



# Preference-Approval Structures and Opportunity Sets

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## Abstract

This paper first proposes the combination of preference-approval structures with ranking opportunity sets (i.e., collections of options), a relevant topic in welfare analysis. We suggest that various possibilities emerge, and we illustrate them with specific examples in finite contexts. Characterizations of three rankings of opportunity sets defined from preference-approval structures on the individual options are proven. Their representability by utilities is considered too. Various preference-approval structures on opportunity sets are defined from the information contained in a preference-approval structure on the options, that supplement these rankings of opportunity sets with appropriate collections of “approved” opportunity sets. Finally, preference-approval structures on opportunity sets are defined using the information derived from a ranking of the options. As a secondary contribution, we compute the number of preference-approval structures on a set with finite cardinality, thereby enriching the fundamental theory of this enhanced type of preferences.

**Keywords** Opportunity set · Preference-approval · Ranking · Utility representation

## 1 Introduction

The central concern in the problem of ranking sets of objects is understanding how to evaluate and compare different sets of options available to an individual. The challenge lies in establishing a consistent way to rank these sets. Interpretations include the case where it is assumed that a subsequent choice will be made from the options in the selected set, or each choice produces consequences under complete ignorance. Pattanaik and Xu (1990) distinguished non-preference-based and preference-based approaches to the problem. The fact that utility only measures one aspect of well-being is the reason behind this broad distinction. The former position aligns with

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the analysis of freedom of choice and preference for flexibility. It assumes that the availability of choices is inherently valuable, even if not all options are utilized by the individual. The later position takes into account individual preferences. The value of a set of alternatives must necessarily consider the quality of the options that it contains, according to the agent's preferences.

The sets of options are typically referred to as "opportunity sets". In the analysis of the freedom of choice they provide, early discussions were given by Jones and Sugden (1982), Suppes (1987), Pattanaik and Xu (1990), Bossert et al. (1994), and Puppe (1996).

In this strand of the literature, the value of an opportunity set is assessed by the degree of freedom it provides. This is a non-utilitarian approach: sets are valued independently of the utility derived from the choices made from these sets, and freedom of choice itself is an essential aspect of welfare. Even if an individual ultimately selects the same option, a larger set is preferred because it provides more freedom. The archetype is the cardinality rule (Pattanaik and Xu 1990). Most models assume finiteness for tractability. But there are exceptions such as Pattanaik and Xu (2000), who extended the study of finite opportunity sets to compact subsets of the non-negative orthant of a multidimensional space.

Considerations of preference for flexibility motivate another departure from preference-based approaches (Kreps 1979). One possible reason is uncertainty about the agent's preferences when she has to make her choice in multi-period decision models. Therefore flexibility is the ability to adapt to unforeseen circumstances. In addition to emphasizing similarities and difference between preference for freedom and preference for flexibility, Puppe (1996) establishes links between both notions, in the form of conditions ensuring that preference in terms of freedom is equivalent to preference in terms of flexibility.

The indirect-utility ranking (Kreps 1979) is the fundamental example of valuation by the most beneficial component. Ballester et al. (2004), extended its analysis to uncountable sets. Lahiri (2003), relaxed the transitivity assumption, and then Ryan (2014) went beyond the utilization of binary relations: in his model, a path-independent choice function determines the agent's choices. Other examples in this vein are minmax and maxmin (Bossert et al. 1994), which are specific cases of maxmin-based orderings (Barberà et al. 1984). Under the preference-oriented description, the problem is linked to the question of 'reasonably' extending an order on a set to its power set (Gärdenfors 1976; Kannai and Peleg 1984). Nevertheless, already Bossert et al. (1994) and Dutta and Sen (1996) argued that a rank solely defined in terms of comparisons of most preferred elements in the sets would be too simplistic. Both views –utilitarian comparisons and the degree of freedom provided by the opportunity sets– are reconciled by e.g., Bossert et al. (1994) and Potoms and Lauwers (2012), among others. Remarkably, Jones and Sugden (1982) posited that in this combination, the preferences that should be considered to rank opportunity sets are not the individual preferences, but the set of preferences that reasonable people might have in the agent's situation. A valuable review is the (now outdated) Barberà et al. (2004), which includes rankings such as cardinality-first lexicographic, preference-first lexicographic, or leximin and leximax.

This work is the first attempt to introduce preference-approval analysis in this field. Preference-approval structures appeared in Brams and Sanver (2009) as a way to determine social choices using hybrid voting systems. Another contribution in this regard is Kamwa (2019), who studied whether the performance of preference-approval voting was better than that of approval voting, in terms of the election of Condorcet winners and non-election of Condorcet losers. Further studies in the field of voting theory are Barokas (2022, 2023), Barokas and Sprumont (2022), and Kamwa (2023). Conceptually, the improvement of preference-approval structures is the inclusion of a set of “approved” objects that is appropriately related to a preference on the set of objects. Later on they were applied to consensus measurement in Erdamar et al. (2014), and to clustering in Albano et al. (2024). Distances have become important in these fields as well (Albano et al. 2023). They are at the root of strategies that determine a group representative (Dong et al. 2021). Kruger and Sanver (2021) initiated the inspection of Arrovian incompatibilities when using preference-approval structures. Relatedly, Arrow’s characterization of dictatorships has been exported to the subclass of complete preorders that produce preference-approval structures with a fixed set of “approved” objects. In this framework, a pairwise-dictatorial behavior is characterized by independence of irrelevant alternatives and unanimity (Santos-García and Alcantud 2025a, b). A ternary version of preference-approval structures called ternary preferences has recently been proposed (Albano et al. 2025; García-Lapresta and Martínez-Panero 2025), in which the alternatives are divided into three categories.

Our study has two parts, each with its own goals and extensions. First we compute the number of preference-approval structures that can be defined on a set with finite cardinality. This result enriches the fundamental theory of this enhanced type of preferences by providing a fact of importance in computational social choice (Brandt et al. 2016). Therefore it paves the way for future connections between the combinatorial foundations of preference-approval modeling with the computational feasibility of decision-making and aggregation mechanisms. Secondly, we consider three natural procedures that use the knowledge contained in a preference-approval structure on the options to rank the opportunity sets that they define by means of complete preorders. We obtain numerical representations for these rankings. The main contribution of this work is the identification of axioms that allow us to prove characterizations of the three new rankings produced in this way. In addition, preference-approval structures on opportunity sets are defined that supplement these and other known rankings with suitable sets of “approved” opportunity sets. We also point out that it is possible to define preference-approval structures on opportunity sets from rankings of the options.

The main results in this work belong to a strand of the literature on ranking opportunity sets initiated by Puppe (1996), whereby not all alternatives contribute to improving the freedom of an agent equally. As Puppe (1996) does, the use of preference-approval structures also allows us to reject a “universal contribution to freedom” principle (embodied in the Simple Expansion Monotonicity axiom in Pattanaik and Xu (1990) or Simple Monotonicity axiom in Bossert et al. (1994)) that dispenses with preference-based evaluations altogether. For comparison, the distinctive feature of Puppe’s models is Axiom F (also called Preference for Freedom of Choice).

This axiom is weaker than the aforementioned expansion monotonicity principle, as it only requests that in each opportunity set, at least one alternative (though not necessarily all of them) contributes to gaining freedom of choice. Then Axiom F pinpoints a respective set of essential alternatives within each non-empty opportunity set. Appropriate conditions guarantee that this assignment fully determines the induced domination relation on opportunity sets considered by Kreps (1979). Axiom F comes at the cost of assuming that all opportunity sets confer freedom of choice. For this reason Puppe (1996) admits the existence of extreme situations that challenge its applicability. We show that the three rankings defined from preference-approval structures that we study include comparisons of this kind. However when the preferences are linear, two of them obey Puppe's Axiom F.

We distribute the contents of this article as follows. Section 2 gives background. Working on finite sets, the basic question about counting preference-approval structures is solved in Sect. 3. We consider the problem of producing reasonable rankings and preference-approval structures on opportunity sets from either preference-approval structures or rankings on the alternatives in Sect. 4. Concluding remarks are given in Sect. 5.

## 2 Notation

Henceforth  $X$  denotes a finite set of  $n$  options and  $\mathcal{P}$  is the set of its non-empty sets. Elements of  $\mathcal{P}$  (non-empty subsets of  $X$ ) are denoted  $Y, Z, \dots$  and they are named opportunity sets.

### 2.1 Preference-Approval Structures and Their Presentation

When  $R$  is a binary relation on  $X$ , we let its strict part be  $P$ , and its indifference be  $I$ . This means that when  $x, y \in X$ ,  $xPy$  is equivalent to  $xRy$  and not  $yRx$ , and also,  $xIy$  is equivalent to  $xRy$  and  $yRx$ . A complete preorder  $R$  on  $X$  is a complete and transitive binary relation on  $X$ . A linear order is an antisymmetric complete preorder. For each  $Y \subseteq X$ , we denote

$$\max_R(Y) = \{y' \in Y \mid yRz \text{ for each } z \in Y\}.$$

When  $R$  is common knowledge we drop the dependence on it, and we simply write  $\max(Y)$ .

Unless otherwise stated, henceforth  $R$  denotes a complete preorder on  $X$ .

A preference-approval structure on  $X$  is a pair  $(R, A)$  where  $R$  is a complete preorder on  $X$ ,  $A \subseteq X$  (a representation of the "approved" options of  $X$ ), and the following consistency property holds:

$$\text{for all } x, y \in X : xRy \text{ and } y \in A \text{ imply } x \in A. \quad (1)$$

It is implicit that the semantic interpretation of  $R$  is “as good as”. Consequently, Eq. (1) captures the intuition that when an alternative is as good as another one that is “approved”, then the former alternative must be “approved” too.

The original idea is that  $A$  captures the “acceptable” alternatives in  $X$  (Brams and Sanver 2009; Barokas 2022, 2023). We keep the term “approved” as a useful default, although in this work the set  $A$  can capture alternatives with different features. We explore various possibilities depending on the context in Sects. 4.1 and 4.2.

The next example illustrates preference-approval structures and the presentation that is standard in this literature (Erdamar et al. 2014).

**Example 1** Suppose  $X = \{x, y, z, t, w\}$ . Consider the next arrangement of the alternatives in  $X$ :

$$\begin{array}{c} w \\ x \ y \\ \hline z \\ t \end{array}$$

We interpret that  $A = \{x, y, w\}$ , the subset of  $X$  formed by the elements above the line. Also, we interpret  $wPxIyPzPt$ . For example,  $x$  and  $y$  are indifferent because we wrote them in the same row, and  $wPx$  because  $w$  is placed in a row above  $x$ . In this way we have defined a preference-approval structure  $(R, A)$  on  $X$ .

In case that  $R$  is a linear order, then Brams and Sanver (2009), suggest the utilization of left to right (instead of up to bottom) disposition of the alternatives. Thus for example in the situation  $X = \{x, y, z, t, w\}$ , the notation  $yx \mid wtz$  means  $A = \{x, y\}$  (the elements to the left of the vertical bar), and also  $yPxPwPtPz$ .

Erdamar et al. (2014) gave a codification of preference-approval structures. It uses a pair formed by a position vector (of  $R$ ) and an indicator vector (of  $A$ ). We do not use it in this article.

## 2.2 Rules for Ranking Opportunity Sets: The Axiomatic Approach

Indirect utilities and freedom of choice are almost opposed considerations for the comparison of opportunity sets. The focus is on whether freedom has an intrinsic value beyond mere utility. To mathematically discuss this philosophical question, the following fundamental rules respectively exemplify cardinality-based, utility-based, and natural combinations of both approaches. Axiomatic characterizations are available for all these rankings. They contribute to the debate about their applicability. Below we recall the axiomatizations of the first two rankings in this shortlist:

1. The cardinality-based rule  $\succ_C$  (Pattanaik and Xu 1990) is the complete preorder on  $\mathcal{P}$  defined as follows:

$$Y \succ_C Z \text{ if and only if } |Y| \geq |Z|, \text{ for each } Y, Z \in \mathcal{P} \tag{2}$$

According to this rule, the more cardinality an opportunity set has, the higher it ranks.

