Equivalently, Einstein eqns. are

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$$

In vacuum, $\boxed{R_{\mu\nu} = 0}$

Notice that this does not imply flat space. Other components of $R_{\mu\rho\nu}{}^{\sigma}$ may be non-zero.

Geodesically conserved quantities

Suppose that $\partial_{\bar{\sigma}} g_{\mu\nu} = 0$ for ~~some~~ one particular $\bar{\sigma}$, that is that $x^{\bar{\sigma}} \to x^{\bar{\sigma}} + c$ is a symmetry (the metric does not depend on $x^{\bar{\sigma}}$). Then, $p_{\bar{\sigma}} = m u_{\bar{\sigma}}$ is a conserved quantity along the geodesic.

Proof $p^\mu = m u^\mu$, $p^\lambda \nabla_\lambda p^\mu = 0$ (geodesic equation)

Can lower $\mu$ thanks to metric compatibility : $p^\lambda \nabla_\lambda p_\mu = 0$

$$\underbrace{p^\lambda \partial_\lambda p_\mu}_{= m\frac{dx^\lambda}{d\tau}\partial_\lambda p_\mu = m\frac{dp_\mu}{d\tau}} - \Gamma^\rho_{\mu\lambda} p_\rho p^\lambda = 0$$

$$m\frac{dp_\mu}{d\tau} = \Gamma^\rho_{\mu\nu} p_\rho p^\nu = \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) p_\rho p^\nu = \frac{p^\sigma p^\nu}{2}\left(\partial_\mu g_{\nu\sigma} + \cancel{\partial_\nu g_{\mu\sigma}} - \cancel{\partial_\sigma g_{\mu\nu}}\right)$$

$$\Rightarrow \frac{dp_\mu}{d\tau} = \frac{p^\nu p^\sigma}{2m}\partial_\mu g_{\nu\sigma}$$

$$\partial_{\bar{\sigma}} g_{\mu\nu} = 0 \quad \Rightarrow \quad \frac{dp_{\bar{\sigma}}}{d\tau} = 0 \quad \Rightarrow p_{\bar{\sigma}} \text{ is conserved.}$$

Notice that: 1) it is $p_{\bar{\sigma}}$ that is conserved, and not $p^{\bar{\sigma}}$

2) this is not a covariant statement. Only valid in the coordinate system in which the symmetry is explicit (we used $\partial_{\bar{\sigma}}$ and not $\nabla_{\bar{\sigma}}$)

We will see a more geometric way of discussing symmetries later ($\to$ isometries and Killing vectors)

# THE SCHWARZSCHILD SOLUTION

Spherically symmetric vacuum solution. It describes the metric outside of stars and black holes.
Solution of $R_{\mu\nu}=0$ (analogous of $\nabla^2\phi=0$ in Newtonian gravity)
We want a solution that is spherically symmetric and static (these concepts will become more clear when we will study isometries and Killing vectors). Therefore, choosing $\{t,r,\theta,\varphi\}$:

- no explicit dependence on $t$
- no mixed $dt\,dr$, $dt\,d\theta$, $dt\,d\varphi$ terms (odd under t-reversal)

(see Ch. 7 of Carroll's notes for more details).
Because the metric must be Lorentzian, it can be written as

$$ds^2=-e^{2\alpha(r)}dt^2+e^{2\beta(r)}dr^2+r^2d\Omega^2$$

where $d\Omega^2=d\theta^2+\sin^2\theta\, d\varphi^2$ is the metric on a 2-sphere.
Notice that the radius $r$ of the 2-sphere is NOT the proper distance from the center.
Computing $\Gamma^\rho_{\mu\nu}$ and then $R_{\mu\nu}$ from this $g_{\mu\nu}$ (boring but straight-forward) gives

$$R_{tt}=e^{2(\alpha-\beta)}\left(\alpha''+\alpha'^2-\alpha'\beta'+\frac{2}{r}\alpha'\right)$$
$$R_{rr}=-\alpha''-\alpha'^2+\alpha'\beta'+\frac{2}{r}\beta'$$
$$R_{\theta\theta}=e^{-2\beta}\left[r(\beta'-\alpha')-1\right]+1$$
$$R_{\phi\phi}=\sin^2\theta\, R_{\theta\theta}$$

Remembering that $R_{\mu\nu}=0$ (vacuum solution), we have:

$$0=e^{2(\beta-\alpha)}R_{tt}+R_{rr}=\frac{2}{r}(\alpha+\beta)'$$

$\Rightarrow \alpha+\beta=c=\text{const}$, it can be reabsorbed by constant rescaling $t\to te^c$
We can now set $\alpha=-\beta$

$$R_{\theta\theta}=0 \Rightarrow e^{2\alpha}(2r\alpha'+1)=1 \Rightarrow (re^{2\alpha})'=1 \Rightarrow re^{2\alpha}=\mu+r$$

where $\mu$ is a constant

$$\Rightarrow ds^2=-\left(1+\frac{\mu}{r}\right)dt^2+\left(1+\frac{\mu}{r}\right)^{-1}dr^2+r^2d\Omega^2$$

But we saw that in the Newtonian limit we must have

$$g_{00} \simeq -1-2\Phi = -1+2\frac{GM}{r} \quad \Rightarrow \mu = -2GM$$

Defining $\boxed{R_S \equiv 2GM}$ (Schwarzschild radius) we get

$$\boxed{ds^2 = -\left(1-\frac{R_S}{r}\right)dt^2 + \left(1-\frac{R_S}{r}\right)^{-1}dr^2 + r^2 d\Omega^2}$$ Schwarzschild metric

The Newt. l. only used to fix the constant, the metric is exact.

How big is the Schwarzschild radius $R_S$?

Newton's constant $G$ has dimensions $[G] = \frac{L^3}{t^2 m}$ (since $a \sim \frac{L}{t^2} \sim \frac{GM}{r^2}$)

Hence in metric units $R_S = \frac{2GM}{c^2}$, $G = 6{,}67 \times 10^{-8}\, cm^3 g^{-1} s^{-2}$

The Sun has mass $M_\odot = 2\times10^{33} g$, so its Schw. radius is

$$\frac{2GM_\odot}{c^2} = \frac{2\times 6.67\times10^{-8} cm^3 g^{-1} s^{-2} \cdot 2\times10^{33} g}{(3\times10^{10} cm/s)^2} \approx 3\, km$$

For any star, $R_S \approx 3\left(\frac{M}{M_\odot}\right) km$, while the radius of the Sun is $R_\odot \simeq 7\times10^5 km$. For ordinary stars, $R_S$ is well inside the star, where the metric is not even valid (we computed it as a solution in vacuum), so it has no real meaning

- ) asymptotically ($r\to\infty$) the Schwarzschild metric is flat ($\sim$ Minkowski)
- ) it depends only on the total mass within a given radius (just like Newtonian potential). The source may not be static (e.g. pulsating star) but the solution outside it is still the static Schwarzschild metric. It is the only spherically symmetric vacuum solution (this is known as Birkhoff's theorem).

Inside the star, one has still $m(r) = \int_0^r dr' r'^2 \rho(r')$ and

$$ds^2 = -\left(1-\frac{2Gm(r)}{r}\right)dt^2 + \left(1-\frac{2Gm(r)}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

At $r=R$ (external radius of the star), $m(R)=M$ and the solutions match continuosly. Each $m(r)$ has its own $r_S \ll r$, so I never reach it.

