Recap of basic physics

$\vec{r}$ (position) $\quad \vec{v} \equiv \frac{d\vec{r}}{dt}$ (velocity) $\quad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt}$ (acceleration)

$\vec{p} = m\vec{v}$ (linear momentum)

$\dot{\vec{p}} = m\vec{a} = \vec{F}$ (2nd Newton's law) $\quad \ddot{\vec{r}} = \vec{F}/m$

Newton's law ~~is~~ is valid in an inertial system (described by Cartesian coordinates)

If $\vec{F} = 0$ then $\vec{p}$ is conserved

$\ddot{r}_i = F_i/m$

Angular momentum $\quad \vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$

in components, $L_i = \varepsilon_{ijk} r_j p_k$ (repeated indices are summed)

ε_{ijk} = Levi-Civita's symbol (totally antisymmetric)

$\vec{\tau} = \vec{r} \times \vec{F}$ (torque) $\quad \tau_i = \varepsilon_{ijk} r_j F_k$

$= \vec{r} \times \dot{\vec{p}}$

$$\dot{\vec{L}} = \dot{\vec{r}} \times m\vec{v} + \vec{r} \times m\dot{\vec{v}} = m\cancel{\vec{v} \times \vec{v}} + \vec{r} \times \dot{\vec{p}} = \vec{\tau}$$

$$L_i = \varepsilon_{ijk}(\dot{r}_j p_k + r_j \dot{p}_k) = \varepsilon_{ijk}(\cancel{m \underbrace{v_j v_k}_{\text{symm.}}} + r_j \dot{p}_k) = \tau_i$$

If $\vec{\tau} = 0$ then $\vec{L}$ is conserved

Work $\quad W_{12} = \int_1^2 \vec{F} \cdot d\vec{s}$ from 1 to 2

infinitesimal displ.

$$W_{12} = \int m\dot{\vec{v}} \cdot \vec{v}\, dt = \frac{m}{2} \int \frac{d}{dt} \underbrace{(\vec{v} \cdot \vec{v})}_{\equiv v^2} dt = \frac{m}{2}(v_2^2 - v_1^2) \equiv T_2 - T_1$$

W_{12} does not depend on the path from 1 to 2 if $\vec{F}$ is conservative

$\int_1^2$ path $\iff \oint \vec{F} \cdot d\vec{s} = 0 \iff$ there is $V(\vec{r})$ such that $\vec{F} = -\vec{\nabla} V(\vec{r})$

$V(\vec{r})$ = potential

$$W_{12} = -\int_1^2 \vec{\nabla} V \cdot d\vec{s} = -\int_1^2 dV = V_1 - V_2 = T_2 - T_1$$

$$\Rightarrow T_1 + V_1 = T_2 + V_2$$

If $\vec{F}$ is conservative, then $E = T + V$ is conserved

Systems of N particles

$$\vec{F}_n = \dot{\vec{p}}_n = m^{(i)}\ddot{\vec{r}}_n = \vec{F}_n^{(ext)} + \sum_{j \neq n} \vec{F}_{n\ell} \qquad \vec{r}_n \quad n = 1, \dots, N$$

$\vec{F}_{n\ell}$: force on n due to ℓ

m_n: mass of n-th particle

$\vec{F}_{n\ell} = -\vec{F}_{\ell n}$ (Newton's third law)

$$\sum_n m_n \ddot{\vec{r}}_n = \sum_n \vec{F}_n^{(ext)} + \underbrace{\cancel{\sum_{n,\ell} \vec{F}_{n\ell}}}_{=0} = \vec{F}^{(ext)}$$

$$\vec{r}_{cm} \equiv \frac{\sum_n m_n \vec{r}_n}{\sum_n m_n} = \frac{\sum_n m_n \vec{r}_n}{M} \qquad \text{center of mass}$$

$$M\ddot{\vec{r}}_{cm} = \vec{F}^{(ext)}$$

$$\vec{P} = M\vec{v}_{cm} = M\dot{\vec{r}}_{cm} \qquad (\text{total linear momentum})$$

$$\dot{\vec{P}} = M\ddot{\vec{r}}_{cm} = \vec{F}^{(ext)}$$

If $\vec{F}^{(ext)} = 0$ (total exteral force), then $\vec{P}$ is conserved

Total angular momentum $\qquad \vec{L}_{TOT} \equiv \sum_n m_n (\vec{r}^{(i)} \times \vec{v}^{(i)})$

$$\dot{\vec{L}}_{TOT} = \sum_i m^{(i)} \left(\cancel{\vec{v}^{(i)} \times \vec{v}^{(i)}} + \vec{r}^{(i)} \times \dot{\vec{v}}^{(i)} \right) = \sum_i \vec{r}^{(i)} \times \vec{F}_i$$

$$= \sum_i \vec{r}^{(i)} \times \left(\vec{F}_i^{(ext)} + \sum_{j \neq i} \vec{F}_{ij} \right)$$

$$\vec{r}_i \times \vec{F}_{ij} + \vec{r}_j \times \vec{F}_{ji} = (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij} = 0$$

if $\vec{F}_{ij} \parallel \vec{r}_i - \vec{r}_j$

if $\vec{F}_{ij} \parallel \vec{r}_i - \vec{r}_j$ then $\dot{\vec{L}}_{TOT} = \sum_i \vec{r}_i \times \vec{F}_i^{(ext)} \equiv \vec{\tau}^{(ext)}$

if $\vec{\tau}^{(ext)} = 0$ then $\vec{L}_{TOT}$ is conserved

Go to the CoM ref. frame : $\vec{r}_i = \vec{r}_{CM} + \vec{r}_i'$, $\vec{v}_i = \vec{v}_{CM} + \vec{v}_i'$, $\vec{v}_{CM} = \dot{\vec{r}}_{CM}$

$$\vec{r}_{CM} = \frac{1}{M}\sum_i m_i(\vec{r}_{CM} + \vec{r}_i') = r_{CM} + \frac{1}{M}\sum_i m_i \vec{r}_i'$$

$$\Rightarrow \sum_i m_i \vec{r}_i' = 0$$

$$L_{TOT} = \sum_i m_i\left(\vec{r}_{CM}\times\vec{v}_{CM} + \vec{r}_{CM}\times\vec{v}_i' + r_i'\times\vec{v}_{CM} + \vec{r}_i'\times\vec{v}_i'\right)$$

$$= \underbrace{M\vec{r}_{CM}\times v_{CM}}_{\text{ang. mom. of CoM}} + \cancel{\vec{r}_{CM}\times\frac{d}{dt}\left(\sum m_i\vec{r}_i'\right)} + \cancel{\left(\sum m_i\vec{r}_i'\right)\times\vec{v}_{CM}} + \underbrace{\sum_i m_i\vec{r}_i'\times\vec{v}_i'}_{\text{ang. mom. w.r.t. CoM}}$$

<u>Total energy</u>

$$W_{12} = \sum_i \int_1^2 \underbrace{\vec{F}_i\cdot d\vec{s}_i}_{m_i\dot{\vec{v}}_i\cdot\vec{v}_i\,dt = \frac{m_i}{2}\frac{d}{dt}v_i^2\,dt} = \sum_i \frac{m_i}{2}\left(v_{i,2}^2 - v_{i,1}^2\right) \equiv T_2 - T_1$$

$$T \equiv \sum_i \frac{m_i}{2} v_i^2$$

$$= \frac{1}{2}\sum_i m_i\left(v_{CM}^2 + 2\vec{v}_{CM}\cdot\vec{v}_i' + v_i'^2\right) = \frac{M}{2}v_{CM}^2 + \sum_i \frac{m_i}{2}v_i'^2 + \cancel{\vec{v}_{CM}\cdot\frac{d}{dt}\left(\sum_i m_i r_i'\right)}$$

kin. energy of the CoM

kin. energy about the CoM

If all forces are conservative : $\vec{F}_i^{(ext)} = -\vec{\nabla} V_i^{(ext)}$

$$\sum_i \int_1^2 \vec{F}_i^{(ext)}\cdot d\vec{s}_i = \sum_i\left(V_{i,1}^{ext} - V_{i,2}^{ext}\right) \qquad \text{as before}$$

Also, suppose that there is $V_{ij} = V_{ij}(|\vec{r}_i - \vec{r}_j|)$

$$\vec{F}_{ij} = -\nabla_i V_{ij} = +\nabla_j V_{ij} = -\vec{F}_{ji}$$

$$\vec{F}_{ij} = -\nabla_i|\vec{r}_i - \vec{r}_j|\,V_{ij}^{(1)}(|\vec{r}_i - \vec{r}_j|) = -\frac{\vec{r}_i - \vec{r}_j}{|\vec{r}_i - \vec{r}_j|}V_{ij}^{(1)}(|\vec{r}_i - \vec{r}_j|) \;/\!/\; \vec{r}_i - \vec{r}_j$$

$$\sum_{i\neq j}\int_1^2 \vec{F}_{ij}\cdot ds_i = -\sum_{j>i}\int_1^2 \underbrace{\left(\vec{\nabla}_i V_{ij}\cdot d\vec{s}_i + \vec{\nabla}_j V_{ij}\cdot d\vec{s}_j\right)}_{\vec{\nabla}_i V_{ij}\cdot d(\vec{r}_i - \vec{r}_j)} = -\frac{1}{2}\sum_{i\neq j}\int_1^2 \vec{\nabla}_i V_{ij}\cdot d(\vec{r}_i - \vec{r}_j)$$

$$= \frac{1}{2}\sum_{i\neq j}\left(V_{ij,1} - V_{ij,2}\right)$$

$$E \equiv \sum_i \frac{m_i}{2}v_i^2 + \underbrace{\sum_i V_i^{(ext)} + \frac{1}{2}\sum_{i\neq j} V_{ij}}_{\equiv V} \qquad E \text{ is conserved}$$

The Lagrangian formalism

Equivalent to Newtonian mechanics, but more general.

System of N particles:

$$m_a \ddot{\vec{r}}_a = \vec{F}_a \qquad a = 1, \dots, N \quad (\text{labels particles})$$

For conservative forces:

$$\exists V(\vec{r}_1, \dots, \vec{r}_N) \;\Big|\; \vec{F}_a = -\vec{\nabla}_a V = -\frac{\partial V}{\partial \vec{r}_a}$$

Newton's law is valid in Cartesian inertial frames.

Consider a generic change of coordinates:

$$\vec{r}_a = \vec{r}_a(q_1, \dots, q_{3N}; t) \qquad (\text{the system has } 3N \text{ dof's})$$

$$q_\alpha = q_\alpha(\vec{r}_1, \dots, \vec{r}_N; t)$$

$$\left(\sum_a\right) \frac{\partial \vec{r}_a}{\partial q_\alpha} \cdot \frac{\partial q_\beta}{\partial \vec{r}_a} = \frac{\partial q_\beta}{\partial q_\alpha} = \delta_{\alpha\beta} \qquad \left(\Rightarrow \frac{\partial q_\alpha}{\partial \vec{r}_a} = \left[\frac{\partial \vec{r}_a}{\partial q_\alpha}\right]^{-1} \right)$$

$$\dot{\vec{r}}_a = \frac{\partial \vec{r}_a}{\partial q_\alpha} \dot{q}_\alpha + \frac{\partial \vec{r}_a}{\partial t} \qquad \ddot{\vec{r}}_a = \frac{\partial \vec{r}_a}{\partial q_\alpha} \ddot{q}_\alpha + \underbrace{\frac{\partial^2 \vec{r}_a}{\partial q_\alpha \partial q_\beta} \dot{q}_\alpha \dot{q}_\beta + 2 \frac{\partial^2 \vec{r}_a}{\partial t \partial q_\alpha} \dot{q}_\alpha + \frac{\partial^2 \vec{r}_a}{\partial t^2}}_{\equiv \vec{w}_a}$$

$$\ddot{q}_\alpha = \frac{\partial q_\alpha}{\partial \vec{r}_a} \circ \left(-\vec{w}_a + \ddot{\vec{r}}_a \right) = -\left[\frac{\partial \vec{r}_a}{\partial q_\alpha}\right]^{-1} \circ \left(\frac{\partial}{\partial \vec{r}_a} \frac{V}{m_a} + \vec{w}_a \right)$$

$$\ddot{q}_\alpha \neq -\frac{\partial V}{\partial q_\alpha}$$; Newton's equations are <u>not</u> COVARIANT

We need a formalism that behaves better under coordinate transformations.

$$T \equiv \sum_n m_n \frac{\dot{\vec{r}}_n \cdot \dot{\vec{r}}_n}{2} = T(\dot{\vec{r}}_1, \dots, \dot{\vec{r}}_N) \qquad (\text{kinetic energy})$$

$$m_a \dot{\vec{r}}_a = \frac{\partial T}{\partial \dot{\vec{r}}_a} \; (= \vec{p}_a) \quad \left(\text{i.e. } m_a \dot{r}_{a,i} = \frac{\partial T}{\partial \dot{r}_{a,i}}, \; i = 1,2,3 \text{ for Cartesian coordinates} \right)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\vec{r}}_a}\right) + \frac{\partial V}{\partial \vec{r}_a} = 0 \qquad (\text{same as Newton's eq.})$$

Define the "Lagrangian" of the system: $\boxed{\mathcal{L} \equiv T - V}$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_a}\right) - \frac{\partial \mathcal{L}}{\partial \vec{r}_a} = 0$$

In Cartesian coordinates, still equivalent to Newton's law.

However:

$$0 = \left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_a}\right) - \frac{\partial \mathcal{L}}{\partial \vec{r}_a}\right]\cdot\frac{\partial \vec{r}_a}{\partial q_\alpha} = \frac{d}{dt}\left(\frac{\partial \vec{r}_a}{\partial q_\alpha}\cdot\frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_a}\right) - \frac{d}{dt}\left(\frac{\partial \vec{r}_a}{\partial q_\alpha}\right)\cdot\frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_a} - \frac{\partial \mathcal{L}}{\partial \vec{r}_a}\cdot\frac{\partial \vec{r}_a}{\partial q_\alpha}$$

$$\mathcal{L} = \mathcal{L}(\dot{q}_\alpha, q_\alpha, t) \equiv \mathcal{L}\left(\vec{r}_a(q_\alpha;t), \dot{\vec{r}}_a(q_\alpha, \dot{q}_\alpha; t)\right)$$

$$\frac{\partial \mathcal{L}}{\partial q_\alpha} = \frac{\partial \mathcal{L}}{\partial \vec{r}_a}\cdot\frac{\partial \vec{r}_a}{\partial q_\alpha} + \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_a}\cdot\frac{\partial \dot{\vec{r}}_a}{\partial q_\alpha} \qquad \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} = \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_a}\cdot\frac{\partial \dot{\vec{r}}_a}{\partial \dot{q}_\alpha}$$

$$\frac{\partial \dot{\vec{r}}_a}{\partial \dot{q}_\alpha} = \frac{\partial}{\partial \dot{q}_\alpha}\left(\frac{\partial \vec{r}_a}{\partial q_\beta}\dot{q}_\beta + \frac{\partial \vec{r}_a}{\partial t}\right) = \frac{\partial \vec{r}_a}{\partial q_\alpha}$$

$$\frac{d}{dt}\left(\frac{\partial \vec{r}_a}{\partial q_\alpha}\right) = \frac{\partial^2 \vec{r}_a}{\partial q_\alpha \partial q_\beta}\dot{q}_\beta + \frac{\partial^2 \vec{r}_a}{\partial q_\alpha \partial t} = \frac{\partial \dot{\vec{r}}_a}{\partial q_\alpha}$$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\right) - \frac{\partial \mathcal{L}}{\partial q_\alpha} = 0}$$

Euler-Lagrange equations ($\alpha = 1, \dots, 3N$)

Advantages:

1) valid in completely generic coordinate systems
2) natural implementation of constraints
3) useful to generalise Newtonian mechanics (SR, GR, QFT all have Lagrangians)

Constraints

Constraints introduce relations among some degrees of freedom. Not all dof's are independent. Not all EoM's are independent. Implemented by unknown forces (also, usually uninteresting, unless you want to know when it breaks)

Examples: 1) particle on the surface of a sphere, mass sliding on a wedge
2) particles inside a box
3) sphere rolling without sliding (displacement related to rotation)

Constraints are idealizations. In reality, ~~the~~ displacements in some directions are very tiny ($\to$ potential is very steep) and can be neglected.

Three types of constraints:

1) "holonomic" (expressed through equalities)

$$f_A(q_1, \dots, q_{3N}; t) = 0 \qquad A = 1, \dots, m < 3N$$

can also move with time

2) inequalities: $f_A(\dots) \leq 0$

3) equalities involving also velocities: $f_A(q_\alpha, \dot{q}_\alpha; t) = 0$

We will mostly deal with 1)

Choose coordinates such that

$$\underbrace{q_1, \dots, q_n}_{\text{unconstrained}}, \underbrace{q_{n+1}, \dots, q_{3N}}_{\text{constrained}}$$

for constrained ~~dofs~~ $\dot{q}_\alpha \approx \ddot{q}_\alpha = 0$

Euler-Lagrange equations are valid for all variables.

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{\alpha=1}^{3N} \left(\dot{q}_\alpha \frac{\partial}{\partial q_\alpha} + \ddot{q}_\alpha \frac{\partial}{\partial \dot{q}_\alpha} \right) \quad , \text{ but } \sum_{\alpha=1}^{3N} \approx \sum_{\alpha=1}^{n}$$

Same as starting from the EFFECTIVE Lagrangian

$$\mathcal{L}(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n; t) \equiv \mathcal{L}(q_1, \dots, q_n, q_{n+1} = \text{const}, \dots; \dot{q}_1, \dots, \dot{q}_n, \dot{q}_{n+1} = 0, \dots; t)$$

Setting $q_{\alpha > n} = \text{const.}$, $\dot{q}_{\alpha > n} = 0$ before or after taking the derivative does not matter.

Extension Velocity dependence potentials

If force is such that $\vec{F}_a = -\frac{\partial U}{\partial \vec{r}_a} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{\vec{r}}_a}\right)$
(e.g. for EM force)

then Newton's law gives $\frac{d}{dt}\left[\frac{\partial (T-U)}{\partial \dot{\vec{r}}_a} + \frac{\partial U}{\partial \vec{r}_a}\right] = 0$

Can still have $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_a} - \frac{\partial \mathcal{L}}{\partial \vec{r}_a} = 0$, with $\mathcal{L} = T - U$

Comment When is Newton's law covariant?

We need $\vec{w}_a = 0$. This happens if $r_i = M_{i\alpha} q_\alpha + v_i t + c_i$ for $M_{i\alpha}$ and v_i, c_i constant.

