# MIXED FINITE ELEMENT METHODS FOR A CLASS OF NONLINEAR REACTION DIFFUSION PROBLEMS

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**ABSTRACT:** A mixed finite element approximation is presented for a class of nonlinear reaction diffusion problems with a wide applicability. Some results about the existence and uniqueness of weak solutions are resumed. Semidiscrete error estimates are established assuming only Lipschitz regularity of the reactive term and nondecreasing of the diffusive term and demostrated with a new technique. The proofs of this error estimates are described in detail, first in the dual norm, and then in the $L^2$-norm. As an application, a model for numerically simulating wildland fires is presented.

**Keywords** - mixed finite element methods, nonlinear reaction diffusion, error estimates

## 1. INTRODUCTION

In this paper, we present a mixed finite element method for a nonlinear reaction diffusion problem of the form,

$$\begin{array}{ll} u_t - \Delta H(u) = f(u) & in\ \ (0,T)\times\Omega, \\ H(u) = 0 & on\ \ (0,T)\times\partial\Omega, \\ u(0,\boldsymbol{x}) = u_0(\boldsymbol{x}) & in\ \Omega. \end{array} \tag{1}$$

We assume that $H$ is continuous, piecewise smooth and satisfies the following hypotheses,

$$\begin{array}{l} H(0) = 0, \\ 0 < \alpha \le H'(\xi) \le \mu < \infty \quad for\ all\ \xi \in \Re, \end{array} \tag{2}$$

although for most of the numerical analysis of the semidiscrete scheme we only need $\mu \le \infty$.

We also assume that $f$ is continuous, piecewise smooth and satisfies a global Lipschitz condition,

$$|f(\xi) - f(\eta)| \le L_f\,|\xi - \eta| \quad for\ all\ \xi,\eta \in \Re. \tag{3}$$

A mixed finite element approximation of a classical nonlinear reaction diffusion problem, for example, a combustion process with radiation,

$$u_t - div\ \ (K(u)\ \ \boldsymbol{grad}\ u) = f(u)$$

produces a system matrix dependent on the solution, so this matrix is different in each time step with the resulting computational cost. Using the *Kirchoff transformation*,

$$H(u) = \int_0^u K(\xi)\, d\xi$$

the problem to solve is (1) and the new system matrix is independent of the solution.

This work generalizes a previous paper due to J. F. Epperson (Epperson, 1984) in two aspects, we study a more general nonlinear reaction-diffusion equations, with a nonlinearity not only in the diffusive term but also in the reactive term, and we use mixed instead of classical finite element methods. Mixed finite element methods and related cell-centered finite difference methods have become popular in recent years for modelling flow in porous media, due to their inherent conservation properties and the fact that the methods provide flux approximations as part of the formulation, see, for example, (Arbogast et al., 1996; Dawson & Kirby, 1999; Woodward & Dawson, 2000; Li Wu & Allen, 1999). In the particular application to wildland fires simulation that we present in this paper, where there exists strong gaps in the temperature, mixed finite element methods allow discontinuities in the temperature, preserving the continuity of the flux through the inter-element boundaries. Beside the finite differences methods, the lowest order mixed method used in this paper is easy to generalize to higher order methods simply by using higher order Raviart-Thomas elements.

The equations presented here have a wide applicability, not only to heat flow in materials with a temperature dependent conductivity and flow in porous media, but also to the Stefan problem, biological models (Ruuth, 1995), hydro-geological systems (Yifeng Wang et al., 1994; Yutiang Wang et al., 1995), etc.

In section 2, we resume some results about the existence and uniqueness of a weak solution of 1 based on the globally Lipschitz condition of the reaction term, and the monotone property of the diffusion term. We also resume some results about the existence of weak solutions of 4 based on the work of J. I. Díez and I. I. Vrabie (Diaz & Vrabie, 1994).

In section 3 we fix the notation, we describe the spaces of functions, the finite elements and the bilinear forms that we use along the next section. Section 4 is the weighty section, where we establish and prove the main results with a new technique using a discrete Laplacian operator. We prove first the semi-discrete error estimates in the dual norm, and then using this previous estimation, we obtain the corresponding error estimates in the norm $L^2(0,T;L^2(\Omega))$ for $\mu<\infty$.

Finally, in section 5, we present an application to a numerical model of wildland fires that authors have recently developed (Asensio & Ferragut, 2002). The mathematical modelling of reaction-diffusion processes usually involve more than one unknown, in this combustion model, the unknowns are the average temperature $u$ and the unburned solid fuel $v$, which is no diffusive,

$$\begin{array}{ll} u_t-\Delta H(u)=f(u,v) & in\ \ (0,T)\times\Omega, \\ v_t=g(u,v) & in\ \ (0,T)\times\Omega, \\ H(u)=0 & on\ \ (0,T)\times\partial\Omega, \\ u(0)=u_0,\ \ v(0)=v_0 & in\ \Omega. \end{array} \tag{4}$$

## 2. ABOUT EXISTENCE AND UNIQUENESS OF WEAK SOLUTIONS

First, let us consider the nonlinear diffusion equation,

$$\begin{array}{ll} u_t-\Delta H(u)=f & in\ \ (0,T)\times\Omega, \\ H(u)=0 & on\ \ (0,T)\times\partial\Omega, \\ u(0,\boldsymbol{x})=u_0(\boldsymbol{x}) & in\ \Omega. \end{array} \tag{5}$$

where $H:\Re\rightarrow\Re$ is continuous nondecreasing, $H(0)=0$, $f\in L^1(0,T;L^1(\Omega))$ and $u_0\in L^1(\Omega)$. By a weak solution of (5) we mean a function $u\in\mathcal{C}([0,T];L^1(\Omega))$ such that $H(u)\in L^1(0,T;L^1(\Omega))$ and which satisfies (5) in the sense of distributions over $(0,T)\times\Omega$.