However, if the object is very massive and compact, it could be that $R_S > R$. This happens if the mass within $R$ obeys $M > \frac{c^2 R}{2G}$. In this case (e.g. for a black hole) the solution at $r = R_S$ is formally valid, but becomes singular.

Should I believe the solution or not?

The fact that the metric diverges could simply be an artifact of the coordinate choice. E.g., in flat space but spherical coordinates, the inverse $g^{\mu\nu}$ diverges at $r=0$.

What happens to curvature invariants?

$$R = 0 \ (\text{since } R_{\mu\nu} = 0), \text{ but } R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{48 G^2 M^2}{r^6}$$

Singular at $r=0$, but finite at $r = 2GM$. So $R_S$ is not a real singularity. We will see that geodesics can be continued through it. Still, $r = R_S$ has interesting properties, and we call it <u>event horizon</u>.

## <u>Geodesics in Schwarzschild</u>

High degree of symmetry $\rightarrow$ many conserved quantities. Like in flat space, invariance under rotations and time translation $\longrightarrow$ conservation of ang. mom. and energy along geodesics.

The angular part $d\Omega^2$ of $g_{\mu\nu}$ is the same as Minkowski: we expect motion to happen in a plane ($\theta = \frac{\pi}{2} = \text{const.}$)

$$p_\theta = m r^2 \frac{d\theta}{d\tau} = g_{\theta\theta} m \frac{d\theta}{d\tau} \qquad \partial_\theta g_{\mu\nu} = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 2r^2 \sin\theta\cos\theta \end{bmatrix}_{\theta=\frac{\pi}{2}} = 0$$

$$\Rightarrow p_\theta = \text{const} = p_{\theta,in} = 0$$

$$\Rightarrow \frac{d\theta}{d\tau} = 0 \quad \forall \tau \quad \Rightarrow \quad \theta = \frac{\pi}{2} \ \forall \tau$$

The other conserved quantities are

$$E \equiv -\frac{p_t}{m} = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau} \qquad L = \frac{p_\varphi}{m} = r^2 \sin^2\theta \left.\frac{d\varphi}{d\tau}\right|_{\theta=\frac{\pi}{2}} = r^2 \frac{d\varphi}{d\tau}$$

Same for a massless particle, except that one cannot use proper time $\tau$ but another affine parameter $\lambda$.

Along the geodesic, $\varepsilon = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$ is also conserved, with $\varepsilon = 0$ for massless particles and $\varepsilon = 1$ for massive ones.

Multiplying by $\left(1-\frac{2GM}{r}\right)$ and using conservation laws we get

$$\left(1-\frac{2GM}{r}\right)\varepsilon = E^2 - \left(\frac{dr}{d\lambda}\right)^2 - \left(1-\frac{2GM}{r}\right)r^2\left[\underbrace{\left(\frac{d\theta}{d\lambda}\right)^2}_{=0} + \underbrace{\sin^2\theta}_{=1}\left(\frac{d\phi}{d\lambda}\right)^2\right]$$

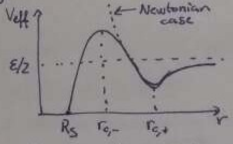
$$\Rightarrow \boxed{\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V_{eff}(r) = \frac{E^2}{2}}$$

where $V_{eff}(r) = \frac{1}{2}\left(1-\frac{2GM}{r}\right)\left(\frac{L^2}{r^2}+\varepsilon\right) = \frac{\varepsilon}{2} - \frac{\varepsilon GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3}$

Effective 1-dimensional classical problem of unit mass particle with "energy" $\frac{E^2}{2}$ in the effective potential $V_{eff}(r)$. Compared to Newtonian gravity (apart for a constant shift by $\varepsilon/2$) we get an additional attractive term $\sim 1/r^3$.

In Newtonian gravity, the centrifugal term $\frac{L^2}{2r^2}$ always forbids falling onto $r=0$, but not in GR. The extra piece is suppressed at large $r$ but dominates at $r \sim R_S$.

At fixed $L$:




$V_{eff}(r=R_S) = 0$

Two circular orbits, at $r = r_{c,-}$ (unstable) and $r = r_{c,+}$ (stable)

$V_{eff}' = 0 \Rightarrow \frac{GM}{r_c^2} - \frac{L^2}{r_c^3} + \frac{3GML^2}{r_c^4} = 0$. (for $\varepsilon = 1$)

$$\boxed{r_{c,\pm} = \frac{L^2 \pm \sqrt{L^2(L^2 - 12G^2M^2)}}{2GM}}$$

The minimum $L^2$ for which a stable orbit exists is $L^2 = 12G^2M^2$, for which

$$r_c = \frac{L^2}{2GM} = 6GM = 3R_S \qquad (r_{c,+} = r_{c,-} = r_c)$$

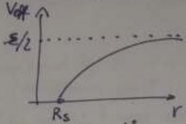
This is also the minimum value of a stable circ. orbit.

For $L^2 \to +\infty$, $r_{c,+} \simeq \frac{L^2}{GM} \to +\infty$ and $r_{c,-} \simeq \frac{L^2}{2GM}\left[1-\left(1-6\frac{G^2M^2}{L^2}\right)\right] \simeq 3GM$

Thus, for $L^2 > 12G^2M^2 \Rightarrow 3GM < r_{c,-} < 6GM < r_{c,+} < L^2/GM$

For $L^2 < 12G^2M^2$, the bump disappears

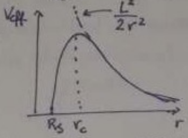
No circular orbits are possible




For a massless particle (photon, $\mathcal{E}=0$):

$$V_{eff}(r) = \frac{L^2}{2r^2} - \frac{GML^2}{r^3}$$

$$V'_{eff}(r_c) = -\frac{L^2}{r_c^3}\left(1-\frac{3GM}{r_c}\right) = 0$$




There can be only unstable circular orbits, at $r_c = 3GM$.

For massive particles, there can be (apart for circular orbits) bound orbits oscillating around $r_{c,+}$ or trapped near $R_s$, or unbound ones bouncing to $+\infty$, ~~and~~ or (for $E^2/2$ larger than max of $V_{eff}$) unbound ones falling from $+\infty$ onto $R_s$ (or escaping). For $L^2 < 12G^2M^2$, only the last two types

For massless particles, trapped orbits near $R_s$ and unbound ones bouncing to $\infty$ (if $E^2/2$ smaller than the max). For larger energies, infalling orbits to $R_s$ and escaping ones to $+\infty$.

## Gravitational redshift

Take an observer at rest in Schwarzschild coords., $u^i = 0$

$$u^\mu u_\mu = -1 \Rightarrow u^0 = +\left(1-\frac{2GM}{r}\right)^{-1/2}$$

A photon with trajectory $x^\mu(\lambda)$ has frequency $\omega$ measured by $u^\mu$:

$$\omega = -g_{\mu\nu}u^\mu\frac{dx^\nu}{d\lambda} = \left(1-\frac{2GM}{r}\right)^{1/2}\frac{dt}{d\lambda} = \left(1-\frac{2GM}{r}\right)^{-1/2}E$$

$E$ is conserved along the photon's path: $E_1 = E_2 = E$

For a photon climbing out of a potential well,

$$\boxed{\frac{\omega_2}{\omega_1} = \left(\frac{1-2GM/r_1}{1-2GM/r_2}\right)^{1/2}}$$

$r_2 > r_1 \Rightarrow \omega_2 < \omega_1$

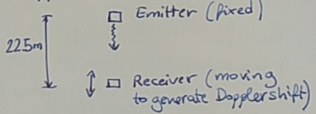
the photon is redshifted.