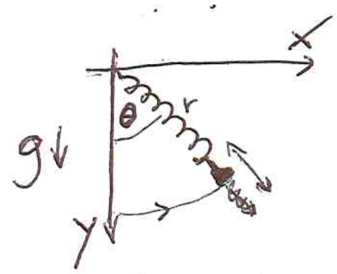
Then $\frac{\partial r_i}{\partial q_\alpha} = M_{i\alpha}$

$$\ddot{q}_\alpha \frac{\partial r_i}{\partial q_\alpha} = \ddot{r}_i = -\frac{1}{m}\frac{\partial V}{\partial r_i} = -\frac{1}{m}\frac{\partial V}{\partial q_\alpha}\frac{\partial q_\alpha}{\partial r_i}$$

$$\ddot{q}_\alpha \underbrace{\frac{\partial r_i}{\partial q_\alpha}\frac{\partial r_i}{\partial q_\beta}}_{= M_{i\alpha}M_{i\beta} = M^T_{\alpha i}M_{i\beta}} = -\frac{1}{m}\frac{\partial V}{\partial q_\alpha}\delta_{\alpha\beta} = -\frac{1}{m}\frac{\partial V}{\partial q_\beta}$$

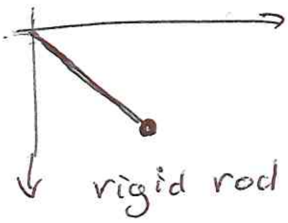
Covariant if $M^T = M^{-1}$, that is if M is a rotation.
This gives ~~g~~ Galilean ~~invariance~~ transformations.

Example (constraints) Pendulum with very rigid spring k and rest length l_0.



$q_1 = \theta$, $q_2 = r - l_0$

$\xrightarrow{\text{as } k \to 0}$



$$\begin{cases} x = (l_0 + q_2)\sin\theta \\ y = (l_0 + q_2)\cos\theta \end{cases} \qquad \begin{cases} \dot{x} = (l_0 + q_2)\cos\theta\,\dot\theta + \dot{q}_2 \sin\theta \\ \dot{y} = -(l_0 + q_2)\sin\theta\,\dot\theta + \dot{q}_2\cos\theta \end{cases}$$

$$T = m\frac{\dot{x}^2 + \dot{y}^2}{2} = \frac{m}{2}\dot{q}_2^2 + \frac{m}{2}(l_0 + q_2)^2\dot\theta^2$$

$$V = -mgy + \frac{1}{2}k(r - l_0)^2 = \frac{k}{2}q_2^2 - mg(l_0 + q_2)\cos\theta$$

$$\mathcal{L} = T - V$$

For q_2: $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_2}\right) - \frac{\partial \mathcal{L}}{\partial q_2} = m\ddot{q}_2 - m(l_0 + q_2)\dot\theta^2 - mg\cos\theta + kq_2 = 0$

as $k \to 0$, $q_2 \sim \frac{m}{k} \to 0$

For q_1: $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\right) - \frac{\partial \mathcal{L}}{\partial q_1} = \frac{d}{dt}\left[m(l_0 + \cancel{q_2})^2\dot\theta\right] + Mg(l_0 + \cancel{q_2})\sin\theta = 0$

Same as starting from $\mathcal{L} = \frac{m l_0^2 \dot\theta^2}{2} + mgl_0\cos\theta$

Equivalent Lagrangians

Newton's equations do not change if V is shifted by a constant.
If $\mathcal{L}' = \mathcal{L} + C$, Euler-Lagrange equations do not change.
Lagrangian formalism has a broader class of equivalence.

Take $F = F(q_1, \dots, q_n; t)$ (any function of coordinates)

Theorem: $\mathcal{L}$ and $\mathcal{L}' \equiv \mathcal{L} + \frac{dF}{dt}$ have the same EoM

proof:

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q_\beta}\dot{q}_\beta$$

The contribution to E-L equations is

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_\alpha}\frac{dF}{dt}\right) - \frac{\partial}{\partial q_\alpha}\frac{dF}{\partial t} = \frac{d}{dt}\left(\frac{\partial F}{\partial q_\alpha}\right) - \frac{\partial^2 F}{\partial q_\alpha \partial t} - \frac{\partial^2 F}{\partial q_\alpha \partial q_\beta}\dot{q}_\beta$$

$$= \frac{\partial^2 F}{\partial q_\beta \partial q_\alpha}\dot{q}_\beta + \frac{\partial^2 F}{\partial t \partial q_\alpha} - \frac{\partial^2 F}{\partial q_\alpha \partial t} - \frac{\partial^2 F}{\partial q_\alpha \partial q_\beta}\dot{q}_\beta = 0$$

$\mathcal{L}$ and $\mathcal{L}'$ are physically equivalent!

We can always add a total time derivative to $\mathcal{L}$

Cyclical coordinates

$p_\alpha \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}$ is called the "conjugate momentum" to q_α
in analogy to ordinary lin. momentum $\vec{p} = m\dot{\vec{r}} = \frac{\partial}{\partial \dot{\vec{r}}}\left(\frac{m\dot{\vec{r}}^2}{2}\right) = \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}}$

Theorem: if $\mathcal{L}$ does not explicitly depend on q_α, then p_α is conserved

$$\frac{\partial \mathcal{L}}{\partial q_\alpha} = 0 \Rightarrow \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} = 0 \Rightarrow \dot{p}_\alpha = 0 \Rightarrow p_\alpha \text{ is constant}$$

Coordinates that do not appear explicitely in $\mathcal{L}$ are said cyclical

$$\mathcal{L} = \mathcal{L}(q_1, \dots, \cancel{q_\alpha}, \dots, q_n; \dot{q}_1, \dots, \dot{q}_n; t)$$

(they still appear through $\dot{q}_\alpha$ ...)

Ex. 1 Free particle $\mathcal{L} = \frac{1}{2} m \dot{\vec{x}}^2$ $\quad \vec{p} = m\dot{\vec{x}}$ $\quad (V = 0)$

$$\frac{\partial \mathcal{L}}{\partial \vec{x}} = 0 \quad \Rightarrow \quad \vec{p} = \text{const}$$

(conservation of linear momentum)

Ex. 2 "central" potential in 2D $V = V(r)$ $\quad \begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases}$

$$\dot{x} = \dot{r}\cos\phi - r\sin\phi\,\dot{\phi} \qquad \dot{y} = \dot{r}\sin\phi + r\cos\phi\,\dot{\phi}$$

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) \qquad \mathcal{L} = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - V(r)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad \Rightarrow \quad p_\phi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m r^2 \dot{\phi} = \text{const}$$

but $L_z = (x\dot{y} - y\dot{x})m = (r^2\cos^2\phi\,\dot{\phi} + r^2\sin^2\phi\,\dot{\phi})m = m r^2 \dot{\phi}$

(conservation of angular momentum)

$$r: \quad p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} \qquad \frac{d}{dt}(m\dot{r}) - m r \dot{\phi}^2 + V'(r) = 0$$

$$m\ddot{r} = -V'(r) + \underbrace{m r \dot{\phi}^2}_{\text{centrifugal force}} = -V'(r) + \frac{L_z^2}{m r^3}$$

Ex. 3 Free particle in 1D with coordinate $q = e^x$

$$x = \log q \qquad \dot{x} = \frac{\dot{q}}{q} \qquad \mathcal{L} = \frac{m}{2}\dot{x}^2 = \frac{m}{2}\frac{\dot{q}^2}{q^2} \qquad \frac{\partial \mathcal{L}}{\partial \dot{q}} = m\frac{\dot{q}}{q^2} \qquad \frac{\partial \mathcal{L}}{\partial q} = -m\frac{\dot{q}^2}{q^3}$$

$$\frac{d}{dt}\left(m\frac{\dot{q}}{q^2}\right) + m\frac{\dot{q}^2}{q^3} = m\left(\frac{\ddot{q}}{q^2} - \frac{2\dot{q}^2}{q^3} + \frac{\dot{q}^2}{q^3}\right) = \frac{m}{q^2}\left(\ddot{q} - \frac{\dot{q}^2}{q}\right) = 0$$

check equivalence: $\ddot{x} = \frac{d}{dt}\left(\frac{\dot{q}}{q}\right) = \frac{\ddot{q}}{q} - \frac{\dot{q}^2}{q^2} = \frac{1}{q}\left(\ddot{q} - \frac{\dot{q}^2}{q}\right) = 0$ OK

The action principle

(or Hamilton's principle)

Consider a Lagrangian system with n unconstrained degrees of freedom (dof's) $\quad q \equiv \{q_1, \dots, q_n\} \quad \mathcal{L} = \mathcal{L}(q, \dot{q}; t)$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\right) - \frac{\partial \mathcal{L}}{\partial q_\alpha} = 0 \qquad \alpha = 1, \dots, n$$

The Euler-Lagrange (E-L) equations can be derived from a minimization problem.

Mathematical interlude (Calculus of variations)

Def: A "functional" is a function of functions. It takes functions as argument, and returns numbers.

function $f(x)$: # $\longrightarrow$ # $\qquad$ (e.g. $\mathbb{R} \to \mathbb{R}$, or $\mathbb{R}^n \to \mathbb{R}^m$)

functional $F[f]$: function $\to$ #

e.g. $F[f] \equiv \int_0^\infty dx\, f(x)$, or $F[f] \equiv f^2(0)$, or $F[f] \equiv \left.\frac{df}{dx}\right|_{x=0}$

One can minimize F with respect to f (just like f w.r.t. x), when f varies in the space of all possible functions. (with some restrictions, like e.g. integrability, continuity ...).

Consider $\quad F[y] \equiv \int_{x_A}^{x_B} f(y(x), y'(x))\, dx \quad$ for a given f

What is the shape of $y(x)$ that minimises $F[y]$, assuming fixed boundary conditions $y(x_A) = y_A$, $y(x_B) = y_B$.

Necessary condition: $F[y]$ is stationary, that is, its value does not change under

$$y(x) \to y(x) + \delta y(x)$$

at 1st order in δy, with fixed b.c. $\delta y(x_A) = \delta y(x_B) = 0$

(same as requesting $f'(x) = 0$ to find minima of f)

The "<u>variation</u>" of F is

$$\delta F \equiv F[y+\delta y] - F[y]$$

$$= \int_{x_A}^{x_B} dx \left[\underbrace{f(y(x)+\delta y(x), y'(x)+\delta y'(x)) - f(y(x), y'(x))}_{\approx \frac{\partial f}{\partial y}\delta y(x) + \frac{\partial f}{\partial y'}\delta y'(x) = \frac{\partial f}{\partial y}\delta y + \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\delta y\right) - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\delta y} \right]$$

$$\approx \int_{x_A}^{x_B} dx \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\delta y(x) + \underbrace{\frac{\partial f}{\partial y'}\delta y(x)\Big|_{x_A}^{x_B}}_{=0}$$

δF must vanish for arbitrary $\delta y(x)$ (with $\delta y_A = \delta y_B = 0$), therefore the integrand must vanish at any x

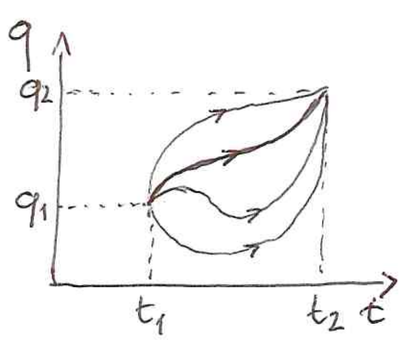
$$\delta F = 0 \iff \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0 \quad \forall x \in [x_A, x_B]$$

Back to Lagrangian systems. Define:

$$S[q] \equiv \int_{t_1}^{t_2} dt\, \mathcal{L}(q(t), \dot{q}(t); t) \qquad \text{"action"}$$

with fixed b.c. $q(t_1) = q_1$, $q(t_2) = q_2$.
(remember that $q = \{q_1, \dots, q_n\}$, so this is valid in n dimensions)

1d



$\delta q_\alpha(t_1) = \delta q_\alpha(t_2) = 0$

Note that:

1) the explicit dependence on t in $\mathcal{L}$ is not affected by the variation. Irrelevant
2) Variational problem is formulatet in terms of b.c. q_1 and q_2. Usually, evolution is solved given q_{in} and $\dot{q}_{in}$. (initial condition). Same number of conditions.

$$\delta S = \int_{t_1}^{t_2} dt \left[\frac{\partial \mathcal{L}}{\partial q_\alpha} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\right)\right]\delta q_\alpha = 0 \iff \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\right) - \frac{\partial \mathcal{L}}{\partial q_\alpha} = 0$$

Euler-Lagrange eqns are a stationary point of the action.
This follows because the δq_α are arbitrary and independent, and $\delta S = 0$ must hold for any choice of δq_α.

Modern viewpoint: the action principle ($\delta S = 0$) is the fundamental formulation of mechanics, and of physics in general. It allows to describe any physical system, not only Newtonian ones. (it is valid for QM, QFT, GR, string theory ...)

Many properties are easier to derive from the action.
For instance, the equivalence of $\mathcal{L}$ and $\mathcal{L}' \equiv \mathcal{L} + \frac{dF}{dt}$

Theorem (2nd derivation) $\mathcal{L}$ and $\mathcal{L}'$ have the same EoM

proof $$S' = \int_{t_1}^{t_2} dt \left[\mathcal{L}(q,\dot{q};t) + \frac{d}{dt}F(q,t)\right] = S + F(q(t),t)\Big|_{t_1}^{t_2}$$

but under $q_\alpha(t) \to q_\alpha(t) + \delta q_\alpha(t)$, $\delta F\Big|_{t_1}^{t_2} = 0$ because of b.c. $\delta q_\alpha(t_1) = \delta q_\alpha(t_2) = 0$

Symmetries and conservation laws

We already saw one example : q_α is cyclic $\Leftrightarrow$ $\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}$ is conserved
$(\partial\mathcal{L}/\partial q_\alpha = 0)$

Let's define a "symmetry" as a transformation of $q_\alpha, \dot{q}_\alpha$ and t that leaves $\mathcal{L}$ invariant ("$\mathcal{L}$ is invariant under the symmetry")

Ex: q cyclic
$$\begin{cases} q(t) \to q(t) + \varepsilon \\ \dot{q}(t) \to \dot{q}(t) \\ t \to t \end{cases}$$
ε small constant

$$\delta\mathcal{L} \equiv \mathcal{L}(q+\varepsilon, \dot{q}, t) - \mathcal{L}(q, \dot{q}, t) \approx \frac{\partial \mathcal{L}}{\partial q}\varepsilon = 0 \quad \Rightarrow \text{symmetry}$$

A much more general statement is the following

Noether's theorem :

If
$$\begin{cases} q(t) \to q(t) + \varepsilon\gamma(t) \\ \dot{q}(t) \to \dot{q}(t) + \varepsilon\dot{\gamma}(t) \\ t \to t \end{cases}$$
for an arbitrary function $\gamma(t)$ is a symmetry, $(\delta\mathcal{L} = 0)$

then
$$\boxed{\frac{\partial \mathcal{L}}{\partial \dot{q}}\gamma(t) = \text{const}}$$
is conserved.

For many dof's $(\alpha = 1, \dots, n)$:

$$\begin{cases} q_\alpha \to q_\alpha + \varepsilon\gamma_\alpha \\ \dot{q}_\alpha \to \dot{q}_\alpha + \varepsilon\dot{\gamma}_\alpha \\ t - t \end{cases}$$
is a symmetry $(\delta\mathcal{L} = 0)$
$$\Longrightarrow \quad \boxed{\sum_\alpha \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\gamma_\alpha = \text{const}}$$

proof. $\delta\mathcal{L} = \mathcal{L}(q+\varepsilon\gamma, \dot{q}+\varepsilon\dot{\gamma}, t) - \mathcal{L}(q, \dot{q}, t)$

$$\approx \frac{\partial \mathcal{L}}{\partial q}\varepsilon\gamma + \frac{\partial \mathcal{L}}{\partial \dot{q}}\varepsilon\dot{\gamma} \underset{\text{E-L}}{=} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\varepsilon\gamma + \frac{\partial \mathcal{L}}{\partial \dot{q}}\varepsilon\dot{\gamma} = \varepsilon\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\gamma\right)$$

$$\delta\mathcal{L} = 0 \quad \Leftrightarrow \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\gamma\right) = 0 \quad \Leftrightarrow \quad \frac{\partial \mathcal{L}}{\partial \dot{q}}\gamma \text{ is conserved}$$

generalization to n dof's is straightforward.

Examples:

1) Euclidean translations $\qquad \mathcal{L} = \sum_a \frac{m_a}{2}\dot{\vec{r}}_a^{\,2} - V$

If $V = V(\vec{r}_1-\vec{r}_2, \dots, \vec{r}_1-\vec{r}_N, \vec{r}_2-\vec{r}_3, \dots, \vec{r}_2-\vec{r}_N, \dots)$ (depends only on differences)

then $\begin{cases} \vec{r}_a \to \vec{r}_a + \varepsilon\hat{n} \\ \dot{\vec{r}}_a \to \dot{\vec{r}}_a \\ t \to t \end{cases}$ is a symmetry

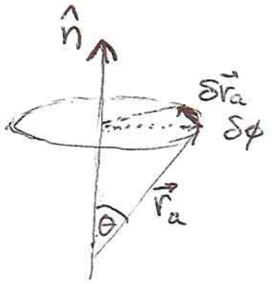
$$\sum_{\alpha=1}^{N} \frac{\partial\mathcal{L}}{\partial\dot{q}_\alpha}\gamma_\alpha = \sum_{a=1}^{N} \frac{\partial\mathcal{L}}{\partial\dot{\vec{r}}_a}\cdot\hat{n} = \hat{n}\cdot\sum_{a=1}^{N}\frac{\partial\mathcal{L}}{\partial\dot{\vec{r}}_a} = \text{const} \qquad (\hat{n} \text{ arbitrary})$$

$$\vec{P}_{TOT} \equiv \sum_a \frac{\partial\mathcal{L}}{\partial\dot{\vec{r}}_a} = \sum_a \vec{p}_a = \text{const} \qquad \text{(total linear momentum)}$$

conservation of $\vec{P}$ follows from transl. invariance

2) Eucl. rotations $\quad$ if $V = V(|\vec{r}_1-\vec{r}_2|, \dots, |\vec{r}_1-\vec{r}_N|, \dots)$ (only on distances)

then rotations are a symmetry



$|\delta\vec{r}_a| = r_a\sin\theta\,\delta\phi \qquad \delta\vec{r}\perp\vec{r} \qquad \delta\vec{r}\perp\hat{n}$

$$\delta\vec{r} = \hat{n}\times\vec{r}\,\delta\phi$$

$\begin{cases} \vec{r}_a \to \vec{r}_a + \hat{n}\times\vec{r}_a\,\delta\phi \\ \dot{\vec{r}}_a \to \dot{\vec{r}}_a + \hat{n}\times\dot{\vec{r}}_a\,\delta\phi \\ t \to t \end{cases} \qquad \delta\phi \sim \varepsilon$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A})$$

$$\Rightarrow \quad \sum_a \frac{\partial\mathcal{L}}{\partial\dot{\vec{r}}_a}\cdot(\hat{n}\times\vec{r}_a) = \text{const.}$$

$$\sum_a \hat{n}\cdot\left(\vec{r}_a\times\frac{\partial\mathcal{L}}{\partial\dot{\vec{r}}_a}\right) = \hat{n}\cdot\sum_a \vec{r}_a\times\frac{\partial\mathcal{L}}{\partial\dot{\vec{r}}_a} = \text{constant}$$

$L = \sum \qquad$ is conserved (consequence of rot. invariance)

3) Time translations. This case is not included in ~~th~~ this formulation of Noether's theorem. Time is special, not a coordinate like the others (it does not get varied). Still, it has ~~an~~ a conservation law associated.