It is well known that, for each $u_0\in L^1(\Omega)$ and $f\in L^1(0,T;L^1(\Omega))$ the problem (5) has a unique solution $u$, see (Brezis, & Crandall, 1979). Moreover, if $u_0\in L^p(\Omega)$ and $f\in L^1(0,T;L^p(\Omega))$ for some $p\in[0,\infty]$, the unique weak solution $u$ of (5) satisfies

$$\|u(t)\|_{0,p,\Omega}\leq\|u_0\|_{0,p,\Omega}+\int_0^t\|f(s)\|_{0,p,\Omega}\,ds$$

for each $t\in[0,T]$, see (Benilan, 1972). Furthermore, if $u_0$ and $f$ are bounded, $u_0\in L^\infty(\Omega)$ and $f\in L^\infty(0,T;L^\infty(\Omega))$, then

$$\begin{array}{rcl} u & \in & W^{1,2}\left(0,T;H^{-1}(\Omega)\right)\cap L^\infty(0,T;L^\infty(\Omega)) \\ H(u) & \in & L^2\left(0,T;H_0^{-1}(\Omega)\right) \end{array} \tag{6}$$

If $f$ depends on the solution, $f = f(u)$, but is bounded, the existence and uniqueness of a weak solution remains true. When $f(u)$ is not bounded but globally Lipschitz, the existence of a weak solution can be proved by standard fixed-point arguments. Uniqueness is simple by the monotone property of $H$. If $u_1$ and $u_2$ are two weak solutions of (1), taking into account that $(v,w)_{H^{-1}(\Omega)} = \left(v, (-\Delta)^{-1} w\right)$ where $(\cdot,\cdot)$ is the inner product of $L^2(\Omega)$, it follows

$$\begin{aligned}((u_1 - u_2)_t, u_1 - u_2)_{H^{-1}(\Omega)} + (H(u_1) - H(u_2), u_1 - u_2) = \\ = (f(u_1) - f(u_2), u_1 - u_2)_{H^{-1}(\Omega)}.\end{aligned}$$

By the monotone property of $H$, $(H(u_1) - H(u_2), u_1 - u_2) \geq 0$. Then, by Lipschitz condition of $f$, it follows from above that $\forall t \in (0,T)$

$$\frac{d}{dt} \|(u_1 - u_2)(t)\|^2_{-1,\Omega} \leq L_f \|(u_1 - u_2)(t)\|^2_{-1,\Omega},$$

so $\forall t \in (0,T)$

$$\|(u_1 - u_2)(t)\|^2_{H^{-1}(\Omega)} \leq \|(u_1 - u_2)(0)\|^2_{H^{-1}(\Omega)} + L_f \int_0^t \|(u_1 - u_2)(s)\|^2_{H^{-1}(\Omega)} ds.$$

By Gronwall's Lemma, being $(u_1 - u_2)(0) = 0$, it concludes $\forall t \in (0,T)$

$$\|(u_1 - u_2)(t)\|^2_{H^{-1}(\Omega)} \leq \|(u_1 - u_2)(0)\|^2_{H^{-1}(\Omega)} e^{L_f t} = 0.$$

For the general case of the semi-diffusive system (4), with $H : \Re \to \Re$ continuous strictly increasing, $H(0) = 0$, $f$ an upper-semicontinuous mapping from $\Re^2$ into $\Re$, and $g : \Re^2 \to \Re$ globally Lipschitz with respect to its second variable, we have the existence of a weak solution using a result from (Diaz & Vrabie, 1994). By a weak solution of (4) we mean a pair $u, v \in \mathcal{C}([0,T]; L^2(\Omega))$ such that $H(u) \in L^1(0,T; L^1(\Omega))$ and which satisfies (4) in the sense of distributions over $(0,T) \times \Omega$. If $H$, $f$ and $g$ verifies the mentioned conditions, then for each $u_0, v_0 \in L^\infty(\Omega)$ there exists $T_0 \in (0,T]$ such that (4) has at least one weak solution $(u,v)$ satisfying (6). This is a local existence result, but with additional hypothesis over $f$ and $g$, (Diaz & Vrabie, 1994) proof a global existence result.

## 3. APPROXIMATION

To fix notation, let $\Omega \subset \Re^2$ be an open bounded domain with piecewise $C^1$ boundary $\Gamma$, for instance polygonal. The letter $C$ will be used to denote a generic positive constant, not always the same, depending only on $\Omega$, $\Gamma$, the data functions, etc. Only when it is important will any dependence on parameter be made explicit.