2. Associated with a complete preorder  $R$  on  $X$ , the indirect-utility ranking  $\succsim_U$  was defined by Kreps (1979), as the following complete preorder on  $\mathcal{P}$ :

$$Y \succsim_U Z \text{ if and only if } \max(Y) R \max(Z), \text{ for each } Y, Z \in \mathcal{P} \quad (3)$$

Note the abuse of notation: what we represent in  $\max(Y) R \max(Z)$  is the fact that  $y R z$  for each (or equivalently, for some)  $y \in \max(Y)$  and  $z \in \max(Z)$ . Hence this ranking determines that the opportunity set  $Y$  is weakly preferred to  $Z$  as long as  $Y$  has preference-maximizing options from  $Y \cup Z$ .

3. The indirect-utility-first lexicographic rule  $\succsim_U^L$  on  $\mathcal{P}$  (Bossert et al. 1994; Dutta and Sen 1996) is defined with the help of a complete preorder  $R$  on  $X$  as follows:

$$Y \succsim_U^L Z \text{ if and only if either } Y \succ_U Z \text{ or } (Y \sim_U Z, Y \succ_C Z) \quad (4)$$

Priority is given to opportunity sets that enable the agent to achieve a higher level of utility. Cardinality acts as a tie-breaking rule.

4. The cardinality-first lexicographic rule  $\succsim_C^L$  on  $\mathcal{P}$  (Bossert et al. 1994; Dutta and Sen 1996) is defined with the help of a complete preorder  $R$  on  $X$  as follows:

$$Y \succsim_C^L Z \text{ if and only if either } Y \succ_C Z \text{ or } (Y \sim_C Z, Y \succ_U Z) \quad (5)$$

Priority is given to opportunity sets that provide the agent with more freedom of choice. Indirect utility acts as a tie-breaking rule.

For further use, here we recall the characterizations of the  $\succsim_C$  and  $\succsim_U$  rankings. The following axioms jointly characterize  $\succsim_C$  (v., Proposition 1 below):

**Strong independence.** For each  $Y, Z \in \mathcal{P}$  and  $x \notin Y \cup Z$ :

$$Y \succ Z \text{ if and only if } Y \cup \{x\} \succ Z \cup \{x\}. \quad (6)$$

**Indifference between no-choice situations.** For all  $x, y \in X, \{x\} \sim \{y\}$ .

**Simple expansion monotonicity.** For all  $x \neq y \in X, \{x, y\} \succ \{y\}$ .

**Proposition 1** (Pattanaik and Xu 1990) *Let  $\succsim$  be a transitive binary relation on  $\mathcal{P}$ . Then  $\succsim = \succsim_C$  if and only if  $\succsim$  satisfies Strong independence, Indifference between no-choice situations, and Simple expansion monotonicity.*

The characterization of indirect-utility rankings requires careful attention. The following axioms are needed for the discussion:

**Extension robustness.**  $Y \succ Z$  implies  $Y \sim Y \cup Z$  for all  $Y, Z \subseteq X$

**Dominance.**  $xRy$  if and only if  $\{x\} \succcurlyeq \{y\}$ .

**Useless freedom of choice.**  $xRy$  implies  $\{x\} \sim \{x, y\}$ .

Consider first the following pioneering statement:

**Proposition 2** (Kreps 1979; Barberà et al. 2004) *Let  $\succcurlyeq$  be a complete preorder on  $\mathcal{P}$ . The following statements are equivalent:*

- (a) *There is a complete preorder  $R$  such that  $Y \succcurlyeq Z \Leftrightarrow \max(Y) R \max(Z)$  (i.e.,  $\succcurlyeq$  coincides with  $\succcurlyeq_U$  defined from  $R$ ).*
- (b)  *$\succcurlyeq$  satisfies Extension robustness.*

It is apparent that  $R$  is not fixed in the Kreps model studied above. This complete preorder is “revealed” from Extension robustness. Below we will need the following rephrasing of his solution for known preferences:

**Proposition 3** (Ballester et al. 2004) *Let  $R$  be a complete preorder on  $X$ . Then the transitive binary relation  $\succcurlyeq$  on  $\mathcal{P}$  coincides with  $\succcurlyeq_U$  defined from  $R$  if and only if  $\succcurlyeq$  satisfies Strong independence, Dominance, and Useless freedom of choice.*

Ballester et al. (2004) obtained this characterization as a by-product of a more general inspection for possibly uncountable sets  $X$  and continuously representable complete preorders. We can easily produce the next consequence of Proposition 2 that does the same job:

**Corollary 1** *Let  $R$  be a complete preorder on  $X$ . Then the transitive binary relation  $\succcurlyeq$  on  $\mathcal{P}$  coincides with  $\succcurlyeq_U$  defined from  $R$  if and only if  $\succcurlyeq$  satisfies Extension robustness and Dominance.*

**Proof** Necessity is immediate. To prove sufficiency, we use Proposition 2 to infer the existence of a complete preorder  $R'$  with  $Y \succcurlyeq Z \Leftrightarrow \max_{R'}(Y) R' \max_{R'}(Z)$ , for all  $Y, Z \subseteq X$ .

Then for any  $x, y \in X$ ,  $xR'y \Leftrightarrow \{x\} \succcurlyeq \{y\} \Leftrightarrow xRy$ , where the last equivalence holds by Dominance. This proves the claim.  $\square$

Section 4.1.4 recalls characterizations of both  $\succcurlyeq_U^L$  and  $\succcurlyeq_C^L$ .

### 3 Counting Preference-Approvals

This section considers the number of preference-approval structures that can be defined on the set  $X$ , whose finite cardinality is  $n$ .

Note that the case where  $R$  is linear is trivial: there are  $n!$  linear orders, each of which generates  $n$  preference-approval structures, so the total sums up to  $n \cdot n!$  preference-approvals. In addition, Santos-García and Alcantud (2025a) have already established the number of preference-approval structures for fixed  $n$  and  $a$ , the cardinality of the set of approved alternatives. In general, the argument can be refined as follows:

**Table 1** The number of preference-approvals for sets with small cardinalities

Cardinality	Number	Cardinality	Number
1	2	6	25.988
2	8	7	296.564
3	44	8	3.816.548
4	308	9	54.667.412
5	2.612	10	862.440.068

**Proposition 4** *The number of preference-approvals on a set  $X$  with finite cardinality  $n$  is*

$$\sum_{k=1}^n (k + 1) \left( \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} i^n \right) \tag{7}$$

**Proof** Let  $\mathcal{S}(n, k)$  denote the Stirling number of the second kind, which can be computed as

$$\mathcal{S}(n, k) = \sum_{i=0}^k \frac{(-1)^{k-i} i^n}{(k - i)! i!} = \frac{1}{k!} \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} i^n$$

This number counts how many different equivalence relations with exactly  $k \in \{1, \dots, n\}$  equivalence classes can be defined on a set with  $n$  elements (Abramowitz and Stegun 1972, 24.1.4). By permuting these classes we obtain all complete preorders on  $X$  whose maximal chains have length  $k$ . Therefore the number of complete preorders on  $X$  is clearly  $\sum_{k=1}^n k! \cdot \mathcal{S}(n, k)$ .

Each complete preorder with maximal chains with length  $k$  produces exactly  $k + 1$  different preference-approvals on  $X$ . Thus by a similar argument,

$$\sum_{k=1}^n k! \cdot (k + 1) \cdot \mathcal{S}(n, k) = \sum_{k=1}^n (k + 1) \left( \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} i^n \right) \tag{8}$$

computes the number of preference-approvals on  $X$ .  $\square$

Table 1 summarizes the number of preference-approvals that can be defined on sets with cardinality up to 10. To fill it, we took advantage of the expression  $\sum_{k=1}^n k! \cdot (k + 1) \cdot \mathcal{S}(n, k)$  that we obtained in the proof of Proposition 4, v. Eq. (8). We used the A008277 ‘sequence’ of the On-line Encyclopedia of Integer Sequences named *Triangle of Stirling numbers of the second kind*.<sup>1</sup>

For illustration, the number of preference-approvals for a set with 10 elements can be computed as follows:

$$1! \cdot 2 \cdot 1 + 2! \cdot 3 \cdot 511 + 3! \cdot 4 \cdot 9330 + 4! \cdot 5 \cdot 34105 + 5! \cdot 6 \cdot 42525 + 6! \cdot 7 \cdot 22827 + 7! \cdot 8 \cdot 5880 + 8! \cdot 9 \cdot 750 + 9! \cdot 10 \cdot 45 + 10! \cdot 11 \cdot 1 = 862440068$$

<sup>1</sup><https://oeis.org/A008277/>

Alternatively, the following *Mathematica* code produces the computation of the formula in Proposition 4:

```

1 ClearAll;
2 (* Function to compute the number of preference-approvals *)
3 preferenceApprovals[n_] := Module[{sum = 0},
4   Do[
5     Do[
6       sum += (k + 1)*(-1)^(k - i)*Binomial[k, i]*i^n, {i, 0, k}],
7     {k, 1, n}];
8   sum];
9 (* Example: Compute the number of preference-approvals for a set with 5 elements *)
10 n = 5;
11 result = preferenceApprovals[n];
12 (* Print the output *)
13 Print["The number of preference-approvals for a set with n = ", n, " elements is: ", result];

```

**Listing 1** *Mathematica* code for the computation of the number of preference-approvals on a set with an arbitrary number of elements  $n$

**Remark 1** Using tools from enumerative combinatorics, Karpov (2019) has given a formula for the number of group-separable preference profiles on a set, and also for the narcissistic version of the concept. This problem is similar to a group extension of ours, but the individual concept studied by Karpov is formally unrelated to preference-approval structures (in the sense that neither of the definitions involved implies the other). Basically, preferences must be linear in Karpov (2019), and the pair of sets of “approved-disapproved” alternatives can be swapped from one agent to another. More details can be found in Santos-García and Alcantud (2025a).

## 4 Preference-Approval Structures and Opportunity Sets

This section is dedicated to investigate the role of preference-approval structures in the analysis of the valuation of opportunity sets. It proposes and studies various approaches that take advantage of preference-approval structures in various ways.

Consequently, henceforth a preference-approval structure  $(R, A)$  on  $X$  is fixed. Section 4.1 shows how we can produce rankings of opportunity sets from  $(R, A)$  with three examples. Afterwards, this section explains which explicit numerical representations they have. Then it proves axiomatic characterizations of these rankings with the help of existing and new axioms, together with respective proofs of independence of the axioms in these results. A discussion clarifies the extent to which the rankings studied in this section express a preference for freedom of choice.

Section 4.2 defines preference-approval structures on opportunity sets from  $(R, A)$ , possibly dispensing with the information embedded in  $R$ . The case where the complete preorder  $R$  defines preference-approval structures on opportunity sets is considered in Sect. 4.3.

### 4.1 Ranking Opportunity Sets from Preference-Approval Structures: Numerical Representations and Characterizations

We proceed to study three rankings on  $\mathcal{P}$  that are established with the help of the preference-approval structure  $(R, A)$ .

