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Aggregation functions defined by Choquet-partitioned functions

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ABSTRACT

Bustince et al. recently introduced a versatile framework for constructing aggregation functions from input-dependent $[0,1]$-valued vectors under appropriate conditions. Here we leverage their framework to propose and investigate a new and structurally different family of operators. Whereas in Choquet-inspired aggregation functions, the weights applied to a difference of successive values depend on the inputs for which they are not weighting, the weights that are applied in the new model remain constant across the inputs that belong to an element of a fixed partition (in the sense of basic set theory) of the set of inputs. Consequently, a key structural distinction is that, in contrast to the former model, the number of weights is inherently finite when the partition is simplicial (a technical concept that aligns with the spirit of Choquet integration). When the partition is the finest (therefore, it is infinite) we obtain the general framework that motivates our investigation, hence the new model generalizes Choquet integrals. We prove that it also encompasses a type of aggregation operators based on weighting vectors that is more general than Ordered Weighted Averaging operators and Induced Ordered Weighted Averaging operators. Other mathematical proofs establish when this new class of idempotent operators has properties such as shift-invariance and continuity. Our proofs identify natural conditions guaranteeing that the new operators are aggregation functions. Numerical examples involving bivariate inputs illustrate their geometric interpretation. This new family of operators highlights the foundational importance and flexibility of the general framework designed by Bustince et al.

1. Introduction

The utilization of weighting vectors to produce aggregation functions has a long tradition. Since the use of weighted average means (WAMs) to ordered weighted averaging (OWAs) aggregation operators [1] and recent generalizations [2], many approaches have provided families of aggregation functions or utility functions from weights, with a variety of properties and purposes [3–9].

Other aggregation functions have proven useful in decision-making and other applications. Fuzzy integrals stand out due to their versatility. In particular, the discrete Choquet integral [10] generalizes statistical measures (inclusive of WAMs) and OWAs. Weighting vectors are replaced with fuzzy measures, allowing for interactions such as synergies and redundancies. The discrete Choquet integral has been used extensively in areas such as decision aid [11], finance and economic disciplines [12–14] or social choice [15]. For this reason, generalizations of this class of integrals abound [16–19].

This paper is directly motivated by a new general model defined by Bustince et al. [20]. Their proposal modifies the design of discrete Choquet integrals by substituting the fuzzy measure with general increasing mappings that are a function of the inputs. Their extension uses input-dependent weights to calculate a convex combination of the inputs. In general, their evaluations do not produce aggregation functions because increasingness fails to hold true. Therefore their primary objective is the identification of a restriction of the model –the "Choquet-inspired aggregation functions"– that simultaneously yields an aggregation function and still includes OWAs and some discrete Choquet integrals as particular cases. Remarkably, Bustince et al. insist that their contribution is a first step towards frameworks that are more general than certain classes of fuzzy integrals.

In continuation of this new line of research, in this work we explore a different type of restriction of their original model whereby the weights are partitioned-dependent. To achieve this goal, the main analytical tool is the new concept of "partitioned weight generator". Constructive mathematical arguments show that the largest partition reduces this proposal to the general case in [20], the new $(\mathbf{w},\mathbf{b})$-weighted average means [2] are a particular case when the partition is the largest, and simplicial partitions (a new technical concept that closely aligns with the spirit of Choquet integration) exactly produce the class of discrete Choquet integrals. Notably, $(\mathbf{w},\mathbf{b})$-weighted average means are more general than OWAs and Induced Ordered Weighted Averaging (IOWA) operators, which are therefore encompassed in the new model. However, this general construction does not necessarily guarantee the production of aggregation functions. The reason is that as in the case of [20], in-

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creasingness of the evaluations is in general not true. To overcome this setback, we propose the concept of increasing partitioned weight generator. Our main theorem proves that in our model (where weights are common to all the inputs belonging to the same element of the partition), when the partitioned weight generator that determines the utility function is increasing, then the evaluation produces an aggregation function.

The contents of this article are distributed as follows. In Section 2 we provide background about aggregation function and average means, plus basic notation. Section 3 recalls the fundamentals of the proposal in [20]. Our new model is defined in Section 4. Here we relate it to existing literature and prove the main results of this article. Numerical examples featuring two-dimensional vectors serve to illustrate the graphical interpretation of the most basic framework. We give concluding remarks in Section 5.

2. Notation and basic background

Let us introduce basic notation and definitions that we need in this work.

(i) Henceforth we fix $N = \{1, \dots, n\}$ with $n \geq 2$. The notation 2^N refers to the powerset of N.
(ii) If $\mathbf{a} = (a_1, \dots, a_n), \mathbf{a}' = (a'_1, \dots, a'_n) \in [0,1]^n$, $\mathbf{a} \geq \mathbf{a}'$ is equivalent to $a_j \geq a'_j$ for all $j \in N$.
(iii) When $\mathbf{x} = (x_1, \dots, x_n) \in [0,1]^n$, $()$ and $[\,]$ denote permutations of N with $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$ and $x_{[1]} \geq x_{[2]} \geq \dots \geq x_{[n]}$. Although $()$ and $[\,]$ are dependent on x, we will omit this dependence if it is implicit from context. Wherever we need to distinguish between permutations, we will use $()_{\mathbf{x}}$ and $[\,]_{\mathbf{x}}$ for clarity. We use the convention $x_{(0)} = 0$ throughout this article.
(iv) For each $i \in N$, $\mathbf{e}_i \in [0,1]^n$ denotes the ith member of the canonical basis of $\mathbb{R}^n$.
(v) The *simplex* of $[0,1]^n$ associated with a permutation $()$ of N is

$$\Delta_{()} = \{(y_1, \dots, y_n) \in [0,1]^n \mid y_{(1)} \leq \dots \leq y_{(n)}\}.$$

The *strict simplex* associated with $()$ is

$$\Delta^{()} = \{(y_1, \dots, y_n) \in [0,1]^n \mid y_{(1)} < \dots < y_{(n)}\}.$$

(vi) The mapping $f : [0,1]^n \to [0,1]$ is *increasing* when $\mathbf{x}, \mathbf{y} \in [0,1]^n$ and $\mathbf{y} \geq \mathbf{x}$ imply $f(\mathbf{y}) \geq f(\mathbf{x})$.

