X} (\tau, \varphi)_{2,p} = 1.$$

Similarly, it is possible to define the symbol $(\sigma, \sigma')_{n,p}$ for commuting matrices $\sigma, \sigma' \in \text{Gl}(n, \Sigma_X)$.

We conjecture that the result of Lemma 4.1 is true for $\text{Gl}(n, \Sigma_X)$ by considering $V_+^n = A_p \oplus \dots \oplus A_p = A_p^n$, $V^n = K_p \oplus \dots \oplus K_p = K_p^n$ and rational points of the group scheme $\text{Gl}(K_p^n)$. Solving this conjecture, the following reciprocity law arises directly:

$$\prod_{p \in X} (\sigma, \sigma')_{n,p} = 1.$$

Recall now from [3] that the commutator $\{ \quad, \quad \}_{V_+^V}$ satisfies the following properties:

- Suppose V is equipped with a direct sum decomposition $V = V_0 \oplus V_1$. Put $V_{i+} := V_i \cap V_+$ for $i = 0, 1$ and assume that $V_+ = V_{0+} \oplus V_{1+}$. Let commuting elements $\tau_0, \tau_1 \in \text{Gl}(V, V_+)$ be given such that

$$\tau_i|_{V_0} \in \text{Gl}(V_0, V_{0+}), \quad \tau_i|_{V_1} = 1$$

for $i = 0, 1$. Then we have

$$\{\tau_0|_{V_0}, \tau_1|_{V_0}\}_{V_{0+}^{V_0}} = \{\tau_0, \tau_1\}_{V_+^V}.$$

- Again suppose V is equipped with a direct sum decomposition $V = V_0 \oplus V_1$, put $V_{i+} := V_i \cap V_+$ for $i = 0, 1$ and assume that $V_+ = V_{0+} \oplus V_{1+}$. Let $\tau_0, \tau_1 \in \text{Gl}(V, V_+)$ be given such that

$$\tau_i|_{V_i} \in \text{Gl}(V_i, V_{i+}), \quad \tau_i|_{V_{1-i}} = 1$$

for $i = 0, 1$ (necessarily τ_0 and τ_1 commute). Then we have

$$\{\tau_0, \tau_1\}_{V_+^V} = (-1)^{\nu_0\nu_1}$$

where

$$\nu_i := i(\tau_i|_{V_i}, V_{i+}) = i(\tau_i, V_+)$$

for $i = 0, 1$.

Example 1. If we consider the commuting elements $\tau, \varphi \in \text{Gl}(2, \Sigma_X)$, where

$$\tau = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} \quad \text{and} \quad \varphi = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix},$$

with $f_i, g_j \in \Sigma_X^*$, we have that

$$\begin{aligned} \{\tau, \varphi\}_{A_p^2}^{K_p^2} &= \left\{ \begin{pmatrix} f_1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} g_1 & 0 \\ 0 & 1 \end{pmatrix} \right\}_{A_p^2}^{K_p^2} \cdot \left\{ \begin{pmatrix} f_1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & g_2 \end{pmatrix} \right\}_{A_p^2}^{K_p^2} \\ &\cdot \left\{ \begin{pmatrix} 1 & 0 \\ 0 & f_2 \end{pmatrix}, \begin{pmatrix} g_1 & 0 \\ 0 & 1 \end{pmatrix} \right\}_{A_p^2}^{K_p^2} \cdot \left\{ \begin{pmatrix} 1 & 0 \\ 0 & f_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & g_2 \end{pmatrix} \right\}_{A_p^2}^{K_p^2} \\ &= (-1)^{\deg(p)[v_p(f_1)v_p(g_2)+v_p(f_2)v_p(g_1)]} \cdot \{f_1, g_1\}_{A_p}^{K_p} \cdot \{f_2, g_2\}_{A_p}^{K_p}, \end{aligned}$$

and Lemma 4.1 shows that the reciprocity law is

$$\begin{aligned} \prod_{p \in X} (-1)^{\deg(p)v_p(\det \tau)v_p(\det \varphi)} \{\tau, \varphi\}_{A_p^2}^{K_p^2} &= \prod_{p \in X} (-1)^{\deg(p)v_p(f_1 \cdot f_2)v_p(g_1 \cdot g_2)} \{\tau, \varphi\}_{A_p^2}^{K_p^2} \\ &= \prod_{p \in X} (f_1, g_1)_p \cdot (f_2, g_2)_p = 1, \end{aligned}$$

where $(\ , \)_p$ is the tame symbol at the closed point $p \in X$. Therefore, in this case, the reciprocity law of the commutator $\{\tau, \varphi\}_{A_p^2}^{K_p^2}$ can be deduced from the Weil reciprocity law.

