

Habréis visto con Jose los símbolos de Christoffel y el tensor de Riemann.

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} (\partial_{\mu} g_{\nu\beta} + \partial_{\nu} g_{\mu\beta} - \partial_{\beta} g_{\mu\nu})$$

En un ~~sistema~~ marco inercial no hay gravedad, entonces $g_{\mu\nu} = \eta_{\mu\nu}$ y $\Gamma^{\alpha}_{\mu\nu} = 0$.

El P.E. dice que se puede siempre pasar a un marco localmente inercial (el marco de caída libre), que es una región suficientemente pequeña en la que bien descrita por la SR. Es decir, en cada $p \in M$ podemos elegir coordenadas tales que (pertenece)

$$\hat{g}_{\mu\nu}(p) = \eta_{\mu\nu}, \quad \left. \frac{\partial \hat{g}_{\mu\nu}}{\partial \hat{x}^{\alpha}} \right|_p = 0$$

(pero $\left. \frac{\partial^2 \hat{g}_{\mu\nu}}{\partial \hat{x}^{\alpha} \partial \hat{x}^{\beta}} \right|_p \neq 0$, de ahí que $\hat{R}_{\mu\nu\alpha\beta} \neq 0$)

Hacemos una expansión de Taylor: $x^{\mu} = \hat{x}^{\mu}|_p + \frac{\partial x^{\mu}}{\partial \hat{x}^{\alpha}} \Big|_p \hat{x}^{\alpha} + \frac{1}{2} \frac{\partial^2 x^{\mu}}{\partial \hat{x}^{\alpha} \partial \hat{x}^{\beta}} \Big|_p \hat{x}^{\alpha} \hat{x}^{\beta} + \dots$
en un entorno de p

$$\frac{\partial x^{\mu}}{\partial \hat{x}^{\alpha}} \approx \frac{\partial x^{\mu}}{\partial \hat{x}^{\alpha}} \Big|_p + \frac{\partial^2 x^{\mu}}{\partial \hat{x}^{\alpha} \partial \hat{x}^{\beta}} \hat{x}^{\beta} + \frac{1}{2} \frac{\partial^3 x^{\mu}}{\partial \hat{x}^{\alpha} \partial \hat{x}^{\beta} \partial \hat{x}^{\gamma}} \hat{x}^{\beta} \hat{x}^{\gamma} + \dots$$

La métrica transforma como $\hat{g}_{\alpha\beta}(\hat{x}) = \frac{\partial x^{\mu}}{\partial \hat{x}^{\alpha}} g_{\mu\nu}(\hat{x}) \frac{\partial x^{\nu}}{\partial \hat{x}^{\beta}}$

$$\hat{g}_{\alpha\beta}(\hat{x}) \approx \hat{g}_{\alpha\beta} \Big|_p + \frac{\partial \hat{g}_{\alpha\beta}}{\partial \hat{x}^{\gamma}} \Big|_p \hat{x}^{\gamma} + \dots \quad g_{\mu\nu}(\hat{x}) \approx g_{\mu\nu} \Big|_p + \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} \hat{x}^{\lambda} + \dots$$

Esquemáticamente: (enchufando)

$$\hat{g}_{\alpha\beta}(\hat{x}) \approx \hat{g}_{\alpha\beta} \Big|_p + \frac{1}{2} \left(\frac{\partial \hat{g}_{\alpha\beta}}{\partial \hat{x}^{\gamma}} \right) \Big|_p \hat{x}^{\gamma} \hat{x}^{\gamma} + \dots = \left(\hat{\partial}_x g \hat{\partial}_x \right) \Big|_p + \left(\hat{\partial}_x \hat{\partial}_g \hat{\partial}_x + 2 \hat{\partial}_x^2 g \hat{\partial}_x \right) \Big|_p \hat{x}^{\gamma} \hat{x}^{\gamma} + \frac{1}{2} \left(\hat{\partial}_x \hat{\partial}_g \hat{\partial}_x + 2 \hat{\partial}_x^2 g \hat{\partial}_x + 2 \hat{\partial}_x^3 g \hat{\partial}_x \right) \Big|_p \hat{x}^{\gamma} \hat{x}^{\gamma} \hat{x}^{\gamma} + \dots$$

1) $\hat{g}_{\alpha\beta} \Big|_p = \eta_{\alpha\beta} = \left(\frac{\partial x^{\mu}}{\partial \hat{x}^{\alpha}} g_{\mu\nu} \frac{\partial x^{\nu}}{\partial \hat{x}^{\beta}} \right) \Big|_p$ 10 ecuaciones para 16 incógnitas (simétrico bajo α, β)

2) $\hat{\partial}_x \hat{g}_{\alpha\beta} \Big|_p = 0 = \left(\frac{\partial x^{\mu}}{\partial \hat{x}^{\alpha}} \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} \frac{\partial x^{\nu}}{\partial \hat{x}^{\beta}} \right) \Big|_p + \dots$ 40 ecuaciones para 40 incógnitas $\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} \Big|_p$

3) $\hat{\partial}_x \hat{\partial}_x \hat{g}_{\alpha\beta} = \text{r.h.s.}$ $10 \times 10 = 100$ ecuaciones para $4 \times 4 \times \frac{5 \times 6}{3!} = 80$ incógnitas $\frac{\partial^3 g_{\mu\nu}}{\partial x^{\lambda} \partial x^{\rho} \partial x^{\sigma}} \Big|_p$

Faltan 20! estas son las componentes del tensor de Riemann, que no puede anularse

Vamos a ~~construir~~ construir la ecuación geodesica directamente dal principio de equivalencia este cambio de coordenatas

$$g_{\mu\nu} = \frac{\partial \hat{x}^\alpha}{\partial x^\mu} \eta_{\alpha\beta} \frac{\partial \hat{x}^\beta}{\partial x^\nu}$$

$$\partial_\rho g_{\mu\nu} = \frac{\partial^2 \hat{x}^\alpha}{\partial x^\rho \partial x^\mu} \eta_{\alpha\beta} \frac{\partial \hat{x}^\beta}{\partial x^\nu} + \frac{\partial \hat{x}^\alpha}{\partial x^\mu} \eta_{\alpha\beta} \frac{\partial^2 \hat{x}^\beta}{\partial x^\rho \partial x^\nu} \Rightarrow \frac{\partial^2 \hat{x}^\alpha}{\partial x^\rho \partial x^\mu} \frac{\partial x^\sigma}{\partial \hat{x}^\alpha} g_{\sigma\rho} + g_{\mu\sigma} \frac{\partial x^\sigma}{\partial \hat{x}^\beta} \frac{\partial^2 \hat{x}^\beta}{\partial x^\rho \partial x^\nu} = 0$$

$$\Rightarrow \frac{\partial^2 \hat{x}^\alpha}{\partial x^\rho \partial x^\mu} \frac{\partial x^\sigma}{\partial \hat{x}^\alpha} g_{\sigma\rho} + g_{\mu\sigma} \frac{\partial x^\sigma}{\partial \hat{x}^\beta} \frac{\partial^2 \hat{x}^\beta}{\partial x^\rho \partial x^\nu} = 0$$

Podemos identificar $\Gamma_{\rho\mu}^\sigma = \frac{\partial x^\sigma}{\partial \hat{x}^\alpha} \frac{\partial^2 \hat{x}^\alpha}{\partial x^\rho \partial x^\mu}$

Una partícula no acelerada ~~sin gravita~~ en el marco localmente inercial habe $\frac{d^2 \hat{x}^\mu}{d\lambda^2} = 0$

$$\frac{d^2 \hat{x}^\mu}{d\lambda^2} = \frac{d}{d\lambda} \left(\frac{d\hat{x}^\mu}{d\lambda} \right) = \frac{d^2 \hat{x}^\mu}{d\lambda^2} + \frac{d\hat{x}^\nu}{d\lambda} \frac{d}{d\lambda} \frac{\partial \hat{x}^\mu}{\partial x^\nu} = \frac{d^2 \hat{x}^\mu}{d\lambda^2} + \frac{\partial x^\sigma}{\partial \hat{x}^\nu} \frac{\partial^2 \hat{x}^\nu}{\partial x^\rho \partial x^\sigma} \frac{d\hat{x}^\rho}{d\lambda} \frac{d\hat{x}^\mu}{d\lambda}$$

$$\frac{d^2 \hat{x}^\mu}{d\lambda^2} = 0 \Leftrightarrow \frac{d^2 \hat{x}^\mu}{d\lambda^2} + \Gamma_{\rho\sigma}^\mu \frac{d\hat{x}^\rho}{d\lambda} \frac{d\hat{x}^\sigma}{d\lambda} = 0$$

Habéis visto el transporte paralelo a lo largo de una curva $x^\mu(\lambda)$
 $t^\mu \nabla_\mu t^\nu = 0$, donde $t^\mu = \frac{dx^\mu}{d\lambda}$ es la tangente

Definición Una geodésica es una curva cuya tangente se transporta paralelamente a sí misma

$$\boxed{t^\mu \nabla_\mu t^\nu = 0}$$

Lo mismo que $\frac{dx^\mu}{d\lambda} \left(\frac{\partial t^\nu}{\partial x^\mu} + \Gamma_{\mu\rho}^\nu t^\rho \right) = \frac{dt^\nu}{d\lambda} + \Gamma_{\mu\rho}^\nu t^\mu t^\rho = \frac{d^2 x^\nu}{d\lambda^2}$
 $\frac{dx^\mu}{d\lambda} \left(\frac{\partial t^\nu}{\partial x^\mu} + \Gamma_{\mu\rho}^\nu t^\rho \right) = \frac{d^2 x^\nu}{d\lambda^2} + \Gamma_{\mu\rho}^\nu \frac{dx^\mu}{d\lambda} \frac{dx^\rho}{d\lambda} = 0$

El tiempo propio de una curva es $\tau = \int_A^B d\lambda \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}$
 (por curva de tipo espacial, la distancia propia ~~el largo~~ es $L = \int_A^B d\lambda \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}$)

Una geodésica ~~tiempo~~ es la curva que maximiza el tiempo propio desde A hasta B (o minimiza la distancia propia)

$$\boxed{t^\mu \nabla_\mu t^\nu = 0 \iff \delta\tau = 0 \forall \delta x^\mu}$$

$$\delta\tau = \int_A^B d\lambda \frac{1}{2\sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}} \left[-2g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{d\delta x^\nu}{d\lambda} + \frac{\partial g_{\mu\nu}}{\partial x^\rho} \delta x^\rho \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right] \quad \frac{d\tau}{d\lambda} = \sqrt{\dots}$$

$$= \int_A^B d\tau \left[-g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d\delta x^\nu}{d\tau} + \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\rho} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \delta x^\rho \right]$$

$$= \int_A^B d\tau \rightarrow \frac{d}{d\tau} \left(g_{\mu\rho} \frac{dx^\mu}{d\tau} \right) \delta x^\rho$$

$$= \int_A^B d\tau \left[\frac{d^2 x^\rho}{d\tau^2} g_{\mu\rho} + \frac{\partial g_{\mu\rho}}{\partial x^\nu} \frac{dx^\nu}{d\tau} \frac{dx^\mu}{d\tau} + \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\rho} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right] \delta x^\rho$$

$$= \int_A^B d\tau \left[\frac{d^2 x^\alpha}{d\tau^2} + \underbrace{g^{\alpha\rho} \left(\frac{\partial g_{\mu\rho}}{\partial x^\nu} + \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\rho} \right) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}_{= \Gamma_{\mu\nu}^\alpha} \right] g_{\alpha\rho} \delta x^\rho$$

Esta es una definición equivalente de geodésica.