In addition to the standard Sobolev spaces, $H^m(\Omega)$ we consider for the vectorial functions the space $H(div;\Omega)$ defined by (Roberts & Thomas, 1991)

$$H(div;\Omega) = \left\{ \boldsymbol{q} = (q_1, q_1) \in \left(L^2(\Omega)\right)^2 : div\ \boldsymbol{q} \in L^2(\Omega) \right\},$$

that is a Hilbert space with the norm

$$\|\boldsymbol{q}\|_{H(div;\Omega)} = \left\{ \|\boldsymbol{q}\|_{0,\Omega}^2 + \|div\ \boldsymbol{q}\|_{0,\Gamma}^2 \right\}^{\frac{1}{2}},$$

where $\|\cdot\|_{0,\Omega}$ denotes the usual scalar or vectorial norms. With the aid of the vectorial trace theorem (Roberts & Thomas, 1991), we have the space

$$\mathcal{H}(div;\Omega) = \left\{ \boldsymbol{q} \in H(div;\Omega) : \boldsymbol{q} \cdot \boldsymbol{n} \in L^2(\Gamma) \right\},$$

where $\mathbf{n}$ denotes the unit exterior normal vector to $\Omega$, which is a Hilbert space with the norm

$$\|\boldsymbol{q}\|_{\mathcal{H}(div;\Omega)} = \left\{ \|\boldsymbol{q}\|_{H(div;\Omega)}^2 + \|\boldsymbol{q} \cdot \boldsymbol{n}\|_{0,\Omega}^2 \right\}^{\frac{1}{2}}.$$

Let $\{\boldsymbol{\tau}_h : h \in \mathcal{H}\}$ be a regular family of triangulations by triangles of $\Omega$. For each natural number $k$, $P_k(\tau)$ denotes the space of restrictions to $\tau$ of polynomial functions of degree $k$ in two variables, and $\sum_\tau^{(k)}$ the principal lattice of order $k$ of the triangle $\tau$. So $\left(\tau, P_k(\tau), \sum_\tau^{(k)}\right)$ is the finite element of Lagrange of degree $k$. Let $\mathcal{L}_h^{(k)}$ be the corresponding Lagrangian interpolation operator, and $L_h^{(k)}$ the space of simplicial Lagrangian interpolants defined by

$$L_h^{(k)} = \left\{ v \in C^0\left(\overline{\Omega}\right) : \forall \tau \in \boldsymbol{\tau}_h, v|_\tau \in P_k(\tau) \right\}$$

It is known (Raviart & Thomas, 1983) that for each positive integer $k$, there exists a constant $C$, independent of $h$, such that,

$$\begin{aligned} \left| v - \mathcal{L}_h^{(k)} v \right|_{1,\Omega} + h^{-1} \left\| v - \mathcal{L}_h^{(k)} v \right\|_{0,\Omega} &\le C h^m |v|_{m+1,\Omega} \\ \text{for each } v \in H^{m+1}(\Omega),\ 1 \le m &\le k \end{aligned} \tag{7}$$

In the case of vectorial functions, for each natural number $k$, we define the space $D_k = (P_{k-1})^2 \oplus \boldsymbol{x} P_{k-1}$. We associate to each triangle $\tau \in \boldsymbol{\tau}_h$ the space $D_k(\tau)$ of restrictions to $\tau$ of functions of $D_k$, and an interpolation operator $\mathcal{E}_{\tau}^{(k)} : \mathcal{H}(div;\tau) \rightarrow D_k(\tau)$ such that $\forall \boldsymbol{q} \in \mathcal{H}(div;\tau)$ the function $\mathcal{E}_{\tau}^{(k)}\boldsymbol{q}$ is the only one in $D_k(\tau)$ verifying

$$\begin{aligned}
\int_{\tau} \mathcal{E}_{\tau}^{(k)}\boldsymbol{q}\cdot\boldsymbol{r}dx &= \int_{\tau}\boldsymbol{q}\cdot\boldsymbol{r}dx \quad \forall\boldsymbol{r}\in\left(P_{k-2}(\tau)\right)^2 \\
\int_{\tau'} \mathcal{E}_{\tau}^{(k)}\boldsymbol{q}\cdot\boldsymbol{n}_{\tau}wdx &= \int_{\tau'}\boldsymbol{q}\cdot\boldsymbol{n}_{\tau}wdx \quad \forall w\in P_{k-1}(\tau') \quad \textit{with } \tau' \textit{ edge of } \tau
\end{aligned}$$

where $\boldsymbol{n}_{\tau}$ is the unit exterior normal to $\partial\tau$, constant on each edge $\tau'$ of $\tau$. The mapping thus defined further satisfies $div\left(\mathcal{E}_{\tau}^{(k)}\boldsymbol{q}\right) = \Pi_{\tau}^{(k)}(div\ \boldsymbol{q})\ \forall\boldsymbol{q}\in\mathcal{H}(div;\tau)$, where $\Pi_{\tau}^{(k)}$ denotes the orthogonal projection in $L^2(\tau)$ onto $P_{k-1}(\tau)$. Denoting by $\mathcal{H}(div;\boldsymbol{\tau}_h)$ the space defined by

$$\mathcal{H}(div;\boldsymbol{\tau}_h) = \left\{\boldsymbol{q}\in\left(L^2(\Omega)\right)^2 : \forall\tau\in\boldsymbol{\tau}_h, \boldsymbol{q}|_{\tau}\in\mathcal{H}(div;\tau)\right\},$$