Example 2. Let us now consider the commuting elements

$$\tau = \begin{pmatrix} 0 & f \\ 1 & g \end{pmatrix} \quad \text{and} \quad \varphi = \begin{pmatrix} 1 & f \\ 1 & 1 + g \end{pmatrix},$$

with $f, g \in \Sigma_X^* \cap A_p$; that is, $v_p(f) = \alpha \geq 0$, $v_p(g) = \beta \geq 0$ and $1 + g - f \neq 0$. To simplify, we assume in this example that the ground field k is algebraically closed.

Let x and y be two elements of A_p .

If $x = c_x f + r_x$, one has that $(x, y) = (y - c_x g) \cdot (0, 1) + c_x \cdot (f, g) + (r_x, 0)$, and there exists an isomorphism of k -vector spaces

$$A_p^2 / \tau A_p^2 \simeq A_p / \langle f \rangle.$$

Also, setting $x - y = \bar{c}(f - g - 1) + \bar{r}$, it follows from the equality $(x, y) = (x - \bar{c}f)(1, 1) + \bar{c}(f, 1 + g) + (0, \bar{r})$ that

$$A_p^2 / \varphi A_p^2 \simeq A_p / \langle f - g - 1 \rangle.$$

Moreover, since

$$\tau\varphi = \varphi\tau = \begin{pmatrix} f & f(1 + g) \\ 1 + g & f + g + g^2 \end{pmatrix},$$

by setting $x = c_x f + r_x$ and $y - c_x(1 + g) = c_{\bar{y}}(f - g - 1) + r_{\bar{y}}$, from the expression

$$(x, y) = [c_x - c_{\bar{y}}(1 + g)](f, 1 + g) + c_{\bar{y}}(f(1 + g), f + g + g^2) + (r_x, r_{\bar{y}}),$$

we deduce that

$$A_p^2 / \varphi\tau A_p^2 \simeq A_p / \langle f \rangle \oplus A_p / \langle f - g - 1 \rangle.$$

Bearing in mind the statements of [4] and [11], we have that the group $\widetilde{\text{Gl}}(V)(\text{Spec } k)$ is defined as

$$\widetilde{\text{Gl}}(V)(\text{Spec } k) = \{(f, s) \text{ with } f \in \text{Gl}(V)(\text{Spec } k), s \in \text{Det } \mathcal{C}_{f(V_+)}^\bullet, s \neq 0\},$$

together with the multiplication $(f, s) \cdot (g, s') = (fg, s \cdot \bar{f}(s'))$, where $\text{Det } \mathcal{C}_{f(V_+)}^\bullet$ is the k -vector space:

$$\text{Det } \mathcal{C}_{f(V_+)}^\bullet := \Lambda V_+ / [V_+ \cap fV_+] \otimes_k \Lambda(fV_+ / [V_+ \cap fV_+])^*,$$

Λ again being the maximal exterior power, and

$$\bar{f}(s') \in \Lambda fV_+ / [fV_+ \cap fgV_+] \otimes_k \Lambda(fgV_+ / [fV_+ \cap fgV_+])^*$$

being determined by the k -morphisms

$$\begin{aligned} f: V_+ / [V_+ \cap gV_+] &\longrightarrow fV_+ / [fV_+ \cap fgV_+], \\ f^{-1}: fgV_+ / [fV_+ \cap fgV_+] &\longrightarrow gV_+ / [V_+ \cap gV_+]. \end{aligned}$$

Thus, $\{f, g\}_{V_+}^V = [(f, s) \cdot (g, s')] \cdot [(g, s') \cdot (f, s)]^{-1} \in k^*$.