$\begin{cases} q_\alpha(t) \to q_\alpha(t) \\ \dot{q}_\alpha(t) \to \dot{q}_\alpha(t) \\ t \to t+\varepsilon \end{cases} \qquad \begin{aligned} &\delta\mathcal{L} \equiv \mathcal{L}(q_\alpha, \dot{q}_\alpha; t+\varepsilon) - \mathcal{L}(q_\alpha, \dot{q}_\alpha; t) \simeq \frac{\partial\mathcal{L}}{\partial t}\varepsilon \\ &\delta\mathcal{L} = 0 \iff \frac{\partial\mathcal{L}}{\partial t} = 0 \end{aligned}$

That is, time translation is a symmetry of $\mathcal{L}$, ~~then~~ iff $\mathcal{L}$ does not depend explicitely on t: $\mathcal{L} = \mathcal{L}(q_\alpha, \dot{q}_\alpha; \cancel{t})$.

To get the consrvation law we cannot use the simple version of Noether's theorem we proved (notice that $\partial\mathcal{L}/\partial t \neq d\mathcal{L}/dt$, so $\mathcal{L}$ is <u>not</u> conserved).

$$\frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\ddot{q}_\alpha + \frac{\partial \mathcal{L}}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial \mathcal{L}}{\partial t} \overset{\text{use E-L}}{=} \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\ddot{q}_\alpha + \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\right)\dot{q}_\alpha + \frac{\partial \mathcal{L}}{\partial t} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\dot{q}_\alpha\right) + \frac{\partial \mathcal{L}}{\partial t}$$

so
$$\frac{d}{dt}\underbrace{\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\dot{q}_\alpha - \mathcal{L}\right]}_{\equiv \mathcal{H}\ (\text{"Hamiltonian"})} = -\frac{\partial \mathcal{L}}{\partial t} \qquad \boxed{\frac{d\mathcal{H}}{dt} = -\frac{\partial \mathcal{L}}{\partial t}}$$

<u>If</u> time translations are a symmetry of $\mathcal{L}$ (that is, if $\partial\mathcal{L}/\partial t = 0$), then $d\mathcal{H}/dt = 0$. The Hamiltonian is the conserved quantity associated to time-translational invariance.

In most cases, $\mathcal{H}$ is the energy (but <u>not</u> in <u>all</u> cases).

<u>If</u> $T = T(\dot{q}_\alpha, q_\alpha)$ and $V = V(q_\alpha)$, then

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\dot{q}_\alpha - \mathcal{L} = \frac{\partial T}{\partial \dot{q}_\alpha}\dot{q}_\alpha - T + V$$

If T is <u>quadratic</u> in $\dot{q}_\alpha$'s (usually the case, but <u>not</u> always), then

$$\frac{\partial T}{\partial \dot{q}_\alpha}\dot{q}_\alpha = 2T, \quad \text{then} \quad \mathcal{H} = T + V \equiv E$$

(if $T = M_{\gamma\beta}(\{q\})\dfrac{\dot{q}_\gamma \dot{q}_\beta}{2}$ with $M_{\gamma\beta} = M_{\beta\gamma}$, then $\dfrac{\partial T}{\partial \dot{q}_\alpha} = M_{\alpha\beta}\dot{q}_\beta$, $\dfrac{\partial T}{\partial \dot{q}_\alpha}\dot{q}_\alpha = 2T$)

Conservation of energy follows from time-transl. invariance.

<u>Ex.</u> $T = \frac{m}{2}\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 \qquad V = V(r) \qquad \mathcal{L} = T - V$

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{r}}\dot{r} + \frac{\partial \mathcal{L}}{\partial \dot{\phi}}\dot{\phi} - \mathcal{L} = m\dot{r}^2 + mr^2\dot{\phi}^2 - \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + V = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + V(r)$$
$$= T + V = E$$

Generalization of Noether's theorem

$$\begin{cases} q_\alpha \to q_\alpha + \varepsilon \gamma_\alpha \\ \dot{q}_\alpha \to \dot{q}_\alpha + \varepsilon \dot{\gamma}_\alpha \\ t \to t \end{cases} \quad \text{is such that} \quad \mathcal{L}' = \mathcal{L} + \varepsilon \frac{d}{dt} F(q,t)$$

Taylor-expand: $$\delta \mathcal{L} = \varepsilon \left(\frac{\partial \mathcal{L}}{\partial q_\alpha} \gamma_\alpha + \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} \dot{\gamma}_\alpha \right) = \varepsilon \frac{d}{dt} F(q,t)$$

Use EoM: $$\delta \mathcal{L} = \varepsilon \left[\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} \right) \gamma_\alpha + \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} \dot{\gamma}_\alpha \right] = \varepsilon \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} \gamma_\alpha \right)$$

$$\Rightarrow \frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} \gamma_\alpha - F \right] = 0 \qquad \boxed{\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} \gamma_\alpha - F(q,t) = \text{const}}$$

Galilean transformations

~~N~~ interacting particles : $\mathcal{L} = \sum_a \frac{m_a}{2}\dot{\vec{r}}_a^{\,2} - V(\vec{r}_1, \dots, \vec{r}_n)$

If $V = V(\vec{r}_1 - \vec{r}_2, \dots, \vec{r}_1 - \vec{r}_N, \vec{r}_2 - \vec{r}_3, \dots)$, then $\mathcal{L}$ is invariant under

$$\begin{cases} \vec{r}_a \to \vec{r}_a + \varepsilon \vec{v}_R t \\ \dot{\vec{r}}_a \to \dot{\vec{r}}_a + \varepsilon \vec{v}_R \\ t \to t \end{cases}$$

"Galilean transformations" (or boosts) $(\varepsilon \ll 1)$

$$\delta T = \sum_a \frac{m_a}{2}(\dot{\vec{r}}_a + \varepsilon\vec{v}_R)\cdot(\dot{\vec{r}}_a + \varepsilon\vec{v}_R) - \sum_a \frac{m_a}{2}\dot{\vec{r}}_a^{\,2} \approx \varepsilon \sum_a m_a \dot{\vec{r}}_a \cdot \vec{v}_R$$

$$= \frac{d}{dt}\left(\varepsilon \sum_a m_a \vec{r}_a \cdot \vec{v}_R\right) \qquad \text{(not zero but total derivative)}$$

$$\delta V = 0 \qquad \delta\mathcal{L} = \delta T$$

BUT adding a total time derivative to $\mathcal{L}$ does not change the EoM. The ~~new~~ transformed $\mathcal{L}' = \mathcal{L} + \delta\mathcal{L}$ is equivalent. (not invariant). Still, there is an associated conservation law.

$$\delta\mathcal{L} = \varepsilon\vec{v}_R \cdot \frac{d}{dt}\underbrace{\left(\sum_a m_a \vec{r}_a\right)}_{M\vec{r}_{CM}}$$

but also $\delta\mathcal{L} = \varepsilon \sum_a \frac{\partial\mathcal{L}}{\partial\vec{r}_a}\cdot\vec{v}_R t + \varepsilon\sum_a \frac{\partial\mathcal{L}}{\partial\dot{\vec{r}}_a}\cdot\vec{v}_R$

$$= \varepsilon\left[\sum_a m_a\dot{\vec{r}}_a\cdot\vec{v}_R + \sum_a \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\vec{r}}_a}\right)\cdot\vec{v}_R t\right] = \varepsilon\frac{d}{dt}\Big(\underbrace{\sum_a \frac{\partial\mathcal{L}}{\partial\dot{\vec{r}}_a}}_{\vec{P}_{TOT}}\cdot\vec{v}_R t\Big)$$

$$= \varepsilon\vec{v}_R\cdot\frac{d}{dt}\left(\vec{P}_{TOT}t\right)$$

$$\Rightarrow \varepsilon\vec{v}_R\cdot\frac{d}{dt}\left(M\vec{r}_{CM} - \vec{P}_{TOT}t\right) = 0 \qquad \text{for arbitrary } \vec{v}_R$$

$$\Rightarrow \frac{d}{dt}\left(M\vec{r}_{CM} - \vec{P}_{TOT}t\right) = 0 \qquad \vec{r}_{CM} = \frac{\vec{P}_{TOT}t}{M} + \text{const.}$$

Independent ~~of~~ the conservation of $\vec{P}_{TOT}$. It follows from Galilean invariance

Exercises

1] Particles in 3D, central potential.

$$\mathcal{L} = \frac{m}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2) - V(\sqrt{x^2+y^2+r^2})$$

$$\begin{cases} x = r\cos\phi\sin\theta \\ y = r\sin\phi\sin\theta \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \dot{x} = \dot{r}\cos\phi\sin\theta + r\cos\phi\cos\theta\dot{\theta} - r\sin\phi\dot{\phi}\sin\theta \\ \dot{y} = \dot{r}\sin\phi\sin\theta + r\sin\phi\cos\theta\dot{\theta} + r\cos\phi\dot{\phi}\sin\theta \\ \dot{z} = \dot{r}\cos\theta - r\sin\theta\dot{\theta} \end{cases}$$

$$\begin{aligned} \dot{x}^2+\dot{y}^2+\dot{z}^2 &= \dot{r}^2(\cos^2\phi\sin^2\theta + \sin^2\phi\sin^2\theta + \cos^2\theta) \\ &+ \dot{\theta}^2(\cos^2\phi\cos^2\theta + \sin^2\phi\cos^2\theta + \sin^2\theta)r^2 \\ &+ \dot{\phi}^2 r^2(\cos^2\phi\sin^2\theta + \sin^2\phi\sin^2\theta) \\ &+ 2\dot{r}\dot{\theta}\underbrace{(\cos^2\phi\sin\theta\cos\theta + \sin^2\phi\sin\theta\cos\theta - \cos\theta\sin\theta)}_{=0}r \quad \text{(the rest cancels)} \\ &= \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2 \end{aligned}$$

$$\mathcal{L} = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) - V(r)$$

E-L for θ: $\frac{d}{dt}(mr^2\dot{\theta}) - mr^2\sin\theta\cos\theta\,\dot{\phi}^2 = 0$

Initial conditions: $\vec{r}(t_{in}) = \vec{r}_{in}$, $\dot{\vec{r}}(t_{in}) = \vec{v}_{in}$. Choose $\hat{z}$ to be orthogonal to that plane: $z_{in} = 0$, $\dot{z}_{in} = 0$. $\Rightarrow \cos\theta_{in} = 0$, $\sin\theta_{in}\dot{\theta}_{in} = 0$

$\Rightarrow \theta_{in} = \pi/2$, $\dot{\theta}_{in} = 0$

$\theta = \pi/2$ is a solution at all times (conservation of $\vec{L}$)

Motion takes place in the x-y plane only:

$$\mathcal{L} = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - V(r) \qquad \text{(effective 2D Lagrangian)}$$

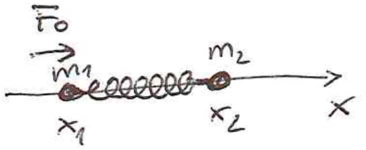
E-L for ϕ: $\frac{d}{dt}(mr^2\dot{\phi}) = 0$ (conservation of $|\vec{L}|$)

$mr^2\dot{\phi} \equiv L = \text{const}$ $\qquad \dot{\phi} = \frac{L}{mr^2}$

E-L for r: $m\ddot{r} + V'(r) - mr\dot{\phi}^2 = m\ddot{r} - \left(\frac{L^2}{mr^3} - V'(r)\right) = 0$

$$\boxed{m\ddot{r} = -\frac{\partial}{\partial r}V_{eff}(r)} \qquad \boxed{V_{eff}(r) \equiv V(r) + \frac{L^2}{2mr^2}}$$

2|



spring with k, l_0 (and k very large) and constant force F_0 on m_1

$$\mathcal{L} = \frac{m_1 \dot{x}_1^2}{2} + \frac{m_2 \dot{x}_2^2}{2} - \frac{k}{2}(x_2 - x_1 - l)^2 + F_0 x_1$$

$q_1 = x_1$, $q_2 = x_2 - x_1 - l$ $\qquad \dot{q}_1 = \dot{x}_1 \qquad \dot{q}_2 = \dot{x}_2 - \dot{x}_1$

$\dot{x}_2 = \dot{q}_2 + \dot{q}_1$

$$\mathcal{L} = \frac{m_1 \dot{q}_1^2}{2} + \frac{m_2}{2}(\dot{q}_1^2 + \dot{q}_2^2 + 2\dot{q}_1\dot{q}_2) - \frac{k}{2} q_2^2 + F_0 q_1$$

$$= m_1 \dot{q}_1^2 + \frac{m_2}{2}\dot{q}_2^2 + m_2 \dot{q}_1 \dot{q}_2 - \frac{k}{2} q_2^2 + F_0 q_1$$

E-L for q_1: $(m_1 + m_2)\ddot{q}_1 + m_2 \ddot{q}_2 - F_0 = 0$

E-L for q_2: $m_2(\ddot{q}_2 + \ddot{q}_1) + k q_2 = 0$

When $k \to 0$, $q_2 = -\frac{m_2}{k}(\ddot{q}_2 + \ddot{q}_1) \to 0 \qquad \dot{q}_2 \approx 0 \qquad \ddot{q}_2 \approx 0$

then $(m_1 + m_2)\ddot{q}_1 - F_0 = 0$

Same as plugging $q_2 = 0$ in $\mathcal{L}$ from the start:

$$\mathcal{L} \to \frac{m_1 + m_2}{2}\dot{q}_1^2 + F_0 q_1.$$

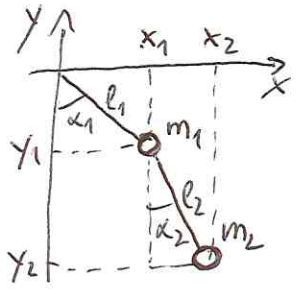
Reaction force F_R:

$\ddot{q}_1 = \frac{F_0}{m_1 + m_2}$ from the solution, but also $\ddot{q}_1 = \frac{F_0}{m_1} - \frac{F_R}{m_1}$

$$\Rightarrow \quad F_R = F_0 - \frac{m_1}{m_1 + m_2} F_0 = \frac{m_2}{m_1 + m_2} F_0$$

$\ddot{x}_2 = \frac{F_R}{m_2} = \frac{F_0}{m_1 + m_2}$ (same as $\ddot{x}_1$)

3| Double pendulum with masses m_1 and m_2, lengths l_1 l_2



$$\begin{cases} x_1 = l_1 \sin\alpha_1 \\ y_1 = -l_1 \cos\alpha_1 \end{cases} \qquad \begin{cases} x_2 = l_1 \sin\alpha_1 + l_2 \sin\alpha_2 \\ y_2 = -l_1\cos\alpha_1 - l_2 \cos\alpha_2 \end{cases}$$

$$T = \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2) = T_1 + T_2$$

$$V = m_1 g y_1 + m_2 g y_2$$

$$\begin{cases}\dot x_1 = l_1\cos\alpha_1\,\dot\alpha_1\\ \dot y_1 = l_1\sin\alpha_1\,\dot\alpha_1\end{cases}\qquad \begin{cases}\dot x_2 = l_1\cos\alpha_1\,\dot\alpha_1 + l_2\cos\alpha_2\,\dot\alpha_2\\ \dot y_2 = l_1\sin\alpha_1\,\dot\alpha_1 + l_2\sin\alpha_2\,\dot\alpha_2\end{cases}$$

$$T_1 = \frac{m_1}{2} l_1^2\dot\alpha_1^2 \qquad T_2 = \frac{m_2}{2}\Big[l_1^2\dot\alpha_1^2 + l_2^2\dot\alpha_2^2 + 2l_1l_2\dot\alpha_1\dot\alpha_2\underbrace{(\cos\alpha_1\cos\alpha_2+\sin\alpha_1\sin\alpha_2)}_{\cos(\alpha_1-\alpha_2)}\Big]$$

$$V = -gm_1l_1\cos\alpha_1 - gm_2(l_1\cos\alpha_1 + l_2\cos\alpha_2)$$

$$\mathcal{L} = \frac{m_1+m_2}{2} l_1^2\dot\alpha_1^2 + \frac{m_2}{2} l_2^2\dot\alpha_2^2 + m_2 l_1 l_2\dot\alpha_1\dot\alpha_2\cos(\alpha_1-\alpha_2) + g(m_1+m_2)l_1\cos\alpha_1 + gm_2l_2\cos\alpha$$

$$\frac{\partial\mathcal{L}}{\partial\dot\alpha_1} = (m_1+m_2)l_1^2\dot\alpha_1 + m_2l_1l_2\dot\alpha_2\cos(\alpha_1-\alpha_2)$$

$$\frac{\partial\mathcal{L}}{\partial\alpha_1} = -m_2l_1l_2\dot\alpha_1\dot\alpha_2\sin(\alpha_1-\alpha_2) - g(m_1+m_2)l_1\sin\alpha_1$$

$$\frac{\partial\mathcal{L}}{\partial\dot\alpha_2} = m_2l_2^2\dot\alpha_2 + m_2l_1l_2\dot\alpha_1\cos(\alpha_1-\alpha_2)$$

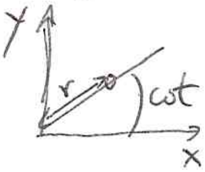
$$\frac{\partial\mathcal{L}}{\partial\alpha_2} = m_2l_1l_2\dot\alpha_1\dot\alpha_2\sin(\alpha_1-\alpha_2) - gm_2l_2\sin\alpha_2$$

$\Rightarrow$ for α_1
$$(m_1+m_2)l_1\ddot\alpha_1 + m_2l_2\ddot\alpha_2\cos(\alpha_1-\alpha_2) + m_2l_2\dot\alpha_2(\dot\alpha_2-\cancel{\dot\alpha_1})\sin(\alpha_1-\alpha_2) + \cancel{m_2l_2\dot\alpha_1\dot\alpha_2\sin(\alpha_1-\alpha_2)} + g(m_1+m_2)\sin\alpha_1 = 0$$

$\Rightarrow$ for α_2
$$m_2l_2\ddot\alpha_2 + m_2l_1\ddot\alpha_1\cos(\alpha_1-\alpha_2) + m_2l_1\dot\alpha_1(\cancel{\dot\alpha_2}-\dot\alpha_1)\sin(\alpha_1-\alpha_2) - \cancel{m_2l_1\dot\alpha_1\dot\alpha_2\sin(\alpha_1-\alpha_2)} + gm_2\sin\alpha_2 = 0$$

$$l_2\ddot\alpha_2 + l_1\ddot\alpha_1\cos(\alpha_1-\alpha_2) - l_1\dot\alpha_1^2\sin(\alpha_1-\alpha_2) + g\sin\alpha_2 = 0$$

4| Ring sliding freely on a uniformly rotating thin rod.



$$\begin{cases} x = r\cos\omega t\\ y = r\sin\omega t\end{cases} \qquad T = \frac{m}{2}(\dot r^2 + r^2\omega^2) = \mathcal{L}$$

$$\frac{\partial\mathcal{L}}{\partial\dot r} = m\dot r \qquad \frac{\partial\mathcal{L}}{\partial r} = mr\omega^2$$

