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ORIGINAL ARTICLE

On New Properties of the Drazin-Star and the Star-Drazin Inverses

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Abstract

The aim of this work is to study new properties of the Drazin-Star and the Star-Drazin inverses of a bounded finite potent operator on a Hilbert space. Given a bounded finite potent operator $\varphi \in \mathrm{End}_k(\mathcal{H})$, we prove that the pseudo-characteristic polynomials of $\varphi^{D,*}$ and $\varphi^{*,D}$ coincide. Accordingly, we obtain that $\sigma(\varphi^{D,*}) = \sigma(\varphi^{*,D})$, $\mathrm{Tr}_{\mathcal{H}}(\varphi^{D,*}) = \mathrm{Tr}_{\mathcal{H}}(\varphi^{*,D})$ and $\det_{\mathcal{H}}(\mathrm{Id} + \varphi^{D,*}) = \det_{\mathcal{H}}(\mathrm{Id} + \varphi^{*,D})$. In particular, these results hold for a finite square complex matrix A. Moreover, we offer the explicit characterization of the AST-decompositions of $\mathcal{H}$ induced by the Group-Star and the Star-Group inverses of a bounded linear operator ψ on $\mathcal{H}$ with $i(\psi) \leq 1$.

Keywords Spectrum · Trace · Determinant · Star-Drazin inverse · Drazin-Star inverse · Bounded operator · Hilbert space · Finite potent endomorphism · Square matrix

Mathematics Subject Classification 46C05 · 47L30 · 15A09

1 Introduction

For an arbitrary $(n \times n)$-matrix A with entries in the complex field, the index of A, $i(A) \geq 0$, is the smallest integer such that $\mathrm{rk}(A^{i(A)}) = \mathrm{rk}(A^{i(A)+1})$.

In 1958, Drazin in [2] showed the existence of a unique $n \times n$ complex matrix A^D, called the Drazin inverse, satisfying the equations:

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- $A^{r+1}A^D = A^r$ for $r = i(A)$;
- $A^D A A^D = A^D$;
- $A^D A = A A^D$.

When $i(A) \leq 1$, it is known that the Drazin inverse A^D coincides with the group inverse $A^\#$.

Moreover, if $\mathrm{Mat}_{n\times n}(\mathbb{C})$ is the set of $(n \times n)$-matrices A with entries in the complex field, given again $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$, the second-named author has introduced in [4] the notions of Drazin-Star and Star-Drazin matrices.

The Drazin-Star matrix of a finite square complex matrix A is $A^{D,*} = A^D A A^*$ and is defined as the unique solution of the system of equations:

$$\begin{aligned} X(A^\dagger)^* X &= X, \\ A^r X &= A^r A^*, \\ X(A^\dagger)^* &= A^D A\,, \end{aligned}$$

where $i(A) = r$, $A^\dagger$ is the Moore-Penrose inverse of A and A^* is the conjugate transpose of A.

Analogously, the Star-Drazin matrix of A is the unique solution of

$$\begin{aligned} X(A^\dagger)^* X &= X \\ X A^r &= A^* A^r \\ (A^\dagger)^* X &= A^D A \end{aligned}$$

and its explicit expression is $A^{*,D} = A^* A A^D$. By the equation $X(A^\dagger)^* X = X$, the Drazin-Star (or Drazin-Star) matrix of A is an outer inverse of $(A^\dagger)^*$ and it is called the Drazin-Star (or Drazin-Star) inverse of $(A^\dagger)^*$ [4].

When $i(A) = 1$, Mosić also studied in [4] the notions of Group-Star $A^{\#,*}$ and Star-Group $A^{*,\#}$ inverses of A, such $A^{\#,*} = A^\# A A^*$ and $A^{*,\#} = A^* A A^\#$. All of these matrices are useful for solving different systems of linear equations.

On the other hand, the notion of finite potent endomorphism on an arbitrary vector space was introduced by Tate in [11] as a basic tool for his elegant definition of Abstract Residues. This concept will be defined in Sect. 2.1.

Recently, the first-named author of this work has studied the properties of bounded finite potent operators on Hilbert spaces in [9] and has extended the notions of Drazin-Star and Star-Drazin inverses to these linear maps in [7].

If $\mathcal{H}$ is a Hilbert space and φ is a bounded finite potent operator on $\mathcal{H}$ with closed $\mathrm{Im}\,\varphi$ and $i(\varphi) = r$, we say that a linear map $\varphi^{D,*} \in \mathrm{End}_{\mathbb{C}}(\mathcal{H})$ is a Drazin-Star inverse of φ when it satisfies that

- $\varphi^{D,*} \circ (\varphi^\dagger)^* \circ \varphi^{D,*} = \varphi^{D,*}$;
- $\varphi^r \circ \varphi^{D,*} = \varphi^r \circ \varphi^*$;
- $\varphi^{D,*} \circ (\varphi^\dagger)^* = \varphi^D \circ \varphi$.

Moreover, with the above hypothesis on $\mathcal{H}$ and φ, we say that a linear map $\varphi^{*,D} \in \mathrm{End}_{\mathbb{C}}(\mathcal{H})$ is a Star-Drazin inverse of φ when it satisfies that

- $\varphi^{*,D} \circ (\varphi^{\dagger})^* \circ \varphi^{*,D} = \varphi^{*,D}$;
- $\varphi^{*,D} \circ \varphi^r = \varphi^* \circ \varphi^r$;
- $(\varphi^{\dagger})^* \circ \varphi^{*,D} = \varphi \circ \varphi^D$.

The existence and uniqueness of the Drazin-Star and the Star-Drazin inverses of a bounded finite potent operator have been proved in [7, Theorem 3.3] and [7, Theorem 3.7] respectively. It is known that $\varphi^{D,*} = \varphi^D \circ \varphi \circ \varphi^*$ and $\varphi^{*,D} = \varphi^* \circ \varphi \circ \varphi^D$, where φ^* is the adjoint operator of φ. Moreover, when $i(\varphi) \leq 1$, the notions of Group-Star $\varphi^{\#,*}$ and Star-Group $\varphi^{*,\#}$ inverses of φ was also introduced in [7].

With the previous notation, it is important to recall, from [7, Proposition 3.9], that $(\varphi^{D,*})^* = (\varphi^*)^{*,D}$ and $(\varphi^{*,D})^* = (\varphi^*)^{D,*}$. Therefore, we have that $(\varphi^{\#,*})^* = (\varphi^*)^{*,\#}$ and $(\varphi^{*,\#})^* = (\varphi^*)^{\#,*}$.

