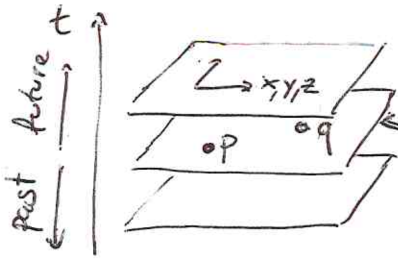
Recap of SR

SR is theory of 4-dim space-time: set of <u>events</u> labeled $(t, \vec{x})$

<u>Galilean space-time</u>:

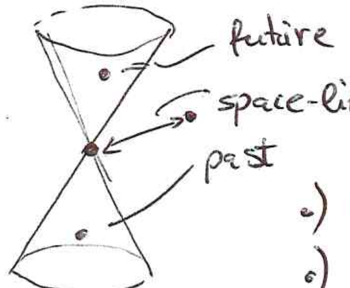


$p = (t, x, y, z)$ $q = (t, x', y', z')$

flat 3d surface of simultaneity (simultaneous events)

cannot be at p AND q

<u>Relativistic space-time</u>:



- Fundametal structure is the light cone (4d set of events)
- No concept of simultaneity
- Locally true also in GR

<u>Inertial frame</u> (or observer): non-accelerated coordinate system made of rigid (Cartesian) 3d grid with syncronized clocks (in GR, it may exist only locally)

Two inertial frames in relative motion are related.

Call $(t, \vec{x})$ and $(t', \vec{x}')$ the cooord. of <u>same</u> event in the 2 frames:

Galilean transf. $\begin{cases} t' = t \\ x' = x - vt \\ y' = y \\ z' = z \end{cases}$ vs SR boost $\begin{cases} t' = \gamma(t - vx/c^2) \\ x' = \gamma(x - vt) \\ y' = y \\ z' = z \end{cases}$ $\gamma^2 \equiv \dfrac{1}{1 - v^2/c^2}$

$w \equiv \dfrac{\Delta x}{\Delta t}$ $\boxed{w' = \dfrac{\Delta x'}{\Delta t'} = w - v \, , \; a' = a}$ vs $w' = \dfrac{\Delta x - v\Delta t}{\Delta t - v\Delta x/c^2} = \dfrac{w - v}{1 - wv/c^2}$

Leaves invariant: $\Delta \vec{x}^2$ (for simult. events only!), Δt

$w' = c \iff w = c$ $\boxed{c = \text{const}}$

SR boosts leave invariant $\boxed{\Delta s^2 \equiv -c^2 \Delta t^2 + \Delta \vec{x}^2}$ (check!)

Truly 4d theory: all coordinates (incl. t) can be transformed

<u>Notation</u>: x^μ; $x^0 = ct$, $\underbrace{x^1 = x, x^2 = y, x^3 = z}_{x^i \text{ for } i = 1,2,3}$ $\mu = 0,1,2,3$ (<u>can set $c = 1$</u>)

$\Delta s^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu$ (with summation convention)

$\eta \equiv \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$ (<u>METRIC</u>) replaces Cartesian metric $\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$

Separation between events can be

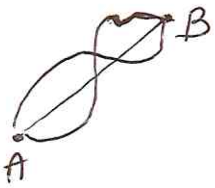
1) Time-like ($\Delta s^2 < 0$) 2) null ($\Delta s^2 = 0$) 3) space-like ($\Delta s^2 > 0$)

Proper time: $\Delta\tau^2 = -\Delta s^2$ is the time interval measured by an observer moving from A to B on a straight line (i.e. with ~~constant~~ $x^i_A = x^i_B$)

$(\Rightarrow \Delta x^i = 0,\ \Delta\tau^2 = \Delta t^2)$

For a general curve $x^\mu(\lambda)$, $\Delta x^\mu \approx \frac{dx^\mu}{d\lambda}\Delta\lambda$

$$\Delta\tau = \int_A^B d\lambda \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}$$



In GR, $\eta_{\mu\nu}$ will be replaced by a more general expression

Poincaré transformations

Two types of transf. leaving Δs^2 invariant

$$x^\mu \to \begin{cases} x^{\mu'} = x^\mu + a^\mu & \text{(translations, both in time and space)} \\ x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu & \text{(Lorentz transf.)} \end{cases}$$

$$\Delta s'^2 = (\Delta x')^T \eta (\Delta x') = \Delta x^T (\Lambda^T \eta \Lambda)\Delta x = \Delta s^2 \iff \boxed{\Lambda^T \eta \Lambda = \eta}$$

$\boxed{\Lambda^{\mu'}{}_\rho \eta_{\mu'\nu'} \Lambda^{\nu'}{}_\sigma = \eta_{\rho\sigma}}$ (similar to $R^T R = \mathbb{1}$ for rotations)

They form a group: $O(3,1)$

E.g. $\Lambda^{\mu'}{}_\nu = \begin{pmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix}$ rotation around $\hat{z}$ $(0 < \theta < 2\pi)$

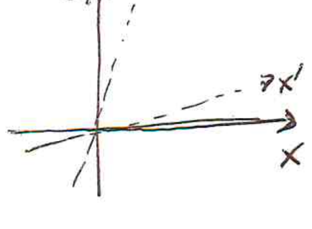
$\Lambda^{\mu'}{}_\nu = \begin{pmatrix} \cosh\phi & -\sinh\phi & & \\ -\sinh\phi & \cosh\phi & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ boost in x direction $(-\infty < \phi < +\infty)$

(check that it obeys $\Lambda^T \eta \Lambda = \eta$)

Rotations and boosts form the group $SO(3,1)$ (special Lorentz tr.)

For boosts: $\begin{cases} t' = \cosh\phi\, t - \sinh\phi\, x = \cosh\phi(t - \tanh\phi\, x) \\ x' = -\sinh\phi\, t + \cosh\phi\, x = \cosh\phi(x - \tanh\phi\, t) \end{cases}$

$\Rightarrow v = \tanh\phi$, $\phi = \tanh^{-1}(v)$, $\gamma = \cosh\phi$ $(v < 1)$



x' axis $(t' = 0)$: $t = vx$

t' axis $(x' = 0)$: $t = x/v$

The light cone $t = \pm x$ is left invariant (check!)

TENSORS IN MINKOWSKI SPACE-TIME

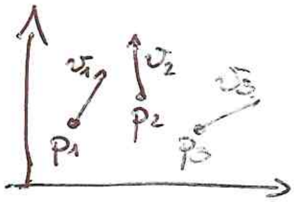
We need to construct vector and tensor fields. For this, we need to attach a vector space to each point of space-time. (these vector spaces must not be confused with the space-time itself). We want to do it in a way that is ready to generalize to curved space-times.

Vectors:

For each p, we construct a real vector space T_p (tangent space at p)

$$\forall v, w \in T_p \text{ and } \forall a, b \in \mathbb{R} \implies av + bw \in T_p$$

The set of all T_p's is the tangent bundle.
A set of vectors containing one vector $\in T_p \ \forall p$ is called a vector field



$v_1 \in T_{p_1}, v_2 \in T_{p_2}, \ldots$

Basis of T_p: complete set of n linearly indep. vectors, with $n = \dim(T_p)$

In Minkowski, there is a natural basis $\hat{e}_{(\mu)}$ aligned with the 4 axes (the same for all p's):

$$v(p) = \underbrace{v^\mu(p)}_{\text{components}} \underbrace{\hat{e}_{(\mu)}}_{\text{basis vectors}}$$

Example:

Tangent vector to a curve $x^\mu(\lambda)$: $\quad V^\mu = \dfrac{dx^\mu}{d\lambda}, \quad V = V^\mu \hat{e}_{(\mu)}$

Under Lorentz transf., $V^\mu \to V^{\mu'} = \Lambda^{\mu'}{}_\nu V^\nu$, but V must not transform

$$V = V^{\mu'} \hat{e}_{(\mu')} = \Lambda^{\mu'}{}_\nu V^\nu \hat{e}_{(\mu')} \implies \hat{e}_{(\mu)} = \Lambda^{\mu'}{}_\mu \hat{e}_{(\mu')}$$

$\Lambda_{\nu'}{}^\mu$ is the inverse transf. of $\Lambda^{\nu'}{}_\mu$ (e.g. when $\phi \to -\phi$, $\vec{v} \to -\vec{v}$)$^{(*)}$

$$\Lambda_{\nu'}{}^\mu \hat{e}_{(\mu)} = \underbrace{\Lambda_{\nu'}{}^\mu \Lambda^{\mu'}{}_\mu}_{\delta_{\nu'}^{\mu'}} \hat{e}_{(\mu')} = \hat{e}_{(\nu')}$$

$\cancel{(*) \text{ because } \dfrac{\partial x^\mu}{\partial x^{\nu'}} \dfrac{\partial x^{\nu'}}{\partial x^\lambda} = \delta^{\mu}_{\lambda}}$

$$\implies \boxed{\hat{e}_{(\mu')} = \Lambda_{\mu'}{}^\mu \hat{e}_{(\mu)}}$$

Transformation of the basis

(*) $\boxed{\Lambda_{\nu'}{}^\mu \equiv (\Lambda^{-1})^\mu{}_{\nu'}}$

(the index position will become clear later)

One-forms (or dual vectors):

Define T_p^* (cotangent space) as the dual vector space of T_p

space of all linear applications $\omega : T_p \to \mathbb{R}$

$$\omega(aV + bW) = a\omega(V) + b\omega(W) \qquad \forall a,b \in \mathbb{R}$$

It is a vector space with $(a\omega + b\eta)(V) = a\omega(V) + b\eta(V)$

Basis such that $\hat{\theta}^{(\nu)}(\hat{e}_{(\mu)}) = \delta^\nu_\mu$, $\omega = \omega_\mu \hat{\theta}^{(\mu)}$

components (with lower indices!)

Action of 1-forms on vectors:

$$\omega(V) = \omega_\mu \hat{\theta}^{(\mu)}(V^\nu \hat{e}_{(\nu)}) = \omega_\mu V^\nu \delta^\mu_\nu = \omega_\mu V^\mu \in \mathbb{R}$$

($\sim$ scalar product of components)

Lorentz transf.:

$$\omega(V) = \omega_\mu V^\mu = \omega_{\mu'} V^{\mu'} = \omega_{\mu'} \Lambda^{\mu'}{}_\nu V^\nu \Rightarrow \omega_\mu = \omega_{\mu'} \Lambda^{\mu'}{}_\mu$$

~~Like~~ Therefore,

$$\boxed{\omega_{\nu'} = \Lambda^\mu{}_{\nu'} \omega_\mu} \quad \text{and} \quad \boxed{\hat{\theta}^{(\mu')} = \Lambda^{\mu'}{}_\nu \hat{\theta}^{(\nu)}}$$

(finish the proof!)

Objects with upper (lower) indices transform with $\Lambda^{\nu'}{}_\mu$ ($\Lambda_{\nu'}{}^\mu$)

Sometimes called contravariant (covariant) vectors.

Since $\omega(V) = \omega_\mu V^\mu$, I can also interpret it as $V(\omega)$.

Hence, $T_p^{**} = T_p$. Cannot proceed further.

Example: the gradient of a scalar function is covariant $d\phi \equiv \frac{\partial \phi}{\partial x^\mu} \hat{\theta}^{(\mu)}$

$$\frac{\partial \phi}{\partial x^{\mu'}} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial \phi}{\partial x^\mu} = \Lambda_{\mu'}{}^\mu \frac{\partial \phi}{\partial x^\mu}$$

Notation: $\frac{\partial \phi}{\partial x^\mu} \equiv \partial_\mu \phi \equiv \phi_{,\mu}$

The gradient acting on the tangent vector $\to \frac{\partial \phi}{\partial x_\mu} \frac{dx^\mu}{d\lambda} = \frac{d\phi}{d\lambda}$

Tensors

Generalize vectors and one-forms to arbitrary numbers of indices.
Tensor of rank (k,l) is a multilinear map from vectors and one-forms into $\mathbb{R}$:

$$T : \underbrace{T_p^* \times \dots \times T_p^*}_{k} \times \underbrace{T_p \times \dots \times T_p}_{l} \longrightarrow \mathbb{R}$$

Multilinear means linear in each argument:

$$T(a\omega + b\eta, cV + dW) = cT(a\omega+b\eta, V) + dT(a\omega+b\eta, W)$$
$$= acT(\omega,V) + adT(\omega,W) + bcT(\eta,V) + bdT(\eta,W)$$

Scalar is $(0,0)$ tensor, vector is $(1,0)$ tensor, co-vector is $(0,1)$

Prove that ~~it is a~~ tensors are a vector space.

~~Basis~~ Tensor product is a new operation ~~defined as~~ that given T of rank (k,l) and S of rank (m,n) gives a tensor of rank $(k+m, l+n)$:

$$(T \otimes S)(\omega^{(1)}, \dots, \omega^{(k+m)}, V^{(1)}, \dots, V^{(l+n)})$$
$$\equiv T(\omega^{(1)}, \dots, \omega^{(k)}, V^{(1)}, \dots, V^{(l)})\, S(\omega^{(k+1)}, \dots, \omega^{(k+m)}, V^{(l+1)}, \dots, V^{(l+n)})$$

In general, $T \otimes S \neq S \otimes T$

Basis is the ~~set~~ of all tensors $(4^{k+l}$ of them!)

$$\hat{e}_{(\mu_1)} \otimes \dots \otimes \hat{e}_{(\mu_k)} \otimes \hat{\theta}^{(\nu_1)} \otimes \dots \otimes \hat{\theta}^{(\nu_l)}$$

> dimension of space of (k,l) tensors is 4^{k+l}

For any tensor T,

$$T = T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}\, \hat{e}_{(\mu_1)} \otimes \dots \otimes \hat{\theta}^{(\nu_l)}$$

or equivalently $T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} = T(\hat{\theta}^{(\mu_1)}, \dots, \hat{\theta}^{(\mu_k)}, \hat{e}_{(\nu_1)}, \dots, \hat{e}_{(\nu_l)})$

(action of T on the basis). T completely specified by components.
Action of T on ~~vector~~ its arguments:

$$T(\omega^{(1)}, \dots, \omega^{(k)}, V^{(1)}, \dots, V^{(l)}) = T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}\, \omega^{(1)}_{\mu_1} \dots \omega^{(k)}_{\mu_k} V^{(1)\nu_1} \dots V^{(l)\nu_l}$$

The order matters!!

