Theoretical advancements for the use of d-Choquet integrals with differential privacy

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Abstract

This manuscript overviews the theoretical contribution of the author to the utilization of d-Choquet integrals with a guarantee for differential privacy. This topic has gained traction in 2025.

1 Introducción

These notes encompass the presentation of the contents of the book chapter titled "The differentially private d-Choquet integral: an extension of differentially private Choquet integrals". The talk was delivered by the author at the MDAI 2025 conference held in Valencia, Spain, in September 2025. MDAI stands for "Modeling Decisions for Artificial Intelligence". The research has been published in [1], and it is supplemented with an experimental analysis reported in [2].

This book chapter initiates the theoretical investigation of differential privacy with aggregation operators that generalize the Choquet integral [10]. In turn, Choquet integrals extend weighted average means and OWA (for ordered weighted averaging) aggregation operators [5, 20]. Their applications are multiple [4, 12]. Other theoretical results, plus a few experiments, supplement its contents in [2]. By doing so these research works amplify the scope of application of the pioneering Torra [18] and its sequel by Alcantud [3] (which imaugurate the research of differential privacy issues with the Choquet integral), to the d-Choquet integrals defined by Bustince *et al.* [9]. The innovation of [9] was the utilization of an axiomatic notion of dissimilarity studied in Bustince *et al.* [8] as a replacement of the naive subtraction in the formula of the discrete Choquet integral.

Our two works pave the way to the investigation of data privacy for other extensions of the discrete Choquet integral such as the inclusion-exclusion integral [13], the d-CC integrals [14, 16] that extend CC-integrals [15], d_G-Choquet integrals [17], and d-XC integrals [19].

2 The presentation and materials

The research corresponding to the slides printed below have been published in https://mdai.cat/mdai2025/proc.lnai.mdai2025.pdf

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The differentially private d-Choquet integral:

an extension of differentially private Choquet integrals

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Motivation and goal of this presentation

▷ Choquet integrals extend both Weighted average means (WAM) and Ordered weighted average (OWA) operators.

They have been generalized by Bustince *et al.* (2021) with the use of restricted dissimilarity functions (Bustince *et al.*, 2008).

▷ Within the study of differential privacy (a mathematical expression of privacy preservation for data analysis from databases):

Torra (2025) and A. (2025a) are concerned with Choquet integrals.

This work initiates the investigation of differentially private d-Choquet integrals. A short continuation is available in A. (2025b).

Basic definitions

Preliminary notation

We work with an arbitrary set $N = \{e_1, \ldots, e_T\}$ with T elements.

For any $\mathbf{a} = (a_1, \ldots, a_T) \in \mathbb{R}^T$, a vector $\mathbf{a}_{\nearrow} = (a_{\sigma(1)}, \ldots, a_{\sigma(T)})$ is presented with $a_{\sigma(1)} \leqslant \ldots \leqslant a_{\sigma(T)}$ for a bijective mapping $\sigma : \{1, \ldots, T\} \longrightarrow \{1, \ldots, T\}$ (i.e., a permutation of the indices).

Associated with $\mathbf{a}$ and j we refer to $L_j^{\mathbf{a}} = \{\sigma(j), \ldots, \sigma(T)\}$, the set of indices corresponding to the largest $T - j + 1$ components of $\mathbf{a}$.

As in Torra (2025) and A. (2025a,b), we consider the space of databases $\mathcal{D} = [0,1]^T = \{f | f : N \longrightarrow [0,1]\}$. We let $f_i = f(e_i) \in [0,1]$.

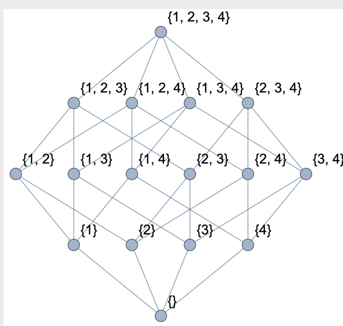
We write $f \sim \bar{f}$ when $f, \bar{f} \in \mathcal{D}$ differ in at most 1 element.

We denote by $rg(M)$ the range of any mapping M.

Capacities

We identify $N = \{1, \ldots, T\}$, it represents the set of indices too.