The aim of this work is to prove that the pseudo-characteristic polynomials of $\varphi^{D,*}$ and $\varphi^{*,D}$ coincide. Accordingly, we have that $\sigma(\varphi^{D,*}) = \sigma(\varphi^{*,D})$, $\mathrm{Tr}_{\mathcal{H}}(\varphi^{D,*}) = \mathrm{Tr}_{\mathcal{H}}(\varphi^{*,D})$ and $\det_{\mathcal{H}}(\mathrm{Id} + \varphi^{D,*}) = \det_{\mathcal{H}}(\mathrm{Id} + \varphi^{*,D})$. In particular, these results hold for a finite square complex matrix A. Also, we offer new properties of the Drazin-Star and the Star-Drazin inverses of a bounded linear operator on a Hilbert space with an arbitrary index. Moreover, the explicit characterization of the AST-decompositions of $\mathcal{H}$ induced by the Group-Star and the Star-Group inverses of a bounded linear operator ψ on $\mathcal{H}$ with $i(\psi) \leq 1$ is given.

The paper is organized as follows. In Sect. 2, we recall the definitions and main properties of finite potent endomorphisms, the Drazin inverse, the pseudo-characteristic polynomial of a finite potent endomorphism, the bounded finite potent operators and the Drazin inverse of a bounded finite potent endomorphism. Moreover, as new results, we provide in this section the proofs of some properties of the adjoint operator of a bounded finite potent endomorphism on a Hilbert space.

Section 3 is devoted to offer new properties of the Drazin-Star and the Star-Drazin inverses of a bounded finite potent linear operator on an arbitrary Hilbert space and the characterization of the AST-decompositions of a Hilbert space induced by $\psi^{\#,*}$ and $\psi^{*,\#}$, respectively, when $i(\psi) \leq 1$.

Finally, Sect. 4 contains the main results of this work. Indeed, in this section we prove that the pseudo-characteristic polynomial, the spectrum, the trace and the determinant of the Drazin-Star and the Star-Drazin inverses of a bounded finite potent operator on an arbitrary Hilbert space or of a square complex matrix coincide.

2 Preliminaries

This section is added for the sake of completeness.

2.1 Finite Potent Endomorphisms

Let k be an arbitrary field, and let V be a k-vector space.

Let us now consider an endomorphism φ of V. We say that φ is *finite potent* if $\varphi^n V$ is finite dimensional for some n. This definition was introduced by Tate in [11] as a basic tool for his elegant definition of Abstract Residues.

In 2007, Argerami et al. showed in [1] that an endomorphism φ is finite potent if and only if V admits a φ-invariant decomposition $V = U_\varphi \oplus W_\varphi$ such that $\varphi_{|U_\varphi}$ is nilpotent, W_φ is finite dimensional, and $\varphi_{|W_\varphi} \colon W_\varphi \xrightarrow{\sim} W_\varphi$ is an isomorphism.

Indeed, if $k[x]$ is the algebra of polynomials in the variable x with coefficients in k, we may view V as an $k[x]$-module via φ, and the explicit definition of the above φ-invariant subspaces of V is:

- $U_\varphi = \{v \in V \text{ such that } \varphi^m(v) = 0 \text{ for some } m\}$,
- $W_\varphi = \left\{ \begin{array}{l} v \in V \text{ such that } p(\varphi)(v) = 0 \text{ for some } p(x) \in k[x] \\ \text{relatively prime to } x \end{array} \right\}$.

Note that if the annihilator polynomial of φ is $x^m \cdot p(x)$ with $(x, p(x)) = 1$, then $U_\varphi = \operatorname{Ker} \varphi^m$ and $W_\varphi = \operatorname{Ker} p(\varphi)$.

Hence, this decomposition is unique. We shall call this decomposition the φ-invariant AST-decomposition of V.

Basic examples of finite potent endomorphisms are all endomorphisms of a finite-dimensional vector spaces and finite rank or nilpotent endomorphisms of infinite-dimensional vector spaces.

For a finite potent endomorphism φ, a trace $\operatorname{Tr}_V(\varphi) \in k$ may be defined as $\operatorname{Tr}_V(\varphi) = \operatorname{Tr}_{W_\varphi}(\varphi_{|W_\varphi})$.

This trace has the following properties:

(1) if V is finite dimensional, then $\operatorname{Tr}_V(\varphi)$ is the ordinary trace;
(2) if W is a subspace of V such that $\varphi W \subset W$, then

$$\operatorname{Tr}_V(\varphi) = \operatorname{Tr}_W(\varphi) + \operatorname{Tr}_{V/W}(\varphi)\,;$$

(3) if φ is nilpotent, then $\operatorname{Tr}_V(\varphi) = 0$.

Usually, Tr_V is named “Tate’s trace”.

Moreover, Hernández Serrano and the author of this paper have offered in [3] a definition of a determinant for finite potent endomorphisms satisfying the following properties:

- if V is finite dimensional, then $\det_V(1 + \varphi)$ is the ordinary determinant;
- if W is a subspace of V such that $\varphi W \subset W$, then

$$\det_V(1 + \varphi) = \det_W(1 + \varphi) \cdot \det_{V/W}(1 + \varphi)\,;$$

- if φ is nilpotent, then $\det_V(1 + \varphi) = 1$.

If $\lambda_1, \ldots, \lambda_N$ are all the nonzero eigenvalues of φ counted up to algebraic multiplicity, from [3, Proposition 3.18] we know that

$$\operatorname{Tr}_V(\varphi) = \sum_{i=1}^{N} \lambda_i \quad \text{and} \quad \det_V(1 + \varphi) = \prod_{i=1}^{N} (1 + \lambda_i)\,. \tag{2.1}$$

2.2 Drazin Inverse of Finite Potent Endomorphisms

Let V be an arbitrary k-vector space and let $\varphi \in \mathrm{End}_k(V)$ be a finite potent endomorphism of V. Let us consider the AST-decomposition $V = U_\varphi \oplus W_\varphi$ induced by φ.

