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Multi criteria decision-making model using the circular Pythagorean fuzzy soft set model

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ABSTRACT

Lack of certainty is a major issue in decision-making, data analysis and modeling in many fields. Soft set theory has evolved into an effective method for addressing this issue, both due to its adaptability and its ability to blend with other models for the representation of uncertainty (inclusive of fuzzy sets and its extensions). Membership-based theories alone are incompatible with slackness in the evaluations, such as those that stem from measurement errors. This issue led to the appearance of interval-valued fuzzy sets (with only membership), and afterwards interval-valued intuitionistic fuzzy sets and circular intuitionistic fuzzy sets (with both membership and non-membership), plus other extensions that allow for wider domains of the evaluations. Circular Pythagorean fuzzy sets were defined with this motivation, and here we first combine them with soft set theory. In this setting, we propose novel score and accuracy functions based on optimistic and pessimistic perspectives. This is done with the help of a decision-maker-controlled parameter $\lambda \in [0, 1]$ that captures her/his approach to the problem. Furthermore, inspired by related models we define essential operations for the new framework, including complement, min-OR, max-OR, min-AND, max-AND, min-union, max-union, min-intersection, and max-intersection. All these tools allow us to extend decision-making strategies designed for Pythagorean fuzzy soft sets to the circular domain motivated by considerations of slackness. Finally, the study compares the proposed approach with circular intuitionistic fuzzy soft set based decision-making models to evaluate its effectiveness.

1. Introduction

Ambiguity and imprecision in data pose major problems in many real-world fields, including engineering, economics, business, and management. Approaches such as fuzzy set (FS) theory, introduced by Zadeh [1] and subsequently extended in many ways, have emerged as useful alternatives to probability theory, which has historically been the preferred method for handling uncertainty. Current extensions include interval-valued FS theory by Gorzalczany [2], intuitionistic fuzzy set (IFS) theory by Atanassov [3], Pythagorean fuzzy set (PFS) theory by Yager [4], Fermatean fuzzy set (FFS) by Senapati and Yager [5], and q-rung orthopair fuzzy set (q-ROFS) theory by Yager [6]. This line of research remains highly active, with recent advancements focusing on geometric interpretations and hybrid models to capture more complex uncertainty, such as the circular Pythagorean fuzzy set (C_rPFS) [7] and disc Pythagorean fuzzy set (DPFS) [8] models, and their applications in modern decision-making frameworks [9,10]. While FSs describe partial membership of elements in a set representing a 'class', IFSs and their subsequent extensions additionally introduce evaluations of partial non-membership, which are to some extent independent of the membership degree. For example, in an IFS, the sum of the membership and non-membership degrees for any given element cannot exceed 1. Alternatively, and also departing from the IFS model, two competing ideas allow for slackness: interval-valued intuitionistic fuzzy set (IVIFS) theory and circular intuitionistic fuzzy set (C_rIFS) theory launched by Atanassov [11,12]. Slackness is fixed for both memberships and non-memberships in IVIFS, whereas it is fixed for the distance to the pair membership/non-membership in C_rIFS. Alternative approaches (i.e., that do not associate partial memberships/non-memberships) include the rough set theory proposed by Pawlak [13] and the soft set (S_tS) theory proposed by Molodtsov [14]. The later is inspired by the idea that belongingness to a "class" may be multiply interpreted, depending on which attribute of the "class" is considered. The semantic roots of S_tS theory have been examined by Yang and Yao [15] and Alcantud [16]. A recent and thorough review of the expanding research on this theory is presented by Alcantud et al. [17], whereas John [18] delivers a detailed exposition of the topic.

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Other mathematical models have been effectively included into generalized S_tS theory. If belongingness is not only multiply defined, but also partially evaluated by respective memberships as in FS theory, then we obtain a fuzzy soft set (FS_tS) [19]. This model has its own decision-making strategies [20,21] and specialized forecasting tools [22]. If in addition, non-memberships are considered with the same characteristics as memberships and the restrictions imposed by IFS, then one has an intuitionistic fuzzy soft set (IFS_tS) [23]. There are many similar extensions for hybridized S_tS, such as Pythagorean fuzzy soft set (PFS_tS) [24,25], cubic Pythagorean fuzzy soft sets (C_uPFS_tS) [26], Fermatean fuzzy soft sets (FFS_tS) [27], interval-valued intuitionistic fuzzy soft set (IVIFS_tS) [28], and (a, b)-fuzzy soft set [29]. The integration of the structures of soft set with rough set and probability gives rise to models known as rough soft set [30] (see also Zhu et al. [31] and Akram et al. [32] for more sophisticated models combining soft and rough set theories) and probabilistic soft set [33]. In the realm of practical applications, Kovkov et al. [34] extended S_tS theory to optimization problems by introducing object approximation techniques. For an advanced overview of foundational and related research, see Shahzadi et al. [35–37].

This study originates from a detailed exploration of the limitations of the C_rIFS model. In Atanassov's C_rIFS framework, uncertainty in decision making is captured by enclosing the membership and non-membership degrees within a circular region, rather than treating them as fixed scalar values. This circle is defined by a center comprising a pair of non-negative real numbers whose sum does not exceed one. This geometric approach enables a more flexible representation of uncertainty, allowing for a range of possible values rather than a single point estimate. Building on this idea, recent research has explored various applications and extensions of the C_rIFS model. For instance, Chen [38,39] proposed several multi-criteria decision making (MCDM) methods based on C_rIFS, demonstrating their utility in situations involving conflicting or imprecise criteria. In another development, Khan et al. [10] introduced the concept of circular intuitionistic fuzzy preference relations, and studied their aggregation mechanisms, entropy measures, and similarity metrics. These tools were subsequently applied in a group decision-making context using a TOPSIS (Technique for order of preference by similarity to ideal solution) based selection process, which ranks alternatives based on their relative closeness to an ideal solution. Further advancing this line of research, Olgun et al. [7] extended Atanassov's concept to define the C_rPFS. Unlike the C_rIFS, which adheres to the constraint that the sum of the membership degree (MD) and non-membership degree (NMD) must not exceed one, the C_rPFS adopts a relaxed version of the Pythagorean condition, requiring that the squared membership and non-membership degrees together do not exceed one (see Alcantud [40] for semantic justification). This modification not only preserves the circular structure for representing uncertainty but also increases the expressive power of the model, making it better suited for handling more complex and ambiguous decision-making scenarios. In parallel, related concepts such as disc Pythagorean fuzzy set (DPFS) [8] and circular Fermatean fuzzy set (C_rFFS) [41] have been introduced, further enriching the landscape of FS theory with geometrically bounded models. Moreover, the C_rPFS model has been successfully integrated into various MCDM frameworks [9], enhancing the precision and adaptability of decision analysis by allowing decision-makers to work with bounded regions defined by circles instead of relying solely on fixed numerical values. In light of this ongoing development, the present work contributes to the field by embedding the C_rPFS framework into S_tS theory. With this combination, the benefit that we obtain is the ability to consider multiple perspectives of the elements that define the problem. This integration results in a novel, more sensitive, and adaptable approach for configuring decision-making processes and conducting data analysis, particularly in environments characterized by high levels of uncertainty or imprecision. Concurrently, new researches in uncertainty modeling have explored ideas beyond traditional fuzzy sets to capture the analytic complexities of group decision-making. A significant trend involves integrating quantum decision theory with linguistic and fuzzy frameworks to model interference effects and belief biases among experts. For instance, a fuzzy quantum group decision-making model was developed for meteorological disaster emergencies, using a quantum-like Bayesian network to handle the probabilistic uncertainty and mutual interference between decision-makers, with Deng entropy discussing these effects [42]. In a similar manner, a quantum-linguistic multi-attribute group consensus method based on trust networks was proposed, where linguistic Z-numbers are extended into quantum states and trust paths are aggregated using quantum probability to characterize interference and promote consensus [43]. While these quantum-inspired models are good at capturing the superposition and interaction of cognitive states in social contexts, our proposed C_rPFS_tS model offers a complementary, geometrically grounded approach. We focus on representing the evaluative slackness and cyclic imprecision in the data itself, providing a flexible parameterized framework for problems where uncertainty is inherent in the attributes and their periodic or conflicting nature, rather than primarily in the cognitive interactions between decision-makers.

1.1. Motivations and contributions

This article is motivated by the following facts:

1. The need for a more robust model is evident from the shortcomings of the FS and S_tS models in managing complicated, uncertain, and circular data. By providing a more realistic, adaptable, and expressive framework for MCDM situations, C_rPFS_tS closes this gap.
2. A potent extension of PFS and S_tS is the C_rPFS_tS model. The inclusion of circular functions in C_rPFS_tS enhances expressiveness and improves uncertainty modeling. This circular interpretation preserves the fundamental constraint $MD^2 + NMD^2 \leq 1$.
3. The recently suggested score and accuracy functions provide a more comprehensive method that shows remarkable efficacy for both C_rPFS_tS and C_rIFS_tS, surpassing the drawbacks of previous models.
4. The C_rPFS_tS environment allows for a more flexible and realistic depiction of uncertainty by modeling decision values using circular regions. With C_rPFS_tS, decision makers can express their preferences within a restricted zone, capturing both the MD and NMD in a circular form, as opposed to being given a single precise value. In MCDM scenarios, this deeper structure enhances the capacity to manage ambiguous, imprecise, and contradicting information.

The contributions of this article are given as follows:

1. By modeling decision values using circular regions, this study introduces a novel approach termed the C_rPFS_tS, which combines the concepts of C_rPFS and S_tS to provide a more flexible and realistic depiction of uncertainty.
2. This novel model's primary goal is to address decision-making issues when conventional models fall short because of high levels of uncertainty or complicated problem circumstances.
3. The article outlines the primary characteristics of C_rPFS_tS and emphasizes some of their possible uses. It describes algebraic operations that enable these sets to be joined logically, like intersection and union. Because of this, it is more adaptable and useful for resolving issues in everyday life.
4. The paper introduces an efficient transformation mechanism that enables the conversion of standard PFS_tS into C_rPFS_tS, facilitating seamless integration with existing data. This demonstrates that the incorporation of the C_rPFS_tS framework not only enhances the problem-solving capabilities of fuzzy soft set models but also significantly broadens their applicability to complex and cyclic decision-making scenarios.

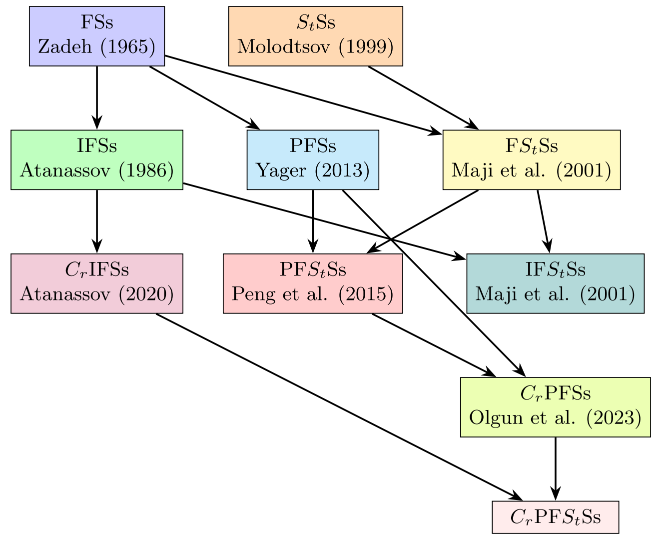

Fig. 1. The development of models leading to C_rPFS_tS.

5. C_rPFS_tS theory is introduced to address the limitations of existing models like IFS_tS and PFS_tS. IFS_tS restricts the sum of membership and non-membership to 1, limiting their ability to represent uncertainty. PFS_tS improves this with squared terms but still lack the ability to model circular or cyclic relationships in data. C_rPFS_tSs overcome these issues by incorporating a circular representation, allowing for more flexible and accurate modeling of hesitation, conflict, and ambiguity. This makes C_rPFS_tSs especially useful in complex decision-making scenarios where traditional models fall short.

To summarize, the development and enhancement of models that lead to C_rPFS_tS are shown in Fig. 1. Table 1 summarizes the abbreviations corresponding to models that are mentioned in this article.

1.2. Structure of the study

First, the definitions of PFS, PFS_tS, and C_rPFS are given in Section 2. In order for the readers to comprehend the origins and importance of C_rPFS_tS, this section gives the necessary background information too. In Section 3, C_rPFS_tS is formally defined and thoroughly examined, including its characteristics, mathematical operations, and attributes. The concepts behind the practical implementation of C_rPFS_tS are presented in Section 4, especially in situations that are complex and ambiguous. A decision-making algorithm is presented in Algorithm 1. It is then used in Section 4.2 in a circumstance taken from related literature, and it is compared to current decision-making techniques in Section 4.3. The sensitivity of the suggested algorithm is examined in Section 4.4. The nature of uncertainty, membership assignment, representation, applications, and all restrictions are covered in Section 5, which also examines this theoretical approach. This approach has also contrasted with earlier hybrid theories that link S_tS with different extended versions of FS. Conclusions, limits of the proposed work, and future research directions are covered in Section 6.

2. Preliminaries

This section recalls some basic definitions including PFS, PFS_tS and C_rPFS.

Definition 2.1 ([44]). Let $\mathfrak{V}$ be a universal set. A PFS denoted by $\mathcal{A}$ in $\mathfrak{V}$ is defined as:

$$\mathcal{A} = \{(\theta, \phi_{\mathcal{A}}(\theta), \psi_{\mathcal{A}}(\theta)) : \theta \in \mathfrak{V}\},$$

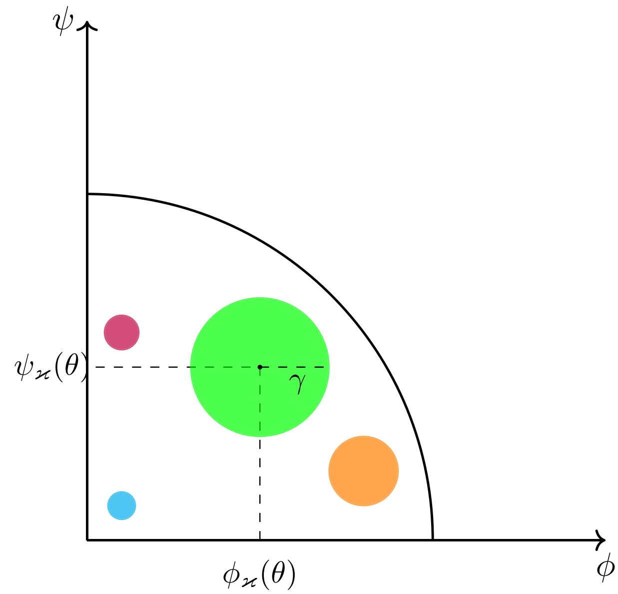

Fig. 2. Graphical visualization of a C_rPFS, with the elements defining one of its constituents C_rPFV, namely, $(\phi_{\varkappa}(\theta), \psi_{\varkappa}(\theta); \gamma)$.

where the mappings $\phi_{\mathcal{A}}, \psi_{\mathcal{A}} : \mathfrak{V} \to [0, 1]$ satisfy the condition:

$$0 \le (\phi_{\mathcal{A}}(\theta))^2 + (\psi_{\mathcal{A}}(\theta))^2 \le 1, \quad \forall \theta \in \mathfrak{V}.$$

Here, $\phi_{\mathcal{A}}(\theta)$ and $\psi_{\mathcal{A}}(\theta)$ represent the MD and NMD of the element θ in $\mathcal{A}$, respectively. The indeterminacy degree of θ is given by $\sqrt{1 - (\phi_{\mathcal{A}}(\theta))^2 - (\psi_{\mathcal{A}}(\theta))^2}$. Additionally, for any $\theta \in \mathfrak{V}$, each $(\phi_{\mathcal{A}}(\theta), \psi_{\mathcal{A}}(\theta))$ is a Pythagorean fuzzy value (PFV).