In the weak field limit ($r \gg 2GM$),

$$\frac{\omega_2}{\omega_1} \approx 1 - \frac{2GM}{r_1} + \frac{2GM}{r_2} = 1 + \Phi_1 - \Phi_2$$

we recover the effect implied by the equivalence principle for a Newtonian potential.

Gravitational redshift experimentally confirmed by Pound and Rebka (1960), using $\gamma$-rays emitted by $^{57}Fe$ (Mössbauer effect).
The photons emitted from top of the 22.5 meters tower reached the ground blue-shifted, and would be absorbed by another $^{57}Fe$ only if this was moving, so that Doppler shift compensates the gravitational blue-shift.




Agreement within 10%
(later improved)

## Precession of orbits

We will study it in the nearly-~~spherical~~ circular approximation. (See Carroll or Weinberg for full details). Small oscillations around the min of

$$V_{eff}(r) = \frac{1}{2} - \frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3}$$

From $V'_{eff}(r_c) = 0$, we get $L^2 = \frac{GMr_c^2}{r_c - 3GM}$. The frequency of radial oscillations around $r_c$ is

$$\omega_r^2 = \left.\frac{\partial^2 V_{eff}}{\partial r^2}\right|_{r=r_c} = -\frac{2GM}{r_c^3} + \frac{3L^2}{r_c^4} - \frac{12GML^2}{r_c^5} = \frac{GM}{r_c^3}\,\frac{r_c - 6GM}{r_c - 3GM}$$

This frequency is in the proper time $\tau$, in which $V_{eff}$ was defined.
Since $L = r_c^2 \omega_\varphi$, the angular frequency is

$$\omega_\varphi^2 = \frac{L^2}{r_c^4} = \frac{GM}{r_c^2(r_c - 3GM)}$$

In the $r_c \gg GM$ limit, $\omega_r \approx \omega_\varphi$ and the orbit closes (this is actually true for any bound orbit in Newtonian gravity, not just nearly circular ones). In general, $\omega_\varphi > \omega_r$ and it does not close.

$$\omega_{precession} \equiv \omega_\varphi - \omega_r \approx \sqrt{\frac{GM}{r_c^3}}\left[\frac{3GM}{r_c} + \mathcal{O}\left(\left(\frac{GM}{r_c}\right)^2\right)\right] \approx \frac{3GM}{r_c}\,\omega_\varphi$$

For Mercury, $\frac{\omega_{precession}}{\omega_\varphi} \approx \frac{3}{2}\frac{R_S}{r_c} \approx \frac{3}{2}\frac{3km}{5{,}7\times10^7 km} \approx 10^{-7}$

GR solves the problem of the anomalous precession of Mercury, known since 1859.

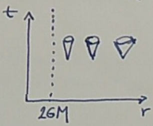
The bending of light geodesics, gravitational redshift and precession of perihelia of planetary orbits are the 3 "classical" tests of GR.

## Nearing the event horizon

What happens when particles approach the Schwarzschild radius $R_S = 2GM$ ? What happens to the black hole solution at $r < R_S$ ?

A light ray travelling in radial direction ($d\theta = d\phi = 0$) has

$$ds^2 = 0 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 \Rightarrow \boxed{\frac{dt}{dr} = \pm\left(1-\frac{2GM}{r}\right)^{-1}}$$




Light cones "close up" as one approaches $R_S$ ($dt/dr \to \pm\infty$)

The light ray seems to "slow down" ($\frac{dr}{dt} \to 0$) as it approaches $R_S$, and never reach (in these coordinates!)

A fixed interval of proper time $\Delta\tau_{emitted}$ (e.g. the proper time separation between pulses emitted by a beacon falling towards $R_S$) is measured by an observer at $r_{obs}$ as $\Delta\tau_{obs} = \frac{1-2GM/r_{obs}}{1-2GM/r}\Delta\tau_{emitted}$ (same calculation as for grav. redshift) and $\Delta\tau_{obs} \to \infty$ as $r \to R_S$. The observer never sees the beacon crossing $R_S$.

But how much proper time does it take to reach the horizon?

A massive body which is free-falling radially ($L=0$) obeys

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \left(\frac{1}{2} - \frac{GM}{r}\right) = \frac{E^2}{2} \Rightarrow \boxed{\Delta\tau = \int \frac{dr}{\sqrt{E^2 - 1 + 2GM/r}}}$$

The integrand is continuous as $r \to 2GM$ ($\to 1/E$), nothing happens. The singularity at $r = R_S$ is just an accident of the coordinates we use. Also, the metric determinant is $g = -r^4\sin^2\theta$, continuous throug $R_S$.

Let us construct coordinates that are not singular at the horizon. We do this in steps (forgetting about angular coordinates)

Tortoise coordinates: change in the radial coord. so that the light cones no longer close up.

$$\cancel{r_*}\ \Delta t = \int_{r_0}^{r} dr' \frac{1}{1-R_s/r'} = \int_{r_0}^{r} dr' \left(1 + \frac{R_s}{r'-R_s}\right) = r - r_0 + R_s\left(\log\frac{r-R_s}{R_s} - \log\frac{r_0-R_s}{R_s}\right)$$

$$\Rightarrow \boxed{r_* \equiv r + R_s \log\left(\frac{r}{R_s} - 1\right)} \qquad \Delta t = \Delta r_*$$

Light rays now obey $t = \pm r_* + c$ (like in flat space). This also makes it explicit that it takes an infinite coord. time to reach $R_s$.

The metric (neglecting angles) becomes

$$ds^2 = \left(1 - \frac{R_s}{r}\right)\left(-dt^2 + dr_*^2\right) \qquad \left(dr = \left(1 - \frac{R_s}{r}\right) dr_*\right)$$

We moved the problematic region ($R_s$) to $-\infty$, since $r_*(R_s) = -\infty$

We can also define new coordinates adapted to the light cone:

$$u \equiv t + r_* \ , \ v \equiv t - r_*$$

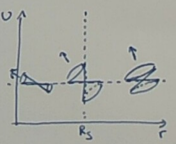
so that in-going null geodesics are described by $u = \text{const}$ $(t = -r_* + \text{const})$ and out-going ones by $v = \text{const}$ $(t = r_* + \text{const})$.

Eddington-Finkelstein coordinates: they correspond to choosing $(u, r)$ instead of $(t, r) \Rightarrow dt^2 = du^2 + dr_*^2 - 2du\,dr_*$ $\Rightarrow dr_*^2 - dt^2 = 2du\,dr\left(1 - \frac{R_s}{r}\right)^{-1} - du^2$

$$ds^2 = -\left(1 - \frac{R_s}{r}\right) du^2 + 2du\,dr$$

Null geodesics ~~are now~~ ($ds^2 = 0$) are now described by

$$\frac{du}{dr} = \begin{cases} 0 & \text{(in-going, } u = \text{const)} \\ 2\left(1 - R_s/r\right)^{-1} & \text{(out-going)} \quad (= \infty \text{ at } R_s) \text{ (vertical slope)} \end{cases}$$




The evolution of the light cone can be followed through $R_s$ in these coordinates. It closes up only at $r = 0$, but it "tilts over" at $r = R_s$. At $R_s$, the future light cone is entirely contained in the $r < R_s$ region. Light rays cannot escape!!