En general, sería bastante $t^\mu \nabla_\mu t^\nu = f(\lambda) t^\nu$. En este caso, hacemos una reparametrización $\lambda \rightarrow \alpha(\lambda)$

$$\frac{d^2 x^\alpha}{d\lambda^2} = \frac{d^2 \alpha}{d\lambda^2} \frac{dx^\mu}{d\alpha} + \left(\frac{d\alpha}{d\lambda} \right) \frac{d^2 x^\mu}{d\alpha^2} \Rightarrow \left[\frac{d^2 \alpha}{d\lambda^2} + \Gamma_{\rho\sigma}^\mu \frac{d\alpha}{d\lambda} \frac{d\alpha}{d\lambda} \right] \left(\frac{d\alpha}{d\lambda} \right)^2 = \left(f(\lambda) - \frac{d^2 \alpha}{d\lambda^2} \right) \frac{dx^\mu}{d\alpha}$$

Need $d^2 \alpha / d\lambda^2 = f(\lambda)$ } El mismo, con $S = \frac{1}{2} \int d\tau \left[-g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right]$

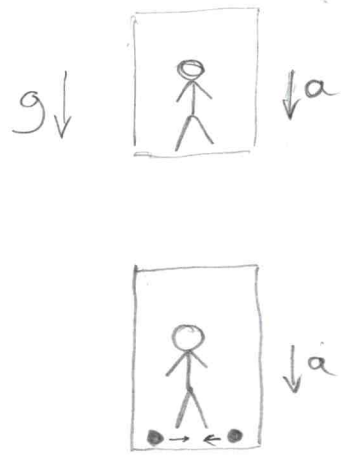
Riemann: $(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) \omega_\alpha = R_{\mu\nu\alpha}{}^\beta \omega_\beta - R^\beta{}_{\alpha\mu\nu} \omega_\beta$
 $(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) v^\alpha = R^\alpha{}_{\beta\mu\nu} v^\beta$

$$R^\alpha{}_{\beta\mu\nu} \equiv \partial_\mu \Gamma^\alpha{}_{\nu\beta} - \partial_\nu \Gamma^\alpha{}_{\mu\beta} + \Gamma^\alpha{}_{\mu\lambda} \Gamma^\lambda{}_{\nu\beta} - \Gamma^\alpha{}_{\nu\lambda} \Gamma^\lambda{}_{\mu\beta}$$

$$R_{\mu\nu\alpha\beta} = g_{\alpha\lambda} R_{\mu\nu}{}^\lambda{}_\beta$$

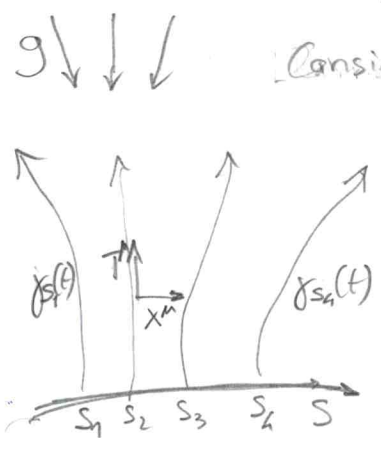
Commutador (Echelle de Lie): $[v, w]$ es un vector
 $[X^\alpha, X^\beta] = 0$
 $\uparrow \uparrow$ coordinate vectors $\left(\frac{\partial}{\partial x^\alpha}\right) \left(\frac{\partial}{\partial x^\beta}\right)$

Desviación geodésica



La ~~aca~~ El P.E. dice que la aceleración de caída puede cancelar la gravedad localmente (en el punto). ~~Per~~ En dos puntos cercanos pero diferentes pueden pasarse cosas diferentes

En el marco inercial todavía hay curvatura. Dos ~~objetos~~ partículas en caída libre siguiendo geodésicas diferentes pueden acercarse.



Consideremos una familia de geodésicas $\gamma_s(t)$:
 s indica la coordenada de la posición inicial
 $T^\mu \equiv \left(\frac{\partial}{\partial t}\right)^\mu$ $X^\mu = \left(\frac{\partial}{\partial s}\right)^\mu$ ← desplazamiento a la siguiente geodésica
 t y s son las coordenadas de una subvariedad bidimensional.

$v^\mu \equiv T^\nu \nabla_\nu X^\mu$ dice como la distancia entre geodésicas cambia con t .

Velocidad relativa entre ~~las~~ geodésicas
 $a^\alpha \equiv T^\mu \nabla_\mu v^\alpha = T^\mu \nabla_\mu (T^\nu \nabla_\nu X^\alpha)$ aceleración relativa

$$= T^\mu \nabla_\mu (X^\nu \nabla_\nu T^\alpha + [T, X]^\alpha) = (T^\mu \nabla_\mu X^\nu) \nabla_\nu T^\alpha + T^\mu X^\nu \nabla_\mu \nabla_\nu T^\alpha$$

$$= (X^\mu \nabla_\mu T^\nu + [T, X]^\nu) \nabla_\nu T^\alpha + T^\mu X^\nu (\nabla_\nu \nabla_\mu T^\alpha + R^\alpha{}_{\beta\mu\nu} T^\beta)$$

$$= X^\mu \nabla_\mu (T^\nu \nabla_\nu T^\alpha) + R^\alpha{}_{\beta\mu\nu} T^\beta T^\mu X^\nu$$

$$T^\mu T^\nu \nabla_\mu \nabla_\nu X^\alpha = T^\mu T^\nu \nabla_\mu \nabla_\nu X^\alpha + R^\alpha{}_{\beta\mu\nu} X^\beta$$

$$\frac{d^2 x^\mu}{dt^2} + \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta$$

$$\frac{d p^\mu}{dt} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = 0$$

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

$$G = R - 2R = -R \quad -R = 8\pi G T$$

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{g_{\mu\nu} T}{2} \right)$$

En el vacío
In vacuum, $R_{\mu\nu} = 0$

Geodesically conserved quantities

If $\partial_{\bar{\alpha}} g_{\mu\nu} = 0$ for a particular $\bar{\alpha}$ (if the metric does not depend on $\bar{\alpha}$), then $p_{\bar{\alpha}} \equiv m u_{\bar{\alpha}}$ is conserved along the geodesic.

Proof ~~$p^\mu \equiv m u^\mu$~~ $p^\mu \equiv m u^\mu = m \frac{dx^\mu}{dt}$

$$p^\alpha \nabla_\rho p^\mu = 0 \quad (\text{geodesic}) \quad \Rightarrow \quad p^\rho \nabla_\rho p_\mu = 0$$

$$p^\rho \nabla_\rho p_\mu = m \frac{d p_\mu}{dt} - \Gamma^\rho_{\mu\sigma} p^\sigma p_\rho$$

$$m \frac{d p_\mu}{dt} = \Gamma^\nu_{\rho\mu} p^\rho p_\nu = \frac{g^{\nu\sigma}}{2} (\partial_\rho g_{\mu\sigma} + \partial_\mu g_{\rho\sigma} - \partial_\sigma g_{\rho\mu}) p^\rho p_\nu$$

$$= \frac{1}{2} (\partial_\rho g_{\mu\sigma} + \partial_\mu g_{\rho\sigma} - \partial_\sigma g_{\rho\mu}) p^\rho p^\sigma = \frac{1}{2} \partial_\mu g_{\rho\sigma} p^\rho p^\sigma$$

$$\partial_{\bar{\alpha}} g_{\mu\nu} = 0 \quad \Rightarrow \quad \frac{d p_{\bar{\alpha}}}{dt} = 0 \quad (p_{\bar{\alpha}} \text{ is conserved})$$

Note that: 1) $p_{\bar{\alpha}}$ is not conserved

2) not a covariant statement. Only valid in the coordinate system in which the symmetry is explicit

METRICA DE SCHWARZSCHILD

Soluciones de las ecuaciones de Einstein en simetría esférica en el vacío. Describe la métrica en el exterior de una estrella o agujero negro.

~~Solución~~ $R_{\mu\nu} = 0$ (vacío) (como $\nabla^2\phi = 0$ en gravedad Newtoniana)

Buscamos solución estática y con simetría esférica

Si tomamos $\{t, r, \theta, \varphi\}$;

Coordenadas explícitas

- 1) no hay dependencia en t
- 2) no hay términos mixtos $dt dr, dt d\theta, dt d\varphi$ en la métrica

$$\frac{d}{d\lambda} \quad g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = g_{00} \left(\frac{dt}{d\lambda}\right)^2 + g_{0i} \frac{dt}{d\lambda} \frac{dx^i}{d\lambda} + g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}$$

↑
impar bajo inversión temporal

$$\left(\frac{dx^i}{d\lambda} \rightarrow -\frac{dx^i}{d\lambda} \text{ si } \frac{dt}{d\lambda} \rightarrow -\frac{dt}{d\lambda} \right)$$

Porque la métrica sea Lorentziana:

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2$$

en general sería $ds^2 = -e^{2\tilde{\alpha}(\tilde{r})} dt^2 + e^{2\tilde{\beta}(\tilde{r})} d\tilde{r}^2 + e^{2\tilde{\gamma}(\tilde{r})} \tilde{r}^2 d\Omega^2$

$$\left(e^{2\tilde{\beta}(\tilde{r})} \frac{d\tilde{r}}{d\tilde{r}} \right)^2 d\tilde{r}^2 \equiv r^2$$

$$\equiv e^{2\beta(r)}$$

r es el radio de la esfera, pero no es la distancia desde el centro

$$g_{\mu\nu} \rightarrow T^{\alpha}_{\mu\nu} \rightarrow R_{\mu\nu} \quad (\text{sencillo pero feo})$$

$$R_{tt} = e^{2(\alpha-\beta)} \left(\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{2}{r}\alpha' \right)$$

$$R_{\theta\theta} = e^{-2\beta} [r(\beta' - \alpha') - 1] + 1$$

$$R_{rr} = -\alpha'' - \alpha'^2 + \alpha'\beta' + \frac{2}{r}\beta'$$

$$R_{\phi\phi} = \sin^2\theta R_{\theta\theta}$$

Tenemos que poner $R_{\mu\nu} = 0$ (vacío)

$$e^{2(\beta-\alpha)} R_{tt} + R_{rr} = -\frac{2}{r}(\alpha' + \beta') = 0 \Rightarrow \alpha = \beta + c$$