Definition 2.2 ([45]). Let $\mathcal{A} = (\phi, \psi)$ be a PFV, then its score and accuracy functions are defined as

$$\eta_{\mathcal{A}} = \phi^2 - \psi^2 \in [-1, 1].$$

$$\zeta_{\mathcal{A}} = \phi^2 + \psi^2 \in [0, 1].$$

Definition 2.3 ([25]). Let $\mathfrak{V}$ and $\mathfrak{P}$ be the universal set and set of parameters, respectively. Suppose $\mathfrak{I}^{\mathfrak{V}}$ denote the power set of $\mathfrak{V}$, representing the collection of all Pythagorean fuzzy subsets of $\mathfrak{V}$ and $\mathcal{A} \subseteq \mathfrak{P}$. A pair $(\Theta, \mathcal{A})$ is called a PFS_tS over $\mathfrak{V}$ if $\Theta : \mathcal{A} \to \mathfrak{I}^{\mathfrak{V}}$.

Definition 2.4 ([7]). Let $\mathfrak{V}$ be a universal set. A C_rPFS $\varkappa$ in $\mathfrak{V}$ is defined as:

$$\varkappa = \{(\theta, \phi_{\varkappa}(\theta), \psi_{\varkappa}(\theta); \gamma) : \theta \in \mathfrak{V}\},$$

where $\gamma \in [0, \sqrt{2}]$ indicates the radius of the circle whose center is located at the given point $(\phi_{\varkappa}(\theta), \psi_{\varkappa}(\theta))$ on the plane. The coordinates $(\phi_{\varkappa}(\theta), \psi_{\varkappa}(\theta))$ define mappings such that $\phi_{\varkappa}, \psi_{\varkappa} : \mathfrak{V} \to [0, 1]$, and they satisfy the condition:

$$0 \le (\phi_{\varkappa}(\theta))^2 + (\psi_{\varkappa}(\theta))^2 \le 1, \quad \forall\ \theta \in \mathfrak{V}.$$

Here, $\phi_{\varkappa}(\theta)$ and $\psi_{\varkappa}(\theta)$ represent the MD and NMD associated with the element θ in $\varkappa$, respectively.

In addition, for any C_rPFS $\varkappa$ and any $\theta \in \mathfrak{V}$, we say that $(\phi_{\varkappa}(\theta), \psi_{\varkappa}(\theta); \gamma)$ is a circular Pythagorean fuzzy value (C_rPFV) and $\pi_{\varkappa}(\theta) = \sqrt{1 - (\phi_{\varkappa}(\theta))^2 - (\psi_{\varkappa}(\theta))^2}$ is called the indeterminacy degree of θ with respect to $\varkappa$.

Graphically, a C_rPFS is illustrated in Fig. 2. A comparison between C_rIFS and C_rPFS is shown in Fig. 3.

The information represented in a PFS can be formally condensed into a C_rPFV, in the following manner proposed in [7]:

Table 1
Abbreviations and their descriptions.

Item	Description
FS	Fuzzy set
FSs	Fuzzy sets
MD	Membership degree
NMD	Non-membership degree
IFS	Intuitionistic fuzzy set
IFSs	Intuitionistic fuzzy sets
PFS	Pythagorean fuzzy set
PFSs	Pythagorean fuzzy sets
PFV	Pythagorean fuzzy value
PFVs	Pythagorean fuzzy values
FFS	Fermatean fuzzy set
FFSs	Fermatean fuzzy sets
IVIFS	Interval-valued intuitionistic fuzzy set
q-ROFS	Q-rung orthopair fuzzy set
DPFS	Disk Pythagorean fuzzy set
S_tS	Soft set
S_tSs	Soft sets
IFS_tS	Intuitionistic fuzzy soft set
IFS_tSs	Intuitionistic fuzzy soft sets
FS_tS	Fuzzy soft set
PFS_tS	Pythagorean fuzzy soft set
PFS_tSs	Pythagorean fuzzy soft sets
C_uPFS_tS	Cubic Pythagorean fuzzy soft set
FFS_tS	Fermatean fuzzy soft set
IVIFS_tS	Interval-valued intuitionistic fuzzy soft set
MCDM	Multi-criteria decision making
TOPSIS	Technique for order of preference by similarity to ideal solution
C_rIFS	Circular intuitionistic fuzzy set
C_rIFSs	Circular intuitionistic fuzzy sets
C_rIFV	Circular intuitionistic fuzzy value
C_rIFVs	Circular intuitionistic fuzzy values
C_rPFS	Circular Pythagorean fuzzy set
C_rPFSs	Circular Pythagorean fuzzy sets
C_rPFV	Circular Pythagorean fuzzy value
C_rPFVs	Circular Pythagorean fuzzy values
C_rFFS	Circular Fermatean fuzzy set
C_rPFS_tS	Circular Pythagorean fuzzy soft set
C_rPFS_tSs	Circular Pythagorean fuzzy soft sets
C_rPFS_tV	Circular Pythagorean fuzzy soft value
C_rPFS_tVs	Circular Pythagorean fuzzy soft values

Definition 2.5 ([7]). Suppose that a PFS $\mathcal{A}$ in $\mathfrak{V}$ is given by

$$\mathcal{A} = \{(\theta_1, \phi_{\mathcal{A}}(\theta_1) = \phi_1, \psi_{\mathcal{A}}(\theta_1) = \psi_1), (\theta_2, \phi_{\mathcal{A}}(\theta_2) = \phi_2, \psi_{\mathcal{A}}(\theta_2) = \psi_2), (\theta_3, \phi_{\mathcal{A}}(\theta_3) = \phi_3, \psi_{\mathcal{A}}(\theta_3) = \psi_3), \dots, (\theta_k, \phi_{\mathcal{A}}(\theta_k) = \phi_k, \psi_{\mathcal{A}}(\theta_k) = \psi_k) : \theta_1, \theta_2, \theta_3, \dots, \theta_k \in \mathfrak{V}\}.$$

For simplicity, we represent its corresponding PFVs as

$$\{(\phi_1, \psi_1), (\phi_2, \psi_2), (\phi_3, \psi_3), \dots, (\phi_k, \psi_k)\}.$$

Then a C_rPFV $(\phi, \psi; \gamma)$ is defined from $\mathcal{A}$ by the following parameters:

$$\phi = \sqrt{\frac{\sum_{i=1}^{k} \phi_i^2}{k}}, \quad \psi = \sqrt{\frac{\sum_{i=1}^{k} \psi_i^2}{k}}, \text{ and}$$

$$\gamma = \min\left\{ \max_{1 \le i \le k} \sqrt{(\phi - \phi_i)^2 + (\psi - \psi_i)^2}, \sqrt{2} \right\}.$$

3. Circular Pythagorean fuzzy soft set

The present section introduces the new setting investigated in this paper and discusses its interpretation.

Definition 3.1. Let $\mathfrak{V}$ and $\mathfrak{P}$ represent the universal set alongside its associated parameter domain, respectively. Let $\mathfrak{I}^{\mathfrak{V}}$ denote the power set of $\mathfrak{V}$, that is $\mathfrak{I}^{\mathfrak{V}}$ is the set of all C_rPF subsets of $\mathfrak{V}$. Fix $\varkappa \subseteq \mathfrak{P}$. A pair $(\Theta, \varkappa)$ is a circular Pythagorean fuzzy soft set (C_rPFS_tS) over $\mathfrak{V}$ where $\Theta : \varkappa \to \mathfrak{I}^{\mathfrak{V}}$. In other words, a C_rPFS_tS means a way of associating subsets of $\mathfrak{V}$ with respect to parameters in $\varkappa$. When it comes to each parameter $\epsilon \in \varkappa$ and each $\theta \in \mathfrak{V}$, $\Theta(\epsilon)$ describes those elements of $\mathfrak{V}$ that are close to the desired parameter ϵ. We write the circular Pythagorean fuzzy soft value (C_rPFS_tV) $\Theta(\epsilon)$ as

$$\Theta(\epsilon) = \{(\theta, \phi_{\Theta(\theta)}, \psi_{\Theta(\theta)}; \gamma) \mid \theta \in \mathfrak{V}\}$$

where $\phi_{\Theta(\theta)}$ is known as the circular-valued fuzzy MD of the element θ with respect to the parameter ϵ, and $\psi_{\Theta(\theta)}$ is known as the circular-valued fuzzy NMD of the element θ with respect to the parameter ϵ, when we assume a radius $\gamma \in [0, \sqrt{2}]$.

There are two general types of uncertainty in this formal model: fuzziness and imprecision. In classical set theory, an element either belongs to a set (we say that its membership to the set is 1) or does not belong to the set (and then we say that its membership is 0). In fuzzy set theory, fuzziness allows for partial memberships, which means

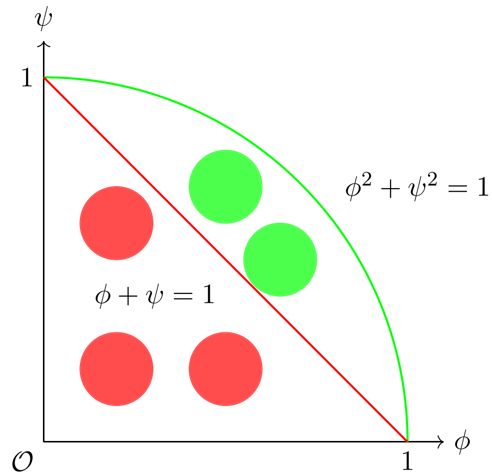

Fig. 3. Comparison of C_rIFS with C_rPFS.

that an element can belong to a set having a degree of membership between 0 and 1. Imprecision refers to the uncertainty or slackness in explaining the exact membership degree of an element in a fuzzy set. It shows decision maker's lack of confidence in the precise value of membership, often arising from errors of measurements, ambiguity or incomplete information. This setback repeats itself in the more convenient PFS model discussed above. While standard PFSs handle fuzziness through a restriction on the squared membership grades of the alternatives, they lack a mechanism to represent this evaluative imprecision. The proposed circular Pythagorean fuzzy soft set model directly addresses this gap while introducing a multi-attribute point of view. The circular region around the central point explicitly models imprecision by allowing the true value to lie anywhere within a radius, thus capturing the slackness in the expert's evaluation. Furthermore, the soft set parameterization handles the fuzziness and imprecision associated with multiple attributes or perspectives on the data.

We illustrate the elements in this new concept with the following situation.

Example 3.1. Let us consider a $C_r\text{PF}S_t\text{S}$ $(\Theta, \varkappa)$, where $\mathfrak{U}$ is a set of three cars under discussion for purchase, denoted by $\mathfrak{U} = \{w_1, w_2, w_3\}$, and $\varkappa$ is the set of parameters, where $\varkappa = \{\theta_1, \theta_2, \theta_3, \theta_4\}$ = {expensive, fuel-efficient, good suspension, accidental}. For the decision-maker, the $C_r\text{PF}S_t\text{S}$ $(\Theta, \varkappa)$ characterizes the "attractiveness of the cars". Suppose for illustration that the parameterized values are as follows:

$$\Theta(\theta_1) = \{(w_1, 0.3, 0.8; 0.2), (w_2, 0.6, 0.4; 0.2), (w_3, 0.51, 0.69; 0.2)\}$$

$$\Theta(\theta_2) = \{(w_1, 0.11, 0.25; 0.3), (w_2, 0.5, 0.2; 0.3), (w_3, 0.12, 0.6; 0.3)\}$$

$$\Theta(\theta_3) = \{(w_1, 0.8, 0.1; 0.17), (w_2, 0.8, 0.3; 0.17), (w_3, 0.5, 0.7; 0.17)\}$$

$$\Theta(\theta_4) = \{(w_1, 0.73, 0.4; 0.21), (w_2, 0.1, 0.4; 0.21), (w_3, 0.8, 0.3; 0.21)\}$$

The $C_r\text{PF}S_t\text{S}$ $(\Theta, \varkappa)$ can be regarded as a parameterized collection of C_rPFSs on $\mathfrak{U}$. We can summarize it by the following descriptions, one for each parameter:

Expensive cars: $\{(w_1, 0.3, 0.8; 0.2), (w_2, 0.6, 0.4; 0.2), (w_3, 0.51, 0.69; 0.2)\}$,

Fuel-efficient cars: $\{(w_1, 0.11, 0.25; 0.3), (w_2, 0.5, 0.2; 0.3), (w_3, 0.12, 0.6; 0.3)\}$,

Good suspension cars: $\{(w_1, 0.8, 0.1; 0.17), (w_2, 0.8, 0.3; 0.17), (w_3, 0.5, 0.7; 0.17)\}$,

Accidental cars: $\{(w_1, 0.73, 0.4; 0.21), (w_2, 0.1, 0.4; 0.21), (w_3, 0.8, 0.3; 0.21)\}$.

If we have a $C_r\text{PF}S_t\text{S}$ $(\Theta, \varkappa)$ over $\mathfrak{U}$ and $\Theta(\epsilon)$ is a C_rPFS value set (C_rPFV) for the parameter ϵ, we can define a class C_rPFV of $(\Theta, \varkappa)$, denoted by $C_{(\Theta,\varkappa)}$. This is given as:

$$C_{(\Theta,\varkappa)} = \{\Theta(\epsilon) \mid \epsilon \in \varkappa\}.$$

This class $C_{(\Theta,\varkappa)}$ contains all C_rPFVs associated with the parameters in $\varkappa$ by the $C_r\text{PF}S_t\text{S}$.

To compare two $C_r\text{PF}S_t\text{Ss}$, we can utilize the following definitions provided below:

Definition 3.2. Let $\mathfrak{U}$ and $\varkappa$ represent the universal set alongside its associated parameter domain respectively. Let $\mathfrak{X}, \mathfrak{Y} \subseteq \varkappa$, and let

$$(\Theta_1, \mathfrak{X}) = \{(v, \phi_{\Theta_1(\theta)}(v), \psi_{\Theta_1(\theta)}(v); \gamma_{\Theta_1(\theta)}) \mid v \in \mathfrak{U}, \theta \in \mathfrak{X}\}$$

and

$$(\Theta_2, \mathfrak{Y}) = \{(v, \phi_{\Theta_2(q)}(v), \psi_{\Theta_2(q)}(v); \gamma_{\Theta_2(q)}) \mid v \in \mathfrak{U}, q \in \mathfrak{Y}\}$$

be two $C_r\text{PF}S_t\text{Ss}$. Then, we declare $(\Theta_1, \mathfrak{X}) \subset_v (\Theta_2, \mathfrak{Y})$ if and only if the following conditions hold true:

1. $\mathfrak{X} \subseteq \mathfrak{Y}$.
2. $\forall\ \theta \in \mathfrak{X}$, $\Theta_1(\theta) \subset_v \Theta_2(\theta)$. That is, for all $v \in \mathfrak{U}$ and $\theta \in \mathfrak{X}$, the following conditions are fulfilled:
   (a) $\gamma_{\Theta_1(\theta)} = \gamma_{\Theta_2(\theta)}$.
   (b) One of the following holds:
      i. $\phi_{\Theta_1(\theta)}(v) < \phi_{\Theta_2(\theta)}(v)$ and $\psi_{\Theta_1(\theta)}(v) \geq \psi_{\Theta_2(\theta)}(v)$.
      ii. $\phi_{\Theta_1(\theta)}(v) \leq \phi_{\Theta_2(\theta)}(v)$ and $\psi_{\Theta_1(\theta)}(v) > \psi_{\Theta_2(\theta)}(v)$.
      iii. $\phi_{\Theta_1(\theta)}(v) < \phi_{\Theta_2(\theta)}(v)$ and $\psi_{\Theta_1(\theta)}(v) > \psi_{\Theta_2(\theta)}(v)$.