We shall call *index of* φ, $i(\varphi)$, to the nilpotent order of $\varphi_{|U_\varphi}$, which coincides with the smaller $n \in \mathbb{N}$ such that $\mathrm{Im}\,\varphi^n = W_\varphi$. One has that $i(\varphi) = 0$ if and only if V is a finite-dimensional vector space and φ is an automorphism. In [10, Lemma 3.2] is proved that for finite-dimensional vector spaces, this definition of index coincides with the index of the matrix associated with φ.

For each finite potent endomorphism φ there exists a unique finite potent endomorphism φ^D that satisfies that:

(1) $\varphi^{r+1} \circ \varphi^D = \varphi^r$;
(2) $\varphi^D \circ \varphi \circ \varphi^D = \varphi^D$;
(3) $\varphi^D \circ \varphi = \varphi \circ \varphi^D$,

where r is the index of φ.

The map φ^D is the Drazin inverse of φ and is the unique linear map such that

$$\varphi^D(v) = \begin{cases} (\varphi_{|W_\varphi})^{-1}(v) & \text{if} \quad v \in W_\varphi \\ 0 & \text{if} \quad v \in U_\varphi \end{cases}. \tag{2.2}$$

Moreover, φ^D satisfies the following properties:

- $(\varphi^D)^D = \varphi$ if and only if $i(\varphi) \leq 1$;
- $\varphi = \varphi^D$ if and only if $\varphi_{|U_\varphi} = 0$ and $(\varphi_{|W_\varphi})^2 = \mathrm{Id}_{|W_\varphi}$;
- if ψ is a projection finite potent endomorphism, then $\psi^D = \psi$.

2.3 CN Decomposition of a Finite Potent Endomorphism

Let V be again an arbitrary k-vector space. Given a finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$, there exists a unique decomposition $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1, \varphi_2 \in \mathrm{End}_k(V)$ are finite potent endomorphisms satisfying that

- $i(\varphi_1) \leq 1$;
- φ_2 is nilpotent;
- $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1 = 0$.

According to [6, Theorem 3.2], if φ^D is the Drazin inverse of φ, one has that $\varphi_1 = \varphi \circ \varphi^D \circ \varphi$ is the *core part* of φ. Also, φ_2 is named the *nilpotent part* of φ and one has that

$$\varphi = \varphi_1 \Longleftrightarrow U_\varphi = \mathrm{Ker}\,\varphi \Longleftrightarrow W_\varphi = \mathrm{Im}\,\varphi \Longleftrightarrow (\varphi^D)^D = \varphi \Longleftrightarrow i(\varphi) \leq 1\,. \tag{2.3}$$

Moreover, if $V = W_\varphi \oplus U_\varphi$ is the AST-decomposition of V induced by φ, then φ_1 and φ_2 are the unique linear maps such that

$$\varphi_1(v) = \begin{cases} \varphi(v) & \text{if } v \in W_\varphi \\ 0 & \text{if } v \in U_\varphi \end{cases} \quad \text{and} \quad \varphi_2(v) = \begin{cases} 0 & \text{if } v \in W_\varphi \\ \varphi(v) & \text{if } v \in U_\varphi \end{cases}. \tag{2.4}$$

By definition of Tate's trace, for every finite potent endomorphism $\varphi \in \operatorname{End}_k(V)$, one has that

$$\operatorname{Tr}_V(\varphi) = \operatorname{Tr}_V(\varphi_1).$$

2.4 Bounded Finite Potent Endomorphisms on Hilbert Spaces

Let $\mathcal{H}$ be again an arbitrary Hilbert space. Recently, the first-named author of this work has studied in [9] the set of bounded finite potent endomorphisms on $\mathcal{H}$, which will be denoted by $B_{fp}(\mathcal{H})$. This section is devoted to summarize the main results of [9].

If $\varphi \in B_{fp}(\mathcal{H})$, $\mathcal{H} = W_\varphi \oplus U_\varphi$ is the AST-decomposition induced by φ and $\varphi = \varphi_1 + \varphi_2$ is the CN-decomposition, then the following properties hold:

(1) φ is quasi-compact;
(2) $\varphi_1, \varphi_2 \in B_{fp}(\mathcal{H})$ and φ_1 is of trace class;
(3) φ is compact if and only if φ_2 is compact;
(4) if $\operatorname{Tr}(\varphi_1)$ is the trace of φ_1 as a trace class operator, then $\operatorname{Tr}(\varphi_1) = \operatorname{Tr}_{\mathcal{H}}(\varphi)$;
(5) given a non-zero $\lambda \in \mathbb{C}$, one has that λ is an eigenvalue of φ if and only if λ is an eigenvalue of $\varphi_{|W_\varphi}$;
(6) if $i(\varphi) \geq 1$, then $\sigma(\varphi) = \{0, \lambda_1, \dots, \lambda_n\}$, where $\{\lambda_1, \dots, \lambda_n\}$ are the eigenvalues of $\varphi_{|W_\varphi}$;
(7) $\operatorname{Tr}_{\mathcal{H}}(\varphi) = \operatorname{Tr}_{W_\varphi}(\varphi_{|W_\varphi}) = \operatorname{Tr}(\varphi_1) = \operatorname{Tr}^L_{\mathcal{H}}(\varphi) = \operatorname{Tr}^R_{\mathcal{H}}(\varphi)$,

where $\operatorname{Tr}_{\mathcal{H}}(\varphi)$ is the Tate's trace of φ as a finite potent endomorphism; $\operatorname{Tr}_{W_\varphi}(\varphi_{|W_\varphi})$ is the trace of the endomorphism $\varphi_{|W_\varphi}$ on the finite-dimensional $\mathbb{C}$-vector space W_φ; $\operatorname{Tr}(\varphi_1)$ is the trace of φ_1 of a trace class operator; $\operatorname{Tr}^L_{\mathcal{H}}(\varphi)$ is the Leray trace and $\operatorname{Tr}^R_{\mathcal{H}}(\varphi)$ is the trace of φ as a Riesz trace class operator.

Moreover, given again a bounded finite potent endomorphism $\varphi \in B_{fp}(\mathcal{H})$, the adjoint operator φ^* is defined as

$$\langle h, \varphi(h') \rangle_{\mathcal{H}} = \langle \varphi^*(h), h' \rangle_{\mathcal{H}}$$

for every $h, h' \in \mathcal{H}$, and it satisfies that:

(1) $i(\varphi^*) = i(\varphi)$;
(2) $\varphi^* = (\varphi_1)^* + (\varphi_2)^*$ is the CN-decomposition of φ^*;
(3) if $\mathcal{H} = W_{\varphi^*} \oplus U_{\varphi^*}$ is the AST-decomposition induced by φ^*, then one has that $W_{\varphi^*} = U_\varphi^\perp$ and $U_{\varphi^*} = W_\varphi^\perp$;
(4) $\sigma(\varphi^*) = \overline{\sigma(\varphi)}$.