Definition. A discrete fuzzy measure (or a **capacity**) is a set function $\mu : 2^N \longrightarrow [0, 1]$ that is monotonic (i.e., $\mu(A) \leqslant \mu(B)$ whenever $A \subseteq B \subseteq N$) and satisfies $\mu(\varnothing) = 0$, $\mu(N) = 1$.



Domain of a capacity with $N = \{1, 2, 3, 4\}$.

The discrete Choquet integral: two formulas

Definition. The discrete Choquet integral $C^\mu : \mathbb{R}_+^T \longrightarrow \mathbb{R}_+$ associated with a fuzzy measure μ on N evaluates $\mathbf{a} = (a_1, \ldots, a_T) \in \mathbb{R}_+^T$ by

$$C^\mu(\mathbf{a}) = \sum_{i=1}^{T} \left(a_{\sigma(i)} - a_{\sigma(i-1)}\right) \mu(L_i^{\mathbf{a}}) = \sum_{i=1}^{T} \Delta_{\sigma(i)}\mu(L_i^{\mathbf{a}}) a_{\sigma(i)},$$

(1) the permutation σ ensures $\mathbf{a}_{\nearrow} = \left(a_{\sigma(1)}, \ldots, a_{\sigma(T)}\right)$ and the convention $a_{\sigma(0)} = 0$ applies,
(2) $\Delta_i\mu(A \cup \{i\}) = \mu(A \cup \{i\}) - \mu(A)$, for each $A \subseteq N \setminus \{i\}$ and $i \in N$.

Torra (2025) considers the Choquet integral on $\mathcal{D}$ –the space of databases– associated with μ: for any $f \in \mathcal{D}$, $C^\mu(f) = C^\mu(f_1, \ldots, f_T)$.

Basic capacities and Choquet integrals

- The smallest (or null, sometimes degenerate) fuzzy measure $\mu_\perp$ assigns $\mu_\perp(N) = 1$, and $\mu_\perp(N') = 0$ when $N' \subsetneq N$.
- The largest (also called universal or maximal) fuzzy measure $\mu_\top$ assigns $\mu_\top(\varnothing) = 0$, and $\mu_\top(N') = 1$ when $\varnothing \neq N' \subseteq N$.
- The discrete Choquet integral associated with $\mu_\perp$, resp., $\mu_\top$, concides with the minimum, resp., maximum, function.

The d-discrete Choquet integral – preliminaries

It replaces the arithmetic subtraction of values with a restricted dissimilarity function:

Definition (Bustince *et al.*, 2008). The mapping $\delta : [0,1]^2 \to [0,1]$ is a *restricted dissimilarity function* when for all $a_1, a_2, a_3 \in [0,1]$:

a) $\delta(a_1, a_2) = \delta(a_2, a_1)$,

b) $\delta(a_1, a_2) = 0 \Leftrightarrow a_1 = a_2$, and $\delta(a_1, a_2) = 1 \Leftrightarrow \{a_1, a_2\} = \{0, 1\}$,

c) $a_1 \leqslant a_2 \leqslant a_3 \Rightarrow \delta(a_1, a_2) \leqslant \delta(a_1, a_3)$ and $\delta(a_2, a_3) \leqslant \delta(a_1, a_3)$.

Examples. $\delta_1(a, b) = |\sqrt{a} - \sqrt{b}|$, $\delta_2(a, b) = |a^2 - b^2|$, $\delta_3(a, b) = |a - b|$, for all $a, b \in [0, 1]$.

The d-discrete Choquet integral

Definition (Bustince *et al.*, 2021). Fix a discrete fuzzy measure μ on N and $\delta : [0,1]^2 \to [0,1]$, a restricted dissimilarity function. The *discrete d-Choquet integral* (with T variables) with respect to μ and δ is

$$C^{\mu,\delta}(a_1,\ldots,a_T) = \sum_{i=1}^{T} \delta(a_{\sigma(i)}, a_{\sigma(i-1)})\mu(L_i^{\mathbf{a}}),\ \forall \mathbf{a} = (a_1,\ldots,a_T) \in \mathbb{R}_+^T,$$

where the permutation σ ensures $\mathbf{a}_{\nearrow} = (a_{\sigma(1)},\ldots,a_{\sigma(T)})$ and the convention $a_{\sigma(0)} = 0$ applies.