In components:

$$T \otimes S \sim T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}\, S^{\mu_{k+1} \dots \mu_{k+m}}{}_{\nu_{l+1} \dots \nu_{l+1}}$$

Under Lorentz, $T^{\mu_1'\dots\mu_k'}{}_{\nu_1'\dots\nu_\ell'} = \Lambda^{\mu_1'}{}_{\mu_1}\dots\Lambda^{\mu_k'}{}_{\mu_k}\Lambda_{\nu_1'}{}^{\nu_1}\dots\Lambda_{\nu_\ell'}{}^{\nu_\ell}T^{\mu_1\dots\mu_k}{}_{\nu_1\dots\nu_\ell}$

Each upper/lower index transforms as vector/covector.
This ~~is~~ can be an alternative definition of tensor.

Consider a $(1,1)$ tensor T. Defined as a map $T: T_p^*\otimes T_p \to \mathbb{R}$.
Let it act on $V\in T_p$. It is now a map $T(\dots,V): T_p^*\to\mathbb{R}$.
It maps $\omega\in T_p^* \longrightarrow T(\omega,V)\in\mathbb{R}$.
Therefore, $T(\dots,V)$ is a $(1,0)$ tensor (i.e, a vector).
Thus, T is also a map from T_p into T_p.

$$\begin{aligned} T:\ T_p &\longrightarrow T_p \\ V &\longrightarrow T(\dots,V) \\ V^\mu &\longrightarrow T^\mu{}_\nu V^\nu \end{aligned}$$

Under Lorentz, $T^{\mu'}{}_{\nu'}V^{\nu'} = \Lambda^{\mu'}{}_\rho \underbrace{\Lambda_{\nu'}{}^\sigma\Lambda^{\nu'}{}_\tau}_{\delta^\sigma_\tau} T^\rho{}_\sigma V^\tau = \Lambda^{\mu'}{}_\nu T^\nu{}_\sigma V^\sigma$.

It transforms as a vector! It actually is a vector

Def.: "Contraction" Each pair of repeated upper/lower indices is said to be contracted (= summed over from 0 to 3)

Each pair of contracted indices is Lorentz-invariant (their direct and inverse transf. cancel away): it is a scalar

Ex. $U^\mu{}_\nu \equiv \underset{(1,2)}{T^\mu{}_{\nu\rho}}\ \underset{(2,1)}{S^{\rho\sigma}{}_\tau}\ \underset{(0,1)}{\omega_\sigma}\ \underset{(1,0)}{V^\tau}$ is a $(1,1)$ tensor

Def. Symmetric/antisymmetric if exchanging two same-type indices does not change the tensor / returns a $-$ sign.

Ex $S_{\mu\nu\rho} = S_{\nu\mu\rho}$ ($S_{\mu\nu\rho}$ is symmetric under $\mu\leftrightarrow\nu$)
$S_{\mu\nu\rho} = S_{\nu\mu\rho} = S_{\mu\rho\nu}$ ($S_{\mu\nu\rho}$ is totally symmetric)
$A_{\mu\nu\rho} = -A_{\nu\mu\rho}$ ($A_{\mu\nu\rho}$ is antisymmetric under $\mu\leftrightarrow\nu$)
$T_{\mu\nu}+T_{\nu\mu}$ is symmetric ; $T_{\mu\nu}-T_{\nu\mu}$ is antisymmetric

Def Symmetrization / Antisymmetrization

$$T_{(\mu_1\ldots\mu_n)\rho\ldots}{}^{\sigma\ldots} \equiv \frac{1}{n!}\left(T_{\mu_1\ldots\mu_n\rho\ldots}{}^{\sigma\ldots} + \text{all permutations of } \mu_1,\ldots,\mu_n\right)$$

$$T_{[\mu_1\ldots\mu_n]\rho\ldots}{}^{\sigma\ldots} \equiv \frac{1}{n!}\left(T_{\mu_1\mu_2\ldots\mu_n\rho\ldots}{}^{\sigma\ldots} - T_{\mu_2\mu_1\ldots\mu_n\rho\ldots}{}^{\sigma\ldots} + \text{all } \mu_1,\ldots,\mu_n \text{ permutations with sign}\right)$$

Ex.

$$T_{(\mu\nu)\rho}{}^{\sigma} = \frac{1}{2}\left(T_{\mu\nu\rho}{}^{\sigma} + T_{\nu\mu\rho}{}^{\sigma}\right)$$

$$T_{(\mu|\nu|\rho)}{}^{\sigma} = \frac{1}{2}\left(T_{\mu\nu\rho}{}^{\sigma} + T_{\rho\nu\mu}{}^{\sigma}\right)$$

$$T_{\mu\nu}{}^{[\rho\sigma]} = \frac{1}{2}\left(T_{\mu\nu}{}^{\rho\sigma} - T_{\mu\nu}{}^{\sigma\rho}\right)$$

$$T_{[\mu\nu\rho]}{}^{\sigma} = \frac{1}{6}\left(T_{\mu\nu\rho}{}^{\sigma} - T_{\nu\mu\rho}{}^{\sigma} - T_{\mu\rho\nu}{}^{\sigma} + T_{\nu\rho\mu}{}^{\sigma} - T_{\rho\nu\mu}{}^{\sigma} + T_{\rho\mu\nu}{}^{\sigma}\right)$$

Ex. M is a (1,1) tensor acting on ω and V

$$M(\omega,V) = M^{\mu}{}_{\nu}\,\omega_\mu V^\nu = (\omega_0,\omega_1,\omega_2,\omega_3)\begin{bmatrix} M^0{}_0 & M^0{}_1 & M^0{}_2 & M^0{}_3 \\ M^1{}_0 & M^1{}_1 & & \vdots \\ \vdots & & \ddots & \vdots \\ M^3{}_0 & \cdots & \cdots & M^3{}_3 \end{bmatrix}\begin{pmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{pmatrix}$$

Ex. The EM field strength $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = A_{\nu,\mu} - A_{\mu,\nu}$ is an antisymmetric (0,2) tensor.

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}$$

The metric: $\eta_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$ is a (0,2) tensor

$$\eta : T_p \times T_p \to \mathbb{R} \quad \text{or} \quad \eta : T_p \to T_p^*$$

$\eta(V,W) = \eta_{\mu\nu}V^\mu W^\nu \equiv V\cdot W$ (scalar or inner or dot product)

$\eta(V,V) = \eta_{\mu\nu}V^\mu V^\nu \equiv V\cdot V$ (norm) can be time-like (<0), space-like (>0) or null $(=0)$

The inverse metric $\eta^{\mu\nu} \equiv \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$ is a (2,0) tensor with the same components as $\eta_{\mu\nu}$

$$\eta^{\mu\nu}\eta_{\nu\rho} = \delta^\mu_\rho = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \quad \text{is a (1,1) tensor}$$

We can use $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$ to get new tensors from existing ones. E.g., given V^μ and ω_μ we define

$$V_\mu \equiv \eta_{\mu\nu} V^\nu \quad \text{and} \quad \omega^\mu \equiv \eta^{\mu\nu}\omega_\nu$$

Same for any tensor. E.g., $T_\mu{}^\nu{}_\rho = \eta_{\mu\alpha}\eta^{\nu\beta}T^\alpha{}_{\beta\rho}$.
This operation is said "raising and lowering indices"
In this form, the scalar product "looks" more Euclidean:

$$V\cdot V = V_\mu V^\mu = V_0V^0 + V_iV^i = -(V^0)^2 + \sum_i (V^i)^2$$
$$\omega\cdot\omega = \omega_\mu\omega^\mu = \omega_0\omega^0 + \omega_i\omega^i$$

We can also have $\Lambda_{\mu'}{}^{\nu} = \eta_{\mu'\nu'}\eta^{\mu\nu}\Lambda^{\nu'}{}_\mu$ by raising and lowering.
Remember that Lorentz transf. are defined by $\Lambda^{\mu'}{}_\mu \eta_{\mu'\nu'}\Lambda^{\nu'}{}_\nu = \eta_{\mu\nu}$
Multiply this by $\eta^{\rho\mu}$ and get

$$\delta^\rho{}_\nu = \eta^{\rho\mu}\eta_{\mu\nu} = \eta^{\rho\mu}\Lambda^{\mu'}{}_\mu\eta_{\mu'\nu'}\Lambda^{\nu'}{}_\nu = \Lambda_{\nu'}{}^\rho\Lambda^{\nu'}{}_\nu$$

This is why we defined $\Lambda_{\nu'}{}^\rho \equiv (\Lambda^{-1})^\rho{}_{\nu'}$ (like Carroll, but unlike Paolo and Schutz)

<u>Notice</u> $\eta_{\mu\nu}$, $\eta^{\mu\nu}$ and $\delta^\mu{}_\nu$, $\delta_\mu{}^\nu$ are tensors, so their indices are co/contravariant. ~~However,~~ They transform with Λ's. Howevere, they are defined to be constant everywhere. The components of $\eta_{\mu'\nu'}$ are still $\begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$.
They are actually <u>invariant</u> tensors.
This actually defines the Lorentz transformations.

Energy-momentum tensor

Also called stress-energy tensor.

Recall the proper time $\tau(\lambda) \equiv \int d\lambda \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}$, where $\frac{dx^\mu}{d\lambda}$ is the tangent vector to the time-like curve $x^\mu(\lambda)$. Can re-parametrize the curve as $x^\mu(\tau)$.

Def. The four-velocity of a fluid element following $x^\mu(\tau)$ is

$$\boxed{U^\mu \equiv \frac{dx^\mu}{d\tau}}$$ - normalized so that $\eta_{\mu\nu}U^\mu U^\nu = -1$

In the rest frame, U^μ has components $(1,0,0,0)$.

In the frame in which the particle has velocity $\vec{v}$,

$$U^{\mu'} = \begin{pmatrix} \gamma \\ \gamma v_x \\ \gamma v_y \\ \gamma v_z \end{pmatrix}$$ obtained applying $\Lambda^{\mu'}{}_{\mu}(-\vec{v}) = \begin{pmatrix} \gamma & \gamma v_x & & \\ \gamma v_x & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ to U^μ (for boost in x direction)

If $n = N/\Delta V$ is the number density of dust particles (*) at rest, in the moving frame $n \to \gamma n$, since $\Delta V \to \Delta V/\gamma$ (contraction of lengths)

The volume crossing a surface $\Delta y \Delta z$ in time Δt (at x=constant) is $v_x \Delta t \Delta y \Delta z$. Hence, the flux in x^i direction per unit area and time is $\gamma n v_i$ (divide by $\Delta t \Delta y \Delta z$). At rest, $(\text{flux})_i = 0$.

Def The number-flux four vector is

$$\boxed{N^\mu \equiv n U^\mu}$$

$N^i = (\text{flux})_i$. Can interpret N^0 as flux in t direction (= across a t = const. surface)

Def The quadrimomentum vector is $\boxed{p^\mu \equiv m U^\mu}$

The energy density in the rest frame is $\rho = mn$. In the moving frame, $\rho \to \gamma^2 \rho = \gamma^2 mn$, since $n \to \gamma n$ and $m \to \gamma m$.

In the rest frame ρ completely describes the dust particles, and $\rho = mn = mn(U^0)^2$. In the moving frames, it is desribed by

$$\boxed{T^{\mu\nu}_{dust} \equiv mnU^\mu U^\nu = p^\mu N^\nu = N^\mu p^\nu} \qquad (T_{dust} = U \otimes U)$$

(*) dust is a collection of particles at rest with each other, or a perfect fluid with no pressure.

Def The stress-energy tensor $T^{\mu\nu}$ of a fluid is the flux of μ-momentum across a surface of constant x^ν

T^{00} = energy density = energy flux across $t=$const surface
$T^{0i} = \gamma^2 \rho v^i$ = energy flux across $x^i=$ const surface
$T^{i0} = T^{0i} = \gamma^2 n m v^i$ = density of i-momentum
$T^{ij} = \gamma^2 \rho v^i v^j$ = flux of i-momentum across $x^j=$ const surface

A generic perfect fluid is described by ρ and p (=pressure).
In the rest frame, it must be spatially isotropic. Thus,

$$T^{\mu\nu} = \begin{bmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{bmatrix} \qquad (\text{at rest})$$

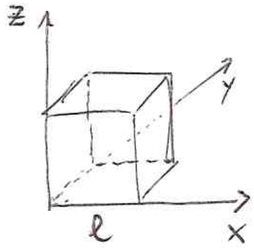
In the generic frame, this must be

$$T^{\mu\nu} = (\rho+p)U^\mu U^\nu + p\eta^{\mu\nu} \qquad \left[T = (\rho+p)U\otimes U + p\eta^{-1}\right]$$

(since it is a tensor and it reduces to the desired rest frame expression)

A real fluid has non-zero off-diagonal components also in rest frame.
T^{0i} is heat flux, T^{i0} the assoc. momentum density, T^{ij} anisotr. stress.

Conservation of $T^{\mu\nu}$



Energy in the cube $= \ell^3 T^{00}$
μ-momentum in cube $= \ell^3 T^{\mu 0}$

Variation $\frac{\partial}{\partial t}(\ell^3 T^{\mu 0})$ occurs because of flux across the surface: $\ell^2 \sum_{i=1}^{3} \Big[\underbrace{T^{\mu i}(x^i=0) - T^{\mu i}(x^i=\ell)}_{\simeq -\ell \frac{\partial}{\partial x^i} T^{\mu i}}\Big]$

$$\cancel{\ell^2[T^{\mu}} \Rightarrow \frac{\partial}{\partial t} T^{\mu 0} = -\frac{\partial}{\partial x^i} T^{\mu i} \quad \Rightarrow \boxed{\partial_\nu T^{\mu\nu} = 0} \quad (T^{\mu\nu}{}_{,\nu} = 0)$$

In practice, any conserved symmetric tensor can be interpreted as an energy-momentum tensor. $\partial_\nu T^{\mu\nu} = 0$ gives the EoM's.