An increasing mapping $f : [0,1]^n \to [0,1]$ is called an *aggregation function* [21,22] whenever $f(0, \dots, 0) = 0$, $f(1, \dots, 1) = 1$. And f is a *symmetric* function when for each permutation $()$ of N and each $(y_1, \dots, y_n) \in [0,1]^n$,

$$f(y_1, \dots, y_n) = f(y_{(1)}, \dots, y_{(n)}).$$

We say that $\mathbf{w} = (w_1, \dots, w_n) \in [0,1]^n$ is a *weighting vector* when $\sum_{j=1}^n w_j = 1$. The weighted average mean associated with such $\mathbf{w}$ is $\mathrm{WAM}_{\mathbf{w}}(\mathbf{x}) = w_1 x_1 + \dots + w_n x_n$. The *OWA operator* [1] defined with the weighting vector $\mathbf{w} = (w_1, \dots, w_n)$ is $\mathrm{OWA}_{\mathbf{w}} : [0,1]^n \to [0,1]$ such that when $\mathbf{x} = (x_1, \dots, x_n) \in [0,1]^n$:

$$\mathrm{OWA}_{\mathbf{w}}(\mathbf{x}) = \sum_{j=1}^n w_j x_{[j]}.$$

The IOWA operator is similar, but the permutation in the formula above is always induced by a fixed vector [4,9]. All in all, it is a weighted average mean associated with a fixed weighting vector obtained from the data.

Recently Halaš et al. [2] have introduced the $(\mathbf{w}, \mathbf{b})$-weighted average means. Each function $\mathrm{WAM}_{\mathbf{w},\mathbf{b}}$ in this class is defined from a weighting vector $\mathbf{w}$ plus $\mathbf{b} : [0,1]^n \to [0,1]^n$, an order-inducing mapping. When $\mathbf{x} \in [0,1]^n$, it assigns the value

$$\mathrm{WAM}_{\mathbf{w},\mathbf{b}}(\mathbf{x}) = \mathrm{WAM}_{\mathbf{w}_{\mathbf{b}(\mathbf{x})}}(\mathbf{x}). \tag{1}$$

In this formula $\mathbf{w}_{\mathbf{b}(\mathbf{x})}$ denotes the simple average mean of all vectors $\mathbf{w}_{\sigma^{-1}}$ that satisfy $\mathbf{b}(\mathbf{x})_{\sigma(1)} \geq \ldots \geq \mathbf{b}(\mathbf{x})_{\sigma(n)}$. This class of functions extends both WAMs (when all $\mathbf{b}(\mathbf{x})$'s are strictly decreasing), OWAs (when $\mathbf{b}$ is the identity mapping), and IOWAs (when $\mathbf{b}$ is constant), owing to [2, Theorem 3.1]. They are not necessarily increasing, therefore not all $(\mathbf{w}, \mathbf{b})$-weighted average means are aggregation functions.

A mapping $m : 2^N \to [0,1]$ is a *fuzzy measure* [23] on N when (i) $m(N) = 1$, $m(\emptyset) = 0$, and (ii) $X \subseteq Y \subseteq N$ implies $m(X) \leq m(Y)$. The *Choquet integral* defined by m is $C_m : [0,1]^n \to [0,1]$ such that when $\mathbf{x} = (x_1, \dots, x_n) \in [0,1]^n$,

$$C_m(\mathbf{x}) = \sum_{j=1}^n (x_{(j)} - x_{(j-1)}) \cdot m(A^{\mathbf{x}}_{(j)}), \tag{2}$$

where $A^{\mathbf{x}}_{(j)} = \{(j), (j+1), \dots, (n)\}$ for all $j \in N$. Equivalently, one can write

$$C^{\mu}(\mathbf{x}) = \sum_{j=1}^n \Delta_{(j)} m(A^{\mathbf{x}}_{(j)}) x_{(j)},$$

where $\Delta_k m(A \cup \{k\}) = m(A \cup \{k\}) - m(A)$ are called *discrete derivatives* of m for each $A \subseteq N \setminus \{k\}$ and $k \in N$.

Remark 2.1. The discrete Choquet integral associates every $\mathbf{x} \in [0,1]^n$ with a number through a linear combination of its components in a context-dependent manner: the weighting vector that uses depends on the *simplex* that $\mathbf{x}$ lies in, and it is extracted from a fuzzy measure.

Bustince et al. [20] utilized the preceding observation to establish a generalized methodology for the numerical evaluation of vectors; the fundamentals of this approach are detailed in the subsequent section.

3. Advanced background: General framework containing Choquet integrals

The model designed in [20] operates as follows to compute utility functions on $[0,1]^n$. When $\mathbf{x} \in [0,1]^n$, a vector $W^{\mathbf{x}} \in \{1\} \times [0,1]^{n-1} \times \{0\} \subseteq [0,1]^{n+1}$ is associated that satisfies

$$W^{\mathbf{x}}_1 = 1 \geq W^{\mathbf{x}}_2 \geq \dots \geq W^{\mathbf{x}}_n \geq W^{\mathbf{x}}_{n+1} = 0.$$

A mapping $W : [0,1]^n \longrightarrow [0,1]^{n+1}$ has been defined, making the elements of W dependent on $\mathbf{x}$. The model uses the weighting vectors $w^{\mathbf{x}} \in [0,1]^n$ produced from W as follows: when $\mathbf{x} \in [0,1]^n$,

$$w^{\mathbf{x}}_i = W^{\mathbf{x}}_i - W^{\mathbf{x}}_{i+1} \text{ for each } i \in N.$$

Thus another mapping $w : [0,1]^n \longrightarrow [0,1]^n$ has been defined, making the elements of w dependent on $\mathbf{x}$ too (with each $\mathbf{x} \in [0,1]^n$, we have $w^{\mathbf{x}}$). Now we are ready to define a utility function $A^* : [0,1]^n \to [0,1]$ through the expression

$$A^*(\mathbf{x}) = \sum_{i=1}^n w^{\mathbf{x}}_i \cdot x_{(i)}, \text{ for each } \mathbf{x} \in [0,1]^n. \tag{3}$$

This formula can be rearranged as follows: for all $\mathbf{x} \in [0,1]^n$, using the maintained convention $x_{(0)} = 0$,

$$A^*(\mathbf{x}) = x_{(1)} + \sum_{i=1}^{n-1} (x_{(i+1)} - x_{(i)}) W^{\mathbf{x}}_{i+1} = \sum_{i=0}^{n-1} (x_{(i+1)} - x_{(i)}) W^{\mathbf{x}}_{i+1}. \tag{4}$$

The later expression is formally similar to Eq. (2). Not suprisingly, all discrete Choquet integrals defined from a fuzzy measure can be obtained in this way [20]. In particular, all OWA operators are defined by taking appropriate assignments that are independent of $\mathbf{x}$ as follows. Suppose that the OWA operator is associated with the weighting vector $(w_1, \dots, w_n) \in [0,1]^n$, then we need $W_1 = 1$, $W_j = W_{j-1} - w_{j-1}$ for each $j \in N \setminus \{1\}$, and $W_{n+1} = 0$.