Now, we recall a statement of [9] that shall be useful for the present work.

Thus, it follows from [9, Proposition 4.1] that

Proposition 2.1 *If $\mathcal{H}$ is a Hilbert space and we consider $\varphi \in B_{fp}(\mathcal{H})$, then the adjoint φ^* is also a bounded finite potent endomorphism.*

2.4.1 Drazin Inverse of a Bounded Finite Potent Operator

Our task is now to recall different properties of the Drazin inverse of a bounded finite potent endomorphism on a Hilbert space offered in [8]. Thus, if $\mathcal{H}$ is a Hilbert space and $\varphi \in B_{fp}(\mathcal{H})$, then the Drazin inverse φ^D is also a bounded finite potent endomorphism satisfying that

(1) $(\varphi^D)^* = (\varphi^*)^D$;
(2) $([\varphi^*]^D)^D = \varphi^*$ if and only if $i(\varphi) \leq 1$;
(3) $\varphi^* = (\varphi^*)^D$ if and only if $\varphi_{|U_\varphi} = 0$ and $(\varphi_{|W_\varphi})^2 = \mathrm{Id}_{|W_\varphi}$;
(4) if ψ is a projection finite potent endomorphism, then $(\psi^*)^D = \psi^*$.

2.5 Pseudo-Characteristic Polynomial of a Finite Potent Endomorphism

Recently, the first-named author has introduced in [5] the notion of *pseudo-characteristic polynomial* $\tilde{c}_\varphi(x)$ of a finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$ as follows:

$$\tilde{c}_\varphi(x) = \begin{cases} c_{\varphi_{|W_\varphi}}(x) & \text{if } W_\varphi \neq 0 \\ 1 & \text{if } W_\varphi = 0 \end{cases}, \tag{2.5}$$

where $c_{\varphi_{|W_\varphi}}(x)$ is the characteristic polynomial of $\varphi_{|W_\varphi} \in \mathrm{End}_k(W_\varphi)$. If $\varphi = \varphi_1 + \varphi_2$ is the CN-decomposition of φ, one has that $\tilde{c}_\varphi(x) = \tilde{c}_{\varphi_1}(x)$.

If $\mathcal{H}$ is a Hilbert space and $\varphi \in B_{fp}(\mathcal{H})$ is a bounded finite potent linear operator with spectrum $\sigma(\varphi)$, according to [5, Lemma 3.19], we know that $0 \neq \lambda \in \sigma(\varphi)$ if and only if λ is a root of $\tilde{c}_\varphi(x)$.

Furthermore, if $\lambda_1, \dots, \lambda_N$ are all the nonzero eigenvalues of φ counted up to algebraic multiplicity, it is clear that

$$\tilde{c}_\varphi(x) = \prod_{i=1}^{N}(x - \lambda_i)\,. \tag{2.6}$$

2.6 Drazin-Star and Star-Drazin Inverses of a Bounded Finite Potent Linear Operator

Let us consider again a Hilbert space $\mathcal{H}$. Given $\varphi \in B_{fp}(\mathcal{H})$, with closed $\mathrm{Im}\,\varphi$ and $i(\varphi) = r$, a linear map $\varphi^{D,*} \in \mathrm{End}_{\mathbb{C}}(\mathcal{H})$ is the *Drazin-Star inverse* of φ when it satisfies that

- $\varphi^{D,*} \circ (\varphi^\dagger)^* \circ \varphi^{D,*} = \varphi^{D,*}$;
- $\varphi^r \circ \varphi^{D,*} = \varphi^r \circ \varphi^*$;
- $\varphi^{D,*} \circ (\varphi^\dagger)^* = \varphi^D \circ \varphi$.

Moreover, a linear operator $\varphi^{*,D} \in \mathrm{End}_{\mathbb{C}}(\mathcal{H})$ is the *Star-Drazin inverse* of φ when the following properties hold:

- $\varphi^{*,D} \circ (\varphi^\dagger)^* \circ \varphi^{*,D} = \varphi^{*,D}$;
- $\varphi^{*,D} \circ \varphi^r = \varphi^* \circ \varphi^r$;
- $(\varphi^\dagger)^* \circ \varphi^{*,D} = \varphi \circ \varphi^D$.

With the previous hypothesis on $\varphi \in B_{fp}(\mathcal{H})$, it is known, from [7, Theorem 3.3] and [7, Theorem 3.7], the existence and uniqueness of $\varphi^{D,*}$ and $\varphi^{*,D}$, respectively. The explicit expressions of these inverses are $\varphi^{D,*} = \varphi^D \circ \varphi \circ \varphi^*$ and $\varphi^{*,D} = \varphi^* \circ \varphi \circ \varphi^D$ and it is clear that $(\varphi^*)^{D,*} = (\varphi^{*,D})^*$.

When $\widetilde{\varphi} \in B_{fp}(\mathcal{H})$ and $i(\widetilde{\varphi}) \leq 1$, we have that the Group-Star of $\widetilde{\varphi}$ is $\widetilde{\varphi}^{\#,*} = \widetilde{\varphi}^\# \circ \widetilde{\varphi} \circ \widetilde{\varphi}^*$ and the Star-Group inverses of this linear operator is $\widetilde{\varphi}^{*,\#} = \widetilde{\varphi}^* \circ \widetilde{\varphi} \circ \widetilde{\varphi}^\#$.

2.7 Some Properties of the Adjoint Operator of a Bounded Finite Potent Endomorphism of a Hilbert Space

The final part of this section is devoted to offer some properties of the adjoint operator of a bounded finite potent endomorphism of Hilbert space $\mathcal{H}$. In particular, we shall characterize the structure of the kernel and the image of φ^*, where $\varphi \in B_{fp}(\mathcal{H})$ has closed image. When $\varphi \in B_{fp}(\mathcal{H})$ and $i(\varphi) = 1$, it is clear that $\mathrm{Im}\,\varphi$ is always closed because it is a finite-dimensional $\mathbb{C}$-vector space.