Conservation of # of particles

# of particles in the cube $= \ell^3 N^0$
(same derivation as above) $\Rightarrow \boxed{\partial_\mu N^\mu = 0} \quad \left[(nU^\mu)_{,\mu} = 0\right]$

Perfect fluid equations of motion

$$\partial_\nu T^{\mu\nu} = \partial_\nu\left[(\rho+p)U^\mu U^\nu + p\eta^{\mu\nu}\right] = 0 \quad \text{and} \quad \partial_\mu(\underbrace{nU^\mu}_{N^\mu}) = 0$$

Show first that $U_\mu \partial_\nu U^\mu = 0$: $(*)$

$$U_\mu U^\mu = -1 \Rightarrow \partial_\nu(U_\mu U^\mu) = 0 \Rightarrow 0 = (U^\mu U^\rho \eta_{\mu\rho})_{,\nu} \overset{\eta_{\mu\rho} = \text{const}}{=} \eta_{\mu\rho}\partial_\nu(U^\mu U^\rho)$$

$$= 2\eta_{\mu\rho}U^\rho \partial_\nu U^\mu \underset{\eta_{\mu\rho}=\eta_{\rho\mu}}{=} 2U_\mu \partial_\nu U^\mu \quad \checkmark$$

Then, $\partial_\nu\left[(\rho+p)U^\mu U^\nu\right] = \partial_\nu\left[\frac{\rho+p}{n}U^\mu N^\nu\right] = N^\nu \partial_\nu\left(\frac{\rho+p}{n}U^\mu\right)$

$$\Rightarrow \partial_\nu T^{\mu\nu} = nU^\nu \partial_\nu\left(\frac{\rho+p}{n}U^\mu\right) + \eta^{\mu\nu}\partial_\nu p = 0$$

Now contract with U_μ : $U^\nu U_\mu \partial_\nu\left(\frac{\rho+p}{n}U^\mu\right) = \cancel{U^\nu \frac{\rho+p}{n} U_\mu \partial_\nu U^\mu}_{(*)} - U^\nu \partial_\nu\left(\frac{\rho+p}{n}\right)$

$$\Rightarrow U_\mu \partial_\nu T^{\mu\nu} = U^\nu\left[-n\partial_\nu\left(\frac{\rho+p}{n}\right) + \partial_\nu p\right] = -U^\nu\left[\partial_\nu \rho - \frac{\rho+p}{n}\partial_\nu n\right] = 0$$

But $\partial_\nu(nU^\nu) = 0 \Rightarrow (\partial_\nu n)U^\nu = -n\partial_\nu U^\nu$

$$\Rightarrow \boxed{U^\nu \partial_\nu \rho + (\rho+p)\partial_\nu U^\nu = 0}$$ continuity equation

Equivalent to $\partial_\nu(\rho U^\nu) + p\partial_\nu U^\nu = 0$. In the rest frame, it is the 0 component of $\partial_\nu T^{\mu\nu}$.

Now look at $\partial_\nu T^{i\nu}$ (remember $U^i = 0$ but $\partial_\nu U^i \neq 0$)

$$nU^\nu\left[\cancel{U^i \partial_\nu \frac{\rho+p}{n}} + \frac{\rho+p}{n}\partial_\nu U^i\right] + \eta^{i\nu}\partial_\nu p = 0$$

$$\boxed{(\rho+p)U^\nu \partial_\nu U^i + \partial^i p = 0}$$

It reduces to Euler's eqn. when $p \ll \rho$ (but $\partial^i p \neq 0$)

Notice that $U^\nu \partial_\nu = \frac{d}{d\tau}$.

The equivalence principle

Newtonian physics and SR are built around the concept of inertial observers (= on which no forces are acting).
For mechanical forces, it's easy to say when they are acting or not.
Also for EM, one can build shielded neutral observers which are inertial. Acceleration is with respect to these.
However, gravity cannot be shielded, no neutral observers.
How can inertial observers be defined? Acceleration w.r.t. what?

$$\vec{F} = m_i \vec{a} \qquad \vec{F}_g = -m_g \vec{\nabla}\phi$$

But we know since Galileo that $\boxed{m_i = m_g}$. All bodies feel the same acceleration regardless of mass. There are no inertial observers!

The <u>weak equivalence principle</u> simply states that inertial and gravitational mass are equal.

verified to great accuracy: Eötvös (1908) 10^{-9}
Eöt-wash (1987) 10^{-13}

For a system of N particles with mutual pairwise forces,

$$m_a \frac{d^2\vec{x}_a}{dt^2} = m_a g + \sum_{b \neq a} \vec{F}(\vec{x}_b - \vec{x}_a)$$

With a non-Galilean coordinate transf. $\vec{x}' = \vec{x} - \frac{1}{2} g t^2$ I get

$$m_a \frac{d^2\vec{x}'_a}{dt} = \sum_{b \neq a} \vec{F}(\vec{x}_b - \vec{x}_a)$$

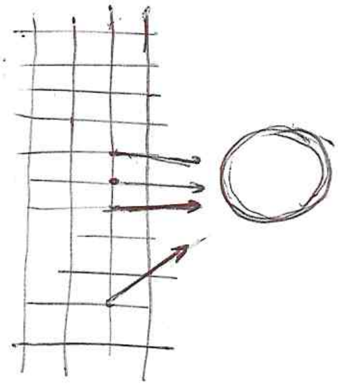
In the new frame (accelerated) the laws of mechanics are the same, but there is no gravity. Possible if $\vec{g}$ is uniform.
If $g = g(\vec{r})$, there are residual tidal forces.
Obs. in free-falling elevator sees objects moving towards each other
Ocean tides are due to inhom. sun-moon grav. attraction.
Also, in SR other forces contribute to mass (binding energy).
Do they contribute equally to m_i and m_g?
Yes, according to the <u>strong</u> equivalence principle.

Strong equivalence principle: at every space time-point in a gravitational field there is a reference frame in which, in a small enough region, ALL laws of physics reduce to SR.

This is called "locally inertial frame".

In a small enough region, $g(\vec{r}) \sim$ constant. Tidal forces are of second order. Locally, gravity can be erased.

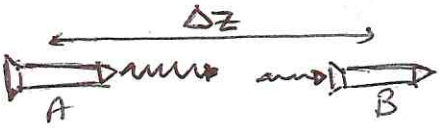
We will see that geometrically this means that space-time is locally flat. On larger scales, curvature of space-time induces accelerations w.r.t. the locally inertial frame.



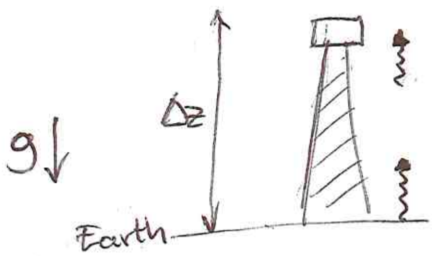
Gravitational redshift

A consequence of EP.

Imagine two rockets with ~~constant~~ constant acc. a



A photon emitted by A reaches B after $\Delta t = z/c$, when velocity has increased by $\Delta v \approx a \Delta t = za/c$. Doppler shift: $\frac{\Delta\lambda}{\lambda_0} \approx \frac{\Delta v}{c} \approx \frac{az}{c^2}$



According to EP, indistinguishable from photon reaching the top of a tower of height Δz in a gravitational field ~~[illegible]~~ $|\vec{g}| = a$

$$\frac{\Delta\lambda}{\lambda_0} \approx \Delta\phi \quad (c=1)$$

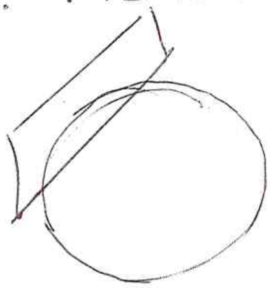
At the tower's top, rods are shorter and clocks are slower!

Differentiable manifolds

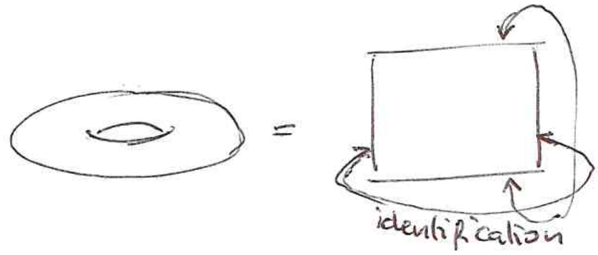
Need a mathematical structure to describe curved space-time, replacing flat Minkowski, which still "looks like" Minkowski locally. This is what is called a "manifold"

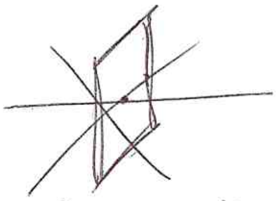
Ex.

sphere S^2



torus T^2



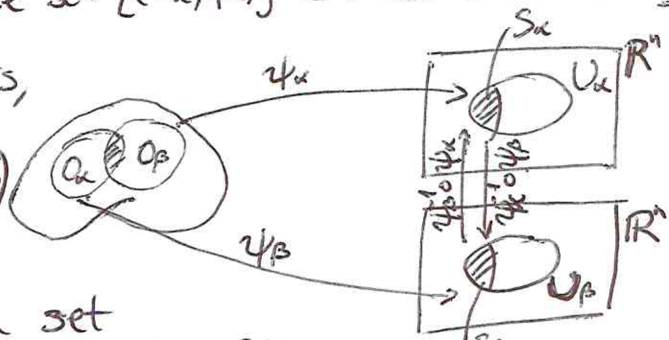


This is NOT a manifold. It changes dimensions.

A manifold is a set M together with a set of charts (or coordinate systems) mapping it into $\mathbb{R}^n$, which are sewn together smoothly. This set of charts is called an "atlas".

Def. An n-dimensional C^∞ real manifold M is a set together with a collection of subsets $\{O_\alpha\}$ such that:

1) each $p \in M$ is in at least one O_α (i.e., $\{O_\alpha\}$ covers M)
2) $\forall \alpha$, there is an invertible map $\psi_\alpha : O_\alpha \to U_\alpha \subset \mathbb{R}^n$, where $U_\alpha = \psi_\alpha(O_\alpha)$ (the image of O_α) is an open subset of $\mathbb{R}^n$
   (the pair (O_α, ψ_α) is called a chart, or coord. system.)
3) the charts are smoothly sewn together: if two charts overlap (if $O_\alpha \cap O_\beta \neq \emptyset$), then $S_\alpha \equiv \psi_\alpha(O_\alpha \cap O_\beta) \subset U_\alpha$ and $S_\beta \equiv \psi_\beta(O_\alpha \cap O_\beta)$ (the images of the intersection) are open sets, and the map $(\psi_\beta \circ \psi_\alpha^{-1}) : S_\alpha \to S_\beta$ is C^∞ (the set $\{(O_\alpha, \psi_\alpha)\}$ is then called an atlas)
4) the atlas is maximal, that is, it contains all possible charts compatible with 2) and 3)



In short, a manifold M is a set with a maximal atlas mapping it into $\mathbb{R}^n$ smoothly. Each portion $O_\alpha \subset M$ locally "looks like" $\mathbb{R}^n$.

The atlas gives M a differential structure (here, C^∞) inherited from $\mathbb{R}^n$.

Examples

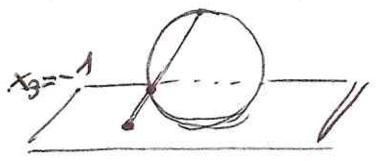
- $\mathbb{R}^n$ is a manifold. Can be covered by a single chart with $\mathbb{1}$ as map. However, we can also choose another coordinate system (e.g. spherical coordinates)
- S^1 (circle). The map $\psi: S^1 \to [0,2\pi)$ (angle) is NOT a good chart ($\psi(S_1)$ must be open). We need at least two charts to cover S^1. E.g. with images $(0,2\pi)$ and $(-\pi,\pi)$
- S^2 (sphere). Not with one chart either. Can use standard polar coordinates (4 charts), or six hemispheric caps $O_i^{\pm} = \{(x^1,x^2,x^3) \in S^2 \mid \pm x^i > 0\}$ mapped into the open disk, or 2 stereographic projections

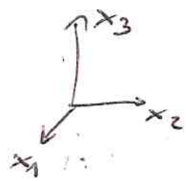
$U_1 = \{(x_2,x_3) \in \mathbb{R}^2 \mid x_2^2 + x_3^2 < 1\}$ $\quad U_2 = \{(x_1,x_3) \in \mathbb{R}^2 \mid x_1^2 + x_3^2 < 1\}$ $\quad U_3 = \dots$

Need to check that the mapping between the images of the intersection is C^∞

$$U_1 \to U_2 \quad \begin{cases} x_3 \to x_3 \\ x_2 \to x_1 = \pm\sqrt{1-x_3^2-x_2^2} \end{cases} \qquad C^\infty \text{ for } x_3^2 + x_2^2 < 1$$

For stereographic projection:

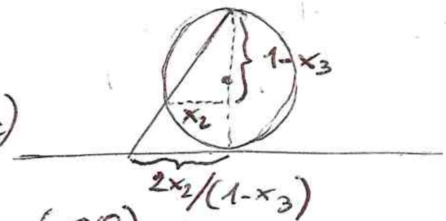




$$\psi_1(x_1,x_2,x_3) = \left(\frac{2x_1}{1-x_3}, \frac{2x_2}{1-x_3}\right) \equiv (y_1,y_2)$$

from the north pole, and

$$\psi_2(x_1,x_2,x_3) = \left(\frac{2x_1}{1+x_3}, \frac{2x_2}{1+x_3}\right) \equiv (z_1,z_2)$$



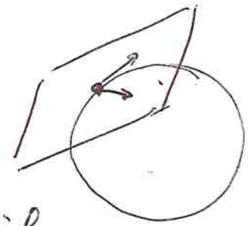
Check that it is a good atlas (C^∞)

$$(z_1,z_2) = \frac{1-x_3}{1+x_3}(y_1,y_2) \qquad y_1^2+y_2^2 = \frac{4(x_1^2+x_2^2)}{(1-x_3)^2} = \frac{4(1+x_3)}{1-x_3}$$

$$\Rightarrow x_3 = \frac{y_1^2+y_2^2-4}{y_1^2+y_2^2+4} < 1 \qquad \Rightarrow (z_1,z_2) = \frac{4}{y_1^2+y_2^2}(y_1,y_2) \quad (y_1^2+y_2^2>0)$$

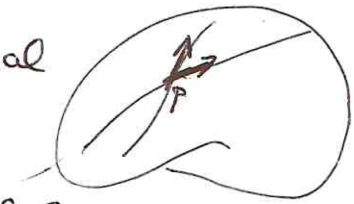
VECTORS ON A MANIFOLD

In GR, space-time is no longer a vector space like Minkowski / $\mathbb{R}^4$. E.g., on a sphere, there is no way to add two vectors, or multiply by a scalar, such that the result is still in the sphere.
But we should recover a vector space structure locally (because locally, physics must be SR).



