Contributions to the use of preference-approval analysis for ranking opportunity sets

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Abstract

This manuscript overviews the theoretical contribution of the author to the utilization of preference-approval structures in the analysis of rankings of opportunity sets and social welfare functions. The topic benefits from several contributions by the author in 2025-2026.

1 Introduction

These notes document the presentation given by the author at the 15th Scientific Meeting CLADAG 2025 held in Napoli, Italy, in September 2025. CLADAG stands for "Classification and Data Analysis Group". This presentation synthesizes the core findings of a research program focused on preference-approval structures, which are further supported by related results reported in [15, 16].

The presentation provides a comprehensive overview of these findings, organized around the central theme of preference-approval structures.

2 The presentation and other materials

The accompanying slides detail the initiation of a theoretical investigation into preference-approval structures [6] within the context of ranking opportunity sets [5, 9, 12, 13].

Preference-approval structures were initially introduced within a voting framework [6]. Subsequent research has demonstrated the broad applicability of these structures [2, 3, 4, 7, 8, 10, 11].

- The focus of [1] is welfare analysis. As reported in this presentation, we first introduce preference-approval structures in the investigation of ranking opportunity sets. The main contribution is the definition of three rankings of opportunity sets defined from preference-approval structures on the alternatives. They are axiomatically characterized.

- From another perspective, we note that [16] bears comparison with the preceding [14]. Both papers are concerned with Arrow-type impossibilities. Approbatory social welfare functions are defined by [16]. In our article, the domain of preference-approvals is not universal, however population is variable. Whereas, roughly speaking, the focus of [14] is the type of conclusions deriving from Arrow's extended assumptions, in [16] we investigate which properties lead to conclusions similar to those of Arrow's theorem. In this way, we characterize pairwise dictatorial approbatory social welfare functions.

- Other theoretical aspects that we investigated include aggregation of preference-approval structures. This important problem had been ignored until [15].

Our works pave the way to the axiomatic investigation of other criteria with similar informational bases, as well as to new aggregation procedures.

The research corresponding to the slides printed below has been published in [1]. In fact, the published version includes more material pertaining to the same topic.

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Challenges in preference-approval of opportunity sets

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Motivation

First utilization of **preference-approval** analysis in the field of **ranking sets of objects** (opportunity sets).

▷ A problem with long tradition: **How to evaluate and compare different sets of options available to an individual?**

- Valuation by the most beneficial component (since Kreps, 1979): linked to the question of ‘reasonably’ extending an order on a set to its power set (Gärdenfors, 1976; Kannai and Peleg, 1984).
- Non-utilitarian approach: preference for freedom and preference for flexibility. Archetype: cardinality rule (Pattanaik and Xu, 1990).
- Mixed arguments. Bossert et al (1994).

▷ **Preference-approval structures** since Brams and Sanver (2009).

Goals

1. Two basic (however unnoticed) questions:
   - Number of preference-approval structures that exist on a set with finite cardinality.
   - Natural type of representability for preference-approval structures.

2. Define and characterize three rankings that use the knowledge contained in a preference-approval structure on a set to rank its opportunity sets.

   Secondary goal: numerical representations for them.

Basic definitions

Basic definitions

X is a **finite** set of n options — $\mathcal{P}$ is the set of its non-empty sets.

Elements of $\mathcal{P}$ (**opportunity sets**) are denoted Y, Z, ...

- R is a binary relation on X, its strict part is P, and its indifference is I.

Unless otherwise stated, henceforth R denotes a complete preorder on X (complete and transitive). Its semantic interpretation is "as good as".

For each $Y \subseteq X$, $\max_R(Y) = \{y \in Y \mid yRz \text{ for each } z \in Y\}$.

A linear order is an antisymmetric complete preorder.

- The notation of type $\succcurlyeq$ is associated with binary relations on $\mathcal{P}$.

Two benchmark rankings of opportunity sets

- **Indirect-utility** ranking $\succcurlyeq_U$ on $\mathcal{P}$ (Kreps, 1979):

$$Y \succcurlyeq_U Z \text{ if and only if } \max(Y)\, R \max(Z), \text{ for each } Y, Z \in \mathcal{P}.$$

The opportunity set Y is weakly preferred to Z as long as Y has preference-maximizing options from $Y \cup Z$.

- **Cardinality-based rule** $\succcurlyeq_C$ on $\mathcal{P}$ (Pattanaik and Xu, 1990):

$$Y \succcurlyeq_C Z \text{ if and only if } |Y| \geqslant |Z|, \text{ for each } Y, Z \in \mathcal{P}$$

The more cardinality an opportunity set has, the higher it ranks.