Lemma 2.2 *If $\mathcal{H}$ is a Hilbert space and $\varphi \in B_{fp}(\mathcal{H})$, then* $\mathrm{Ker}\,\varphi^* = [\mathrm{Im}\,\varphi]^\perp$.

Proof The statement is a direct consequence of the following equivalences:

$$\begin{aligned} v \in \mathrm{Ker}\,\varphi^* &\iff \langle \varphi^*(v), h\rangle_{\mathcal{H}} = 0 \text{ for every } h \in \mathcal{H} \iff \\ &\iff \langle v, \varphi(h)\rangle_{\mathcal{H}} = 0 \text{ for every } h \in \mathcal{H} \iff \\ &\iff v \in [\mathrm{Im}\,\varphi]^\perp . \end{aligned}$$

□

Note that the statement of Lemma 2.2 holds for an arbitrary $\varphi \in B_{fp}(\mathcal{H})$, that is: it is not necessary to assume that $\mathrm{Im}\,\varphi$ is closed. Moreover, from the well-known Banach Closed Range Theorem, if $\mathrm{Im}\,\varphi$ is closed, then one has that $\mathrm{Im}\,\varphi^*$ is also closed.

Lemma 2.3 *If $\mathcal{H}$ is a Hilbert space and $\varphi \in B_{fp}(\mathcal{H})$ with closed* $\mathrm{Im}\,\varphi$*, then* $\mathrm{Im}\,\varphi^* = [\mathrm{Ker}\,\varphi]^\perp$.

Proof Since $\mathrm{Im}\,\varphi^*$ is closed, bearing in mind Lemma 2.2, the claim is immediately deduced from the following equalities:

$$\mathrm{Im}\,\varphi^* = [(\mathrm{Im}\,\varphi^*)^\perp]^\perp = [\mathrm{Ker}\,(\varphi^*)^*]^\perp = [\mathrm{Ker}\,\varphi]^\perp .$$

□

Given $\varphi \in B_{fp}(\mathcal{H})$, let us consider the AST-decompositions $\mathcal{H} = W_\varphi \oplus U_\varphi$ and $\mathcal{H} = W_{\varphi^*} \oplus U_{\varphi^*}$ determined by φ and φ^*, respectively.

Proposition 2.4 *We have that the dimensions of W_φ and W_{φ^*} as $\mathbb{C}$-vector spaces coincide.*

Proof Since the degree of the pseudo-characteristic polynomial $\tilde{c}_\varphi(x)$ is the dimension of W_φ, the assertion is a direct consequence of [5, Lemma 3.20] that shows that the degree of $\tilde{c}_\varphi(x)$ is equal to the degree of $\tilde{c}_{\varphi^*}(x)$. □

Remark 2.5 Bearing in mind that $W_{\varphi^*} = U_\varphi^\perp$, it is clear that the statement of Proposition 2.4 can also be proved from the isomorphisms of $\mathbb{C}$-vector spaces:

$$W_\varphi \simeq \mathcal{H}/U_\varphi \simeq U_\varphi^\perp = W_{\varphi^*} .$$

3 Study of the Kernel and the Image of the Drazin-Star and the Star-Drazin Inverses of a Bounded Finite Potent Operator

The goal of this section is to study of the kernel and the image of the Drazin-Star and the Star-Drazin inverses of a bounded finite potent operator as basic tools to obtain the main results of this work. Moreover, we shall determine the AST-decompositions induced by the Group-Star and the Star-Group inverses of a bounded finite potent operator $\psi \in B_{fp}(\mathcal{H})$ with $i(\psi) \leq 1$.

3.1 Kernel and Image of the Star-Drazin and the Drazin-Star Inverses

Let $\mathcal{H}$ be a Hilbert space, let φ be a bounded finite potent operator on $\mathcal{H}$, with closed $\operatorname{Im}\varphi$ and $i(\varphi) = r$, and let us denote again by $\varphi^{*,D}$ the Star-Drazin inverse of φ. We shall now study $\operatorname{Ker}\varphi^{*,D}$ and $\operatorname{Im}\varphi^{*,D}$.

Lemma 3.1 *Given a Hilbert space $\mathcal{H}$ and a bounded finite operator $\varphi \in B_{fp}(\mathcal{H})$ with adjoint operator φ^*, if $\mathcal{H} = W_\varphi \oplus U_\varphi$ is the AST-decomposition induced by φ and we consider $w \in W_\varphi$ such that $w \neq 0$, then $\varphi^*(w) \neq 0$.*

Proof If $0 \neq w = \varphi(w')$ with $w' \in W_\varphi$, we have that

$$0 \neq \langle w, w\rangle_{\mathcal{H}} = \langle w, \varphi(w')\rangle_{\mathcal{H}} = \langle \varphi^*(w), w'\rangle_{\mathcal{H}}$$

and we deduce that $\varphi^*(w) \neq 0$. □

Proposition 3.2 *If $\mathcal{H}$ is a Hilbert space and $\varphi \in B_{fp}(\mathcal{H})$ with closed* $\operatorname{Im}\varphi$*, that induces the AST-decomposition $\mathcal{H} = W_\varphi \oplus U_\varphi$, then* $\operatorname{Ker}\varphi^{*,D} = U_\varphi$.

Proof Since $(\varphi^D)_{|U_\varphi} = 0$ and $\varphi^{*,D} = \varphi^* \circ \varphi \circ \varphi^D$, it is clear that $U_\varphi \subseteq \operatorname{Ker}\varphi^{*,D}$. On the other hand, given $h \in \operatorname{Ker}\varphi^{*,D}$ such that $h = w + u$ with $w \in W_\varphi$ and $u \in U_\varphi$, bearing in mind that $\varphi^{*,D}(w) = \varphi^*(w)$, one has that

$$0 = \varphi^{*,D}(h) = \varphi^{*,D}(w + u) = \varphi^*(w)$$

and it follows from Lemma 3.1 that $w = 0$.

Accordingly, we obtain that Ker $\varphi^{*,D} \subseteq U_\varphi$, from where the claim is proved. □

Lemma 3.3 *If $\mathcal{H}$ is a Hilbert space and $\varphi \in B_{fp}(\mathcal{H})$ is a bounded finite potent operator with closed* Im φ*, that induces the AST-decomposition $\mathcal{H} = W_\varphi \oplus U_\varphi$, then* Im $\varphi^{*,D} = \varphi^*(W_\varphi)$.