Definition 3.3. Let $\mathfrak{U}$ and $\mathfrak{P}$ represent the universal set alongside its associated parameter domain respectively. Let $\mathfrak{X}, \mathfrak{Y} \subseteq \varkappa$, and define two $C_r\text{PF}S_t\text{Ss}$ as:

$$(\Theta_1, \mathfrak{X}) = \{(v, \phi_{\Theta_1(\theta)}(v), \psi_{\Theta_1(\theta)}(v); \gamma_{\Theta_1(\theta)}) \mid v \in \mathfrak{U}, \theta \in \mathfrak{X}\},$$

$$(\Theta_2, \mathfrak{Y}) = \{(v, \phi_{\Theta_2(q)}(v), \psi_{\Theta_2(q)}(v); \gamma_{\Theta_2(q)}) \mid v \in \mathfrak{U}, q \in \mathfrak{Y}\}.$$

Then, we declare $(\Theta_1, \mathfrak{X}) \subseteq_v (\Theta_2, \mathfrak{Y})$ if and only if the following conditions are fulfilled:

1. $\mathfrak{X} \subseteq \mathfrak{Y}$.
2. $\forall\ \theta \in \mathfrak{X}$, $\Theta_1(\theta) \subseteq_v \Theta_2(\theta)$, i.e., $\forall v \in \mathfrak{U}$ and $\theta \in \mathfrak{X}$:
   (a) $\gamma_{\Theta_1(\theta)} = \gamma_{\Theta_2(\theta)}$,
   (b) $\phi_{\Theta_1(\theta)}(v) \leq \phi_{\Theta_2(\theta)}(v)$,
   (c) $\psi_{\Theta_1(\theta)}(v) \geq \psi_{\Theta_2(\theta)}(v)$.

Definition 3.4. Let $\mathfrak{U}$ and $\mathfrak{P}$ represent the universal set alongside its associated parameter domain respectively. Let $\mathfrak{X}, \mathfrak{Y} \subseteq \varkappa$, and define two $C_r\text{PF}S_t\text{Ss}$ as:

$$(\Theta_1, \mathfrak{X}) = \{(v, \phi_{\Theta_1(\theta)}(v), \psi_{\Theta_1(\theta)}(v); \gamma_{\Theta_1(\theta)}) \mid v \in \mathfrak{U}, \theta \in \mathfrak{X}\},$$

$$(\Theta_2, \mathfrak{Y}) = \{(v, \phi_{\Theta_2(q)}(v), \psi_{\Theta_2(q)}(v); \gamma_{\Theta_2(q)}) \mid v \in \mathfrak{U}, q \in \mathfrak{Y}\}.$$

Then, $(\Theta_1, \mathfrak{X}) \subset_\rho (\Theta_2, \mathfrak{Y})$ if and only if the following conditions are fulfilled:

1. $\mathfrak{X} \subseteq \mathfrak{Y}$.
2. $\forall \theta \in \mathfrak{X}$, $\Theta_1(\theta) \subset_\rho \Theta_2(\theta)$, i.e., $\forall v \in \mathfrak{U}$ and $\theta \in \mathfrak{X}$:
   (a) $\gamma_{\Theta_1(\theta)} < \gamma_{\Theta_2(\theta)}$,
   (b) $\phi_{\Theta_1(\theta)}(v) = \phi_{\Theta_2(\theta)}(v)$,
   (c) $\psi_{\Theta_1(\theta)}(v) = \psi_{\Theta_2(\theta)}(v)$.

Definition 3.5. Let $\mathfrak{V}$ and $\mathfrak{P}$ represent the universal set alongside its associated parameter domain respectively. Let $\mathfrak{X}, \mathfrak{Y} \subseteq \varkappa$, and define two C_rPFS_tSs as:

$$(\Theta_1, \mathfrak{X}) = \{(v, \phi_{\Theta_1(\theta)}(v), \psi_{\Theta_1(\theta)}(v); \gamma_{\Theta_1(\theta)}) \mid v \in \mathfrak{V}, \theta \in \mathfrak{X}\},$$

$$(\Theta_2, \mathfrak{Y}) = \{(v, \phi_{\Theta_2(q)}(v), \psi_{\Theta_2(q)}(v); \gamma_{\Theta_2(q)}) \mid v \in \mathfrak{V}, q \in \mathfrak{Y}\}.$$

Then, $(\Theta_1, \mathfrak{X}) \subseteq_\rho (\Theta_2, \mathfrak{Y})$ if and only if the following conditions are fulfilled:

1. $\mathfrak{X} \subseteq \mathfrak{Y}$.
2. $\forall\, \theta \in \mathfrak{X}$, $\Theta_1(\theta) \subseteq_\rho \Theta_2(\theta)$, i.e., $\forall v \in \mathfrak{V}$ and $\theta \in \mathfrak{X}$:
   (a) $\gamma_{\Theta_1(\theta)} \leq \gamma_{\Theta_2(\theta)}$,
   (b) $\phi_{\Theta_1(\theta)}(v) = \phi_{\Theta_2(\theta)}(v)$,
   (c) $\psi_{\Theta_1(\theta)}(v) = \psi_{\Theta_2(\theta)}(v)$.

Definition 3.6. Let $\mathfrak{V}$ and $\mathfrak{P}$ represent the universal set alongside its associated parameter domain respectively. Let $\mathfrak{X}, \mathfrak{Y} \subseteq \varkappa$, and define two C_rPFS_tSs as:

$$(\Theta_1, \mathfrak{X}) = \{(v, \phi_{\Theta_1(\theta)}(v), \psi_{\Theta_1(\theta)}(v); \gamma_{\Theta_1(\theta)}) \mid v \in \mathfrak{V}, \theta \in \mathfrak{X}\},$$

$$(\Theta_2, \mathfrak{Y}) = \{(v, \phi_{\Theta_2(q)}(v), \psi_{\Theta_2(q)}(v); \gamma_{\Theta_2(q)}) \mid v \in \mathfrak{V}, q \in \mathfrak{Y}\}.$$

Then, $(\Theta_1, \mathfrak{X}) \subset (\Theta_2, \mathfrak{Y})$ if and only if the following conditions are fulfilled:

1. $\mathfrak{X} \subseteq \mathfrak{Y}$.
2. $\forall\, \theta \in \mathfrak{X}$, $\Theta_1(\theta) \subset \Theta_2(\theta)$, i.e., $\forall\, v \in \mathfrak{V}$ and $\theta \in \mathfrak{X}$:
   (a) $\gamma_{\Theta_1(\theta)} < \gamma_{\Theta_2(\theta)}$, and
   (b) One of the following holds:
      i. $\phi_{\Theta_1(\theta)}(v) < \phi_{\Theta_2(\theta)}(v)$ and $\psi_{\Theta_1(\theta)}(v) \geq \psi_{\Theta_2(\theta)}(v)$,
      ii. $\phi_{\Theta_1(\theta)}(v) \leq \phi_{\Theta_2(\theta)}(v)$ and $\psi_{\Theta_1(\theta)}(v) > \psi_{\Theta_2(\theta)}(v)$,
      iii. $\phi_{\Theta_1(\theta)}(v) < \phi_{\Theta_2(\theta)}(v)$ and $\psi_{\Theta_1(\theta)}(v) > \psi_{\Theta_2(\theta)}(v)$.

Definition 3.7. Let $\mathfrak{V}$ and $\mathfrak{P}$ represent the universal set alongside its associated parameter domain respectively. Suppose $\mathfrak{X}, \mathfrak{Y} \subseteq \varkappa$, and define two C_rPFS_tSs as:

$$(\Theta_1, \mathfrak{X}) = \{(v, \phi_{\Theta_1(\theta)}(v), \psi_{\Theta_1(\theta)}(v); \gamma_{\Theta_1(\theta)}) \mid v \in \mathfrak{V}, \theta \in \mathfrak{X}\},$$

$$(\Theta_2, \mathfrak{Y}) = \{(v, \phi_{\Theta_2(q)}(v), \psi_{\Theta_2(q)}(v); \gamma_{\Theta_2(q)}) \mid v \in \mathfrak{V}, q \in \mathfrak{Y}\}.$$

Then, $(\Theta_1, \mathfrak{X}) \subseteq (\Theta_2, \mathfrak{Y})$ if and only if the following conditions are fulfilled:

1. $\mathfrak{X} \subseteq \mathfrak{Y}$.
2. $\forall\, \theta \in \mathfrak{X}$, $\Theta_1(\theta) \subseteq \Theta_2(\theta)$, i.e., $\forall\, v \in \mathfrak{V}$ and $\theta \in \mathfrak{X}$:
   (a) $\gamma_{\Theta_1(\theta)} \leq \gamma_{\Theta_2(\theta)}$,
   (b) $\phi_{\Theta_1(\theta)}(v) \leq \phi_{\Theta_2(\theta)}(v)$,
   (c) $\psi_{\Theta_1(\theta)}(v) \geq \psi_{\Theta_2(\theta)}(v)$.

Definition 3.8. Suppose $\mathfrak{V}$ and $\mathfrak{P}$ represent the universal set alongside its associated parameter domain respectively. Let $\mathfrak{X}, \mathfrak{Y} \subseteq \varkappa$, and define two C_rPFS_tSs as:

$$(\Theta_1, \mathfrak{X}) = \{(v, \phi_{\Theta_1(\theta)}(v), \psi_{\Theta_1(\theta)}(v); \gamma_{\Theta_1(\theta)}) \mid v \in \mathfrak{V}, \theta \in \mathfrak{X}\},$$

$$(\Theta_2, \mathfrak{Y}) = \{(v, \phi_{\Theta_2(q)}(v), \psi_{\Theta_2(q)}(v); \gamma_{\Theta_2(q)}) \mid v \in \mathfrak{V}, q \in \mathfrak{Y}\}.$$

Then, $(\Theta_1, \mathfrak{X}) =_v (\Theta_2, \mathfrak{Y})$ if and only if the following conditions are fulfilled:

1. $\mathfrak{X} = \mathfrak{Y}$.
2. $\forall\, \theta \in \mathfrak{X}$, $\Theta_1(\theta) =_v \Theta_2(\theta)$, i.e., $\forall\, v \in \mathfrak{V}$ and $\theta \in \mathfrak{X}$:
   (a) $\phi_{\Theta_1(\theta)}(v) = \phi_{\Theta_2(\theta)}(v)$,
   (b) $\psi_{\Theta_1(\theta)}(v) = \psi_{\Theta_2(\theta)}(v)$.

Definition 3.9. Suppose $\mathfrak{V}$ and $\mathfrak{P}$ represent the universal set alongside its associated parameter domain respectively. Let $\mathfrak{X}, \mathfrak{Y} \subseteq \varkappa$, and define two C_rPFS_tSs as:

$$(\Theta_1, \mathfrak{X}) = \{(v, \phi_{\Theta_1(\theta)}(v), \psi_{\Theta_1(\theta)}(v); \gamma_{\Theta_1(\theta)}) \mid v \in \mathfrak{V}, \theta \in \mathfrak{X}\},$$

$$(\Theta_2, \mathfrak{Y}) = \{(v, \phi_{\Theta_2(q)}(v), \psi_{\Theta_2(q)}(v); \gamma_{\Theta_2(q)}) \mid v \in \mathfrak{V}, q \in \mathfrak{Y}\}.$$

Then, $(\Theta_1, \mathfrak{X}) =_\rho (\Theta_2, \mathfrak{Y})$ if and only if the following conditions are fulfilled:

1. $\mathfrak{X} = \mathfrak{Y}$.
2. $\forall\, \theta \in \mathfrak{X}$, $\gamma_{\Theta_1(\theta)} = \gamma_{\Theta_2(\theta)}$.

Definition 3.10. Suppose $\mathfrak{V}$ and $\mathfrak{P}$ represent the universal set alongside its associated parameter domain respectively. Let $\mathfrak{X}, \mathfrak{Y} \subseteq \varkappa$, and define two C_rPFS_tSs as:

$$(\Theta_1, \mathfrak{X}) = \{(v, \phi_{\Theta_1(\theta)}(v), \psi_{\Theta_1(\theta)}(v); \gamma_{\Theta_1(\theta)}) \mid v \in \mathfrak{V}, \theta \in \mathfrak{X}\},$$

$$(\Theta_2, \mathfrak{Y}) = \{(v, \phi_{\Theta_2(q)}(v), \psi_{\Theta_2(q)}(v); \gamma_{\Theta_2(q)}) \mid v \in \mathfrak{V}, q \in \mathfrak{Y}\}.$$

Then, $(\Theta_1, \mathfrak{X}) = (\Theta_2, \mathfrak{Y})$ if and only if the following conditions are fulfilled:

1. $(\Theta_1, \mathfrak{X}) =_v (\Theta_2, \mathfrak{Y})$,
2. $(\Theta_1, \mathfrak{X}) =_\rho (\Theta_2, \mathfrak{Y})$.

Once the C_rPFS_tS model has been put forward, explaining how it can be manipulated for applications is equally essential. The following definitions outline the operations and procedures for effectively using C_rPFS_tSs in decision-making tasks.

Definition 3.11. Given $\mathfrak{P} = \{\theta_1, \theta_2, \theta_3, \ldots, \theta_k\}$ as the set of parameters, the negation of $\mathfrak{P}$, denoted by $\sim \mathfrak{P}$, is defined as:

$$\sim \mathfrak{P} = \{\sim \theta_1, \sim \theta_2, \sim \theta_3, \ldots, \sim \theta_k\}.$$

Definition 3.12. Suppose $\mathfrak{V}$ represents the universal set, and let $(\Theta, \varkappa)$ be a C_rPFS$_t$S. The complement of $(\Theta, \varkappa)$, denoted by $(\Theta, \varkappa)^c$, is defined as:

$$(\Theta, \varkappa)^c = (\Theta^c, \sim \varkappa),$$

where the mapping $\Theta^c : \sim \varkappa \to C_r\text{PFS}(\mathfrak{V})$ is given by:

$$\Theta^c(\epsilon) = \left\{ \left(v, \psi_{\Theta(\sim\epsilon)}(v), \phi_{\Theta(\sim\epsilon)}(v); \gamma_{\Theta(\sim\epsilon)}\right) \mid v \in \mathfrak{V} \right\}, \quad \forall \epsilon \in \sim \varkappa.$$

We illustrate previous definitions with the next example:

Example 3.2. Given that $(\Theta_1, \varkappa_1)$, $(\Theta_2, \varkappa_2)$, $(\Theta_3, \varkappa_3)$, $(\Theta_4, \varkappa_3)$, and $(\Theta_5, \varkappa_1)$ are C_rPFS_tSs over the universal set $\mathfrak{V} = \{w_1, w_2, w_3, \ldots, w_6\}$, where $\mathfrak{V}$ is the set of cars, the parameter sets are as follows:

$$\varkappa_1 = \{\theta_1, \theta_2\} = \{\text{fuel efficient, expensive}\},$$

$$\varkappa_2 = \{\theta_1, \theta_2, \theta_3\} = \{\text{fuel efficient, expensive, good suspension}\},$$

$$\varkappa_3 = \{\theta_1, \theta_3\} = \{\text{fuel efficient, good suspension}\}.$$

The values of the C_rPFS_tSs are given in Table 2.