Podemos eliminar c con una redefinición $t \rightarrow e^{-c}t \Rightarrow \beta = -\alpha$

$$R_{\theta\theta} = 0 \Rightarrow e^{2\alpha} [2r\alpha' + 1] = 1 \Rightarrow (e^{2\alpha})' = 1 \Rightarrow re^{2\alpha} = r + \mu$$

$$ds^2 = -\left(1 + \frac{\mu}{r}\right) dt^2 + \left(1 + \frac{\mu}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

En el límite Newtoniano, sabemos que $g_{00} \approx -1 - 2\phi \approx -1 + \frac{2GM}{r}$

$$\Rightarrow \mu = -\frac{2GM}{c^2} \equiv -R_s \quad \text{Radio de Schwarzschild}$$

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

¿Cuál es el tamaño de R_s ?

$$[G] = \left[\frac{L^3}{E^2 M}\right]$$

$$\ddot{r} = -\frac{GM}{r^2}$$

$$R_s = \frac{2GM}{c^2}$$

$$G = 6,67 \cdot 10^{-8} \text{ cm}^3 \text{ g}^{-1} \text{ s}^2$$

$$\Rightarrow M_{\odot} = 2 \cdot 10^{33} \text{ g}$$

$$R_s = \frac{2M}{M_{\odot}} \frac{2M_{\odot}}{c^2} = \frac{M}{M_{\odot}} \frac{2 \cdot 6,67 \cdot 10^{-8} \cdot 2 \cdot 10^{33}}{9 \cdot 10^{20}} \text{ cm} \approx 3 \text{ km} \frac{M}{M_{\odot}}$$

$$R_{\odot} \approx 7 \cdot 10^5 \text{ km} \gg R_s$$

En el interior es válida con $m(r) = 4\pi \int_0^r dr' r'^2 \rho(r')$

$$\text{at } R_s \quad R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{48G^2 M^2}{r^6} \quad \text{no es singular}$$

Geodésicas en Schwarzschild

Muchas simetrías \rightarrow muchas cantidades conservadas
 Invariancia bajo traslaciones temporales y rotaciones.

\rightarrow conservación de momento angular y energía a lo largo de las geodésicas. (como en Minkowski)

Parte angular de \mathbb{R}^2 de $g_{\mu\nu}$ es la misma que $\eta_{\mu\nu}$:
 el movimiento se produce en un plano. ($\theta = \frac{\pi}{2} = \text{const}$)

$$p_\theta = m g_{\theta\theta} \frac{d\theta}{dt} = m r^2 \frac{d\theta}{dt} \quad \frac{\partial}{\partial \theta} g_{\mu\nu} = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 2r^2 \sin\theta \cos\theta & \\ & & & \end{bmatrix}_{\theta=\pi/2} = 0$$

$$\Rightarrow p_\theta = \text{const}; \text{ si } \frac{d\theta}{dt} = 0 \text{ a } t = t_0, \Rightarrow \frac{d\theta}{dt} = 0 \forall t \Rightarrow \theta = \frac{\pi}{2} \forall t$$

Otras cantidades conservadas (porque $\frac{\partial}{\partial \phi} g_{\mu\nu} = \frac{\partial}{\partial t} g_{\mu\nu} = 0$):

$$L \equiv \frac{p_\phi}{m} = m g_{\phi\phi} \frac{d\phi}{dt} = m r^2 \sin^2\theta \left. \frac{d\phi}{dt} \right|_{\theta=\pi/2} = m r^2 \frac{d\phi}{dt}$$

$$E \equiv -\frac{p_t}{m} = -g_{tt} \frac{dt}{dt} = \left(1 - \frac{2GM}{r}\right) \frac{dt}{dt}$$

Lo mismo para una partícula sin masa (excepto que no podemos utilizar τ sino un ^{otro} parámetro afín λ).

Parametrización afín $\Rightarrow \epsilon \equiv -g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = \text{constante}$
 con $\epsilon = 0$ (partícula sin masa) o $\epsilon = 1$ (y $\lambda = \tau$, partícula masiva)

$$\epsilon = \left(1 - \frac{2GM}{r}\right) \left(\frac{dt}{d\lambda}\right)^2 - \left(1 - \frac{2GM}{r}\right)^{-1} \left(\frac{dr}{d\lambda}\right)^2 - r^2 \left[\underbrace{\left(\frac{d\theta}{d\lambda}\right)^2}_{=0} + \underbrace{\sin^2\theta}_{=1} \left(\frac{d\phi}{d\lambda}\right)^2 \right]$$

Multiplicamos por $\left(1 - \frac{2GM}{r}\right)$

$$\left(1 - \frac{2GM}{r}\right) \epsilon = E^2 - \left(\frac{dr}{d\lambda}\right)^2 - \left(1 - \frac{2GM}{r}\right) r^2 \left(\frac{d\phi}{d\lambda}\right)^2$$

$$\boxed{\frac{1}{2} \left(\frac{dr}{d\lambda}\right)^2 + V_{\text{eff}}(r) = \frac{E^2}{2}} \quad \text{con } V_{\text{eff}}(E) \equiv \frac{1}{2} \left(1 - \frac{2GM}{r}\right) \left(\frac{L^2}{r^2} + \epsilon\right)$$

Problema unidimensional clásico de partícula con "masa" = 1 y "energía" $\frac{E^2}{2}$ en el potencial efectivo

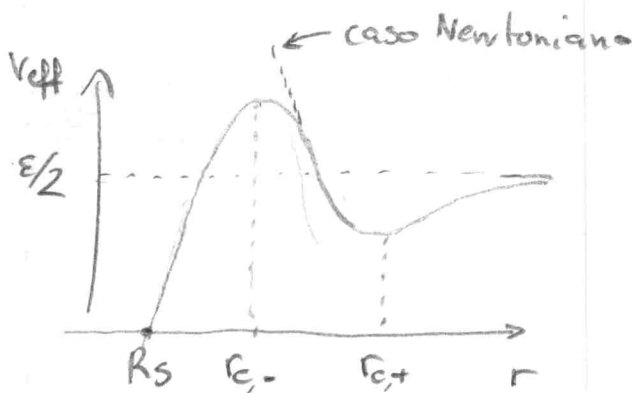
$$V_{\text{eff}}(r) = \frac{\epsilon}{2} - \frac{2GM}{r} + \frac{L^2}{2r^2} - \frac{6ML^2}{r^3}$$

Hay una constante $\epsilon/2$ y un término atractivo $\sim -1/r^3$

En gravedad Newtoniana, el término $L^2/2r^2$ siempre prohíbe que $r \rightarrow 0$. En GR el nuevo término es dominante por $r \lesssim R_s$

Por L fijado:

$$V_{\text{eff}}(r=R_s) = 0$$



Hay 2 orbitas circulares por $r=r_{c-}$ (inestable) y $r=r_{c+}$ (stable)

$$V'_{\text{eff}} = \frac{\epsilon GM}{r_c^2} - \frac{L^2}{r_c^3} + \frac{3GM L^2}{r_c^4} = 0$$

(por $\epsilon=1$)

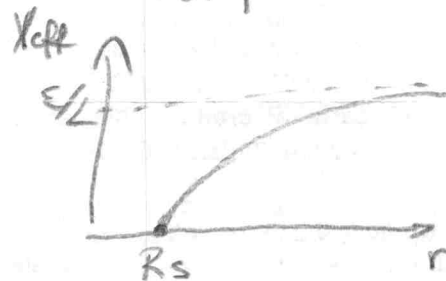
$$r_{c,\pm} = \frac{L^2 \pm \sqrt{L^2(L^2 - 12G^2M^2)}}{2GM}$$

Solo hay una orbita estable si $L^2 > 12G^2M^2$ ($r_c > 6GM = 3R_s$)

Por $L^2 \rightarrow \infty$, $r_{c+} \approx \frac{L^2}{GM} \rightarrow +\infty$ and $r_{c-} \rightarrow 3GM$

En general, $3GM < r_{c-} < 6GM < r_{c+} < L^2/GM$

Por $L^2 < 12G^2M^2$ no hay orbitas circulares

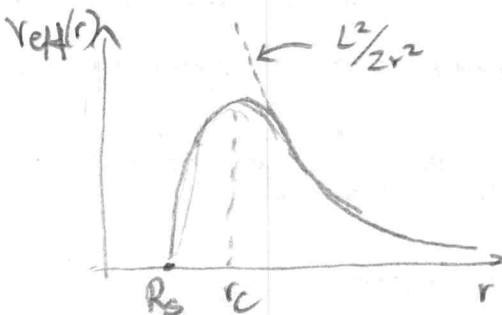


Por una partícula sin masa ($\epsilon=0$)

$$V_{\text{eff}}(r) = \frac{L^2}{2r^2} - \frac{GM L^2}{r^3}$$

$$V'_{\text{eff}}(r_c) = -\frac{L^2}{r_c^3} \left(1 - \frac{3GM}{r_c}\right) = 0$$

Solo hay una orbita circular inestable con $r = r_c = 3GM$



Por $L=0$ ($\frac{d\phi}{dx}=0$), orbita radial

$$\frac{dr}{dt} = \pm \left(1 - \frac{2GM}{r}\right) \quad (\epsilon=0)$$

Un fotón no puede cruzar R_s ($\frac{dr}{dt}=0$) en coordenadas de Schwarzschild.