Preference-approval structures

Preference-approval structure on X: (R, A) where R is a complete preorder on X, $A \subseteq X$ (the "approved" options of X), and

$$\text{for all } x, y \in X: \quad xRy \text{ and } y \in A \text{ imply } x \in A. \tag{1}$$

Interpretation: when an alternative is as good as another one that is "approved", the former alternative must be "approved".

Example: standard presentation. With X={x,y,z,t,w} consider

$$\begin{array}{c} \text{w} \\ \text{x y} \\ \hline \text{z} \\ \text{t} \end{array}$$

We interpret $A = \{x, y, w\}$ (subset of X formed by the elements above the line).

Also, we interpret $w\,P\,x\,I\,y\,P\,z\,P\,t$.

First results

Counting preference-approval structures

Proposition. The number of preference-approvals on a X is

$$\sum_{k=0}^{n} k! \cdot k \cdot \mathcal{S}(n,k) = \sum_{k=1}^{n} (k+1)\Big(\sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n \Big)$$

where $\mathcal{S}(n,k)$ denotes the Stirling number of the second kind.

The number of preference-approvals for sets with small cardinalities

Cardinality	Number	**Cardinality**	Number
1	2	**6**	25.988
2	8	**7**	296.564
3	44	**8**	3.816.548
4	308	**9**	54.667.412
5	2.612	**10**	862.440.068

Representing preference-approval structures

Proposition. Let R be a binary relation on the finite set X, and let $A \subseteq X$.

Then (R, A) is a preference-approval structure on X if and only if there are $u : X \longrightarrow \mathbb{R}$ and $\alpha \in \mathbb{R}$ such that

$$xRy \Leftrightarrow u(x) \geqslant u(y), \text{ and}$$
$$a \in A \Leftrightarrow u(a) \geqslant \alpha.$$

New rankings

Ranking opportunity sets from preference-approvals

A preference-approval structure (R, A) on X is fixed.

The following **rankings** on $\mathcal{P}$ can be established:

1. **Preference-approval cardinality-based rule**

$$Y \succcurlyeq_C^{pa} Z \Leftrightarrow |Y \cap A| \geqslant |Z \cap A|, \text{ each } Y, Z \in \mathcal{P}.$$

2. **Preference-approval indirect-utility-first lexicographic rule**

$$Y \succcurlyeq_U^{L,pa} Z \Leftrightarrow \text{either } Y \succ_U Z \text{ or } \left(Y \sim_U Z, Y \succcurlyeq_C^{pa} Z\right), \text{ each } Y, Z \in \mathcal{P}.$$

3. **Preference-approval cardinality-first lexicographic rule**

$$Y \succcurlyeq_C^{L,pa} Z \Leftrightarrow \text{either } Y \succ_C^{pa} Z \text{ or } \left(Y \sim_C^{pa} Z, Y \succcurlyeq_U Z\right), \text{ each } Y, Z \in \mathcal{P}.$$

Numerical representations

Numerical representations of these rankings

$\succcurlyeq_C^{pa}$ is represented by $c(Y) = |Y \cap A|$ for each $Y \in \mathcal{P}$.

Recall $u(x) = |\{y \in X : xRy\}|$ for all $x \in X$ represents R. Then

- $v(Y) = \max_{y \in Y} u(y) + \dfrac{|Y \cap A|}{|A| + 1}$, for all $Y \in \mathcal{P}$,

  defines a numerical representation of $\succcurlyeq_U^{L,pa}$, and

- $w(Y) = |Y \cap A| + \dfrac{\max_{y \in Y} u(y)}{|X| + 1}$, for all $Y \in \mathcal{P}$,

  defines a numerical representation of $\succcurlyeq_C^{L,pa}$.

Gist of the argument: $\dfrac{|Y \cap A|}{|A| + 1}, \dfrac{\max_{y \in Y} u(y)}{|X| + 1} \subseteq [0, 1)$.

Characterizations

Characterization of $\succcurlyeq_C^{pa}$

Characterization of the preference-approval cardinality-based rules $\succcurlyeq_C^{pa}$ defined from **any** preference-approval structure:

Proposition. Let $\succcurlyeq$ be a transitive relation on $\mathcal{P}$.

Then there is a preference-approval structure (R, A) such that $\succcurlyeq = \succcurlyeq_C^{pa}$ if and only if $\succcurlyeq$ satisfies Preference of no-choice approval situations, Approval-expansion monotonicity, Indifference under non-approval expansions, and Strong independence.