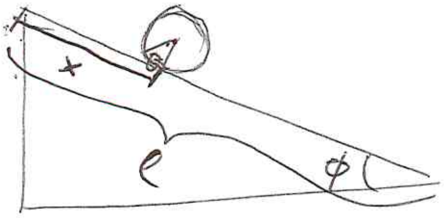
$$\Rightarrow m\ddot r - mr\omega^2 = 0 \qquad \ddot r = r\omega^2$$

$$\boxed{r = r_0e^{\omega t}}$$

Constraint $\quad L = mrv = mr^2\omega \qquad \dot L = 2m\dot r r\omega = \tau = rF_R$

$$F_R = 2m\dot r\omega = 2mr\omega^2$$

Ex 5 | Hoop of mass M rolling down an incline, without slipping



$$T_{CM} = \frac{M}{2}\dot{x}^2$$

$$T_{ROT} = \sum \frac{dm}{2}(r\dot{\theta})^2 = \frac{M}{2} r^2\dot{\theta}^2 \quad \left(\text{same } \dot{\theta} \text{ for all } dm\right)$$

$$V = Mg(\ell - x)\sin\phi$$

Constraint: $x = r\theta$ (here, holonomic) $\Rightarrow \dot{x} = r\dot{\theta}$

$$\mathcal{L} = \frac{M}{2}\dot{x}^2 + \frac{M}{2}r^2\dot{\theta}^2 + Mg(x-\ell)\sin\phi \rightarrow M\dot{x}^2 + Mg(x-\ell)\sin\phi$$

$$\frac{\partial\mathcal{L}}{\partial\dot{x}} = 2M\dot{x} \qquad \frac{\partial\mathcal{L}}{\partial x} = Mg\sin\phi$$

E-L: $2M\ddot{x} - Mg\sin\phi = 0 \Rightarrow \ddot{x} = \frac{g\sin\phi}{2}$

Motion in 1D

$$\mathcal{L} = \frac{m}{2}\dot{x}^2 - V(x) \qquad m\ddot{x} + V'(x) = 0$$

We know that E is conserved ($\partial\mathcal{L}/\partial t = 0$ and T is quadratic in $\dot{x}$):

$$\dot{x}\underbrace{(m\ddot{x} + V'(x))}_{=0} = \frac{d}{dt}\left(\frac{m}{2}\dot{x}^2 + V(x)\right) = \dot{E} = 0$$

$$E \equiv \frac{m}{2}\dot{x}^2 + V(x) = \text{const.} \qquad \dot{x} = \sqrt{\frac{2(E-V(x))}{m}} \qquad \frac{dt}{dx} = \frac{1}{\dot{x}}$$

Can find $t(x)$ with $t - t_0 = \int_{x_0}^{x} dx' \frac{dt}{dx'} = \int_{x_0}^{x} dx' \sqrt{\frac{m}{2(E-V(x'))}}$

__If__ the integral can be done and __if__ $t(x)$ can be inverted, then this gives ~~t~~ $x(t)$

__Ex.__ $V(x) = \frac{k}{2}x^2$ (harmonic oscillator)

$$t - t_0 = \int_0^x dx' \sqrt{\frac{m}{2(E - \frac{k}{2}x'^2)}} = \sqrt{m}\int_0^x \frac{dx'}{\sqrt{2E}} \frac{1}{\sqrt{1 - \frac{kx'^2}{2E}}} = \left[\xi \equiv \sqrt{\frac{k}{2E}}\,x\right]$$

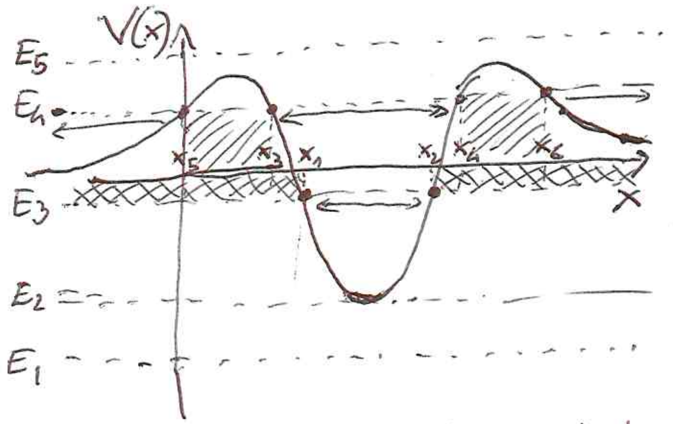
time of $x=0$

$$= \sqrt{\frac{m}{k}}\int_0^{\xi} d\xi' \frac{1}{\sqrt{1-\xi'^2}} = \sqrt{\frac{m}{k}}\left[\arcsin\left(\sqrt{\frac{k}{2E}}\,x\right) \cancel{-\arcsin\left(\sqrt{\frac{k}{2E}}\,x_0\right)}\right]$$

$$(t-t_0)\omega = \arcsin\left(\sqrt{\frac{k}{2E}}\,x\right) \qquad \omega^2 \equiv k/m$$

$$x = \sqrt{\frac{2E}{k}}\sin(\omega(t-t_0)) \qquad \text{OK}$$

In general, the solution requires $E\left(= \frac{m}{2}\dot{x}^2 + V\right) > V(x)$



E_1: $E < V$, no solution

E_2: $E = V$, $\dot{x} = 0$, $x(t) = x_{min} = \text{const}$

E_3: $E > V_{min}$, oscillations between "turning points" x_1 and x_2.

Period: $T = 2\int_{x_1}^{x_2} dx \sqrt{\frac{m}{2(E-V(x))}}$

E_4: oscillations between "turning points" x_3 and x_4
OR
bounce at x_5 towards $-\infty$
bounce at x_6 towards $+\infty$

E_5: $E > V_{max}$, no turning points
Solution from $-\infty$ to $+\infty$, or viceversa

2-body problem

2 particles in 3D $\quad \mathcal{L} = \frac{m_1}{2}\dot{\vec{r}}_1^{\,2} + \frac{m_2}{2}\dot{\vec{r}}_2^{\,2} - V(\vec{r}_1 - \vec{r}_2)$

$V = V(\vec{r}_1 - \vec{r}_2)$

Convenient change of coordinates: $\vec{R} \equiv \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$ (center of mass)

$\vec{r} \equiv \vec{r}_2 - \vec{r}_1$ (relative position)

$$(m_1+m_2)\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2 = -m_1\vec{r} + (m_1+m_2)\vec{r}_2$$

$$\vec{r}_2 = \vec{R} + \frac{m_1}{m_1+m_2}\vec{r} \qquad \vec{r}_1 = \vec{r}_2 - \vec{r} = \vec{R} - \frac{m_2}{m_1+m_2}\vec{r} \qquad M \equiv m_1+m_2$$

$$\mathcal{L} = \frac{m_1}{2}\left(\dot{\vec{R}} - \frac{m_2\dot{\vec{r}}}{M}\right)^2 + \frac{m_2}{2}\left(\dot{\vec{R}} + \frac{m_1\dot{\vec{r}}}{M}\right)^2 - V(\vec{r})$$

$$= \frac{M}{2}\dot{\vec{R}}^2 + \underbrace{m_1 m_2 \frac{m_1+m_2}{M^2}}_{= \frac{m_1 m_2}{m_1+m_2} \equiv \mu \text{ "reduced mass"}} \frac{\dot{\vec{r}}^2}{2} - V(\vec{r}) = \mathcal{L}_{CM} + \mathcal{L}_{rel}$$

$$\mathcal{L}_{CM}(\vec{R},\dot{\vec{R}}) = \frac{M}{2}\dot{\vec{R}}^2 \qquad \mathcal{L}_{rel}(\vec{r},\dot{\vec{r}}) \equiv \frac{\mu}{2}\dot{\vec{r}}^2 - V(\vec{r})$$

$\mathcal{L}$ decomposes as the sum of two independent Lagrangians which can be treated independently

$\vec{R}$: free particle, all components are cyclic

$$\vec{P} = \frac{\partial \mathcal{L}_{CM}}{\partial \dot{\vec{R}}} = M\dot{\vec{R}} = \text{const} \qquad \text{(uniform CoM motion)}$$

$\vec{r}$: equivalent to single particle of mass μ and pos. $\vec{r}$

If $V = V(r)$ ("central potential"), spherical coords are more convenient

$$\mathcal{L}_{rel} = \frac{\mu}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) - V(r)$$

Solutions (seen already): choose $\hat{z} \parallel \vec{L}$

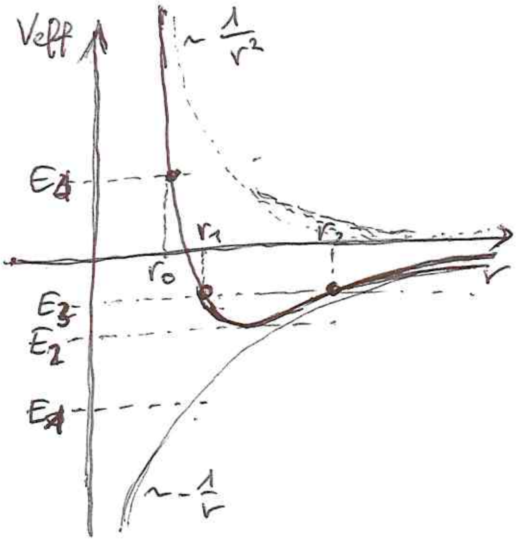
$$\theta = \frac{\pi}{2} \text{ constant}, \quad \dot{\phi} = \frac{L}{\mu r^2}, \quad \mu\ddot{r} = -V'(r) + \frac{L^2}{\mu r^3}$$

$$\mu\ddot{r} = -\frac{\partial}{\partial r}\left(V(r) + \frac{L^2}{2\mu r^2}\right) \equiv -\frac{\partial}{\partial r}V_{eff}(r) \qquad \boxed{V_{eff} \equiv V(r) + \frac{L^2}{\mu r^2}}$$

The radial coordinate is equivalent to a 1D problem with potential $V_{eff}(r) = V(r) + \frac{L^2}{\mu r^2}$ ← adds a centrifugal force

$$E = \frac{\mu}{2}\dot{r}^2 + V_{eff}(r) = \text{const}$$

<u>Ex 1</u> $V(r) = -\frac{c}{r}$ (gravitational, Coulomb)



E_1: $E < \min_r V_{eff}$ impossible

E_2: $E = V_{eff,min}$ $r = r_{min}$ $\forall t$

circular orbits with $\dot{\phi} = \frac{L^2}{\mu r_{min}^2}$

$$\phi = \phi_0 + \frac{L^2}{\mu r_{min}^2} t$$

E_3: oscillations between r_1 and r_2

bound orbits

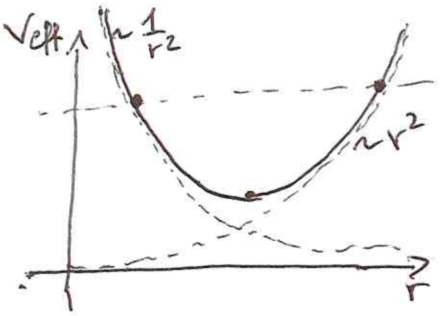
E_4: $E > 0$, unbound orbits

from $+\infty$, to r_0, back to $+\infty$

$\dot{\phi} = \frac{L}{\mu r^2} > 0$ ϕ always grows, no turning points

Orbits are parametrized by constants L, E

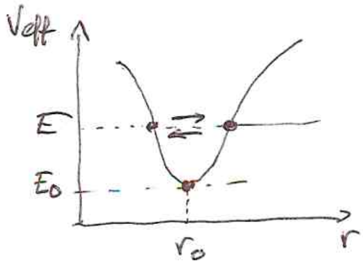
<u>Ex 2</u> $V(r) = \frac{k}{2} r^2$ (harmonic oscillator)



Only bound orbits (for $E > V_{eff,min}$)

Closed orbits

Goldstein, ch. 3.6



$$E = \frac{\mu}{2}\dot{r}^2 + V_{eff}(r) \qquad V_{eff}(r) = V(r) + \frac{L^2}{2\mu r^2}$$

$$\theta(t) = \frac{\pi}{2} \ \forall t \qquad \dot\phi = \frac{L}{\mu r^2} > 0 \ \forall t \qquad \mu\ddot{r} + V'_{eff}(r) = 0$$

minimum at $V'_{eff}(r_0) = 0 \quad \Rightarrow \quad \boxed{V'(r_0) = \frac{L^2}{\mu r_0^3}}$

- if $E = E_0 = V_{eff}(r_0) \Rightarrow \dot r = 0 \Rightarrow r(t) = \text{const} = r_0 \ \forall t$

$$\dot\phi = \frac{L}{\mu r_0^2} = \text{const} \quad \Rightarrow \quad \phi = \omega_0 (t - t_0) \equiv \phi_0(t) \qquad \boxed{\omega_0 \equiv \frac{L}{\mu r_0^2}}$$

circular orbit ($r = $ const.) with linearly growing ϕ, and frequency ω_0

- if $E > E_0$ (slightly!), oscillations around r_0 and $\phi_0(t)$

$$\begin{cases} r(t) = r_0 + \delta r(t) \\ \phi(t) = \phi_0(t) + \delta\phi \end{cases}$$

$$\dot\phi = \frac{L}{\mu(r_0+\delta r)^2} \approx \frac{L}{\mu r_0^2} - \frac{2L}{\mu r_0^3}\delta r = \omega_0 - \frac{2L}{\mu r_0^3}\delta r \qquad \boxed{\delta\dot\phi = -\frac{2L}{\mu r_0^3}\delta r}$$

δr and $\delta\phi$ have <u>same</u> frequency ω. ϕ has frequency $\omega_0 + \omega$

Periods: $T = \frac{2\pi}{\omega}$ and $T' = \frac{2\pi}{\omega_0 + \omega}$

Orbits close if δr completes n oscillations while ϕ completes $n+m$, with n, m integers: $nT = (n+m)T' \Rightarrow \frac{n}{\omega} = \frac{n+m}{\omega_0+\omega} \Rightarrow \frac{\omega_0+\omega}{\omega} = \frac{n+m}{n}$

$$\boxed{\frac{\omega_0}{\omega} = \frac{m}{n}}$$

ω and ω_0 must be "<u>commensurate</u>"

What is ω? $\quad \mu\ddot{r} + \cancel{V'_{eff}(r_0)} + V''_{eff}(r_0)\delta r = 0 \Rightarrow \delta\ddot{r} + \frac{V''_{eff}(r_0)}{\mu}\delta r = 0$

$$\boxed{\omega^2 \equiv \frac{V''_{eff}(r_0)}{\mu}} \qquad \omega^2 = \frac{V''(r_0)}{\mu} + \frac{1}{\mu}\left(\frac{L^2}{2\mu r_0^2}\right)'' = \frac{V''(r_0)}{\mu} + \frac{3L^2}{\mu^2 r_0^4} = \frac{V''(r_0)}{\mu} + 3\omega_0^2$$

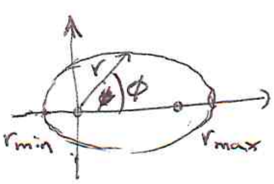
Power-law potentials: $V = Ar^\alpha$, $V' = \alpha A r^{\alpha-1}$, $V'' = \alpha(\alpha-1)Ar^{\alpha-2} = \frac{\alpha-1}{r}V'$

$$\frac{V''(r_0)}{\mu} = \frac{\alpha-1}{r_0}\frac{V'(r_0)}{\mu} = (\alpha-1)\frac{L^2}{\mu^2 r_0^4} = (\alpha-1)\omega_0^2 \quad \Rightarrow \quad \omega^2 = (\alpha+2)\omega_0^2$$

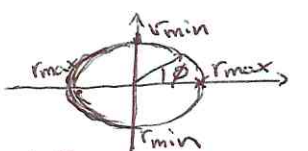
$$\alpha = \frac{\omega^2}{\omega_0^2} - 2 = \left(\frac{m}{n}\right)^2 - 2$$

1) $\frac{m}{n} = 1 \quad \alpha = -1 \quad V(r) \propto 1/r$ (Coulomb, Newton)

2) $\frac{m}{n} = 2 \quad \alpha = 2 \quad V(r) \propto r^2$ (harmonic osc.)



one oscillation of r for each turn

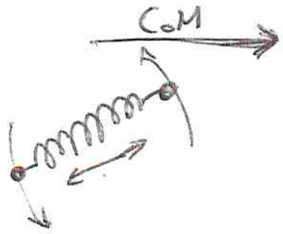


2 oscillations in r every turn

both are elliptical orbits

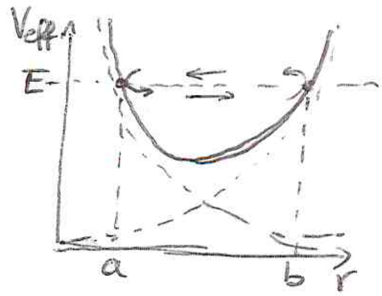
Only these two cases remain true beyond linear order

3D Harmonic Oscillator



$$V(r) = \frac{k}{2} r^2 \qquad \dot{\phi} = \frac{L}{\mu r^2}$$

$$\theta(t) = \frac{\pi}{2} \quad \forall t \qquad (z=0)$$

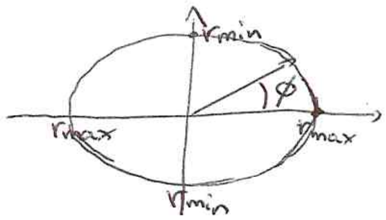


Let us study the effective 2D problem in Cartesian coordinates

$$V = \frac{k}{2}(x^2+y^2) \qquad T = \frac{\mu}{2}(\dot{x}^2+\dot{y}^2)$$

$$\mathcal{L} = T - V = \left(\frac{\mu}{2}\dot{x}^2 - \frac{k}{2}x^2\right) + \left(\frac{\mu}{2}\dot{y}^2 - \frac{k}{2}y^2\right) \equiv \mathcal{L}_x(x,\dot{x}) + \mathcal{L}_y(y,\dot{y})$$

Two independent 1D oscillators. The orientation of $\hat{x}$-$\hat{y}$ axes is arbitrary (rotational invariance). Convenient choice: choose t_0 so that $r(t_0) = a = r_{max}$ (outer turning point) and r_{max} on $\hat{x}$ axi.



$$\begin{cases} \phi(t_0) = 0 \\ \dot{r}(t_0) = 0 \end{cases} \qquad \begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases} \qquad \begin{cases} x(t_0) = a \\ y(t_0) = 0 \end{cases}$$

$$\begin{cases} \dot{x} = \dot{r}\cos\phi - \dot{\phi} r \sin\phi \\ \dot{y} = \dot{r}\sin\phi + \dot{\phi} r\cos\phi \end{cases} \Rightarrow \begin{cases} \dot{x}(t_0) = 0 \\ \dot{y}(t_0) = \dot{\phi} a = \frac{L}{\mu a} \end{cases}$$

$$\begin{cases} x(t) = a\cos(\bar{\omega}(t-t_0)) \\ y(t) = \underbrace{\frac{L}{\bar{\omega}\mu a}}_{\equiv b} \sin(\bar{\omega}(t-t_0)) \end{cases}$$

with $\bar{\omega} = \sqrt{\frac{k}{\mu}}$

$$b \equiv \frac{L}{\bar{\omega}\mu a}$$

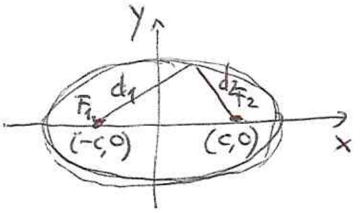
$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b} = 1}$$ ellipse