Proof Bearing in mind that $\operatorname{Im}(\varphi \circ \varphi^D) = W_\varphi$, the claim is immediately deduced from the explicit expression of the Star-Drazin inverse. □

Proposition 3.4 *With the previous notation, one has that*

$$[\operatorname{Im} \varphi^{*,D}] \cap [\operatorname{Ker} \varphi] = \{0\}\,.$$

Proof Let us consider $h \in [\operatorname{Im} \varphi^{*,D}] \cap [\operatorname{Ker} \varphi]$. Since Lemma 3.3 shows that $h = \varphi^*(w)$ with $w \in W_\varphi$, we have that

$$\langle h, h\rangle_{\mathcal{H}} = \langle \varphi^*(w), h\rangle_{\mathcal{H}} = \langle h, \varphi(h)\rangle_{\mathcal{H}} = \langle h, 0\rangle_{\mathcal{H}} = 0\,,$$

from where we deduce that $h = 0$ and the claim is proved. □

Our task is now to characterize the Kernel and the image of the Drazin-Star inverse $\varphi^{D,*}$.

Proposition 3.5 *If $\mathcal{H}$ is a Hilbert space and $\varphi \in B_{fp}(\mathcal{H})$ is a bounded finite potent operator with closed* Im φ*, that induces the AST-decomposition $\mathcal{H} = W_\varphi \oplus U_\varphi$, then one has that* Im $\varphi^{D,*} = W_\varphi$ *and* Ker $\varphi^{D,*} = [\varphi(U_\varphi^\perp)]^\perp$.

Proof From Proposition 3.2, one has that

$$\begin{aligned}\operatorname{Im} \varphi^{D,*} &= \operatorname{Im}((\varphi^*)^*)^{D,*} = \operatorname{Im}((\varphi^*)^{*,D})^* = \\ &= [\operatorname{Ker}(\varphi^*)^{*,D}]^\perp = [U_{\varphi^*}]^\perp = [W_\varphi^\perp]^\perp = W_\varphi\,,\end{aligned}$$

from where the first assertion is deduced.

Moreover, bearing in mind Lemma 3.3, one obtains that

$$\begin{aligned}\operatorname{Ker} \varphi^{D,*} &= \operatorname{Ker}((\varphi^*)^*)^{D,*} = \operatorname{Ker}((\varphi^*)^{*,D})^* = \\ &= [\operatorname{Im}(\varphi^*)^{*,D}]^\perp = [\varphi(W_{\varphi^*})]^\perp = [\varphi(U_\varphi^\perp)]^\perp\end{aligned}$$

and the proof is completed. □

3.1.1 Characterization of Finite Potent Endomorphisms with Index Less or Equal to One

Given an arbitrary k-vector space, we shall now characterize a finite potent linear map $\varphi \in \operatorname{End}_k(V)$ with $i(\varphi) \leq 1$ from its kernel and image.

Lemma 3.6 *If V is an arbitrary k-vector space and $\varphi \in \text{End}_k(V)$ is a finite potent endomorphism, then $i(\varphi) \leq 1$ if and only if* $\text{Ker}\,\varphi \cap Im\varphi = \{0\}$.

Proof If $\varphi \in \text{End}_k(V)$ is a finite potent endomorphism such that $i(\varphi) \leq 1$, it is known that $\text{Ker}\,\varphi = U_\varphi$ and $\text{I}m\varphi = W_\varphi$. Then, since $U_\varphi \cap W_\varphi = \{0\}$, we deduce that $\text{Ker}\,\varphi \cap \text{Im}\varphi = \{0\}$.

Conversely, let us assume that φ is finite potent and satisfies that $\text{Ker}\,\varphi \cap \text{I}m\varphi = \{0\}$. Given $v \in U_\varphi$, if there exists $r_v \in \mathbb{N}$ such that $r_v > 1$, $\varphi^{r_v}(v) = 0$ and $\varphi^{r_v - 1}(v) \neq 0$, then $\varphi^{r_v - 1}(v) \in Ker\varphi \cap \text{I}m\varphi = \{0\}$, from which contradiction is obtained. Accordingly, one has that $U_\varphi \subseteq \text{Ker}\,\varphi$, from where we deduce that $U_\varphi = \text{Ker}\,\varphi$ and $i(\varphi) \leq 1$. □

Proposition 3.7 *Given an arbitrary Hilbert space $\mathcal{H}$ and a bounded finite potent operator $\varphi \in B_{fp}(\mathcal{H})$ with closed* $\text{Im}\,\varphi$ *and AST-decomposition $\mathcal{H} = W_\varphi \oplus U_\varphi$, if $\varphi^*(W_\varphi) \cap U_\varphi = \{0\}$, then $i(\varphi^{*,D}) \leq 1$, $W_{\varphi^{*,D}} = \varphi^*(W_\varphi)$ and $U_{\varphi^{*,D}} = U_\varphi$.*

Proof The statement is immediately deduced from Proposition 3.2, Lemma 3.3 and Lemma 3.6. □

Let us now consider two closed $\mathbb{C}$-subspaces $V_1, V_2 \subset \mathcal{H}$. It is clear that

$$V_1 \cap V_2^\perp = \{0\} \Longleftrightarrow V_1^\perp \cap V_2 = \{0\}\,. \tag{3.1}$$

Proposition 3.8 *With the previous notation, if $\varphi(W_{\varphi^*}) \cap U_{\varphi^*} = \{0\}$, one has that $i(\varphi^{D,*}) \leq 1$, $W_{\varphi^{D,*}} = W_\varphi$ and $U_{\varphi^{D,*}} = [\varphi(W_{\varphi^*})]^\perp$.*

Proof Bearing in mind that (3.1) implies that

$$W_\varphi \cap [\varphi(U_\varphi^\perp)]^\perp = \{0\} \Longleftrightarrow W_\varphi^\perp \cap \varphi(U_\varphi^\perp) = \{0\}\,,$$

since $W_{\varphi^*} = U_\varphi^\perp$ and $U_{\varphi^*} = W_\varphi^\perp$, the claim is obtained from Proposition 3.5 and Lemma 3.6. □

3.1.2 AST-Decomposition of the Star-Group Inverse of a Bounded Finite Potent Endomorphism

Our goal is now to apply the previous results to obtain the explicit expression of the AST-decomposition of the Group-Star inverse of a bounded finite potent endomorphism.