Hence, fixing an isomorphism $A_p \simeq k[[t]]$, we can consider the elements $\{\tilde{v}_1 = (1, 0), \tilde{v}_2 = (t, 0), \dots, \tilde{v}_\alpha = (t^{\alpha-1}, 0)\} \subseteq A_p^2/\varphi\tau A_p^2$ such that their images in $A_p^2/\tau A_p^2$ determine a basis $\langle v_i \rangle$ of this k -vector space. Similarly, we have a basis $\langle w_j \rangle$ of $A_p^2/\varphi A_p^2$ consisting of the images of the elements $\{\tilde{w}_1 = (0, 1), \tilde{w}_2 = (0, t), \dots, \tilde{w}_\beta = (0, t^{\beta-1})\} \subseteq A_p^2/\varphi\tau A_p^2$ in $A_p^2/\varphi A_p^2$ via the natural projection.

Then, $\text{Det } \mathcal{C}_{\tau A_p^2}^\bullet = \Lambda A_p^2/\tau A_p^2 = \langle v_1 \wedge \dots \wedge v_\alpha \rangle$ and $\text{Det } \mathcal{C}_{\varphi A_p^2}^\bullet = \Lambda A_p^2/\varphi A_p^2 = \langle w_1 \wedge \dots \wedge w_\beta \rangle$, and an easy computation shows that

$$\bar{\tau}(w_1 \wedge \dots \wedge w_\beta) = (-\tilde{w}_1) \wedge \dots \wedge (-\tilde{w}_\beta) = (-1)^{v(f-g-1)}(\tilde{w}_1 \wedge \dots \wedge \tilde{w}_\beta),$$

and that

$$\bar{\varphi}(v_1 \wedge \dots \wedge v_\alpha) = (\tilde{v}_1 + \tilde{w}_1) \wedge \dots \wedge (\tilde{v}_\alpha + \tilde{w}_\alpha),$$

with $\tilde{w}_j = 0$ when $\beta < j \leq \alpha$.

Thus, denoting $s_1 = v_1 \wedge \dots \wedge v_\alpha$ and $s_2 = w_1 \wedge \dots \wedge w_\beta$, from the exact sequence of k -vector spaces

$$0 \rightarrow \tau A_p^2/\tau\varphi A_p^2 \rightarrow A_p^2/\tau\varphi A_p^2 \rightarrow A_p^2/\tau A_p^2 \rightarrow 0,$$

bearing in mind the explicit definition of the group law of $\widetilde{\text{Gl}}(V)$ ([4], [11]), one has that

$$\begin{aligned} (\tau, s_1) \cdot (\varphi, s_2) &= (\tau\varphi, s_1 \cdot \bar{\tau}(s_2)) \\ &= (\tau\varphi, (-1)^{v_p(f-g-1)} \cdot \tilde{w}_1 \wedge \dots \wedge \tilde{w}_\beta \wedge \tilde{v}_1 \wedge \dots \wedge \tilde{v}_\alpha). \end{aligned}$$

Analogously, from the exact sequence:

$$0 \rightarrow \varphi A_p^2/\tau\varphi A_p^2 \rightarrow A_p^2/\tau\varphi A_p^2 \rightarrow A_p^2/\varphi A_p^2 \rightarrow 0,$$

we have that

$$\begin{aligned} (\varphi, s_2) \cdot (\tau, s_1) &= (\tau\varphi, s_2 \cdot \bar{\varphi}(s_1)) \\ &= (\tau\varphi, (\tilde{v}_1 + \tilde{w}_1) \wedge \dots \wedge (\tilde{v}_\alpha + \tilde{w}_\alpha) \wedge \tilde{w}_1 \wedge \dots \wedge \tilde{w}_\beta) \\ &= (\tau\varphi, (-1)^{v_p(f)v_p(f-g-1)} \cdot \tilde{w}_1 \wedge \dots \wedge \tilde{w}_\beta \wedge \tilde{v}_1 \wedge \dots \wedge \tilde{v}_\alpha). \end{aligned}$$

Then,

$$\{\tau, \varphi\}_{A_p^2}^{K_p^2} = (-1)^{v_p(f)v_p(f-g-1)+v_p(f-g-1)},$$

and since $\det \tau = -f$ and $\det \varphi = 1 + g - f$, Theorem 4.2 shows that

$$\prod_{p \in X} (-1)^{v_p(\det \varphi)} = 1,$$

which is equivalent to the well-known formula $\sum_p v_p(\det \varphi) = 0$ when X is a complete curve over an algebraically closed field. Moreover, since $(-1)^{\det \tau \det \varphi} \{\tau, \varphi\}_{A_p^2}^{K_p^2} = (-1, \det \varphi)_p$, the reciprocity law can also be deduced from the Weil reciprocity law in this example.