we define the equilibrium interpolation operator $\mathcal{E}_h^{(k)}$ of order $k$ on $\mathcal{H}(div;\boldsymbol{\tau}_h)$ constructed from operators $\mathcal{E}_{\tau}^{(k)}$ as follows, $\left(\mathcal{E}_h^{(k)}\boldsymbol{q}\right)|_{\tau} = \mathcal{E}_{\tau}^{(k)}(\boldsymbol{q}|_{\tau}) \quad \forall\tau\in\boldsymbol{\tau}_h$. This operator verifies

$$\begin{aligned}
div\ \boldsymbol{q} &= 0 \Rightarrow div\ \left(\mathcal{E}_{\tau}^{(k)}\boldsymbol{q}\right) = 0 \quad \forall\tau\in\boldsymbol{\tau}_h,\ \forall\boldsymbol{q}\in\mathcal{H}(div;\tau), \\
\mathcal{E}_h^{(k)}\boldsymbol{q} &\in H(div;\Omega) \quad \forall\boldsymbol{q}\in H(div;\Omega)\cap\mathcal{H}(div;\boldsymbol{\tau}_h).
\end{aligned}$$

We denote $E_h^{(k)}$ the finite-dimensional space of equilibrium interpolants,

$$E_h^{(k)} = \left\{\boldsymbol{q}\in H(div;\Omega) : \forall\tau\in\boldsymbol{\tau}_h, \boldsymbol{q}|_{\tau}\in D_k(\tau)\right\},$$

which is the space of functions $\boldsymbol{q}\in\left(L^2(\Omega)\right)^2$ for which the restriction $\boldsymbol{q}|_{\tau}$ may be identified with a function in $D_k(\tau)$ for each $\tau\in\boldsymbol{\tau}_h$ and for which $\boldsymbol{q}|_{\tau_1}\cdot\boldsymbol{n}_{\tau_1}+\boldsymbol{q}|_{\tau_2}\cdot\boldsymbol{n}_{\tau_2} = 0$ on $\tau'$ for all $\tau' = \tau_1\cap\tau_2$ with $\tau_1,\tau_2\in\tau_h$. It is known too (Roberts & Thomas, 1991) that there exists a constant $C$, independent of $h$, such that,

$$\begin{aligned}
&\left\|\boldsymbol{q}-\mathcal{E}_h^{(k)}\boldsymbol{q}\right\|_{0,\Omega} \leq Ch^l\left|\boldsymbol{q}\right|_{l,\Omega}, \\
&\qquad \forall\boldsymbol{q}\in\left(H^l(\Omega)\right)^2,\ 1\leq l\leq k, \\
&\left\|div\left(\boldsymbol{q}-\mathcal{E}_h^{(k)}\boldsymbol{q}\right)\right\|_{0,\Omega} \leq Ch^l\left|div\ \boldsymbol{q}\right|_{l,\Omega}, \\
&\qquad \forall\boldsymbol{q}\in\left(H^1(\Omega)\right)^2\ \textit{with}\ div\ \boldsymbol{q}\in H^l(\Omega),\ 1\leq l\leq k
\end{aligned} \tag{8}$$

If $\{\tau_h : h\in\mathcal{H}\}$ is a regular family of "triangulations" by rectangles having sides parallel to the axes, we can construct approximations of order $k$, for each natural number $k$, of functions in $L^2(\Omega)$ and functions in $H(div;\Omega)$. For any two nonnegative integers $k$ and $l$, we denote by $P_{k,l}$ the space of polynomials in two variables, of degree $k$ in the first variable and of degree $l$ in the second variable, and by $P_{k,l}(\tau)$ the space of restrictions to $\tau$ the rectangle of polynomials in $P_{k,l}$. To a rectangle $\tau$, and a positive integer $k$, is associated the set $\sum_{\tau}^{(k)}$ of points of the grid on $\tau$ obtained subdividing each edge of $\tau$ into $k$ equal parts. Let $\mathcal{L}_h^{(k)}$ be the corresponding Lagrangian interpolation operator, and $L_h^{(k)}$ the space of simplicial Lagrangian interpolants,

$$L_h^{(k)} = \left\{ v \in C^0\left(\overline{\Omega}\right) : \forall \tau \in \boldsymbol{\tau}_h, v|_{\tau} \in P_{k,k}(\tau) \right\}.$$

The error estimate for Lagrangian interpolation given by (7) remains valid when the family of triangulations is a regular family of triangulations by rectangles. The demonstrations for this case may be found for example in (Raviart & Thomas, 1983).