Gravitational redshift

U^μ = cuadri-velocidad del observador radial ($U^\mu U_\mu = +1$)

$x^\mu(\lambda)$ trayectoria del fotón

$$\omega = -g_{\mu\nu} U^\mu \frac{dx^\nu}{d\lambda}$$

En coordenadas de Schw. un observador en reposo tiene

$$U^i = 0, \quad U^0 = \left(1 - \frac{2GM}{r}\right)^{-1/2}$$

$$\omega = -\sqrt{1 - \frac{2GM}{r}} \frac{dt}{d\lambda} = \left(1 - \frac{2GM}{r}\right)^{1/2} E \quad (E \text{ conservado})$$

$$\frac{\omega_2}{\omega_1} = \left(\frac{1 - 2GM/r_2}{1 - 2GM/r_1}\right)^{1/2}$$

$$r_2 > r_1 \Rightarrow \omega_2 < \omega_1$$

El fotón es más rojo en r_2

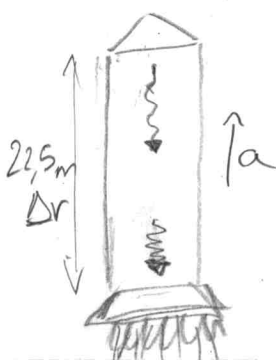
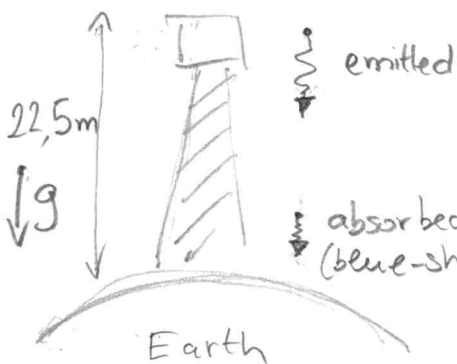
En el régimen de weak field

$$\frac{\omega_2}{\omega_1} \approx 1 - \frac{GM}{r_1} + \frac{GM}{r_2} = 1 + \phi_1 - \phi_2$$

For $r_2 - r_1 \ll r_1, r_2$, $\frac{\omega_2}{\omega_1} \approx 1 - \frac{GM}{r_1^2} (r_2 - r_1)$

Measured by Pound y Rebka (1960) con rayos γ emitidos y absorbidos por ^{57}Fe (Mössbauer effect) desde una torre.

Mismo que el derivado a partir del P.E. si el equipo experimental estuviese en un cohete acelerado



El fotón alcanza después $\Delta t = \Delta r/c$, cuando la velocidad se ha incrementado de $\Delta v = a \Delta t = a \Delta r/c$

$$\frac{\Delta \omega}{\omega} \approx \frac{\Delta v}{c} \approx \frac{a \Delta r}{c^2}$$

Precesión de las órbitas

En aproximación casi-circular (pequeñas oscilaciones alrededor del mínimo r_c de V_{eff})

$$V_{\text{eff}}(r) = \frac{1}{2} - \frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3}$$

$$V'_{\text{eff}}(r_c) = 0 \Rightarrow L^2 = \frac{GM r_c^2}{r_c - 3GM}$$

Frecuencia de oscilación radial:

$$\omega_r^2 = \left. \frac{\partial^2 V_{\text{eff}}}{\partial r^2} \right|_{r_c} = -\frac{2GM}{r_c^3} + \frac{3L^2}{r_c^4} - \frac{12GML^2}{r_c^5} = \frac{GM}{r_c^3} \left(\frac{3r_c - 12GM}{r_c - 3GM} - 2 \right) = \frac{GM}{r_c^3} \frac{r_c - 6GM}{r_c - 3GM}$$

Frecuencia angular (ya que $L = r_c^2 \omega_\phi$):

$$\omega_\phi^2 = \frac{L^2}{r_c^4} = \frac{GM}{r_c^2(r_c - 3GM)}$$

En el límite $r_c \gg GM$, $\omega_r^2 \approx \frac{GM}{r_c^3} \approx \omega_\phi^2$ y la órbita se cierra. (como en gravedad Newtoniana, las órbitas son elípticas)

En RG en contrario hay que $\omega_\phi > \omega_r$ y no se cierra.

$$\omega_{\text{precession}} \equiv \omega_\phi - \omega_r \approx \sqrt{\frac{GM}{r_c^3}} \left[\frac{3GM}{r_c} + \mathcal{O}\left(\frac{GM}{r_c}\right)^2 \right] \approx \frac{3GM}{r_c} \omega_\phi \approx \frac{3}{2} \frac{R_s}{r_c} \omega_\phi$$

For Mercury $\frac{\omega_{\text{precession}}}{\omega_\phi} \approx \frac{3}{2} \frac{3 \text{ km}}{5.7 \times 10^7 \text{ km}} \approx 10^{-7}$

La RG soluciona el problema de la precesión anómala de Mercurio, observada en 1859 por Le Verrier.

Deflexión de la luz

Métrica de Schwarzschild linearizada:

$$ds^2 \approx -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 + \frac{2GM}{r}\right) dr^2 + r^2 d\Omega^2$$

$$\approx -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 + \frac{2GM}{r}\right) (dr^2 + r^2 d\Omega^2) \approx -(1 + 2\phi) dt^2 + (1 - 2\phi) d\vec{x}^2$$

Los símbolos de Christoffel por esta métrica son:

$$\Gamma_{00}^i = \Gamma_{0i}^0 = \partial_i \phi, \quad \Gamma_{jk}^i = \delta_{jk} \partial_i \phi + \delta_{ik} \partial_j \phi - \delta_{ij} \partial_k \phi$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$h_{\mu\nu} = \begin{bmatrix} -2\phi & & & 0 \\ & 2\phi & & \\ & & 2\phi & \\ 0 & & & 2\phi \end{bmatrix}$$

$$\left(\phi = -\frac{GM}{r} \right)$$

$$r \equiv \tilde{r} \left(1 + \frac{GM}{r} \right), \quad dr = d\tilde{r}$$

Trayectoria de un fotón: $x^\mu(\lambda) = x^{(0)\mu}(\lambda) + x^{(1)\mu}(\lambda) + \dots$

$k^\mu \equiv \frac{dx^{(0)\mu}}{d\lambda}$ $l^\mu \equiv \frac{dx^{(1)\mu}}{d\lambda}$

$k^\mu = (k, \vec{k})$ (imperturbada) $(\eta_{\mu\nu} k^\mu k^\nu = 0)$ $k^2 = \vec{k}^2$
 (pequeña desviación)

$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^{\mu}_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0$

$\Rightarrow \frac{dl^\mu}{d\lambda} \approx -\Gamma^{\mu}_{\rho\sigma} k^\rho k^\sigma$

2) $\frac{dl^i}{d\lambda} \approx -\Gamma^i_{00} k^2 - \Gamma^i_{j\ell} k^j k^\ell \approx -2\partial\phi k^2 + 2k^i (\vec{k} \cdot \vec{\nabla}\phi) \equiv -2k^2 \nabla_\perp^i \phi$

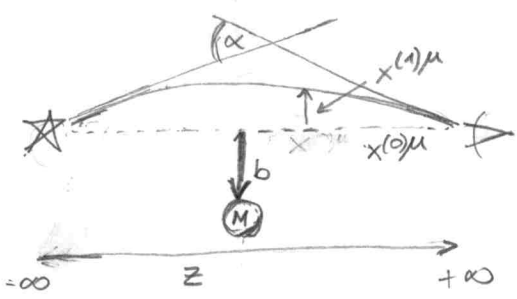
$\vec{\nabla}_\perp \phi = \vec{\nabla}\phi - \hat{k}(\hat{k} \cdot \vec{\nabla}\phi)$

Deflexión: $\Delta \vec{\ell} = -2k^2 \int \vec{\nabla}_\perp \phi d\lambda = -2k \int_{-\infty}^{+\infty} \vec{\nabla}_\perp \phi dz$

$\vec{\alpha} \equiv \frac{\vec{\ell}_{in} - \vec{\ell}_{out}}{k} = 2 \int_{-\infty}^{+\infty} dz \nabla_\perp \phi$

1) $0 = (\eta_{\mu\nu} + h_{\mu\nu})(k^\mu + l^\mu)(k^\nu + l^\nu) \Rightarrow 2\eta_{\mu\nu} k^\mu l^\nu + h_{\mu\nu} k^\mu k^\nu = 0 \Rightarrow \vec{k} \cdot \vec{l} = k l^0 + 2k^2 \phi$
 $-2\phi(k^2 + \vec{k}^2)$

$\frac{dl^0}{d\lambda} \approx -2\Gamma^0_{0i} k^0 k^i = -2k \vec{k} \cdot \vec{\nabla}\phi \Rightarrow l^0 = -2k \int \vec{k} \cdot \vec{\nabla}\phi d\lambda = -2k\phi$ (por $\phi=0$)
 $\frac{d\vec{x}}{d\lambda} \cdot \vec{\nabla}\phi = \frac{d\phi}{d\lambda}$
 $\Rightarrow \boxed{\vec{k} \cdot \vec{l} = 0}$



$\phi = -\frac{GM}{r} \approx -\frac{GM}{\sqrt{z^2 + b^2}}$

$\vec{\nabla}_\perp \phi \approx \frac{\partial \phi}{\partial \vec{b}} \approx \frac{GM \vec{b}}{(z^2 + b^2)^{3/2}}$

$y = \sqrt{z^2 + b^2}$
 $dy = \frac{z}{y} dz$

$\vec{\alpha} = 2GM \vec{b} \int_{-\infty}^{+\infty} \frac{dz}{(z^2 + b^2)^{3/2}} = \frac{4GM \vec{b}}{b^2} \int_0^\infty \frac{dy}{(1+y^2)^{3/2}}$

$= \frac{4GM}{b} \hat{b} \int_0^\infty dy \frac{d \frac{y}{\sqrt{1+y^2}}}{dy \sqrt{1+y^2}}$

$= \frac{1}{\sqrt{1+y^2}} - \frac{y^2}{(1+y^2)^{3/2}} = \frac{1}{(1+y^2)^{3/2}}$
 $= 1$

$|\vec{\alpha}| = \frac{4GM}{b}$

En grandad Newtoniana, el angulo de deflección por una partícula de masa m estaba $\alpha \approx \frac{2GM}{v_\infty^2 r_{min}}$ y es independiente de m.

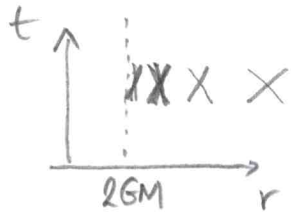
Considerando el límite $m \rightarrow 0$ y $v_\infty \rightarrow c$, obtendría 1/2 del resultado de RG. Esto es un test de RG vs Newton. Diferencia de un factor 2 sería un test de RG vs ~~otras~~ vs otros modelos de gravitación.

Medurada en 1919 por Eddington al principio y al final de una ocultación durante una eclipse:

$\alpha \approx \frac{2R_s}{R_0} \approx \frac{2.3km}{7 \times 10^5 km} \approx 10^{-5}$

Acercas de R_s

Fotón en dirección radial : $\frac{dt}{dr} = \pm \left(1 - \frac{2GM}{r}\right)^{-1}$



Los conos de luz se cierran a medida que r tiende a R_s .

$$\frac{dr}{dt} \rightarrow 0$$

Un intervalo fijo de tiempo propio $\Delta\tau_{emitted}$ (e.g. entre los impulsos emitidos por una baliza) es medido a r_{obs} como $\Delta t_{obs} = \frac{1 - R_s/r_{obs}}{1 - R_s/r_{emitted}} \Delta\tau_{emitted}$.