A remaining question is to characterize the commutator $\{\tau, p(\tau)\}_{A_p^2}^{K_p^2}$ for two commuting elements $\tau, p(\tau)$, where $\tau \in \text{Gl}(2, \Sigma_X)$ and $p(x)$ is a polynomial with coefficients in k .

Finally, regarding Σ_X^* as a subgroup of $\text{Gl}(2, \Sigma_X)$ by means of the diagonal embedding $f \mapsto \sigma_f := \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}$, one has that $\Sigma_X^* = Z(\text{Gl}(2, \Sigma_X))$ and, therefore, there exists a commutator map

$$\{ \cdot, \cdot \}_{A_p^2}^{K_p^2}: \Sigma_X^* \times \text{Gl}(2, \Sigma_X) \rightarrow k^*.$$

We shall compute the explicit expression of this commutator map.

Lemma 4.4. *If $f \in \Sigma_X^*$, then $i(\sigma_f, A_p^2) = 2 \deg(p)v_p(f)$.*

Proof. The claim is a direct consequence of Lemma 4.1. □

Lemma 4.5. *Setting*

$$J^1 := \begin{pmatrix} \pm 1 & 0 \\ 1 & \pm 1 \end{pmatrix} \quad \text{and} \quad J^2 := \begin{pmatrix} 0 & -1 \\ 1 & h \end{pmatrix}$$

with $h \in A_p$, one has that $\{\sigma_f, J^1\}_{A_p^2}^{K_p^2} = \{\sigma_f, J^2\}_{A_p^2}^{K_p^2} = 1$ for all $f \in \Sigma_X^*$.

Proof. With the notation of Example 2, if $g \in \text{Aut}_k(V_+)$, then $\text{Det } \mathcal{C}_{g(V_+)}^\bullet = k$ is trivial, and $\bar{f}: \text{Det } \mathcal{C}_{g(V_+)}^\bullet \rightarrow \text{Det } \mathcal{C}_{g(V_+)}^\bullet$ is the identity on k for all $f \in \text{Gl}(V, V_+)$.

Let us assume that $v_p(f) \geq 1$. Then,

$$A_p^2/[A_p^2 \cap \sigma_f A_p^2] \simeq k(p)[[t]]/t^n \oplus k(p)[[t]]/t^n,$$

with $n = v_p(f)$. Let $\{e_1, \dots, e_d\}$ be a basis of $k(p)$ as a k -vector space, $d = \deg(p)$, and let us consider the basis $\{e_j \cdot t^i\}_{i \in \{0, \dots, n-1\}}^{j \in \{1, \dots, d\}}$ of $k(p)[[t]]/t^n$ as a vector space over k .

Thus, $\{(e_j \cdot t^i, 0), (0, e_j \cdot t^i)\}_{i \in \{0, \dots, n-1\}}^{j \in \{1, \dots, d\}}$ is a basis of $k(p)[[t]]/t^n \oplus k(p)[[t]]/t^n$ as a k -vector space.

It is clear that $J^i \in \text{Aut}_k(A_p^2)$ and, therefore, $\bar{\sigma}_f: \text{Det } \mathcal{C}_{J^i(A_p^2)}^\bullet \rightarrow \text{Det } \mathcal{C}_{J^i(A_p^2)}^\bullet$ is the identity.

We set \widetilde{J}^1 and \widetilde{J}^2 to denote the k -automorphisms of $A_p^2/[A_p^2 \cap \sigma_f A_p^2]$ induced by J^1 and J^2 respectively. We shall now compute $\det \widetilde{J}^1$ and $\det \widetilde{J}^2$ in order to determine $\widetilde{J}^1, \widetilde{J}^2 \in \text{Aut}_k(\Lambda A_p^2/[A_p^2 \cap \sigma_f A_p^2])$.