~~Kepler's problem~~

$$\cancel{\boxed{V(r) = -\frac{k}{r}}} \qquad \cancel{k = Gm_1m_2 = G\mu M > 0} \qquad \cancel{V_{eff} = -\frac{k}{r} + \frac{L^2}{2\mu r^2}}$$

<u>Some geometry</u> (properties of ellipses)

Ellipse $\equiv$ locus of points whose <u>sum</u> of distances from foci is consta



$$c \equiv \frac{\overline{F_1F_2}}{2} \qquad a \equiv \frac{d_1+d_2}{2} \qquad d_1+d_2 > 2c \quad (a>c)$$

These points satisfy $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with $\boxed{b^2 \equiv a^2 - c^2}$

Proof. $d_1 = \sqrt{(x+c)^2+y^2} \qquad d_2 = \sqrt{(c-x)^2+y^2} \qquad d_1 + d_2 = 2a$

$$d_1^2 = (2a \underset{(+)}{-} d_2)^2 = 4a^2 \underset{(+)}{-} 4ad_2 + d_2^2$$

$$(-)\ 4ad_2 = 4a^2 + d_2^2 - d_1^2 = 4a^2 - 4xc$$

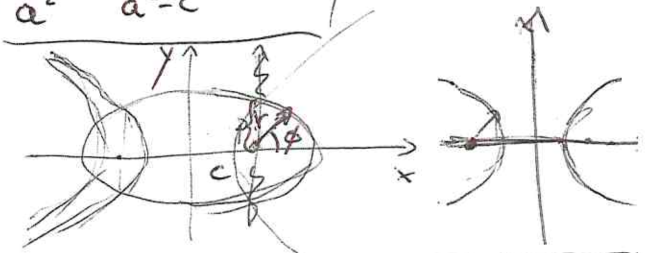
$$a^2d_2^2 = a^2[(c-x)^2+y^2] = (a^2 - xc)^2 = a^4 - 2a^2xc + x^2c^2$$

$$a^2(c^2+x^2+y^2) = a^4 + x^2c^2$$

$$(a^2-c^2)x^2 + a^2y^2 = a^2(a^2-c^2) \quad \Longrightarrow \quad \boxed{\frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1}$$

Replace <u>sum</u> with difference to get hyperbola ($|d_1 - d_2| = 2a$, and $a<c$)

<u>Polar form relative to focus</u>



$x = c + r\cos\phi \qquad y = r\sin\phi$

$$\cancel{\frac{c^2 + 2r\cos\phi + r^2\cos^2\phi}{a^2} + \frac{r^2\sin^2\phi}{b^2} = 1}$$

$$d_2 = r \qquad d_1 = \sqrt{(2c)^2 + r^2 - \underbrace{\cos(\pi-\phi)}_{-\cos\phi}4c}$$

$$d_1 + d_2 = 2a \qquad d_1^2 = (2a \underset{(+)}{-} d_2)^2 = 4a^2 - 4ar + r^2$$

$$4c^2 + \cancel{r^2} + 4cr\cos\phi = 4a^2 + \cancel{r^2} \underset{(+)}{-} 4ar$$

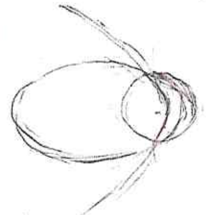
$$(a + c\cos\phi)r = a^2 - c^2$$

$$r = \frac{a^2 - c^2}{{}_{(-)}a + c\cos\phi} \quad \Longrightarrow \quad \boxed{r = \frac{a(1-e^2)}{1+e\cos\phi}} \qquad \boxed{e^2 \equiv \frac{c^2}{a^2} = 1 - \frac{b^2}{a^2}}$$

"eccentricity"

$\ell \equiv a(1-e^2) = \frac{b^2}{a}$ "semi-latus rectum" (r as $\phi = \frac{\pi}{2}$)

$$\boxed{\frac{\ell}{r} = 1 + e\cos\phi} \qquad 0<e<1$$



For hyperbola, $\frac{\ell}{r} = 1 \pm e\cos\phi$ with $e>1$ $\left(e \equiv \frac{c}{a}\right)$ $\ell = a(e^2-1$

Kepler's problem

$$\boxed{V(r) = -\frac{k}{r}} \qquad k \equiv G m_1 m_2 = G\mu M > 0 \qquad V_{eff}(r) = -\frac{k}{r} + \frac{L^2}{2\mu r^2}$$

(effective 1D problem)

Recall that $\dot{r} = \sqrt{\frac{2}{\mu}\left(E - V_{eff}(r)\right)} = \frac{dr}{dt}$ $\qquad \dot{\phi} = \frac{L}{\mu r^2} = \frac{d\phi}{dt}$

$$t = \int dt = \int \frac{dr}{\dot{r}} = \int dr \sqrt{\frac{\mu}{2(E - V_{eff}(r))}}$$

Can also write $\phi = \int \dot{\phi}\, dt = \int dr \frac{\dot{\phi}}{\dot{r}}$

$$\Rightarrow \boxed{\phi(r) = \int \frac{dr}{r^2} \frac{L}{\sqrt{2\mu(E - V_{eff}(r))}}}$$

Enough to study the shape of the orbits

$$E - V_{eff}(r) = E + \frac{k}{r} - \frac{L^2}{2\mu r^2}$$

$$2\mu \frac{E - V_{eff}(r)}{L^2} = \frac{2\mu}{L^2}\left(E + \frac{k}{r}\right) - \frac{1}{r^2} = \underbrace{\frac{2\mu E}{L^2} + \frac{\mu^2 k^2}{L^4}}_{\equiv K} - \left(\frac{1}{r} - \frac{\mu k}{L^2}\right)^2 = K(1 - u^2)$$

$$u \equiv \frac{1}{\sqrt{K}}\left(\frac{1}{r} - \frac{\mu k}{L^2}\right) \qquad du = -\frac{1}{\sqrt{K}} \frac{dr}{r^2}$$

$$\phi = -\int \frac{du}{\sqrt{1-u^2}} = -\arcsin\left(\frac{1}{\sqrt{K}}\left(\frac{1}{r} - \frac{k\mu}{L^2}\right)\right) + \text{const}$$

Choose const. to be $\pi/2$ and use $\frac{\pi}{2} - \arcsin(x) = \arccos(x)$

$$\boxed{\phi(r) = \arccos\left(\frac{L/r - \mu k/L}{\sqrt{2\mu E + \mu^2 k^2/L^2}}\right)}$$

$$\sqrt{2\mu E + \frac{\mu^2 k^2}{L^2}}\cos\phi = \frac{L}{r} - \frac{\mu k}{L} \qquad \frac{L^2}{\mu k}\frac{1}{r} = 1 + \sqrt{\frac{2EL^2}{\mu k^2} + 1}\cos\phi$$

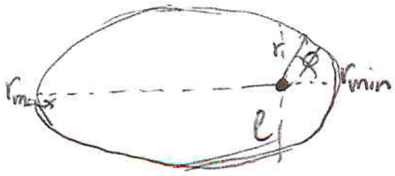
Elliptical orbit with $\ell = \frac{L^2}{\mu k}$ and $e^2 = 1 + \frac{2EL^2}{\mu k^2}$ if $E < 0$

Four cases:

- circular orbit: $V'_{eff}(r_0) = \frac{k}{r_0^2} - \frac{L^2}{\mu r_0^3} = 0 \Rightarrow r_0 = \frac{L^2}{\mu k}$

$$E_0 = V_{eff}(r_0) = -k\left(\frac{\mu k}{L^2}\right) + \frac{L^2}{2\mu}\left(\frac{\mu k}{L^2}\right)^2 = -\frac{\mu k^2}{2L^2} < 0 \qquad e^2 = 1 + \frac{2E_0 L^2}{\mu k^2} = 0$$

- elliptical orbit: $0 < e < 1$, $E < 0$

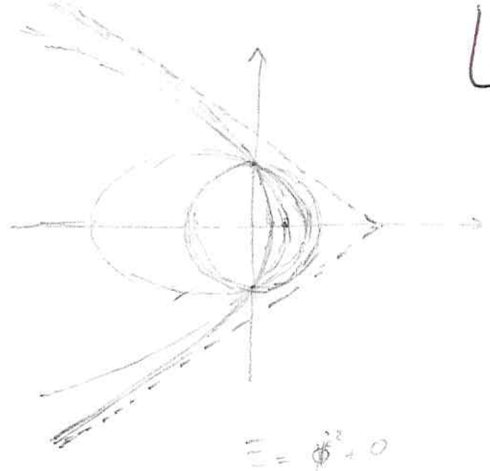


$$\boxed{r_{min} = \frac{\ell}{1+e}} \qquad \boxed{r_{max} = \frac{\ell}{1-e}}$$

"perihelion" "aphelion"

- parabolic orbits : $e = 1$, $E = 0$

$$r_{min} = \frac{\ell}{2} = \frac{L^2}{2\mu k} \quad , \quad r_{max} = \infty$$



- hyperbolic orbits : $e > 1$, $E > 0$

$$r_{min} = \frac{\ell}{1+e} \quad , \text{ asymptotes: } \boxed{\cos\phi_a = -\frac{1}{e}}$$

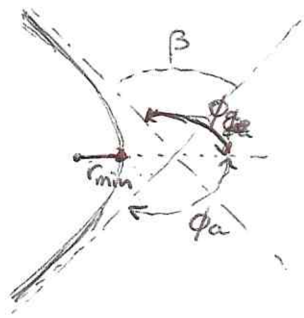
$$\phi_a = \arccos\left(-\frac{1}{e}\right) = \frac{\pi}{2} - \arccos\left(\frac{1}{e}\right)$$

Deflection angle for hyperbolic orbits:

$$\beta = 2\phi_a - \pi = 2\arccos\left(-\frac{1}{e}\right) - \pi$$

$$= \pi - 2\arccos\left(\frac{1}{e}\right) = 2\arcsin\left(\frac{1}{e}\right)$$

$$\boxed{\beta = 2\arcsin(1/e)}$$



$$e = \sqrt{1 + \frac{2EL^2}{\mu k^2}} = \sqrt{1 + \frac{E}{|E_0|}} \qquad e \gg 1 \text{ for } E \gg |E_0| \qquad \boxed{\beta \approx \frac{2}{e} \ll 1}$$

Highly energetic orbits have small deflection angles. Usually parametrize by r_{min} ($\approx$ impact parameter) and v_∞ (speed at r_∞). Because $E \gg |V(r_{min})|$, $v_\infty \approx v_{min}$.

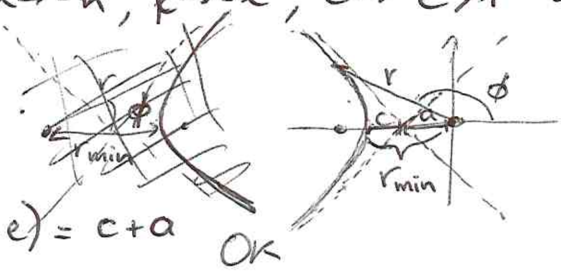
$$L = \mu r^2 \dot{\phi} = \mu r_{min} v_{min} \approx \mu r_{min} v_\infty$$

$$e \approx \sqrt{\frac{E}{|E_0|}} \approx \sqrt{\frac{v_\infty^2 L^2}{k^2}} \approx \frac{\mu v_\infty^2 r_{min}}{|k|} \quad \Rightarrow \quad \boxed{\beta \approx \frac{2|k|}{\mu v_\infty^2 r_{min}}}$$

For gravity, ~~$\beta \approx 2G$~~ $k = G\mu M$, $\beta \approx \dfrac{2GM}{\mu v_\infty^2 r_{min}}$

For repulsive potential $k \to -k$, $\ell \to -\ell$, $e \to e > 1$ $\quad -\dfrac{\ell}{r} = 1 + e\cos\phi$

Opposite branch:



r_{min} for $\phi = \pi$

$$r_{min} = -\frac{\ell}{1-e} = \frac{\ell}{e-1}$$

$$r_{min} = \frac{a(e^2-1)}{e-1} = a(1+e) = c + a \quad \text{OK}$$

Kepler's problem - Time dependence

Want $r(t)$ and $\phi(t)$, not just $r(\phi)$. $\quad \ell = \frac{L^2}{\mu k} \quad e^2 = 1 - \frac{2|E|L^2}{\mu k^2}$

$$t(r) = \int dr \sqrt{\frac{\mu}{2\left(E + \frac{k}{r} - \frac{L^2}{2\mu r^2}\right)}}$$

Assume $E = -|E| < 0$ (bound elliptical orbit)

$$t(r) = \sqrt{\frac{\mu}{2|E|}} \int dr \frac{r}{\sqrt{-r^2 + \frac{kr}{|E|} - \frac{L^2}{2\mu|E|}}}$$

major semiaxis:

$$a = \frac{r_{max} + r_{min}}{2} = \frac{\ell}{2}\left(\frac{1}{1-e} + \frac{1}{1+e}\right) = \frac{\ell}{1-e^2} = \frac{L^2}{\mu k}\frac{\mu k^2}{2|E|L^2} = \frac{k}{2|E|}$$

$$a^2(1-e^2) = \frac{k^2}{4E^2}\frac{2|E|L^2}{\mu k^2} = \frac{L^2}{2\mu|E|}$$

$$= \sqrt{\frac{\mu}{2|E|}} \int dr \frac{r - a + a}{\sqrt{\underbrace{-r^2 + 2ar - a^2(1-e^2)}_{a^2e^2 - (r-a)^2}}}$$

$$= \begin{bmatrix} u = -(r-a)/ae \\ du = -dr/ae \end{bmatrix} = -a\sqrt{\frac{\mu}{2|E|}} \int du \frac{1 - eu}{\sqrt{1-u^2}} = -\sqrt{\frac{\mu a^3}{k}}\left[\arcsin(u) - e\int \frac{du\, u}{\sqrt{1-u^2}}\right]$$

$$= \begin{bmatrix} s = 1 - u^2 \\ ds = -u\, du \end{bmatrix} = -\sqrt{\frac{\mu a^3}{k}}\left[\arcsin(u) + e\int \frac{ds}{\sqrt{s}}\right] = -\sqrt{\frac{\mu a^3}{k}}\left[\arcsin(u) + e\sqrt{s} + \text{const.}\right]$$

choose to so that it is $-\pi/2$

$$= \sqrt{\frac{\mu a^3}{k}}\left[\arccos(u) - e\sqrt{1-u^2}\right]$$

Define $\boxed{u = \cos\psi}$

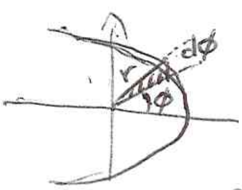
$$\Rightarrow \begin{cases} t(\psi) = \sqrt{\frac{\mu a^3}{k}}(\psi - e\sin\psi) \\ r(\psi) = a(1 - eu) = a(1 - e\cos\psi) \end{cases}$$

As $\psi \to \psi + 2\pi$, $r \to r$. Period is $T = t(\psi + 2\pi) - t(\psi) = 2\pi\sqrt{\frac{\mu a^3}{k}}$

$$\boxed{T^2 = \frac{4\pi^2 a^3}{G\mu}}$$

Kepler's 3rd law

Kepler's 2nd law (constancy of "area velocity") is a trivial consequence of conservation of L.



$$dA = \frac{r^2}{2}d\phi \qquad \boxed{\frac{dA}{dt} = \frac{r^2}{2}\dot{\phi} = \frac{L}{2\mu}}$$

What is the relation between ϕ and ψ?

$$(1 + e\cos\phi) = \frac{\ell}{r} = \frac{\cancel{a}(1-e^2)}{\cancel{a}(1 - e\cos\psi)} \qquad \cos\phi = \frac{1}{e}\left(\frac{1-e^2}{1 - e\cos\psi} - 1\right) = \frac{\cos\psi - e}{1 - e\cos\psi}$$

Same periodicity (2π) but complicated relation!

Small oscillations

Expand the Lagrangian around a known solution.

$$\mathcal{L}(q+\delta q, \dot q+\delta\dot q, t) \simeq \mathcal{L}(q,\dot q,t) + \frac{\partial\mathcal{L}}{\partial\dot q}\delta\dot q + \frac{\partial\mathcal{L}}{\partial q}\delta q + \frac{\partial^2\mathcal{L}}{\partial\dot q^2}\frac{\delta\dot q^2}{2} + \frac{\partial^2\mathcal{L}}{\partial q\partial\dot q}\delta q\delta\dot q + \frac{\partial^2\mathcal{L}}{\partial q^2}\frac{\delta q^2}{2} + \dots$$

$$= \mathcal{L} + \underbrace{\left(-\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot q} + \frac{\partial\mathcal{L}}{\partial q}\right)}_{=0}\delta q + \frac{\partial^2\mathcal{L}}{\partial\dot q^2}\frac{\delta\dot q^2}{2} + \left(\frac{\partial^2\mathcal{L}}{\partial q^2} - \frac{d}{dt}\frac{\partial^2\mathcal{L}}{\partial q\partial\dot q}\right)\frac{\delta q^2}{2} + \underbrace{\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot q}\delta q + \frac{\partial^2\mathcal{L}}{\partial q\partial\dot q}\frac{\delta q^2}{2}\right)}_{\text{irrelevant}}$$

E-L $\quad \frac{d}{dt}\left(\frac{\partial^2\mathcal{L}}{\partial\dot q^2}\delta\dot q\right) + \left(\frac{d}{dt}\frac{\partial^2\mathcal{L}}{\partial q\partial\dot q} - \frac{\partial^2\mathcal{L}}{\partial q^2}\right)\delta q = 0$

Has general solution if ~~its~~ coefficients are constant, that is if $q = q^{(0)}$, $\dot q = \ddot q = 0$ and $\frac{\partial\mathcal{L}}{\partial t} = 0$: if $q^{(0)}$ is an equilibrium solution

Several variables: $\quad \mathcal{L} = \frac{m_{\alpha\beta}}{2}\delta\dot q_\alpha\delta\dot q_\beta - \frac{k_{\alpha\beta}}{2}\delta q_\alpha\delta q_\beta$

$\boxed{m_{\alpha\beta} \equiv \left.\frac{\partial^2\mathcal{L}}{\partial\dot q_\alpha\partial\dot q_\beta}\right|_{\vec q=\vec q^{\,0},\ \dot{\vec q}=0}}$ "mass matrix" $\boxed{k_{\alpha\beta} \equiv -\left.\frac{\partial^2\mathcal{L}}{\partial q_\alpha\partial q_\beta}\right|_{\vec q=\vec q^{\,(0)},\ \dot{\vec q}=0}}$

Matrix form: $\quad \mathcal{L} = \frac{\delta\dot{\vec q}\cdot\hat m\cdot\delta\dot{\vec q}}{2} - \frac{\delta\vec q\cdot\hat k\cdot\delta\vec q}{2} \qquad \vec\eta \equiv \delta\vec q$

E-L: $\quad \hat m\cdot\ddot{\vec\eta} + \hat k\cdot\vec\eta = 0 \qquad$ like harmonic oscillator

~~Eigenvalue problem~~
Assume $\vec\eta = \vec a\, e^{-i\omega t}$: $\quad \boxed{(-\omega^2\hat m + \hat k)\cdot\vec a = 0} \quad (*)$

~~Eigenvalues~~ frequencies: it has solutions only if $\boxed{\det(\hat k - \omega^2\hat m) = 0}$

n-th order algebraic equation for ω^2, with n solutions ω_α^2 ($\alpha = 1,\dots,n$) called "eigenfrequencies" (or "characteristic freqs").