1. When the informational basis for comparing opportunity sets in terms of freedom of choice is cardinality, but not all alternatives are freedom-giving, we can resort to the preference-approval cardinality-based rule  $\succ_C^{pa}$  on  $\mathcal{P}$ : under the assumption that  $A$  captures the freedom-giving alternatives,

$$Y \succ_C^{pa} Z \text{ if and only if } |Y \cap A| \geq |Z \cap A|, \text{ for each } Y, Z \in \mathcal{P} \quad (9)$$

This ranking dispenses with the information in  $R$ , which aligns with the spirit of Pattanaik and Xu’s cardinality-based rule.

In addition, “approval” is an exogenously given unique selection, which sets the evaluations apart from those based on the endogenously defined notion of essentiality within opportunity sets defined in Puppe (1996) and Puppe and Xu (2010). An interpretation that introduces naive utility-based components of the well-being enjoyed by the agent considers that  $A$  captures the set of “sufficiently good” alternatives. Then  $\succ_C^{pa}$  compares the opportunity sets by the number of “sufficiently good” alternatives that they contain. Adding alternatives whose desirability is below a certain threshold does not extend the freedom of choice of the agent.

2. A more sophisticated rule combining preference-based evaluations with cardinality-based preference for freedom of choice is the preference-approval indirect-utility-first lexicographic rule  $\succ_U^{L,pa}$  on  $\mathcal{P}$ : it is defined as

$$Y \succ_U^{L,pa} Z \text{ if and only if either } Y \succ_U Z \text{ or } (Y \sim_U Z, Y \succ_C^{pa} Z), \text{ for each } Y, Z \in \mathcal{P} \quad (10)$$

This rule gives priority to indirect utility à-la-Kreps, and uses the preference-approval cardinality-based rule as a tie-breaker.

3. Similarly, if we refine the preference-approval cardinality-based rule with comparisons based on indirect-utilities, then we obtain the preference-approval cardinality-first lexicographic rule  $\succ_C^{L,pa}$  on  $\mathcal{P}$ :

$$Y \succ_C^{L,pa} Z \text{ if and only if either } Y \succ_C^{pa} Z \text{ or } (Y \sim_C^{pa} Z, Y \succ_U Z), \text{ for each } Y, Z \in \mathcal{P} \quad (11)$$

For further use, note that when  $A = X$  the expressions above define known preferences over opportunity sets that we recalled in Sect. 2.2, namely,  $\succ_C^{pa} = \succ_C$ ,  $\succ_U^{L,pa} = \succ_U^L$ , and  $\succ_C^{L,pa} = \succ_C^L$ .

We proceed to illustrate these three rules in situations with small cardinalities. Then we briefly discuss the compatibility of evaluations based on preferences enhanced with approved/non approved options, with Pattanaik and Xu (1990)’s idea that freedom of choice is an intrinsic component of welfare.

**Example 2** Suppose  $X = \{x, y, z\}$ . Consider the next preference-approval on  $X$ :  $xPyPz$  and  $A = \{x, y\}$ . As the preference is linear, we summarize the information with  $xy \mid z$  in the notation by Brams and Sanver (2009).

Then the rankings  $\succ_U^{L,pa}$  and  $\succ_C^{L,pa}$  coincide: both are the complete preorder on  $\mathcal{P}$  represented as follows

$$\begin{array}{l}
 X \quad \{x, y\} \\
 \{x\} \quad \{x, z\} \\
 \{y\} \quad \{y, z\} \\
 \{z\}
 \end{array}$$

However  $\succ_C^{pa}$  is the complete preorder on  $\mathcal{P}$  represented as follows

$$\begin{array}{l}
 X \quad \{x, y\} \\
 \{x\} \quad \{y\} \quad \{x, z\} \quad \{y, z\} \\
 \{z\}
 \end{array}$$

It is not always the case that  $\succ_U^{L,pa} = \succ_C^{L,pa}$ , not even when  $R$  is linear. To prove it, suppose now  $X = \{x, y, z, t\}$ . Consider the next arrangement of the alternatives in  $X$ , which represents another preference-approval on  $X$  where the preference is linear, in the notation by Brams and Sanver (2009):  $xyz \mid t$ .

Then  $\succ_U^{L,pa}$  is the complete preorder on  $\mathcal{P}$  represented as follows:

$$\begin{array}{l}
 X \quad \{x, y, z\} \\
 \{x, y\} \quad \{x, y, t\} \quad \{x, z\} \quad \{x, z, t\} \\
 \{x\} \quad \{x, t\} \\
 \{y, z\} \quad \{y, z, t\} \\
 \{y\} \quad \{y, t\} \\
 \{z\} \quad \{z, t\} \\
 \{t\}
 \end{array}$$

However  $\succ_C^{L,pa}$  is the complete preorder on  $\mathcal{P}$  represented as follows:

$$\begin{array}{l}
 X \quad \{x, y, z\} \\
 \{x, y\} \quad \{x, y, t\} \quad \{x, z\} \quad \{x, z, t\} \\
 \{y, z\} \quad \{y, z, t\} \\
 \{x\} \quad \{x, t\} \\
 \{y\} \quad \{y, t\} \\
 \{z\} \quad \{z, t\} \\
 \{t\}
 \end{array}$$

**Remark 2** The first preference-approval structure defined in Example 2 produces three rankings  $\succ_C^{pa}$ ,  $\succ_U^{L,pa}$  and  $\succ_C^{L,pa}$  on opportunity sets, each of which contradicts the Simple Expansion Monotonicity axiom in Pattanaik and Xu (1990) (for example, when the opportunity set is  $Y = \{x, z\}$ ). This confirms that the models in this section do not imply that all alternatives are equally valuable when it comes to improving the freedom of choice of a decision maker, in the sense that the property  $Y \succ Y \setminus \{y\}$  for all  $Y \subseteq X$  and  $y \in Y$  can be contradicted.

**Remark 3** Likewise, all  $\succsim_C^{pa}$ ,  $\succsim_U^{L,pa}$  and  $\succsim_C^{L,pa}$  are incompatible with Axiom F in Puppe (1996):

1. The preference-approval structure on  $X = \{x, y, z\}$  with  $x \mid yz$  satisfies  $\{y, z\} \sim_C^{pa} \{y\} \sim_C^{pa} \{z\}$ . This proves that  $\succsim_C^{pa}$  can contradict Axiom F, even if the preference is linear.
2. Consider the preference-approval structure  $(R, A)$  on  $X = \{x, y, z\}$  with  $x P y I z$  and  $A = \{x, y, z\}$ , which is presented as follows:

$$\frac{x}{yz}$$

Because  $\{y, z\}$ ,  $\{y\}$  and  $\{z\}$  have equal indirect utility and equal number of elements in  $A$ , one has both  $\{y, z\} \sim_U^{L,pa} \{y\} \sim_U^{L,pa} \{z\}$  and  $\{y, z\} \sim_C^{L,pa} \{y\} \sim_C^{L,pa} \{z\}$ . This proves that  $\succsim_U^{L,pa}$  and  $\succsim_C^{L,pa}$  can contradict Axiom F.

However, if  $(R, A)$  is a preference-approval structure whose preference is linear on  $X \setminus A$ , then it is straightforward to prove that both  $\succsim_U^{L,pa}$  and  $\succsim_C^{L,pa}$  satisfy Axiom F.

Based on Remarks 2 and 3, we conclude that if we use Axiom F as a “substantial description of preference for freedom of choice” (Puppe 1996, p. 178) then for rankings defined from preference-approval structures with linear preferences, the preference-approval indirect-utility-first lexicographic/cardinality-first lexicographic rules consider freedom of choice as a component of the value of the opportunity sets. Although this conclusion is not as applicable to the preference-approval cardinality-based rule, the separation between approved/non approved options is compatible with Pattanaik and Xu (1990)’s idea of the intrinsic value of freedom of choice, when it is shaped by the agent’s ordinal assessment of the value of the alternatives relative to other alternatives.

### 4.1.1 Numerical Representations

As the rankings defined above are complete preorders on the finite set  $\mathcal{P}$ , they must be representable by a utility function. We proceed to construct them in this section.

It is clear that the complete preorder  $\succsim_C^{pa}$  is represented by the utility function  $f(Y) = |Y \cap A|$  for each  $Y \in \mathcal{P}$ .

Representations of  $\succsim_U^{L,pa}$  and  $\succsim_C^{L,pa}$  defined from  $(R, A)$  can be explicitly given with the help of  $u(x) = |\{y \in X : xRy\}|$  for all  $x \in X$ . This expression defines  $u : X \rightarrow \mathbb{N}$  that represents the complete preorder  $R$ . It is important to bear in mind that  $u(x)$  is always an integer value.

Let us now define

$$v(Y) = \max_{y \in Y} u(y) + \frac{|Y \cap A|}{|A| + 1}, \text{ for all } Y \in \mathcal{P}, \tag{12}$$

and

$$w(Y) = |Y \cap A| + \frac{\max_{y \in Y} u(y)}{|X| + 1}, \text{ for all } Y \in \mathcal{P}. \tag{13}$$

Noting  $\frac{|Y \cap A|}{|A| + 1} \in \left[0, \frac{|A|}{|A| + 1}\right] \subseteq [0, 1)$  and  $\frac{\max_{y \in Y} u(y)}{|X| + 1} \in \left[0, \frac{|X|}{|X| + 1}\right] \subseteq [0, 1)$ , we can show that when  $Y, Z \subseteq X$ ,

$$\begin{aligned} v(Y) \geq v(Z) &\Leftrightarrow \text{either } \max_{y \in Y} u(y) > \max_{z \in Z} u(z) \text{ or } \left(\max_{y \in Y} u(y) = \max_{z \in Z} u(z), \frac{|Y \cap A|}{|A| + 1} > \frac{|Z \cap A|}{|A| + 1}\right) \\ &\Leftrightarrow \text{either } Y \succ_U Z \text{ or } (Y \sim_U Z, |Y \cap A| > |Z \cap A|) \Leftrightarrow Y \succ_U^{L,pa} Z, \end{aligned}$$

and

$$\begin{aligned} w(Y) \geq w(Z) &\Leftrightarrow \text{either } |Y \cap A| > |Z \cap A| \text{ or } \left(|Y \cap A| = |Z \cap A|, \frac{\max_{y \in Y} u(y)}{|X| + 1} > \frac{\max_{z \in Z} u(z)}{|X| + 1}\right) \\ &\Leftrightarrow \text{either } Y \succ_C Z \text{ or } (Y \sim_C Z, \max_{y \in Y} u(y) > \max_{z \in Z} u(z)) \Leftrightarrow Y \succ_C^{L,pa} Z. \end{aligned}$$

In conclusion,  $v$  defines a numerical representation of  $\succ_U^{L,pa}$ , and  $w$  defines a numerical representation of  $\succ_C^{L,pa}$ .

### 4.1.2 Further Axioms

In order to characterize the rankings constructed above from preference-approval structures, we need some axioms. They are versions of similar axioms that are adapted to the current context. We present them in three groups. First we have axioms stated exclusively on the ranking of opportunity sets. Then we give axioms that connect it with a preference on the individual options. Finally, we present axioms linking the ranking with a preference-approval structure on the options.

To help state the axioms of the first type, associated with  $\succ$ , a transitive relation on  $\mathcal{P}$ , we define

$$V_{\succ} = \{y \in X \mid \text{there is } z \in X \text{ with } \{y\} \succ \{z\}\}. \tag{14}$$

When this set is non-empty, we witness the existence of elements that are inherently more freedom-giving than others. Or under the indirect-utility interpretation, one has the natural case of elements that convey more utility than others.

The options in  $V_{\succ}$  define singletons that cover other singletons in the corresponding hierarchy of no-choice opportunity sets. They will be important in our axiomatic characterization of  $\succ_C^{pa}$  and at that point, they will reveal the ‘‘approved’’ alternatives. For lack of a better name, we use the term ‘covering’ in the axioms concerned with options belonging to  $V_{\succ}$ .