Bearing in mind that

$$\begin{aligned} \widetilde{J}^1(e_j \cdot t^i, 0) &= (\pm e_j \cdot t^i, e_j \cdot t^i), \\ \widetilde{J}^1(0, e_j \cdot t^i) &= (0, \pm e_j \cdot t^i), \end{aligned}$$

the automorphism \widetilde{J}^1 is determined by the matrix:

$$\widetilde{J}^1 = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_d \end{pmatrix} \quad \text{with} \quad A_j = \begin{pmatrix} \pm 1 & & 0 & \vdots \\ & \ddots & & \vdots \\ 0 & & \pm 1 & \vdots \\ \dots & \dots & \dots & \vdots \\ 1 & & 0 & \pm 1 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{pmatrix}.$$

Thus, $\det \widetilde{J}^1 = 1$ and $\bar{J}^1(s) = s$ for all $s \in \text{Det } \mathcal{C}_{\sigma_f A_p^2}^\bullet = \Lambda A_p^2 / [A_p^2 \cap \sigma_f A_p^2]$.
 Moreover, since

$$\begin{aligned} \widetilde{J}^2(e_j \cdot t^i, 0) &= (0, e_j \cdot t^i), \\ \widetilde{J}^2(0, e_j \cdot t^i) &= (-e_j \cdot t^i, e_j \cdot [ht^i \text{ mod. } t^n]), \end{aligned}$$

$\widetilde{J}^2 \in \text{Aut}_k(A_p^2/[A_p^2 \cap \sigma_f A_p^2])$ is characterized by the matrix:

$$\widetilde{J}^2 = \begin{pmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_d \end{pmatrix} \quad \text{where} \quad B_j = \begin{pmatrix} & & \vdots & -1 & & 0 \\ & 0 & & \vdots & & \ddots \\ & & & \vdots & 0 & & -1 \\ \dots & \dots & \dots & \vdots & \dots & \dots & \dots \\ 1 & & 0 & \vdots & h^1 & & 0 \\ & \ddots & & \vdots & \vdots & \ddots & \\ 0 & & 1 & \vdots & h^n & \dots & h^1 \end{pmatrix},$$

and, again, $\det \widetilde{J}^2 = 1$ and $\bar{J}^2(s) = s$ for all

$$s \in \text{Det } \mathcal{C}_{\sigma_f A_p^2}^\bullet = \Lambda A_p^2 / [A_p^2 \cap \sigma_f A_p^2].$$

Thus, when $v_p(f) \geq 1$, if $0 \neq s \in \text{Det } \mathcal{C}_{\sigma_f A_p^2}^\bullet$ and $0 \neq \lambda_i \in \text{Det } \mathcal{C}_{J^i A_p^2}^\bullet$, then

$$\begin{aligned} \{\sigma_f, J^i\}_{A_p^2}^{K_p^2} &= [(\sigma_f, s) \cdot (J^i, \lambda_i)] \cdot [(J^i, \lambda_i) \cdot (\sigma_f, s)]^{-1} \\ &= (\sigma_f \cdot J^i, s \cdot \bar{\sigma}_f(\lambda_i)) \cdot (J^i \sigma_f, \lambda_i \cdot \bar{J}^i(s))^{-1} \\ &= (\sigma_f \cdot J^i, \lambda_i s) \cdot (\sigma_f \cdot J^i, \lambda_i s)^{-1} = 1. \end{aligned}$$

Bearing in mind that

$$\{\text{Id}, J^i\}_{A_p^2}^{K_p^2} = 1 = \{\sigma_f, J^i\}_{A_p^2}^{K_p^2} \{\sigma_{f^{-1}}, J^i\}_{A_p^2}^{K_p^2},$$

we conclude the proof. □

Proposition 4.6. *For all $f \in \Sigma_X^*$ and $\varphi \in \text{Gl}(2, \Sigma_X)$, one has that*

$$\{\sigma_f, \varphi\}_{A_p^2}^{K_p^2} = (-1)^{\deg(p)v_p(f)v_p(\det \varphi)} N_{k(p)/k} \left(\frac{f^{v_p(\det \varphi)}}{[\det \varphi]^{v_p(f)}}(p) \right) \in k^*.$$

Proof. If $\varphi = \begin{pmatrix} \det \varphi & 0 \\ 0 & 1 \end{pmatrix} \cdot \widetilde{\varphi}$ is the decomposition referred to above, it follows from the general properties of a commutator that

$$\{\sigma_f, \varphi\}_{A_p^2}^{K_p^2} = \left\{ \sigma_f, \begin{pmatrix} \det \varphi & 0 \\ 0 & 1 \end{pmatrix} \right\}_{A_p^2}^{K_p^2} \cdot \{\sigma_f, \widetilde{\varphi}\}_{A_p^2}^{K_p^2}.$$