From Table 2, we clearly see the inclusion $\varkappa_1, \varkappa_3 \subseteq \varkappa_2$. We conclude the following facts:

$$(\Theta_3, \varkappa_3) \subset_v (\Theta_2, \varkappa_2), \quad (\Theta_1, \varkappa_1) \subseteq_v (\Theta_2, \varkappa_2), \quad (\Theta_1, \varkappa_1) \subset_\rho (\Theta_2, \varkappa_2),$$

Table 2
The C_rPFS_tS used in Example 3.2.

	$(\Theta_1, \varkappa_1)$		$(\Theta_2, \varkappa_2)$			$(\Theta_3, \varkappa_3)$		$(\Theta_4, \varkappa_3)$		$(\Theta_5, \varkappa_1)$	
	θ_1	θ_2	θ_1	θ_2	θ_3	θ_1	θ_3	θ_1	θ_3	θ_1	θ_2
w_1	(0.3, 0.3; 0.3)	(0.6, 0.15; 0.35)	(0.3, 0.3; 0.35)	(0.6, 0.15; 0.4)	(0.5, 0.13; 0.21)	(0.2, 0.4; 0.35)	(0.3, 0.15; 0.22)	(0.2, 0.4; 0.3)	(0.3, 0.15; 0.2)	(0.35, 0.3; 0.3)	(0.6, 0.15; 0.35)
w_2	(0.5, 0.25; 0.3)	(0.5, 0.15; 0.35)	(0.5, 0.25; 0.35)	(0.5, 0.15; 0.4)	(0.55, 0.22; 0.21)	(0.1, 0.4; 0.35)	(0.5, 0.15; 0.22)	(0.1, 0.4; 0.3)	(0.5, 0.15; 0.2)	(0.2, 0.4; 0.3)	(0.5, 0.25; 0.35)
w_3	(0.45, 0.2; 0.3)	(0.4, 0.1; 0.35)	(0.45, 0.2; 0.35)	(0.4, 0.1; 0.4)	(0.33, 0.4; 0.21)	(0.4, 0.25; 0.35)	(0, 0.6; 0.21)	(0.4, 0.25; 0.3)	(0, 0.6; 0.2)	(0.15, 0.29; 0.3)	(0.25, 0.18; 0.35)
w_4	(0.55, 0.11; 0.3)	(0.65, 0.07; 0.35)	(0.55, 0.11; 0.35)	(0.65, 0.07; 0.4)	(0.67, 0.02; 0.21)	(0.4, 0.21; 0.35)	(0.65, 0.07; 0.21)	(0.4, 0.21; 0.3)	(0.65, 0.07; 0.2)	(0.5, 0.11; 0.3)	(0.4, 0.17; 0.35)
w_5	(0.46, 0.2; 0.3)	(0.5, 0.25; 0.35)	(0.46, 0.2; 0.35)	(0.5, 0.25; 0.4)	(0.56, 0.06; 0.21)	(0.2, 0.5; 0.35)	(0.5, 0.25; 0.21)	(0.2, 0.5; 0.3)	(0.5, 0.25; 0.2)	(0.6, 0.1; 0.3)	(0.5, 0.25; 0.35)
w_6	(0.72, 0.01; 0.3)	(0.75, 0.02; 0.35)	(0.72, 0.01; 0.35)	(0.75, 0.02; 0.4)	(0.1, 0.26; 0.21)	(0.1, 0.31; 0.35)	(0.04, 0.32; 0.21)	(0.1, 0.31; 0.3)	(0.04, 0.32; 0.2)	(0.7, 0; 0.3)	(0.75, 0.02; 0.35)

Table 3
The complement of $(\Theta_1, \varkappa_1)$ in Example 3.2.

$\mathfrak{V}$	θ_1	θ_2
w_1	(0.3, 0.3; 0.3)	(0.15, 0.6; 0.35)
w_2	(0.25, 0.5; 0.3)	(0.15, 0.5; 0.35)
w_3	(0.2, 0.45; 0.3)	(0.1, 0.4; 0.35)
w_4	(0.11, 0.55; 0.3)	(0.07, 0.65; 0.35)
w_5	(0.2, 0.46; 0.3)	(0.25, 0.5; 0.35)
w_6	(0.01, 0.72; 0.3)	(0.02, 0.75; 0.35)

$$(\Theta_1, \varkappa_1) \subseteq_\rho (\Theta_2, \varkappa_2), \quad (\Theta_4, \varkappa_3) \subset (\Theta_2, \varkappa_2), \quad (\Theta_4, \varkappa_3) \subseteq (\Theta_2, \varkappa_2),$$

$$(\Theta_4, \varkappa_3) =_\upsilon (\Theta_2, \varkappa_2), \quad (\Theta_1, \varkappa_1) =_\rho (\Theta_5, \varkappa_1).$$

The complement of $(\Theta_1, \varkappa_1)$ is given in Table 3.

The ideas of null and whole C_rPFS_tS are defined as follows:

Definition 3.13. The null C_rPFS_tS is denoted by Φ over the universal set $\mathfrak{V}$ and is defined as:

$$\forall\ \epsilon \in \varkappa, \quad \phi_{\Phi(\epsilon)}(v) = 0, \quad \psi_{\Phi(\epsilon)}(v) = 1, \quad \gamma_{\Phi(\epsilon)} = 0, \quad \forall\ v \in \mathfrak{V}.$$

Definition 3.14. The whole C_rPFS_tS is denoted by Δ over the universal set $\mathfrak{V}$ and is defined as:

$$\forall\ \epsilon \in \varkappa, \quad \phi_{\Delta(\epsilon)}(v) = 1, \quad \psi_{\Delta(\epsilon)}(v) = 0, \quad \gamma_{\Delta(\epsilon)} = 1, \quad \forall\ v \in \mathfrak{V}.$$

We proceed to develop various logical operations between C_rPFS_tSs.

Definition 3.15. Define $\mathfrak{V}$ as the universal set and $\mathfrak{P}$ as a set of parameters. Let $\mathfrak{X}, \mathfrak{Y} \subseteq \mathfrak{P}$, and consider two C_rPFS_tSs:

$$(\Theta_1, \mathfrak{X}) = \left\{ (v, \phi_{\Theta_1(\theta)}(v), \psi_{\Theta_1(\theta)}(v); \gamma_{\Theta_1(\theta)}) \mid v \in \mathfrak{V},\ \theta \in \mathfrak{X} \right\},$$

$$(\Theta_2, \mathfrak{Y}) = \left\{ (v, \phi_{\Theta_2(q)}(v), \psi_{\Theta_2(q)}(v); \gamma_{\Theta_2(q)}) \mid v \in \mathfrak{V},\ q \in \mathfrak{Y} \right\}.$$

The min-AND operation of $(\Theta_1, \mathfrak{X})$ and $(\Theta_2, \mathfrak{Y})$, denoted by $(\Theta_1, \mathfrak{X}) \bigwedge_{\min} (\Theta_2, \mathfrak{Y})$, is defined as another C_rPFS_tS:

$$(\Theta_1, \mathfrak{X}) \bigwedge_{\min} (\Theta_2, \mathfrak{Y}) = (\Theta', \mathfrak{X} \times \mathfrak{Y}),$$

where

$$\Theta'(\theta, q) = \Theta_1(\theta) \cap_{\min} \Theta_2(q), \quad \forall (\theta, q) \in \mathfrak{X} \times \mathfrak{Y}.$$

Specifically,

$$\Theta'(\theta, q)(v) = \Big(v, \min\big(\phi_{\Theta_1(\theta)}(v), \phi_{\Theta_2(q)}(v)\big), \max\big(\psi_{\Theta_1(\theta)}(v), \psi_{\Theta_2(q)}(v)\big), \min\big(\gamma_{\Theta_1(\theta)}, \gamma_{\Theta_2(q)}\big) \Big), \quad \forall\ v \in \mathfrak{V}.$$

Definition 3.16. Define $\mathfrak{V}$ as the universal set and $\mathfrak{P}$ as a set of parameters. Let $\mathfrak{X}, \mathfrak{Y} \subseteq \mathfrak{P}$, and consider two C_rPFS_tSs:

$$(\Theta_1, \mathfrak{X}) = \left\{ (v, \phi_{\Theta_1(\theta)}(v), \psi_{\Theta_1(\theta)}(v); \gamma_{\Theta_1(\theta)}) \mid v \in \mathfrak{V},\ \theta \in \mathfrak{X} \right\},$$

$$(\Theta_2, \mathfrak{Y}) = \left\{ (v, \phi_{\Theta_2(q)}(v), \psi_{\Theta_2(q)}(v); \gamma_{\Theta_2(q)}) \mid v \in \mathfrak{V},\ q \in \mathfrak{Y} \right\}.$$

The max-AND operation of $(\Theta_1, \mathfrak{X})$ and $(\Theta_2, \mathfrak{Y})$, denoted by $(\Theta_1, \mathfrak{X}) \bigwedge_{\max} (\Theta_2, \mathfrak{Y})$, is defined as another C_rPFS_tS:

$$(\Theta_1, \mathfrak{X}) \bigwedge_{\max} (\Theta_2, \mathfrak{Y}) = (\Theta', \mathfrak{X} \times \mathfrak{Y}),$$

where

$$\Theta'(\theta, q) = \Theta_1(\theta) \cap_{\max} \Theta_2(q), \quad \forall (\theta, q) \in \mathfrak{X} \times \mathfrak{Y}.$$

Specifically,

$$\Theta'(\theta, q)(v) = \Big(v, \min\big(\phi_{\Theta_1(\theta)}(v), \phi_{\Theta_2(q)}(v)\big), \max\big(\psi_{\Theta_1(\theta)}(v), \psi_{\Theta_2(q)}(v)\big), \max\big(\gamma_{\Theta_1(\theta)}, \gamma_{\Theta_2(q)}\big) \Big), \quad \forall\ v \in \mathfrak{V}.$$

Definition 3.17. Define $\mathfrak{V}$ as the universal set and $\mathfrak{P}$ as a set of parameters. Let $\mathfrak{X}, \mathfrak{Y} \subseteq \mathfrak{P}$, and consider two C_rPFS_tSs:

$$(\Theta_1, \mathfrak{X}) = \left\{ (v, \phi_{\Theta_1(\theta)}(v), \psi_{\Theta_1(\theta)}(v); \gamma_{\Theta_1(\theta)}) \mid v \in \mathfrak{V},\ \theta \in \mathfrak{X} \right\},$$

$$(\Theta_2, \mathfrak{Y}) = \left\{ (v, \phi_{\Theta_2(q)}(v), \psi_{\Theta_2(q)}(v); \gamma_{\Theta_2(q)}) \mid v \in \mathfrak{V},\ q \in \mathfrak{Y} \right\}.$$

The min-OR operation of $(\Theta_1, \mathfrak{X})$ and $(\Theta_2, \mathfrak{Y})$, denoted by $(\Theta_1, \mathfrak{X}) \bigvee_{\min} (\Theta_2, \mathfrak{Y})$, is defined as another C_rPFS_tS:

$$(\Theta_1, \mathfrak{X}) \bigvee_{\min} (\Theta_2, \mathfrak{Y}) = (\Theta', \mathfrak{X} \times \mathfrak{Y}),$$

where

$$\Theta'(\theta, q) = \Theta_1(\theta) \cup_{\min} \Theta_2(q), \quad \forall (\theta, q) \in \mathfrak{X} \times \mathfrak{Y}.$$

Specifically,

$$\Theta'(\theta, q)(v) = \Big(v, \max\big(\phi_{\Theta_1(\theta)}(v), \phi_{\Theta_2(q)}(v)\big), \min\big(\psi_{\Theta_1(\theta)}(v), \psi_{\Theta_2(q)}(v)\big), \min\big(\gamma_{\Theta_1(\theta)}, \gamma_{\Theta_2(q)}\big) \Big), \quad \forall\ v \in \mathfrak{V}.$$

Definition 3.18. Define $\mathfrak{V}$ as the universal set and $\mathfrak{P}$ as a set of parameters. Let $\mathfrak{X}, \mathfrak{Y} \subseteq \mathfrak{P}$, and consider two C_rPFS_tSs:

$$(\Theta_1, \mathfrak{X}) = \left\{ (v, \phi_{\Theta_1(\theta)}(v), \psi_{\Theta_1(\theta)}(v); \gamma_{\Theta_1(\theta)}) \mid v \in \mathfrak{V},\ \theta \in \mathfrak{X} \right\},$$

$$(\Theta_2, \mathfrak{Y}) = \left\{ (v, \phi_{\Theta_2(q)}(v), \psi_{\Theta_2(q)}(v); \gamma_{\Theta_2(q)}) \mid v \in \mathfrak{V},\ q \in \mathfrak{Y} \right\}.$$

The max-OR operation of $(\Theta_1, \mathfrak{X})$ and $(\Theta_2, \mathfrak{Y})$, denoted by $(\Theta_1, \mathfrak{X}) \bigvee_{\max} (\Theta_2, \mathfrak{Y})$, is defined as another C_rPFS_tS:

$$(\Theta_1, \mathfrak{X}) \bigvee_{\max} (\Theta_2, \mathfrak{Y}) = (\Theta', \mathfrak{X} \times \mathfrak{Y}),$$

where

$$\Theta'(\theta, q) = \Theta_1(\theta) \cup_{\max} \Theta_2(q), \quad \forall (\theta, q) \in \mathfrak{X} \times \mathfrak{Y}.$$

Specifically,

$$\Theta'(\theta, q)(v) = \Big(v, \max\big(\phi_{\Theta_1(\theta)}(v), \phi_{\Theta_2(q)}(v)\big), \min\big(\psi_{\Theta_1(\theta)}(v), \psi_{\Theta_2(q)}(v)\big), \max\big(\gamma_{\Theta_1(\theta)}, \gamma_{\Theta_2(q)}\big) \Big), \quad \forall v \in \mathfrak{V}.$$

Theorem 3.1. *Let $\mathfrak{V}$ be a universal set and let $(\Theta_1, \mathfrak{X}_1)$ and $(\Theta_2, \mathfrak{X}_2)$ be two C_rPFS_tSs over $\mathfrak{V}$. Then:*

1. $\big((\Theta_1, \mathfrak{X}_1) \wedge_{\min} (\Theta_2, \mathfrak{X}_2)\big)^c = (\Theta_1, \mathfrak{X}_1)^c \vee_{\min} (\Theta_2, \mathfrak{X}_2)^c$
2. $\big((\Theta_1, \mathfrak{X}_1) \wedge_{\max} (\Theta_2, \mathfrak{X}_2)\big)^c = (\Theta_1, \mathfrak{X}_1)^c \vee_{\max} (\Theta_2, \mathfrak{X}_2)^c$

3. $((\Theta_1,\mathfrak{X}_1)\vee_{\min}(\Theta_2,\mathfrak{X}_2))^c=(\Theta_1,\mathfrak{X}_1)^c\wedge_{\min}(\Theta_2,\mathfrak{X}_2)^c$
4. $((\Theta_1,\mathfrak{X}_1)\vee_{\max}(\Theta_2,\mathfrak{X}_2))^c=(\Theta_1,\mathfrak{X}_1)^c\wedge_{\max}(\Theta_2,\mathfrak{X}_2)^c$

Proof. It is obvious. □

Definition 3.19. Let $\mathfrak{V}$ be the universal set, and let $\mathfrak{P}$ be a set of parameters. Let $\mathfrak{X},\mathfrak{Y}\subseteq\mathfrak{P}$. Consider the following two C_rPFS_tSs:

$$(\Theta_1,\mathfrak{X})=\{(\nu,\phi_{\Theta_1(\theta)}(\nu),\psi_{\Theta_1(\theta)}(\nu);\gamma_{\Theta_1(\theta)})\mid\nu\in\mathfrak{V}\text{ and }\theta\in\mathfrak{X}\},$$

$$(\Theta_2,\mathfrak{Y})=\{(\nu,\phi_{\Theta_2(q)}(\nu),\psi_{\Theta_2(q)}(\nu);\gamma_{\Theta_2(q)})\mid\nu\in\mathfrak{V}\text{ and }q\in\mathfrak{Y}\}.$$

Then, the union $(\Theta_1,\mathfrak{X})\cup_{\min}(\Theta_2,\mathfrak{Y})$ is again a C_rPFS_tS and it is defined as:

$$(\Theta',\mathfrak{X}\cup\mathfrak{Y})=(\Theta_1,\mathfrak{X})\cup_{\min}(\Theta_2,\mathfrak{Y}),$$

where, $\forall\ \epsilon\in\mathfrak{X}\cup\mathfrak{Y}$:

$$\phi_{\Theta'(\epsilon)}(\nu)=\begin{cases}\phi_{\Theta_1(\epsilon)}(\nu), & \text{if }\epsilon\in\mathfrak{X}-\mathfrak{Y},\\ \phi_{\Theta_2(\epsilon)}(\nu), & \text{if }\epsilon\in\mathfrak{Y}-\mathfrak{X},\\ \max\left(\phi_{\Theta_1(\epsilon)}(\nu),\phi_{\Theta_2(\epsilon)}(\nu)\right), & \text{if }\epsilon\in\mathfrak{X}\cap\mathfrak{Y}.\end{cases}$$

$$\psi_{\Theta'(\epsilon)}(\nu)=\begin{cases}\psi_{\Theta_1(\epsilon)}(\nu), & \text{if }\epsilon\in\mathfrak{X}-\mathfrak{Y},\\ \psi_{\Theta_2(\epsilon)}(\nu), & \text{if }\epsilon\in\mathfrak{Y}-\mathfrak{X},\\ \min\left(\psi_{\Theta_1(\epsilon)}(\nu),\psi_{\Theta_2(\epsilon)}(\nu)\right), & \text{if }\epsilon\in\mathfrak{X}\cap\mathfrak{Y}.\end{cases}$$

The radius $\gamma_{\Theta'(\epsilon)}$ is defined as $\gamma_{\Theta'(\epsilon)}=\min\left(\gamma_{\Theta_1(\epsilon)},\gamma_{\Theta_2(\epsilon)}\right)$.