Hay un retraso gravitatorio (mismo calcul que para $\Delta\omega$)
 $\Delta t_{obs} \rightarrow \infty$ a medida que $r \rightarrow R_s$

El observador nunca ve la baliza cruzando el horizonte.

Cuanto tiempo propio tarda? Trayectoria radial con $L=0$:

$$\frac{1}{2} \left(\frac{dr}{dt}\right)^2 + \left(\frac{1}{2} - \frac{GM}{r}\right) = \frac{E^2}{2} \Rightarrow \Delta\tau = \int \frac{dr}{\sqrt{E^2 - 1 + 2GM/r}}$$

El integrando es continuo en R_s , no pasa nada!

Tambien $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{48G^2M^2}{r^6}$ no es singular.

Tampoco es singular el determinante $g = -r^4 \sin^2\theta$.

Se pueden construir coordenadas que ~~no~~ ^{hay un singularidad} ~~están~~ ^{son} singulares en R_s .

$$\Delta t = \int_{r_0}^r dr' \frac{1}{1 - R_s/r'} = \int_{r_0}^r dr' \left(1 + \frac{R_s}{r' - R_s}\right) = r - r_0 + R_s \left(\log \frac{r - R_s}{R_s} - \log \frac{r_0 - R_s}{R_s}\right)$$

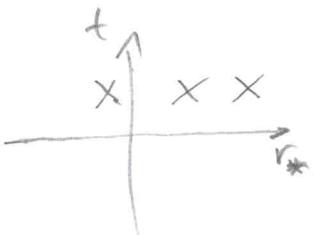
$$r_* \equiv r + R_s \log\left(\frac{r - R_s}{R_s}\right)$$

$$\Delta t = \Delta r_*$$

"Tortoise coordinates"

Los rayones de luz obedecen a $t = \pm r_*$. Necesitan un tiempo infinito Δt (en coordenadas) para llegar a R_s . $r_*(R_s) = -\infty$

La métrica se convierte en $ds^2 = \left(1 - \frac{R_s}{r}\right) (-dt^2 + dr_*^2)$



De manera similar, se puede definir (para $r > R_s$)

$$T \equiv \sqrt{\frac{r}{R_s} - 1} e^{r/2R_s} \sinh\left(\frac{t}{2R_s}\right), \quad R \equiv \sqrt{\frac{r}{R_s} - 1} e^{r/2R_s} \cosh\left(\frac{t}{2R_s}\right)$$

"coordenadas de Kruskal-Szekeres"

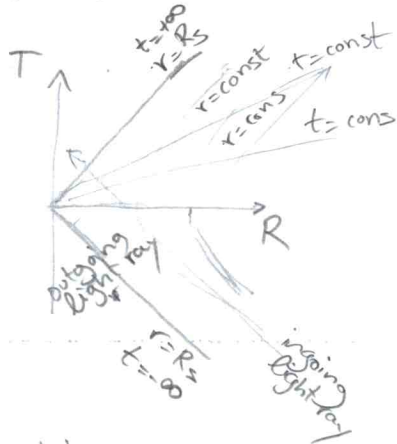
$$ds^2 = \frac{4R_s^3}{r} e^{-r/R_s} (-dT^2 + dR^2) + r^2 d\Omega^2$$

con r definido implícitamente desde

$$T^2 - R^2 = \left(1 - \frac{r}{R_s}\right) e^{r/R_s} \quad \text{y} \quad \frac{T}{R} = \tanh\left(\frac{t}{2R_s}\right)$$

-) r constante ($r > R_s$) $\Rightarrow T^2 - R^2 = \text{const.} (< 0)$ (Hyperbolas)
-) t constante $\Rightarrow T/R = \text{const}$ (lineas rectas)
-) rayones luminosos: $\frac{dT}{dR} = \pm 1 \Rightarrow T = \pm R + C$
-) ~~para~~ ^{si} $r \rightarrow R_s$, $T^2 = R^2 \Rightarrow T = \pm R$
tambien ~~para~~ ^{si} $t \rightarrow \pm\infty \Rightarrow$

$r = R_s = \text{const}$ es un cono de luz en (t, r) , y tarda $t = \infty$ para llegar a i



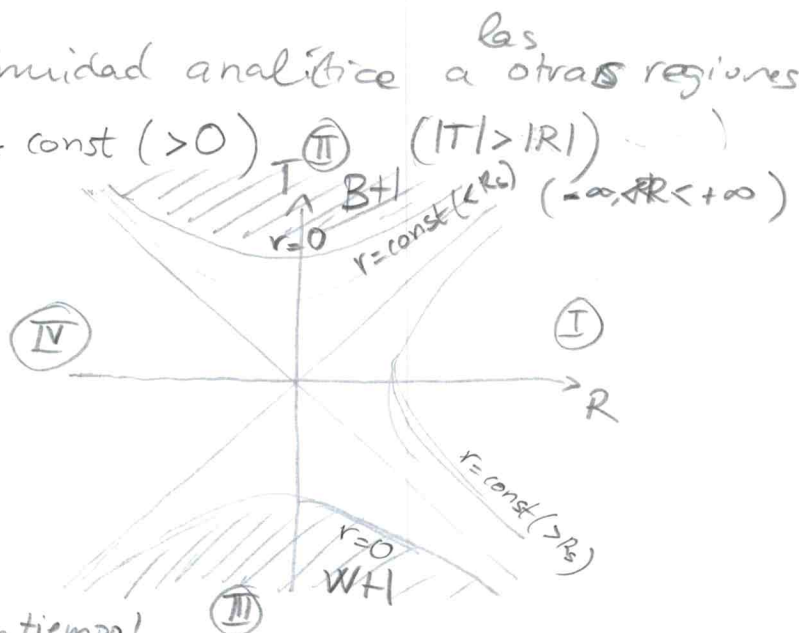
Se pueden extender para continuidad analítica a ^{las} otras regiones

-) $r = \text{const.} (r < R_s)$: $T^2 - R^2 = \text{const.} (> 0)$ (II) ($|T| > |R|$)
 -) $r = 0$: $T^2 - R^2 = 1$
- $$T \equiv \sqrt{1 - \frac{r}{R_s}} e^{r/2R_s} \cosh\left(\frac{t}{2R_s}\right)$$
- $$R \equiv \sqrt{1 - \frac{r}{R_s}} e^{r/2R_s} \sinh\left(\frac{t}{2R_s}\right)$$

Todas curvas de tipo tiempo en (II) acaban en $r = 0$

La singularidad es en el futuro de todas geodesicas !! Para $r < R_s$

El tiempo se convierte en esp., y el esp. en tiempo!



Ondas gravitacionales

EM: Φ Coulomb (estática) ^{distribución de cargas} \rightarrow Maxwell (dinámica/ondas)

GR: Poisson \rightarrow Einstein

Primero paso ~~para~~ ^{de} una descripción de la gravedad como un campo, en el que el gravitón es el mediador de la interacción (como el fotón en EM)

Perturbaciones en el régimen de weak field:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad |h_{\mu\nu}| \ll 1 \quad \Rightarrow \quad g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu}$$

$$g_{\mu\nu} g^{\nu\rho} = (\eta_{\mu\nu} + h_{\mu\nu})(\eta^{\nu\rho} - h^{\nu\rho}) \approx \eta_{\mu\nu} \delta^{\nu\rho} + h_{\mu\nu} \delta^{\nu\rho} - h_{\mu\nu} \delta^{\nu\rho} + \mathcal{O}(h^2)$$

Los índices se bajan y suben con $\eta^{\mu\nu}$ y $\eta_{\mu\nu}$: $h_{\mu}^{\nu} = \eta^{\nu\sigma} h_{\mu\sigma}$, $h \equiv h_{\mu}^{\mu}$

$$T_{\mu\nu}^p \approx \frac{1}{2} \eta^{\rho\sigma} (\partial_{\mu} h_{\nu\rho} + \partial_{\nu} h_{\mu\rho} - \partial_{\rho} h_{\mu\nu}) \quad T_{\rho\nu}^p \approx \frac{1}{2} \eta^{\rho\sigma} \partial_{\nu} h_{\rho\sigma} \approx \frac{1}{2} \partial_{\nu} h \quad \uparrow \text{trace}$$

$$R_{\mu\nu} \approx \partial_{\rho} T_{\mu\nu}^p - \partial_{\nu} T_{\mu\rho}^p + \mathcal{O}(h^2) \approx \partial^{\rho} \partial_{\mu} h_{\nu\rho} - \frac{1}{2} \partial_{\rho} \partial^{\rho} h_{\mu\nu} - \frac{1}{2} \partial_{\mu} \partial_{\nu} h \quad \square \equiv \partial_{\mu} \partial^{\mu}$$

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R \approx \partial^{\rho} \partial_{\mu} h_{\nu\rho} - \frac{1}{2} \square h_{\mu\nu} - \frac{1}{2} \partial_{\mu} \partial_{\nu} h - \frac{\eta_{\mu\nu}}{2} (\partial^{\rho} \partial^{\sigma} h_{\rho\sigma} - \square h)$$

Está conveniente ~~de~~ introducir $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{\eta_{\mu\nu}}{2} h$ (con $\bar{h} = h - 2h = -h$)

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{\eta_{\mu\nu}}{2} \bar{h}$$

~~G_{μν}~~

$$\Rightarrow G_{\mu\nu} \approx \partial^{\rho} \partial_{\mu} \bar{h}_{\nu\rho} - \frac{1}{2} \square \bar{h}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \partial^{\rho} \partial^{\sigma} \bar{h}_{\rho\sigma} = 8\pi G T_{\mu\nu}$$

~~Toro pert~~ Dos perturbaciones $h_{\mu\nu}$ y $\tilde{h}_{\mu\nu}$ relacionadas por un cambio de coordenadas deben describir la misma física. (un diffeomorfismo)

Difeomorfismo tiene que ser también pequeño: $\tilde{x}^{\mu} \equiv x^{\mu} - \xi^{\mu}$
 $\partial \tilde{x}^{\mu} / \partial x^{\nu} = \partial x^{\mu} / \partial x^{\nu} - \partial_{\nu} \xi^{\mu}$
 $\partial x^{\mu} / \partial \tilde{x}^{\nu} = \partial x^{\mu} / \partial x^{\nu} + \partial_{\nu} \xi^{\mu}$

$$\tilde{g}_{\mu\nu} \equiv \eta_{\mu\nu} + \tilde{h}_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} (\eta_{\alpha\beta} + h_{\alpha\beta}) \approx \eta_{\mu\nu} + h_{\mu\nu} + \partial_{\mu} \xi_{\nu} + \partial_{\nu} \xi_{\mu}$$

$$\Rightarrow \boxed{h_{\mu\nu} \rightarrow \tilde{h}_{\mu\nu} = h_{\mu\nu} + \partial_{\mu} \xi_{\nu} + \partial_{\nu} \xi_{\mu}}$$

bajo diffeomorfismo infinitesimal

Como la física es la misma, esta es la libertad de gauge de la GR linealizada. Similar a lo que pasa en EM cuando

$$A_\mu \rightarrow A_\mu + \partial_\mu \chi$$

$A_\mu + \partial_\mu \chi$ obedece a la misma ecuación que A_μ . Si A_μ es una solución, también es $A_\mu + \partial_\mu \chi$