In SR, the vector space structure of T_p was inherited ~~be~~ from space-time (~~a~~ aligned basis $\hat{e}_{(\mu)}$). The same woulbe possible if manifold M is embedded in $\mathbb{R}^n$. But we need intrinsic method.

$\forall p \in M$, there are curves going through it. We can define T_p if we know how to compute tangent vectors intrinsically.



In SR, $V = \frac{dx^\mu}{d\lambda}\hat{e}_{(\mu)}$. In GR, we don't know $\hat{e}_{(\mu)}$. But we can build directional derivatives.

~~The~~ Any function $f(p): M \to \mathbb{R}$ defines a function $F = f\circ\psi^{-1}: \mathbb{R}^n \to \mathbb{R}$ through the chart. This function is now diffentiable (it inherits $\mathbb{R}^n$'s structure).
Each curve through p ~~in~~ M is mapped into a curve $x^\mu(\lambda)$ in $\mathbb{R}^n$

$$\frac{d}{d\lambda}F(x^\mu(\lambda)) = \frac{\partial F}{\partial x^\mu}\frac{dx^\mu}{d\lambda}$$

A different curve $\tilde{x}^\mu(\eta)$ gives $\frac{dF}{d\eta} = \left.\frac{\partial F}{\partial x^\mu}\right|_{\tilde{x}^\mu}\frac{d\tilde{x}^\mu}{d\eta}$, but both are evaluated at $\tilde{x}^\mu = x^\mu = \psi(p)$, that is $\left.\frac{\partial F}{\partial x^\mu}\right|_p$. The two curves are associated to the derivatives $\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu}$ and $\frac{d}{d\eta} = \frac{d\tilde{x}^\mu}{d\eta}\frac{\partial}{\partial x^\mu}$. Both act on $f\circ\psi^{-1}$ linearly: $\frac{d}{d\lambda}(aF+bG) = a\frac{dF}{d\lambda} + b\frac{dG}{d\lambda}$, where $G = g\circ\psi^{-1}$, and $aF+bG = (af+bg)\circ\psi^{-1}$.
They map f into $\mathbb{R}$.
We choose this as definition of tangent vector.

<u>Def</u> The <u>tangent space</u> T_p is the space of all directional derivative operators along curves through p.

Each directional derivative is a <u>tangent</u> <u>vector</u>.

Tangent vector v is a linear map $v: \mathcal{F} \to \mathbb{R}$ (where $\mathcal{F}$ is the space of C^∞ functions from $\mathcal{M} \to \mathbb{R}$).

<u>Basis of T_p</u>: the partial derivatives ∂_μ are derivative operators along a curve in which only one x^μ is changing. They are thus tangent vectors. Since each curve ~~is expressed~~ through p is expressed in coordinates, each directional derivative takes the form $\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu}$. Hence, ∂_μ is a basis of T_p (called <u>coordinate basis</u>)

Each vector $v \in T_p$ has the expression $v = \underset{\text{components}}{v^\mu}\, \underset{\text{basis}}{\partial_\mu}$

In curved space, the set $\{\partial_\mu\}$ of vectors replaces $\{\hat{e}_{(\mu)}\}$ which we used in flat space.

<u>Note</u> that some texts define tangent vectors directly as linear applications $v: \mathcal{F} \to \mathbb{R}$ (e.g. Wald) which also obey Leibnitz rule:

$$v(fg) = f(p)v(g) + v(f)g(p)$$

This definition is coordinate-independent (more elegant) but equivalent.

<u>Transformation under coordinate change</u>. $x^\mu \to x^{\mu'}$ (generic!)

$$\boxed{\partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\partial_\mu}$$ but V must remain invariant

$$V = V^\mu \partial_\mu = V^{\mu'}\partial_{\mu'} = V^{\mu'}\frac{\partial x^\mu}{\partial x^{\mu'}}\partial_\mu \quad \Rightarrow \quad V^\mu = \frac{\partial x^\mu}{\partial x^{\mu'}}V^{\mu'} \qquad \boxed{V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu}V^\mu}$$

Generalizes the <u>transf.</u> rule of vector basis and components under Lorentz transf. in SR. <u>If</u> $x^{\mu'} = \Lambda^{\mu'}{}_\mu x^\mu$, then $\hat{e}_{(\mu')} = \Lambda_{\mu'}{}^\mu \hat{e}_{(\mu)}$

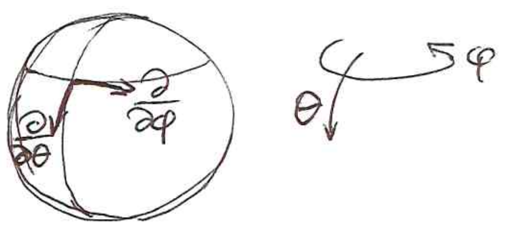
$$\frac{\partial x^{\mu'}}{\partial x^\mu} = \Lambda^{\mu'}{}_\mu \text{ (direct)}. \qquad \frac{\partial x^\mu}{\partial x^{\mu'}} = \left(\frac{\partial x^{\mu'}}{\partial x^\mu}\right)^{-1} = \Lambda_{\mu'}{}^\mu \text{ (inverse)}$$

Note that:

1) Lorentz transformations have a special role in SR, they leave the metric invariant (they are a symmetry). Generic coordinate transf. in GR change the metric. But we will study isometries, which do not.

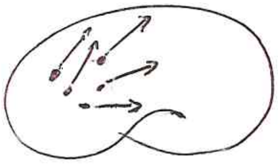
2) Components of T_p vectors change if we change the basis of T_p. In general not direct consequence of change of coords. ~~At~~ It happens IF we choose the coordinate basis $\left\{\frac{\partial}{\partial x^\mu}\right\}$. ~~Whe~~ We could have chosen a different basis.

Ex. sphere S^2 with (θ, φ)



3) Every T_p is independent from the others. No unique coordinate-indep. way of relating $v \in T_p$ to $w \in T_q$

In spite of last point, we can still define a vector field on M as a set of vectors $v|_p \in T_p \;\; \forall p \in M$.



The ~~fiel~~ vector field is smooth if $\forall f \in F \; (C^\infty)$ $\;\; v|_p(f): M \to \mathbb{R}$ is C^∞

Choosing $\{\partial_\mu\}$ as basis, if $V^\mu(x^\mu)$ are C^∞

Dual vectors

As in Minkowski, a dual vector $\omega \in T_p^*$ is a linear map $\omega: T_p \to \mathbb{R}$.

The vector space T_p^* of all linear maps is the cotangent space.

Example: given a function $f \in F$, the gradient df is a one-form associating to each vector the derivative of f along that vector.

$$f: M \to \mathbb{R} \qquad df: T_p \to \mathbb{R} \qquad df\left(\frac{d}{d\lambda}\right) \equiv \frac{df}{d\lambda}$$

The gradients dx^μ of the coordinate functions x^μ (curves in M where only one coord. is changing) are a natural basis of T_p^*:

$$\boxed{dx^\mu(\partial_\nu) = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu} \qquad \left(\text{like } \hat\theta^{(\mu)}(\hat e_{(\nu)}) = \delta^\mu_\nu \text{ in SR}\right)$$

Any one-form $\omega \in T_p^*$ is expanded as $\omega = \underset{\text{component}}{\omega_\mu} \underset{\text{basis}}{dx^\mu}$

The action on $v \in T_p$ is

$$\omega(v) = \omega_\mu dx^\mu \left(v^\nu \frac{\partial}{\partial x^\nu} \right) = \omega_\mu v^\nu \underbrace{dx^\mu(\partial_\nu)}_{\delta^\mu_\nu} = \omega_\mu v^\mu$$

and must be invariant. This implies (like Minkowski)

$$\boxed{\omega_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \omega_\mu} \quad (\text{so that } \omega_{\mu'} v^{\mu'} = \omega_\mu v^\mu) \quad \text{and} \quad \boxed{dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} dx^\mu}$$

For the gradient, $df = \frac{\partial f}{\partial x^\mu} dx^\mu$

$$df\left(\frac{d}{d\lambda}\right) = \frac{\partial f}{\partial x^\mu} dx^\mu \left(\frac{dx^\nu}{d\lambda} \partial_\nu \right) = \frac{\partial f}{\partial x^\mu} \frac{dx^\mu}{d\lambda} = \frac{df}{d\lambda}$$

A vector is also a map $v: T_p^* \to \mathbb{R}$ such that $v(\omega) = \omega_\mu v^\mu = \omega(v)$

Tensors

Same as Minkowski. A tensor T is a linear map (rank (k,l))

$$T: \underbrace{T_p^* \times \dots T_p^*}_{k} \times \underbrace{T_p \times \dots \times T_p}_{l} \to \mathbb{R}$$

Can be expanded as

$$T = T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} \partial_{\mu_1} \otimes \dots \otimes \partial_{\mu_k} \otimes dx^{\nu_1} \otimes \dots \otimes dx^{\nu_l}$$

and transforms under change of coordinates as

$$T^{\mu_1' \dots \mu_k'}{}_{\nu_1' \dots \nu_l'} = \frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}} \dots \frac{\partial x^{\nu_1}}{\partial x^{\nu_1'}} T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}$$

Also for tensors we can have tensor fields, with comp. $T^{\mu_1 \dots}{}_{\nu_1 \dots}(x)$. (position dependent).

What about derivatives? We have seen already $\frac{\partial f}{\partial x^\mu}$ (der. of scalar).

<u>BUT</u>:

$$\partial_{\mu'} \omega_{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} \left(\frac{\partial x^\nu}{\partial x^{\nu'}} \omega_\nu \right) = \underbrace{\frac{\partial x^\mu \partial x^\nu}{\partial x^{\mu'} \partial x^{\nu'}} \partial_\mu \omega_\nu}_{\text{standard piece}} + \underbrace{\frac{\partial^2 x^\nu}{\partial x^{\mu'} \partial x^{\nu'}} \omega_\nu}_{???}$$

The extra piece vanishes for Lorentz transformations (constant $\Lambda_{\nu'}{}^\nu$), but not here. This is because the basis at x^μ and at $x^\mu + \Delta x^\mu$ are not aligned. Tensors at x^μ and $x^\mu + \Delta x^\mu$ transform differently.

<u>HOWEVER</u>:

$$\partial_{\mu'} \omega_{\nu'} - \partial_{\nu'} \omega_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} (\partial_\mu \omega_\nu - \partial_\nu \omega_\mu)$$

The extra piece is symmetric and cancels.

So: the partial derivative of a (k,ℓ) tensor does not transform like a $(k, \ell+1)$ tensor. More precisely, it still is a tensor but the new $\partial'_{\nu'}$ is not the same as $\partial_{\nu'}$ (the transf. of the old derivative operator) when it acts on non-scalar objects. We will see more about covariant derivatives.

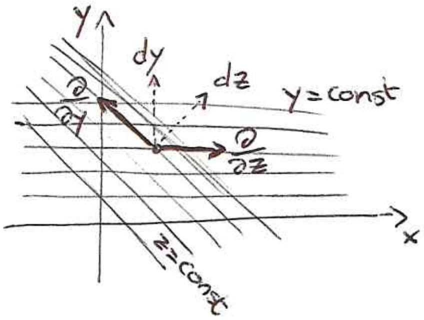
Also x^μ is <u>NOT</u> a vector: $x^{\mu'}(x^\mu) \neq \frac{\partial x^{\mu'}}{\partial x^\mu} x^\mu$ (unlike for Lorentz transformations). The coordinate transf. is non-linear.

However, the <u>exterior derivative</u> of a p-form field $A_{\mu_1 \cdots \mu_p}$

$$(dA)_{\mu_1 \cdots \mu_{p+1}} \equiv (p+1) \partial_{[\mu_1} A_{\mu_2 \cdots \mu_{p+1}]}$$

(totally antisymmetrized) is a good tensor.
E.g. the EM field strength.

<u>Example</u>: describe $\mathbb{R}^2$ with y and $z = x + y$.



$$dz\left(\frac{\partial}{\partial y}\right) = 0 \quad dy\left(\frac{\partial}{\partial z}\right) = 0$$

$\frac{\partial}{\partial z}$ at $y = \text{const}$, $\frac{\partial}{\partial y}$ at $z = \text{const}$

<u>Example</u>: $S_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & x^2 \end{pmatrix}$ (0,2) tensor at (x,y)

Change of coordinate: $x' = \frac{2x}{y}$ $\quad y' = y/2$

$$S = S_{\mu\nu}\, dx^\mu \otimes dx^\nu = (dx)^2 + x^2 (dy)^2$$

$$\begin{cases} x = x'y' \\ y = 2y' \end{cases} \Rightarrow \begin{cases} dx = dx'y' + x'dy' \\ dy = 2dy' \end{cases} \Rightarrow S = (dx'y' + x'dy')^2 + 4(x'y')^2 (dy')^2$$

$$S_{\mu'\nu'} = \begin{pmatrix} y'^2 & x'y' \\ x'y' & x'^2(1+4y'^2) \end{pmatrix}$$

THE METRIC

$g_{\mu\nu}$ (0,2) tensor. Corresponds to $\eta_{\mu\nu}$ in Minkowski.
Assumed to be non-degenerate ($\det(g_{\mu\nu}) \neq 0$):

$$g(v,w) = 0 \quad \forall v \in T_p \quad \text{only if } w = 0$$

Inverse metric: $g^{\mu\nu} = (g_{\mu\nu})^{-1}$.
The notation $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ is ~~a short~~ replacement for $g = g_{\mu\nu} dx^\mu \otimes dx^\nu$.

Examples $\mathbb{R}^3$ with Euclidean metric: $ds^2 = dx^2 + dy^2 + dz^2$
Changing to sph. coordinates: $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$,
$z = r\cos\theta$

$$dx = dr\sin\theta\cos\varphi + r\cos\theta d\theta\cos\varphi + r\sin\theta\sin\varphi d\varphi \qquad dy = \dots \quad dz = \dots$$

$$\Rightarrow ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\varphi^2$$

For a sphere S^2 embedded in $\mathbb{R}^3$ $(r=1)$: $ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$

Length of curve (Euclidean): $\Delta l = \int d\lambda \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}$

In GR $(\eta_{\mu\nu} \to g_{\mu\nu})$: $\Delta\tau = \int d\lambda \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}$ (timelike curve)

Coordinate dep. metric does not imply curvature. But the opposite is true.