Definition 3.20. Let $\mathfrak{V}$ be the universal set, and let $\mathfrak{P}$ be a set of parameters. Let $\mathfrak{X},\mathfrak{Y}\subseteq\mathfrak{P}$. Consider the following two C_rPFS_tSs:

$$(\Theta,\mathfrak{X})=\{(\nu,\phi_{\Theta(\theta)}(\nu),\psi_{\Theta(\theta)}(\nu);\gamma_{\Theta(\theta)})\mid\nu\in\mathfrak{V}\text{ and }\theta\in\mathfrak{X}\},$$

$$(\omega,\mathfrak{Y})=\{(\nu,\phi_{\omega(q)}(\nu),\psi_{\omega(q)}(\nu);\gamma_{\omega(q)})\mid\nu\in\mathfrak{V}\text{ and }q\in\mathfrak{Y}\}.$$

Then, the union $(\Theta,\mathfrak{X})\cup_{\max}(\omega,\mathfrak{Y})$ is again a C_rPFS_tS and is defined as:

$$(\Theta',\mathfrak{X}\cup\mathfrak{Y})=(\Theta,\mathfrak{X})\cup_{\max}(\omega,\mathfrak{Y}),$$

where, $\forall\ \epsilon\in\mathfrak{X}\cup\mathfrak{Y}$:

$$\phi_{\Theta'(\epsilon)}(\nu)=\begin{cases}\phi_{\Theta(\epsilon)}(\nu), & \text{if }\epsilon\in\mathfrak{X}-\mathfrak{Y},\\ \phi_{\omega(\epsilon)}(\nu), & \text{if }\epsilon\in\mathfrak{Y}-\mathfrak{X},\\ \max\left(\phi_{\Theta(\epsilon)}(\nu),\phi_{\omega(\epsilon)}(\nu)\right), & \text{if }\epsilon\in\mathfrak{X}\cap\mathfrak{Y}.\end{cases}$$

$$\psi_{\Theta'(\epsilon)}(\nu)=\begin{cases}\psi_{\Theta(\epsilon)}(\nu), & \text{if }\epsilon\in\mathfrak{X}-\mathfrak{Y},\\ \psi_{\omega(\epsilon)}(\nu), & \text{if }\epsilon\in\mathfrak{Y}-\mathfrak{X},\\ \min\left(\psi_{\Theta(\epsilon)}(\nu),\psi_{\omega(\epsilon)}(\nu)\right), & \text{if }\epsilon\in\mathfrak{X}\cap\mathfrak{Y}.\end{cases}$$

The radius $\gamma_{\Theta'(\epsilon)}$ is defined as:

$$\gamma_{\Theta'(\epsilon)}=\max\left(\gamma_{\Theta(\epsilon)},\gamma_{\omega(\epsilon)}\right).$$

Definition 3.21. Let $\mathfrak{V}$ be the universal set, and let $\mathfrak{P}$ be a set of parameters. Let $\mathfrak{X},\mathfrak{Y}\subseteq\mathfrak{P}$. Consider the following two C_rPFS_tSs:

$$(\Theta,\mathfrak{X})=\{(\nu,\phi_{\Theta(\theta)}(\nu),\psi_{\Theta(\theta)}(\nu);\gamma_{\Theta(\theta)})\mid\nu\in\mathfrak{V}\text{ and }\theta\in\mathfrak{X}\},$$

$$(\omega,\mathfrak{Y})=\{(\nu,\phi_{\omega(q)}(\nu),\psi_{\omega(q)}(\nu);\gamma_{\omega(q)})\mid\nu\in\mathfrak{V}\text{ and }q\in\mathfrak{Y}\}.$$

Then, the intersection $(\Theta,\mathfrak{X})\cap_{\min}(\omega,\mathfrak{Y})$ is again a C_rPFS_tS and is defined as:

$$(\Theta',\mathfrak{X}\cup\mathfrak{Y})=(\Theta,\mathfrak{X})\cap_{\min}(\omega,\mathfrak{Y}),$$

where, $\forall\ \epsilon\in\mathfrak{X}\cup\mathfrak{Y}$:

$$\phi_{\Theta'(\epsilon)}(\nu)=\begin{cases}\phi_{\Theta(\epsilon)}(\nu), & \text{if }\epsilon\in\mathfrak{X}-\mathfrak{Y},\\ \phi_{\omega(\epsilon)}(\nu), & \text{if }\epsilon\in\mathfrak{Y}-\mathfrak{X},\\ \min\left(\phi_{\Theta(\epsilon)}(\nu),\phi_{\omega(\epsilon)}(\nu)\right), & \text{if }\epsilon\in\mathfrak{X}\cap\mathfrak{Y}.\end{cases}$$

$$\psi_{\Theta'(\epsilon)}(\nu)=\begin{cases}\psi_{\Theta(\epsilon)}(\nu), & \text{if }\epsilon\in\mathfrak{X}-\mathfrak{Y},\\ \psi_{\omega(\epsilon)}(\nu), & \text{if }\epsilon\in\mathfrak{Y}-\mathfrak{X},\\ \max\left(\psi_{\Theta(\epsilon)}(\nu),\psi_{\omega(\epsilon)}(\nu)\right), & \text{if }\epsilon\in\mathfrak{X}\cap\mathfrak{Y}.\end{cases}$$

The radius $\gamma_{\Theta'(\epsilon)}$ is defined as:

$$\gamma_{\Theta'(\epsilon)}=\min\left(\gamma_{\Theta(\epsilon)},\gamma_{\omega(\epsilon)}\right).$$

Definition 3.22. Let $\mathfrak{V}$ be the universal set, and let $\mathfrak{P}$ be a set of parameters. Let $\mathfrak{X},\mathfrak{Y}\subseteq\mathfrak{P}$. Consider the following two C_rPFS_tSs:

$$(\Theta,\mathfrak{X})=\{(\nu,\phi_{\Theta(\theta)}(\nu),\psi_{\Theta(\theta)}(\nu);\gamma_{\Theta(\theta)})\mid\nu\in\mathfrak{V}\text{ and }\theta\in\mathfrak{X}\},$$

$$(\omega,\mathfrak{Y})=\{(\nu,\phi_{\omega(q)}(\nu),\psi_{\omega(q)}(\nu);\gamma_{\omega(q)})\mid\nu\in\mathfrak{V}\text{ and }q\in\mathfrak{Y}\}.$$

Then, the intersection $(\Theta,\mathfrak{X})\cap_{\max}(\omega,\mathfrak{Y})$ is again a C_rPFS_tS and is defined as:

$$(\Theta',\mathfrak{X}\cup\mathfrak{Y})=(\Theta,\mathfrak{X})\cap_{\max}(\omega,\mathfrak{Y}),$$

where, $\forall\ \epsilon\in\mathfrak{X}\cup\mathfrak{Y}$:

$$\phi_{\Theta'(\epsilon)}(\nu)=\begin{cases}\phi_{\Theta(\epsilon)}(\nu), & \text{if }\epsilon\in\mathfrak{X}-\mathfrak{Y},\\ \phi_{\omega(\epsilon)}(\nu), & \text{if }\epsilon\in\mathfrak{Y}-\mathfrak{X},\\ \min\left(\phi_{\Theta(\epsilon)}(\nu),\phi_{\omega(\epsilon)}(\nu)\right), & \text{if }\epsilon\in\mathfrak{X}\cap\mathfrak{Y}.\end{cases}$$

$$\psi_{\Theta'(\epsilon)}(\nu)=\begin{cases}\psi_{\Theta(\epsilon)}(\nu), & \text{if }\epsilon\in\mathfrak{X}-\mathfrak{Y},\\ \psi_{\omega(\epsilon)}(\nu), & \text{if }\epsilon\in\mathfrak{Y}-\mathfrak{X},\\ \max\left(\psi_{\Theta(\epsilon)}(\nu),\psi_{\omega(\epsilon)}(\nu)\right), & \text{if }\epsilon\in\mathfrak{X}\cap\mathfrak{Y}.\end{cases}$$

The radius $\gamma_{\Theta'(\epsilon)}$ is defined as:

$$\gamma_{\Theta'(\epsilon)}=\max\left(\gamma_{\Theta(\epsilon)},\gamma_{\omega(\epsilon)}\right).$$

The following Theorems explain the theoretical performance of the operators defined above in this section.

Theorem 3.2. *Let $\mathfrak{V}$ be the universal set, and let $\mathfrak{P}$ be a set of parameters. Let Φ and Δ denote the null and whole C_rPFS_tSs, respectively. Suppose $(\Theta,\varkappa)$ and $(\Theta,\mathfrak{P})$ are C_rPFS_tSs over $\mathfrak{V}$. Then the following properties hold:*

1. $(\Theta,\varkappa)\cup_{\min}(\Theta,\varkappa)=(\Theta,\varkappa)$
2. $(\Theta,\varkappa)\cup_{\max}(\Theta,\varkappa)=(\Theta,\varkappa)$
3. $(\Theta,\varkappa)\cap_{\min}(\Theta,\varkappa)=(\Theta,\varkappa)$
4. $(\Theta,\varkappa)\cap_{\max}(\Theta,\varkappa)=(\Theta,\varkappa)$
5. $(\Theta,\mathfrak{P})\cup_{\max}\Phi=(\Theta,\mathfrak{P})$
6. $(\Theta,\mathfrak{P})\cap_{\min}\Phi=\Phi$
7. $(\Theta,\mathfrak{P})\cup_{\max}\Delta=\Delta$
8. $(\Theta,\mathfrak{P})\cap_{\min}\Delta=(\Theta,\mathfrak{P})$

Theorem 3.3. *Let $\mathfrak{V}$ be a universal set, and let $(\Theta_1,\varkappa_1)$ and $(\Theta_2,\varkappa_2)$ be two C_rPFS_tSs. Then:*

1. $((\Theta_1,\varkappa_1)\cup_{\min}(\Theta_2,\varkappa_2))^c=(\Theta_1,\varkappa_1)^c\cap_{\min}(\Theta_2,\varkappa_2)^c$
2. $((\Theta_1,\varkappa_1)\cap_{\min}(\Theta_2,\varkappa_2))^c=(\Theta_1,\varkappa_1)^c\cup_{\min}(\Theta_2,\varkappa_2)^c$

Theorem 3.4. *Let $\mathfrak{V}$ be a universal set, and let $(\Theta_1,\varkappa_1)$ and $(\Theta_2,\varkappa_2)$ be two C_rPFS_tSs. Then:*

1. $((\Theta_1,\varkappa_1)\cup_{\max}(\Theta_2,\varkappa_2))^c=(\Theta_1,\varkappa_1)^c\cap_{\max}(\Theta_2,\varkappa_2)^c$
2. $((\Theta_1,\varkappa_1)\cap_{\max}(\Theta_2,\varkappa_2))^c=(\Theta_1,\varkappa_1)^c\cup_{\max}(\Theta_2,\varkappa_2)^c$

Theorem 3.5. *Let $\mathfrak{V}$ be a universal set, and let $(\Theta_1,\varkappa_1)$, $(\Theta_2,\varkappa_2)$, and $(\Theta_3,\varkappa_3)$ be three C_rPFS_tSs. Then:*

1. $(\Theta_1,\varkappa_1)\cap_{\min}\left((\Theta_2,\varkappa_2)\cap_{\min}(\Theta_3,\varkappa_3)\right)=\left((\Theta_1,\varkappa_1)\cap_{\min}(\Theta_2,\varkappa_2)\right)\cap_{\min}(\Theta_3,\varkappa_3)$
2. $(\Theta_1,\varkappa_1)\cup_{\min}\left((\Theta_2,\varkappa_2)\cup_{\min}(\Theta_3,\varkappa_3)\right)=\left((\Theta_1,\varkappa_1)\cup_{\min}(\Theta_2,\varkappa_2)\right)\cup_{\min}(\Theta_3,\varkappa_3)$

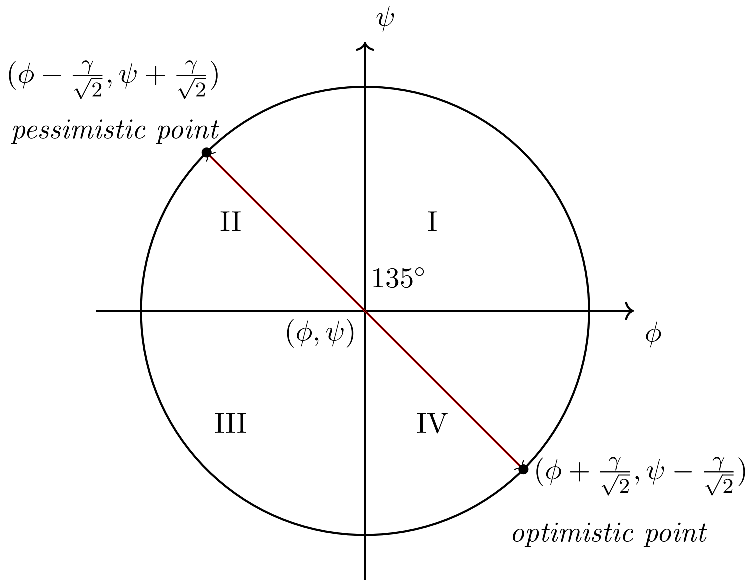

Fig. 4. The optimistic and pessimistic points of a C_rPFV.