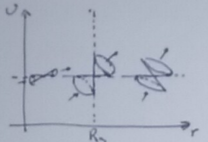
What about out-going geodesics? Like in-going ones, they take an infinite coordinate time to reach from $r=R_s$ to anywhere at $r>R_s$, but a finite proper time. They can be completed towards the past. This would recover the symmetry of space-time under t-reversal.

This can be done by choosing $(u,r)$: $-dt^2+dr_*^2 = -du^2 - 2\left(1-\frac{R_s}{r}\right)^{-1} du\,dr$

$$\boxed{ds^2 = -\left(1-\frac{R_s}{r}\right)du^2 - 2\,du\,dr}$$

In these coordinates, null geodesics ($ds^2=0$) are described by

$$\frac{du}{dr} = \begin{cases} -2\left(1-\frac{R_s}{r}\right)^{-1} & \text{(in-going, } =\infty \text{ at } r=R_s\text{, vertical slope)} \\ 0 & \text{(out-going, } u=\text{const)} \end{cases}$$




At $r=R_s$, the future light cone lies outside of the $r=R_s$ null surface. ~~This black~~ Light signals can only go out. This solution is called a white hole.

The $(u,r)$ coordinates provide a past extension of the $(R_s,+\infty)$ region. It also covers the $r<R_s$ region, but it is not the same extension provided by $(v,r)$. The two extensions are disconnected, in spite of having the same r coordinate. In the $(v,r)$ plane, $u=$ constant curves cannot penetrate in the $r<R_s$ region, nor can $v=$ constant curves in the $(u,r)$ plane:

$$\left.\begin{aligned} v(u=\text{const}) &= t+r_* = u+2r_* \to -\infty \\ u(v=\text{const}) &= t-r_* = v-2r_* \to +\infty \end{aligned}\right\} \text{ as } r\to R_s$$

In order to find a maximal extension, we can try to use $(u,v)$:

$$du\,dv = (dt-dr_*)(dt+dr_*) = dt^2-dr_*^2 \Rightarrow ds^2 = -\left(1-\frac{R_s}{r}\right)du\,dv$$

$$\frac{v-u}{2} = r_* = r + R_s\log\left(\frac{r}{R_s}-1\right) \Rightarrow e^{-\frac{r}{R_s}+\frac{v-u}{2R_s}} = e^{\log\left(\frac{r}{R_s}-1\right)} = \frac{r-R_s}{R_s}$$

$$ds^2 = -\frac{r-R_s}{r}du\,dv = -\frac{R_s}{r}e^{-\frac{r}{R_s}}e^{\frac{v-u}{2R_s}}du\,dv$$

Defining $U \equiv -e^{-\frac{u}{2R_s}}$ and $V \equiv e^{v/2R_s}$ $\Rightarrow ds^2 = -\frac{4R_s^3}{r}e^{-r/R_s}dU\,dV$

In these coordinates, nothing happens to $g_{\mu\nu}$ at $r=R_s$

U and V are light-cone coordinates: $ds^2=0$ if $dU=0$ or $dV=0$. We can go back to "standard" coordinates with

$$T=\frac{U+V}{2}\,,\quad R\equiv\frac{V-U}{2} \qquad \boxed{ds^2=\frac{4R_S^3}{r}e^{-r/R_S}\left(-dT^2+dR^2\right)+r^2d\Omega^2}$$

(reintroducing angles). <u>Kruskal-Szekeres coordinates</u>

In terms of the original variables $(t,r)$, they are (for $r>R_S$)

$$T=\sqrt{\frac{r}{R_S}-1}\,e^{r/2R_S}\sinh\left(\frac{t}{2R_S}\right)\,,\quad R=\sqrt{\frac{r}{R_S}-1}\,e^{r/2R_S}\cosh\left(\frac{t}{2R_S}\right)$$

so that r in the expression for $ds^2$ above is obtained inverting

$$T^2-R^2=\left(1-\frac{r}{R_S}\right)e^{r/R_S}\,,\quad\text{and } t=2R_S\,\text{arctanh}\left(\frac{T}{R}\right)$$

Properties of the K-S coordinates:

- Light rays are at 45° ($U=$ const or $V=$ const):

$$T=\frac{U+V}{2}=U+R=R+\text{const} \quad\text{or}\quad T=V-R=-R+\text{const}$$

- Horizon $(r=R_S)$ is $T^2-R^2=0 \Rightarrow T=\pm R$ (null surface)
- Surface of constant r: $T^2-R^2=$ const (hyperbolae)
- Singularity $r=0$: $T^2-R^2=1$
- $\left(1-\frac{r}{R_S}\right)e^{r/R_S}$ is monotonically decreasing function of r

$$\Rightarrow T^2-R^2\leq 1 \quad\Rightarrow -\sqrt{R^2+1}\leq T\leq\sqrt{R^2+1} \quad (-\infty\leq R\leq+\infty)$$

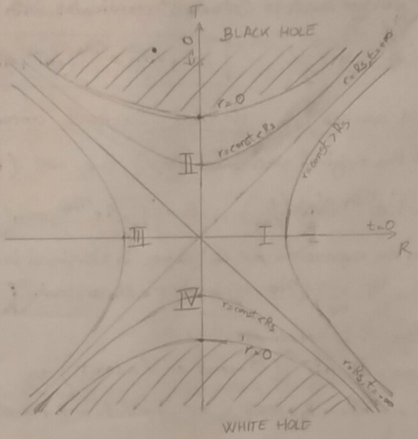
- Surface of constant t: $\frac{T}{R}=\tanh\left(\frac{t}{2R_S}\right)=$ const (straight lines through origin)

as $t\to\pm\infty$, $T\to\pm R$ (horizon again...)

In the $r<R_S$ region, the relations become (by analytical continuation)

$$T=\sqrt{1-\frac{r}{R_S}}\,e^{r/2R_S}\cosh\left(\frac{t}{2R_S}\right)\,,\quad R=\sqrt{1-\frac{r}{R_S}}\,e^{r/2R_S}\sinh\left(\frac{t}{2R_S}\right)$$

so that $t=2R_S\,\text{arctanh}\left(\frac{R}{T}\right)$




II is the future extension of region I $(r > R_s)$

IV is its past extension (with a white hole at $r=0$)

III is a parallel exterior region

# Light deflection

Third "classical" test of GR (after precession of orbits and grav. redshift), measured by Eddington first in 1919 eclipse.
Today, "it is the foundation of an important astrophysical and cosmological probe known as gravitational lensing (deflection of light from distant galaxies and quasars) which allows to measure the mass content of the Universe.