From inspection of Simple expansion monotonicity, it can be observed that the next axiom relaxes it:

**Covering-expansion monotonicity.** For each  $y \in X$  such that there is  $z \in X$  with  $\{y\} \succ \{z\}$ :

$$\{x, y\} \succ \{x\} \text{ when } y \neq x \in X. \tag{15}$$

The axiom below considers a variation of Simple expansion monotonicity adapted to our context. It relaxes Useless freedom of choice:

**Indifference under non-covering expansions.** For each  $y \in X$  such that there is no  $z \in X$  with  $\{y\} \succ \{z\}$ :

$$\{x, y\} \sim \{x\} \text{ for all } x \in X. \tag{16}$$

We also need the following axiom that for elementary situations, establishes a weak priority of certain cases:

**Preference of no-choice covering situations.** For each  $y \in X$  such that there is  $z \in X$  with  $\{y\} \succ \{z\}$ :

$$\{y\} \succcurlyeq \{x\} \text{ for all } x \in X. \tag{17}$$

The next auxiliary results will prove helpful:

**Lemma 1** *If  $\succcurlyeq$  is a transitive relation on  $\mathcal{P}$ , then:*

(a) *When  $\succcurlyeq$  satisfies Indifference under non-covering expansions,*

$$\{x\} \sim \{y\} \text{ for all } x \notin V_{\succcurlyeq}, y \notin V_{\succcurlyeq}. \tag{18}$$

(b) *When  $\succcurlyeq$  satisfies Preference of no-choice covering situations,*

$$\{x\} \sim \{y\} \text{ for all } x, y \in V_{\succcurlyeq} \tag{19}$$

and

$$\{x\} \succ \{y\} \text{ for all } x \in V_{\succcurlyeq}, y \notin V_{\succcurlyeq}. \tag{20}$$

(c) *When  $\succcurlyeq$  satisfies Covering-expansion monotonicity, Indifference under non-covering expansions, and Strong independence,*

$$Y \succcurlyeq \{y\} \text{ whenever } y \in Y \in \mathcal{P}. \tag{21}$$

**Proof** In case (a), we have  $\{x\} \sim \{x, y\} \sim \{y\}$ , hence the conclusion follows from transitivity. In case (b), when  $x, y \in V_{\succcurlyeq}$  we have  $\{x\} \succcurlyeq \{y\} \succcurlyeq \{x\}$ , hence the conclusion (19). When  $x \in V_{\succcurlyeq}$  and  $y \notin V_{\succcurlyeq}$ , we have  $\{x\} \succcurlyeq \{y\}$  by Preference of no-choice covering situations, and the case  $\{y\} \succcurlyeq \{x\}$  would entail the contradiction  $y \in V_{\succcurlyeq}$ . This proves (20).

To prove (c), suppose  $Y = \{y_1, \dots, y_k\}$ . Using either Covering-expansion monotonicity or Indifference under non-covering expansions,  $\{y_1, y_2\} \succcurlyeq \{y_1\}, \dots, \{y_1, y_k\} \succcurlyeq \{y_1\}$ . By Strong independence,  $\{y_1, y_2, y_3\} \succcurlyeq \{y_1, y_3\} \succcurlyeq \{y_1\}$  thus  $\{y_1, y_2, y_3\} \succcurlyeq \{y_1\}$  by transitivity. A recursive argument proves the claim.  $\square$

**Lemma 2** *If  $\succcurlyeq$  is a transitive relation on  $\mathcal{P}$  that satisfies Indifference under non-covering expansions and Strong independence, then*

$$Y \cup \{x\} \sim \{x\} \text{ for all } Y \subseteq X \setminus V_{\succcurlyeq} \text{ and } x \in X. \tag{22}$$

**Proof** We use a recursive argument on  $|Y|$ .

The case  $|Y| = 1$  is exactly Indifference under non-covering expansions.

Suppose  $Y = \{y_1, y_2\}$ . Then Indifference under non-covering expansions implies  $Y \sim \{y_1\}$ , and now Strong independence followed by the induction hypothesis (or Indifference under non-covering expansions) entail

$$Y \cup \{x\} \sim \{y_1, x\} \sim \{x\} \text{ for all } x \in X.$$

The conclusion follows from transitivity.

Suppose that the conclusion holds for all  $Y \subseteq X \setminus V_{\succcurlyeq}$  with  $|Y| < k$ .

Consider the case  $Y = \{y_1, \dots, y_k\} \subseteq X \setminus V_{\succcurlyeq}$ . We apply the induction hypothesis to  $\{y_1, \dots, y_{k-1}\}$  in order to guarantee  $Y = \{y_1, \dots, y_{k-1}, y_k\} \sim \{y_k\}$ . Now Strong independence followed by Indifference under non-covering expansions entail

$$Y \cup \{x\} \sim \{y_k, x\} \sim \{x\} \text{ for all } x \in X.$$

The conclusion follows from transitivity.  $\square$

We proceed to recall three axioms linking  $\succcurlyeq$ , a ranking of  $\mathcal{P}$ , with  $R$ , a fixed complete preorder on  $X$ .

They are all defined in Bossert et al. (1994).

**Best-element conditional independence.** For each  $Y, Z \in \mathcal{P}$  and  $x \notin Y \cup Z$  such that  $\max(Y)Px, \max(Z)Px$ :

$$Y \succcurlyeq Z \text{ if and only if } Y \cup \{x\} \succcurlyeq Z \cup \{x\}. \tag{23}$$

**Indirect preference principle.** For each  $Y \in \mathcal{P}$  such that  $|Y| > 1$ :  $\{\max(Y)\} \succ Y \setminus \{\max(Y)\}$ .

**Simple dominance.** For each  $x, y \in X, xPy \Rightarrow \{x\} \succ \{y\}$ .

Best-element conditional independence appears in Bossert et al. (1994) under the name Weak independence; here we adopt the terminology in Barberà et al. (2004). Simple dominance is called Weak extension rule in Barberà et al. (2004). Best-element conditional independence relaxes Strong independence. The Indirect preference principle implies Simple dominance, which is weaker than Dominance defined in Ballester et al. (2004). Best-element conditional independence and Simple domi-

nance are satisfied by both  $\succsim_U^L$  and  $\succsim_C^L$ . While  $\succsim_U^L$  satisfies the Indirect preference principle,  $\succsim_C^L$  contradicts it.

Now we finish this section by defining axioms that describe possible properties that link rankings of  $\mathcal{P}$  with  $(R, A)$ , a fixed preference-approval structure on  $X$ .

Our next definition reconsiders Simple expansion monotonicity in Barberà et al. (2004), by adapting it to this situation in a naive manner. Simple expansion monotonicity does not refer to any prior information about  $X$ . Our version creates a relaxed form that is better suited to our current needs. Note that only the information about  $A$  is needed to define it.

**Preference-approval-expansion monotonicity.** For each  $x \in X$ :

$$\{x, y\} \succ \{x\} \text{ when } x \neq y \text{ and } y \in A. \tag{24}$$

For preference-approval structures of the form  $(R, X)$ , Preference-approval-expansion monotonicity and Simple expansion monotonicity establish the same property.

We adapt two other axioms to rankings motivated by preference-approval structures too.

**Preference-approval indirect indifference principle.** For each  $x, y, z \in X$ :

$$\{x, y\} \sim \{x, z\} \text{ when } xPyPz, \text{ and either } z \in A \text{ or } y \notin A. \tag{25}$$

The Preference-approval indirect indifference principle relaxes the Simple indirect indifference principle, defined in Barberà et al. (1984) for rankings obtained from  $R$  alone. The Simple indirect indifference principle means  $\{x, y\} \sim \{x, z\}$  when  $xPyPz$ , for each  $x, y, z \in X$ . Therefore, for preference-approval structures of the form  $(R, X)$ , both axioms establish the same property.

**Preference-approval weight of freedom.** For each  $x, y, z \in X$ :

$$\{y, z\} \succ \{x\} \text{ when } xPyPz \text{ and } z \in A, \text{ and} \tag{26}$$

$$\{y, z\} \sim \{y\} \text{ when } yPz \text{ and } z \notin A. \tag{27}$$

Preference-approval weight of freedom is a variation of the axiom that Barberà et al. (1984) called Simple priority of freedom. This is an axiom for rankings defined from  $R$  alone, and it establishes that  $xPyPz$  implies  $\{y, z\} \succ \{x\}$ . The first part of the new axiom, Eq. (26), makes full utilization of the preference-approval structure. The second, Eq. (27), disregards the order  $R$ . It means that adding a non-approved alternative to a set formed by another alternative leaves the agent indifferent. For preference-approval structures of the form  $(R, X)$ , Preference-approval weight of freedom and Simple priority of freedom capture the same property.

Inspired by Lemma 3.3 in Bossert et al. (1994), we prove the following implication:

**Lemma 3** Fix  $(R, A)$ , a preference-approval structure on  $X$  such that  $|A| \geq 3$ . Suppose that  $\succsim$  is a transitive relation on  $\mathcal{P}$  that with reference to  $(R, A)$ , satisfies Best-element conditional independence, Simple dominance, the Preference-approval

indirect indifference principle, and Preference-approval weight of freedom. Then  $\succsim$  satisfies Preference-approval-expansion monotonicity.

**Proof** Fix  $y \in A$  and  $x \neq y, x \in X$  Let us prove  $\{x, y\} \succ \{x\}$ .

We select  $z \in A$  with  $x \neq z \neq y$ .

First we argue when  $xPy$ , which entails  $x \in A$ . Three cases can be distinguished.

Case 1:  $zPxPy$ . Then  $\{x, y\} \succ \{z\} \succ \{x\}$ , due to Preference-approval weight of freedom (26) plus Simple dominance. Transitivity yields the conclusion.

Case 2:  $xPzPy$ . Then  $\{x, y\} \succ \{z, y\}$  due to the utilization of Best-element conditional independence for  $\{x\} \succ \{z\}$ , which holds by Simple dominance. Preference-approval weight of freedom (26) guarantees  $\{z, y\} \succ \{x\}$ . Transitivity yields the conclusion.

Case 3:  $xPyPz$ . Now  $\{x, y\} \sim \{x, z\}$  is true by the Preference-approval indirect indifference principle. Besides,  $\{x\} \succ \{y\}$  holds by Simple dominance, therefore  $\{x, z\} \succ \{y, z\}$  by Best-element conditional independence. By Preference-approval weight of freedom (26),  $\{y, z\} \succ \{x\}$ . Transitivity yields the conclusion.

Let us now argue when  $yPx$ . Three cases can be distinguished.

Case 1:  $zPyPx$ . When  $x \in A$ ,  $\{y, x\} \succ \{z\}$  and  $\{z\} \succ \{x\}$ , respectively owing to Preference-approval weight of freedom (26) and Simple dominance. When  $x \notin A$ , using Preference-approval weight of freedom (27) and Simple dominance one has  $\{y, x\} \sim \{y\} \succ \{x\}$ . In each case, transitivity yields the conclusion.

Case 2:  $yPzPx$ . As  $\{y\} \succ \{z\}$  holds by Simple dominance, it is the case that  $\{y, x\} \succ \{z, x\}$  by Best-element conditional independence. Besides, when  $x \in A$ ,  $\{x, z\} \succ \{y\}$ , owing to Preference-approval weight of freedom (26). When  $x \notin A$ ,  $\{x, z\} \sim \{z\}$  owing to Preference-approval weight of freedom (27). In each case, transitivity yields  $\{x, y\} \succ \{z\}$  hence the conclusion because  $\{z\} \succ \{x\}$  by Simple dominance.