The ω_α^2 are real: $\quad \vec a^{\dagger}\cdot(\hat k - \omega^2\hat m)\cdot\vec a = 0 \quad \Rightarrow \quad \boxed{\omega^2 = \frac{\vec a^{\dagger}\cdot\hat k\cdot\vec a}{\vec a^{\dagger}\cdot\hat m\cdot\vec a}}$

and both $\hat k$ and $\hat m$ are symmetric...

For standard kinetic term, $m_{\alpha\beta}$ is pos. def. and $a_\alpha m_{\alpha\beta} a_\beta > 0$.

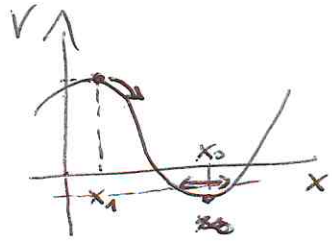
ω_α^2 are all positive iff $k_{\alpha\beta}$ is also pos. def.

For $\mathcal{L} = T - V(\vec q)$, iff $\vec q^{\,(0)}$ is a minimum of $V(\vec q)$.

If $\vec q^{\,(0)}$ is max or saddle point, then some ω_α are imaginary.
$\Rightarrow$ exponential solutions, runaway. Can still use it, but ~~they~~ for very short time!

Eigenvectors:

Example:



$V'(x_0) = V'(x_1) = 0$ $\quad V''(x_0) > 0$ $\quad k \equiv V''(x_1) < $

$-\omega^2 m + k = 0 \Rightarrow \omega^2 = \frac{k}{m} < 0$

$\omega = \pm i\sqrt{\frac{|k|}{m}}$

$x = x_0 + \eta(t) = x_0 + Ae^{-\sqrt{|k|/m}\,t} + Be^{\sqrt{|k|/m}\,t}$

runaway

Eigenvectors: Plug ω_α^2 back in (*) and solve for $\vec{a}_\alpha$. Get n vectors $\vec{a}_1, \dots, \vec{a}_n$. Each has n components. $\boxed{(\hat{k} - \omega_\alpha^2 \hat{m}) \cdot \vec{a}_\alpha = 0}$ (***)

For $\alpha \neq \beta$, one has $\hat{k} \cdot \vec{a}_\alpha = \omega_\alpha^2 \hat{m} \cdot \vec{a}_\alpha$ and $\vec{a}_\beta^T \cdot \hat{k} = \omega_\beta^2 \vec{a}_\beta^T \cdot \hat{m}$.

Multiply by $\vec{a}_\beta^T$ and $\vec{a}_\alpha$, and subtract: ($\vec{a}_\beta^T \cdot \hat{k} \cdot \vec{a}_\alpha$ cancels)

$$(\omega_\alpha^2 - \omega_\beta^2)\vec{a}_\beta^T \cdot \hat{m} \cdot \vec{a}_\alpha = \cancel{\vec{a}_\beta^T \hat{k} \vec{a}_\alpha - \vec{a}_\beta^T \hat{k} \vec{a}_\alpha} = 0$$

If $\omega_\alpha^2 \neq \omega_\beta^2$, then $\boxed{\vec{a}_\beta^T \cdot \hat{m} \cdot \vec{a}_\alpha = 0}$ for $\alpha \neq \beta$.

(all non-degenerate eigenfrequencies)

For $\alpha = \beta$, $\vec{a}_\alpha^T \cdot \hat{m} \cdot \vec{a}_\alpha > 0$ ($\hat{m}$ is pos. def.)

But $\vec{a}$ has arbitrary normalization. Can rescale it so that

$$\vec{a}_\alpha^T \cdot \hat{m} \cdot \vec{a}_\alpha = 1$$

Each $\vec{a}_\alpha$ has n components: $a_{\beta\alpha}$ for $\beta = 1, \dots, n$, $\vec{a}_\alpha^T$ has $a_{\alpha\beta}$

$$\boxed{a_{\beta\gamma} m_{\gamma\delta} a_{\delta\alpha} = \delta_{\beta\alpha}}$$

In matrix notation: $\boxed{\hat{A}^T \cdot \hat{m} \cdot \hat{A} = \mathbb{1}}$, $\hat{A} = \left(\left(\vec{a}_1\right), \dots, \left(\vec{a}_n\right) \right)$ label of vector $(\hat{A})_{\alpha\beta} = a_{\alpha\beta}$ component of vector

Define: $\hat{\omega}^2 \equiv \begin{pmatrix} \omega_1^2 & & 0 \\ & \ddots & \\ 0 & & \omega_n^2 \end{pmatrix}$ diagonal matrix

Equation (***) is $\sum_\gamma (k_{\beta\gamma} - \omega_\alpha^2 m_{\beta\gamma}) a_{\gamma\alpha} = 0$ (fixed α) $\cancel{\Rightarrow \sum_\gamma k_{\beta\gamma} a_{\gamma\alpha} = \sum_\gamma m_{\beta\gamma} a_{\gamma\alpha} \omega_\alpha}$

$(\hat{\omega})_{\alpha\beta} = \omega_\alpha^2 \delta_{\alpha\beta} \equiv \omega_{\alpha\beta}$ not summed $\Rightarrow k_{\beta\gamma} a_{\gamma\alpha} = m_{\beta\gamma} a_{\gamma\delta} \omega_{\delta\alpha}$ summed $\hat{k} \cdot \hat{A} =$

$$\hat{k} \cdot \hat{A} = \hat{m} \cdot \hat{A} \cdot \hat{\omega} \Rightarrow \hat{A}^T \cdot \hat{k} \cdot \hat{A} = \underbrace{\hat{A}^T \cdot \hat{m} \cdot \hat{A}}_{=\mathbb{1}} \cdot \hat{\omega} = \hat{\omega}$$

$$\Rightarrow \begin{cases} \hat{A}^T \cdot \hat{m} \cdot \hat{A} = 1 \\ \hat{A}^T \cdot \hat{k} \cdot \hat{A} = \hat{\omega}^2 \text{ (diagonal)} \end{cases}$$

both $\hat{m}$ and $\hat{k}$ are diagonalize by the same "congruence" transform.

General solution: $\vec{\eta} = \sum_\alpha C_\alpha \vec{a}_\alpha e^{i\omega_\alpha t} = A\cdot\vec{Q}$

$$\vec{Q} = \{C_1 e^{-i\omega_1 t}, \dots, C_n e^{-i\omega_n t}\} \qquad \dot{\vec{\eta}} = A\cdot\dot{\vec{Q}}$$

C_α complex numbers, take real part at the end:

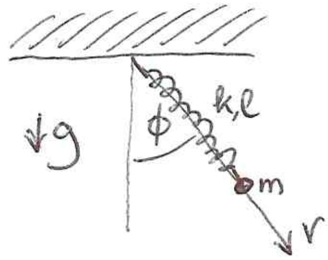
$$C_\alpha = \rho_\alpha e^{i\phi_\alpha} \qquad \mathrm{Re}\left(C_\alpha e^{-i\omega_\alpha t}\right) = \underset{\text{amplitude}}{\rho_\alpha} \cos(\omega_\alpha t - \underset{\text{phase}}{\phi_\alpha})$$

Same as ~~hav~~ starting from $A_\alpha e^{i\omega_\alpha t} + B_\alpha e^{-i\omega_\alpha t}$ with A_α, B_α real

$$\mathcal{L}_2 = \frac{\dot{\vec{Q}}^T\cdot A^T \hat{m} A\cdot\dot{\vec{Q}}}{2} - \frac{\vec{Q}^T\cdot A^T\hat{k}A\cdot\vec{Q}}{2} = \frac{1}{2}\dot{\vec{Q}}\cdot\dot{\vec{Q}} - \frac{1}{2}\vec{Q}\cdot\hat{\omega}^2\cdot\vec{Q}$$

"normal modes" of the oscillations

Ex



dof's: ϕ, r $\qquad T = \frac{m}{2}(\dot{x}^2+\dot{y}^2) = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2)$

$$V = \frac{k}{2}(r-\ell)^2 - mgr\cos\phi$$

$$\mathcal{L} = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{k}{2}(r-\ell)^2 - mgr\cos\phi$$

Equilibrium: $\begin{cases} \frac{\partial V}{\partial r} = k(r_0-\ell) - mg\cos\phi_0 = 0 \\ \frac{\partial V}{\partial \phi} = mg\, r_0 \sin\phi_0 = 0 \end{cases} \Rightarrow \begin{cases} \phi_0 = 0 \\ r_0 = \ell + \frac{mg}{k} \end{cases}$

$$\frac{\partial^2 V}{\partial r^2} = k \qquad \frac{\partial^2 V}{\partial r\partial\phi} = mg\sin\phi_0 = 0 \qquad \frac{\partial^2 V}{\partial\phi^2} = mgr_0\cos\phi_0 = mg\left(\ell + \frac{mg}{k}\right)$$

$$\hat{k} = \begin{pmatrix} mgr_0 & 0 \\ 0 & k \end{pmatrix} \text{ pos. def.} \qquad \eta_1 \equiv \phi - \phi_0 = \phi \qquad \eta_2 \equiv r - r_0 \equiv \delta r \quad \dot{r} = \dot{\delta r}$$

$$\hat{m} = \begin{pmatrix} \frac{\partial^2 T}{\partial\dot{\phi}^2} & 0 \\ 0 & \frac{\partial^2 T}{\partial \dot{r}^2} \end{pmatrix} = m\begin{pmatrix} r_0^2 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{System is already diagonal}$$

$$\det\left(\hat{k} - \omega^2\hat{m}\right) = (mgr_0 - \omega^2 m r_0^2)(k - m\omega^2) = 0$$

$$\Rightarrow \omega_1^2 = \frac{g}{r_0}, \quad \omega_2^2 = \frac{k}{m}$$

$\leftarrow$ pendulum with length $= r_0 = \ell + mg/k$

Eigenvectors: $\begin{pmatrix} a \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b \end{pmatrix}$ $\qquad$ Normal modes $\quad \rho_1\cos(\omega_1 t - \phi_1)$

$a = \frac{1}{\sqrt{m}\, r_0} \qquad b = \frac{1}{\sqrt{m}} \qquad\qquad \rho_2\cos(\omega_2 t - \phi_2)$

Ex <u>Double pendulum</u> expand $\mathcal{L}$ around $\alpha_1 = \alpha_2 = 0$

$$\cos\alpha_{1,2} \approx 1 - \frac{\alpha_{1,2}^2}{2} \qquad \cos(\alpha_1 - \alpha_2) \approx 1$$

$$\mathcal{L} = \frac{m_1+m_2}{2} l_1^2 \dot{\alpha}_1^2 + \frac{m_2}{2} l_2^2 \dot{\alpha}_1^2 + m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 - g(m_1+m_2) l_1 \frac{\alpha_1^2}{2} - g m_2 l_2 \frac{\alpha_2^2}{2}$$

$$\hat{m} = \begin{bmatrix} (m_1+m_2) l_1^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 l_2^2 \end{bmatrix} \qquad \hat{k} = \begin{bmatrix} g(m_1+m_2) l_1 & 0 \\ 0 & g m_2 l_2 \end{bmatrix}$$

$$0 = \det\left(\hat{k} - \omega^2 \hat{m}\right) = \det\begin{pmatrix} (m_1+m_2) l_1 (g - \omega^2 l_1) & -\omega^2 l_1 l_2 m_2 \\ -\omega^2 l_1 l_2 m_2 & m_2 l_2 (g - \omega^2 l_2) \end{pmatrix}$$

$$= (m_1+m_2) m_2 l_1 l_2 (g - \omega^2 l_1)(g - \omega^2 l_2) - \omega^4 m_2^2 l_1^2 l_2^2$$

$$= (m_1+m_2) m_2 l_1 l_2 \left(g^2 - \omega^2 g (l_1 + l_2)\right) + \omega^4 m_1 m_2 l_1^2 l_2^2$$

$$\Rightarrow \omega^4 - \underbrace{\frac{g(l_1+l_2)(m_1+m_2)}{m_1 l_1 l_2}}_{\frac{g m_2}{\mu\lambda}} \omega^2 + \underbrace{\frac{g^2(m_1+m_2)}{m_1 l_1 l_2}}_{\frac{g^2 m_2}{\mu l_1 l_2}} = 0 \qquad \lambda \equiv \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}$$

ω^2 ~~coeff~~

$$\omega_{1,2}^2 = \frac{g m_2}{2\mu\lambda} \pm \sqrt{\left(\frac{g m_2}{2\mu\lambda}\right)^2 - \frac{g^2 m_2}{\mu l_1 l_2}} = \frac{g m_2}{2\mu\lambda}\left[1 \pm \sqrt{1 - \frac{4\lambda^2}{l_1 l_2}\frac{\mu}{m_2}}\right]$$

$$\begin{pmatrix} \frac{(m_1+m_2) g l_1^2 m_2}{2\mu\lambda}\left(\frac{2\mu\lambda}{m_2 l_1} - 1 \mp \sqrt{1 - \frac{4\lambda^2}{l_1 l_2}\frac{\mu}{m_2}}\right) & -\frac{g l_1 l_2 m_2^2}{2\mu\lambda}\left(1 \pm \sqrt{}\right) \\ -\frac{g l_1 l_2 m_2^2}{2\mu\lambda}\left(1 \pm \sqrt{}\right) & \frac{g m_2^2 l_2^2}{2\mu\lambda}\left(\frac{2\mu\lambda}{m_2 l_2} - 1 \mp \sqrt{}\right) \end{pmatrix} \vec{a}_{1,2} = 0$$

$$\begin{pmatrix} \frac{m_1+m_2}{m_2}\frac{l_1}{l_2}\left(C - 1 \mp \sqrt{1 - \frac{2\lambda}{l_2} C}\right) & 1 \pm \sqrt{1 - \frac{2\lambda}{l_2} C} \\ 1 \pm \sqrt{1 - \frac{2\lambda}{l_2} C} & \frac{l_2}{l_1}\left(C \frac{l_1}{l_2} - 1 \mp \sqrt{1 - \frac{2\lambda}{l_2} C}\right) \end{pmatrix} \vec{a}_{1,2} = 0$$

Vanishing eigenfrequencies

Sometimes one of the solutions has $\omega_1^2 = 0$.

This corresponds to a normal mode with $\ddot{Q}_1 = 0$

$$\Rightarrow Q_1(t) = c_1 + d_1 t \qquad \text{(uniform motion)}$$

This is not a small oscillation.

In the presence of transational invariance, one expects this to happen ($\Rightarrow$ center of mass motion). The equilibrium condition can only fix the differences, not the overall position of the system.

N.B. the eigenvectors $\vec{a}_\alpha$ are linearly independent.
If they were not, one could write $\vec{a}_n = \sum_{\alpha=1}^{n-1} C_\alpha \vec{a}_\alpha$

$$1 = \vec{a}_n^T \cdot \hat{m} \cdot \vec{a}_n = \sum_{\alpha=1}^{n-1} C_\alpha \underbrace{\vec{a}_n^T \cdot \hat{m} \cdot \vec{a}_\alpha}_{\delta_{\alpha n}} = 0 \quad , \text{ but it must be } \neq 0$$

Degenerate eigenfrequencies

It happens when $\omega_\alpha^2 = \omega_\beta^2$ for some $\alpha \neq \beta$.
Then $(\omega_\alpha^2 - \omega_\beta^2)\vec{a}_\beta^T \cdot \hat{m} \cdot \vec{a}_\alpha = 0$ does not imply $\vec{a}_\beta^T \cdot \hat{m} \cdot \vec{a}_\alpha = 0$.
However, both $\vec{a}_\alpha$ and $\vec{a}_\beta$ are solutions of

$$(\hat{k} - \omega_\alpha^2 \hat{m})\vec{a} = 0 \qquad \text{(with same eigenfreq. } \omega_\alpha^2)$$

The "kernel" (or "null eigenspace") of $\hat{k} - \omega_\alpha^2 m$ has dimension 2. $\vec{a}_\alpha$ and $\vec{a}_\beta$ can still be taken as independent.
Can make them orthogonal w.r.t. $\hat{m}$ withe the Gram-Schmidt method:

$$\vec{a}_\alpha' = \frac{\vec{a}_\alpha}{\sqrt{\vec{a}_\alpha \cdot \hat{m} \cdot \vec{a}_\alpha}} \quad \text{and} \quad \vec{a}_\beta' = \frac{\vec{a}_\beta - (\vec{a}_\alpha'^T \cdot \hat{m} \cdot \vec{a}_\beta)\vec{a}_\alpha'}{\sqrt{\;\ldots\;}}$$

so that $\vec{a}_\alpha'^T \cdot \hat{m} \cdot \vec{a}_\beta' = 0$

The same procedure works for higher multiplicities.
Also for degenerate eigenfreqs we can find a set $\{\vec{a}_1, \ldots, \vec{a}_n\}$ of orthonormal vectors w.r.t $\hat{m}$, so that

$$\mathcal{L} = \sum_\alpha \left(\frac{\dot{Q}_\alpha^2}{2} - \frac{\omega_\alpha^2 Q_\alpha^2}{2} \right) = \sum_{\alpha=1}^{p} \left(\frac{\dot{Q}_\alpha^2}{2} - \frac{\omega_1^2 Q_\alpha^2}{2} \right) + \sum_{\alpha > p} \left(\frac{\dot{Q}_\alpha^2}{2} - \frac{\omega_\alpha^2 Q_\alpha^2}{2} \right)$$

$\omega_\alpha^2 = \omega_1^2$

Any p-dim. rotation of the $\vec{a}_\alpha'$ is also leaving $\mathcal{L}$ in normal form

The Hamiltonian formalism

General system with Lagrangian $\mathcal{L}(q,\dot q;t)$, $q \equiv \{q_1,\dots,q_n\}$

Equations of motion : $\frac{d}{dt}p_\alpha - \frac{\partial \mathcal{L}}{\partial q_\alpha} = 0$, where $\boxed{p_\alpha \equiv \frac{\partial \mathcal{L}}{\partial \dot q_\alpha} = p_\alpha(q,\dot q,t)}$: the conjugate momentum to q_α.

In principle, one can invert the n relations $p_\alpha = p_\alpha(q,\dot q;t)$ as:

$$\dot q_\alpha = \dot q_\alpha(q,p;t) \qquad (p \equiv \{p_1,\dots,p_n\})$$

and parametrise the system in terms of q_α and p_α (not q_α and $\dot q_\alpha$)

Ex: particle in 1D , $\mathcal{L} = \frac{m}{2}\dot x^2 - V(x)$, $p = \frac{\partial \mathcal{L}}{\partial \dot x} = m\dot x \Rightarrow \dot x = \frac{p}{m}$

The 2n-dimensional space parametrized by the "canonical variables" $\{q_\alpha, p_\alpha\}_{\alpha=1,\dots,n}$ is called the "phase space".