Accordingly, one has that

$$\begin{aligned} \{\sigma_f, \begin{pmatrix} \det \varphi & 0 \\ 0 & 1 \end{pmatrix}\}_{A_p^2}^{K_p^2} &= \left\{ \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}, \begin{pmatrix} \det \varphi & 0 \\ 0 & 1 \end{pmatrix} \right\}_{A_p^2}^{K_p^2} \\ &= \left\{ \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \det \varphi & 0 \\ 0 & 1 \end{pmatrix} \right\}_{A_p^2}^{K_p^2} \cdot \left\{ \begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix}, \begin{pmatrix} \det \varphi & 0 \\ 0 & 1 \end{pmatrix} \right\}_{A_p^2}^{K_p^2} \\ &= \{f, \det \varphi\}_{A_p}^{K_p} \cdot (-1)^{i(f, A_p) \cdot i(\det \varphi, A_p)} \\ &= (-1)^{\deg(p)v_p(f)v_p(\det \varphi)} N_{k(p)/k} \left(\frac{fv_p(\det \varphi)}{[\det \varphi]v_p(f)}(p) \right). \end{aligned}$$

Moreover, $\{\sigma_f, \tilde{\varphi}\}_{A_p^2}^{K_p^2} = \{\sigma_f, J_{\tilde{\varphi}}\}_{A_p^2}^{K_p^2}$, where $J_{\tilde{\varphi}}$ is again the Jordan matrix associated with $\tilde{\varphi}$, and

$$J_{\tilde{\varphi}} \in \{J^1 := \begin{pmatrix} \pm 1 & 0 \\ 1 & \pm 1 \end{pmatrix}, J^2 := \begin{pmatrix} 0 & -1 \\ 1 & \text{tr } \tilde{\varphi} \end{pmatrix}, J^3 := \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}\}.$$

Our aim now is to see that $\{\sigma_f, J^i\}_{A_p^2}^{K_p^2} = 1$ for $i \in \{1, 2, 3\}$.

Indeed, this assertion is a direct consequence of Lemma 4.5 for J^1 and J^2 with $\text{tr } \tilde{\varphi} \in A_p$.

Furthermore,

$$\begin{aligned} \{\sigma_f, J^3\}_{A_p^2}^{K_p^2} &= \left\{ \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \right\}_{A_p^2}^{K_p^2} \cdot \left\{ \begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix}, \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \right\}_{A_p^2}^{K_p^2} \\ &\quad \cdot \left\{ \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & g^{-1} \end{pmatrix} \right\}_{A_p^2}^{K_p^2} \cdot \left\{ \begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & g^{-1} \end{pmatrix} \right\}_{A_p^2}^{K_p^2} \\ &= \{f, g\}_{A_p}^{K_p} \cdot \{f, g^{-1}\}_{A_p}^{K_p} \cdot (-1)^{i(f, A_p) \cdot i(g, A_p)} \cdot (-1)^{i(f, A_p) \cdot i(g^{-1}, A_p)} \\ &= 1. \end{aligned}$$

Finally, since the general expression of J^2 is $\begin{pmatrix} 0 & -1 \\ 1 & h \cdot g^{-1} \end{pmatrix}$, with $h, g \in A_p$ and $g \neq 0$, then

$$\begin{aligned} J^2 &= \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 & -g^{-1} \\ g & h \end{pmatrix} \\ &= \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \cdot \left[\begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & h \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & g^{-1} \end{pmatrix} \right], \end{aligned}$$

and, therefore,

$$\{\sigma_f, J^2\}_{A_p^2}^{K_p^2} = \{\sigma_f, \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}\}_{A_p^2}^{K_p^2} \cdot \{\sigma_f, \begin{pmatrix} 0 & -1 \\ 1 & h \end{pmatrix}\}_{A_p^2}^{K_p^2} = 1.$$

Hence, $\{\sigma_f, \tilde{\varphi}\}_{A_p^2}^{K_p^2} = 1$ for every $\tilde{\varphi} \in \text{Gl}(2, \Sigma_X)$ such that $\det \tilde{\varphi} = 1$, whence the proposition is deduced. □