Bustince et al. [20] showed that A^* defined as in this section is always idempotent. Therefore it satisfies the boundary conditions that are required for being an aggregation function. When $W^{\mathbf{x}} = W^{\mathbf{x}+\mathbf{k}}$ for all $\mathbf{x}, \mathbf{k} \in [0,1]^n$ such that $\mathbf{k}$ is a constant vector that satisfies $\mathbf{x} + \mathbf{k} \in [0,1]^n$,

A^* is shift-invariant too [20, Remark 3.1 (ii)]. However these authors prove by example that increasingness is not guaranteed.

The natural question that arises is: which additional conditions enable us to guarantee that A^* is an aggregation function? As a first answer, Bustince et al. [20] produced a particular model that necessitates preliminary notation. Let $n \geq 3$. For $i, j \in N$, $i \neq j$, and $\mathbf{a} = (a_1, \ldots, a_n) \in [0,1]^n$, $\hat{a}_{i,j}$ represents the $(n-2)$-dimentional vector that eliminates a_i and a_j from $\mathbf{a}$.

Proposition 3.1. *[20] Let $n \geq 3$. If $F : [0,1]^{n-2} \to [0,1]$ is an increasing and symmetric function, then it defines an aggregation function $\Gamma_F : [0,1]^n \to [0,1]$ as follows: if $\mathbf{x} = (x_1, \ldots, x_n) \in [0,1]^n$,*

$$\Gamma_F(\mathbf{x}) = x_{(1)} + \sum_{j=1}^{n-1} (x_{(j+1)} - x_{(j)}) F(\hat{\mathbf{x}}_{(j),(j+1)}). \tag{5}$$

Note that this model stems from the case where we set $W^{\mathbf{x}}_{j+1} = F(\hat{\mathbf{x}}_{(j),(j+1)})$, throughout, F having the properties described in Proposition 3.1. In addition, the conditions of Proposition 3.1 guarantee that Γ_F is an idempotent and symmetric aggregation function [20, Prop. 4.1, 4.2] called *Choquet-inspired aggregation function*.

In the next section the question raised by the structural problems of the general framework described above is answered in a completely different way. We define and investigate another class of utility functions that pertain to the framework defined by [20] and described in this section, for which the vector $W^{\mathbf{x}}$ is partition-dependent. Natural properties of this assignment suffice to ensure that the resulting A^* is an aggregation function.

4. New framework and aggregation functions defined by Choquet-partitioned functions

Remark 2.1 explained how the discrete Choquet integral calculates the evaluations of vectors by looking at a simplex, which determines the numbers used in the corresponding summation. We generalize this idea in the framework of the model recalled in Section 3 as follows:

Definition 4.1. Let $\mathcal{A} = \{P_i \mid i \in I\}$ be a partition of $[0,1]^n$, indexed by a set of indices I.

Suppose that for each $i \in I$, a vector $W^{P_i} \in \{1\} \times [0,1]^{n-1} \times \{0\} \subseteq [0,1]^{n+1}$ is given that satisfies

$$W^{P_i}_1 = 1 \geq W^{P_i}_2 \geq \cdots \geq W^{P_i}_n \geq W^{P_i}_{n+1} = 0.$$

We say that $\{W^{P_i} \mid i \in I\}$ is a *partitioned weight generator*. We denote it as $W_{\mathcal{A}}$.

The vectors in $W_{\mathcal{A}} = \{W^{P_i} \mid i \in I\}$ let us define a utility function as in Section 3. First we define a mapping from $[0,1]^n$ to $[0,1]^{n+1}$ as follows:

$$\text{when } \mathbf{x} \in [0,1]^n, W^{\mathbf{x}}_{\mathcal{A}} = W^{P_i} \text{ with } \mathbf{x} \in P_i \in \mathcal{A}. \tag{6}$$

With this mapping we routinely compute for all $j \in N$ and $\mathbf{x} \in [0,1]^n$:

$w^{\mathbf{x}}_j = W^{P_i}_j - W^{P_i}_{j+1}$ such that $\mathbf{x} \in P_i$,

and finally we resort to Eq. (3) to write down

$$A^*_{\mathcal{A}}(\mathbf{x}) = \sum_{j=1}^{n} w^{\mathbf{x}}_j \cdot x_{(j)}, \text{ for each } \mathbf{x} \in [0,1]^n. \tag{7}$$

This is the *Choquet-partitioned function defined by* $W_{\mathcal{A}}$. A rearrangement of the terms lets us define this function in the equivalent form: for each $\mathbf{x} \in [0,1]^n$ and $i \in I$ with $\mathbf{x} \in P_i$,

$$A^*_{\mathcal{A}}(\mathbf{x}) = \sum_{j=0}^{n-1} (x_{(j+1)} - x_{(j)}) W^{P_i}_{j+1}. \tag{8}$$

In the rest of this section, we first position this model in relation with the literature. Examples show that in general, it produces functions that lack basic properties such as increasingness, shift-invariance, and continuity. Then we study conditions that ensure these properies. Particularly, Section 4.3 proves that the model produces aggregation functions under certain testable conditions.

4.1. Relationship with the literature and examples

Intuitively, we describe the structural addition of the model described above to the flexible framework proposed by Bustince et al. [20] as follows. Vectors are associated with a partition of the n-dimensional cube in such way that in the final computations, the elements of W are dependent on each vector $\mathbf{x}$ *through the member of the partition* $\mathbf{x}$ *belongs to*. Thanks to this two-step dependence we define $W_{\mathcal{A}} : [0,1]^n \longrightarrow [0,1]^{n+1}$, and we denote $W^{\mathbf{x}}_{\mathcal{A}} = W_{\mathcal{A}}(\mathbf{x})$ throughout for simplicity.

The next particular examples position this model in relation with known cases:

1. When the partition $\mathcal{A}$ is the finest (i.e., we use $\mathcal{A}^{\infty} = \{P_{\mathbf{x}} = \{\mathbf{x}\} | \mathbf{x} \in [0,1]^n\}$), then we have the model reported in the previous section.
   This instance overrides the role of the partition and returns the model in [20]. For this reason, it assures that OWAs and Choquet integrals, and IOWAs, are particular cases of Choquet-partitioned functions, owing to [20, Section 3]. We revisit this ability below in this section.
2. When the partition $\mathcal{A}$ is the smallest (i.e., we use $\mathcal{A}^0 = \{[0,1]^n\}$), then we have a constant vector (i.e., irrespectively of $\mathbf{x}$, we use $W^{\mathbf{x}}_{\mathcal{A}} = (1, W_2, \ldots, W_n, 0)$ for some fixed values with $1 \geq W_2 \geq \cdots \geq W_n \geq 0$), leading to an OWA operator too [20].
   The same is true when the partition $\mathcal{A}$ is arbitrary and $W^{P_i} = W^{P_j}$ for all $i, j \in I$.
3. We have the standard definition of a discrete Choquet integral when the partition $\mathcal{A}$ is *simplicial*. This means $\mathcal{A} = \{P_{()} \mid () \text{ is a permutation of } N\}$ with the conditions (a) $\Delta^{()} \subseteq P_{()}$ for each permutation $()$ of N, and (b) $\mathbf{x} \in P_{()}$ implies $x_{(1)} \leq \cdots \leq x_{(n)}$. This case is examined in full detail below.