Similarmente, $\bar{h}_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \frac{\eta_{\mu\nu}}{2} (\partial_\rho \xi^\rho) = \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \frac{\eta_{\mu\nu}}{2} \partial_\rho \xi^\rho$

$$\partial^\nu \bar{h}_{\mu\nu} \rightarrow \partial^\nu \bar{h}_{\mu\nu} + \cancel{\partial_\mu \partial^\nu \xi_\nu} + \partial^\nu \partial_\nu \xi_\mu - \cancel{\partial_\mu \partial^\nu \xi^\rho}$$

Buscando ξ_μ de ahí que $\square \xi_\mu = -\partial^\nu \bar{h}_{\mu\nu}$, podemos imponer que

$$\boxed{\partial^\nu \bar{h}_{\mu\nu} = 0}$$

Similar al gauge de Lorenz $\partial^\nu A_\nu = 0$ en EM, obtenido resolviendo

$$\square \chi = -\partial^\nu A_\nu$$

En este gauge las ecuaciones de Einstein simplifican como

$$\boxed{\square \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}}$$

lo que es una ecuación de onda con un source. En el exterior, donde $T_{\mu\nu} = 0$, hay propagación en el vacío.

$$\bar{h}_{\mu\nu} = \underset{\substack{\uparrow \\ \text{tensor de polarización}}}{\epsilon_{\mu\nu}} e^{ik_\alpha x^\alpha}$$

$$\square \bar{h}_{\mu\nu} = 0 \Rightarrow \boxed{k_\alpha k^\alpha = 0}$$

se propagan a la velocidad de la luz!

Gauge de Lorenz: $\boxed{k^\nu \epsilon_{\mu\nu} = 0}$

Cuatro ecuaciones ^{algebraicas} para 10 grados de libertad. Quedan 6

Todavía ~~puedo~~ hay libertad residual de gauge para hacer difeomorfismos

con $\square \xi_\mu = 0$. Bajo este dif., $\delta \bar{h}_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial_\rho \xi^\rho \Rightarrow \square \delta \bar{h}_{\mu\nu} = 0$

No violan el gauge de Lorenz (como en EM hay libertad residual con $\square \chi = 0$, que preserva $\partial_\mu A^\mu = 0$) $\Rightarrow 10 - 4 - 4 = 2$

$$\xi_\mu = \hat{\xi}_\mu e^{ik_\alpha x^\alpha} \Rightarrow \delta \epsilon_{\mu\nu} = i(k_\mu \hat{\xi}_\nu + k_\nu \hat{\xi}_\mu - \eta_{\mu\nu} k_\rho \hat{\xi}^\rho)$$

$$\delta \epsilon \equiv \delta \epsilon_{\mu\nu} \eta^{\mu\nu} = -2ik \cdot \hat{z}$$

$$\delta \epsilon_{0i} = i(k_0 \hat{z}_i + k_i \hat{z}_0)$$

Esta libertad ~~ad~~residual puede utilizarse para imponer

$$\boxed{\epsilon = 0} \quad (\Rightarrow \bar{h}_{\mu\nu} = h_{\mu\nu}) \quad \text{y} \quad \boxed{\epsilon_{0i} = 0}$$

De esta condición y el ~~gauge~~ gauge de Lorentz se obtiene

$$0 = k^\mu \epsilon_{0\mu} = k^0 \epsilon_{00} \quad \Rightarrow \quad \boxed{\epsilon_{00} = 0}$$

Consideremos una onda en dirección de \hat{z} : $k^\mu = (\omega, 0, 0, \omega)$

$$0 = k^\mu \epsilon_{\mu\nu} = \omega(\epsilon_{0\nu} + \epsilon_{3\nu}) \quad \Rightarrow \quad \epsilon_{3\nu} = 0$$

$$\epsilon_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon_{11} & \epsilon_{12} & 0 \\ 0 & \epsilon_{12} - \epsilon_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

"transverse traceless (TT) gauge"

Hay dos componentes independientes para los 2 grados de libertad ("polarización")

$$\epsilon_{+, \mu\nu} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\epsilon_{x, \mu\nu} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

La solución general es una combinación lineal (parte real)

$$h_{\mu\nu}^{TT} = \bar{h}_{\mu\nu}^{TT} = \text{Re} \left[(c_+ \epsilon_{+, \mu\nu} + c_x \epsilon_{x, \mu\nu}) e^{ik_\alpha x^\alpha} \right] \quad c_+, c_x \in \mathbb{C}$$

Dado que $c_+ = h_+ e^{i\phi_1}$, $c_x = h_x e^{i\phi_2}$ y $k_\alpha x^\alpha = \omega(t-z)$

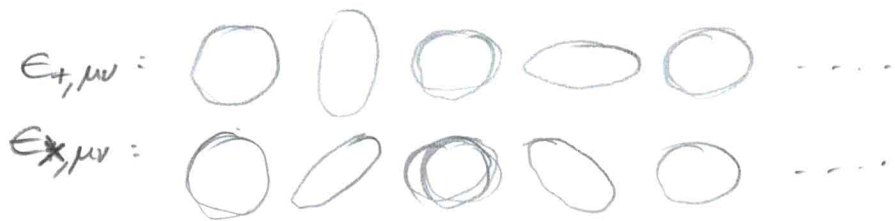
$$\bar{h}_{\mu\nu}^{TT} = h_+ \epsilon_{+, \mu\nu} \cos(\omega(t-z) + \phi_1) + h_x \epsilon_{x, \mu\nu} \cos(\omega(t-z) + \phi_2)$$

Ponemos por simplicidad $\phi_1 = \phi_2 = 0$, $ds^2 = (\eta_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu$

$$ds^2 = -dt^2 + \left[1 + h_+ \cos(\omega(t-z)) \right] dx^2 + \left[1 - h_+ \cos(\omega(t-z)) \right] dy^2 + 2h_x \cos(\omega(t-z)) dx dy + dz^2$$

La perturbación solo actúa sobre vectores en el ^{plano} $x-y$, sin afectar a los en dirección \hat{z} .

Los vectores propios de $E_{+,\mu\nu}$ son $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, y los de $E_{x,\mu\nu}$ son $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
 La acción sobre el círculo unitario es



sin cambiar el área, dado que $\bar{E}_{\mu\nu} = 0$.
 Se pueden introducir polarizaciones circulares:

$$E_{\mu\nu}^L = \frac{E_{+,\mu\nu} + iE_{x,\mu\nu}}{\sqrt{2}} \quad E_{\mu\nu}^R = \frac{E_{+,\mu\nu} - iE_{x,\mu\nu}}{\sqrt{2}}$$

de ahí que la restricción a x-y parezca $E_{\mu\nu}^L = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$, $E_{\mu\nu}^R = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$

Bajo una rotación $R^\mu_\nu = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, se ~~lee~~ obtiene

$$E_{x,\mu\nu} \rightarrow R^\alpha_\mu R^\beta_\nu E_{x,\alpha\beta} = (R^T E_x R)_{\mu\nu} = \begin{pmatrix} -\sin 2\theta & \cos 2\theta \\ \cos 2\theta & \sin 2\theta \end{pmatrix}$$

$$E_{+,\mu\nu} \rightarrow (R^T E_+ R)_{\mu\nu} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

$$\Rightarrow E_{\mu\nu}^L \rightarrow e^{-2i\theta} E_{\mu\nu}^L, \quad E_{\mu\nu}^R \rightarrow e^{2i\theta} E_{\mu\nu}^R$$

Vuelven a si mismas ~~después~~ con $\theta = \pi \Rightarrow$ spin 2!

GR es una teoría de partículas sin masa y spin 2, cuya invariancia de gauge es la invariancia bajo difeomorfismos.

Nota $R^\alpha_{\beta\mu\nu}$ (y por lo tanto $R_{\mu\nu}$) es invariante bajo transf. infinitesimales $h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$. Es como en EM, donde

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ es invariante bajo $A_\mu \rightarrow A_\mu + \partial_\mu \chi$.

Tampoco $T_{\mu\nu}$ muda, porque tiene que ser evaluado con $g_{\mu\nu} \times \eta_{\mu\nu}$

Producción de GW

$$\square \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$$

No se puede utilizar la misma procedura porque $e^{ik_\mu x^\mu}$ no es una solución con $T_{\mu\nu} \neq 0$. Pero ~~es lo mismo~~ que en EM. tiene la misma función de Green

$$\square_x G(x^\mu - y^\mu) = \underbrace{\delta_D^{(4)}(x^\mu - y^\mu)}_{\delta_D(x^0 - y^0) \delta_D^{(3)}(\vec{x} - \vec{y})}$$

La solución general es

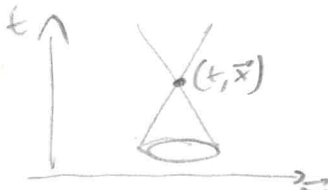
$$\bar{h}_{\mu\nu}(x) = -16\pi G \int d^4y G(x-y) T_{\mu\nu}(y)$$

La función de Green retardada es

$$G(x-y) = - \frac{\overset{\leftarrow \text{Heaviside}}{\Theta_4(x^0 - y^0)}}{4\pi |\vec{x} - \vec{y}|} \delta_D(x^0 - y^0 - |\vec{x} - \vec{y}|)$$

$$\Rightarrow \bar{h}_{\mu\nu} = 4G \int d^3y \frac{T_{\mu\nu}(t - |\vec{x} - \vec{y}|, \vec{y})}{|\vec{x} - \vec{y}|}$$

Recibe contribución de todo el cono de luz pasado:



Consideremos fuentes que sean pequeñas, aisladas y lejanas.