Proposition 4.7. *If X is a complete, smooth and connected curve, for all $f \in \Sigma_X^*$ and $\varphi \in \text{Gl}(2, \Sigma_X)$, one has that*

$$\prod_{p \in X} \{\sigma_f, \varphi\}_{A_p^2}^{K_p^2} = 1.$$

Proof. The claim is a direct consequence of Theorem 4.2 and Lemma 4.4, because

$$1 = \prod_{p \in X} (-1)^{2 \deg(p) v_p(f) v_p(\det \varphi)} \{ \sigma_f, \varphi \}_{A_p^2}^{K_p^2} = \prod_{p \in X} \{ \sigma_f, \varphi \}_{A_p^2}^{K_p^2}.$$

□

Remark 4.8. If $(\ , \)_p: \Sigma_X^* \times \Sigma_X^* \rightarrow k^*$ is the tame symbol, there exists a commutative diagram

$$\begin{array}{ccc} \Sigma_X^* \times \mathrm{Gl}(2, \Sigma_X) & \xrightarrow{\{ \ , \ \}_{A_p^2}^{K_p^2}} & k^* \\ \downarrow i & \searrow & \uparrow \\ \Sigma_X^* \times \Sigma_X^* & & (\ , \)_p \end{array}$$

i being the morphism $i(\sigma_f) = f$. A similar expression has recently been obtained by A. Beilinson, S. Bloch and H. Esnault ([5], 3.9.2) using super-extensions of Σ_X^* .

In particular, the restricted commutator map:

$$\{ \ , \ \}_{A_p^2}^{K_p^2}: \Sigma_X^* \times \mathrm{Sl}(2, \Sigma_X) \rightarrow k^*$$

is trivial, $\mathrm{Sl}(2, \Sigma_X)$ being the special linear group of Σ_X .

Therefore, the reciprocity law of Proposition 4.7 follows again from the Weil reciprocity law. A remaining problem is to determine whether the reciprocity law of $\mathrm{Gl}(2, \Sigma_X)$ (Theorem 4.2) can always be deduced from the Weil reciprocity law.

Remark 4.9 (Contou-Carrère Symbol on $\mathrm{Gl}(2, \Sigma_X \otimes_k B)$). Keeping the assumptions on X , if B is a local Artinian k -algebra, the group $\mathrm{Gl}(2, \Sigma_X \otimes_k B)$ consists of the matrices:

$$\tau \in \mathrm{Mat}_{2 \times 2}(\Sigma_X \otimes_k B),$$

such that $\det \tau$ is invertible in $\Sigma_X \otimes_k B$, and $(\Sigma_X \otimes_k B)^*$ is also a subgroup of $\mathrm{Gl}(2, \Sigma_X \otimes_k B)$ by means of the diagonal embedding $f \mapsto \sigma_f$. One has that $\mathrm{Gl}(2, \Sigma_X \otimes_k B) \subseteq \mathrm{Gl}(K_p^2)(\mathrm{Spec} B)$ and there exists a central extension of groups

$$1 \rightarrow B^* \rightarrow \widetilde{\mathrm{Gl}}(2, \Sigma_X \otimes_k B) \rightarrow \mathrm{Gl}(2, \Sigma_X \otimes_k B) \rightarrow 1,$$

whose commutator is denoted by $\{ \varphi, \phi \}_{A_{p,B}^2}^{K_{p,B}^2}$, for all $\varphi, \phi \in \mathrm{Gl}(2, \Sigma_X \otimes_k B)$, such that $\varphi \cdot \phi = \phi \cdot \varphi$.

We have a map $\{ \ , \ \}_{A_{p,B}^2}^{K_{p,B}^2}: (\Sigma_X \otimes_k B)^* \times \mathrm{Gl}(2, \Sigma_X \otimes_k B) \rightarrow B^*$ and, similar to the above and using the statements of [3], a computation shows that $\{ \sigma_f, \varphi \}_{A_{p,B}^2}^{K_{p,B}^2} = \langle f, \det \varphi \rangle_p$, where $\langle \ , \ \rangle_p$ is the Contou-Carrère symbol ([7]), f and $\det \varphi$ being Laurent series with coefficients in B via an immersion $(\Sigma_X \otimes_k B)^* \hookrightarrow B((t))^*$.

Moreover, when X is complete, using the method of this work a hypothetical generalization of the abstract reciprocity law (Theorem 3.5) to $\mathrm{Spec} B$ -valued points will allow us to recover the reciprocity law for the Contou-Carrère symbol of an algebraic curve over a perfect field (see [10]) without assuming that B is a finite k -algebra.

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