Case 3:  $yPxPz$ . By the Preference-approval indirect indifference principle,  $\{x, y\} \sim \{y, z\}$ . As  $\{y\} \succ \{x\}$  by Simple dominance, Best-element conditional independence assures  $\{y, z\} \succ \{x, z\}$ . Using Preference-approval weight of freedom (26) and Simple dominance one has  $\{x, z\} \succ \{y\} \succ \{z\}$ . Transitivity yields the conclusion.  $\square$

### 4.1.3 Characterizations

We are ready to use the axioms defined in Sect. 4.1.2 to characterize the rankings constructed from preference-approval structures above in Sect. 4.1.

The next proposition characterizes the preference-approval cardinality-based rules  $\succsim_C^{pa}$  defined from any preference-approval structure, by means of four axioms.

**Proposition 5** *Let  $\succsim$  be a transitive relation on  $\mathcal{P}$ . Then there is a preference-approval structure  $(R, A)$  such that  $\succsim = \succsim_C^{pa}$  if and only if  $\succsim$  satisfies Covering-expansion monotonicity, Indifference under non-covering expansions, Strong independence, and Preference of no-choice covering situations.*

**Proof** For the necessity part, one can immediately check that  $\succsim_C^{pa}$  defined from a preference-approval structure  $(R, A)$  satisfies the axioms. Conversely, define  $V_{\succsim}$  by Eq. (14). We claim that any preference-approval structure  $(R, V_{\succsim})$  produces a preference-approval cardinality-based rule  $\succsim_C^{pa}$  that coincides with  $\succsim$ .

To prove this claim, we take advantage of the fact that  $\succsim_C^{pa}$  is complete in order to transform the conclusion  $\succsim = \succsim_C^{pa}$  into the following two statements: for each  $Y, Z \in \mathcal{P}$ ,

$$|Y \cap V_{\succsim}| = |Z \cap V_{\succsim}| \text{ (i.e., } Y \sim_C^{pa} Z) \text{ implies } Y \sim Z, \tag{28}$$

$$|Y \cap V_{\succsim}| > |Z \cap V_{\succsim}| \text{ (i.e., } Y \succ_C^{pa} Z) \text{ implies } Y \succ Z. \tag{29}$$

We prove (28) by induction on  $|Y \cap V_{\succsim}| = |Z \cap V_{\succsim}|$ .

The case  $|Y \cap V_{\succsim}| = |Z \cap V_{\succsim}| = 0$  is the case  $Y, Z \subseteq X \setminus V_{\succsim}$ . We use Lemma 2 twice with  $y \in Y$  and then  $z \in Z$ , and also Lemma 1 (a):

$$Y = (Y \setminus \{y\}) \cup \{y\} \sim \{y\} \sim \{z\} \sim (Z \setminus \{z\}) \cup \{z\} = Z.$$

The conclusion  $Y \sim Z$  follows from transitivity.

The case  $|Y \cap V_{\succsim}| = |Z \cap V_{\succsim}| = 1$  is the case  $Y \cap V_{\succsim} = \{y\}$ ,  $Z \cap V_{\succsim} = \{z\}$ . Lemma 1 (b) guarantees  $\{y\} \sim \{z\}$ . We can use Lemma 2 twice with  $y \in Y$  and then  $z \in Z$ , because  $Y \setminus \{y\} \subseteq X \setminus V_{\succsim}$  and  $Z \setminus \{z\} \subseteq X \setminus V_{\succsim}$ :

$$Y = (Y \setminus \{y\}) \cup \{y\} \sim \{y\} \sim \{z\} \sim (Z \setminus \{z\}) \cup \{z\} = Z.$$

The conclusion  $Y \sim Z$  follows from transitivity.

Now suppose that the conclusion holds when  $|Y \cap V_{\succsim}| = |Z \cap V_{\succsim}| < k$ .

In case that  $|Y \cap V_{\succsim}| = |Z \cap V_{\succsim}| = k \geq 2$ , we select  $y \in Y \cap V_{\succsim}$  and  $z \in Z \cap V_{\succsim}$ , with the condition  $y = z$  if  $Y \cap Z \cap V_{\succsim} \neq \emptyset$ . The induction hypothesis guarantees  $Y \setminus \{y\} \sim Z \setminus \{z\}$ . Strong independence implies  $Z \sim (Y \setminus \{y\}) \cup \{z\}$ .

The conclusion follows when  $Y \cap Z \cap V_{\succsim} \neq \emptyset$  (because in this case  $y = z$ ). Suppose on the contrary that  $Y \cap Z \cap V_{\succsim} = \emptyset$ . We pick  $y' \in V_{\succsim} \cap (Y \setminus \{y\})$ , which exists because  $|Y \cap V_{\succsim}| = k \geq 2$ . We can apply the induction hypothesis to  $(Y \setminus \{y, y'\}) \cup \{z\}$  and  $Y \setminus \{y'\}$ , because both sets have exactly  $k - 1$  elements in  $V_{\succsim}$ , therefore  $(Y \setminus \{y, y'\}) \cup \{z\} \sim Y \setminus \{y'\}$ . Strong independence implies  $(Y \setminus \{y\}) \cup \{z\} \sim Y$ . Coupled with  $Z \sim (Y \setminus \{y\}) \cup \{z\}$ , transitivity yields the conclusion.

Let us now prove (29).

First suppose  $|Z \cap V_{\succsim}| = 0$ . With  $y \in Y \cap V_{\succsim}$  and  $z \in Z$ , we can assure  $\{y\} \succ \{z\}$  by Lemma 1 (b). An appeal to Lemma 1 (c) and then Lemma 2 proves  $Y \succ \{y\} \succ \{z\} \sim Z$  hence  $Y \succ Z$ .

Let us now assume  $|Z \cap V_{\succsim}| > 0$ . Since  $|Y \cap V_{\succsim}| > |Z \cap V_{\succsim}|$ , we can find  $Y' \subset Y$  with  $|Y' \cap V_{\succsim}| = |Z \cap V_{\succsim}| > 0$  and  $Y' \cap (X \setminus V_{\succsim}) = Y \cap (X \setminus V_{\succsim})$ . As proven above,  $Y' \sim Z$ .

Let us select  $y \in (Y \setminus Y') \cap V_{\succsim}$  and  $y' \in Y' \cap V_{\succsim}$ . Then  $\{y, y'\} \succ \{y'\}$  and a sequential application of Strong independence and transitivity (adding the remaining elements of  $Y'$ ) gives

$$Y' \cup \{y\} \succ Y'.$$

An appeal to Lemma 1 (c) guarantees  $Y \setminus Y' \succcurlyeq \{y\}$ . A sequential addition of the elements of  $Y'$  gives, by Strong independence and transitivity, the conclusion  $Y \succ Z$  because

$$Y \succcurlyeq Y' \cup \{y\} \succ Y' \sim Z.$$

□

For the characterizations of  $\succcurlyeq_U^{L,pa}$  and  $\succcurlyeq_C^{L,pa}$ , our inspiration is the approach in Bossert et al. (1994). Consequently, both criteria are axiomatized by properties stated in terms of a fixed preference-approval structure whose preference is linear. Firstly, the next proposition characterizes the preference-approval indirect-utility-first lexicographic rule  $\succcurlyeq_U^{L,pa}$  defined from a given preference-approval structure, by means of four axioms:

**Proposition 6** Fix  $(R, A)$ , a preference-approval structure where  $R$  is a linear order. Let  $\succcurlyeq$  be a reflexive and transitive relation on  $\mathcal{P}$ .

Then  $\succcurlyeq = \succcurlyeq_U^{L,pa}$  if and only if  $\succcurlyeq$  satisfies Best-element conditional independence, Indirect preference principle, Preference-approval expansion monotonicity, and the Preference-approval indirect indifference principle.

**Proof** For the necessity part, one can immediately check that  $\succcurlyeq_U^{L,pa}$  defined in Eq. (10) satisfies the axioms.

To prove sufficiency, we take advantage of the fact that  $\succcurlyeq_U^{L,pa}$  is complete in order to transform the conclusion  $\succcurlyeq = \succcurlyeq_U^{L,pa}$  into the following two statements: for each  $Y, Z \in \mathcal{P}$ ,

$$\max(Y) = \max(Z), |Y \cap A| = |Z \cap A| \text{ (i.e., } Y \sim_U^{L,pa} Z) \text{ imply } Y \sim Z, \tag{30}$$

$$\max(Y) \succ \max(Z), \text{ or } \left( \max(Y) = \max(Z), |Y \cap A| > |Z \cap A| \right) \tag{31}$$

(i.e.,  $Y \succ_U^{L,pa} Z$ ) imply  $Y \succ Z$ .

We first argue under the conditions of (30). When  $Y = Z$ , reflexivity assures  $Y \sim Z$ . Otherwise, we let  $\max(Y) = \max(Z) = x$ . When  $C = (Y \cap Z) \setminus \{x\}$ , and  $Y' = Y \setminus C$ ,  $Z' = Z \setminus C$ , then  $|Y' \cap A| = |Z' \cap A|$ ,  $Y' \cap Z' = \{x\}$  and  $\max(Y') = \max(Z') = x$ . We can repeatedly use the Preference-approval indirect

indifference principle (25) and Best-element conditional independence as follows. First, we can write  $Y' = \{x, y_1, \dots, y_t, \dots, y_k\}$  and  $Z' = \{x, z_1, \dots, z_t, \dots, z_k\}$  so that all  $y_1, \dots, y_t, z_1, \dots, z_t$  are in  $A$  (this collection might be empty) and all other elements of  $Y'$  and  $Z'$  are not in  $A$ . Using (25),  $\{x, y_1\} \sim \{x, z_1\}$ , then  $\{x, z_1, y_2, \dots, y_k\} \sim Y'$  by (23). Similarly,  $\{x, y_2\} \sim \{x, z_2\}$  by (25) and then  $\{x, z_1, y_2\} \sim \{x, z_1, z_2\}$  by (23), therefore  $\{x, z_1, z_2, y_3, \dots, y_k\} \sim \{x, z_1, y_2, y_3, \dots, y_k\} \sim Y'$ . We end up with  $Y' \sim Z'$ . To conclude, we use Best-element conditional independence to add the elements of  $C$  and obtain  $Y' \cup C = Y \sim Z = Z' \cup C$ .

Let us now prove the claim under the conditions of (31). We argue separately for each of the two assumptions.

First suppose  $\max(Y) = \max(Z) = x$  and  $|Y \cap A| > |Z \cap A|$ . Select  $Y' \subset Y \cap A$  with  $Y' \cap Z = \emptyset$ ,  $|Y'| = |Y \cap A| - |Z \cap A| \geq 1$ . With every  $y \in Y'$ , either  $y = \max(Y')$  and then Preference-approval expansion monotonicity (24) yields  $\{x, y\} \succ \{x\}$ , or  $\max(Y')Py$ . For every  $y \in Y'$  in the later situation, noting  $xPy$ , we get  $\{x, y, \max(Y')\} \succ \{x, \max(Y')\} \sim \{x, y\} \succ \{x\}$ , using Best-element conditional independence (23), the Preference-approval indirect indifference principle (25), and Preference-approval expansion monotonicity (24). A sequential application of this argument to each element of  $Y'$  guarantees  $Y' \cup \{x\} \succ \{x\}$  (for illustration, adding another  $y' \in Y'$  gives  $\{x, y, y', \max(Y')\} \succ \{x, y', \max(Y')\} \succ \{x\}$  using Best-element conditional independence plus the previous claim).

Now we can apply Best-element conditional independence sequentially to the elements of  $Z \setminus \{x\}$  to get  $Y' \cup Z \succ Z$ .