Transformada de Fourier en el tiempo:

$$f(t, \vec{x}) = \frac{1}{\sqrt{2\pi}} \int d\omega e^{-i\omega t} \hat{f}(\omega, \vec{x}), \quad \hat{f}(\omega, \vec{x}) = \frac{1}{\sqrt{2\pi}} \int dt e^{i\omega t} f(t, \vec{x})$$

$$\hat{h}_{\mu\nu}(\omega, \vec{x}) = \frac{4G}{\sqrt{2\pi}} \int dt d^3y e^{i\omega t} \frac{T_{\mu\nu}(t - |\vec{x} - \vec{y}|, \vec{y})}{|\vec{x} - \vec{y}|} = 4G \int d^3y \frac{e^{i\omega |\vec{x} - \vec{y}|}}{|\vec{x} - \vec{y}|} \hat{T}_{\mu\nu}(\omega, \vec{y})$$

Fuente pequeña y lejana: $|\vec{x} - \vec{y}| \approx r$

$$\hat{h}_{\mu\nu} \approx \frac{4G e^{i\omega r}}{r} \int d^3y \hat{T}_{\mu\nu}(\omega, \vec{y})$$

Gauge de Lorenz: $\partial_\mu \bar{h}^{\mu\nu} = -\partial^\nu \bar{h}^{\mu\mu} \Rightarrow -i\omega \hat{h}^{0\nu} = -\partial^i \hat{h}^{i\nu} \Rightarrow \boxed{\hat{h}^{0\nu} = -\frac{i}{\omega} \partial^i \hat{h}^{i\nu}}$

Solo necesitamos la parte espacial de \bar{h}_{ij} .

Dado que $\bar{h}^{\mu\nu} = \eta^{\mu\alpha} \eta^{\nu\beta} \bar{h}_{\alpha\beta} \Rightarrow \bar{h}^{00} = \bar{h}_{00}, \bar{h}^{ij} = \bar{h}_{ij}, \bar{h}^{i0} = -\bar{h}_{i0}$, y el mismo para $T_{\mu\nu}$.

Conservación de $T_{\mu\nu}$, $\partial_\mu T^{\mu\nu} = 0 \Rightarrow \boxed{\partial_i \hat{T}^{vi} = i\omega \hat{T}^{v0}}$

$$\int d^3y \hat{T}^{ij}(\omega, \vec{y}) = \int d^3y \left[\underbrace{\partial_k (y^i \hat{T}^{kj})}_{\text{boundary, no contribuye para fuente localizada}} - y^i \underbrace{\partial_k \hat{T}^{kj}}_{= i\omega \hat{T}^{0j}} \right] = -i\omega \int d^3y y^i \hat{T}^{0j}$$

\hat{T}_{ij} es simétrico:

$$\int d^3y \hat{T}^{ij} = -\frac{i\omega}{2} \int d^3y (y^i \hat{T}^{0j} + y^j \hat{T}^{0i}) = +\frac{i\omega}{2} \int d^3y y^i y^j \partial_k \hat{T}^{0k} = -\frac{\omega^2}{2} \int d^3y y^i y^j \hat{T}^{00}$$

$$= \partial_k (y^i y^j \hat{T}^{0k}) - y^i y^j \partial_k \hat{T}^{0k}$$

$$\hat{h}_{ij} = -2G\omega^2 \frac{e^{i\omega r}}{r} \hat{I}_{ij}(\omega)$$

con $\boxed{I_{ij}(t) \equiv \int d^3y y^i y^j T^{00}(t, \vec{y})}$

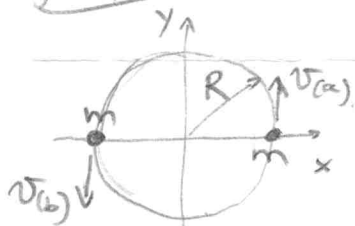
Momento de cuadrupolo de la densidad de energía

y $\hat{I}_{ij} \stackrel{\sim}{=} \frac{1}{\sqrt{2\pi}} \int dt e^{+i\omega t} I_{ij}(t)$

Tras una otra transformada de Fourier:

$$\bar{h}_{\mu\nu} = \frac{2G}{r} \frac{d^2 I_{ij}(t-r)}{dt^2} \quad t_{ret} = t-r$$

Ejemplo Sistema binario (BHs, estrellas de neutrones)



$$x(a) = R \cos \Omega t = -x(b)$$

$$y(a) = R \sin \Omega t = -y(b)$$

$$z(a) = z(b) = 0$$

$$I_{11} = 2m x_{(a)}^2 = 2m R^2 \cos^2 \Omega t = m R^2 (1 + \cos 2\Omega t)$$

$$I_{22} = 2m y_{(a)}^2 = 2m R^2 \sin^2 \Omega t = m R^2 (1 - \cos 2\Omega t)$$

$$I_{12} = 2m x_{(a)} y_{(a)} = 2m R^2 \cos \Omega t \sin \Omega t = m R^2 \sin 2\Omega t$$

$$(I_{3i} = I_{i3} = 0)$$

$$\ddot{I}_{11} = -4m\Omega^2 R^2 \cos 2\Omega t = -\ddot{I}_{22}, \quad \ddot{I}_{12} = -4m\Omega^2 R^2 \sin 2\Omega t$$

$$\bar{h}_{ij}(t, \vec{x}) = \frac{8Gm}{r} \Omega^2 R^2 \begin{bmatrix} -\cos 2\Omega t_{ret} & -\sin 2\Omega t_{ret} & 0 \\ -\sin 2\Omega t_{ret} & \cos 2\Omega t_{ret} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Utilizemos la aproximación Newtoniana por el sistema binario y órbitas circulares: ($M_{TOT} = 2m$, $|\vec{x}_a - \vec{x}_b| = 2R$)

$$\frac{GM_{TOT}}{|\vec{x}_a - \vec{x}_b|^2} = \frac{L^2}{|\vec{x}_a - \vec{x}_b|^3} = \Omega^2 |\vec{x}_a - \vec{x}_b| \Rightarrow \Omega^2 = \frac{Gm}{4R^3}$$

Interferómetros de GW eg. LIGO, Virgo

Estudian BH-BH sistemas binarios.

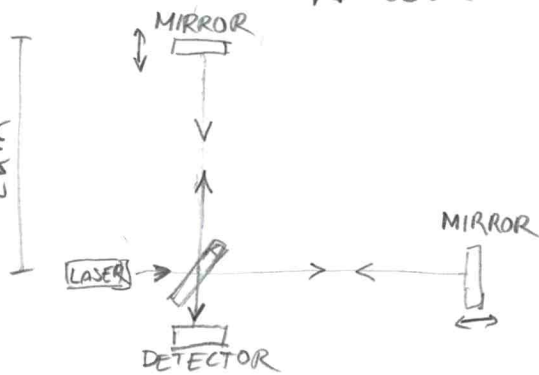
$$m \times 10^6 M_\odot \Rightarrow R_S \approx 10^6 \text{ cm}$$

Gravedad Newtoniana ok para $R \approx 10R_S \approx 10^7 \text{ cm}$

Distancia típica: $r \approx 100 \text{ Mpc} \approx 10^{26} \text{ cm}$

$$\text{Frecuencia: } \Omega \approx \sqrt{\frac{Gm}{4R^3}} \approx \sqrt{\frac{R_S}{R}} \frac{c}{2\sqrt{2}R} \approx 100 \text{ Hz}$$

Tamaño de h esta $h \sim \frac{8GM}{r} \Omega^2 R^2 \approx \frac{R_S^2}{rR} \approx 10^{-21} !!$



Los espejos se mueven al pasar de la onda.

Las posiciones en coordenadas NO mudan, si están inicialmente en reposo (~~de~~ con $dx^i/dt = 0$)

$$\frac{dx^i}{dt^2} = -\Gamma_{00}^i \left(\frac{dx^0}{dt}\right)^2, \text{ pero } \Gamma_{00}^i = 0 \text{ en TT gauge.}$$

$$\Rightarrow dx^i/dt = 0 \text{ siempre}$$

La distancia en coordenadas Δx y Δy entre espejos y detector permanece constante. Lo que cambia es la distancia propia

$$\Delta s_1 = \sqrt{1 + h \cos(\omega(t-z))} \Delta x, \quad \Delta s_2 = \sqrt{1 - h \cos(\omega(t-z))} \Delta y$$

Estamos asumiendo que la onda tiene polarización +, y el detector ^{se encuentra} ~~está~~ perpendicular a la dirección de propagación.

Los fotones del laser viajan por trayectorias rectas, y su tiempo de viaje es $\Delta t_1 = 2\Delta s_1$ o $\Delta t_2 = 2\Delta s_2$.

$$\left(\frac{2\Delta s_1}{c} \approx \frac{2\Delta x}{c} \approx \frac{4 \text{ km}}{3 \times 10^8 \text{ km/s}} \approx 10^{-5} \text{ s} \ll \Omega^{-1} \approx 10^{-2} \text{ s. los espejos permanecen casi fijos}\right)$$

Diferencia en tiempos de viaje:

$$\Delta t_1 - \Delta t_2 \approx 2\Delta x \left(1 + \frac{h}{2} \cos(\dots)\right) - 2\Delta y \left(1 - \frac{h}{2} \cos(\dots)\right) \approx 2(\Delta x - \Delta y) + (\Delta x + \Delta y) h \cos(\dots)$$

El interferómetro está calibrado de ahí que $2(\Delta x - \Delta y) = n\lambda$
($\lambda =$ longitud de onda del laser) y no hayan franjas de interferencia
cuando no hay GW. Como la GW pasa, el máximo señal se
(interferencia destructiva) se ~~hay~~ ^{alcanza} para $\delta L \equiv (\Delta x + \Delta y)h \approx \lambda/2$

El desplazamiento propio del espejo es

$$\delta L \sim 10^{-16} \text{ cm} \left(\frac{h}{10^{-21}} \right) \left(\frac{\Delta x}{\text{km}} \right) \quad \text{muy pequeño !!}$$

Mucho ~~mas~~ ^{menor} pequeño que el tamaño de un átomo ($\sim 10^{-8} \text{ cm}$)

Δx ~~tiene~~ ^{hay} que ser lo más grande posible para compensar.

En LIGO hay un espejo adicional de ahí que los fotones reboten
 ~ 300 veces en cada brazo antes de salir.

La longitud efectiva es $\sim 300 \otimes \times 4 \text{ km} \approx 1200 \text{ km}$!

~~Los~~ En el futuro, se construirán detectores en el espacio (e.g. LISA)

Cosmologia

Solutions based on the assumption that the Universe is spatially homogeneous and isotropic.

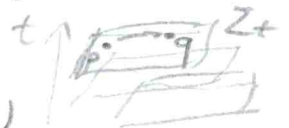
Isotropy: experimentally justified by isotropic distribution of galaxies, α and γ ray emission, CMB

Homogeneity: Copernican principle. ^{There is no special location.} More of a philosophical assumption than an experimental fact. We cannot go somewhere else to check it!

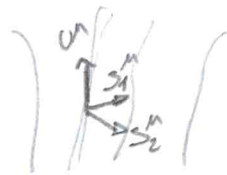
This means invariance under space translations and rotations. NOT time translations.

These assumptions are justified only on very large scales. Locally the Universe is very inhomogeneous and anisotropic.