3. $(\Theta_1, \varkappa_1) \cap_{\min} \left((\Theta_2, \varkappa_2) \cup_{\min} (\Theta_3, \varkappa_3)\right) = \left((\Theta_1, \varkappa_1) \cap_{\min} (\Theta_2, \varkappa_2)\right) \cup_{\min} \left((\Theta_1, \varkappa_1) \cap_{\min} (\Theta_3, \varkappa_3)\right)$
4. $(\Theta_1, \varkappa_1) \cup_{\min} \left((\Theta_2, \varkappa_2) \cap_{\min} (\Theta_3, \varkappa_3)\right) = \left((\Theta_1, \varkappa_1) \cup_{\min} (\Theta_2, \varkappa_2)\right) \cap_{\min} \left((\Theta_1, \varkappa_1) \cup_{\min} (\Theta_3, \varkappa_3)\right)$

Theorem 3.6. *Let $\mathfrak{U}$ be a universal set, and let $(\Theta_1, \varkappa_1)$, $(\Theta_2, \varkappa_2)$, and $(\Theta_3, \varkappa_3)$ be three $C_r PFS_t Ss$. The following equalities hold:*

1. $(\Theta_1, \varkappa_1) \cap_{\max} \left((\Theta_2, \varkappa_2) \cap_{\max} (\Theta_3, \varkappa_3)\right) = \left((\Theta_1, \varkappa_1) \cap_{\max} (\Theta_2, \varkappa_2)\right) \cap_{\max} (\Theta_3, \varkappa_3)$
2. $(\Theta_1, \varkappa_1) \cup_{\max} \left((\Theta_2, \varkappa_2) \cup_{\max} (\Theta_3, \varkappa_3)\right) = \left((\Theta_1, \varkappa_1) \cup_{\max} (\Theta_2, \varkappa_2)\right) \cup_{\max} (\Theta_3, \varkappa_3)$
3. $(\Theta_1, \varkappa_1) \cap_{\max} \left((\Theta_2, \varkappa_2) \cup_{\max} (\Theta_3, \varkappa_3)\right) = \left((\Theta_1, \varkappa_1) \cap_{\max} (\Theta_2, \varkappa_2)\right) \cup_{\max} \left((\Theta_1, \varkappa_1) \cap_{\max} (\Theta_3, \varkappa_3)\right)$
4. $(\Theta_1, \varkappa_1) \cup_{\max} \left((\Theta_2, \varkappa_2) \cap_{\max} (\Theta_3, \varkappa_3)\right) = \left((\Theta_1, \varkappa_1) \cup_{\max} (\Theta_2, \varkappa_2)\right) \cap_{\max} \left((\Theta_1, \varkappa_1) \cup_{\max} (\Theta_3, \varkappa_3)\right)$

Proof. It is trivial. □

It is important to note that the operators utilized in this work, such as min-OR and max-AND, are inspired by the corresponding operators for the context of the C_rIFS_tS model defined in [46]. These operators are adapted to our setting in order to ensure consistency and comparability with existing frameworks.

Now we proceed to introduce novel score and accuracy functions specifically designed to address the decision-making problem outlined in Section 4. Unlike the score and accuracy measures proposed by Çakır and Taş [47], which are primarily intended for the defuzzification of C_rPFS, our proposed functions are tailored to the decision-making context. To provide motivation, we begin by detailing the structure of C_rPFVs, which naturally extends the description of CIFVs presented in [47].

A C_rPFS consists of C_rPFVs (one for each alternative) where the pair formed by uncertainly assessed MD and NMD defines a point that serves as the center of a circle, with γ representing the margin of error that guarantees that the real pair (MD,NMD) lies in the circle so defined. Each point in a C_rPFV corresponds to a PFV that is a candidate for being the real value of the pair MD-NMD corresponding to the alternative under inspection. Fig. 4 represents these items, together with two special points, namely, the pessimistic point (where the MD is the lowest and the NMD is the highest among the pairs in the C_rPFV) and the optimistic point (where the MD is the highest and the NMD is the lowest among the pairs in the C_rPFV).

We define the new scores and accuracies with the help of these items and a parameter λ that reflects the decision-maker's perspective, indicating whether they are optimistic, pessimistic, or neutral. Its exact role is described below:

Definition 3.23. The novel score function $\eta_{C_r PF}$ and accuracy function $\zeta_{C_r PF}$ for C_rPFSs are defined as follows on its constituents C_rPFVs. Let $\mho_{C_r PF} = (\phi_{C_r PF}, \psi_{C_r PF}; \gamma)$ be a C_rPFV whose associated pessimistic and optimistic points are

$$(\phi_{C_r PF} - \frac{\gamma}{\sqrt{2}}, \psi_{C_r PF} + \frac{\gamma}{\sqrt{2}}) \quad \text{and} \quad (\phi_{C_r PF} + \frac{\gamma}{\sqrt{2}}, \psi_{C_r PF} - \frac{\gamma}{\sqrt{2}})$$

respectively. The numbers $\eta_{C_r PF}$ (score of $\mho_{C_r PF}$) and $\zeta_{C_r PF}$ (accuracy of $\mho_{C_r PF}$) with respect to the decision maker's choice $\lambda \in [0,1]$ are defined as follows:

$$\eta_{C_r PF} = \frac{\lambda \cdot \eta_{PF}\left((\phi_{C_r PF} + \frac{\gamma}{\sqrt{2}}, \psi_{C_r PF} - \frac{\gamma}{\sqrt{2}})\right) + (1-\lambda) \cdot \eta_{PF}\left((\phi_{C_r PF} - \frac{\gamma}{\sqrt{2}}, \psi_{C_r PF} + \frac{\gamma}{\sqrt{2}})\right)}{3}$$

$$= \frac{\lambda(\phi_{C_r P} + \psi_{C_r P})(\phi_{C_r P} - \psi_{C_r P} + \gamma\sqrt{2}) + (1-\lambda)(\phi_{C_r P} + \psi_{C_r P})(\phi_{C_r P} - \psi_{C_r P} - \gamma\sqrt{2})}{3} \tag{1}$$

$$\zeta_{C_r P} = \lambda \cdot \zeta_{PF}\left((\phi_{C_r P} + \frac{\gamma}{\sqrt{2}}, \psi_{C_r P} - \frac{\gamma}{\sqrt{2}})\right) + (1-\lambda) \cdot \zeta_{PF}\left((\phi_{C_r P} - \frac{\gamma}{\sqrt{2}}, \psi_{C_r P} + \frac{\gamma}{\sqrt{2}})\right)$$

$$= 2\sqrt{2}\lambda\gamma(\phi_{C_r P} - \psi_{C_r P}) + \phi^2_{C_r P} + \psi^2_{C_r P} + \gamma^2 + \sqrt{2}\gamma(\psi_{C_r P} - \phi_{C_r P}) \tag{2}$$

Note $\eta_{C_r P} \in [-1,1]$ and $\zeta_{C_r P} \in [0,1]$. The parameter $\lambda \in [0,1]$ quantitatively determines a decision-maker's risk preference by weighting possible outcomes between extreme pessimism ($\lambda = 0$) and extreme optimism ($\lambda = 1$). When $\lambda = 0$, decisions are based solely on the worst case scenario (Score = min(outcomes)), representing complete risk aversion. In contrast, when $\lambda = 1$, only the best case scenario is considered (Score = max(outcomes)), reflecting pure risk seeking behavior. For intermediate values of λ, the decision score is a weighted combination of the best and worst outcomes.

$$\text{Score} = \lambda \cdot \max(\text{outcomes}) + (1-\lambda) \cdot \min(\text{outcomes})$$

Here, $\lambda < 0.5$ emphasizes pessimism (e.g., $\lambda = 0.3$ gives 30% weight to the best and 70% to the worst outcome), $\lambda = 0.5$ represents a balanced approach, and $\lambda > 0.5$ favors optimism (e.g., $\lambda = 0.8$ gives

80% weight to the best outcome). This framework allows for systematic risk calibration. For example, with outcomes $\{2, 5, 10\}$:

1. $\lambda = 0.2$ yields a score of 3.6 (cautious),
2. $\lambda = 0.7$ yields a score of 7.6 (ambitious).

Thus, λ acts as a tunable parameter that aligns decisions with a desired level of risk tolerance arranging from conservative ($\lambda \to 0$), to neutral ($\lambda = 0.5$), to aggressive ($\lambda \to 1$).

Definition 3.24. Let $\mho_1 = (\phi_1, \psi_1; \gamma_1)$ and $\mho_2 = (\phi_2, \psi_2; \gamma_2)$ be two C_rPFVs. The comparison technique is defined in the following manner:

1. If $\eta(\mho_1) < \eta(\mho_2)$, then $\mho_1 < \mho_2$.
2. If $\eta(\mho_1) > \eta(\mho_2)$, then $\mho_1 > \mho_2$.
3. If $\eta(\mho_1) = \eta(\mho_2)$, then:
   (a) If $\zeta(\mho_1) < \zeta(\mho_2)$, then $\mho_1 < \mho_2$.
   (b) If $\zeta(\mho_1) > \zeta(\mho_2)$, then $\mho_1 > \mho_2$.
   (c) If $\zeta(\mho_1) = \zeta(\mho_2)$, then $\mho_1 = \mho_2$.

Definitions 3.2 to 3.10 establish subset relations, like ν-subset for MD, NMD comparisons, ρ-subset for radius precision. Definitions 3.11 and 3.12 define the negation and complement of a C_rPFS_tV respectively. Definitions 3.13 and 3.14 define the null and whole C_rPFS_tS respectively. Definitions 3.15, 3.16, 3.17 and 3.18 establish the min-AND, max-AND, min-OR and max-OR operations for two C_rPFS_tSs. Theorem 3.1 provides the important relationship between two C_rPFS_tSs in terms of min-AND, max-AND, min-OR and max-OR operations. Definitions 3.19, 3.20, 3.21 and 3.22 define the min-union, max-union, min-intersection and max-intersection operations for two C_rPFS_tSs. Theorems 3.3, 3.4, 3.5 and 3.6 provide the important relationship between two C_rPFS_tSs in terms of min-union, max-union, min-intersection and max-intersection operations. These ensure C_rPFS_tS operations behave consistently, extending classical fuzzy logic with radius-aware rules for applications like decision-making under cyclic uncertainty.

4. Decision-making problem

The streamlined problem addressed here involves, in mathematical terms, selecting an object from a set, where each object satisfies certain properties to varying degrees based on a defined list of requirements. An algorithm is introduced to assist in identifying the most suitable object using information provided by multiple observers. The input focuses on characteristic elements such as color, geometry, and surface roughness which are represented using the corresponding C_rPFS. For precision, let $\mathfrak{V} = \{w_1, w_2, w_3, \dots, w_k\}$ be the objects, together with the family of parameter sets $\{\varkappa_1, \varkappa_2, \varkappa_3, \dots, \varkappa_k\}$ that describes the objects in $\mathfrak{V}$. Let $\mathfrak{P}$ be the set of parameters such that $\bigcup_{i=1}^{k} \varkappa_i \subseteq \mathfrak{P}$. For a fixed i, the ith class is represented by $\varkappa_i$, and its elements represent particular property sets. In this framework, we present an algorithm to calculate the decisive score functions and conclude the decision-making process. Afterwards, its computational complexity is studied. Then we perform a numerical exercise with a comparison with existing techniques. We finish this section with a sensitivity analysis.

4.1. Algorithm

Using the data described above, which are expressed in the form of various PFS_tSs, we first carry out the necessary operations to derive their resultant PFS_tS with the specified parameters. Next, we transform this PFS_tS into a C_rPFS_tS using Definition 2.5. Ultimately, the score function is applied to the corresponding C_rPFVs to effectively rank the available options. The procedural steps for the algorithm are given as follows (see Fig. 5):

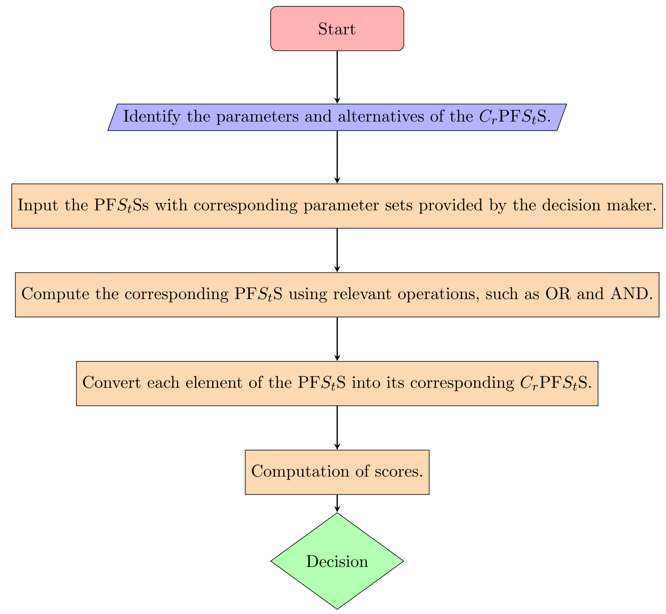

Fig. 5. Flow chart of Algorithm 1.

Algorithm 1 Procedure for deriving C_rPFS_tSs and computing scores

Require: Data in the form of PFS_tSs and a set of parameters $\mathfrak{P}$.
Ensure: Resultant C_rPFS_tS with computed scores.
Step 1: Define Parameters and Alternatives
Specify the parameters and the set of alternatives for C_rPFS_tSs.
Step 2: Input data
Input the PFS_tSs provided by decision-makers and associate them with the parameters in $\mathfrak{P}$.
Step 3: Apply operations on PFS_tSs
Perform relevant operations (e.g., OR operation) on the given PFS_tSs to derive a resultant PFS_tS.
Step 4: Convert this PFS_tS to a C_rPFV
Transform the resultant PFS_tS into a C_rPFV using Definition 2.5.
Step 5: Calculate scores
Apply the score function outlined in Equation (1) to evaluate the scores of each alternative.
Step 6: Determine optimality
Select the unique alternative with the highest score, if one exists, as the optimal choice.
Step 7: Handle Ties
If multiple alternatives have the same maximum score, then use Definition 3.24 for choosing a best optimal choice (otherwise, break ties arbitrarily).

4.1.1. Analysis of computational complexity

The computational complexity of Algorithm 1 is analyzed step-by-step to assess its scalability. Let n be the number of alternatives, m be the number of parameters in the initial sets, and k be the number of PFVs aggregated to form a single C_rPFV (see Definition 2.5).

- **Step 3 (Apply operations):** The OR operation on two PFS_tSs generates a new set with up to $O(m^2)$ parameters. The subsequent operations have a similar combinatorial cost in the worst case, leading to $O(m^x)$ parameters, where x depends on the number of operations chained. This is the most significant cost driver and highlights that the number of parameters should be managed carefully.

Table 4
Tabular representation of the PFS_tS $(\Theta_1, \varkappa_1)$.

$\mathfrak{V}$	θ_1	θ_2	θ_3	θ_4
w_1	(0.22, 0.51)	(0.33, 0.49)	(0.46, 0.30)	(0.66, 0.07)
w_2	(0.22, 0.51)	(0.74, 0.08)	(0.23, 0.53)	(0.365, 0.365)
w_3	(0.29, 0.44)	(0.29, 0.44)	(0.61, 0.15)	(0.51, 0.22)
w_4	(0.58, 0.15)	(0.16, 0.66)	(0.30, 0.46)	(0.58, 0.15)
w_5	(0.51, 0.22)	(0.25, 0.57)	(0.46, 0.30)	(0.365, 0.365)
w_6	(0.66, 0.07)	(0.16, 0.66)	(0.30, 0.46)	(0.22, 0.51)

Table 5
Tabular representation of the PFS_tS $(\Theta_2, \varkappa_2)$.