Finally, we resort to (30) because  $Y \sim_U^{L,pa} Y' \cup Z$ . We conclude because  $Y \sim Y' \cup Z \succ Z$  and transitivity yield  $Y \succ Z$ .

Now suppose  $\max(Y)P \max(Z)$ . We consider two subcases.

- (i) In the subcase  $|Y \cap A| \leq |Z \cap A|$ , if  $Z \cap A = \emptyset$  then  $Y \cap A = \emptyset$  too. We deduce  $\{\max(y)\} \sim Y$  and  $\{\max(Z)\} \sim Z$  because  $\{\max(Y)\} \sim_U^{L,pa} Y$  and  $\{\max(Z)\} \sim_U^{L,pa} Z$ , and we can resort to (30). Using Simple dominance, which holds due to the Indirect preference principle, we know  $\{\max(Y)\} \succ \{\max(Z)\}$ , and the conclusion follows from transitivity.

When  $Z \cap A \neq \emptyset$ , let  $C = \{z_1, \dots, z_r\}$  be the subset formed exactly by the top  $|Z \cap A| - |Y \cap A| + 1$  elements of  $Z \cap A$ . Then  $\{\max(Y)\} \succ C$ : we apply the Indirect preference principle to  $M = \{\max(Y), z_1, \dots, z_r\}$ , noting  $\max(M) = \max(Y)$  by the assumption  $\max(Y)P \max(Z)$ . By Best-element conditional independence,  $\{\max(Y)\} \cup (Z \setminus C) \succ Z$ . The subsets  $Y$  and  $\{\max(Y)\} \cup (Z \setminus C)$  satisfy the conditions of (30): their respective maxima are  $\{\max(Y)\}$ , and  $|Y \cap A| = |(\{\max(Y)\} \cup (Z \setminus C)) \cap A|$ . Whence  $Y \sim \{\max(Y)\} \cup (Z \setminus C) \succ Z$  by transitivity.

- (ii) In the subcase  $|Y \cap A| > |Z \cap A|$ , we argue separately when  $|Z \cap A| = 0$  and  $|Z \cap A| > 0$ .

Suppose  $|Z \cap A| = 0$ . Then  $\{\max(Z)\} \sim Z$  because  $\{\max(Z)\} \sim_U^{L,pa} Z$  so we can resort to (30). Using Simple dominance, which holds due to the Indirect

preference principle, we know  $\{\max(Y)\} \succ \{\max(Z)\}$ . We note  $Y \succcurlyeq \{\max(Y)\}$  because  $\max(Y) = \max(\{\max(Y)\})$  and  $|Y \cap A| \geq |\{\max(Y)\} \cap A|$ : we apply either (30) when  $|Y \cap A| = |\{\max(Y)\} \cap A|$ , or the statement we have previously proven when  $|Y \cap A| > |\{\max(Y)\} \cap A|$ . The conclusion follows from transitivity, as  $Y \succcurlyeq \{\max(Y)\} \succ \{\max(Z)\} \sim Z$ .

Suppose now  $|Z \cap A| > 0$ . Let  $Y'$  be the subset formed exactly by the  $|Z \cap A|$  top elements of  $Y$  according to  $R$ , whence  $\max(Y') = \max(Y)$ . Then  $|Y' \cap A| \leq |Z \cap A| < |Y \cap A|$ , and  $\max(Y') P \max(Z)$ . We conclude by transitivity, because  $Y \succ Y'$  owing to the fact that the conclusion under the first assumption of (31) has been proven above, and  $Y' \succ Z$  owing to subcase (i).  $\square$

The next proposition characterizes the preference-approval cardinality-first lexicographic rule  $\succcurlyeq_C^{L,pa}$  defined from a given preference-approval structure, by means of four axioms. Its proof borrows argument from the proof of Proposition 6 due to the fact that for any preference-approval structure  $(R, A)$ ,  $\sim_C^{L,pa} = \sim_U^{L,pa}$ . The reason is that whenever  $Y, Z \in \mathcal{P}$ ,  $Y \sim_C^{L,pa} Z$  if and only if  $(|Y \cap A| = |Z \cap A|$  and  $\max(Y) = \max(Z))$  if and only if  $Y \sim_U^{L,pa} Z$ .

**Proposition 7** Fix  $(R, A)$ , a preference-approval structure where  $R$  is a linear order. Let  $\succcurlyeq$  be a reflexive and transitive relation on  $\mathcal{P}$ .

Then  $\succcurlyeq = \succcurlyeq_C^{L,pa}$  if and only if  $\succcurlyeq$  satisfies Best-element conditional independence, Simple dominance, Preference-approval weight of freedom, and the Preference-approval indirect indifference principle.

**Proof** For the necessity part, one can immediately check that  $\succcurlyeq_C^{L,pa}$  defined in Eq. (11) satisfies the axioms.

To prove sufficiency, the fact that  $\succcurlyeq_C^{L,pa}$  is complete allows us to transform the conclusion  $\succcurlyeq = \succcurlyeq_C^{L,pa}$  into the following two statements: for each  $Y, Z \in \mathcal{P}$ ,

$$|Y \cap A| = |Z \cap A|, \max(Y) = \max(Z) \text{ (i.e., } Y \sim_C^{L,pa} Z) \text{ imply } Y \sim Z, \tag{32}$$

$$|Y \cap A| > |Z \cap A|, \text{ or } \left( |Y \cap A| = |Z \cap A|, \max(Y) P \max(Z) \right) \tag{33}$$

(i.e.,  $Y \succ_C^{L,pa} Z$ ) imply  $Y \succ Z$ .

Proposition 6 proves that under the conditions of (32), Best-element conditional independence and the Preference-approval indirect indifference principle guarantee  $Y \sim Z$ .

To prove (33), we consider two cases.

Case 1. When  $|Y \cap A| = |Z \cap A|$  and  $\max(Y) P \max(Z)$ , we know  $\{\max(Y)\} \succ \{\max(Z)\}$  due to Simple dominance. We distinguish two subcases.

In case  $|Y \cap A| = |Z \cap A| \leq 1$ ,  $Z \sim \{\max(Z)\}$  and  $Y \sim \{\max(Y)\}$  using (32). Thus we conclude by transitivity.

When  $|Y \cap A| = |Z \cap A| > 1$ , we note  $\{\max(Y)\} \cup ((Z \setminus \{\max(Z)\}) \cap A) \succ Z \cap A$  due to repeated application of Best-element conditional independence. We can apply (32) to prove both  $\{\max(Y)\} \cup ((Z \setminus \{\max(Z)\}) \cap A) \sim Y$ , and  $Z \sim Z \cap A$ . The conclusion follows from transitivity.

Case 2. Suppose now  $|Y \cap A| > |Z \cap A|$ , three subcases arise.

- (i) If  $\max(Y) = \max(Z)$  then because  $Y \succ_U^{L,pa} Z$ , the argument proven in Proposition 6 applies, which guarantees  $Y \succ Z$ . Note that we can use that piece of proof due to Lemma 3.
- (ii) If  $\max(Y)P \max(Z)$  it is also the case that  $Y \succ_U^{L,pa} Z$ .

When  $|Z \cap A| = 0$ ,  $Z \sim \{z_1\}$  using (32), also  $\{y_1\} \succ \{z_1\}$  by Simple dominance. And we conclude  $Y \succ Z$  by transitivity because  $Y \succ \{y_1\}$ , both if  $|Y \cap A| = 1$  by (32), or if  $|Y \cap A| > 1$  invoking (i).

Otherwise, letting  $Y'$  be formed by the top  $|Z \cap A|$  elements of  $Y$ , which include  $\max(Y)$ , we have  $Y \succ Y'$  by (i), and also  $Y' \succ Z$  invoking Case 1. Transitivity yields the conclusion.

- (iii) If  $\max(Z)P \max(Y)$ , we write  $Y = \{y_1, \dots, y_m\}$  with  $y_1 P y_2 P \dots P y_m$ . It is impossible that  $m = 1$ , because in this case  $y_1 \in A$  and  $\max(Z) = z_1 P y_1 \notin A$  owing to  $|Y \cap A| > |Z \cap A|$ , contradicting the fact that  $(R, A)$  is a preference-approval structure. Thus we are assuming  $\max(Z) = z_1 P y_1 P y_2$ , hence  $y_1, y_2, z_1 \in A$  because  $|Y \cap A| > |Z \cap A|$ . Therefore  $\{y_1, y_2\} \succ \{z_1\}$  by Preference-approval weight of freedom. Let  $q = |Y \cap A| - |Z \cap A| + 1 \leq |Y \cap A|$ . If  $|Y \cap A| = 2$  then  $Y \sim_C^{L,pa} \{y_1, y_2\}$ , whence  $Y \sim \{y_1, y_2\} \succ \{z_1\}$  using Eq. (32). If  $|Y \cap A| > 2$ , case (i) proves  $\{y_1, y_2, \dots, y_q\} \succ \{y_1, y_2\} \succ \{z_1\}$ . Whatever the case, we can repeatedly apply Best-element conditional independence to  $\{y_1, y_2, \dots, y_q\} \succ \{z_1\}$  to get  $Y \succ \{z_1\} \cup \{y_{q+1}, \dots, y_m\}$ . Noting  $|(\{z_1\} \cup \{y_{q+1}, \dots, y_m\}) \cap A| = |Z \cap A|$  and  $\max(\{z_1\} \cup \{y_{q+1}, \dots, y_m\}) = \max(Z)$ , we deduce  $\{z_1\} \cup \{y_{q+1}, \dots, y_m\} \sim Z$  therefore we conclude  $Y \succ Z$ .  $\square$

#### 4.1.4 Independence of the Axioms

In this section we prove that the axioms in the Propositions stated in the previous section are independent. We give appropriate examples for the case of Proposition 5. Then we provide indirect arguments that prove the independence of the axioms in Propositions 6 and 7.

**Example 3** The next situations guarantee that the axioms in Proposition 5 are independent.

- a) Consider  $X = \{x, y, z\}$  and the indirect-utility ranking  $\succ_U$  defined by the complete preorder such that  $xIyPz$ . It satisfies Indifference under non-covering

- expansions, Strong independence, and Preference of no-choice covering situations. However it fails to satisfy Covering-expansion monotonicity because  $\{y\} \succ_U \{z\}$  but  $\{x, y\} \succ_U \{x\}$  does not hold.
- b) The cardinality-based rule  $\succ_C$  satisfies Covering-expansion monotonicity (vacuously), Strong independence, and Preference of no-choice covering situations (vacuously). However it fails to satisfy Indifference under non-covering expansions because for each  $y \in X$ , there is no  $x \in X$  with  $\{y\} \succ_C \{z\}$ , but  $\{x, y\} \sim_C \{x\}$  is never true.
  - c) Consider the ranking  $\succ$  on  $X = \{x, y, z, t\}$  such that  $X \sim \{x, y, z\} \succ Y \sim Y'$  if  $Y, Y' \subseteq X, Y, Y' \notin \{X, \{x, y, z\}\}$ . Then  $\succ$  satisfies Covering-expansion monotonicity (vacuously), Indifference under non-covering expansions, and Preference of no-choice covering situations (vacuously). However it fails to satisfy Strong independence, because  $\{x, z\} \sim \{z, t\}$ , but  $\{x, y, z\} \succ \{y, z, t\}$ .
  - d) Consider the following version  $\succ'_L$  of the lexicimax ranking  $\succ_L$  defined from  $R$  on  $X = \{x, y, z\}$  with  $xPyPz$ : when  $Y, Z \subseteq X, Y \succ'_L Z$  if and only if  $Y \cap \{x, y\} \succ_L Z \cap \{x, y\}$ . Then  $\succ'_L$  satisfies Covering-expansion monotonicity, Indifference under non-covering expansions, and Strong independence. However it fails to satisfy Preference of no-choice covering situations, because  $\{y\} \succ'_L \{z\}$ , but  $\{x\} \succ'_L \{y\}$ .