Let $\mathcal{H}$ be a Hilbert space. If $\psi \in B_{fp}(\mathcal{H})$ is a bounded finite potent operator with $i(\psi) \leq 1$ with AST-decomposition $\mathcal{H} = W_\psi \oplus U_\psi$, recall that $\text{Im}\,\psi = W_\psi$ is a finite-dimensional $\mathbb{C}$-vector space and the Star-Group inverse of ψ is $\psi^{*,\#} = \psi^* \circ \psi \circ \psi^\#$.

Proposition 3.9 *With the previous notation, we have that $i(\psi^{*,\#}) \leq 1$, $W_{\psi^{*,\#}} = \psi^*(W_\psi)$ and $U_{\psi^{*,\#}} = \text{Ker}\,\psi$.*

Proof Since $i(\psi) \leq 1$, it follows from Proposition 3.2 that Ker $\psi^{*,\#} = U_\psi =$ Ker ψ and from Lemma 3.3 that Im $\psi^{*,\#} = \psi^*(W_\psi)$.

Hence, bearing in mind that Ker $\psi^{*,\#} \cap$ Im $\psi^{*,\#} = \{0\}$ (Proposition 3.4), one has that $i(\psi^{*,\#}) \leq 1$.

Thus, we have that $W_{\psi^{*,\#}} =$ Im $\psi^{*,\#} = \psi^*(W_\psi)$ and $U_{\psi^{*,\#}} =$ Ker $\psi^{*,\#} =$ Ker ψ. □

3.1.3 AST-Decomposition of the Group-Star Inverse of a Bounded Finite Potent Endomorphism

We shall now offer the explicit expression of the AST-decomposition of the Star-Group inverse of a bounded finite potent endomorphism.

Let $\mathcal{H}$ be again a Hilbert space. If $\psi \in B_{fp}(\mathcal{H})$ is a bounded finite potent operator with $i(\psi) \leq 1$, then the Group-Star inverse of ψ is $\psi^{\#,*} = \psi^\# \circ \psi \circ \psi^*$.

Lemma 3.10 *One has that* $i(\psi^{\#,*}) \leq 1$.

Proof Bearing in mind that $i(\varphi) = i(\varphi^*)$ for every $\varphi \in B_{fp}(\mathcal{H})$, the claim is deduced from Proposition 3.9 and the following equalities:

$$i(\psi^{\#,*}) = i([(\psi^*)^*]^{\#,*}) = i([(\psi^*)^{*,\#}]^*) = i((\psi^*)^{*,\#}) = 1\,.$$

□

Proposition 3.11 *If $\mathcal{H}$ is a Hilbert space and $\psi \in B_{fp}(\mathcal{H})$ is a bounded finite potent operator with $i(\psi) \leq 1$, one has that $W_{\psi^{\#,*}} = W_\psi =$ Im ψ and $U_{\psi^{\#,*}} = [\text{Im}\,\psi]^\perp = U_{\psi^*}$.*

Proof Bearing in mind that $\psi([\text{Ker}\,\psi]^\perp) =$ Im ψ, the statement is a direct consequence of Proposition 3.5 and Lemma 3.10. □

4 Coincidence of the Pseudo-Characteristic Polynomials of the Drazin-Star and the Star-Drazin Inverses

The aim of this final section is to prove that the pseudo-characteristic polynomials of the Drazin-Star and the Star-Drazin inverses coincide and to apply this property to obtain new results about these inverses.

Let us again consider an arbitrary Hilbert space $\mathcal{H}$ and a bounded finite potent linear operator $\varphi \in B_{fp}(\mathcal{H})$ with closed image.

Recall from Lemma 3.3 that Im $\varphi^{*,D} = \varphi^*(W_\varphi)$ and from Proposition 3.5 that Im $\varphi^{D,*} = W_\varphi$. Hence, one has that $W_{\varphi^{*,D}} \subseteq \varphi^*(W_\varphi)$ and $W_{\varphi^{D,*}} \subseteq W_\varphi$.

Lemma 4.1 *With the previous notation, if $\{e_1, \dots, e_n\}$ is a family of vectors of W_φ such that $\varphi^{D,*}(e_1) = \lambda e_1$ and $\varphi^{D,*}(e_i) = \lambda e_i + e_{i-1}$ for every $i \in \{2, \dots, n\}$ and a certain $\lambda \in \mathbb{C}$, denoting $v_j = \varphi^*(e_j)$ for all $j \in \{1, \dots, n\}$, we have that $\{v_1, \dots, v_n\}$ is a family of vectors of $\varphi^*(W_\varphi)$ such that $\varphi^{*,D}(v_1) = \lambda v_1$ and $\varphi^{*,D}(v_i) = \lambda v_i + v_{i-1}$ for every $i \in \{2, \dots, n\}$.*

Proof The statement is immediately deduced from the equalities

$$\varphi^{*,D}(v_1) = (\varphi^* \circ \varphi \circ \varphi^D)(\varphi^*(e_1)) = \varphi^*(\varphi^{D,*}(e_1)) = \varphi^*(\lambda e_1) = \lambda v_1$$

and

$$\varphi^{*,D}(v_i) = (\varphi^* \circ \varphi \circ \varphi^D)(\varphi^*(e_i)) = \varphi^*(\varphi^{D,*}(e_i)) = \varphi^*(\lambda e_i + e_{i-1}) = \lambda v_1 + v_{i-1} \,,$$

for every $i \in \{2, \dots, n\}$. □

Lemma 4.2 *Given a Hilbert space $\mathcal{H}$ and a bounded finite potent linear operator $\varphi \in B_{fp}(\mathcal{H})$ with closed $\operatorname{Im}\varphi$, if $B = \{e_1, \dots, e_n\}$ is a Jordan basis of $W_{\varphi^{D,*}}$ for $\varphi^{D,*}$, then $\varphi^*(B) = \{\varphi^*(e_1), \dots, \varphi^*(e_n)\}$ is a linearly independent family of vectors of $W_{\varphi^{*,D}}$. Moreover, one has that $\dim_{\mathbb{C}} W_{\varphi^{D,*}} \leq \dim_{\mathbb{C}} W_{\varphi^{*,D}}$.*

Proof If V is an arbitrary k-vector space, $\tilde{\varphi} \in \operatorname{End}_k(V)$ is a finite potent endomorphism with AST-decomposition $V = W_{\tilde{\varphi}} \oplus U_{\tilde{\varphi}}$ and $L \subset V$ is a φ-invariant subspace such that $\varphi_{|_L} \in \operatorname{Aut}_k(L)$, it is known that $L \subseteq W_{\tilde{\varphi}}$. Accordingly, the claim is a direct consequence of Lemma 4.1. □

Lemma 4.3 *If $\mathcal{H}$ is a Hilbert space and $\varphi \in B_{fp}(\mathcal{H})$ is a bounded finite potent linear operator with closed $\operatorname{Im}\varphi$, then $\dim_{\mathbb{C}} W_{\varphi^{*,D}} = \dim_{\mathbb{C}} W_{\varphi^{D,*}}$.*

Proof If follows from Lemma 4.2 that $\dim_{\mathbb{C}} W_{\varphi^{D,*}} \leq \dim_{\mathbb{C}} W_{\varphi^{*,D}}$.