A simplicial partition distributes the inputs of $[0,1]^n$ in subsets indexed by the permutations of N. The inputs in a given member of the partition $P_{()}$ are comonotonic: the permutation $()$ orders their components from lowest to highest values. When all the components of $\mathbf{x} \in [0,1]^n$ are different ($\mathbf{x}$ belongs to a strict simplex), then it is forcefully a member of $P_{()}$ where $()$ is the unique permutation such that $\mathbf{x} \in \Delta^{()}$. When $\mathbf{x} \in [0,1]^n$ is not in a strict simplex (because at least two of its components are equal) then the partition assigns it consistently, in the sense that it belongs to a member of the partition where the inputs are comonotonic with it.

Any partitioned weight generator whose underlying partition is simplicial produces a Choquet-partitioned function that is a discrete Choquet integral.

Conversely, suppose that we have a discrete Choquet integral defined from a fuzzy measure m on N. Then we borrow arguments from [20] to state that it can be obtained by any simplicial partition of $[0,1]^n$ with the assignment $W^{P_{()}}_i = m(\{(i), \ldots, (n)\})$ for all $i = 1, 2, \ldots, n+1$ and permutation $()$ of N.

Concerning the $(\mathbf{w}, \mathbf{b})$-weighted average means defined by Eq. (1), it is also the case that it must be a particular Choquet-partitioned function defined by $W_{\mathcal{A}^{\infty}}$ defined as follows. Observe that Eq. (1) can be rewritten as

$$\text{WAM}_{\mathbf{w},\mathbf{b}}(\mathbf{x}) = \text{OWA}_{\sigma^{-1}(\mathbf{w}_{\mathbf{b}(\mathbf{x})})}(\mathbf{x}), \tag{9}$$

where $\sigma(x_1) \geq \ldots \geq \sigma(x_n)$. Both the permutation σ and the vector $\mathbf{w}_{\mathbf{b}(\mathbf{x})}$ depend on $\mathbf{x}$. Therefore if we use $\mathcal{A}^{\infty}$ and for each $\mathbf{x}$ we compute the vector $W^{P_{\mathbf{x}}}$ such that

$W^{P_{\mathbf{x}}}_1 = 1$,
$W^{P_{\mathbf{x}}}_2 = W^{P_{\mathbf{x}}}_1 - (\mathbf{w}_{\mathbf{b}(\mathbf{x})})_{\sigma^{-1}(1)}$,
$\ldots$
$W^{P_{\mathbf{x}}}_n = W^{P_{\mathbf{x}}}_{n-1} - (\mathbf{w}_{\mathbf{b}(\mathbf{x})})_{\sigma^{-1}(n)}$, and
$W^{P_{\mathbf{x}}}_{n+1} = 0$,

we obtain a partitioned weight generator $W_{\mathcal{A}^{\infty}}$ that guarantees $A^*_{\mathcal{A}^{\infty}}(\mathbf{x}) = \text{WAM}_{\mathbf{w},\mathbf{b}}(\mathbf{x})$ when $\mathbf{x} \in [0,1]^n$.

Example 4.1. When $n = 2$, [2, Example 3.1] calculated the following $(\mathbf{w}, \mathbf{b})$-weighted average mean: for each $\mathbf{x} = (x, y) \in [0,1]^2$,

$$\mathrm{WAM}_{\mathbf{w},\mathbf{b}}(\mathbf{x}) = \begin{cases} x\,w_1 + y\,w_2 & \text{when } x - y > \frac{1}{3}, \\ x\,\frac{1}{2} + y\,\frac{1}{2} & \text{when } x - y = \frac{1}{3}, \\ x\,w_2 + y\,w_1 & \text{when } x - y < \frac{1}{3}. \end{cases}$$

Let us write this function differently to how the authors obtained it: it is the $(\mathbf{w}, \mathbf{b})$-weighted average mean defined from the weighting vector $\mathbf{w} = (w_1, w_2)$ and the order-inducing function

$$\mathbf{b}(\mathbf{x}) = \begin{cases} (1,0) & \text{when } x - y > \frac{1}{3}, \\ (1,1) & \text{when } x - y = \frac{1}{3}, \text{ and} \\ (0,1) & \text{when } x - y < \frac{1}{3}. \end{cases}$$

Observe that there are only 2 permutations on $N = \{1, 2\}$, namely, id and id^{-1}. We are now ready to replicate the argument above, taking into account that

$$\mathbf{w}_{\mathbf{b}(\mathbf{x})} = \begin{cases} \mathbf{w}_{id} = (w_1, w_2) & \text{when } x - y > \frac{1}{3}, \\ \frac{1}{2}(\mathbf{w}_{id} + \mathbf{w}_{id^{-1}}) = (\frac{1}{2}, \frac{1}{2}) & \text{when } x - y = \frac{1}{3}, \\ \mathbf{w}_{id^{-1}} = (w_2, w_1) & \text{when } x - y < \frac{1}{3}. \end{cases}$$

New examples of utility functions emerge from the use of Eq. (7) with vectors stemming from Definition 4.1, in addition to the familiar situations that we have mentioned so far. First we proceed to illustrate the new model with a shortlist of low-dimensional examples. Then we study basic properties and situations conducive to aggregation functions in the rest of this section.

Example 4.2. We fix $n = 2$.

1. Let $\mathcal{A}_1 = \{P_y \mid y \in [0,1]\}$ with $P_y = \{(x, y) \mid x \in [0,1]\}$ for each $y \in [0,1]$. We also let $W^{P_y} = (1, y, 0)$ when $y \in [0,1]$.