Example. If the dissimilarity is $\delta_3(a,b) = |a-b|$ then $C^{\mu,\delta_3} = C^{\mu}$.

Following Torra (2025), we define d-Choquet integrals on $\mathcal{D}$ by

$$C^{\mu,\delta}(f) = C^{\mu,\delta}(f(e_1),\ldots,f(e_T)) = C^{\mu,\delta}(f_1,\ldots,f_T) \in [0,T],\ \text{each } f \in \mathcal{D}.$$

The d-discrete Choquet integral: basic facts

To guarantee that the range of $C^{\mu,\delta}$ is in $[0,1]$, Bustince *et al.* (2008) defined: the restricted dissimilarity function δ satisfies (P1) when

$$a_1,\dots,a_T \in [0,1], a_1 \leqslant \cdots \leqslant a_T, \text{ imply } \sum_{j=1}^{T} \delta(a_{j-1},a_j) \leqslant 1 \ (a_0 = 0).$$

Proposition (improvement of result in Bustince *et al.*, 2008.) Fix a restricted dissimilarity function δ. The next statements are equivalent:

(a) The d-Choquet integral with respect to every fuzzy measure μ on $\{1,\dots,T\}$ is such that $C^{\mu,\delta} : [0,1]^T \longrightarrow [0,1]$.

(b) $C^{\mu_\top,\delta} : [0,1]^T \longrightarrow [0,1]$ with $\mu_\top$ defined on $\{1,\dots,T\}$.

(c) δ satisfies condition (P1).

Differential privacy

Differential privacy

Dwork (2006) defined differential privacy in terms of a "privacy budget" ϵ. Smaller values of $\epsilon \Rightarrow$ better privacy; absolute privacy if $\epsilon = 0$.

The implementation of this concept by the Laplace mechanism uses the **sensitivity** of a query or function (Dwork, 2006).

Differential privacy for our model

We are concerned with disclosure of the **output of an aggregation of databases by a d-Choquet integral** $C^{\mu,\delta}$. Taking advantage of a fundamental result in Torra (2025), we find that its sensitivity is

$$\Delta_{\mathcal{D}}(C^{\mu,\delta}) = \sup\{|C^{\mu,\delta}(f) - C^{\mu,\delta}(\bar{f})| : f, \bar{f} \in \mathcal{D} \text{ and } f \sim \bar{f}\}.$$

Then $L_{C^{\mu,\delta}}(f) = C^{\mu,\delta}(f) + L\left(0, \frac{\Delta_{\mathcal{D}}(C^{\mu,\delta})}{\epsilon}\right)$ is the **differentially private d-Choquet integral defined from μ and δ**.

$L\left(0, \frac{\Delta_{\mathcal{D}}(C^{\mu,\delta})}{\epsilon}\right)$: Laplacian noise, 0 mean, scale parameter $\frac{\Delta_{\mathcal{D}}(C^{\mu,\delta})}{\epsilon}$.

$\Rightarrow Pr(L_{C^{\mu,\delta}}(f) \in S) \leqslant e^{\epsilon} \cdot Pr(L_{C^{\mu,\delta}}(\bar{f}) \in S)$ when $f \sim \bar{f}$, $S \subseteq rg(L_{C^{\mu,\delta}})$.

Main results

Main results (I)

For each restricted dissimilarity function δ, $\Delta_{\mathcal{D}}(C^{\mu_\perp,\delta}) = 1$.

(Note first that $C^{\mu_\perp,\delta} : [0,1]^T \longrightarrow [0,1]$, which entails $\Delta_{\mathcal{D}}(C^{\mu_\perp,\delta}) \leqslant 1$),

If in addition δ satisfies condition (P1), then $\Delta_{\mathcal{D}}(C^{\mu_\top,\delta}) = 1$. We cannot dispense with the restriction that δ satisfies (P1):

Example. $\Delta_{\mathcal{D}}(C^{\mu_\top,\delta}) > 1$ for

$$\delta(x,y) = \begin{cases} 1, & \text{if } \{x,y\} = \{0,1\}, \\ 0, & \text{if } x = y, \\ \text{otherwise,} & \begin{cases} 0.1, & \text{if } |x-y| < 0.3, \\ 0.9, & \text{if } |x-y| \geqslant 0.3. \end{cases} \end{cases}$$

We use the databases $f, \overline{f} : N = \{1,2,3\} \longrightarrow [0,1]$ with $f(1) = 0.2$, $f(2) = 0.3$, $f(3) = 0.65$, $\overline{f}(1) = 1$, $\overline{f}(2) = f(2)$, $\overline{f}(3) = f(3)$.