For the approximation of vector functions on a rectangle $\tau$, we define the space $D_k(\tau)$ to be the product space $P_{k,k-1}(\tau) \times P_{k-1,k}(\tau)$, and the corresponding equilibrium interpolant $\mathcal{E}_{\tau}^{(k)}$ of order $k$, $k \geq 1$ on $\tau$, verifying,

$$\begin{aligned} \int_{\tau} \mathcal{E}_{\tau}^{(k)} \boldsymbol{q} \cdot \boldsymbol{r} dx &= \int_{\tau} \boldsymbol{q} \cdot \boldsymbol{r} dx \quad \forall \boldsymbol{r} \in P_{k-2,k-1}(\tau) \times P_{k-1,k-2}(\tau) \\ \int_{\tau'} \mathcal{E}_{\tau}^{(k)} \boldsymbol{q} \cdot \boldsymbol{n}_{\tau} w dx &= \int_{\tau'} \boldsymbol{q} \cdot \boldsymbol{n}_{\tau} w dx \quad \forall w \in P_{k-1}(\tau') \quad with\ \tau'\ edge\ of\ \tau \end{aligned}$$

As in the previous case, $div\left(\mathcal{E}_{\tau}^{(k)} \boldsymbol{q}\right) = \Pi_{\tau}^{(k-1)}(div\ \boldsymbol{q})\ \ \forall \boldsymbol{q} \in \mathcal{H}(div; \tau)$ where $\Pi_{\tau}^{(k-1)}$ is the orthogonal $L^2$ projection on $L^2(\Omega)$ onto $P_{k-1,k-1}(\tau)$. We define the interpolation operator $\mathcal{E}_h^{(k)}$ from $H(div; \Omega) \cap \mathcal{H}(div; \boldsymbol{\tau}_h)$ onto $E_h^{(k)}$, where $E_h^{(k)}$ is defined by

$$E_h^{(k)} = \left\{ \boldsymbol{q} \in H(div; \Omega) : \forall \tau \in \boldsymbol{\tau}_h, \boldsymbol{q}|_{\tau} \in D_k(\tau) \right\},$$

by requiring as before that $\mathcal{E}_h^{(k)} \boldsymbol{q}$ agree on $\tau$ with $\mathcal{E}_{\tau}^{(k)}(\boldsymbol{q}|_{\tau})$ for each $\tau \in \boldsymbol{\tau}_h$. The error estimate (8) remains valid in the rectangular case. For demonstration in dimension 2, see (Thomas, 1977).

We denote for simplicity $M = L^2(\Omega)$ and $X = H(div; \Omega)$, but maintain the classical notation for norms to avoid confusion. In order to obtain the corresponding mixed variational formulation of the problem (1) we introduce a new variable $\boldsymbol{p} = \boldsymbol{grad}\ H(u)$ . In a thermal problem, $u$ denotes the temperature, $\boldsymbol{p}$ is the heat flow, and $\boldsymbol{p} \cdot \boldsymbol{n}$, $\boldsymbol{n}$ a unit vector normal to $\Gamma$, is the flux across $\Gamma$.

Assuming the solution is sufficiently regular, the corresponding mixed variational formulation of (1) is: find $u(t) \in M$, such that $\forall\ t \in (0, T)$

$$\begin{aligned} &(u_t(t), v) - (div\ \boldsymbol{p}(t), v) = (f(u(t)), v) && \forall\ v \in M, \\ &(\boldsymbol{p}(t), \boldsymbol{q}) + (H(u(t)), div\ \boldsymbol{q}) = 0 && \forall\ \boldsymbol{q} \in X, \end{aligned} \tag{9}$$

with initial conditions $u(0) = u_0$.

We introduce the following bilinear forms, with $(\cdot,\cdot)$ the corresponding scalar product,

$$\begin{array}{rccl} a(\cdot,\cdot): & X \times X & \longrightarrow & \Re \\ & \boldsymbol{p},\ \boldsymbol{q} & \longrightarrow & a(\boldsymbol{p},\boldsymbol{q}) = (\boldsymbol{p},\boldsymbol{q}), \end{array}$$

$$\begin{array}{rccl} b(\cdot,\cdot): & X \times M & \longrightarrow & \Re \\ & \boldsymbol{p},\ v & \longrightarrow & b(\boldsymbol{p},v) = (div\ \boldsymbol{p},v). \end{array}$$

If $X'$ and $M'$ are the corresponding dual spaces, and $\langle\cdot,\cdot\rangle_M$ $\langle\cdot,\cdot\rangle_X$ the duality products, we have the following linear operators, $\mathcal{A} \in \mathcal{L}(X,X')$ and $\mathcal{B} \in \mathcal{L}(X,M')$,

$$\begin{array}{rccl} \mathcal{A}: & X & \longrightarrow & X' \\ & \boldsymbol{p} & \longrightarrow & \mathcal{A}\boldsymbol{p} \ \text{ such that } \langle \mathcal{A}\boldsymbol{p},\boldsymbol{q}\rangle_X = a(\boldsymbol{p},\boldsymbol{q}) \quad \forall \boldsymbol{q} \in X, \end{array}$$

$$\begin{array}{rccl} \mathcal{B}: & X & \longrightarrow & M' \\ & \boldsymbol{p} & \longrightarrow & \mathcal{B}\boldsymbol{p} \ \text{ such that } \langle \mathcal{B}\boldsymbol{p},v\rangle_M = b(\boldsymbol{p},v) \quad \forall v \in M, \end{array}$$

and the transpose operator,

$$\begin{array}{rccl} \mathcal{B}^\star: & M & \longrightarrow & X' \\ & v & \longrightarrow & \mathcal{B}^\star v \ \text{ such that } \langle \mathcal{B}^\star v,\boldsymbol{q}\rangle_M = b(\boldsymbol{q},v) \quad \forall \boldsymbol{q} \in X. \end{array}$$