Sylvester's theorem (or Sylvester's law of inertia).
For any symmetric matrix $A \in \mathbb{R}^{n\times n}$ there is an invertible matrix $P \in \mathbb{R}^{n\times n}$ such that $P^T A P = D$, where $D = \mathrm{diag}(\underbrace{-1,\dots,-1}_{t}, \underbrace{1,\dots,1}_{s}, 0,\dots,0)$.
and $t + s = \mathrm{rank}(A)$. (*)

The quantity $s - t$ is called the signature of the matrix (sometimes instead the signature is (t,s)).

If the metric is everywhere continuous and non-degenerate (invertible), then $t+s = n$ and the signature is the same at each $p \in \mathcal{M}$. It is coordinate-independent.

$g_{\mu\nu}$ is Euclidean (or Riemannian) if it is positive def. $(t=0)$
$g_{\mu\nu}$ is Lorentzian (or pseudo-Riemannian) if $t = 1$, $s = n-1$

In GR, Lorentzian metric (signature (1,3))

(*) $P = O|\Lambda|^{-1}$, with $O \in SO(n)$ and $|\Lambda| = \mathrm{diag}(|\lambda_1|, \dots, |\lambda_n|)$

At each $p \in \mathcal{M}$ we can choose coordinates such that

$$g_{\hat\mu\hat\nu}(p) = \eta_{\hat\mu\hat\nu} \quad \text{and} \quad \partial_{\hat\alpha} g_{\hat\mu\hat\nu}\big|_p = 0$$

(but we still have $\partial_{\hat\alpha}\partial_{\hat\beta} g_{\hat\mu\hat\nu} \neq 0$).

This is a <u>locally inertial frame</u> (sufficiently small region well described by SR). Curvature (and gravity) is encoded in the 2nd derivatives ($\rightarrow$ Riemann tensor).

<u>Sketch of proof</u> (see Schutz page 149-150, "local flatness th.")

Change of coordinate from generic x^μ to $x^{\hat\mu}$: $g_{\hat\mu\hat\nu} = \frac{\partial x^\mu}{\partial x^{\hat\mu}}\frac{\partial x^\nu}{\partial x^{\hat\nu}} g_{\mu\nu}$ (*)

Taylor expansion (for simplicity around $x^\mu(p) = x^{\hat\mu}(p) = 0$):

$$x^\mu = \frac{\partial x^\mu}{\partial x^{\hat\mu}}\Big|_p x^{\hat\mu} + \frac{1}{2}\frac{\partial^2 x^\mu}{\partial x^{\hat\mu}\partial x^{\hat\nu}} x^{\hat\mu}x^{\hat\nu} + \dots$$

$$\frac{\partial x^\mu}{\partial x^{\hat\mu}} = \frac{\partial x^\mu}{\partial x^{\hat\mu}}\Big|_p + \frac{\partial^2 x^\mu}{\partial x^{\hat\mu}\partial x^{\hat\nu}}\Big|_p x^{\hat\nu} + \frac{1}{2}\frac{\partial^3 x^\mu}{\partial x^{\hat\mu}\partial x^{\hat\nu}\partial x^{\hat\rho}}\Big|_p x^{\hat\nu}x^{\hat\rho}$$

$$g_{\hat\mu\hat\nu} = g_{\hat\mu\hat\nu}\big|_p + \partial_{\hat\rho} g_{\hat\mu\hat\nu}\big|_p x^{\hat\rho} + \dots \qquad g_{\mu\nu} = g_{\mu\nu}\big|_p + \partial_{\hat\rho} g_{\mu\nu}\big|_p x^{\hat\rho} + \dots$$

(note that $g_{\mu\nu}$ must be expressed in $x^{\hat\mu}$ coordinates).

Plugging these in (*) gives schematically

$$\hat g_p + (\hat\partial \hat g)_p \hat x + \frac{1}{2}(\hat\partial\hat\partial \hat g)_p \hat x \hat x + \dots = (\hat\partial x\, g\, \hat\partial x)_p + (\hat\partial x \hat\partial g \hat\partial x + 2\hat\partial x\, g\, \hat\partial^2 x)_p \hat x$$
$$+ \frac{1}{2}(\hat\partial x \hat\partial^2 g \hat\partial x + 2\hat\partial x\, g\, \hat\partial^3 x + 2\hat\partial^2 x\, g\, \hat\partial^2 x)_p \hat x \hat x + \dots$$

Leading order: $g_{\hat\mu\hat\nu}(p) = \eta_{\hat\mu\hat\nu} = \left(\frac{\partial x^\mu}{\partial x^{\hat\mu}}\frac{\partial x^\nu}{\partial x^{\hat\nu}} g_{\mu\nu}\right)_p$

10 equations (symmetric in $\hat\mu \leftrightarrow \hat\nu$) for 16 unknowns $\left(\frac{\partial x^\mu}{\partial x^{\hat\mu}}\right)_p$

Can always be solved. Remaining 6 dof's are the parameters of a residual Lorentz transf $\Lambda^\mu{}_{\hat\mu}$ (which leaves $\eta_{\hat\mu\hat\nu}$ invariant)

First order: $\partial_{\hat\alpha} g_{\hat\mu\hat\nu}\big|_p = 0 = \left(\frac{\partial x^\mu}{\partial x^{\hat\mu}}\frac{\partial x^\nu}{\partial x^{\hat\nu}}\partial_{\hat\alpha} g_{\mu\nu} + \dots\right)_p$

4x10 equations (symm. in $\hat\mu \leftrightarrow \hat\nu$ but not $\hat\alpha$) for 40 unknowns $\left(\frac{\partial^2 x^\mu}{\partial x^{\hat\mu}\partial x^{\hat\nu}}\right)_p$

Can always be solved.

Second order: $\partial_{\hat\alpha}\partial_{\hat\beta} g_{\hat\mu\hat\nu}\big|_p = \text{r.h.s.}$ 4x20=80, with 20=4x5x6/3!

$10 \times 10 = 100$ eqns. (symm. in $\hat\mu,\hat\nu$ and $\hat\alpha,\hat\beta$) for 80 unknowns $\left(\frac{\partial^3 x^\mu}{\partial x^{\hat\mu}\partial x^{\hat\nu}\partial x^{\hat\rho}}\right)_p$

Cannot be solved! The 20 missing dof's are the components of Riemann tensor (which describes curvature)

Number of independent components of totally symm. tensors

A tensor with m totally symmetric indices on dimension n manifold has $\binom{n+m-1}{m}$ ~~comps~~ independent components.

Same problem as putting m equal objects in n boxes.

$\binom{n}{k} \equiv \frac{n!}{k!(n-k)!}$ $(n \geq k \geq 0)$ binomial coefficient

(number of inequivalent ways to pick k objects from a set of n, or to place k objects in n slots)

Proved with the "stars and bars" method:

$$\overbrace{\underbrace{\star\star}_{k_0}|\underbrace{\star}_{k_1}|\underbrace{\,}_{k_2}|\underbrace{\star\star\cdots\cdots}_{k_i}|\underbrace{\star}_{k_{n-1}}}^{m}$$

m stars and $n-1$ separators

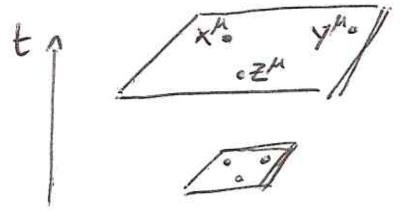
$k_\mu = \#$ of indices w/ value $\mu = 0, \dots, n-1$

$$\sum_{\mu=0}^{n-1} k_\mu = m$$

Total of $m+n-1$ slots, m occupied by stars. Inequivalent ways of placing m stars in $n+m-1$ slots is $\binom{n+m-1}{m} = \binom{n+m-1}{n-1}$ (also $n-1$ separators in $n+m-1$ slots)

Example Expanding Universe ($-+++$ metric)

$$ds^2 = -dt^2 + a^2(t)\left[dx^2 + dy^2 + dz^2\right] \qquad g_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & a^2 & & \\ & & a^2 & \\ & & & a^2 \end{bmatrix}$$



The physical distance between fixed ("comoving") coordinate points grows with $a(t)$ ("scale factor")

Consider $a(t) = t^q$ (we will see that perfect fluids with different p have different q. Dust ($p=0$, $q=\frac{1}{2}$) or radiation ($p=\frac{1}{3}$, $q=\frac{2}{3}$))

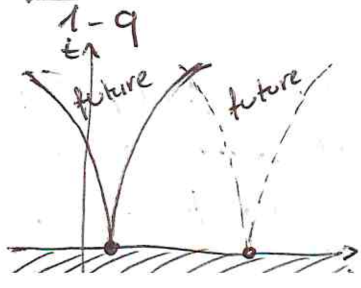
$a(t) \to 0$ as $t \to 0$ ("Big Bang", initial singularity)

Light rays: $ds^2 = 0$, $dt^2 = a^2 dx^2 \Rightarrow \frac{dx}{dt} = \pm t^{-q}$ (sloppy)

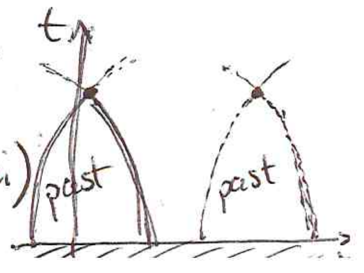
Rigorous derivation: $ds^2 = g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = 0$, where $\frac{dx^\mu}{dt}$ are ~~the~~ components of tangent vector to $x^\mu(t)$ (curve parametrized by $\lambda = t$)

Solution: $x - x_0 = \pm \frac{t^{1-q}}{1-q}$ $\qquad t - t_0 = \pm\left[(1-q)(x-x_0)\right]^{\frac{1}{1-q}}$ $(0<q<1)$

Future light cones of points at $t=0$



Past light cones may not intersect (unlike Minkowski)



A ~~point~~ event at $t = t_0$ receives information from (= is causally connected to) only a finite region, whose boundary is called horizon. This region grows with time.

Exercise: Show that the change of coordinates $t' = t + \frac{1}{2} H \vec{x}^2 + \mathcal{O}((\vec{x}^2)^2)$ and $\vec{x}' = a\vec{x}\left(1 + \frac{H^2 a^2 \vec{x}^2}{4} + \mathcal{O}((\vec{x}^2)^2)\right)$ brings $g_{\mu\nu} = \mathrm{diag}(-1, a^2, a^2, a^2)$ to a locally inertial frame form.

COVARIANT DERIVATIVES

We saw that $\partial_\mu \omega_\nu$ is not a "good" (0,2) tensor since it does not transform covariantly. This is because $\partial'_{\mu'}$ (the derivative w.r.t. $x^{\mu'}$) does not coincide with $\partial_{\mu'}$ (the transformation of ∂_μ) when it acts on an object with indices (but $\partial'_{\mu'} = \partial_{\mu'}$ on scalars).

This makes us unable to relate tensors at two nearby points in a coordinate-independent manner. Need a "better" derivative.

Definition: A ~~covariant~~ derivative ∇ is an ~~maps~~ operator mapping a smooth type (k,l) tensor field into a smooth tensor field of type $(k, l+1)$, which satisfies:

1) linearity: $\nabla(aT_1 + bT_2) = a\nabla T_1 + b\nabla T_2$
2) Leibnitz rule: $\nabla(T_1 T_2) = T_1 \nabla T_2 + \nabla T_1 T_2$
3) Commutativity with contraction:
$$\nabla_\alpha \left(T^{\mu_1 \dots \lambda \dots \mu_k}{}_{\nu_1 \dots \lambda \dots \nu_l} \right) = \nabla_\alpha T^{\mu_1 \dots \lambda \dots \mu_k}{}_{\nu_1 \dots \lambda \dots \nu_l}$$
4) it reduces to ordinary derivative ∂ on scalar functions
5) it commutes on scalar functions: $\nabla_a \nabla_b f = \nabla_b \nabla_a f$ (in this case it is said to be torsion-free)

In a given coordinate system ∂_μ satisfy these properties. However, in ~~a~~ different coordinates ∂'_μ is a different map. Not covariantly related. Which one do we choose?

Note: we defined ∂_μ as a vector (basis of T_p) at $p \in M$, since it maps $f \in F$ into $\frac{\partial f \circ x^{-1}}{\partial x^\mu}\Big|_p \in \mathbb{R}$ (at fixed p). However, as p varies it defines a vector field, hence a map from F $(=(0,0)) \rightsquigarrow (0,1)$ tensor fields

Transformation from locally inertial to generic frame:

$$\partial_\mu \omega_\nu = \frac{\partial x^{\hat\mu}}{\partial x^\mu}\frac{\partial x^{\hat\nu}}{\partial x^\nu}\partial_{\hat\mu}\omega_{\hat\nu} + \frac{\partial^2 x^{\hat\nu}}{\partial x^\mu \partial x^\nu}\omega_{\hat\nu} \quad , \quad \text{but} \quad \omega_{\hat\nu} = \frac{\partial x^\nu}{\partial x^{\hat\nu}}\omega_\nu$$

We can thus rewrite this as

$$\partial_\mu \omega_\nu - \Gamma^\rho_{\mu\nu}\omega_\rho = \frac{\partial x^{\hat\mu}}{\partial x^\mu}\frac{\partial x^{\hat\nu}}{\partial x^\nu}\left(\partial_{\hat\mu}\omega_{\hat\nu} - \underbrace{\Gamma^{\hat\rho}_{\hat\mu\hat\nu}}_{=0}\omega_{\hat\rho}\right)$$

where $\boxed{\Gamma^\rho_{\mu\nu} \equiv \frac{\partial x^\rho}{\partial x^{\hat\nu}}\frac{\partial^2 x^{\hat\nu}}{\partial x^\mu \partial x^\nu}}$ is the "Christoffel symbol", ~~and~~ which by definition vanishes in the ~~locally~~ locally inertial frame: $\Gamma^{\hat\rho}_{\hat\mu\hat\nu} = 0$

The same holds between $x^{\hat\mu}$ and x^μ. Then,

$$\partial_{\mu'}\omega_{\nu'} - \Gamma^{\rho'}_{\mu'\nu'}\omega_{\rho'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\left(\partial_\mu \omega_\nu - \Gamma^\rho_{\mu\nu}\omega_\rho\right)$$

between two generic frames x^μ and $x^{\mu'}$. We then <u>define</u>

$$\boxed{\nabla_\mu \omega_\nu \equiv \partial_\mu \omega_\nu - \Gamma^\rho_{\mu\nu}\omega_\rho}$$

as the <u>covariant derivative</u> of ω_ν. $\Gamma^\rho_{\mu\nu}$ is <u>not</u> a tensor, but is exactly what is needed to remove the extra piece from the transformation of $\partial_\mu \omega_\nu$, so that $\nabla_\mu \omega_\nu$ is a well behaved $(0,2)$ tensor.