Any $\mathbf{x} = (x, y) \in [0,1]^2$ belongs to $P_y \in \mathcal{A}_1$ thus the mapping defined by Eq. (6) computes

$$W^{\mathbf{x}}_{\mathcal{A}_1} = W^{P_y} = (1, y, 0),$$

therefore $w^{\mathbf{x}}_1 = 1 - y, \quad w^{\mathbf{x}}_2 = y - 0.$

Then our model defines A^*_1 given by:

$$A^*_1(\mathbf{x}) = \begin{cases} x\,w^{\mathbf{x}}_1 + y\,w^{\mathbf{x}}_2 & \text{when } x \le y \\ y\,w^{\mathbf{x}}_1 + x\,w^{\mathbf{x}}_2 & \text{when } x \ge y. \end{cases}$$

which in this case, produces the utility function

$$A^*_1(\mathbf{x}) = \begin{cases} x + y^2 - xy & \text{when } x \le y \\ y - y^2 + xy & \text{when } x \ge y. \end{cases}$$

2. Let $\mathcal{A}_2 = \{P_x \mid x \in [0,1]\}$ with $P_x = \{(x, y) \mid y \in [0,1]\}$ for each $x \in [0,1]$. We also let $W^{P_x} = (1, x, 0)$ when $x \in [0,1]$.

Repeating the computations in the case described above, now any $\mathbf{x} = (x, y) \in [0,1]^2$ belongs to $P_x \in \mathcal{A}_2$ thus

$$W^{\mathbf{x}}_{\mathcal{A}_2} = W^{P_x} = (1, x, 0),$$

$$w^{\mathbf{x}}_1 = 1 - x, \quad w^{\mathbf{x}}_2 = x - 0.$$

Therefore, our model defines the utility function

$$A^*_2(\mathbf{x}) = \begin{cases} x - x^2 + xy & \text{when } x \le y \\ y + x^2 - xy & \text{when } x \ge y. \end{cases}$$

From inspection, both $A^*_1(\mathbf{x})$ and $A^*_2(\mathbf{x})$ are aggregation functions. Fig. 1 shows their graphical representations.

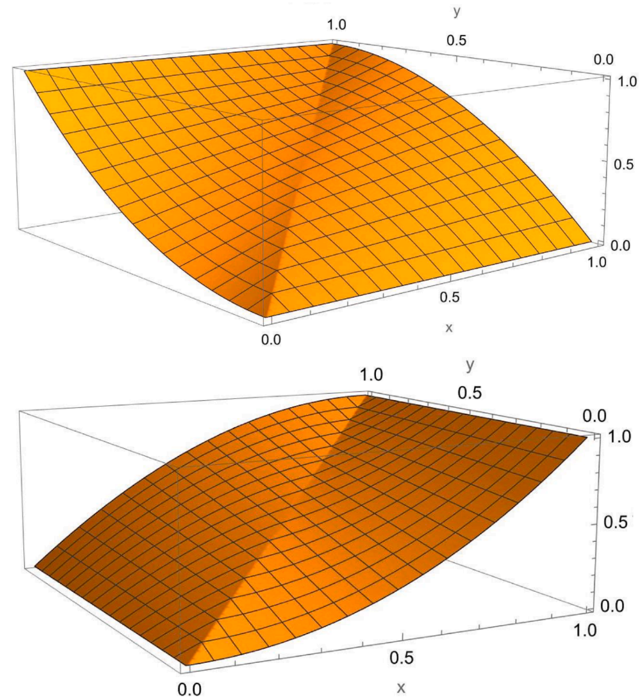

Fig. 1. The aggregation functions $A^*_1(x, y)$ and $A^*_2(x, y)$ in Example 4.2.

The next example shows two cases where the utility function is not continuous:

Example 4.3. We continue our examination of cases where $n = 2$.

1. Let $\mathcal{A}_3 = \{P_1, P_2\}$ with $P_1 = \{(x, y) \mid y \in [0.5, 1]\}$, $P_2 = \{(x, y) \mid y \in [0, 0.5)\}$. We also let $W^{P_1} = (1, 1, 0)$ and $W^{P_2} = (1, 0, 0)$.

Thus for each $\mathbf{x} = (x, y) \in [0,1]^2$,

$$W^{\mathbf{x}}_{\mathcal{A}_3} = \begin{cases} W^{P_1} = (1, 1, 0) & \text{when } y \ge 0.5, \\ W^{P_2} = (1, 0, 0) & \text{when } y < 0.5. \end{cases}$$

This means that

$$w^{\mathbf{x}}_1 = 0, w^{\mathbf{x}}_2 = 1 \text{ when } y \ge 0.5, \text{ and}$$

$$w^{\mathbf{x}}_1 = 1, w^{\mathbf{x}}_2 = 0 \text{ when } y < 0.5.$$

Routine computations show that our model defines A^*_3 given by:

$$A^*_3(\mathbf{x}) = \begin{cases} y & \text{if either } (x \le y, y \ge 0.5) \text{ or } (x \ge y, y < 0.5), \\ x & \text{if either } (x \ge y \ge 0.5) \text{ or } (x \le y < 0.5). \end{cases}$$

2. Let $\mathcal{A}_4 = \{P_1, P_2\}$ with $P_1 = \{(x, y) \mid x \in [0.5, 1]\}$ and $P_2 = \{(x, y) \mid x \in [0, 0.5)\}$. We also let $W^{P_1} = (1, 1, 0)$ and $W^{P_2} = (1, 0, 0)$.

Thus for each $\mathbf{x} = (x, y) \in [0,1]^2$,

$$W^{\mathbf{x}}_{\mathcal{A}_4} = \begin{cases} W^{P_1} = (1, 1, 0) & \text{when } x \ge 0.5, \\ W^{P_2} = (1, 0, 0) & \text{when } x < 0.5. \end{cases}$$

This means that

$$w^{\mathbf{x}}_1 = 0, w^{\mathbf{x}}_2 = 1 \text{ when } x \ge 0.5, \text{ and}$$

$$w^{\mathbf{x}}_1 = 1, w^{\mathbf{x}}_2 = 0 \text{ when } x < 0.5.$$

Routine computations show that our model defines A^*_4 given by:

$$A^*_4(\mathbf{x}) = \begin{cases} y & \text{if either } (y \ge x \ge 0.5) \text{ or } (y \le x < 0.5), \\ x & \text{if either } (x \ge y, x \ge 0.5) \text{ or } (y \ge x, x < 0.5). \end{cases}$$

From inspection, both $A^*_3(\mathbf{x})$ and $A^*_4(\mathbf{x})$ are aggregation functions. Fig. 2 shows their graphical representations. Observe that they are not continuous in the points with $y = 0.5$, respectively, $x = 0.5$.