Now, since from Proposition 2.4, we know that

$$\dim_{\mathbb{C}} W_{\varphi^{*,D}} = \dim_{\mathbb{C}} W_{((\varphi^*)^{D,*})^*} = \dim_{\mathbb{C}} W_{(\varphi^*)^{D,*}}$$

and

$$\dim_{\mathbb{C}} W_{\varphi^{D,*}} = \dim_{\mathbb{C}} W_{((\varphi^*)^{*,D})^*} = \dim_{\mathbb{C}} W_{(\varphi^*)^{*,D}} \,,$$

we conclude bearing in mind that Lemma 4.2 also shows that

$$\dim_{\mathbb{C}} W_{(\varphi^*)^{D,*}} \leq \dim_{\mathbb{C}} W_{(\varphi^*)^{*,D}} \,.$$

□

Proposition 4.4 *If $\mathcal{H}$ is a Hilbert space, $\varphi \in B_{fp}(\mathcal{H})$ is a bounded finite potent operator with closed $\operatorname{Im}\varphi$ and $B = \{e_1, \dots, e_n\}$ is a Jordan basis of $W_{\varphi^{D,*}}$ for $\varphi^{D,*}$, then we have that $\dim_{\mathbb{C}} W_{\varphi^{D,*}} = \dim_{\mathbb{C}} W_{\varphi^{*,D}}$, $\varphi^*(B) = \{\varphi^*(e_1), \dots, \varphi^*(e_n)\}$ is a Jordan basis of $W_{\varphi^{*,D}}$ for $\varphi^{*,D}$ and the matrix associated to $(\varphi^{D,*})_{|_{W_{\varphi^{D,*}}}}$ in B coincides with the matrix associated to $(\varphi^{*,D})_{|_{W_{\varphi^{*,D}}}}$ in $\varphi^*(B)$.*

Proof It is a direct consequence of Lemmas 4.1 and 4.3. □

Theorem 4.5 *With the previous notation, one has that $\tilde{c}_{\varphi^{*,D}}(x) = \tilde{c}_{\varphi^{D,*}}(x)$.*

Proof The statement is obtained from Proposition 4.4 and the explicit expression (2.5) of the pseudo-characteristic polynomial of a finite potent endomorphism. □

Corollary 4.6 *Given a Hilbert space $\mathcal{H}$ and a bounded finite potent linear operator $\varphi \in B_{fp}(\mathcal{H})$ with closed* $\operatorname{Im} \varphi$*, the following properties hold:*

- $\sigma(\varphi^{D,*}) = \sigma(\varphi^{*,D})$;
- $\operatorname{Tr}_{\mathcal{H}}(\varphi^{D,*}) = \operatorname{Tr}_{\mathcal{H}}(\varphi^{*,D})$;
- $\det_{\mathcal{H}}(Id + \varphi^{D,*}) = \det_{\mathcal{H}}(Id + \varphi^{*,D})$.

Proof These assertions are direct consequence of the characterization of the spectrum of a bounded finite potent operator, the equalities (2.1) and Theorem 4.5, because from (2.6) we know that the set of eigenvalues of $(\varphi^{D,*})_{|W_{\varphi^{D,*}}}$ coincides with the set of eigenvalues of $(\varphi^{*,D})_{|W_{\varphi^{*,D}}}$, counted both up to algebraic multiplicity. □

4.1 Coincidence of the Characteristic Polynomial of $A^{*,D}$ and $A^{D,*}$ and Applications

The final part of this work is devoted to offer new properties of the Drazin-Star and the Star-Drazin inverses of a complex square matrix from the previous results on bounded finite potent operators.

Given a square complex matrix $A \in \operatorname{Mat}_{n\times n}(\mathbb{C})$ with core-nilpotent decomposition $A = A_1 + A_2$, if $c_A(x)$ is the characteristic polynomial of A, one has that

$$c_A(x) = |x\operatorname{Id} - A| = x^r \cdot q(x) \text{ with } (x, q(x)) = 1 \text{ and } r = n - \operatorname{rk}(A_1)\,. \tag{4.1}$$

Lemma 4.7 *If $A \in \operatorname{Mat}_{n\times n}(\mathbb{C})$, then* $\operatorname{rk}((A^{*,D})_1) = \operatorname{rk}((A^{D,*})_1)$.

Proof If $T \in \operatorname{End}_{\mathbb{C}}(\mathbb{C}^n)$ is an endomorphism such that $T \equiv A$ in a certain basis of $\mathbb{C}^n$ and $\mathbb{C}^n = W_T \oplus U_T$ is the AST-decomposition induced by T, since the dimension of W_T is the rank of A_1, the statement is a direct consequence of Lemma 4.3. □

Proposition 4.8 *If $A \in \operatorname{Mat}_{n\times n}(\mathbb{C})$, then $c_{A^{D,*}}(x) = c_{A^{*,D}}(x)$.*

Proof Bearing in mind (4.1), the assertion is a direct consequence of Theorem 4.5 and Lemma 4.7. □

From this proposition we have the following corollary:

Corollary 4.9 *Given an arbitrary complex square matrix $A \in \operatorname{Mat}_{n\times n}(\mathbb{C})$, the following properties hold:*

(1) $\sigma(A^{D,*}) = \sigma(A^{*,D})$;
(2) $\operatorname{Tr}(A^{D,*}) = \operatorname{Tr}(A^{*,D})$;
(3) $\det(Id + A^{D,*}) = \det(Id + A^{*,D})$.

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