$\mathfrak{V}$	θ_5	θ_6	θ_7	θ_8	θ_9
w_1	(0.33, 0.49)	(0.18, 0.74)	(0.66, 0.16)	(0.55, 0.37)	(0.325, 0.325)
w_2	(0.66, 0.16)	(0.55, 0.37)	(0.25, 0.57)	(0.09, 0.83)	(0.455, 0.195)
w_3	(0.49, 0.33)	(0.37, 0.55)	(0.25, 0.57)	(0.09, 0.83)	(0.455, 0.195)
w_4	(0.74, 0.08)	(0.74, 0.18)	(0.16, 0.66)	(0.09, 0.83)	(0.39, 0.26)
w_5	(0.16, 0.66)	(0.09, 0.83)	(0.74, 0.08)	(0.74, 0.18)	(0.455, 0.195)
w_6	(0.25, 0.57)	(0.18, 0.75)	(0.66, 0.16)	(0.55, 0.37)	(0.325, 0.325)

Table 6
Tabular representation of the PFS_tS $(\Theta_3, \varkappa_3)$.

$\mathfrak{V}$	θ_{10}	θ_{11}	θ_{12}	θ_{13}
w_1	(0.22, 0.51)	(0.29, 0.43)	(0.09, 0.84)	(0.66, 0.07)
w_2	(0.44, 0.29)	(0.36, 0.36)	(0.37, 0.56)	(0.365, 0.365)
w_3	(0.365, 0.365)	(0.43, 0.29)	(0.28, 0.65)	(0.44, 0.29)
w_4	(0.51, 0.22)	(0.43, 0.29)	(0.56, 0.37)	(0.22, 0.51)
w_5	(0.44, 0.29)	(0.43, 0.29)	(0.465, 0.465)	(0.29, 0.44)
w_6	(0.66, 0.07)	(0.50, 0.22)	(0.65, 0.28)	(0.66, 0.07)

- **Step 4 (Convert to C_rPFV):** For each of the $O(m^x)$ parameters and n alternatives, we compute the aggregate C_rPFV. This involves calculating the root mean square for MD and NMD and the radius γ over k values, which is an $O(k)$ process per alternative parameter pair. Thus, this step is $O(n \cdot m^x \cdot k)$.
- **Step 5 (Calculate scores):** Calculating the novel score and accuracy functions for each of the n alternatives is an $O(1)$ operation per alternative, leading to a total of $O(n)$.

Therefore, the overall time complexity of the algorithm is dominated by $O(n \cdot m^x \cdot k)$. This indicates that the method is highly sensitive to the number of parameters m and the number of operations performed, while it scales linearly with the number of alternatives n. For practical applications with a large number of parameters, feature selection is recommended to keep m manageable. The algorithm is well suited for problems where the number of core attributes is not excessively large, which is typical in many MCDM scenarios.

4.2. Numerical example

To demonstrate the application of Algorithm 1, consider the scenario previously discussed in [46,48]. Let the universal set of options be defined as $\mathfrak{U} = \{w_1, w_2, w_3, w_4, w_5, w_6\}$, and the set of parameters as $\mathfrak{P} = \{\theta_j : 1 \le j \le 13\}$. The elements of $\mathfrak{P}$ correspond to the following attributes: {blackish, dark brown, yellowish, reddish, large, small, very small, average, very large, coarse, moderately coarse, fine, extra fine}. Define the following parameter subsets: $\varkappa_1 = \{\theta_1, \theta_2, \theta_3, \theta_4\}$ = {blackish, dark brown, yellowish, reddish}, $\varkappa_2 = \{\theta_5, \theta_6, \theta_7, \theta_8, \theta_9\}$ = {large, small, very small, average, very large}, and $\varkappa_3 = \{\theta_{10}, \theta_{11}, \theta_{12}, \theta_{13}\}$ = {coarse, moderately coarse, fine, extra fine}. The tabular representation of the PFS_tSs $(\Theta_1, \varkappa_1)$, $(\Theta_2, \varkappa_2)$, and $(\Theta_3, \varkappa_3)$ is given in Tables 4, 5, and 6.

Next, we apply the operation "OR" on $(\Theta_1, \varkappa_1)$ and $(\Theta_2, \varkappa_2)$, obtaining a PFS_tS $(\Theta_4, \varkappa_4)$, which has a total of $4 \times 5 = 20$ parameters from which we can take a few for representation. Specifically,

$$\varkappa_4 = \{v_{15}, v_{19}, v_{25}, v_{28}, v_{37}, v_{48}, v_{49}\}.$$

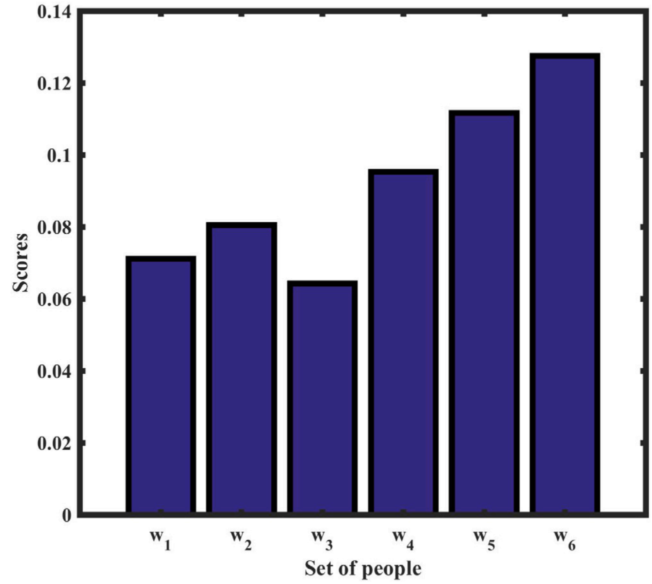

Fig. 6. Ranking of people.

produces the PFS_tS $(\Theta_4, \varkappa_4)$ represented in Table 7.

Now, let

$$\varkappa_5 = \{v_{15} \vee \theta_{10},\ v_{19} \vee \theta_{12},\ v_{25} \vee \theta_{11},\ v_{28} \vee \theta_{13}, v_{37} \vee \theta_{12},\ v_{48} \vee \theta_{12},\ v_{49} \vee \theta_{13}\}.$$

This is the set of parameters provided by the expert or observer. The corresponding PFS_tS $(\Theta_5, \varkappa_5)$ is presented in Table 8.

Next, we convert the evaluation of each element of the PFS_tS $(\Theta_5, \varkappa_5)$ into its corresponding C_rPFV with the help of Definition 2.5. The decision maker's parameter $\lambda = 0.5$ is taken during the calculations of scores. The results are shown in Table 9. The assignment of scores can be seen in Fig. 6. We clearly choose w_6 as the optimal choice.

4.3. Comparative study

The algorithm presented in this paper produces results equivalent to those obtained by Singh et al. [48] and Unni et al. [46] using the C_rIFS_tS approach (see Fig. 7). Table 10 compares the score values from [48] with the corresponding scores generated by the proposed algorithm. It is evident that both calculation methods assign the highest score to w_6, indicating that the final choice remains the same. In the work of Unni et al. [46], the variables w_1 and w_5 had the same score; however, the proposed algorithm was able to differentiate and rank them with distinct scores. The value of λ can vary within $[0, 1]$, which may result in different score outcomes. The proposed algorithm offers an advantage over [46], as it enables the decision-maker to select a perspective either pessimistic or optimistic for the decision-making process.

4.3.1. Comparison with C_rIFS_tS

Here is a thorough tabular comparison of the C_rPFS_tS and C_rIFS_tS decision-making models given in Table 11, emphasizing their advantages, improvements, and appropriate application cases:

4.4. Sensitivity analysis

This section assesses the robustness of Algorithm 1 by introducing small perturbations to the original data. The modified datasets under consideration are detailed in Tables 12, 13, and 14. For consistency and ease of comparison, the notation used in the original experiment is preserved throughout. Upon applying Algorithm 1 to the modified

Table 7
Tabular representation of the PF$\mathcal{S}_t$S $(\Theta_4, \varkappa_4)$.

$\mathfrak{V}$	v_{15}	v_{19}	v_{25}	v_{28}	v_{37}	v_{48}	v_{49}
w_1	(0.33, 0.49)	(0.325, 0.325)	(0.33, 0.49)	(0.55, 0.37)	(0.66, 0.16)	(0.66, 0.07)	(0.66, 0.07)
w_2	(0.66, 0.16)	(0.455, 0.195)	(0.74, 0.08)	(0.74, 0.08)	(0.25, 0.53)	(0.365, 0.365)	(0.455, 0.195)
w_3	(0.49, 0.33)	(0.455, 0.195)	(0.49, 0.33)	(0.29, 0.44)	(0.61, 0.15)	(0.51, 0.22)	(0.51, 0.195)
w_4	(0.74, 0.08)	(0.58, 0.15)	(0.74, 0.08)	(0.16, 0.66)	(0.30, 0.46)	(0.58, 0.15)	(0.58, 0.15)
w_5	(0.51, 0.22)	(0.51, 0.195)	(0.25, 0.57)	(0.74, 0.18)	(0.74, 0.08)	(0.74, 0.18)	(0.455, 0.195)
w_6	(0.66, 0.07)	(0.66, 0.07)	(0.25, 0.57)	(0.55, 0.37)	(0.66, 0.16)	(0.55, 0.37)	(0.325, 0.325)

Table 8
Tabular representation of the PF$\mathcal{S}_t$S $(\Theta_5, \varkappa_5)$.

$\mathfrak{V}$	$v_{15} \vee \theta_{10}$	$v_{19} \vee \theta_{12}$	$v_{25} \vee \theta_{11}$	$v_{28} \vee \theta_{13}$	$v_{37} \vee \theta_{12}$	$v_{48} \vee \theta_{12}$	$v_{49} \vee \theta_{13}$
w_1	(0.33, 0.49)	(0.325, 0.325)	(0.33, 0.43)	(0.66, 0.07)	(0.66, 0.16)	(0.66, 0.07)	(0.66, 0.07)
w_2	(0.66, 0.16)	(0.455, 0.195)	(0.74, 0.08)	(0.74, 0.08)	(0.37, 0.53)	(0.37, 0.365)	(0.455, 0.195)
w_3	(0.49, 0.33)	(0.455, 0.195)	(0.49, 0.29)	(0.44, 0.29)	(0.61, 0.15)	(0.51, 0.22)	(0.51, 0.195)
w_4	(0.74, 0.08)	(0.58, 0.15)	(0.74, 0.08)	(0.22, 0.51)	(0.56, 0.37)	(0.58, 0.15)	(0.58, 0.15)
w_5	(0.51, 0.20)	(0.51, 0.19)	(0.43, 0.22)	(0.74, 0.08)	(0.74, 0.08)	(0.74, 0.18)	(0.455, 0.19)
w_6	(0.66, 0.07)	(0.66, 0.07)	(0.50, 0.22)	(0.66, 0.07)	(0.66, 0.16)	(0.65, 0.28)	(0.66, 0.07)

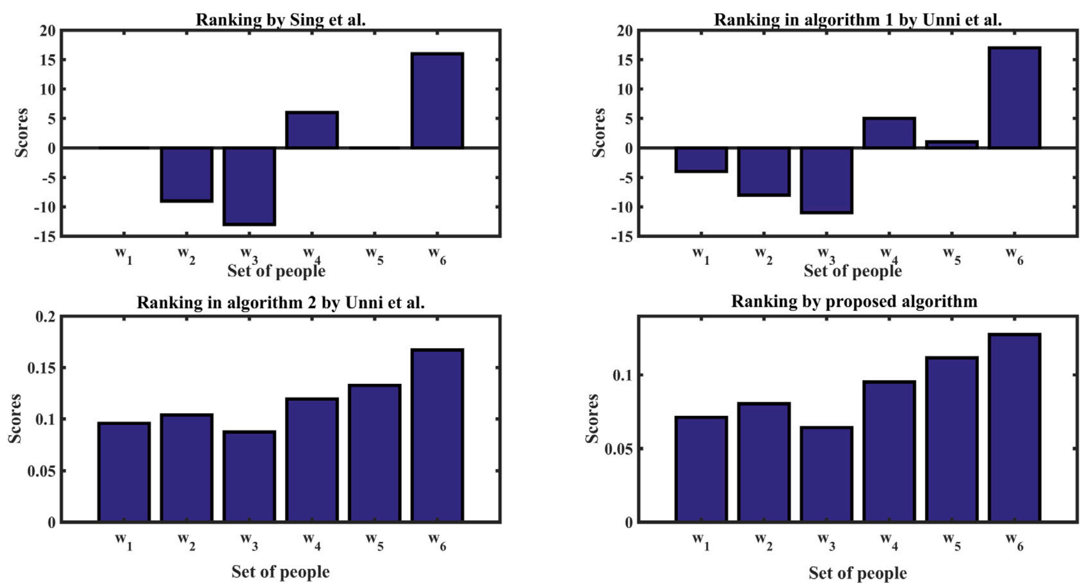

Fig. 7. Comparative analysis with existing methods.

Table 9
C_rPFVs corresponding to $(\Theta_5, \mathfrak{B}_5)$ and their score values.

$\mathfrak{V}$	C_rPFV	Score
w_1	(0.5432, 0.2856; 0.2954)	0.0712
w_2	(0.5630, 0.2748; 0.3200)	0.0805
w_3	(0.5033, 0.2461; 0.1436)	0.0642
w_4	(0.5936, 0.2577; 0.4508)	0.0953
w_5	(0.6041, 0.1730; 0.1804)	0.1117
w_6	(0.6381, 0.1568; 0.1519)	0.1275

Table 10
Comparison of score values.

$\mathfrak{V}$	Score values in [48]	Score values in Algorithm 1 [46]	Score values in Algorithm 2 [46]	Proposed Algorithm
w_1	0	−4	0.0957143	0.0712
w_2	−9	−8	0.104048	0.0805
w_3	−13	−11	0.087381	0.0642
w_4	6	5	0.119524	0.0953
w_5	0	1	0.132619	0.1117
w_6	16	17	0.167143	0.1275

data (see Table 15), the resulting scores demonstrate a strong alignment with those derived from the original dataset. Fig. 8 presents a side-by-side comparison of the scores and rankings for both the original and modified data. Notably, the optimal choice remains unchanged as w_6, indicating the stability of the algorithm. These findings confirm that minor variations in the input data do not significantly influence the decision-making outcome, thereby highlighting the reliability and robustness of the proposed method.

5. Discussion

The traditional fuzzy set models like IFSs, C_rIFSs, PFSs, C_rPFSs, FFSs, C_rFFSs have been widely used in medical diagnosis and pattern recognition [49–55]. Such models provide an efficient way to handle uncertainty and vagueness of complex systems and have ended up being fairly practical in the real-life situation of decision-making. The C_rPF$\mathcal{S}_t$S structure yields an advanced model that enhances models such as IFS, PFS, and $\mathcal{S}_t$S by integrating their strengths and introducing a circular structure. This novelty is particularly effective to handle data involving periodicity or cyclic behavior. C_rPF$\mathcal{S}_t$S offers greater accuracy and flexibility, where professionals often face overlapping and imprecise information. In most of the medical situations, doctors must

Table 11

Comparison of proposed model with C_rIFS_tS.