Still, our definition of $\Gamma^\rho_{\mu\nu}$ in ~~the~~ a generic frame depends on the inertial frame. We want to get rid of this. Remember that $g_{\mu\nu} = \frac{\partial x^{\hat\mu}}{\partial x^\mu} g_{\hat\mu\hat\nu}\frac{\partial x^{\hat\nu}}{\partial x^\nu}$, where $g_{\hat\mu\hat\nu} = \eta_{\hat\mu\hat\nu}$ and $\frac{\partial g_{\hat\mu\hat\nu}}{\partial x^{\hat\alpha}} = 0$.

Differentiating, since $\partial_\alpha g_{\hat\mu\hat\nu} = \frac{\partial x^{\hat\alpha}}{\partial x^\alpha}\partial_{\hat\alpha} g_{\hat\mu\hat\nu} = 0$

$$\partial_\alpha g_{\mu\nu} = \left[\frac{\partial^2 x^{\hat\mu}}{\partial x^\alpha \partial x^\mu}\frac{\partial x^{\hat\nu}}{\partial x^\nu} + \frac{\partial x^{\hat\mu}}{\partial x^\mu}\frac{\partial^2 x^{\hat\nu}}{\partial x^\alpha \partial x^\nu}\right] g_{\hat\mu\hat\nu} = \Gamma^\rho_{\alpha\mu} g_{\rho\nu} + \Gamma^\rho_{\alpha\nu} g_{\mu\rho}$$

$$\frac{\partial x^\rho}{\partial x^{\hat\mu}}\frac{\partial x^\sigma}{\partial x^{\hat\nu}} g_{\rho\sigma}$$

Note that $\boxed{\Gamma^\rho_{\alpha\mu} = \Gamma^\rho_{\mu\alpha}}$.

Hence, $\partial_\alpha g_{\mu\nu} + \partial_\mu g_{\alpha\nu} = 2\Gamma^\rho_{\alpha\mu} g_{\rho\nu} + \Gamma^\rho_{\alpha\nu} g_{\mu\rho} + \Gamma^\rho_{\mu\nu} g_{\alpha\rho} = 2\Gamma^\rho_{\alpha\mu} g_{\rho\nu} + \partial_\nu g_{\alpha\mu}$

Solving for $\Gamma^\rho_{\alpha\mu}$ gives the desired expression

$$\boxed{\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right)}$$

independently of the locally inertial frame. This <u>follows from the EP</u>, (which requests $\partial_{\hat\alpha} g_{\hat\mu\hat\nu} = 0$).

Similarly for vectors, but here $V^{\hat\nu} = \frac{\partial x^{\hat\nu}}{\partial x^{\nu}} V^{\nu}$, $V^{\nu} = \frac{\partial x^{\nu}}{\partial x^{\hat\nu}} V^{\hat\nu}$

$$\partial_\mu V^\nu = \frac{\partial x^{\hat\mu}}{\partial x^\mu}\frac{\partial x^\nu}{\partial x^{\hat\nu}}\partial_{\hat\mu}V^{\hat\nu} + \underbrace{\frac{\partial x^{\hat\mu}}{\partial x^\mu}\frac{\partial^2 x^\nu}{\partial x^{\hat\mu}\partial x^{\hat\nu}}}_{=\frac{\partial^2 x^\nu}{\partial x^\mu \partial x^{\hat\nu}}} V^{\hat\nu} \qquad (\text{notice } \nu \leftrightarrow \hat\nu)$$

But $\frac{\partial x^{\hat\nu}}{\partial x^\alpha}\frac{\partial x^\nu}{\partial x^{\hat\nu}} = \delta_\alpha^\nu$. Differentiating, $\frac{\partial x^{\hat\nu}}{\partial x^\alpha}\frac{\partial^2 x^\nu}{\partial x^\mu \partial x^{\hat\nu}} = -\frac{\partial^2 x^{\hat\nu}}{\partial x^\mu \partial x^\alpha}\frac{\partial x^\nu}{\partial x^{\hat\nu}} = -\Gamma^\nu_{\mu\alpha}$ (since $\partial_\mu \delta_\alpha^\nu = 0$). Thus the extra piece is $-\Gamma^\nu_{\mu\alpha}V^\alpha$:

$$\partial_\mu V^\nu + \Gamma^\nu_{\mu\alpha}V^\alpha = \frac{\partial x^{\hat\mu}}{\partial x^\mu}\frac{\partial x^\nu}{\partial x^{\hat\nu}}\Big(\partial_{\hat\mu}V^{\hat\nu} + \underbrace{\Gamma^{\hat\nu}_{\hat\mu\hat\alpha}}_{=0}V^{\hat\alpha}\Big)$$

We then define the <u>covariant derivative</u> of V^ν as

$$\boxed{\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\alpha}V^\alpha}$$

which transforms as a good $(1,1)$ tensor.

Requesting that ∇_μ obeys Leibnitz rule and commutes with contraction of indices, the action on scalar $\omega_\nu V^\nu$ is

$$\nabla_\mu(\omega_\nu V^\nu) = (\nabla_\mu \omega_\nu)V^\nu + \omega_\nu \nabla_\mu V^\nu = (\partial_\mu\omega_\nu)V^\nu - \cancel{\Gamma^\rho_{\mu\nu}\omega_\rho V^\nu} + \omega_\nu\partial_\mu V^\nu + \cancel{\omega_\nu \Gamma^\nu_{\mu\rho}V^\rho}$$
$$= \partial_\mu(\omega_\nu V^\nu)$$

On a scalar, ∇_μ reduces to ∂_μ as it should (already covariant)

But also $T^{\mu_1\dots\mu_k}{}_{\nu_1\dots\nu_\ell}\,\omega_{\mu_1}\cdots\omega_{\mu_k}V^{\nu_1}\cdots V^{\nu_\ell}$ is a scalar. Requesting Leibnitz and that $\nabla_\mu \to \partial_\mu$ on it, one gets for a (k,ℓ) tensor

$$\boxed{\nabla_\alpha T^{\mu_1\dots\mu_k}{}_{\nu_1\dots\nu_\ell} = \partial_\alpha T^{\mu_1\dots\mu_k}{}_{\nu_1\dots\nu_\ell} + \sum_{i=1}^{k}\Gamma^{\mu_i}_{\alpha\rho}T^{\mu_1\dots\rho\dots\mu_k}{}_{\nu_1\dots\nu_\ell} - \sum_{i=1}^{\ell}\Gamma^{\rho}_{\alpha\nu_i}T^{\mu_1\dots\mu_k}{}_{\nu_1\dots\rho\dots\nu_\ell}}$$

What about commutativity on scalar functions?

$$\nabla_\alpha\nabla_\beta f = \partial_\alpha\partial_\beta f - \Gamma^\rho_{\alpha\beta}\partial_\rho f = \partial_\beta\partial_\alpha f - \Gamma^\rho_{\beta\alpha}\partial_\rho f = \nabla_\beta\nabla_\alpha f \qquad \text{OK}$$

∇_μ obeys properties 1)-5) (it is obviously linear) and is thus a good derivative operator. It is also a covariant map.

Metric compatibility:

What does ∇_μ do on $g_{\alpha\beta}$?

$$\nabla_\mu g_{\alpha\beta} = \partial_\mu g_{\alpha\beta} - \Gamma^\rho_{\mu\alpha} g_{\rho\beta} - \Gamma^\rho_{\mu\beta} g_{\rho\alpha}$$

$$= \partial_\mu g_{\alpha\beta} - \frac{1}{2}\left(\partial_\mu g_{\alpha\beta} + \cancel{\partial_\alpha g_{\mu\beta}} - \cancel{\partial_\beta g_{\mu\alpha}}\right) - \frac{1}{2}\left(\partial_\mu g_{\beta\alpha} + \cancel{\partial_\beta g_{\mu\alpha}} - \cancel{\partial_\alpha g_{\mu\beta}}\right) = 0$$

Our covariant derivative is called metric-compatible, since $\boxed{\nabla_\alpha g_{\mu\nu} = 0}$

This follows from the EP (which dictates the expression for $\Gamma^\alpha_{\mu\nu}$), and generalizes $\partial_\alpha \eta_{\mu\nu} = 0$ of flat space-time.

There actually are infinite covariant derivatives. Each $\tilde{\nabla}_\mu$ is related to ∇_μ by some tensor $C^\alpha_{\mu\nu}$ (called connection) in the same way as ∇_μ to ∂_μ by $\Gamma^\alpha_{\mu\nu}$ (which is not a tensor). However, only one satisfies $\boxed{\nabla_\alpha g_{\mu\nu} = 0}$

Contraction of Christoffel symbol. Another useful property is

$$\Gamma^\alpha_{\mu\alpha} = \Gamma^\alpha_{\alpha\mu} = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g} \quad , \quad \text{where } g = \det(g_{\mu\nu})$$

Proof

$$\Gamma^\mu_{\mu\lambda} = \frac{1}{2} g^{\mu\rho}\left(\cancel{\partial_\mu g_{\lambda\rho}} + \partial_\lambda g_{\mu\rho} - \cancel{\partial_\rho g_{\mu\lambda}}\right) = \frac{1}{2} g^{\mu\rho} \frac{\partial}{\partial x^\lambda} g_{\mu\rho}$$

for symm. matrix M, $\log \det M = \mathrm{Tr} \log M \quad \Rightarrow \quad \frac{1}{|M|}\frac{d}{dx}|M| = \mathrm{Tr}\, M^{-1}\frac{d}{dx}M$

(obvious if M is diagonal)

$$\Gamma^\mu_{\mu\lambda} = \frac{1}{2}\mathrm{Tr}\left(g^{-1}\frac{\partial}{\partial x^\lambda} g\right) = \frac{1}{2}\frac{1}{|\det(g)|}\frac{\partial}{\partial x^\lambda}|\det(g)| = \frac{1}{\sqrt{|\det(g)|}}\frac{\partial}{\partial x^\lambda}\sqrt{|\det(g)|}$$

Parallel transport

In flat space-time, a ~~vector~~ tensor remains constant along a curve iff $\frac{dT}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial T}{\partial x^\mu} = 0$

Its components in the "global" basis aligned with the axes do not change. In curved space, the coordinate basis changes from point to point, and is not covariant (based on ∂_μ).

Introduce a coordinate-indep. notion of parallel transport along a curve with tangent t^μ by asking that

$$\boxed{t^\mu \nabla_\mu T^{\alpha_1 \dots \alpha_k}{}_{\beta_1 \dots \beta_\ell} = 0}$$

For a vector V^α, $t^\mu \nabla_\mu V^\alpha = \underbrace{\frac{dx^\mu}{d\lambda}\partial_\mu V^\alpha}_{= dV^\alpha/d\lambda} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\lambda}V^\nu \quad \left(t^\mu = \frac{dx^\mu}{d\lambda}\right)$

System of first-order differential equations (formally similar to Schrödinger's eqn), always a local solution.
Path dependent (but coordinate indep.) connection between tangent spaces, $T_p \rightsquigarrow T_q$. $\nwarrow \Gamma^\alpha_{\mu\nu}$

With a metric-compatible connection ($\nabla_\alpha g_{\mu\nu} = 0$), if V^α and W^α are ~~paral~~ parallel-transported along $t^\mu = \frac{dx^\mu}{d\lambda}$, then

$$t^\mu \nabla_\mu (V_\alpha W^\alpha) = t^\mu \nabla_\mu (g_{\alpha\beta} V^\alpha V^\beta) = 0$$

Parallel transport preserves norms and scalar products.

Ex. $\mathbb{R}^2$ in polar coordinates (in Cartesian $g_{\mu\nu} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$ constant, $\Gamma^\alpha_{\mu\nu} = 0$)

$$ds^2 = dr^2 + r^2 d\theta^2 \qquad g^{ij} = \begin{bmatrix} 1 & \\ & r^{-2} \end{bmatrix} \qquad g_{ij} = \begin{bmatrix} 1 & \\ & r^2 \end{bmatrix}$$

$$\Gamma^r{}_{rr} = \frac{1}{2} g^{r\sigma} (2\partial_r g_{r\sigma} - \cancel{\partial_\sigma g_{rr}}) = g^{rr} \underbrace{\partial_r g_{rr}}_{=0} + \cancel{g^{r\theta} \partial_r g_{r\theta}} = 0$$

$$\Gamma^r{}_{\theta\theta} = \frac{1}{2} g^{rr} (\cancel{2\partial_\theta g_{\theta r}} - \underbrace{\partial_r g_{\theta\theta}}_{2r}) = -r$$

Check that $\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{1}{r}$ and $\Gamma^r{}_{r\theta} = \Gamma^r{}_{\theta r} = \Gamma^\theta{}_{rr} = \Gamma^\theta{}_{\theta\theta} = 0$

What is the expression of the scalar (= invariant) $\nabla_\mu V^\mu$ in (r, θ)?

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu{}_{\mu\rho} V^\rho = \partial_\mu V^\mu + \frac{1}{\sqrt{|g|}} \partial_\rho \sqrt{|g|}\, V^\rho = \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|}\, V^\mu)$$

$$\Rightarrow \nabla_\mu V^\mu = \frac{1}{r} \partial_\mu (r V^\mu) = \partial_r V^r + \frac{1}{r} V^r + \partial_\theta V^\theta$$

Ex. Klein-Gordon eqn. in curved space:

$$\Box \phi \equiv \nabla_\mu \nabla^\mu \phi = \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|}\, g^{\mu\nu} \partial_\nu \phi) = 0$$

GEODESICS

~~Cor~~ ~~In~~
Curved space generalisation of straight lines. In flat space, a particle propagates on straight lines in absence of forces. In curved space, thanks to EP, a test particle subject only to gravity obeys $d^2x^{\hat{\mu}}/d\lambda^2 = 0$ locally inertial frame.