Identifying $M$ with $M'$, that is to say, identifying the duality product with the scalar product in $M$, (9) can be written in the form

$$(u_t(t),v) - (\mathcal{B}\boldsymbol{p}(t),v) = (f(u(t)),v) \qquad \forall\ v \in M \tag{10}$$

$$\langle \mathcal{A}\boldsymbol{p}(t),\boldsymbol{q}\rangle_X + \langle \mathcal{B}^\star H(u(t)),\boldsymbol{q}\rangle_X = 0 \qquad \forall\ \boldsymbol{q} \in X, \tag{11}$$

finding $\boldsymbol{p}(t) = \mathcal{A}^{-1}\mathcal{B}^\star H(u(t))$ from (11) and replacing in (10),

$$(u_t(t),v) - \left(\mathcal{B}\mathcal{A}^{-1}\mathcal{B}^\star H(u(t)),v\right) = (f(u(t)),v) \qquad \forall\ v \in M.$$

so, in the space of functions $M \approx M'$ is realized

$$u_t - \mathcal{B}\mathcal{A}^{-1}\mathcal{B}^\star H(u(t)) = f(u(t)),$$

where $\mathcal{B}\mathcal{A}^{-1}\mathcal{B}^\star = -\Delta$ is an operator from $M$ onto $M$. Denoting $\mathcal{D} = (-\Delta)^{-1}$ as an operator from $M$ onto $M$, the continuous problem to solve is: *find* $u(t) \in M$, *such that* $\forall t \in (0,T)$,

$$\begin{array}{l} \mathcal{D}u_t(t) - H(u(t)) = \mathcal{D}f(u(t)), \\ u(0) = u_0. \end{array} \tag{12}$$

Let $M_h = L_h^{(k-1)} \subset M$ and $X_h = E_h^{(k)} \subset X$ the finite dimensional subspaces described before, for any of the two kinds of triangulations mentioned. The corresponding semidiscrete formulation of the problem (9) is: *find* $u_h(t) \in M_h$, *such that* $\forall\ t \in (0,T)$

$$\begin{array}{ll} (u_{h,t}(t), v_h) - (div\ \boldsymbol{p}_h(t), v_h) = (f(u_h(t)), v_h) & \forall\ v_h \in M_h, \\ (\boldsymbol{p}_h(t), \boldsymbol{q}_h) + (H(u_h(t)), div\ \boldsymbol{q}_h) = 0 & \forall\ \boldsymbol{q}_h \in X_h, \end{array} \tag{13}$$

*with initial conditions* $u_h(0) = u_{0,h}$, *being* $u_{0,h}$ *an approximation of* $u_0$ *on* $M_h$.

*Remark.* Functions $f(u_h)$ and $H(u_h)$ have not to be in $M_h$, however, as $div \boldsymbol{q}_h \in M_h \quad \forall \boldsymbol{q}_h \in M_h$, equations (13) can be written as follow

$$\begin{array}{ll} (u_{h,t}(t), v_h) - (div\ \boldsymbol{p}_h(t), v_h) = (\mathcal{P}_h f(u_h(t)), v_h) & \forall\ v_h \in M_h, \\ (\boldsymbol{p}_h(t), \boldsymbol{q}_h) + (\mathcal{P}_h H(u_h(t)), div\ \boldsymbol{q}_h) = 0 & \forall\ \boldsymbol{q}_h \in X_h, \end{array}$$

where $\mathcal{P}_h$ is the orthogonal projection operator on $M$ onto $M_h$.

As in the continuous case, the corresponding linear operators are

$$\begin{array}{rrcl} \mathcal{A}_h : & X_h & \longrightarrow & X_h' \\ & \boldsymbol{p}_h & \longrightarrow & \mathcal{A}_h \boldsymbol{p}_h \quad \textit{such that } \langle \mathcal{A}_h \boldsymbol{p}_h, \boldsymbol{q}_h \rangle_{X_h} = (\boldsymbol{p}_h, \boldsymbol{q}_h) \quad \forall \boldsymbol{q}_h \in X_h, \end{array}$$

$$\begin{array}{rrcl} \mathcal{B}_h : & X_h & \longrightarrow & M_h' \\ & \boldsymbol{q}_h & \longrightarrow & \mathcal{B}_h \boldsymbol{q}_h \quad \textit{such that } \langle \mathcal{B}_h \boldsymbol{q}_h, v_h \rangle_{M_h} = (div\ \boldsymbol{q}_h, v_h) \quad \forall v_h \in M_h, \end{array}$$

and the transpose operator,

$$\begin{array}{rrcl} \mathcal{B}_h^t : & M_h & \longrightarrow & X_h' \\ & v_h & \longrightarrow & \mathcal{B}_h^t v_h \quad \textit{such that } \langle \mathcal{B}_h^t v_h, \boldsymbol{q}_h \rangle_{M_h} = (div\ \boldsymbol{q}_h, v_h) \quad \forall \boldsymbol{q}_h \in X_h. \end{array}$$

Identifying $M_h$ with $M_h'$, and proceeding as in the continuous case, it follows in $M_h$ the equation

$$u_{h,t}(t) - \mathcal{B}_h \mathcal{A}_h^{-1} \mathcal{B}_h^t \mathcal{P}_h H(u_h(t)) = \mathcal{P}_h f(u(t)). \tag{14}$$