Equations of motion :
$$\begin{cases} \frac{dp_\alpha}{dt} = \left.\frac{\partial \mathcal{L}}{\partial q_\alpha}\right|_{\dot q_\alpha = \dot q_\alpha(q,p,t)} \\ \frac{dq_\alpha}{dt} = \dot q_\alpha(q,p,t) \end{cases} + \begin{cases} q_\alpha(t=0) = q_\alpha^{(0)} \\ p_\alpha(t=0) = p_\alpha^{(0)} \end{cases}$$

They can be written in a more symmetric form using the Hamiltonian

$$\mathcal{H} \equiv p_\alpha \dot q_\alpha - \mathcal{L} = \mathcal{H}(q,p;t) = p_\alpha \dot q_\alpha(q,p;t) - \mathcal{L}(q,\dot q(q,p;t);t)$$

Take partial derivatives:

$$\frac{\partial \mathcal{H}}{\partial q_\alpha} = \cancel{p_\beta \frac{\partial \dot q_\beta}{\partial q_\alpha}} - \frac{\partial \mathcal{L}}{\partial q_\alpha} - \cancel{\frac{\partial \mathcal{L}}{\partial \dot q_\beta}\frac{\partial \dot q_\beta}{\partial q_\alpha}} \text{ (because of definition of } p_\beta)$$

$$\frac{\partial \mathcal{H}}{\partial p_\alpha} = \dot q_\alpha + \cancel{p_\beta \frac{\partial \dot q_\beta}{\partial p_\alpha}} - \cancel{\frac{\partial \mathcal{L}}{\partial \dot q_\beta}\frac{\partial \dot q_\beta}{\partial p_\alpha}}$$

$$\Rightarrow \boxed{\begin{cases} \dot p_\alpha = -\frac{\partial \mathcal{H}}{\partial q_\alpha} \\ \dot q_\alpha = \frac{\partial \mathcal{H}}{\partial p_\alpha} \end{cases}} \qquad \text{"Canonical (or Hamilton's) equations"}$$

Some maths: Consider a function $f(x)$ of one variable. Set $y \equiv f'(x)$ and invert $\Rightarrow x = x(y)$. Define $g(y) \equiv y\,x(y) - f(x(y))$. Then,

$g'(y) = x(y) + y\,x'(y) - f'(x(y))\,x'(y) = x(y)$. $\Rightarrow \boxed{x = g'(y)}$

$\Rightarrow$ g is the "Legendre transform" of f w.r.t. variable x

$\Rightarrow$ $\mathcal{H}$ is the Legendre transform of $\mathcal{L}$ w.r.t. the $\dot q_\alpha$

$y = f'(x) \leftrightarrow p_\alpha = \frac{\partial \mathcal{L}}{\partial \dot q_\alpha}$ $\qquad x = g'(y) \leftrightarrow \dot q_\alpha = \frac{\partial \mathcal{H}}{\partial p_\alpha}$ $\qquad$ the transform does not affect q_α's and t

Ex 1D particle $\quad \dot{x} = \frac{p}{m} \quad \mathcal{H} = p\dot{x} - \mathcal{L} = \frac{p^2}{m} - \frac{p^2}{2m} + V(x) = \frac{p^2}{2m} + V(x)$ [2

EoM: $\begin{cases} \dot{p} = -\frac{\partial \mathcal{H}}{\partial x} = -V'(x) \\ \dot{x} = \frac{p}{m} \end{cases}$ same as $m\ddot{x} = -V'(x)$!

Remember: whenever T is quadratic in the generalized velocities $\left(T = \frac{1}{2} M_{\alpha\beta}(q)\dot{q}_\alpha \dot{q}_\beta\right)$ then $\mathcal{H} = T + V \equiv E$

$$p_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} = \frac{\partial T}{\partial \dot{q}_\alpha} = M_{\alpha\beta}\dot{q}_\beta \qquad \vec{p} = M \cdot \dot{\vec{q}} \qquad \dot{\vec{q}} = M^{-1} \cdot \vec{p}$$

$$T = \frac{1}{2}\dot{\vec{q}}^T \cdot M \cdot \dot{\vec{q}} = \frac{1}{2}\vec{p}^T \cdot M^{-1} M M^{-1} \cdot \vec{q} = \frac{1}{2}\vec{p}^T \cdot M^{-1} \cdot \vec{p}$$

$$\boxed{\mathcal{H} = \frac{1}{2}\vec{p}^T \cdot M^{-1} \cdot \vec{p} + V(\vec{q})}$$

We ~~had~~ also showed that $\frac{d}{dt}\mathcal{H} = -\frac{\partial \mathcal{L}}{\partial t}\Big|_{q,\dot{q}}$. This can be shown directly in the new formalism:

$$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t}\Big|_{q,p} + \cancel{\underbrace{\frac{\partial \mathcal{H}}{\partial q_\alpha}}_{-\dot{p}_\alpha}\dot{q}_\alpha} + \cancel{\underbrace{\frac{\partial \mathcal{H}}{\partial p_\alpha}}_{\dot{q}_\alpha}\dot{p}_\alpha} = \frac{\partial}{\partial t}\Big(p_\alpha \dot{q}_\alpha(q,p;t) - \mathcal{L}(q, \dot{q}(q,p;t);t)\Big)$$

$$= \cancel{\left(p_\alpha - \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\right)}\frac{\partial \dot{q}_\alpha}{\partial t} - \frac{\partial \mathcal{L}}{\partial t}\Big|_{q,\dot{q}(q,p;t)}$$

$$\boxed{\frac{d\mathcal{H}}{dt}\Big|_{q,p} = \frac{\partial \mathcal{H}}{\partial t}\Big|_{q,p} = -\frac{\partial \mathcal{L}}{\partial t}\Big|_{q,\dot{q}}}$$

If $\mathcal{L}$ does not depend explicitly on time (time-translational invariance) then $\mathcal{H}$ is conserved.

Hamiltonian formulation is equivalent to Lagrangian, but more useful for some (esp. theoretical) applications. E.g. for quantisation. Also, more intuitive ($\mathcal{H} \sim$ energy) than $\mathcal{L}$.

The canonical variables (q, p) must be thought as independent (unlike q and $\dot{q}$). System with $2n$ (and not n) dof's.
In time, the system describes a trajectory in phase space: $(q(t), p(t))$

Ex. 2D motion with central potential

Cartesian $\mathcal{L} = \frac{m}{2}(\dot{x}^2+\dot{y}^2) - V(\sqrt{x^2+y^2})$

$$\begin{cases} p_x = m\dot{x} \\ p_y = m\dot{y} \end{cases} \quad \begin{cases} \dot{x} = p_x/m \\ \dot{y} = p_y/m \end{cases} \quad \mathcal{H} = p_x\dot{x} + p_y\dot{y} - \mathcal{L} = \frac{p_x^2+p_y^2}{2m} + V(\sqrt{x^2+y^2})$$

Polar $\mathcal{L} = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - V(r)$

$$\begin{cases} p_r = m\dot{r} \\ p_\phi = mr^2\dot{\phi} \end{cases} \quad \begin{cases} \dot{r} = \frac{p_r}{m} \\ \dot{\phi} = \frac{p_\phi}{mr^2} \end{cases} \quad \mathcal{H} = p_r\dot{r} + p_\phi\dot{\phi} - \mathcal{L} = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + V(r)$$

EoM
$$\begin{cases} \dot{r} = \frac{\partial \mathcal{H}}{\partial p_r} = \frac{p_r}{m} \quad \checkmark \\ \dot{p}_r = -\frac{\partial \mathcal{H}}{\partial r} = -\frac{\partial}{\partial r}\underbrace{\left[V(r) + \frac{p_\phi^2}{2mr^2}\right]}_{V_{eff}(r)} \end{cases} \quad \begin{cases} \dot{\phi} = \frac{\partial \mathcal{H}}{\partial p_\phi} = \frac{p_\phi}{mr^2} \quad \checkmark \\ \dot{p}_\phi = -\frac{\partial \mathcal{H}}{\partial \phi} = 0 \quad p_\phi = \text{const} \end{cases}$$

(cons. of ang. momentum)

Poisson brackets

The Poisson brackets offer a method to write canonical ~~equa~~ Hamilton's equations in a more symmetric way. Complete equivalence of q_α's and p_α's.

Recall:
$$\begin{cases} \dot{q}_\alpha = \frac{\partial \mathcal{H}}{\partial p_\alpha} \\ \dot{p}_\alpha = -\frac{\partial \mathcal{H}}{\partial q_\alpha} \end{cases} \quad \alpha = 1, \dots, n$$

Definition: For any two functions of phase-space and time

$$f(q,p,t) \text{ and } g(q,p,t)$$

their Poisson braket is defined as

$$\boxed{[f,g] \equiv \frac{\partial f}{\partial q_\alpha}\frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial g}{\partial q_\alpha}}$$

(Notice that Landau-Lifschitz uses the opposite sign convention)

$[f,g]$ is another function of q, p and t

Properties

1) antisymmetry : $[g,f] = -[f,g]$; $[f,f] = 0$

2) $[f, \text{const}] = 0$

3) linearity : $[f_1 + f_2, g] = [f_1, g] + [f_2, g]$

$[\alpha f, g] = \alpha [f, g]$ $\qquad [f, \alpha g] = \alpha [f,g]$ (α const)

4) Leibnitz rule : $[fg, h] = f[g,h] + g[f,h]$

5) Jacobi identity : $[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = 0$

Can be shown by explicit calculation. Sum of $8 \times 3 = 24$ terms that all cancel.

$$[f,[g,h]] = \frac{\partial f}{\partial q_\alpha}\frac{\partial [g,h]}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial [g,h]}{\partial q_\alpha} = \frac{\partial f}{\partial q_\alpha}\frac{\partial}{\partial p_\alpha}\left[\frac{\partial g}{\partial q_\beta}\frac{\partial h}{\partial p_\beta} - \frac{\partial g}{\partial p_\beta}\frac{\partial h}{\partial q_\beta}\right] - \ldots$$

$$= \frac{\partial f}{\partial q_\alpha}\frac{\partial^2 g}{\partial p_\alpha \partial q_\beta}\frac{\partial h}{\partial p_\beta} + \frac{\partial f}{\partial q_\alpha}\frac{\partial g}{\partial q_\beta}\frac{\partial^2 h}{\partial p_\alpha \partial p_\beta} - \ldots \quad = 8 \text{ terms}$$

~~Combining with solutions of Hamilton's equations:~~

6) $$[q_\alpha, f] = \underbrace{\frac{\partial q_\alpha}{\partial q_\beta}}_{=\delta_{\alpha\beta}}\frac{\partial f}{\partial p_\beta} - \underbrace{\frac{\partial q_\alpha}{\partial p_\beta}}_{=0}\frac{\partial f}{\partial q_\beta} = \frac{\partial f}{\partial p_\alpha}$$

$$[p_\alpha, f] = -\frac{\partial f}{\partial q_\alpha}$$ (same proof)

7) $[q_\alpha, q_\beta] = [p_\alpha, p_\beta] = 0$ $\qquad [q_\alpha, p_\beta] = \delta_{\alpha\beta}$

Combining with Hamilton's equations:

$$[q_\alpha, \mathcal{H}] = \frac{\partial \mathcal{H}}{\partial p_\alpha} \qquad [p_\alpha, \mathcal{H}] = -\frac{\partial \mathcal{H}}{\partial q_\alpha} \qquad \Rightarrow \boxed{\begin{cases} \dot{q}_\alpha = [q_\alpha, \mathcal{H}] \\ \dot{p}_\alpha = [p_\alpha, \mathcal{H}] \end{cases}}$$

$\Rightarrow$ The canonical equations are completely symmetric in q_α and p_α when written with Poisson brakets.

<u>Definition</u> A transformation of p's and q's that preserves the form of Hamilton's equations is called a "<u>canonical transformation</u>" ("restricted" if it does not depend on t)

Take the inverse transf $Q_\alpha = Q_\alpha(q,p)$, $P_\alpha = P_\alpha(q,p)$

$$\dot Q_\alpha = \frac{\partial Q_\alpha}{\partial q_\beta}\dot q_\beta + \frac{\partial Q_\alpha}{\partial p_\beta}\dot p_\beta = \frac{\partial Q_\alpha}{\partial q_\beta}\frac{\partial \mathcal{H}}{\partial p_\beta} - \frac{\partial Q_\alpha}{\partial p_\beta}\frac{\partial \mathcal{H}}{\partial q_\beta}$$

$$\frac{\partial \mathcal{H}'}{\partial P_\alpha} = \frac{\partial \mathcal{H}}{\partial q_\beta}\frac{\partial q_\beta}{\partial P_\alpha} + \frac{\partial \mathcal{H}}{\partial p_\beta}\frac{\partial p_\beta}{\partial P_\alpha}$$

$$\dot Q_\alpha = \frac{\partial \mathcal{H}'}{\partial P_\alpha} \iff \boxed{\left.\frac{\partial Q_\alpha}{\partial q_\beta}\right|_{q,p} = \left.\frac{\partial p_\beta}{\partial P_\alpha}\right|_{Q,P} ; \left.\frac{\partial Q_\alpha}{\partial p_\beta}\right|_{p,q} = -\left.\frac{\partial q_\beta}{\partial P_\alpha}\right|_{Q,P}}$$

$$\dot P_\alpha = \frac{\partial P_\alpha}{\partial q_\beta}\dot q_\beta + \frac{\partial P_\alpha}{\partial p_\beta}\dot p_\beta = \frac{\partial P_\alpha}{\partial q_\beta}\frac{\partial \mathcal{H}}{\partial p_\beta} - \frac{\partial P_\alpha}{\partial p_\beta}\frac{\partial \mathcal{H}}{\partial q_\beta}$$

$$\frac{\partial \mathcal{H}'}{\partial Q_\alpha} = \frac{\partial \mathcal{H}}{\partial q_\beta}\frac{\partial q_\beta}{\partial Q_\alpha} + \frac{\partial \mathcal{H}}{\partial p_\beta}\frac{\partial p_\beta}{\partial Q_\alpha}$$

$$\dot P_\alpha = -\frac{\partial \mathcal{H}'}{\partial Q_\alpha} \iff \boxed{\left.\frac{\partial P_\alpha}{\partial q_\beta}\right|_{q,p} = -\frac{\partial p_\beta}{\partial Q_\alpha} ; \left.\frac{\partial P_\alpha}{\partial p_\beta}\right|_{q,p} = \left.\frac{\partial q_\beta}{\partial Q_\alpha}\right|_{Q,P}}$$

("Direct condition" for restricted canonical transf.)

· What happens to the Poisson brackets in the new coordinates?

$$[Q_\alpha, Q_\beta]_{q,p} = \frac{\partial Q_\alpha}{\partial q_\gamma}\frac{\partial Q_\beta}{\partial p_\gamma} - \frac{\partial Q_\alpha}{\partial p_\gamma}\frac{\partial Q_\beta}{\partial q_\gamma} = \frac{\partial p_\gamma}{\partial P_\alpha}\frac{\partial Q_\beta}{\partial p_\gamma} + \frac{\partial q_\gamma}{\partial P_\alpha}\frac{\partial Q_\beta}{\partial q_\gamma} = \frac{\partial Q_\beta}{\partial P_\alpha} = 0$$

$$[P_\alpha, P_\beta]_{q,p} = \frac{\partial P_\alpha}{\partial q_\gamma}\frac{\partial P_\beta}{\partial p_\gamma} - \frac{\partial P_\alpha}{\partial p_\gamma}\frac{\partial P_\beta}{\partial q_\gamma} = -\frac{\partial p_\gamma}{\partial Q_\alpha}\frac{\partial P_\beta}{\partial p_\gamma} - \frac{\partial q_\gamma}{\partial Q_\alpha}\frac{\partial P_\beta}{\partial q_\gamma} = -\frac{\partial P_\beta}{\partial Q_\alpha} = 0$$

$$[Q_\alpha, P_\beta]_{q,p} = \frac{\partial Q_\alpha}{\partial q_\gamma}\frac{\partial P_\beta}{\partial p_\gamma} - \frac{\partial Q_\alpha}{\partial p_\gamma}\frac{\partial P_\beta}{\partial q_\gamma} = \frac{\partial p_\gamma}{\partial P_\alpha}\frac{\partial P_\beta}{\partial p_\gamma} + \frac{\partial q_\gamma}{\partial P_\alpha}\frac{\partial P_\beta}{\partial q_\gamma} = \frac{\partial P_\beta}{\partial P_\alpha} = \delta_{\alpha\beta}$$

$$\Rightarrow \begin{cases} [Q_\alpha, Q_\beta]_{q,p} = 0 \\ [P_\alpha, P_\beta]_{q,p} = 0 \\ [Q_\alpha, P_\beta]_{q,p} = \delta_{\alpha\beta} \end{cases}$$

<u>Canonical transformations preserve the fundamental Poisson brackets</u>

For <u>any</u> function $f(q,p,t)$:

$$\frac{d}{dt}f = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial f}{\partial p_\alpha}\dot{p}_\alpha = \frac{\partial f}{\partial t} + \underbrace{\frac{\partial f}{\partial q_\alpha}\frac{\partial \mathcal{H}}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial \mathcal{H}}{\partial q_\alpha}}_{[f,\mathcal{H}]}$$

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial t} + [f,\mathcal{H}]}$$

For $f=\mathcal{H}$, $\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t} + \cancel{[\mathcal{H},\mathcal{H}]} = \frac{\partial \mathcal{H}}{\partial t}$, as we already saw.

Canonical transformations

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G 9.4

We ~~are~~ know that in the Lagrangian formalism any "point transformation" of the coordinates $q_\alpha = q_\alpha(\{Q_\beta\},t)$ preserves the form of the Euler-Lagrange eqns:

new coords.

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}'}{\partial \dot{Q}_\alpha}\right) - \frac{\partial \mathcal{L}'}{\partial Q_\alpha} = 0 \iff \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\right) - \frac{\partial \mathcal{L}}{\partial q_\alpha} = 0$$

where $\mathcal{L}'(Q,\dot{Q};t) \equiv \mathcal{L}(q(Q,t), \dot{q}(Q,\dot{Q},t);t)$

In the Hamiltonian formalism, the p_α's and the q_α's are <u>all</u> coordinates in phase space. I could consider the more general transformation $q_\alpha = q_\alpha(\{Q_\beta,P_\beta\}_{\beta=1,\dots,n},t)$ and $p_\alpha = p_\alpha(\{Q_\beta,P_\beta\}_{\beta=1,\dots,n},t)$, which mixes P and Q.