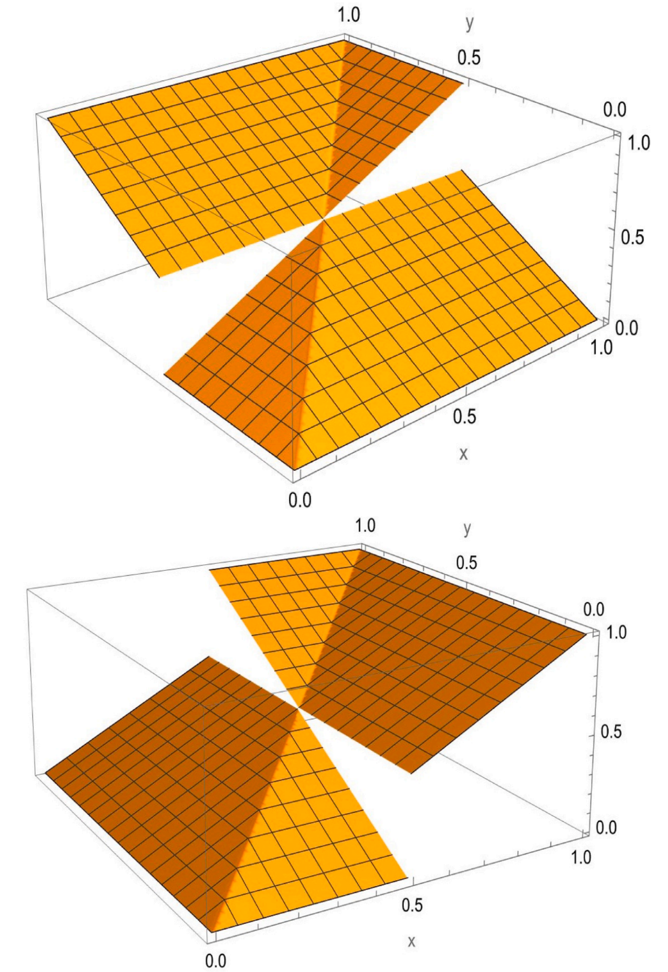

Fig. 2. The aggregation functions $A_3^*(x,y)$ and $A_4^*(x,y)$ in Example 4.3.

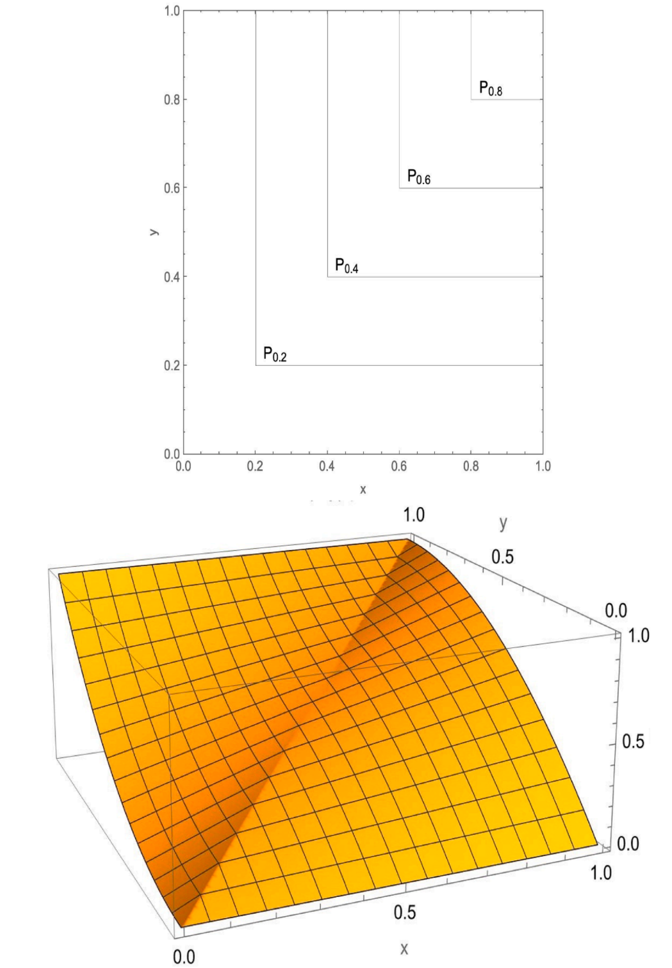

Fig. 3. A visualization of the partition $\mathcal{A}_5$ and the aggregation function $A_5^*(x,y)$ in Example 4.4.

Example 4.4. Another interesting case where $n = 2$ stems from $\mathcal{A}_5 = \{P_z \mid z \in [0,1]\}$ with $P_z = \{(x,y) \mid \min(x,y) = z\}$ for each $z \in [0,1]$. This partition is inspired by Leontieff preferences. We also let $W^{P_z} = (1, z, 0)$ for each $z \in [0,1]$. Now our model defines A_5^* given by:

$$A_5^*(\mathbf{x}) = \begin{cases} x + xy - x^2 & \text{if } x \leq y, \\ y + xy - y^2 & \text{if } x > y. \end{cases}$$

From inspection, $A_5^*(\mathbf{x})$ is an aggregation function. Fig. 3 shows its graphical representation, plus a visualization of the partition $\mathcal{A}_5$.

Section 4.3 unifies the examples above within a common structure. Prior to this, we put forward basic properties of the functions defined by the model that we have proposed.

4.2. First properties: Analysis of shift-invariance and continuity

Since the new structured description includes the model designed in [20], which does not guarantee increasingness of $A_{\mathcal{A}}^*$, we can assure that $A_{\mathcal{A}}^*$ computed by the model that we have presented is not always an aggregation function. In Section 4.3 we investigate this issue in more depth.

For similar reasons, another property of the Choquet integral that does not always obtain is shift-invariance. To guarantee this property, we consider the following conditions:

Definition 4.2. Fix a partition $\mathcal{A}$ of $[0,1]^n$. Then $\mathcal{A}$ is *shift-invariant* when it is the case that $\mathbf{x} \in P_i \in \mathcal{A}$, $\mathbf{k} = (k, \dots, k) \in [0,1]^n$, $\mathbf{x} + \mathbf{k} \in [0,1]^n$, imply $\mathbf{x} + \mathbf{k} \in P_i$.

For example, $\mathcal{A}^0$ is a shift-invariant partition of $[0,1]^2$. $\mathcal{A}^d = \{P_1^d, P_2^d, P_3^d\}$ with $P_1^d = \{(x,y) \mid x > y\}$, $P_2^d = \{(x,y) \mid x < y\}$, $P_3^d = \{(x,y) \mid x = y\}$, is a shift-invariant partition of $[0,1]^2$, and so is its variation $\mathcal{A}^e$ defined in Example 4.5 below.