We introduce now the operator $\mathcal{D}_h$ defined as

$$\begin{array}{rrcl} \mathcal{D}_h : & M & \longrightarrow & M_h \\ & \varphi & \longrightarrow & v_h = \mathcal{D}_h \varphi \end{array}$$

where $v_h$ is the mixed finite element approximation of the solution to the problem

$$\begin{array}{rcll} -\Delta v & = & \varphi & \textit{in } \Omega, \\ v|_\Gamma & = & 0 & \textit{on } \Gamma, \end{array} \tag{15}$$

that is, $v_h$ is the element from the pair $(\boldsymbol{z}_h, v_h)$ solution to the problem

$$\begin{aligned} (\boldsymbol{z}_h, \boldsymbol{q}_h) + (v_h, div\boldsymbol{q}_h) = 0 \quad & \forall \boldsymbol{q}_h \in X_h, \\ (div\boldsymbol{z}_h, w_h) = -(\varphi, w_h) \quad & \forall w_h \in M_h, \end{aligned} \tag{16}$$

these equations can be written using the notation of the discrete linear operators described above,

$$\begin{aligned} \mathcal{A}_h \boldsymbol{z}_h + \mathcal{B}_h^t v_h = 0 \quad & in\ X_h, \\ \mathcal{B}_h \boldsymbol{z}_h = -\mathcal{P}_h \varphi \quad & in\ M_h, \end{aligned}$$

so $v_h = \left(\mathcal{B}_h \mathcal{A}_h^{-1} \mathcal{B}_h^t\right)^{-1} \mathcal{P}_h \varphi$, and we can write $\mathcal{D}_h = \left(\mathcal{B}_h \mathcal{A}_h^{-1} \mathcal{B}_h^t\right)^{-1} \mathcal{P}_h$.

*Remark.* Observe that if $\varphi \in M_h$, then the orthogonal projection operator can be removed, that is $\left(\mathcal{B}_h \mathcal{A}_h^{-1} \mathcal{B}_h^t\right)^{-1} \mathcal{P}_h \varphi = \left(\mathcal{B}_h \mathcal{A}_h^{-1} \mathcal{B}_h^t\right)^{-1} \varphi$.

Multiplying (14) by $\left(\mathcal{B}_h \mathcal{A}_h^{-1} \mathcal{B}_h^t\right)^{-1}$ and taking into account that $u_{h,t} \in M_h$, the semidiscrete problem to solve is: find $u_h(t) \in M_h$, such that $\forall\ t \in (0, T)$

$$\begin{aligned} & \mathcal{D}_h u_{h,t} - \mathcal{P}_h H(u_h) = \mathcal{D}_h f(u_h), \\ & u_h(0) = u_{0,h}. \end{aligned} \tag{17}$$

*Remark.* There is a particular case in which the orthogonal projection operator $\mathcal{P}_h$ is not necessary. For $k = 1$ functions of $M_h$ are piecewise constants, so $H(u_h) \in M_h$ and then $\mathcal{P}_h H(u_h) = H(u_h)$.

## 3. SEMI-DISCRETE ERROR ESTIMATES

Equations of the mixed variational formulation of the auxiliary elliptic problem (15) are

$$\begin{aligned} (\boldsymbol{z}, \boldsymbol{q}) - (v, div\ \boldsymbol{q}) = 0 \quad & \forall\ \boldsymbol{q} \in X, \\ (div\ \boldsymbol{z}, w) + (\varphi, w) = 0 \quad & \forall\ w \in M. \end{aligned} \tag{18}$$

We introduce the space $V = \{\boldsymbol{q} \in X : b(\boldsymbol{q}, w) = 0\ \forall w \in M\}$. It is clear that the bilinear form $a(\cdot,\cdot)$ is $V$-elliptic, and the bilinear form $b(\cdot,\cdot)$ satisfies the "inf-sup" continuous condition of Babuska-Brezzi,

$$\exists \beta > 0 :\ \sup \boldsymbol{q} \in X \frac{b(\boldsymbol{q}, w)}{\|\boldsymbol{q}\|_{H(div;\Omega)}} \geq \beta \|w\|_{0,\Omega} \quad \forall w \in M, \tag{19}$$

so the problem (18) has a unique solution.

For the corresponding equations of semi-discrete problem (16) associated to the elliptic problem, we define $V_h = \{\boldsymbol{q}_h \in X_h : b(\boldsymbol{q}_h, w_h) = 0\ \forall w_h \in M_h\}$. It is easy to see that for the finite dimensional subspaces $M_h$ and $X_h$ described before, $V_h \subset V$, so the bilinear application $a(\cdot,\cdot)$ is $V_h$-elliptic.