Aspect	C_rPFS_tS	C_rIFS_tS	Key benefits
Uncertainty	High ($MD^2 + NMD^2 \leq 1$)	Moderate ($MD + NMD \leq 1$)	C_rPFS_tS better for handling vague and imprecise data
Flexibility	High and supports more diverse scenarios	Limited flexibility	C_rPFS_tS suits complex and unstructured decision problems
Decision Accuracy	High and better discrimination among options	Moderate	C_rPFS_tS improves precision in MCDM
Hesitation Modeling	Effectively captures hesitation	Limited capability	C_rPFS_tS better reflects human decision behavior
Generalization	Generalizes C_rIFS_tS	Cannot generalize C_rPFS_tS	C_rPFS_tS offers broader applicability

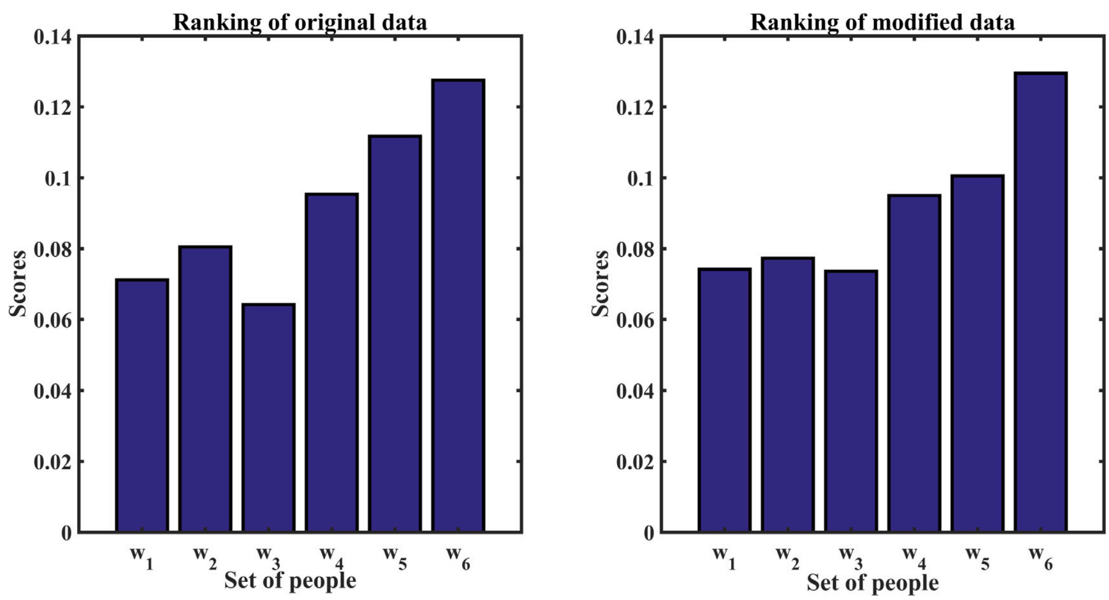

Fig. 8. Comparison of score-based rankings for original and modified data sets.

Table 12

Modified values of the PFS_tS $(\Theta_1, \varkappa_1)$.

$\mathfrak{V}$	e_1	e_2	e_3	e_4
w_1	(0.23, 0.51)	(0.33, 0.49)	(0.41, 0.30)	(0.66, 0.099)
w_2	(0.22, 0.51)	(0.7, 0.077)	(0.235, 0.05)	(0.365, 0.365)
w_3	(0.291, 0.44)	(0.28, 0.44)	(0.61, 0.15)	(0.51, 0.22)
w_4	(0.58, 0.15)	(0.16, 0.662)	(0.30, 0.46)	(0.581, 0.152)
w_5	(0.51, 0.22)	(0.25, 0.57)	(0.46, 0.32)	(0.365, 0.365)
w_6	(0.68, 0.09)	(0.16, 0.66)	(0.30, 0.462)	(0.22, 0.51)

Table 15

C_rPFVs corresponding to $(\Theta_5, \varkappa_5)$, and score values.

$\mathfrak{V}$	C_rPFV	Score
w_1	(0.5485, 0.2802; 0.2817)	0.0741
w_2	(0.5497, 0.2654; 0.2925)	0.0773
w_3	(0.5260, 0.2362; 0.1203)	0.0736
w_4	(0.5834, 0.2354; 0.4495)	0.0950
w_5	(0.6071, 0.2590; 0.2397)	0.1005
w_6	(0.6441, 0.1629; 0.1550)	0.1294

Table 13

Modified values of the PFS_tS $(\Theta_2, \varkappa_2)$.

$\mathfrak{V}$	e_5	e_6	e_7	e_8	e_9
w_1	(0.35, 0.48)	(0.19, 0.742)	(0.67, 0.18)	(0.55, 0.38)	(0.325, 0.325)
w_2	(0.67, 0.16)	(0.55, 0.37)	(0.35, 0.575)	(0.095, 0.85)	(0.45, 0.19)
w_3	(0.49, 0.34)	(0.37, 0.555)	(0.25, 0.59)	(0.092, 0.84)	(0.455, 0.195)
w_4	(0.745, 0.087)	(0.74, 0.184)	(0.16, 0.6)	(0.09, 0.08)	(0.391, 0.3)
w_5	(0.165, 0.66)	(0.09, 0.84)	(0.74, 0.082)	(0.75, 0.18)	(0.45, 0.3)
w_6	(0.27, 0.57)	(0.18, 0.77)	(0.66, 0.19)	(0.55, 0.375)	(0.32, 0.325)

Table 14

Modified values of the PFS_tS $(\Theta_3, \varkappa_3)$.

$\mathfrak{V}$	e_{10}	e_{11}	e_{12}	e_{13}
w_1	(0.24, 0.5)	(0.29, 0.4)	(0.09, 0.85)	(0.66, 0.075)
w_2	(0.44, 0.291)	(0.36, 0.36)	(0.375, 0.56)	(0.365, 0.36)
w_3	(0.36, 0.365)	(0.4, 0.29)	(0.28, 0.65)	(0.45, 0.29)
w_4	(0.51, 0.225)	(0.43, 0.29)	(0.56, 0.37)	(0.22, 0.5)
w_5	(0.44, 0.29)	(0.43, 0.29)	(0.465, 0.465)	(0.29, 0.44)
w_6	(0.66, 0.07)	(0.50, 0.22)	(0.65, 0.28)	(0.66, 0.07)

make decisions even in the presence of changing or uncertain information. Different diseases can cause the same symptom, the symptom can appear and disappear, or appear again over time. Symptoms can interact with each other in complicated ways. The existing models like PFS_tSs help doctors to assign only two values to each symptom. First is MD, that shows how much a symptom supports a certain condition. Second is NMD, that shows how much that symptom argues against that certain condition. But this approach assumes the things to stay stable or change in a linear way. It does not handle situations where symptoms go up or down, or repeat over time. At this point, the C_rPFS_tS becomes useful. It introduces a circular structure which means that it can model those things or patterns that change in cycles. For example, a disease is cured well at first, then less, then cured again. A symptom might appear, disappear, and come back later. C_rPFS_tSs handle these ups and downs in more realistic and natural ways. So, instead of giving one-time judgment like "this symptom means flu", C_rPFS_tS allows the doctors to see how the relationship between diagnosis and symptoms change over time. This paper explores how C_rPFS_tSs can be used to approach decision-making problems limited by this issue.

There are now a number of alternatives to conventional set theoretic methods in uncertainty modeling. FS, IFS, PFS, q-ROFS, C_rIFS, C_rPFS, rough set, and others are among them. Many of these have been expanded to soft set versions to address multi-attribute problems, which involve the simultaneous interpretation of data from multiple angles. Of these, the evolution from S_tS to FS_tS and from FS_tS to IFS_tS and PFS_tS which ultimately result in C_rPFS_tS has received more and more attention. The shortcomings of previous models, including IFS_tS and PFS_tS, are addressed by C_rPFS_tS, offering a substantial generalization. In particular, PFS_tS provides more freedom by permitting the sum of the squares of MD and NMD to be at most 1, but IFS_tS imposes that the sum of the MD and NMD cannot exceed 1. However, both models depend on exact MD and NMD numerical values, which frequently do not match the way information is given in real-world situations. A specialist may use ranges to convey uncertainty in decision-making activities like medical diagnosis, for instance: "There is a 50%–60% chance of disease", or "There is a 40%–50% chance of no disease". The complexity and integrity of these statements would be lost in IFS_tS or PFS_tS models since they would be condensed into a single, clear value pair, such as (0.55, 0.45).

Instead of using a fixed point to represent an acceptable range of values, C_rPFS_tS introduces a circular zone around the point (MD, NMD). All pairings within that distance from the center are included in this circle, which has radius r, where $0 \leq r \leq \sqrt{2}$. Accordingly, all points in the (MD, NMD) plane that are within 0.05 radius of the center (0.7, 0.3) would be included in the set for a C_rPFS_tS defined with (MD = 0.7, NMD = 0.3) and radius = 0.05. This approach enables a more flexible modeling of uncertainty while capturing the imprecision and variability present in expert-provided ranges. In fields where data is imprecise, this method works especially well. By allowing experts to provide ranges rather than precise numbers or values, C_rPFS_tS helps to make the model more observable and realistic. In fact, as was previously indicated, environmental studies frequently entail situations in which specialists are presented with unclear data that has a margin of error. For example, existing analytical approaches are not always able to predict changes in species abundance due to seasonal changes or track patterns in monthly average water pollution in a river. When the PFS_tS model is used, pollution levels can be assigned membership value, which indicates how much they contribute to the specific sources of pollution, and non-membership value, which indicates how little they do so. In reality, however, the amount of pollution is not constant; it varies seasonally as a result of rainfall, industrial activity, and other natural events. A broader model that can take the volatility element into consideration is required. C_rPFS_tS provides a more accurate depiction that more closely reflects the variation found in the real world. The radius is an important parameter within the framework of C_rPFS_tS, and it is crucially applied in the interpretation of uncertainty and sensitivity towards a cyclical pattern or a fluctuating design. The radius determines how much deviation in the data should be allowed by the next central fuzzy value which is in essence the parameter that controls the extent to which one can deviate. This is particularly crucial in the medical diagnosis. To illustrate, a symptom such as fever might not persist in the same manner all the time; it might increase or decrease with time or recurrence and occasionally, one symptom such as fever might be characterized by a reaction with another such as fatigue or headache depending on timing and extent. An enlarged radius in C_rPFS_tS means the model can bear such uncertain, overlapping or cyclic patterns in order to be more flexible when the conditions of patients are not stable or the symptoms do not remain the same. It can be used to help identify conditions such as malaria where the fever has a cycle in it and interacts with other symptoms, such as sweating and chills, resulting in compounding this symptom. Conversely, a smaller radius will provide a stricter control which will be perfect in a diagnostic environment where the symptoms must conform to a certain set of stringent conditions. This flexibility can be obtained by varying the radius, which permits the model to tolerate inaccuracy, and so makes C_rPFS_tS particularly useful in applications like healthcare, engineering, and pattern recognition. Table 16 summarizes a comparative analysis of the proposed model with existing theories.

6. Conclusions

This paper introduces the hybrid structure named "circular Pythagorean fuzzy soft set", which integrates C_rPFS theory with S_tS theory. With this framework the traditional PFS model is extended by accommodating imprecise representations of membership and non-membership degrees, along with parameterized descriptions of the elements of the problem. To evaluate the C_rPFS_tS components that characterize such elements, new score and accuracy measures are developed. They incorporate the decision-maker's perspective through optimistic and pessimistic viewpoints. A parameter $\lambda \in [0, 1]$ controls this balance: $\lambda = 0$ reflects a fully pessimistic outlook, emphasizing the worst outcomes, while $\lambda = 1$ corresponds to a completely optimistic view, focusing on the best outcomes. Values of λ between these extremes indicate varying degrees of optimism or pessimism: $\lambda \in [0, 0.5)$ biases towards pessimism, whereas $\lambda \in (0.5, 1]$ favors optimism. The definition of operations on C_rPFS_tSs such as complement, min-OR, max-OR, min-AND, max-AND, min-union, max-union, min-intersection, and max-intersection, along with their fundamental properties, establish a solid theoretical foundation that opens avenues for further research and practical applications. Furthermore, this work adapts a decision-making problem initially formulated in FSS theory, proposing an algorithm that extends it into the C_rPFS_tS framework.

The C_rPFS_tS model has a significant potential of applicability in situations where there is uncertainty. It can be applied in the medical diagnostics field to deal with the inconsistency of symptoms in such conditions like cyclical fevers during malaria or changing vital signs with long term illness, enabling more precise judgments. It can further facilitate better predictive maintenance in industrial systems by comparing recurring trends in sensor data that are gauge on manufacturing equipment, leading to enhanced reliability and efficiency of operations.

In financial analytic, C_rPFS_tS provides the opportunity to represent the periodicity of trends and market fluctuations in the prices of stocks, or in the changes in cryptocurrencies, and these phenomena are often subject to some cyclically. The framework is also useful in environments or when conducting environmental monitoring as it can help in assessing seasonal changes in the weather or periodic pollution patterns with greater sensitivity. Also, it can augment the explainability of the models that, in particular, deal with time dependent data with uncertainty. In general, C_rPFS_tS is a versatile and dependable solution that fits more or less any complex or changing decision-making situation.

Future research will focus on exploring various t-norm and t-conorm operators for aggregating C_rPFS_tS, aiming to identify the most effective methods for different applications. Additionally, new distance and similarity metrics can be developed and applied between clusters of PFS_tS by converting them into C_rPFS_tS, particularly in fields like computational image analysis and medical treatment. The circular representation approach may also be extended to generalized fuzzy sets such as FFS and q−ROFS, incorporating parameterization. Finally, the aggregation operators developed for these sets will be applied to enhance group decision-making processes.

CRediT authorship contribution statement

Gulfam Shahzadi: Writing – original draft, Methodology, Formal analysis, Conceptualization. **José Carlos R. Alcantud:** Writing – review & editing, Methodology, Formal analysis, Conceptualization. **Rana Talha Ahmad:** Writing – original draft, Methodology, Formal analysis, Conceptualization.

Table 16
A qualitative comparison of different fuzzy soft set extensions with C_rPFS_tS.

Soft Variants	S_tSs	FS_tSs	PFS_tSs	C_rPFS_tSs
Nature of Uncertainty	Flexible membership degrees for elements in different sets	Allowed graded membership degrees and parameterized approximations to model uncertainty flexibly	Handle uncertainty through MD, NMD, and hesitation degrees, where the hesitation degree represents the degree of uncertainty in decision-making processes	Combine the C_rPFS and S_tS concepts, the MD, NMD and radii have uncertainty
Membership Assignment	Binary relations, an element either in the set or not	Have degrees of membership, parameters can have uncertainty degrees	Have MD, NMD and parameters can have uncertainty degrees	Have MD, NMD similar to C_rPFS and parameters can have uncertainty degrees
Representation	Use matrices or tables with entries as 0 or 1	Use matrices or tables with entries as MD of elements	Use matrices or tables with entries as MD of elements including parameters for NMD and hesitation degrees	Use matrices or tables with entries as MD of elements including parameters for NMD and radii.
Applications	Decision-making and classification problems	Decision-making, classification, and pattern recognition where uncertainty in MD but also in parameters	Decision-making, classification, and pattern recognition where uncertainty in MD,NMD and hesitation degree but also in parameters	Decision-making, classification, and pattern recognition where uncertainty in MD,NMD and radii allowing more comprehensive uncertainty
Limitations	Lack a standard framework for operations, objective parameter selection, intensive computations for large datasets, limited for continuous or high-dimensional data	Complex representation for large number of elements, subjective parameter selection, intensive computations for large datasets, challenging interpretation for high-dimensional data	Complex to represent, subjective parameter selection, intensive computations for large datasets, challenging interpretation because of multiple parameters	Complex to represent, subjective parameter selection, intensive computations for large datasets, challenging interpretation because of multiple parameter

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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