If  $X$  contains more than two elements, the axioms in Proposition 6 must be independent, because when  $A = X$ , the axioms used in this result collapse to those in the characterizations of  $\succ^L_U$  proven in Bossert et al. (1994), Theorem 4.3. Indeed, in such Theorem  $\succ^L_U$  defined from a linear order  $R$  is characterized by Best-element conditional independence, the Indirect preference principle, Simple expansion monotonicity, and the Simple indirect indifference principle. We explained that the Simple indirect indifference principle and the Preference-approval indirect indifference principle establish the same property for preference-approval structures of the form  $(R, X)$  (v., Sect. 4.1.2). And in this case, Preference-approval-expansion monotonicity and Simple expansion monotonicity establish the same property too. Since Bossert et al. (1994) proved independence of the axioms of their Theorem 4.3, the axioms used in Proposition 6 must be independent.

The axioms in Proposition 7 are independent for similar reasons. Now when  $A = X$ , the axioms in this Proposition collapse to those in Bossert et al. (1994), Theorem 4.2, which are independent. Their Theorem 4.2 characterizes  $\succ^L_C$  defined from a linear order  $R$  by the Weak extension rule (here called Simple dominance), Best-element conditional independence, the Simple indirect indifference principle, and Simple priority of freedom.

### 4.2 Preference-Approvals on Opportunity Sets from Preference-Approval Structures

Now we turn our attention to the possibility of using the information contained in a preference-approval structure on a set of options to define preference-approval structures on the collection of its opportunity sets. To this purpose, we proceed to define pairs formed by a complete preorder and a subset (whose elements are opportunity

sets, i.e., subsets of the set of options). Afterwards we shall prove that they meet the requirements to be preference-approval structures on  $\mathcal{P}$ . With this interpretation we can argue that these pairs not only capture a ranking on the opportunity sets, but also they distribute them in two categories, respectively containing the “approved” and “disapproved” opportunity sets.

The following pairs can be defined with the help of the preference-approval structure  $(R, A)$ :

1. The cardinality-based pair  $(\succ_C, \mathcal{A}_C)$  is formed with the help of

$$\mathcal{A}_C = \{Y \in \mathcal{P} \mid |Y| \geq |A|\}. \tag{34}$$

2. The cardinality-first pair is  $(\succ_C^L, \mathcal{A}_C)$ .
3. The preference-approval cardinality-based pair  $(\succ_C^{pa}, \mathcal{A}_I)$  is defined with the help of

$$\mathcal{A}_I = \{Y \in \mathcal{P} \mid Y \cap A \neq \emptyset\}. \tag{35}$$

4. The preference-approval cardinality-first pair is  $(\succ_C^{L,pa}, \mathcal{A}_I)$ .
5. The indirect-utility intersection pair is  $(\succ_U, \mathcal{A}_I)$ .
6. The indirect-utility-first intersection pair is  $(\succ_U^L, \mathcal{A}_I)$ .
7. The preference-approval indirect-utility-first intersection pair is  $(\succ_U^{L,pa}, \mathcal{A}_I)$ .

Note that to define the pairs  $(\succ_C, \mathcal{A}_C)$  and  $(\succ_C^{pa}, \mathcal{A}_I)$ , we dispense with the information provided by  $R$ .

The pairs constructed above appeal to one of two different semantic interpretations of the set  $A$ . Those using  $\mathcal{A}_C$  interpret that  $A$  only serves as a proxy of how many elements we need in order to consider that we have reached “enough freedom of choice”. The pairs defined by  $\mathcal{A}_I$  interpret that  $A$  includes the exact collection of alternatives that are freedom-giving. Consequently, opportunity sets that contain at least one of these alternatives become “acceptable”, in the sense that they give some freedom.

The following lemma may be useful to study the indirect-utility intersection pair and other pairs defined through  $\mathcal{A}_I$ :

**Lemma 4** *The collection of subsets defined by Eq. (35) from  $(R, A)$  coincides with  $\{Y \in \mathcal{P} \mid \max(Y) \in A\}$ .*

**Proof** Obviously, when  $Y \in \mathcal{P}$  satisfies  $\max(Y) \in A$ , it is the case that  $Y \cap A \neq \emptyset$ . Conversely, if  $Y \in \mathcal{P}$  satisfies  $Y \cap A \neq \emptyset$ , then for  $y \in Y \cap A$ , using the definition of preference-approval structure,  $\max(Y)Ry \in A$  implies  $\max(Y) \in A$ .  $\square$

To prove that the pairs defined above are preference-approval structures on  $\mathcal{P}$ , the following arguments are in order.

**Proposition 8** *The cardinality-based pair  $(\succsim_C, \mathcal{A}_C)$  defines a preference-approval structure on  $\mathcal{P}$ .*

**Proof** Suppose  $Y, Z \in \mathcal{P}$  with  $Y \succsim_C Z$  and  $Z \in \mathcal{A}_C$ . This means  $|Y| \geq |Z|$  and also  $|Z| \geq |A|$ . Hence  $Y \in \mathcal{A}_C$  because  $|Y| \geq |A|$ .  $\square$

Owing to the property  $Y \succsim_C^L Z$  implies  $Y \succsim_C Z$  for each  $Y, Z \in \mathcal{P}$ , the next consequence of Proposition 8 follows easily:

**Corollary 2** *The cardinality-first pair  $(\succsim_C^L, \mathcal{A}_C)$  defines a preference-approval structure on  $\mathcal{P}$ .*

**Proposition 9** *The preference-approval cardinality-based pair  $(\succsim_C^{pa}, \mathcal{A}_I)$  defines a preference-approval structure on  $\mathcal{P}$ .*

**Proof** Suppose  $Y, Z \in \mathcal{P}$  with  $Y \succsim_C^{pa} Z$  and  $Z \in \mathcal{A}_I$ . This means  $|Y \cap A| \geq |Z \cap A|$  and also  $Z \cap A \neq \emptyset$ . Hence  $Y \in \mathcal{A}_I$  because  $|Y \cap A| \geq |Z \cap A| \geq 1$  implies  $Y \cap A \neq \emptyset$ .  $\square$

Using the fact that  $Y \succsim_C^{L,pa} Z$  implies  $Y \succsim_C^{pa} Z$  for each  $Y, Z \in \mathcal{P}$ , we prove the next consequence of Proposition 9:

**Corollary 3** *The preference-approval cardinality-first pair  $(\succsim_C^{L,pa}, \mathcal{A}_I)$  defines a preference-approval structure on  $\mathcal{P}$ .*

**Proposition 10** *The indirect-utility intersection pair  $(\succsim_U, \mathcal{A}_I)$  defines a preference-approval structure on  $\mathcal{P}$ .*

**Proof** Suppose  $Y, Z \in \mathcal{P}$  with  $Y \succsim_U Z$  and  $Z \in \mathcal{A}_I$ . This means  $\max(Y) R \max(Z)$  and also  $Z \cap A \neq \emptyset$ . As  $\max(Y) R \max(Z) R a$  for some  $a \in Z \cap A$ , we deduce  $\max(Y) R a$  by transitivity, and  $\max(Y) \in A$  because  $(R, A)$  is a preference-approval structure. Therefore  $Y \in \mathcal{A}_I$ .  $\square$

Now we take advantage of the properties that for each  $Y, Z \in \mathcal{P}$ ,  $Y \succsim_U^L Z$  implies  $Y \succsim_U Z$ , and  $Y \succsim_U^{L,pa} Z$  implies  $Y \succsim_U Z$ , to obtain the following result from Proposition 10:

**Corollary 4** *The indirect-utility-first intersection pair  $(\succsim_U^L, \mathcal{A}_I)$ , and the preference-approval indirect-utility-first intersection pair  $(\succsim_U^{L,pa}, \mathcal{A}_I)$ , define respective preference-approval structures on  $\mathcal{P}$ .*

We illustrate definitions given above in this section with the following Examples 4–6.

**Example 4** Suppose  $X = \{x, y, z\}$ . Consider the next preference-approvals on  $X$ ,  $(R, A)$  and  $(R', A')$ :

$$\frac{xy}{z} \quad \frac{x}{yz}$$

Direct computations show that the cardinality-based and cardinality-first preference-approval structures defined by  $(R, A)$  respectively are

$$\frac{\begin{matrix} X \\ \{x, y\} \{x, z\} \{y, z\} \\ \{x\} \{y\} \{z\} \end{matrix}}{\quad} \quad \frac{\begin{matrix} X \\ \{x, y\} \{x, z\} \{y, z\} \\ \{x\} \{y\} \\ \{z\} \end{matrix}}{\quad}$$

And the cardinality-based and cardinality-first preference-approval structures defined by  $(R', A')$  respectively are

$$\frac{\begin{matrix} X \\ \{x, y\} \{x, z\} \{y, z\} \\ \{x\} \{y\} \{z\} \end{matrix}}{\quad} \quad \frac{\begin{matrix} X \\ \{x, y\} \{x, z\} \\ \{y, z\} \\ \{x\} \\ \{y\} \{z\} \end{matrix}}{\quad}$$

**Example 5** Consider  $(R, A)$  and  $(R', A')$ , the preference-approvals on  $X$  defined in Example 4.

Direct computations show that the preference-approval cardinality-based and preference-approval cardinality-first structures defined by  $(R, A)$  coincide, and they are represented as follows:

$$\frac{\begin{matrix} X \{x, y\} \\ \{x\} \{y\} \{x, z\} \{y, z\} \\ \{z\} \end{matrix}}{\quad}$$

And the preference-approval cardinality-based and preference-approval cardinality-first structures defined by  $(R', A')$  coincide too. They are represented as follows:

$$\frac{\begin{matrix} X \{x, y\} \{x, z\} \{x\} \\ \{y\} \{z\} \{y, z\} \end{matrix}}{\quad}$$

**Example 6** Consider again the two preference-approvals on  $X$ ,  $(R, A)$  and  $(R', A')$ , defined in Example 4.

Direct computations show that the indirect-utility intersection and indirect-utility-first intersection structures defined by  $(R, A)$  respectively are



Finally, suppose  $B \in \mathcal{A}_C$ , i.e.,  $|B| \geq |A|$ . If  $A \subseteq B$ , for any  $a \in A$ , 2.(a) plus 2.(b) guarantee  $(A \setminus \{a\}) \cup (\{a\} \cup (B \setminus A)) \in \mathcal{A}$ . This is  $B \in \mathcal{A}$ . If  $A \subseteq B$  is false, we can write  $A = \{a_1, \dots, a_r, a_{r+1}, \dots, a_k\}$ ,  $B = \{a_1, \dots, a_r, b_{r+1}, \dots, b_l\}$  with  $r < k \leq l$  and  $\{a_{r+1}, \dots, a_k\} \cap B = \emptyset$ . Now  $(A \setminus \{a_{r+1}\}) \cup \{b_{r+1}, b_{k+1}, \dots, b_l\} \in \mathcal{A}$  by 2.(a) and 2.(b). Its cardinality is  $|B|$ . We can iteratively replace each of  $a_{r+2}, \dots, a_k$  (if any) by  $b_{r+2}, \dots, b_k$ , and the successive results are in  $\mathcal{A}$  by 2.(b). At the end of the process,  $B \in \mathcal{A}$ .  $\square$

This statement can be easily modified to avoid reliance on a fixed cardinality. We only need to remove dependence on a fixed  $A$  in the previous statement:

**Corollary 5** *A preference-approval structure  $(\succ, \mathcal{A})$  on  $\mathcal{P}$  is the preference-approval cardinality-based structure  $(\succ_C, \mathcal{A}_C)$  defined from a preference-approval structure on  $X$  if and only if it satisfies conditions 1 and 2.(b) in Proposition 11, and there is  $A \in \mathcal{A}$  such that  $A \setminus \{a\} \notin \mathcal{A}$  for all  $a \in A$ .*

The technique above can be replicated to axiomatize  $(\succ_C^L, \mathcal{A}_C)$  with the help of the characterization of  $\succ_C^L$  given by Bossert et al. (1994).