<u>Question</u>. Under which conditions this transformation preserves the form of Hamilton's eqns?
When do we have that

$$\dot{Q}_\alpha = \frac{\partial \mathcal{H}'}{\partial P_\alpha}, \quad \dot{P}_\alpha = -\frac{\partial \mathcal{H}'}{\partial Q_\alpha}$$

where $\mathcal{H}'(Q,P) \equiv \mathcal{H}(q(Q,P), p(Q,P))$

(for simplicity, assume no explicit t dependence)

For a generic function $f(Q,P)$:

$$[Q_\alpha, f]_{q,p} = \frac{\partial Q_\alpha}{\partial q_\gamma}\frac{\partial f}{\partial p_\gamma} - \frac{\partial Q_\alpha}{\partial p_\gamma}\frac{\partial f}{\partial q_\gamma}$$

$$= \frac{\partial Q_\alpha}{\partial q_\gamma}\left(\frac{\partial f}{\partial Q_\beta}\frac{\partial Q_\beta}{\partial p_\gamma} + \frac{\partial f}{\partial P_\beta}\frac{\partial P_\beta}{\partial p_\gamma}\right) - \frac{\partial Q_\alpha}{\partial p_\gamma}\left(\frac{\partial f}{\partial Q_\beta}\frac{\partial Q_\beta}{\partial q_\gamma} + \frac{\partial f}{\partial P_\beta}\frac{\partial P_\beta}{\partial q_\gamma}\right)$$

$$= \frac{\partial f}{\partial Q_\beta}\cancelto{0}{[Q_\alpha, Q_\beta]_{q,p}} + \frac{\partial f}{\partial P_\beta}\underbrace{[Q_\alpha, P_\beta]_{q,p}}_{\delta_{\alpha\beta}} = \frac{\partial f}{\partial P_\alpha}$$

$$[P_\alpha, f]_{q,p} = [\ldots \text{ same derivation}] = -\frac{\partial f}{\partial Q_\alpha}$$

Same as computing the brackets w.r.t. Q,P directly

For any two functions $f(Q,P)$, $g(Q,P)$:

$$[f,g]_{q,p} = \frac{\partial f}{\partial q_\alpha}\frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial g}{\partial q_\alpha}$$

$$= \left(\frac{\partial f}{\partial Q_\beta}\frac{\partial Q_\beta}{\partial q_\alpha} + \frac{\partial f}{\partial P_\beta}\frac{\partial P_\beta}{\partial q_\alpha}\right)\frac{\partial g}{\partial p_\alpha} - \left(\frac{\partial f}{\partial Q_\beta}\frac{\partial Q_\beta}{\partial p_\alpha} + \frac{\partial f}{\partial P_\beta}\frac{\partial P_\beta}{\partial p_\alpha}\right)\frac{\partial g}{\partial q_\alpha}$$

$$= \frac{\partial f}{\partial Q_\beta}[Q_\beta, g]_{q,p} + \frac{\partial f}{\partial P_\beta}[P_\beta, g]_{q,p} = \frac{\partial f}{\partial Q_\beta}\frac{\partial g}{\partial P_\beta} - \frac{\partial f}{\partial P_\beta}\frac{\partial g}{\partial Q_\beta} = [f,g]_{Q,P}$$

<u>Canonical transformations preserve ALL Poisson brackets!</u>

In summary: canonically transformed variables form a special class of equivalence

1) they preserve the form of Hamilton's eqns

$$\begin{cases} \dot{Q}_\alpha = \dfrac{\partial \mathcal{H}}{\partial P_\alpha} = [Q_\alpha, \mathcal{H}] \\ \dot{P}_\alpha = -\dfrac{\partial \mathcal{H}}{\partial Q_\alpha} = [P_\alpha, \mathcal{H}] \end{cases}$$

2) they preserve all Poisson brackets $[f,g]_{q,p} = [f,g]_{Q,P} = [f,g]$

One can forget about what the original q,p were.

Symplectic matrices

Let's define $\vec{\eta} \equiv \begin{pmatrix} q_1 \\ \vdots \\ q_n \\ p_1 \\ \vdots \\ p_n \end{pmatrix}$ a column vector with $2n$ components

Hamilton's eqns are $$\dot{\vec{\eta}} = \begin{pmatrix} \frac{\partial H}{\partial p_\alpha} \\ -\frac{\partial H}{\partial q_\alpha} \end{pmatrix} = \underbrace{\left[\begin{array}{c|c} 0 & \mathbb{1} \\ \hline -\mathbb{1} & 0 \end{array}\right]}_{\equiv \hat{J}} \frac{\partial H}{\partial \vec{\eta}}$$

$$\hat{J} = \left[\begin{array}{c|c} 0 & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \\ \hline \begin{matrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{matrix} & 0 \end{array}\right]$$ is a $2n \times 2n$ matrix, for which $\hat{J}^T = \left[\begin{array}{c|c} 0 & -\mathbb{1} \\ \hline \mathbb{1} & 0 \end{array}\right]$

$$\Rightarrow \boxed{\hat{J}^T = -\hat{J} = \hat{J}^{-1}}$$

$\hat{J}^2 = -\mathbb{1}_{2n \times 2n}$ $\qquad$ $\mathrm{Det}(\hat{J}) = (-1)^n$

Let's define $\vec{\xi} = \begin{pmatrix} Q_1 \\ \vdots \\ Q_n \\ P_1 \\ \vdots \\ P_n \end{pmatrix}$ $\qquad \hat{M}_{AB} \equiv \frac{\partial \xi^A}{\partial \eta^B} = \begin{array}{c} \begin{matrix} \overbrace{\hphantom{\frac{\partial Q^\alpha}{\partial q^\beta}}}^{n} & \overbrace{\hphantom{\frac{\partial P^\alpha}{\partial q^\beta}}}^{n} \end{matrix} \\ \left[\begin{array}{c|c} \frac{\partial Q^\alpha}{\partial q^\beta} & \frac{\partial P^\alpha}{\partial q^\beta} \\ \hline \frac{\partial Q^\alpha}{\partial p^\beta} & \frac{\partial P^\alpha}{\partial p^\beta} \end{array}\right] \end{array} \begin{array}{l} \}\, n \\ \}\, n \end{array}$

$A, B = 1, \dots, 2n$

$\alpha, \beta = 1, \dots, n$

The 4 direct conditions for canonical transformations are equivalent to

$$\hat{M}^T \cdot \hat{J} \cdot \hat{M} = \hat{J}$$

<u>Proof</u> It can be shown by direct calculation that

$$\hat{M}^T \cdot \hat{J} \cdot \hat{M} = \left[\begin{array}{c|c} [Q_\alpha, Q_\beta] & [Q_\alpha, P_\beta] \\ \hline [P_\alpha, Q_\beta] & [P_\alpha, P_\beta] \end{array}\right] = \left[\begin{array}{c|c} 0 & \delta_{\alpha\beta} \\ \hline -\delta_{\alpha\beta} & 0 \end{array}\right] = \hat{J}$$

$$\Rightarrow \det(\hat{M}^T \cdot \hat{J} \cdot \hat{M}) = (\det(\hat{M}))^2 \det(\hat{J}) = \det(\hat{J})$$

$$\Rightarrow |\det(\hat{M})| = 1$$

Liouville's theorem

· We need two preliminary results

1) Time evolution is a canonical transormation.

$t \to t+dt \qquad (q(t), p(t)) \equiv (q,p) \qquad (q(t+dt), p(t+dt)) \equiv (Q,P)$

$$\begin{cases} Q_\alpha \simeq q_\alpha + \dot{q}_\alpha dt = q_\alpha + \dfrac{\partial \mathcal{H}}{\partial p_\alpha} dt \\ P_\alpha \simeq p_\alpha + \dot{p}_\alpha dt = p_\alpha - \dfrac{\partial \mathcal{H}}{\partial q_\alpha} dt \end{cases}$$

$$[Q_\alpha, Q_\beta] \simeq \underbrace{[q_\alpha, q_\beta]}_{=0} + \left[\frac{\partial \mathcal{H}}{\partial p_\alpha}, q_\beta\right] dt + \left[q_\alpha, \frac{\partial \mathcal{H}}{\partial p_\beta}\right] dt + \mathcal{O}(dt^2)$$

$$\approx -\frac{\partial^2 \mathcal{H}}{\partial p_\beta \partial p_\alpha} dt + \frac{\partial^2 \mathcal{H}}{\partial p_\alpha \partial p_\beta} dt = 0$$

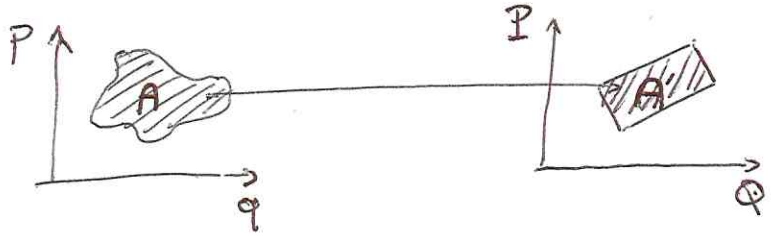
$$[P_\alpha, P_\beta] \approx \ldots = 0$$

$$[Q_\alpha, P_\beta] \approx [q_\alpha, p_\beta] - \left[q_\alpha, \frac{\partial \mathcal{H}}{\partial q_\beta}\right] dt + \left[\frac{\partial \mathcal{H}}{\partial p_\alpha}, p_\beta\right] dt + \mathcal{O}(dt^2)$$

$$\approx \delta_{\alpha\beta} - \cancel{\frac{\partial^2 \mathcal{H}}{\partial p_\alpha \partial q_\beta}} dt + \cancel{\frac{\partial^2 \mathcal{H}}{\partial q_\beta \partial p_\alpha}} dt \approx \delta_{\alpha\beta}$$

Poisson brackets are preserved, so it is canonical

2) Canonical transf. preserve the volume in phase space



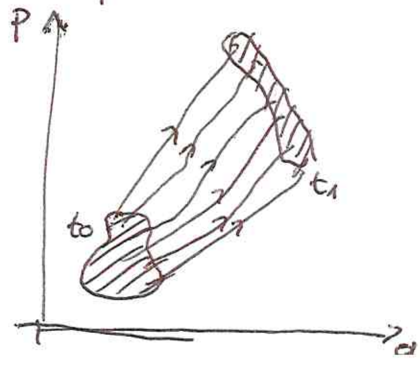
$\mathrm{Vol}(A) = \mathrm{Vol}(A')$

$$\vec{\eta} = \begin{pmatrix} q_1 \\ \vdots \\ q_n \\ p_1 \\ \vdots \\ p_n \end{pmatrix} \qquad \vec{\xi} = \begin{pmatrix} Q_1 \\ \vdots \\ Q_n \\ P_1 \\ \vdots \\ P_n \end{pmatrix}$$

$$\mathrm{Vol}(A') = \int_{A'} dQ_1 \ldots dQ_n dP_1 \ldots dP_n = \int_A |D| dq_1 \ldots dq_n dp_1 \ldots dp_n$$

but $|D| = \left|\det \dfrac{\partial \xi_A}{\partial \eta_B}\right| = |\det \hat{M}| = 1 \implies \mathrm{Vol}(A') = \mathrm{Vol}(A)$

Therefore, 1) + 2) imply that time evolution preserves the volume in phase space (<u>Liouville's theorem</u>)



Symmetries and generators

Time evolution (= time translation) is expressed by Poisson brakets with $\mathcal{H}$, which is the conserved quantity associated to invariance under time translations.

$$\dot{f} = [f, \mathcal{H}] \left(+ \frac{\partial f}{\partial t} \right)$$

The same is true for other transformations

<u>space translations</u> $\begin{cases} \vec{r}_a \to \vec{r}_a + \varepsilon \hat{n} \\ \dot{\vec{r}}_a \to \dot{\vec{r}}_a \end{cases}$ $\quad \vec{p}_a = \dot{\vec{r}}_a m_a \quad \Rightarrow \begin{cases} \vec{r}_a \to \vec{r}_a + \varepsilon \hat{n} \\ \vec{p}_a \to \vec{p}_a \end{cases}$

$$f(\vec{r}_a, \vec{p}_a) \to f(\vec{r}_a + \varepsilon \hat{n}, \vec{p}_a) = f(\vec{r}_a, \vec{p}_a) + \sum_a \frac{\partial f}{\partial \vec{r}_a} \cdot \varepsilon \hat{n}$$

$$\delta f \equiv f(\vec{r}_a + \varepsilon \hat{n}, \vec{p}_a) - f(\vec{r}_a, \vec{p}_a) = \varepsilon \sum_a \frac{\partial f}{\partial \vec{r}_a} \cdot \hat{n} = \varepsilon \sum_a \frac{\partial f}{\partial \vec{r}_a} \cdot \frac{\partial}{\partial \vec{p}_a} \left(\sum_b \vec{p}_b \cdot \hat{n} \right)$$

$$= \varepsilon \Big[f, \underbrace{\sum_b \vec{p}_b}_{\equiv \vec{P}_{TOT}} \cdot \hat{n} \Big] = \varepsilon \hat{n} \cdot [f, \vec{P}_{TOT}]$$

Poisson brakets with $\vec{P}_{TOT}$ (conserved quantity associated with space translations)

<u>Rotations</u> $\begin{cases} \vec{r}_a \to \vec{r}_a + (\hat{n} \times \vec{r}_a) \delta\phi \\ \dot{\vec{r}}_a \to \dot{\vec{r}}_a + (\hat{n} \times \dot{\vec{r}}_a) \delta\phi \end{cases}$ $\quad \vec{p}_a = m_a \dot{\vec{r}}_a \to \vec{p}_a + (\hat{n} \times \vec{p}_a) \delta\phi$

$$\delta f = \delta\phi \left[\sum_a \frac{\partial f}{\partial \vec{r}_a} \cdot (\hat{n} \times \vec{r}_a) + \sum_a \frac{\partial f}{\partial \vec{p}_a} \cdot (\hat{n} \times \vec{p}_a) \right] = \hat{n}\, \delta\phi \cdot \sum_a \left(\vec{r}_a \times \frac{\partial f}{\partial \vec{r}_a} + \vec{p}_a \times \frac{\partial f}{\partial \vec{p}_a} \right)$$

$$[f, \vec{r}_a \times \vec{p}_a]_i = \cancel{\sum_a \left([f, \vec{r}_a] \times \vec{p}_a + \vec{r}_a \times [f, \vec{p}_a] \right)} = \varepsilon_{ijk} [f, r_{aj} p_{ak}]$$

$$= \varepsilon_{ijk} \left([f, r_{aj}] p_{ak} + r_{aj} [f, p_{ak}] \right) = \varepsilon_{ijk} \left(- \frac{\partial f}{\partial p_{aj}} p_{ak} + r_{aj} \frac{\partial f}{\partial r_{ak}} \right)$$

$$= \left(\vec{p}_a \times \frac{\partial f}{\partial \vec{p}_a} \right)_i + \left(\vec{r}_a \times \frac{\partial f}{\partial \vec{r}_a} \right)_i$$

$$\Rightarrow \delta f = \hat{n}\, \delta\phi \cdot \Big[f, \sum_a \vec{r}_a \times \vec{p}_a \Big] = \delta\phi\, \hat{n} \cdot [f, \vec{L}_{TOT}]$$

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Galilean transformations: $\begin{cases} \vec{r}_a \to \vec{r}_a + \varepsilon \vec{v}_R t \\ \vec{p}_a \to \vec{p}_a + \varepsilon m_a \vec{v}_R \end{cases}$

$$\delta f = \varepsilon \sum_a \left(\frac{\partial f}{\partial \vec{r}_a} \cdot \vec{v}_R t + \frac{\partial f}{\partial \vec{p}_a} \cdot m_a \vec{v}_R \right) = \varepsilon \vec{v}_R \cdot \sum_a \left[\frac{\partial f}{\partial \vec{r}_a} \cdot \frac{\partial}{\partial \vec{p}_a} \left(\sum_b \vec{p}_b t \right) + \frac{\partial f}{\partial \vec{p}_a} \cdot \frac{\partial}{\partial \vec{r}_a} \left(\sum_b m_b \vec{r}_b \right. \right.$$

$$= \varepsilon \vec{v}_R \cdot [f, \vec{G}] \qquad \boxed{G \equiv \sum_b \vec{p}_b t - \sum_b m_b \vec{r}_b = \vec{P}_{TOT} t - M \vec{r}_{CM}}$$

$\mathcal{H}, \vec{P}_{TOT}, \vec{L}_{TOT}, \vec{G}$ are the "generators" of time translations, space translations, rotations and boosts. They are not necessarily conserved. They are conserved only if they leave the Hamiltonian invariant. (i.e. if the transformations are symmetries

E.g., for translations:

$$\delta \mathcal{H}_{transl.} = \varepsilon \hat{n} \cdot [\mathcal{H}, \vec{P}_{TOT}] = -\varepsilon \hat{n} \cdot \dot{\vec{P}}_{TOT} \qquad \dot{\vec{P}}_{TOT} = 0 \Leftrightarrow \delta \mathcal{H}_{transl.} = 0$$

Same for $\vec{L}_{TOT}$ and $\vec{G}$

In general, ALL canonical transformations have a generator.

Ex: 1D harmonic oscillator

$$\mathcal{H} = \frac{p^2}{2m} + \frac{k}{2} q^2$$

Suppose that $\begin{cases} p = f(P)\cos Q \\ q = \dfrac{f(P)}{\sqrt{km}} \sin Q \end{cases}$ is a canonical transformation.

We need $1 = [q,p]_{Q,P} = \dfrac{f(P)f'(P)}{\sqrt{km}}\left(\cos^2 Q + \sin^2 Q\right) = \dfrac{1}{2\sqrt{km}} \dfrac{d}{dP} f^2(P)$

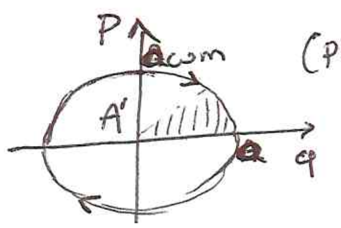
$$\Rightarrow f^2(P) = 2\underbrace{\sqrt{km}}_{\omega m}P \quad \Rightarrow \quad f(P) = \sqrt{2\omega m P} \qquad \omega^2 = \frac{k}{m}$$

$$\mathcal{H}(Q,P) = \frac{f^2(P)}{2m} = \omega P \qquad \text{(does not depend on } Q)$$

$$\dot{P} = -\frac{\partial \mathcal{H}}{\partial Q} = 0 \quad \Rightarrow \quad P = P_0 = \text{const}$$

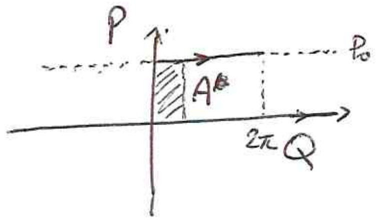
$$\dot{Q} = \frac{\partial \mathcal{H}}{\partial P} = \omega = \text{const} \quad \Rightarrow \quad Q = Q_0 + \omega t$$

$$\Rightarrow \begin{cases} q = \sqrt{\dfrac{2\omega P_0}{k}} \sin(\omega t + Q_0). \\ p = \underbrace{\sqrt{\dfrac{2\omega P_0}{k}}}_{= a} \omega m \cos(\omega t + Q_0) \end{cases} \qquad a = \sqrt{\frac{2\omega P_0}{k}} = \sqrt{\frac{2P_0}{\omega m}}$$



$(p > 0 \Rightarrow \dot{q} > 0)$

$\longrightarrow$



$$\mathrm{Vol}(A') = \int_{A'} dp\,dq = \omega m \int r\,dr\,d\varphi = \omega m \Delta Q \int_0^{\sqrt{\frac{2P_0}{\omega m}}} r\,dr = \frac{\omega m}{2}\left(\sqrt{\frac{2P_0}{\omega m}}\right)^2 \Delta Q$$

$$= P_0 \Delta Q = \mathrm{Vol}(A)$$