- Linearized Schwarzschild metric:

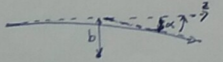
$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \underbrace{\left(1-\frac{2GM}{r}\right)^{-1}}_{\approx 1+\frac{2GM}{r}} dr^2 + r^2 d\Omega^2 \qquad r = \tilde{r}\left(1+\frac{GM}{\tilde{r}}\right) \quad dr = d\tilde{r}$$

$$\approx -\left(1-\frac{2GM}{\tilde{r}}\right)dt^2 + \left(1+\frac{2GM}{\tilde{r}}\right)\left(d\tilde{r}^2 + \tilde{r}^2 d\Omega^2\right) \approx -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1+\frac{2GM}{r}\right)d\vec{x}^2$$

$$\approx -\left(1+2\Phi\right)dt^2 + \left(1-2\Phi\right)d\vec{x}^2 \qquad \left(\Phi = -\frac{GM}{r}\right)$$

$\Phi$ is the result of the linear superimposition of various solutions, obeying $\nabla^2\Phi = 4\pi G\rho$.
The Christoffel symbols for this $g_{\mu\nu}$ are (check!)

$$\Gamma^i_{00} = \Gamma^0_{0i} = \partial_i\Phi \ , \quad \Gamma^i_{jk} = \delta_{jk}\partial_i\Phi - \delta_{ik}\partial_j\Phi - \delta_{ij}\partial_k\Phi$$




$$x^\mu(\lambda) = \underset{\text{unperturbed}}{x^{(0)\mu}(\lambda)} + \underset{\text{small deviation}}{x^{(1)\mu}(\lambda)} + \dots$$

$dz = k\,d\lambda$ $\qquad k^\mu \equiv \frac{dx^{(0)\mu}}{d\lambda} \qquad e^\mu = \frac{dx^{(1)\mu}}{d\lambda}$

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \quad \Rightarrow \quad \frac{de^\mu}{d\lambda} \approx -\Gamma^\mu_{\rho\sigma}k^\rho k^\sigma$$

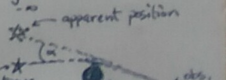
$k^\mu$ is the unperturbed wave-vector, $k^\mu = (k, \vec{k})$ with $k^2 = \vec{k}^2$ ($\eta_{\mu\nu}k^\mu k^\nu = 0$)

$$\frac{de^i}{d\lambda} = -\Gamma^i_{00}k^2 - \Gamma^i_{jl}k^j k^l \approx -2\partial_i\Phi\, k^2 + 2k^i(\vec{k}\cdot\vec{\nabla}\Phi) \approx -2k^2\nabla^i_\perp\Phi$$

$$\vec{\nabla}_\perp\Phi \equiv \vec{\nabla}\Phi - \hat{k}(\hat{k}\cdot\vec{\nabla}\Phi) \qquad \text{(we remove the // component)}$$

So deflection is $\Delta\vec{e} = -2k^2\int\vec{\nabla}_\perp\Phi\, d\lambda = -2k\int_{-\infty}^{+\infty}\vec{\nabla}_\perp\Phi\, dz$

$$\boxed{\vec{\alpha} \equiv \frac{\vec{e}_{in} - \vec{e}_{out}}{k} = 2\int_{-\infty}^{+\infty}\vec{\nabla}_\perp\Phi\, dz}$$



Deflection angle between the apparent and true position : opposite of angle between incoming and outgoing velocities.

If $\Phi$ generated by a single mass: $\Phi = -\frac{GM}{\sqrt{z^2+b^2}}$ , $\vec{\nabla}_\perp \Phi \equiv \frac{\partial \Phi}{\partial \vec{b}} = \frac{GM\vec{b}}{(z^2+b^2)^{3/2}}$

$$\vec{\alpha} = 2GM\vec{b}\int_{-\infty}^{+\infty} dz \frac{1}{(z^2+b^2)^{3/2}} = \frac{4GM\vec{b}}{b^2} = \frac{4GM}{b}\hat{b} = \frac{2R_s}{b}\hat{b}$$

In Newtonian gravity, the deflection angle for a particle of mass m ~~pass~~ by a mass M was $\alpha \approx \frac{2GM}{v_\infty^2 r_{min}}$ , where $r_{min} = b$, independent of m. Considering a photon as a particle with $m \to 0$ and $v_\infty \to c = 1$, I would get half of the deflection. Test for GR versus Newton. Deviations ~~from~~ from the factor of 2 would be a test for GR versus extensions of GR (so far undetected).

In the 1919 observation by Eddington (deflection angle of a star by the sun at the beginning and end of occultation during the solar eclipse), $R_s \approx 3km$ and $b \approx R_\odot \approx 7\times10^5 km$

$$\alpha \approx \frac{2\times 3 km}{7\times10^5 km} \approx 10^{-5}$$ (small number!!)

# GRAVITATIONAL WAVES

In EM, going from static Coulomb's law to dynamical Maxwell eqns lead to EM waves. In GR, moving from Poisson eqn to Einstein eqn will do the same, leading to gravitational waves. First step towards a field description of gravity, in which the graviton (the wave) is the mediator (like photon for EM).

Perturbations around flat space (weak field regime)

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1 \qquad (g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu})$$

Indices are now raised and lowered by $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$:

$$g_{\mu\nu} g^{\nu\rho} = (\eta_{\mu\nu} + h_{\mu\nu})(\eta^{\nu\rho} - h^{\nu\rho}) \approx \delta_\mu{}^\rho + \cancel{h_\mu{}^\rho} - \cancel{h_\mu{}^\rho} + \mathcal{O}(h^2)$$

$$\Gamma^\rho_{\mu\nu} \simeq \frac{1}{2}\eta^{\rho\sigma}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu})$$

$$R_{\mu\nu} \simeq \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\mu \Gamma^\rho_{\rho\nu} + \mathcal{O}(h^2) \approx \partial^\rho \partial_{(\mu} h_{\nu)\rho} - \frac{1}{2}\partial^\rho\partial_\rho h_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\nu h$$

$$\text{since } \Gamma^\rho_{\rho\nu} = \frac{1}{2} g^{\rho\sigma}\partial_\nu g_{\rho\sigma} \approx \frac{1}{2}\eta^{\rho\sigma}\partial_\nu h_{\rho\sigma} = \frac{1}{2}\partial_\nu \underbrace{(\eta^{\rho\sigma} h_{\rho\sigma})}_{= h_\mu{}^\mu \equiv h}$$

(here $h$ is the trace of $h_\mu{}^\nu$)

The Einstein tensor is

$$G_{\mu\nu} \simeq R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R \approx \partial^\rho\partial_{(\mu} h_{\nu)\rho} - \frac{1}{2}\partial^\rho\partial_\rho h_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\nu h - \frac{1}{2}\eta_{\mu\nu}(\partial^\rho\partial^\sigma h_{\rho\sigma} - \partial^\rho\partial_\rho h)$$

It is useful to introduce $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$ (with $\bar{h} = h - 2h = -h$) in terms of which

$$G_{\mu\nu} \simeq \partial^\rho\partial_{(\mu}\bar{h}_{\nu)\rho} - \frac{1}{2}\partial^\rho\partial_\rho \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\partial^\rho\partial^\sigma \bar{h}_{\rho\sigma} = 8\pi G T_{\mu\nu}$$

Two perturbations $h_{\mu\nu}$ and $\tilde{h}_{\mu\nu}$ related by a change of coordinates (a <u>diffeomorphism</u>) must describe the same physics. However, in order not to deform the metric "too much" away from $\eta_{\mu\nu}$, also the change of coordinates must be small. Choosing $\tilde{x}^\mu = x^\mu - \xi^\mu$, so that $\partial x^\mu / \partial \tilde{x}^\nu = \delta^\mu_\nu + \partial_\nu \xi^\mu$, the metric transform as

$$\tilde{g}_{\mu\nu} \equiv \eta_{\mu\nu} + \tilde{h}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu}(\eta_{\alpha\beta} + h_{\alpha\beta}) \Rightarrow \boxed{h_{\mu\nu} \to \tilde{h}_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu}$$

(transf. of metric perturb. under infinitesimal diffeomorphism)

$h_{\mu\nu}$ and $\bar{h}_{\mu\nu}$ describe the same physics. The transformation $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\nu$ encodes the "gauge freedom" of linearized GR. The gauge freedom induced by changes of coordinates is similar to the one of EM, where the physics is unchanged by

$$A_\mu \to A_\mu + \partial_\mu \chi$$

Under diffeomorfism, $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \to \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial_\rho \xi^\rho$.