### 4.3 Preference-Approvals on Opportunity Sets from Rankings

We have mentioned that both the cardinality-based and the preference-approval cardinality-based structures,  $(\succ_C, \mathcal{A}_C)$  and  $(\succ_C^{pa}, \mathcal{A}_I)$ , disregard the preference in  $(R, A)$ . If we dispense with  $A$  to define preference-approvals on opportunity sets from  $R$ , a completely different strand appears. With this information, Definitions 1 to 4 below produce four different constructions of preference-approval structures on  $\mathcal{P}$ . It is admitted that they all use exogenous information to supplement the rankings presented in Sect. 2.2 with respective collections of “acceptable” opportunity sets.

The first case poses no difficulty:

**Definition 1** Let  $R$  be a complete preorder on  $X$ . Each  $\alpha \in \{1, \dots, |X|\}$  defines a preference-approval structure  $(\succ_C, \mathcal{A}^\alpha)$  on  $\mathcal{P}$  with

$$\mathcal{A}^\alpha = \{Y \in \mathcal{P} \mid |Y| \geq \alpha\}. \tag{36}$$

To define  $(\succ_C, \mathcal{A}^\alpha)$ , a fixed number of alternatives determines whether an opportunity set has reached “enough freedom of choice” or not. The case  $\alpha = 2$  is especially appealing, with larger values restricting the cardinality-based crude notion of “having freedom of choice” embodied by  $\succ_C$ .

**Definition 2** Let  $R$  be a complete preorder on  $X$ . Each  $x \in X$  defines a preference-approval structure  $(\succ_U, \mathcal{A}_x)$  on  $\mathcal{P}$  with

$$\mathcal{A}_x^R = \{Y \in \mathcal{P} \mid \exists y \in Y \text{ with } yRx\}. \tag{37}$$

The preference-approval structure  $(\succ_U, \mathcal{A}_x^R)$  supplements the indirect-utility ranking  $\succ_U$  with the help of a benchmark alternative  $x$ . Then it establishes that an opportunity set is “acceptable” if and only if, it contains alternatives as valuable as  $x$ .

To see why  $(\succ_U, \mathcal{A}_x^R)$  is a preference-approval structure on  $\mathcal{P}$ , consider  $Z \succ_U Y$  and  $Y \in \mathcal{A}_x^R$ . We can select  $y \in Y$  with  $yRx$ . As  $\max(Z) R \max(Y) R y R x$ , we conclude  $\max(Z) R x$  from transitivity of  $R$ . This proves  $Z \in \mathcal{A}_x^R$ .

Note that if  $xIy$ , then  $\mathcal{A}_x^R = \mathcal{A}_y^R$ . Then using Definition 2, the complete preorder  $R$  induces a number of different preference-approval structures that coincides with the number of indifference classes derived from  $R$ .

**Example 7** Suppose  $X = \{x, y, z, t\}$ . Consider the complete preorder  $R$  on  $X$  such that  $x P y P z I t$ . It produces three indifference classes, namely,  $\{x\}$ ,  $\{y\}$ , and  $\{z, t\}$ . Consequently, Definition 2 produces three preference-approval structures on  $\mathcal{P}$ . First we have  $(\succ_U, \mathcal{A}_x^R)$  defined as follows:

$$\begin{array}{cccccccc}
 X & \{x\} & \{x, y\} & \{x, z\} & \{x, t\} & \{x, y, z\} & \{x, y, t\} & \{x, z, t\} \\
 \hline
 & & \{y\} & \{y, z\} & \{y, t\} & \{y, z, t\} & & \\
 & & & \{z\} & \{t\} & \{z, t\} & & 
 \end{array}$$

Secondly,  $(\succ_U, \mathcal{A}_y^R)$  defined as follows:

$$\begin{array}{cccccccc}
 X & \{x\} & \{x, y\} & \{x, z\} & \{x, t\} & \{x, y, z\} & \{x, y, t\} & \{x, z, t\} \\
 & & \{y\} & \{y, z\} & \{y, t\} & \{y, z, t\} & & \\
 \hline
 & & & \{z\} & \{t\} & \{z, t\} & & 
 \end{array}$$

And finally  $(\succ_U, \mathcal{A}_z^R) = (\succ_U, \mathcal{A}_t^R)$  defined as follows:

$$\begin{array}{cccccccc}
 X & \{x\} & \{x, y\} & \{x, z\} & \{x, t\} & \{x, y, z\} & \{x, y, t\} & \{x, z, t\} \\
 & & \{y\} & \{y, z\} & \{y, t\} & \{y, z, t\} & & \\
 & & & \{z\} & \{t\} & \{z, t\} & & 
 \end{array}$$

**Definition 3** Let  $R$  be a complete preorder on  $X$ . Each  $x \in X$  and  $\alpha \in \{1, \dots, |X|\}$  define a preference-approval structure  $(\succ_U^\alpha, \mathcal{A}_x^\alpha)$  on  $\mathcal{P}$  with

$$\mathcal{A}_x^\alpha = \{Y \in \mathcal{P} \mid \text{either } \max(Y)Px \text{ or } \max(Y)Ix, |Y| \geq \alpha\}. \tag{38}$$

To see why  $(\succ_U^\alpha, \mathcal{A}_x^\alpha)$  is a preference-approval structure on  $\mathcal{P}$ , consider  $Z \succ_U^\alpha Y$  and  $Y \in \mathcal{A}_x^\alpha$ . Two cases arise. Suppose first  $Z \succ_U Y$ , i.e.,  $\max(Z) P \max(Y)$ . From  $Y \in \mathcal{A}_x^\alpha$ , either  $\max(Y)Px$ , or  $\max(Y)Ix$  plus  $|Y| \geq \alpha$ . In both cases,  $\max(Z) P x$  therefore  $Z \in \mathcal{A}_x^\alpha$ . Now suppose  $Z \sim_U Y$  and  $Z \succ_C Y$ , i.e.,  $\max(Z) I \max(Y)$  and  $|Z| \geq |Y|$ . As in the first case, either  $\max(Y)Px$  (and then  $\max(Z)Px$ ), or

$\max(Y)Ix$  plus  $|Y| \geq \alpha$  (and then  $\max(Z)Ix$  and  $|Z| \geq \alpha$ ). In both cases,  $Z \in \mathcal{A}_x^\alpha$  has been justified.

**Example 8** In the situation of Example 7, Definition 3 produces the preference-approval structures  $(\succ_U^L, \mathcal{A}_y^2)$  and  $(\succ_U^L, \mathcal{A}_x^3)$  on  $\mathcal{P}$  defined as follows:

	$X$			$X$	
$\{x, y, z\}$ $\{x, y\}$  $\{y, z\}$ <hr style="width: 100%;"/> $\{z\}$	$\{x, y, t\}$ $\{x, z\}$ $\{x\}$ $\{y, z, t\}$  $\{y, t\}$ <hr style="width: 100%;"/> $\{y\}$ $\{z, t\}$	$\{x, z, t\}$ $\{x, t\}$  $\{y, t\}$ <hr style="width: 100%;"/> $\{t\}$	$\{x, y, z\}$ <hr style="width: 100%;"/> $\{x, y\}$ $\{y, z\}$  $\{z\}$	$\{x, y, t\}$ $\{x, z\}$ $\{x\}$ $\{y, z, t\}$  $\{y, t\}$ <hr style="width: 100%;"/> $\{y\}$ $\{z, t\}$	$\{x, z, t\}$ $\{x, t\}$  $\{y, t\}$ <hr style="width: 100%;"/> $\{t\}$

**Definition 4** Let  $R$  be a complete preorder on  $X$ . Each  $x \in X$  and  $\alpha \in \{1, \dots, |X|\}$  define a preference-approval structure  $(\succ_C^L, \mathcal{A}_\alpha^x)$  on  $\mathcal{P}$  with

$$\mathcal{A}_\alpha^x = \{Y \in \mathcal{P} \mid \text{either } |Y| > \alpha \text{ or } |Y| = \alpha, \max(Y)Rx\}. \tag{39}$$

To see why  $(\succ_C^L, \mathcal{A}_\alpha^x)$  is a preference-approval structure on  $\mathcal{P}$ , consider  $Z \succ_C^L Y$  and  $Y \in \mathcal{A}_\alpha^x$ . Two cases arise. Suppose first  $Z \succ_C Y$ , i.e.,  $|Z| > |Y|$ . From  $Y \in \mathcal{A}_\alpha^x$ , either  $|Y| > \alpha$ , or  $|Y| = \alpha$  plus  $\max(Y)Rx$ . In both cases,  $|Z| > \alpha$  therefore  $Z \in \mathcal{A}_\alpha^x$ . Now suppose  $Z \sim_C Y$  and  $Z \succ_U Y$ , i.e.,  $|Z| = |Y|$  and  $\max(Z)R \max(Y)$ . As in the first case, either  $|Y| > \alpha$  (and then  $|Z| > \alpha$ ), or  $|Y| = \alpha$  plus  $\max(Y)Rx$  (and then  $|Z| = \alpha$  and  $\max(Z)Rx$ ). In both cases,  $Z \in \mathcal{A}_\alpha^x$  has been justified.

**Example 9** In the situation of Example 7, Definition 4 produces the preference-approval structures  $(\succ_C^L, \mathcal{A}_3^z)$  and  $(\succ_C^L, \mathcal{A}_2^y)$  on  $\mathcal{P}$  respectively presented as follows:

	$X$			$X$	
$\{x, y, z\}$ <hr style="width: 100%;"/> $\{x, y\}$ $\{y, z\}$  $\{z\}$	$\{x, y, t\}$ $\{y, z, t\}$  $\{x, z\}$ $\{x, z\}$ $\{z, t\}$ $\{x\}$ $\{y\}$	$\{x, z, t\}$ $\{x, t\}$ $\{y, t\}$ <hr style="width: 100%;"/> $\{t\}$	$\{x, y, z\}$ <hr style="width: 100%;"/> $\{x, y\}$ $\{y, z\}$  $\{z\}$	$\{x, y, t\}$ $\{y, z, t\}$ $\{x, z\}$ $\{x, z\}$ $\{y, t\}$ <hr style="width: 100%;"/> $\{z, t\}$ $\{x\}$ $\{y\}$	$\{x, z, t\}$ $\{x, t\}$ $\{y, t\}$ <hr style="width: 100%;"/> $\{t\}$

## 5 Conclusions and Future Research

New avenues of research emerge from the introduction of preference-approval structures in the problem of ranking opportunity sets. This article shows that the topic lends itself to many different approaches. We have defined reasonable rankings and preference-approval structures on opportunity sets using the information embedded in a preference-approval structure on the alternatives. The axiomatic basis of these mechanisms for the valuation of opportunity sets has been established. Numerical representability has been briefly explored too.

It has been shown that it is reasonable to produce preference-approval structures on opportunity sets from either preference-approval structures or rankings on the alternatives too. The topic has been treated superficially here, and it may be the subject of further analyses in the future.

In addition, it is possible to extend these ideas with the ternary version of preference-approval structures called ternary preferences (Albano et al. 2025; García-Lapresta and Martínez-Panero 2025).

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### Declarations

**Conflict of interest** The authors declare no conflict of interest.

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