In a generic frame,

$$\frac{d^2x^{\hat{\mu}}}{d\lambda^2} = \frac{d}{d\lambda}\left(\frac{dx^\mu}{d\lambda}\frac{\partial x^{\hat{\mu}}}{\partial x^\mu}\right) = \frac{\partial x^{\hat{\mu}}}{\partial x^\mu}\frac{d^2x^\mu}{d\lambda} + \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\frac{\partial^2 x^{\hat{\mu}}}{\partial x^\mu \partial x^\nu} = \frac{\partial x^{\hat{\mu}}}{\partial x^\mu}\left(\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda}\right)$$

Hence, since $\partial x^{\hat{\mu}}/\partial x^\nu$ is non-singular,

$$\boxed{\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0}$$ <u>geodesic equation</u>

Recalling the def of parallel transport along $t^\mu = \frac{dx^\mu}{d\lambda}$, this is

$$\boxed{t^\mu \nabla_\mu t^\nu = 0}$$

⇒ A geodesic is a curve whose tangent vector is parallel-transported along itself, i.e. it remains "parallel" (in the curved-space sense) to itself.

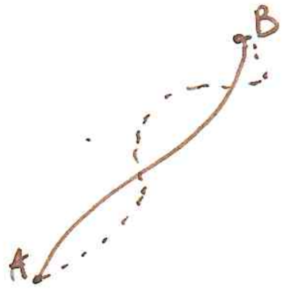
Actually, also $t^\mu \nabla_\mu t^\nu = f(\lambda) t^\nu$ would do. The ~~cha~~ r.h.s. can always be eliminated by a reparametrization $\lambda \to \alpha(\lambda)$:

$$\frac{d^2x^\mu}{d\alpha^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\alpha}\frac{dx^\beta}{d\alpha} = f t^\mu - \frac{d^2\alpha}{d\lambda^2}\left(\frac{d\alpha}{d\lambda}\right)^{-2} t^\mu$$

The choice $t^\mu \nabla_\mu t^\nu = 0$ corresponds to $t_\mu t^\mu = \text{const.}$ (since the ~~so~~ norm of parallel transported vectors is preserved). It is called affine parametrization.

A geodesic is the <u>path of shortest length</u> between two points.

For a timelike curve, <u>maximum</u> proper time.



$$\tau = \int_A^B d\lambda \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}$$

$$\boxed{t^\mu \nabla_\mu t^\nu = 0 \iff \delta\tau = 0}$$

$$\delta\tau = \int_A^B d\lambda \left[-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right]^{-1/2}\left(-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{d\delta x^\beta}{d\lambda} - \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\delta x^\sigma \frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}\right)$$

$$= \int_A^B d\tau \left(-g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{d\delta x^\beta}{d\tau} - \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\delta x^\sigma\right) \qquad \left[\frac{d\tau}{d\lambda} = \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\right]$$

$$= \int_A^B d\tau \left[\frac{d}{d\tau}\left(g_{\alpha\sigma}\frac{dx^\alpha}{d\tau}\right) - \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\right]\delta x^\sigma$$

(int. by parts with fixed end points)

The integrand must vanish.

$$g_{\alpha\sigma}\frac{d^2x^\alpha}{d\tau^2} + \frac{\partial g_{\alpha\sigma}}{\partial x^\beta}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau} - \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

$$\Rightarrow \quad \frac{d^2x^\mu}{d\tau^2} + \underbrace{\frac{1}{2}g^{\mu\sigma}\left(\frac{\partial g_{\alpha\sigma}}{\partial x^\beta} + \frac{\partial g_{\beta\sigma}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\right)}_{\Gamma^\mu_{\alpha\beta}}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \qquad \blacksquare$$

Notice that the same result is obtained from varying $S = \frac{1}{2}\int d\tau\left[-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right]$

Since $\frac{d\lambda}{d\tau} = \frac{1}{\sqrt{-t_\mu t^\mu}} = \text{const}$, then affine parametrization is $\underline{\lambda = a\tau + b}$.

Geodesics in FLRW

Let us infer the Γ's from the variation of S ($ds^2 = -dt^2 + a^2 d\vec{x}^2$).

$$t \to t + \delta t \qquad \vec{x} \to \vec{x} + \delta\vec{x} \qquad (4 \text{ equations}) \qquad \delta g_{\mu\nu} = 2a\dot{a}\begin{bmatrix} 0 & \\ & \delta_{ij}\end{bmatrix}$$

δt $$\delta S = \int d\tau\left[\frac{dt}{d\tau}\frac{d\delta t}{d\tau} - a\dot{a}\,\delta t\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}\delta_{ij}\right] = -\int d\tau\left[\frac{d^2t}{d\tau^2} + a\dot{a}\left(\frac{d\vec{x}}{d\tau}\right)^2\right]\delta t$$

$$\frac{d^2t}{d\tau^2} + a\dot{a}\,\delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = 0 \quad \Leftrightarrow \quad \frac{d^2x^0}{d\tau^2} + \Gamma^0_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

$$\Rightarrow \Gamma^0_{00} = \Gamma^0_{0i} = \Gamma^0_{i0} = 0 \quad , \quad \Gamma^0_{ij} = a\dot{a}\,\delta_{ij}$$

δx^i $\quad \delta g_{\mu\nu} = 0 \qquad \delta S = +\int d\tau\left(-a^2\delta_{ij}\frac{dx^i}{d\tau}\frac{d\delta x^j}{d\tau}\right) = \int d\tau\, a^2\left(\frac{d^2x^i}{d\tau^2} + 2\frac{a'}{a}\frac{dx^i}{d\tau}\right)\delta_{ij}\delta x^j$

$$a' \equiv \frac{da}{d\tau} \qquad \frac{d^2x^i}{d\tau^2} + 2\frac{\dot{a}}{a}\frac{dt}{d\tau}\frac{dx^i}{d\tau} = 0 \quad \longleftrightarrow \quad \frac{d^2x^i}{d\tau^2} + \Gamma^i_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

$$\Rightarrow \Gamma^i_{00} = \Gamma^i_{jk} = 0 \quad , \quad \Gamma^i_{j0} = \Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j$$

Much simpler than direct computation.

CURVATURE

Features of curved space times:

-) parallel transport is path-dependent. Around a closed loop it does not return to itself (not integrable)
-) covariant derivatives do not commute (only on scalars!)
-) initially parallel geodesics may cross

All these properties described by ~~one~~ one tensor.

Covariant derivatives do not commute. On one-form ω_μ,

$$\nabla_\alpha \nabla_\beta \omega_\mu = \partial_\alpha(\nabla_\beta \omega_\mu) - \Gamma^\rho_{\alpha\beta} \nabla_\rho \omega_\mu - \Gamma^\rho_{\alpha\mu} \nabla_\beta \omega_\rho$$

$$= \partial_\alpha \partial_\beta \omega_\mu - \partial_\alpha(\Gamma^\rho_{\beta\mu} \omega_\rho) - \Gamma^\rho_{\alpha\beta} \nabla_\rho \omega_\mu - \Gamma^\rho_{\alpha\mu}(\partial_\beta \omega_\rho - \Gamma^\sigma_{\beta\rho} \omega_\sigma)$$

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\omega_\mu = \underbrace{\left(\partial_\beta \Gamma^\rho_{\alpha\mu} - \partial_\alpha \Gamma^\rho_{\beta\mu} + \Gamma^\sigma_{\alpha\mu}\Gamma^\rho_{\beta\sigma} - \Gamma^\sigma_{\beta\mu}\Gamma^\rho_{\alpha\sigma}\right)}_{\equiv R_{\alpha\beta\mu}{}^\rho} \omega_\rho$$

(Riemann tensor) (1,3)

This expression only depends on ω_μ at p, not near it ($\partial_\alpha \omega_\mu$ drop out). Linear mapping of 1-forms $\to$ (0,3) tensor

On scalars $\nabla_\alpha \nabla_\beta$ commutes (no torsion).

$$0 = (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)(\omega_\mu t^\mu) = \nabla_\alpha(\omega_\mu \nabla_\beta t^\mu + \nabla_\beta \omega_\mu t^\mu) - \alpha \leftrightarrow \beta$$

$$= \omega_\mu(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) t^\mu + t^\mu \underbrace{(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\omega_\mu}_{R_{\alpha\beta\mu}{}^\rho \omega_\rho}$$

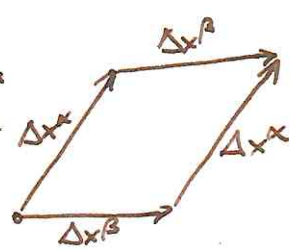
$$\omega_\mu(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) t^\mu = - \omega_\mu R_{\alpha\beta\rho}{}^\mu T^\rho \qquad \forall \omega_\mu$$

$$\boxed{(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) t^\mu = - R_{\alpha\beta\rho}{}^\mu T^\rho}$$

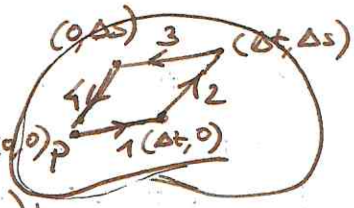
By induction, like for $\nabla_\mu T^{\alpha\ldots}{}_{\beta\ldots}$,

$$\underbrace{(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)}_{\equiv [\nabla_\alpha, \nabla_\beta]} T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_\ell} = - \sum_{i=1}^{k} R_{\alpha\beta\rho}{}^{\mu_i} T^{\mu_1 \ldots \rho \ldots \mu_k}{}_{\nu_1 \ldots \nu_\ell} + \sum_{i=1}^{\ell} R_{\alpha\beta\nu_i}{}^\rho T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \rho \ldots \nu_\ell}$$

∇_α describes parallel transport along the curve where only the α coordinate changes. E.g., $\nabla_t = t^\mu \nabla_\mu$ with $t^\mu = (1,0,0,0)$. So $[\nabla_\alpha, \nabla_\beta]$ is the difference of the parallel transports obtained shifting first by Δx^β followed by Δx^α, and viceversa.



Compute the change of $v^\mu \omega_\mu$, where v^μ is parallel transported and ω_μ is an arbitrary co-vector field, as $\Delta_1+\Delta_2+\Delta_3+\Delta_4$.



Along 1, the tangent vector is $T^\mu = (1,0)$

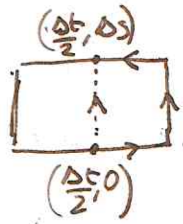
$$\Delta_1 = \underbrace{\Delta t \frac{\partial}{\partial t}}_{\Delta t\, T^\mu \nabla_\mu}(v^\mu\omega_\mu)\Big|_{(\frac{\Delta t}{2},0)}$$

accurate at <u>2nd</u> order in Δt

$$\Delta_1 = \Delta t\, T^\mu \nabla_\mu (v^\mu \omega_\mu)\Big|_{(\frac{\Delta t}{2},0)} = \Delta t\, v^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2},0)}$$

$\cancel{\Delta_1+\Delta_3}$

$$\Delta_1+\Delta_3 = \Delta t \underbrace{\left\{ v^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2},0)} - v^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2},\Delta s)} \right\}}_{O(\Delta s)}$$



$\cancel{\Delta_2+\Delta_4 =}$

At order Δs, parallel transport $(\frac{\Delta t}{2},0)\to(\Delta t,0)\to(\Delta t,\Delta t)\to(\frac{\Delta t}{2},\Delta s)$ is the same as $(\frac{\Delta t}{2},0)\to(\frac{\Delta t}{2},\Delta s)$. Hence, v^μ is still parallel-transported.

$$\Delta_1+\Delta_3 \simeq -\Delta t \Delta s\, \underset{\uparrow\,(0,1)}{S^\rho} \nabla_\rho\left(v^\mu T^\nu \nabla_\nu \omega_\mu\right)\Big|_{(\frac{\Delta t}{2},0)} \simeq -\Delta t\Delta s\, v^\mu S^\rho \nabla_\rho \left(T^\nu \nabla_\nu \omega_\mu\right)\Big|_{(0,0)}$$

$$\Delta_1+\Delta_2+\Delta_3+\Delta_4 \approx \Delta t \Delta s\, v^\mu \underbrace{\left[T^\rho \nabla_\rho (S^\nu \nabla_\nu \omega_\mu) - S^\rho \nabla_\rho (T^\nu \nabla_\nu \omega_\mu)\right]_{(0,0)}}_{T^\nu S^\rho \underbrace{(\nabla_\nu\nabla_\rho - \nabla_\rho\nabla_\nu)\omega_\mu}_{R_{\nu\rho\mu}{}^{\sigma}\omega_\sigma} - \underbrace{(T^\rho\nabla_\rho S^\nu - S^\rho \nabla_\rho T^\nu)}_{T^\rho\partial_\rho S^\nu - S^\rho\partial_\rho T^\nu = 0} \nabla_\nu \omega_\mu}$$

(T's cancel)

$$\boxed{\delta(\omega_\mu v^\mu) \approx \Delta t \Delta s\, v^\mu T^\nu S^\rho R_{\nu\rho\mu}{}^{\sigma} \omega_\sigma} \qquad \forall \omega_\mu$$

$$\Rightarrow \boxed{\delta v^\mu = \Delta t \Delta s\, T^\alpha S^\beta v^\gamma R_{\alpha\beta\gamma}{}^{\mu}}$$

Properties of Riemann tensor

$$\boxed{R_{\mu\nu\rho}{}^{\sigma} = \partial_\nu \Gamma^\sigma_{\mu\rho} - \partial_\mu \Gamma^\sigma_{\nu\rho} + \Gamma^\sigma_{\nu\lambda}\Gamma^\lambda_{\mu\rho} - \Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\nu\rho}} .$$

In the loc. inertial frame, only terms with $\partial_\alpha\partial_\beta g_{\mu\nu}$ survive (in flat space-time, we have $\Gamma^\alpha_{\mu\nu} = 0$ everywhere, $\Rightarrow R_{\mu\nu\rho}{}^\sigma = 0$)