Therefore, $\partial^\nu \bar{h}_{\mu\nu} \to \partial^\nu \bar{h}_{\mu\nu} + \cancel{\partial_\mu \partial^\nu \xi_\nu} + \partial^\nu \partial_\nu \xi_\mu - \cancel{\partial_\mu \partial_\rho \xi^\rho}$

By solving for $\Box \xi_\mu \equiv \partial^\nu \partial_\nu \xi_\mu = -\partial^\nu \bar{h}_{\mu\nu}$, we can thus impose

$$\boxed{\partial^\nu \bar{h}_{\mu\nu} = 0}$$

{This request is similar to the Lorentz gauge condition $\partial^\nu A_\nu = 0$, obtained by solving $\Box \chi = -\partial^\nu A_\nu$

In this gauge, the EoM simplifies to

$$\boxed{\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}}$$

a wave equation with a source.

Outside the source, $T_{\mu\nu} = 0$ and the GW propagates in vacuum.

$$\bar{h}_{\mu\nu} = c_{\mu\nu} e^{i k_\alpha x^\alpha} \qquad \Box \bar{h}_{\mu\nu} = 0 \Rightarrow \boxed{k_\alpha k^\alpha = 0}$$

polarization tensor

GW travel at speed of light!

The Lorentz gauge condition: $\boxed{k^\nu c_{\mu\nu} = 0}$

4 algebraic equations for 10 components of $c_{\mu\nu}$, 6 left.

Gauge condition still leaves a residual gauge freedom to perform diffeomorphisms with $\Box \xi_\mu = 0$. Since $\delta \bar{h}_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial \xi$, then $\Box \delta \bar{h}_{\mu\nu} = 0$ and the diff. does not violate the Lorentz gauge (also in EM, residual gauge freedom with $\Box \chi = 0$,} which preserves the condition $\partial_\mu A^\mu = 0$).

$$\Rightarrow \delta c_{\mu\nu} = i(k_\mu \hat{\xi}_\nu + k_\nu \hat{\xi}_\mu - \eta_{\mu\nu} k \cdot \hat{\xi}) \quad (\text{with } \xi_\mu = \hat{\xi}_\mu e^{i k_\alpha x^\alpha})$$

$$\delta c = \delta c_{\mu\nu} \eta^{\mu\nu} = -2 i k \cdot \hat{\xi} \qquad \delta c_{0i} = i(k_0 \hat{\xi}_i + k_i \hat{\xi}_0)$$

The residual gauge freedom in choosing $\xi^\mu$ can be used to fix

$$\epsilon = 0 \quad (\Rightarrow \bar{h}_{\mu\nu} = h_{\mu\nu}) \quad \text{and} \quad \epsilon_{0,i} = 0$$

The last condition together with the Lorentz gauge imply

$$0 = k^\mu \epsilon_{0\mu} = k_0 \epsilon_{00} \quad \Rightarrow \epsilon_{00} = 0$$

Let us consider a wave directed in $\hat{z}$ direction : $k^\mu = (\omega, 0, 0, \omega)$

$$0 = k^\mu \epsilon_{\nu\mu} = \omega(\cancel{\epsilon_{\nu 0}} + \epsilon_{\nu 3}) \quad \Rightarrow \epsilon_{\nu 3} = 0$$

$$\epsilon_{\mu\nu} = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & \epsilon_{11} & \epsilon_{12} & 0 \\ 0 & \epsilon_{12} & -\epsilon_{11} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

"<u>transvers traceless (TT) gauge</u>"

$$\text{2 remaining degrees of freedom } (10 - \underset{\text{TT}}{\overset{\text{Lorentz}}{4}} - 4 = 2)$$

The wave has therefore 2 independent components ("polarizations")

$$\epsilon_{+,\mu\nu} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \epsilon_{\times,\mu\nu} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The general solution is a linear comb. (must also take real part):

$$h^{TT}_{\mu\nu} = \bar{h}^{TT}_{\mu\nu} = Re\left[\left(\epsilon_{11}\epsilon_{+,\mu\nu} + \epsilon_{12}\epsilon_{\times,\mu\nu}\right)e^{ik_\alpha x^\alpha}\right] \qquad \epsilon_{11}, \epsilon_{12} \in \mathbb{C}$$

Since $\epsilon_{11} = h_+ e^{i\phi_1}$ , $\epsilon_{12} = h_\times e^{i\phi_2}$ with $h_+, h_\times \in \mathbb{R}$, then

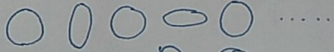
$$h^{TT}_{\mu\nu} = h_+ \epsilon_{+,\mu\nu} \cos(\omega(t-z) + \phi_1) + h_\times \epsilon_{\times,\mu\nu} \cos(\omega(t-z) + \phi_2)$$

Setting for simplicity $\phi_1 = \phi_2 = 0$, the full solution is

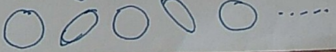
$$ds^2 = -dt^2 + dz^2 + [1 + h_+\cos(\omega(t-z))]dx^2 + [1 - h_+\cos(\omega(t-z))]dy^2 + 2h_\times\cos(\omega(t-z))\,dx\,dy$$

The metric perturbation only acts on vectors in the x-y plane. The $\hat{z}$ direction is untouched. The eigenvectors of $\epsilon_{+,\mu\nu}$ are $\binom{1}{0}, \binom{0}{1}$, those of $\epsilon_{\times,\mu\nu}$ are $\binom{1}{1}, \binom{1}{-1}$ (with eigenvalues $\pm 1$). Their action on a unit circle is then

$\epsilon_{+,\mu\nu}$:



$\epsilon_{\times,\mu\nu}$:



The area does not change under the deformation, since the matrises are traceless.