HOMOGENEITY Spacetime can be foliated by a one-parameter family of space-like hypersurfaces Σ_t , so that $\forall t$ and $\forall p, q \in \Sigma_t$ there is an isometry moving p to q



ISOTROPY There is a set of time-like curves (observers) with tangent u^μ filling the space-time, such that at any point p $\forall s_1^\mu, s_2^\mu \in V_p$ orthogonal to u^μ (space-like) there is an isometry that leaves p, u^μ fixed but rotates s_1^μ into s_2^μ . There is no preferred space-like direction orthogonal to u^μ



At each point, only one observer can see isotropy. A boosted observer would see a preferred direction.

HOM + ISO: Σ_t must be orthogonal to u^μ

Consider the restriction $\Rightarrow ds^2 = -dt^2 + R^2(t) \gamma_{ij}(u) du^i du^j$

(u^1, u^2, u^3) : coordinates on Σ_t "comoving" $R(t)$: scale factor

Construct 3-Riemann from γ_{ij} and raise one index: ${}^{(3)}R_{ik}{}^{jl}$ Antisymm. in ij, kl , symm. in $ij \leftrightarrow kl$.

Linear map from vector space W of two forms (antisymmetric $(0,2)$ tensors) into itself: $L: W \rightarrow W$

L is symmetric, so L has an orthonormal basis of eigenvectors. The eigenvalues must be all equal (or there would be a preferred direction): $L \propto \mathbb{1}$

k cannot depend on position either (homogeneity)

$${}^{(3)}R_{ij}{}^{kl} = k(\delta_i^k \delta_j^l - \delta_i^l \delta_j^k)$$

$${}^{(2)}R_{ijke} = k(\delta_{ik} \delta_{je} - \delta_{ij} \delta_{ke})$$

$${}^{(2)}R_{ijk} = k(3-1)\gamma_{ik} = 2k\gamma_{ik}$$

$${}^{(3)}R = 6k$$

Like in Schwarzschild, $ds^2 \equiv \gamma_{ij} du^i du^j = e^{2\beta(r)} dr^2 + r^2 d\Omega^2$

$${}^{(3)}R_{rr} = \frac{2}{r} \partial_r \beta$$

$${}^{(2)}R_{\theta\theta} = e^{-2\beta} (r \partial_r \beta - 1) + 1$$

and ${}^{(3)}R_{rr} = 2ke^{2\beta}$, ${}^{(3)}R_{\theta\theta} = 2kr^2$ (isotropy)

$$\frac{2}{r} \partial_r \beta = 2ke^{2\beta}, \quad e^{-2\beta} (r \partial_r \beta - 1) + 1 = 2kr^2$$

$$\Rightarrow kr^2 - e^{-2\beta} + 1 = 2kr^2 \quad \Rightarrow \quad e^{2\beta} = \frac{1}{1 - kr^2}$$

$$ds^2 = -dt^2 + R^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right)$$

For $k=0$, $ds^2 = dx^2 + dy^2 + dz^2$. For $k \neq 0$, we can reabsorb $|k|$ with $r \rightarrow \sqrt{|k|} r$, $R(t) \rightarrow R(t)/\sqrt{|k|}$, so that we only consider $k = \{+1, 0, -1\}$ called closed, flat, open.

Another ^{useful} interpretation:

$$d\chi \equiv \frac{dr}{\sqrt{1 - kr^2}}$$

$$\frac{dr}{d\chi} = \sqrt{1 - kr^2}$$

$$r = S_k(\chi) = \begin{cases} \sin(\chi), & k=1 \\ \chi, & k=0 \\ \sinh(\chi), & k=-1 \end{cases}$$

$$ds^2 = -dt^2 + R^2(t) (d\chi^2 + S_k^2(\chi) d\Omega^2)$$

$k=1$ ds^2 is the metric of the ^{unit} 3-sphere, hypersurface in \mathbb{R}^4 defined by $x^2 + y^2 + z^2 + w^2 = R^2$ euclidean

$$w = R \cos \chi, \quad z = R \sin \chi \cos \theta, \quad y = R \sin \chi \sin \theta \sin \phi, \quad x = R \sin \chi \sin \theta \cos \phi$$

$$ds^2 = \frac{dw^2 + dz^2 + dy^2 + dx^2}{R^2} = d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)$$

$k=-1$ ds^2 is the metric of a 3-hyperboloid, hypersurf in Lorentzian \mathbb{R}^4

$$w^2 - x^2 - y^2 - z^2 = +R^2$$

$$w = R \cosh \chi, \quad z = R \sinh \chi \cos \theta, \dots$$

$$ds^2 = \frac{dw^2 - dx^2 - dy^2 - dz^2}{R^2} = d\chi^2 + \sinh^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)$$

It is common to work with $a(t) = R(t)/R_0$, $k = k/R_0^2$ in

$$\boxed{ds^2 = -dt^2 + a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right)} \quad \text{"FLRW metric"}$$

^{4D} See Christoffel symbols on Carroll. Ricci tensor:

$$R_{00} = -\frac{3\ddot{a}}{a} \quad R_{11} = \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1 - kr^2} \quad R_{22} = r^2(a\ddot{a} + 2\dot{a}^2 + 2k) \quad R_{33} = R_{22} \sin^2 \theta$$

$$R = 6 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right] \quad (\text{Ricci scalar})$$

Friedmann equations

The energy-momentum tensor must be homogeneous and isotropic in the same frame. For observer at rest, $u^\mu = (1, 0, 0, 0)$ we need want $T_{\mu\nu} u^\mu u^\nu = \rho$ and $T_{\mu\nu} s^\mu s^\nu = P$ (for s^μ spacelike and \perp to u^μ)

$$T_{\mu\nu} = (\rho + P) u^\mu u^\nu + P g_{\mu\nu}, \quad T = -\rho + 3P$$

or in components

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 \\ 0 & \delta_{ij} P \end{pmatrix} \quad T^\mu{}_\nu = \begin{pmatrix} -\rho & 0 \\ 0 & P \end{pmatrix}$$

$$P + \frac{\rho - 3P}{2} = \frac{\rho + P}{2}$$

Einstein eqns: $R_{\mu\nu} = 8\pi G (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T) = 8\pi G [(\rho + P) u^\mu u^\nu + (P - \frac{\rho + P}{2}) g_{\mu\nu}]$

$$\boxed{\mu\nu=00} \quad -\frac{3\ddot{a}}{a} = 4\pi G(\rho + 3P)$$

$$\boxed{\mu\nu=ij} \quad \frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2k}{a^2} = 4\pi G(\rho - P)$$

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2k}{a^2} = 4\pi G(\rho - P)$$

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2} \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)}$$

"Friedmann eqns"

$$\boxed{H \equiv \frac{\dot{a}}{a}} \quad \text{Hubble parameter}$$

Implications:

- 1) The Universe expands. The physical distance between points separated by $\Delta R = a \Delta x$ grows as $\dot{\Delta R} = \dot{a} \Delta x = \frac{\dot{a}}{a} \Delta R = H \Delta R$
Observationally, $H \approx 70 \text{ km/s/Mpc}$
- 2) The evolution depends on what type of fluid fills the Universe.
 $\ddot{a} < 0$ if $P > -\rho/3$ (deceleration), but $\ddot{a} > 0$ if $P < -\rho/3$ (acceleration!)

Differentiating 1st Friedmann eqn. ($\times a^2$):

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}(\dot{\rho}a^2 + 2\dot{a}\rho a) \quad \bullet \quad 2\dot{a}\ddot{a} = -\frac{8\pi G}{3}\dot{a}a(\rho + 3P)$$

$$\Rightarrow \boxed{\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0}$$

We get the same eqn from $\nabla_{\mu} T^{\mu 0} = 0$.

If we assume an equation of state, $P = w\rho$:

$$\dot{\rho} = -3\frac{\dot{a}}{a}(1+w)\rho \quad \Rightarrow \quad \boxed{\rho = \frac{\rho_0}{a^{3(1+w)}}}$$

For $k=0$, $\left(\frac{\dot{a}}{a}\right)^2 \propto a^{-3(1+w)} \Rightarrow \ddot{a}a^{\frac{1+3w}{2}} = \frac{d}{dt}a^{\frac{3(1+w)}{2}} = \text{const} \Rightarrow a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}$

Four important types of fluids:

"Dust": $w=0, P=0$; used to describe DM; $\rho \propto a^{-3}, a \propto t^{2/3}$

"Radiation": $w=1/3, P=\rho/3$; photons or any relativistic particle

$$\rho \propto a^{-4}, a \propto t^{1/2}$$

"Extra" power of a^{-1} because $\lambda \propto a, w \propto \frac{1}{a}$ ("cosmological redshift")

"Cosm. constant": $w=-1, P=-\rho$; vacuum energy, acceleration! $\rho = \text{const}$
 $a \propto e^{Ht}$

Differentiate 1st Friedmann equation ($\propto a^2$):

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}(\dot{\rho}a^2 + 2\dot{a}\rho a) = -\frac{8\pi G}{3}\dot{a}a(\rho + 3P)$$

$$\Rightarrow \boxed{\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0}$$

Same eq. obtained from $\nabla_{\mu} T^{\mu}_{\nu} = 0$

If we assume an equation of state, $P = w\rho$:

$$\dot{\rho} = -3(1+w)\frac{\dot{a}}{a}\rho \Rightarrow \boxed{\rho \propto a^{-3(1+w)}} \quad (*)$$

- "Dust": $w=0, P=0$ used to describe DM, $\rho \propto a^{-3}, a \propto t^{2/3}$
- "Radiation": $w=1/3, P=\rho/3$ photons, any relativistic particle $\rho \propto a^{-4}, a \propto t^{1/2}$
- "cosm. constant": $w=-1, P=-\rho$ vacuum energy $\rho = \rho_0 = \text{const}, a \propto e^{Ht}$

This ^{continuity} eqn. holds independently for all species

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \sum_i \rho_i$$

where $\rho_{\text{curv}} \equiv -\frac{3\kappa}{8\pi G a^2}$ behaves like a fluid with $w = -1/3, a \propto t$

1) The Universe expands. The physical distance ~~between~~ points separated by $\Delta r = a \Delta x$ grows as $a \Delta x = H a \Delta x = H \Delta r$

Observationally, $H \approx 70 \text{ km/s/Mpc}$

2) The evolution depends on which species fill the Universe. For standard fluids (DM, radiation) $\ddot{a} < 0$.

But if $w < -1/3$ (e.g. cosm. const. Λ) $\ddot{a} > 0$.

This seems to be the case. Why?

$$(*) \left(\frac{\dot{a}}{a}\right)^2 \propto a^{-3(1+w)} \Rightarrow \dot{a} a^{\frac{3(1+w)}{2}} = \frac{d}{dt} a^{\frac{3(1+w)}{2}} = \text{const} \Rightarrow a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}} \quad (k=0)$$