Main results (II)

Comparison of sensitivities of unrestricted Choquet/d-Choquet integrals:

Proposition. For any fuzzy measure μ on N and restricted dissimilarity function δ, $\Delta_{\mathcal{D}}(C^{\mu,\delta}) \geqslant \Delta_{\mathcal{D}}(C^{\mu})$.

The inequality in this Proposition may be strict (by example).

Proposition. For any restricted dissimilarity function δ that satisfies (P1) and fuzzy measure μ on N such that $C^{\mu,\delta}$ is monotonic, $\Delta_{\mathcal{D}}(C^{\mu,\delta}) = \Delta_{\mathcal{D}}(C^{\mu})$.

Neither of the sufficient conditions in the later Proposition is necessary, even if the other holds.

Main results (III)

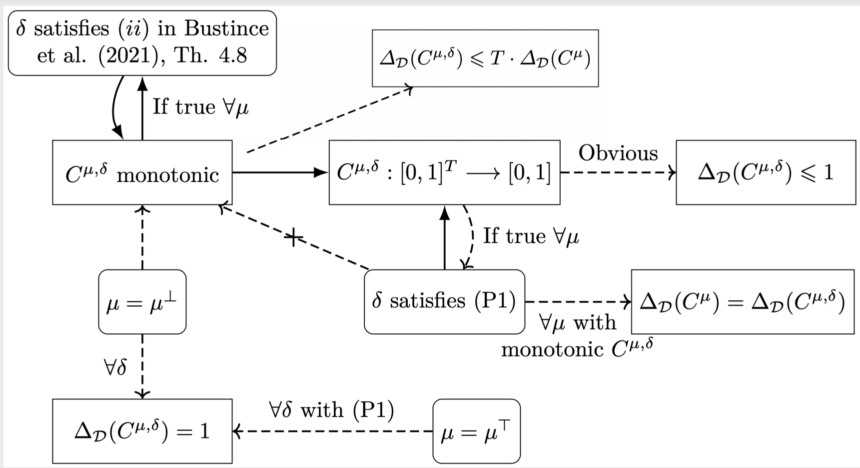
A relationship when we dispense with (P1) in the later Proposition:

Proposition. For any restricted dissimilarity function δ and fuzzy measure μ on N (with $|N| = T$) such that $C^{\mu,\delta}$ is monotonic, $\Delta_{\mathcal{D}}(C^{\mu,\delta}) \leqslant T \cdot \Delta_{\mathcal{D}}(C^{\mu})$.

The upper bound in this Proposition cannot be improved: examples exist with a Dirac fuzzy measure μ_1, such that for each $1 < T' < T = T \cdot \Delta_{\mathcal{D}}(C^{\mu_1})$, there is a sufficiently large ϵ with $\Delta_{\mathcal{D}}(C^{\mu_1,\delta_\epsilon}) \geqslant T'$, where $\delta_\epsilon(x,y) = \begin{cases} 0, & \text{if } x = y, \\ \frac{|x-y|+\epsilon}{1+\epsilon} & \text{otherwise.} \end{cases}$

Summary of results

Graphical summary



Dashed lines mean original results.

Conclusions

Conclusions

The investigation of differential privacy in the context of generalized forms of the Choquet integral is a promising avenue for further research.

Our work paves the way to the investigation of data privacy issues in other extensions of the discrete Choquet integral that have been recently proposed, e.g., the inclusion-exclusion integral (Honda and Okazaki, 2017), the d-CC integrals (Sartori *et al.*, 2023) that extend CC-integrals (Lucca *et al.*, 2017), d_G-Choquet integrals (Takáč *et al.*, 2022), and d-XC integrals (Wieczynski *et al.*, 2022).

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