It is also convenient to introduce circular polarizations as follows:

$$\epsilon_{11} = \frac{\epsilon_R + \epsilon_L}{\sqrt{2}}, \quad \epsilon_{12} = i\frac{\epsilon_L - \epsilon_R}{\sqrt{2}} \qquad \epsilon^L_{\mu\nu} \equiv \frac{\epsilon_{+,\mu\nu} + i\epsilon_{\times,\mu\nu}}{\sqrt{2}}, \quad \epsilon^R_{\mu\nu} \equiv \frac{\epsilon_{+,\mu\nu} - i\epsilon_{\times,\mu\nu}}{\sqrt{2}}$$

So that the x-y restrictions look like $\epsilon^L_{\mu\nu} = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$, $\epsilon^R_{\mu\nu} = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$

Under rotation $R^\mu{}_\nu = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, the polarizations transform as

$$\epsilon_{\times,\mu\nu} \to R^\alpha{}_\mu R^\beta{}_\nu \epsilon_{\times,\alpha\beta} = (R^T \epsilon_\times R)_{\mu\nu} = \begin{pmatrix} -\sin 2\theta & \cos 2\theta \\ \cos 2\theta & \sin 2\theta \end{pmatrix}$$

$$\epsilon_{+,\mu\nu} \to (R^T \epsilon_+ R)_{\mu\nu} = \begin{pmatrix} \cos 2\theta & +\sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

$$\epsilon^L_{\mu\nu} \to e^{-2i\theta} \epsilon^L_{\mu\nu} \qquad \epsilon^R_{\mu\nu} \to e^{2i\theta} \epsilon^R_{\mu\nu}$$

They return into themselves with $\theta = \pi \Rightarrow$ spin-2 !!

Circular polarization will rotate an ellipse:

($\pi$ is enough to go back...)

GR is a theory of massless spin-2 particles whose gauge invariance is the invariance under diffeomorphisms.

~~Production of GW's~~

Remark: $R_{\mu\nu\rho}{}^\sigma$ (and therefore also $R_{\mu\nu}$) is invariant under infinitesimal gauge transf. $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$. This is similar to $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ invariant under $A_\mu \to A_\mu + \partial_\mu \chi$. Thus the gauge condition can be imposed on $\eta_{\mu\nu}$ without changing the Einstein equations. Also the r.h.s. does not change, since $T_{\mu\nu}$ must be evaluated neglecting $h_{\mu\nu}$. The equation $\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$ is therefore consistent and gauge invariant (at first order in $h_{\mu\nu}$). Its divergence simply returns $0 = 0$.

## Production of GWs

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$$

We cannot use the same procedure since $e^{ik_\kappa x^\kappa}$ is not a solution in the presence of $T_{\mu\nu}$. But I can use ~~the~~ the same procedure as for EM, using Green's function. Solution of the associated eqn

$$\Box_x G(x^\sigma - y^\sigma) = \delta_D^{(4)}(x^\sigma - y^\sigma),$$

where $\Box_x$ is the D'Alembertian w.r.t. $x^\sigma$, and $\delta_D^{(4)}$ is the 4D Dirac delta. Then, because of linearity, the gen. solution is

$$\bar{h}_{\mu\nu}(x^\sigma) = -16\pi G \int d^4y\, G(x^\sigma - y^\sigma) T_{\mu\nu}(y^\sigma)$$

The Green's function is the same as for EM (retarded)

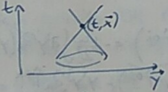
$$G(x^\sigma - y^\sigma) = -\frac{\Theta_H(x^0 - y^0)}{4\pi|\vec{x}-\vec{y}|}\delta_D(x^0 - y^0 - |\vec{x}-\vec{y}|),$$

where $\Theta_H(x)$ is the Heaviside distribution and $\vec{x} = (x^1, x^2, x^3)$, $\vec{y} = (y^1, y^2, y^3)$.

Equivalently,

$$\bar{h}_{\mu\nu}(t,\vec{x}) = 4G \int d^3y \frac{1}{|\vec{x}-\vec{y}|} T_{\mu\nu}(\overbrace{\underbrace{t - |\vec{x}-\vec{y}|}_{\text{retarded time}}}^{\equiv t_{ret}}, \vec{y})$$

It receives contribution from the entire past light cone!




We consider production from small, isolated, non-rel., far away sources, usually periodic in time (e.g. binary systems).

Going to Fourier space in time:

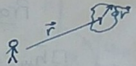
$$\phi(t,\vec{x}) = \frac{1}{\sqrt{2\pi}}\int d\omega\, e^{-i\omega t}\hat{\phi}(\omega,\vec{x}) \qquad \hat{\phi}(\omega,\vec{x}) = \frac{1}{\sqrt{2\pi}}\int dt\, e^{i\omega t}\phi(t,\vec{x})$$

$$\hat{\bar{h}}_{\mu\nu}(\omega,\vec{x}) = \frac{4G}{\sqrt{2\pi}}\int dt\, d^3y\, e^{i\omega t}\frac{T_{\mu\nu}(t-|\vec{x}-\vec{y}|,\vec{y})}{|\vec{x}-\vec{y}|} \quad [t_{ret} = t - |\vec{x}-\vec{y}|,\ dt = dt_{ret}]$$

$$= \frac{4G}{\sqrt{2\pi}}\int dt_{ret}\, d^3y\, e^{i\omega t_{ret}} e^{i\omega|\vec{x}-\vec{y}|}\frac{T_{\mu\nu}(t_r,\vec{y})}{|\vec{x}-\vec{y}|}$$

$$= 4G\int d^3y\, e^{i\omega|\vec{x}-\vec{y}|}\frac{\hat{T}_{\mu\nu}(\omega,\vec{y})}{|\vec{x}-\vec{y}|}$$

We assume that the source is localized and distant: $|\delta\vec{r}| \ll |\vec{r}|$




$$|\vec{x}-\vec{y}| = |\vec{r}+\delta\vec{r}| \approx r$$

$$\hat{\bar{h}}_{\mu\nu} \approx 4G \frac{e^{i\omega r}}{r} \int d^3y\, \hat{T}_{\mu\nu}(\omega,\vec{y})$$

In this mixed Fourier representation, Lorentz gauge is

$$\partial_t \bar{h}^{0\nu} = -\partial_i \bar{h}^{\nu i} \Rightarrow -i\omega\hat{\bar{h}}^{0\nu} = -\partial_i \hat{\bar{h}}^{\nu i} \Rightarrow \boxed{\hat{\bar{h}}^{0\nu} = -\frac{i}{\omega}\partial_i \hat{\bar{h}}^{\nu i}}$$

(Note a wrong sign in Carroll's notes here)

Remember that $\bar{h}^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\alpha}\bar{h}_{\alpha\beta}$, so that $\bar{h}^{00} = \bar{h}_{00}$, $\bar{h}^{ij} = \bar{h}_{ij}$ but $\bar{h}^{i0} = -\bar{h}_{i0}$. Similarly for $T^{\mu\nu}$.

We only need to solve for the space components $\bar{h}_{ij}$. From these, the Lorentz gauge expression gives ~~to~~ all the $\bar{h}_{0i}$, from which (using Lorentz again) we get $\bar{h}_{00}$.

Therefore, we only need to look at space components $T_{ij}$.