Using the aforementioned [20, Remark 3.1 (ii)] we obtain as a corollary:

Proposition 4.1. *Let $W_{\mathcal{A}}$ be a partitioned weight generator associated with $\mathcal{A}$ that is shift-invariant. Then $A_{\mathcal{A}}^*$ is shift-invariant.*

Shift-invariant partitions are *shift-consistent*. This property holds when $\mathbf{x}, \mathbf{y} \in P_i \in \mathcal{A}$, $\mathbf{k} = (k, \dots, k) \in [0,1]^n$, $\mathbf{x} + \mathbf{k}, \mathbf{y} + \mathbf{k} \in [0,1]^n$, imply $\mathbf{x} + \mathbf{k}, \mathbf{y} + \mathbf{k} \in P_j$ for some $j \in I$. $\mathcal{A}^\infty$ proves that shift-consistency does not suffice to obtain the consequence of Proposition 4.1. In particular, this is a shift-consistent partition that is not shift-invariant.

With respect to continuity, recall that Example 4.3 produced two aggregation functions $A_3^*(x,y)$ and $A_4^*(x,y)$ that are not continuous in $[0,1] \times [0,1]$. Hence the design of utility functions that we describe in this section does not guarantee continuity. A natural sufficient condition whose proof is straightforward is presented below:

Proposition 4.2. *Let $W_{\mathcal{A}}$ be a partitioned weight generator associated with $\mathcal{A}$. If the mapping $W_{\mathcal{A}}^{\mathbf{x}}$ defined by Eq. (6) is continuous, then $A_{\mathcal{A}}^*$ is continuous.*

Observe that the assignments $W_{\mathcal{A}_3}^{\mathbf{x}}$ and $W_{\mathcal{A}_4}^{\mathbf{x}}$ computed in Example 4.3 are not continuous in respective segments of $[0,1] \times [0,1]$ (to wit, in $y = 0.5$, respectively, $x = 0.5$). This is the cause of the discontinuities observed in the $A_3^*(x,y)$ and $A_4^*(x,y)$ functions (v., Fig. 2). The constructions in Examples 4.2 and 4.4 illustrate the use of Proposition 4.2.

4.3. Aggregation functions defined by Choquet-partitioned functions

To guarantee the production of aggregation functions defined by the procedure described above in this section, we shall consider the following particular specification of the partitioned weight generator model:

Definition 4.3. Fix $W_{\mathcal{A}}$, a partitioned weight generator (v., Definition 4.1). We say that $W_{\mathcal{A}}$ is *increasing* when $\mathbf{x},\mathbf{y}\in[0,1]^n$ and $\mathbf{x}\geq\mathbf{y}$ imply $W_{\mathcal{A}}^{\mathbf{x}}\geq W_{\mathcal{A}}^{\mathbf{y}}$.

In words, Definition 4.3 requires that the mapping defined from $W_{\mathcal{A}}$ by Eq. (6) be increasing. Note that all $W_{\mathcal{A}_1},\dots,W_{\mathcal{A}_5}$ studied in Examples 4.2 to 4.4 are increasing and they define respective aggregation functions using Eq. (7). We now prove that both facts are not independent: increasingness suffices to guarantee the production of $A_{\mathcal{A}}^*$ that is an aggregation function using Eq. (7). However, Example 4.5 below demonstrates that being increasing is not a necessary condition.

Theorem 4.1. *Let $W_{\mathcal{A}}$ be an increasing partitioned weight generator. Then $A_{\mathcal{A}}^*$ is an aggregation function.*

Proof. As $A_{\mathcal{A}}^*$ is idempotent, it satisfies the boundary conditions $A_{\mathcal{A}}^*(0,\dots,0)=0$, $A_{\mathcal{A}}^*(1,\dots,1)=1$. Therefore we only need to prove monotonicity. It suffices to check a restricted version of this property: the conclusion will follow from a recursive argument, if we prove that for each $\mathbf{x}\in[0,1]^n$, $k\in N$, and $\delta>0$ such that $\mathbf{y}=\mathbf{x}+\delta\mathbf{e}_{(k)}\in[0,1]^n$, it is the case that $A_{\mathcal{A}}^*(\mathbf{y})\geqslant A_{\mathcal{A}}^*(\mathbf{x})$. There exist $P_i,P_{i'}\in\mathcal{A}$ with $\mathbf{x}\in P_i$, $\mathbf{y}\in P_{i'}$. We distinguish two cases.

Case 1: $\mathbf{x}$ is comonotonic with $\mathbf{y}$, i.e., $x_{(1)}=y_{(1)}\leq\dots\leq y_{(k)}=x_{(k)}+\delta\leq x_{(k+1)}=y_{(k+1)}\leq\dots\leq x_{(n)}=y_{(n)}$. We use Eq. (8) to compute

$$A_{\mathcal{A}}^*(\mathbf{y})-A_{\mathcal{A}}^*(\mathbf{x})=\sum_{\substack{j=0\\k-1\neq j\neq k}}^{n-1}(x_{(j+1)}-x_{(j)})(W_{j+1}^{P_{i'}}-W_{j+1}^{P_i})+$$
$$(y_{(k)}-y_{(k-1)})W_k^{P_{i'}}-(x_{(k)}-x_{(k-1)})W_k^{P_i}+$$
$$(y_{(k+1)}-y_{(k)})W_{k+1}^{P_{i'}}-(x_{(k+1)}-x_{(k)})W_{k+1}^{P_i}.$$

Rearranging the summands and substituting values,

$$A_{\mathcal{A}}^*(\mathbf{y})-A_{\mathcal{A}}^*(\mathbf{x})=\sum_{\substack{j=0\\k-1\neq j\neq k}}^{n-1}(x_{(j+1)}-x_{(j)})(W_{j+1}^{P_{i'}}-W_{j+1}^{P_i})+$$
$$(x_{(k)}-x_{(k-1)})W_k^{P_{i'}}+\delta W_k^{P_{i'}}+(x_{(k+1)}-x_{(k)})W_{k+1}^{P_{i'}}$$
$$-\delta W_{k+1}^{P_{i'}}-(x_{(k)}-x_{(k-1)})W_k^{P_i}-(x_{(k+1)}-x_{(k)})W_{k+1}^{P_i}.$$

Therefore we include terms in the summation and obtain

$$A_{\mathcal{A}}^*(\mathbf{y})-A_{\mathcal{A}}^*(\mathbf{x})=\sum_{j=0}^{n-1}(x_{(j+1)}-x_{(j)})(W_{j+1}^{P_{i'}}-W_{j+1}^{P_i})+\delta(W_k^{P_{i'}}-W_{k+1}^{P_{i'}}).$$

A routine utilization of increasingness proves $A_{\mathcal{A}}^*(\mathbf{y})-A_{\mathcal{A}}^*(\mathbf{x})\geq 0$.