To help write down axioms stated exclusively on a transitive ranking of opportunity sets $\succcurlyeq$, we define

$$A_{\succcurlyeq} = \{y \in X \mid \text{ there is } z \in X \text{ with } \{y\} \succ \{z\}\}.$$

Axioms - characterization of $\succcurlyeq_C^{pa}$

Preference of no-choice approval situations. For each $y \in X$ such that there is $z \in X$ with $\{y\} \succ \{z\}$ (i.e., for each $y \in A_{\succcurlyeq}$):

$$\{y\} \succcurlyeq \{x\} \text{ for all } x \in X.$$

Approval-expansion monotonicity. For each $y \in A_{\succcurlyeq}$:

$$\{x, y\} \succ \{x\} \text{ when } y \neq x \in X.$$

Indifference under non-approval expansions. For each $y \in X$ such that there is no $z \in X$ with $\{y\} \succ \{z\}$ (i.e., for each $y \notin A_{\succcurlyeq}$):

$$\{x, y\} \sim \{x\} \text{ for all } x \in X.$$

Strong independence. For each $Y, Z \in \mathcal{P}$ and $x \notin Y \cup Z$:

$$Y \succcurlyeq Z \text{ if and only if } Y \cup \{x\} \succcurlyeq Z \cup \{x\}.$$

Characterization of $\succcurlyeq_U^{L,pa}$

Characterization of the preference-approval indirect-utility-first lexicographic rule $\succcurlyeq_U^{L,pa}$ defined from a given preference-approval structure:

Proposition. Fix (R, A), a preference-approval structure where R is a linear order. Let $\succcurlyeq$ be a reflexive and transitive relation on $\mathcal{P}$.

Then $\succcurlyeq = \succcurlyeq_U^{L,pa}$ if and only if $\succcurlyeq$ satisfies Best-element conditional independence, Indirect preference principle, Preference-approval expansion monotonicity, and the Preference-approval indirect indifference principle.

Axioms - characterization of $\succsim_U^{L,pa}$

Best-element conditional independence. For each $Y, Z \in \mathcal{P}$ and $x \notin Y \cup Z$ such that $\max(Y)Px$, $\max(Z)Px$:

$$Y \succsim Z \text{ if and only if } Y \cup \{x\} \succsim Z \cup \{x\}.$$

Indirect preference principle. For each $Y \in \mathcal{P}$ such that $|Y| > 1$: $\{\max(Y)\} \succ Y \setminus \{\max(Y)\}$.

Preference-approval-expansion monotonicity. For each $x \in X$:

$$\{x, y\} \succ \{x\} \text{ when } x \neq y \in A,$$

Preference-approval indirect indifference principle. For each $x, y, z \in X$:

$$\{x, y\} \sim \{x, z\} \text{ when } xPyPz, \text{ and either } z \in A \text{ or } y \notin A.$$

Characterization of $\succcurlyeq_C^{L,pa}$

Characterization of the preference-approval cardinality-first lexicographic rule $\succcurlyeq_C^{L,pa}$ defined from a given preference-approval structure:

Proposition. Fix (R, A), a preference-approval structure where R is a linear order. Let $\succcurlyeq$ be a reflexive and transitive relation on $\mathcal{P}$.

Then $\succcurlyeq = \succcurlyeq_C^{L,pa}$ if and only if $\succcurlyeq$ satisfies Best-element conditional independence, the Preference-approval indirect indifference principle, Simple dominance, and Preference-approval weight of freedom.

Axioms - characterization of $\succcurlyeq_C^{L,pa}$

Simple dominance. For each $x, y \in X$, $xPy \Rightarrow \{x\} \succ \{y\}$.

Preference-approval weight of freedom. For each $x, y, z \in X$:

$$\{y, z\} \succ \{x\} \text{ when } xPyPz \text{ and } z \in A, \text{ and} \tag{2}$$

$$\{y, z\} \sim \{y\} \text{ when } yPz \text{ and } z \notin A. \tag{3}$$

Conclusions

Conclusions

New avenues of research emerge from the introduction of preference-approval structures in the problem of ranking opportunity sets.

We defined reasonable rankings on opportunity sets using information encoded in a preference-approval structure on the alternatives.

The axiomatic basis of these mechanisms for the valuation of opportunity sets has been established.

Numerical representability has been briefly explored.

Other approaches: preference-approval structures on opportunity sets from preferences/preference-approval structures on the alternatives.

Extensions with the ternary version of preference-approval structures.

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