In Fourier, conservation of $T^{\mu\nu}$ is: $\partial_\mu T^{\mu\nu} = 0 \Rightarrow \boxed{\partial_i \hat{T}^{\nu i} = i\omega \hat{T}^{\nu 0}}$

$$\int d^3y\, \hat{T}^{ij}(\omega,\vec{y}) = \int d^3y \left[\partial_k(y^i \hat{T}^{kj}) - y^i \underbrace{(\partial_k \hat{T}^{kj})}_{=i\omega \hat{T}^{0j}}\right] = \underbrace{\cancel{\int d^3y\, \partial_k(y^i\hat{T}^{kj})}}_{=0 \text{ for a localized source}} - i\omega \int d^3y\, y^i \hat{T}^{0j}$$

But $\hat{T}^{ij}$ is symmetric, so

$$\int d^3y\, \hat{T}^{ij} = -\frac{i\omega}{2}\int d^3y \underbrace{(y^i\hat{T}^{0j} + y^j\hat{T}^{0i})}_{=\partial_k(y^iy^j\hat{T}^{0k}) - y^iy^j\underbrace{\partial_k\hat{T}^{0k}}_{i\omega\hat{T}^{00}}} = \cancel{-\frac{i\omega}{2}\int d^3y\, \partial_k(y^iy^j\hat{T}^{0k})} - \frac{\omega^2}{2}\int d^3y\, y^iy^j\hat{T}^{00}$$

where again the first term is a pure boundary term and does not contribute for a localised source ($\hat{T}^{\mu\nu} \to 0$ as $|\vec{y}| \to \infty$)

$$\Rightarrow \int d^3y\, \hat{T}^{ij}(\omega,\vec{y}) = -\frac{\omega^2}{2}\int d^3y\, y^iy^j\hat{T}^{00}(\omega,\vec{y})$$

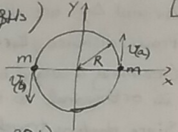
Introduce $\boxed{I_{ij}(t) \equiv \int d^3y\, y^iy^jT^{00}(t,\vec{y})}$ Quadrupole moment of energy density

Then, $\hat{\bar{h}}_{ij} = -2G\omega^2 \frac{e^{i\omega r}}{r}\hat{I}_{ij}(\omega)$, ~~$\hat{h}_{ij}$~~

Fourier-transforming back, ($t_{ret} = t-r$)

$$\boxed{\bar{h}_{ij}(t,\vec{x}) = \frac{2G}{r}\frac{d^2I_{ij}}{dt^2}(t_{ret})}$$

Example Binary system (e.g. neutron stars, BHs)




$x_{(a)} = -x_{(b)} = R\cos\Omega t$

$y_{(a)} = -y_{(b)} = R\sin\Omega t \qquad z_{(a)} = z_{(b)} = 0$

$I_{11} = m(x_{(a)})^2 + m(x_{(b)})^2 = 2mR^2\cos^2\Omega t = mR^2(1+\cos 2\Omega t)$

$I_{22} = m(y_{(a)})^2 + m(y_{(b)})^2 = mR^2(1-\cos 2\Omega t)$

$I_{12} = m x_{(a)} y_{(a)} + m x_{(b)} y_{(b)} = 2mR^2\sin\Omega t\cos\Omega t = mR^2\sin 2\Omega t$

$I_{3i} = I_{i3} = 0 \qquad \ddot I_{11} = -4m\Omega^2R^2\cos 2\Omega t = -\ddot I_{22} \qquad \ddot I_{12} = -4m\Omega^2R^2\sin 2\Omega t$

$$\bar h_{ij}(t,\vec x) = \frac{8Gm}{r}\Omega^2R^2\begin{bmatrix} -\cos 2\Omega t_{ret} & -\sin 2\Omega t_{ret} & 0 \\ -\sin 2\Omega t_{ret} & \cos 2\Omega t_{ret} & 0 \\ 0 & 0 & 0\end{bmatrix}$$

where $r = |\vec x|$ and $t_{ret} = t - |\vec x|$.

Using Newtonian gravity approx for the two-body problem, (reduced mass $\mu = m/2$, separation $|\vec x_a - \vec x_b| = 2R$, total mass $M_{TOT} = 2m$)

circular orbits for $\frac{GM_{TOT}}{|\vec x_a - \vec x_b|^2} = \frac{L^2}{|\vec x_a - \vec x_b|^3} = \Omega^2|\vec x_a - \vec x_b| \Rightarrow \Omega^2 = \frac{Gm}{4R^3}$

Typical size for GW interferometers (e.g. LIGO, Virgo)

They study BH-BH binary systems,

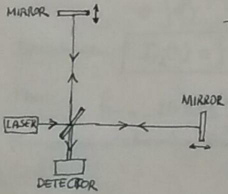
$M \approx 10\,M_\odot \Rightarrow R_s \approx 10^6\,\text{cm}$

Assume Newtonian-gravity OK till $R \approx 10R_s \approx 10^7\,\text{cm}$

and typical distance $r \approx 100\,\text{Mpc} \approx 10^{26}\,\text{cm}$ ($c \approx 3\times10^{10}\,\text{cm/s}$)

Frequency: $\Omega \approx \sqrt{\frac{Gm}{4R^3}} = \sqrt{\frac{R_s}{R}}\frac{c}{2\sqrt{2}R} \approx 100\,\text{Hz}$

Metric perturbation size is $h \approx \frac{8Gm\Omega^2R^2}{r} \approx \frac{R_s^2}{rR} \approx 10^{-21}$ !!




The mirrors move as the GW passes.

Their coordinate position does not change if they are initially at rest ($dx^i/d\tau = 0$):

$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{00}\frac{dx^0}{d\tau}$, but $\Gamma^i_{00} = 0$ in TT gauge.

$\Rightarrow \frac{dx^i}{d\tau} = 0$ at all times (for mirrors)

The coordinate distance $\Delta x$ and $\Delta y$ between mirror and detector remains constant

$\Delta x$ and $\Delta y$ are also the proper distance before GW arrives

Proper distance : $\Delta s_1 = \sqrt{[1 + h\cos(\omega(t-z))]\Delta x^2}$, $\Delta s_2 = \sqrt{[1 - h\cos(\omega(t-z))]\Delta y^2}$

(assuming a wave in pure + polarization and that the detector is orthogonal to the propagation direction...)

Since photons travel along null paths inside the detector, the travel time for photon the laser beam is $\Delta t_1 = 2\Delta s_1$ or $\Delta t_2 = 2\Delta s_2$ in the two arms (from source to detector).

The difference in travel time is

$$\Delta t_1 - \Delta t_2 \approx 2\Delta x\left(1 + \frac{h\cos(\dots)}{2}\right) - 2\Delta y\left(1 - \frac{h\cos(\dots)}{2}\right)$$

$$\approx 2(\Delta x - \Delta y) + (\Delta x + \Delta y)\, h\cos(\dots)$$

The interferometer is calibrated so that $2(\Delta x - \Delta y) = n\lambda$, where $\lambda$ is the laser's wavelength, so that there are no interference fringes when there is no GW. As the GW passes, the max signal (distructive interference) is reached when $\delta L \equiv (\Delta x + \Delta y)h \approx \frac{\lambda}{2}$ (the variation of the proper lenght of the path, given by the mirror's displacement)

The proper displacement of one mirror is

$$\delta L \sim 10^{-16}\ \text{cm} \left(\frac{h}{10^{-21}}\right)\left(\frac{\Delta x}{\text{km}}\right) \quad \text{incredibly small!}$$

Muche less than the size of an atom ($\sim 10^{-8}$ cm)

$\Delta x$ must be as big as possible (arms lenght) in order to compensate for the smallness of $h$.

In LIGO, an additional mirror allows for photons to bounce back and forth $\sim 300$ times in each arm before exiting. The effective arm lenght is $\sim 300 \times 4\,\text{km} \approx 1200$ km !

Future detectors (e.g. LISA) are planned in space.