Case 2: $\mathbf{x}$ is not comonotonic with $\mathbf{y}$. We completely expand the computations when $x_{(1)}=y_{(1)}\leq\dots\leq y_{(k)}=x_{(k+1)}<x_{(k)}+\delta=y_{(k+1)}\leq\dots\leq x_{(n)}=y_{(n)}$, i.e., when the addition of δ to the (k) component makes it overcome exactly one position in the ordered list of components of the vectors. The general case is similar, but notationally cumbersome.

In these circumstances,

$$A_{\mathcal{A}}^*(\mathbf{y})-A_{\mathcal{A}}^*(\mathbf{x})=\sum_{\substack{j=0\\k-1\neq j\neq k}}^{n-1}(x_{(j+1)}-x_{(j)})(W_{j+1}^{P_{i'}}-W_{j+1}^{P_i})+$$
$$(x_{(k+1)}-x_{(k-1)})W_k^{P_{i'}}-(x_{(k)}-x_{(k-1)})W_k^{P_i}+$$
$$(x_{(k)}+\delta-x_{(k+1)})W_{k+1}^{P_{i'}}-(x_{(k+1)}-x_{(k)})W_{k+1}^{P_i}.$$

Because $\sum_{\substack{j=0\\k-1\neq j\neq k}}^{n-1}(x_{(j+1)}-x_{(j)})(W_{j+1}^{P_{i'}}-W_{j+1}^{P_i})\geq 0$ owing to increasingness, the proof will be concluded if we show $(x_{(k+1)}-x_{(k-1)})W_k^{P_{i'}}-(x_{(k)}-x_{(k-1)}))W_k^{P_i}+(x_{(k)}+\delta-x_{(k+1)})W_{k+1}^{P_{i'}}-(x_{(k+1)}-x_{(k)})W_{k+1}^{P_i}\geq 0$. To this purpose, we expand $(x_{(k+1)}-x_{(k-1)})W_k^{P_{i'}}=(x_{(k+1)}-x_{(k)}+x_{(k)}-x_{(k-1)})W_k^{P_{i'}}$. Then after some rearrangement, the sum under inspection becomes

$$(x_{(k+1)}-x_{(k)})W_k^{P_{i'}}+(x_{(k)}-x_{(k-1)})W_k^{P_{i'}}-(x_{(k)}-x_{(k-1)})W_k^{P_i}$$
$$-(x_{(k+1)}-x_{(k)})W_{k+1}^{P_i}+(x_{(k)}+\delta-x_{(k+1)})W_{k+1}^{P_{i'}}.$$

Now we observe that this expression is the sum of three non-negative components, namely:

$$(x_{(k)}-x_{(k-1)})(W_k^{P_{i'}}-W_k^{P_i})\geq 0,$$
$$(x_{(k)}+\delta-x_{(k+1)})W_{k+1}^{P_{i'}}\geq 0,\ \text{and}$$
$$(x_{(k+1)}-x_{(k)})(W_k^{P_{i'}}-W_{k+1}^{P_i})\geq(x_{(k+1)}-x_{(k)})(W_{k+1}^{P_{i'}}-W_{k+1}^{P_i})\geq 0.$$

The first and last inequalities hold true due to the assumption that $W_{\mathcal{A}}$ is an increasing partitioned weight generator. □

Example 4.5. With $n=2$, consider $\mathcal{A}^e=\{P_1,P_2\}$ with $P_1=\{(x,y)\mid y\leq x\}$, $P_2=\{(x,y)\mid y>x\}$. We also let $W^{P_1}=(1,1,0)$ and $W^{P_2}=(1,0,0)$. This partitioned weight generator is not increasing because $\mathbf{x}=(0.5,0)\in P_1$, $\mathbf{y}=(0.5,1)\in P_2$, $\mathbf{y}\geq\mathbf{x}$, and $W^{P_2}\geq W^{P_1}$ is false.

Thus for each $\mathbf{x}=(x,y)\in[0,1]^2$,

$$W_{\mathcal{A}^e}^{\mathbf{x}}=W^{P_1}=(1,1,0),\ w_1^{\mathbf{x}}=0,\ w_2^{\mathbf{x}}=1\ \text{when}\ x\geq y,$$
$$W_{\mathcal{A}^e}^{\mathbf{x}}=W^{P_2}=(1,0,0),\ w_1^{\mathbf{x}}=1,\ w_2^{\mathbf{x}}=0\ \text{when}\ x<y.$$

Routine computations show that our model defines for each $\mathbf{x}\in[0,1]^2$,

$$A_e^*(\mathbf{x})=\begin{cases}0\cdot y+1\cdot x, & \text{if } x\geq y\\ 1\cdot x+0\cdot y, & \text{if } x<y.\end{cases}$$

This means $A_e^*(\mathbf{x})=x$ for each $\mathbf{x}=(x,y)\in[0,1]^2$, which is a shift-invariant aggregation function. The fact that it obeys shift-invariance owes to Proposition 4.1. More specifically, A_e^* is a Choquet integral defined by a Dirac fuzzy measure. The fact that A_e^* is a Choquet integral has been anticipated in [20, Section 3].

Note that $W_{\mathcal{A}}$ is an increasing partitioned weight generator when $\mathcal{A}=\mathcal{A}^0$ is the smallest partition. For this reason, OWAs aggregation functions –which are discrete Choquet integrals defined from symmetric fuzzy measures– satisfy the assumptions of Theorem 4.1. This is not the case of all discrete Choquet integrals (v., Example 4.5).

5. Conclusion

We have defined and studied Choquet-partitioned functions with the help of the new partitioned weight generators. These functions are a class of numerical utility assignments to inputs from the n-dimensional cube $[0,1]^n$. They belong to the general class of utility functions defined by Bustince et al. [20]. Both the discrete Choquet integrals and $(\mathbf{w},\mathbf{b})$-weighted average means [2] (which generalize both OWAs and IOWAs) can be expressed as Choquet-partitioned functions. This achievement is related to a goal expressed in the concluding section of Halaš et al. [2], namely, the combination of the ideas in [20] and [2]. Properties of the partitioned weight generator have been identified, that guarantee the production of shift-invariant and continuous utility functions. Finally, a natural condition of the partitioned weight generators has been proposed that guarantees that Choquet-partitioned functions are aggregation functions.

This work demonstrates that the general construction introduced by Bustince et al. [20] merits further theoretical investigation. Potential applications of this research extend to the broad range of problems where aggregation functions are currently utilized. Specifically, [20] suggested applicability to the fields of image processing, pattern recognition, deep learning, statistical mechanics, and decision-making. Such applications might help test the performance of each approach in the future.

CRediT authorship contribution statement

José Carlos R. Alcantud: Writing – original draft, Methodology, Conceptualization.

Data availability

No datasets were used for the research described in this article.

Declaration of competing interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this article.

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