- $R_{\mu\nu\rho}{}^\sigma = - R_{\nu\mu\rho}{}^\sigma$ (obvius ~~#~~ since $R_{\mu\nu\rho}{}^\sigma \omega_\sigma = (\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)\omega_\rho$)
- $R_{[\mu\nu\rho]}{}^\sigma = 0$. Follows since $R_{\mu\nu\rho}{}^\sigma + R_{\nu\rho\mu}{}^\sigma + R_{\rho\mu\nu}{}^\sigma = 0$, and the same for odd permutations. E.g.,
$$\cancel{\partial_\mu \Gamma^\sigma_{\nu\rho}} - \cancel{\partial_\nu \Gamma^\sigma_{\mu\rho}} + \cancel{\partial_\nu \Gamma^\sigma_{\rho\mu}} - \cancel{\partial_\rho \Gamma^\sigma_{\nu\mu}} + \cancel{\partial_\rho \Gamma^\sigma_{\mu\nu}} - \cancel{\partial_\mu \Gamma^\sigma_{\rho\nu}} = 0$$
and same for the Γ^2 terms.
- for metric-compatible ∇_μ (i.e. $\nabla_\mu g_{\alpha\beta} = 0$), and $R_{\mu\nu\alpha\beta} \equiv R_{\mu\nu\alpha}{}^\gamma g_{\gamma\beta}$:
$$\boxed{R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}} \quad \text{(symmetric under } (\alpha\beta) \leftrightarrow (\mu\nu)\text{)}$$
Easily seen in the ~~ine~~ loc. inertial frame expression for $R_{\mu\nu\alpha\beta}$:
$$R_{\mu\nu\alpha\beta} = \left(-\partial_\mu \Gamma^\gamma_{\nu\alpha} + \partial_\nu \Gamma^\gamma_{\mu\alpha}\right) g_{\gamma\beta} \qquad (\partial_\alpha g_{\mu\nu} = 0)$$
$$= g_{\gamma\beta} g^{\gamma\sigma} \frac{1}{2}\left[\partial_\nu\left(\cancel{\partial_\mu g_{\alpha\sigma}} + \partial_\alpha g_{\mu\sigma} - \partial_\sigma g_{\mu\alpha}\right) - \partial_\mu\left(\cancel{\partial_\nu g_{\alpha\sigma}} + \partial_\alpha g_{\nu\sigma} - \partial_\sigma g_{\nu\alpha}\right)\right]$$
$$= \frac{1}{2}\left(\partial_\nu\partial_\alpha g_{\mu\beta} + \partial_\mu\partial_\beta g_{\nu\alpha} - \partial_\nu\partial_\beta g_{\mu\alpha} - \partial_\mu\partial_\alpha g_{\nu\beta}\right)$$
True in loc. inert. frame + covariant $\Rightarrow$ true in every frame.
Also, $R_{\nu\mu\alpha\beta} = -R_{\mu\nu\alpha\beta}$ and $R_{\mu\nu\beta\alpha} = -R_{\mu\nu\alpha\beta}$. Last, $R_{[\mu\nu\rho]\sigma} = 0 \Rightarrow \boxed{R_{[\mu\nu\rho\sigma]} = 0}$
- Bianchi identity: $\boxed{\nabla_{[\lambda} R_{\alpha\beta]\gamma\delta} = 0}$ (derivatives of Riemann)

  It follows since $\nabla_\lambda R_{\alpha\beta\gamma\delta} + \nabla_\alpha R_{\beta\lambda\gamma\delta} + \nabla_\beta R_{\lambda\alpha\gamma\delta} = 0$, which again can be seen in the loc. in. frame where $\nabla_\mu \to \partial_\mu$ (check!) using the above expression for $R_{\alpha\beta\gamma\delta}$.
- # of independent components:
$$N = \underbrace{\frac{M(M+1)}{2}}_{(\alpha\beta)\leftrightarrow(\gamma\delta)\text{ symm.}} - \underbrace{\frac{n(n-1)(n-2)(n-3)}{4!}}_{\text{totally antisymm.}}, \quad \text{where } M = \underbrace{\frac{n(n-1)}{2}}_{\alpha\leftrightarrow\beta \text{ and } \gamma\leftrightarrow\delta \text{ antisymm.}}$$
$$\Rightarrow \boxed{N = \frac{n^2(n^2-1)}{12}} \qquad \text{For } n=4,\ N=20$$

Contracted forms of Riemann tensor

$$\boxed{R_{\mu\nu} \equiv R_{\mu\alpha\nu}{}^{\alpha}}$$ Ricci tensor

$R_{\mu\nu}$ is symmetric: $R_{\nu\mu} = R_{\nu\alpha\mu\beta}\, g^{\alpha\beta} = R_{\mu\beta\nu\alpha}\, g^{\alpha\beta} = R_{\mu\beta\nu}{}^{\beta} = R_{\mu\nu}$

$$\boxed{R \equiv R_{\mu}{}^{\mu} = R_{\mu\nu} g^{\mu\nu}}$$ Ricci scalar (or scalar curvature)

Contracting the Bianchi identity gives

$$0 = \left(\nabla_{\lambda} R_{\alpha\beta\gamma\delta} + \nabla_{\alpha} R_{\beta\lambda\gamma\delta} + \nabla_{\beta} R_{\lambda\alpha\gamma\delta}\right) g^{\lambda\gamma} g^{\beta\delta} = \nabla^{\gamma} R_{\alpha\gamma} - \nabla_{\alpha} R + \nabla^{\delta} R_{\alpha\delta}$$

$$\Rightarrow \quad \boxed{\nabla^{\mu} R_{\mu\nu} = \frac{1}{2} \nabla_{\nu} R}$$

Can also define $\boxed{G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}$ Einstein tensor

The Einstein tensor is conserved: $\boxed{\nabla^{\mu} G_{\mu\nu} = 0}$

~~However~~, For the moment this is only geometry. No physics in this conservation law. However, $T_{\mu\nu}$ is also conserved. We will postulate that $G_{\mu\nu} \propto T_{\mu\nu}$ ....

Example: sphere of radius a (fixed), $ds^2 = a^2(d\theta^2 + \sin^2\theta\, d\phi^2)$

$$g_{\mu\nu} = a^2 \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}, \quad \Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(\partial_{\mu} g_{\nu\sigma} + \partial_{\nu} g_{\mu\sigma} - \partial_{\sigma} g_{\mu\nu}\right)$$

Only derivative $\neq 0$ is $\partial_{\theta} g_{\phi\phi} = 2a^2 \sin\theta\cos\theta$; $\quad g^{\mu\nu} = \frac{1}{a^2}\begin{bmatrix} 1 & 0 \\ 0 & \sin^{-2}\theta \end{bmatrix}$

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta \qquad \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \frac{\sin\theta\cos\theta}{\sin^2\theta} = \frac{\cos\theta}{\sin\theta}$$ (all other Γ's $= 0$)

$$R_{\theta\phi\theta}{}^{\phi} = \cancel{\partial_{\phi}\Gamma^{\phi}_{\theta\theta}} - \partial_{\theta}\Gamma^{\phi}_{\phi\theta} + \cancel{\Gamma^{\phi}_{\phi\lambda}\Gamma^{\lambda}_{\theta\theta}} - \Gamma^{\phi}_{\theta\lambda}\Gamma^{\lambda}_{\phi\theta} = 1 + \cancel{\frac{\cos^2\theta}{\sin^2\theta}} - \cancel{\left(\frac{\cos\theta}{\sin\theta}\right)^2} = 1$$

$R_{\theta\phi\theta\phi} = R_{\theta\phi\theta}{}^{\phi} g_{\phi\phi} = a^2\sin^2\theta$ (only indep. component: $N = \frac{n^2(n^2-1)}{12} = 1$)

Ricci tensor: $R_{\theta\theta} = R_{\theta\phi\theta\phi}\, g^{\phi\phi} = 1$; $R_{\phi\phi} = R_{\phi\theta\phi\theta}\, g^{\theta\theta} = \sin^2\theta$; $R_{\theta\phi} = 0$

$$R_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix} = \frac{g_{\mu\nu}}{a^2} \qquad R = R_{\mu\nu} g^{\mu\nu} = \frac{g_{\mu\nu} g^{\mu\nu}}{a^2} = \frac{2}{a^2}$$

Scalar curvature is constant and $\to 0$ as $a \to \infty$, as expected.

Principle of general covariance

- Physics in curved space-time -

Any SR equation valid in flat space without gravity will remain true in GR once written in covariant form:

$$\boxed{\eta_{\mu\nu} \to g_{\mu\nu}\ ,\ \partial_\mu \to \nabla_\mu}$$

- this automatically satisfies the EP ($g_{\mu\nu}=\eta_{\mu\nu}$, $\Gamma=0$ locally)
- no new physics yet. It just means writing equations that are valid in any coordinate system (just like the Lagrangian formalism in class. mech.). Can deal with Newton's law in flat space in sperical coordinates, or particle constrained on a sphere:

$$\mathcal{L}^{(\text{free})}_{\text{sph.}} = \frac{m}{2}\left(r^2\dot\theta^2 + r^2\sin^2\theta\,\dot\phi^2\right) = \frac{m}{2} g_{\mu\nu} v^\mu v^\nu \ ,\quad g_{\mu\nu} = r^2\begin{bmatrix}1 & 0\\ 0 & \sin^2\theta\end{bmatrix}$$

- the non-trivial addition is that a non-flat, possibly evolving $g_{\mu\nu}$ will describe gravity:

<u>Ex.</u> $$v^\nu\partial_\nu v^\mu = \frac{d^2x^\mu}{d\tau^2} = 0 \longrightarrow v^\nu\nabla_\nu v^\mu = \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

$$\partial_\mu T^{\mu\nu} = J^\nu \longrightarrow \nabla_\mu T^{\mu\nu} = J^\nu \qquad \partial_\mu F^{\mu\nu} = 0 \longrightarrow \nabla_\mu F^{\mu\nu} = 0$$

$$v^\alpha\partial_\alpha v^\mu = \frac{q}{m}F^\mu{}_\alpha v^\alpha \longrightarrow v^\alpha\nabla_\alpha v^\mu = \frac{q}{m}F^\mu{}_\alpha v^\alpha \qquad [F^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)]$$

This prescription is not entirely unique. We wouldn't know how to promote e.g. $v^\mu\partial_\mu\partial_\nu v^\nu = 0$: should we choose $\nabla_\mu\nabla_\nu$ or $\nabla_\nu\nabla_\mu$? The diffence would be $R_{\mu\nu}v^\mu v^\nu$.

In general, adding a term like e.g. $R^{\mu\nu}J_\nu$ (or any term involving Riemann) would still reproduce SR in flat space ($R_{\alpha\beta\gamma}{}^\delta \to 0$) but not in a locally inertial frame in curved space ($R_{\alpha\beta\gamma}{}^\delta \neq 0$), so it would violate the EP (gravity should be "erased" in free fall).

From the point of view of modern QFT, EP-violating terms coupling matter to curvature are not forbidden by symmetry and thus expected. <u>However</u>, the expected coupling constant is extremely small (since G is very small) and, unless unnaturally amplified, effects are negligible. So far, no violation of EP has been ever detected.

Non-relativistic (Newtonian) limit of geodesic equation:

Geodesic motion reduces to $\vec{a} = -\vec{\nabla}\phi$ when 1) $v \ll c$, 2) gravity is weak, and 3) it is static or slowly varying ($\partial_0 g_{\mu\nu} \approx 0$).

$$\frac{dx^i}{d\tau} \ll \frac{dt}{d\tau} \Rightarrow \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 \approx 0 \quad ; \quad g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu} \text{ and } |\eta_{\mu\nu}| \ll 1 .$$

$$\Gamma^\mu_{00} = \frac{1}{2} g^{\mu\rho}(\underbrace{2\partial_0 g_{0\rho}}_{\approx 0} - \partial_\rho g_{00}) \approx -\frac{1}{2} g^{\mu\rho}\partial_\rho h_{00} \approx -\frac{1}{2}\eta^{\mu i}\partial_i h_{00} \quad (\text{at } 1^{st} \text{ order in } \eta_{\mu\nu})$$

$\mu = 0$: $\quad \frac{d^2t}{d\tau^2} \approx 0 \Rightarrow \frac{dt}{d\tau} \approx \text{const} \equiv A$

$\mu = i$: $\quad \frac{d^2x^i}{d\tau^2} = A^2\frac{d^2x^i}{dt^2} \approx \frac{1}{2}\partial_i h_{00} A^2 \quad \Rightarrow \frac{d^2x^i}{dt^2} \approx \frac{1}{2}\partial_i h_{00}$

It matches $\vec{a} = -\vec{\nabla}\phi$ if $\boxed{h_{00} = -2\Phi}$ (and $\Phi \ll 1$).

Geodesics reproduce Newtonian gravity with $g_{00} \approx -(1+2\Phi)$.

Einstein equations

We seek a covariant generalization of $\nabla^2\Phi = 4\pi G\rho$ (Poisson's law). (flat 3d Laplacian)

Its solution is $\Phi = -\frac{GM}{r}$ for a ~~point~~-like mass M. Natural to promote $\rho \to T_{\mu\nu}$. At the l.h.s. we need 2nd derivatives of $g_{\mu\nu}$, since $\Phi \approx -h_{00}/2$, but it cannot be $\nabla^\rho\nabla_\rho g_{\mu\nu}$ (since $\nabla_\rho g_{\mu\nu} = 0$).

However, $\nabla^\mu T_{\mu\nu} = 0 = \nabla^\mu G_{\mu\nu}$, and $G_{\mu\nu}$ has 2nd derivatives of $g_{\mu\nu}$.

So we postulate $G_{\mu\nu} = \kappa T_{\mu\nu}$, with $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$.

Tracing: $G_{\mu\nu}g^{\mu\nu} = R - 2R = -R \quad \Rightarrow \quad R = -\kappa T_{\mu\nu}g^{\mu\nu} \equiv -\kappa T$

$$G_{\mu\nu} = \kappa T_{\mu\nu} \Rightarrow R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$$

For non-relativistic source: $T_{\mu\nu} = \rho u_\mu u_\nu$, $T = \rho u_\mu u^\mu = -\rho$, $T_{00} \approx \rho$.

We want to reproduce $\nabla^2 h_{00} \approx -8\pi G\rho$.

$$R_{00} = R_{0\lambda 0}{}^{\lambda} = R_{0i0}{}^{i} = \partial_i\Gamma^i_{00} - \underbrace{\partial_0\Gamma^i_{i0}}_{\approx 0 \text{ (static } g_{\mu\nu})} + \mathcal{O}(h^2) \approx -\frac{1}{2}\partial_i\delta^{ij}\partial_j h_{00} = -\frac{1}{2}\nabla^2 h_{00}$$

$$\Rightarrow \nabla^2 h_{00} \approx -2R_{00} = -2\kappa\left(T_{00} - g_{00}\frac{T}{2}\right) \approx -2\kappa\left(\rho - \frac{\rho}{2}\right) = -\kappa\rho$$

We recover the desired Newtonian limit if $\kappa = 8\pi G$.

$$\Rightarrow \boxed{G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}}$$ Einstein equations