Let be $\widetilde{\mathcal{E}}_h^{(k)}$ the following continuous linear operator: given $\boldsymbol{q} \in M$, we define $\Delta\varphi = div\boldsymbol{q}$, $\varphi|_\Gamma = 0$ and set $\widetilde{\mathcal{E}}_h^{(k)} = \mathcal{E}_h^{(k)}(\boldsymbol{grad}\varphi)$. Such a linear operator $\widetilde{\mathcal{E}}_h^{(k)} : X \to X_h$ verifies

$$\begin{array}{ll} i) & b\left(\boldsymbol{q} - \widetilde{\mathcal{E}}_h^{(k)}\boldsymbol{q}, \mathrm{v}_h\right) = 0, \quad \forall v_h \in M_h, \\ ii) & \left\|\widetilde{\mathcal{E}}_h^{(k)}\boldsymbol{q}\right\|_{H(div;\Omega)} \le C\left\|\boldsymbol{q}\right\|_{H(div;\Omega)}, \quad \forall \boldsymbol{q} \in X_h. \end{array}$$

Then, the continuous "inf-sup" condition (19) implies the discrete "inf-sup" condition of Babuska-Brezzi,

$$\exists\beta^\star > 0: \ \sup \boldsymbol{q}_h \in X_h \frac{b(\boldsymbol{q}_h, w_h)}{\|\boldsymbol{q}_h\|_{H(div;\Omega)}} \ge \beta\|w_h\|_{0,\Omega} \quad \forall w_h \in M_h, \tag{20}$$

and so the problem (16) has a unique solution (Roberts & Thomas, 1991).

The error estimate for mixed finite element approximation of Laplace operator gives the following result, with $v = \mathcal{D}\varphi$ and $v_h = \mathcal{D}_h\varphi$, and taking into account the equivalence (Lions & Magenes, 1968; Brezis, 1983)) of the norms $\|\cdot\|_{k+1,\Omega}$ and $\|\Delta\cdot\|_{k-1,\Omega}$ in $H^{k+1}(\Omega) \cap H_0^1(\Omega)$ for sufficient regular conditions on $\Gamma$.

**lemma 1** *If $\mathcal{D}$ and $\mathcal{D}_h$ are the operators described before and $k$ is a positive integer, then for all $\varphi \in M$ exists a constant $C$, independent of $h$, such that,*

$$\|(\mathcal{D} - \mathcal{D}_h)\varphi\|_{0,\Omega} \le Ch^k\|\varphi\|_{k-1,\Omega}.$$

The discrete "inf-sup" condition (20) verified by the auxiliary elliptic problem allows us to proof the following results,

**lemma 2** *If $\mathcal{D}$ and $\mathcal{D}_h$ are the operators described before, then for all $\varphi \in M$*

$$\begin{aligned} \|\mathcal{D}_h\varphi\|_{0,\Omega} &\le \frac{\|\varphi\|_{0,\Omega}}{(\beta^\star)^2}, \\ \left\|\mathcal{D}_h^{\frac{1}{2}}\varphi\right\|_{0,\Omega} &\le \frac{\|\varphi\|_{0,\Omega}}{\beta^\star}. \end{aligned}$$

**lemma 3** *If $\mathcal{D}_h$ is the operator described before, then for all $\varphi \in M$*

$$\left\|\mathcal{D}_h^{\frac{1}{2}}\varphi\right\|_{0,\Omega} \xrightarrow{h\to 0} \|\varphi\|_{-1,\Omega}$$

*Proof of lemma* 2. From the characterization of the norm $\|\cdot\|_{0,\Omega}$ and the inequality $\|\boldsymbol{q}_h\|_{0,\Omega} \leq \|\boldsymbol{q}_h\|_{H(div;\Omega)}$, it follows

$$\|\boldsymbol{z}_h\|_{0,\Omega} = \sup \boldsymbol{q}_h \in X_h \frac{(\boldsymbol{z}_h, \boldsymbol{q}_h)}{\|\boldsymbol{q}_h\|_{0,\Omega}} \geq \sup \boldsymbol{q}_h \in X_h \frac{(\boldsymbol{z}_h, \boldsymbol{q}_h)}{\|\boldsymbol{q}_h\|_{H(div;\Omega)}} = \sup \boldsymbol{q}_h \in X_h \frac{-(\boldsymbol{z}_h, \boldsymbol{q}_h)}{\|\boldsymbol{q}_h\|_{H(div;\Omega)}}.$$

Using the discrete "ins-sup" condition (20) and the first equation of (16) we have,

$$\sup \boldsymbol{q}_h \in X_h \frac{-(\boldsymbol{z}_h, \boldsymbol{q}_h)}{\|\boldsymbol{q}_h\|_{H(div;\Omega)}} = \sup \boldsymbol{q}_h \in X_h \frac{(div\ \boldsymbol{q}_h, v_h)}{\|\boldsymbol{q}_h\|_{H(div;\Omega)}} \geq \beta^\star \|v_h\|_{0,\Omega} = \beta^\star \|\mathcal{D}_h\varphi\|_{0,\Omega}.$$

Thus $\|\mathcal{D}_h\varphi\|_{0,\Omega} \leq \frac{1}{\beta^\star} \|\boldsymbol{z}_h\|_{0,\Omega}$. On the other hand,

$$(\mathcal{D}_h\varphi, \varphi) = (v_h, \varphi) = -(div\ \boldsymbol{z}_h, v_h) = (\boldsymbol{z}_h, \boldsymbol{z}_h),$$

so we get $\|\boldsymbol{z}_h\|_{0,\Omega}^2 = (\mathcal{D}_h\varphi, \varphi) \leq \|\mathcal{D}_h\varphi\|_{0,\Omega} \|\varphi\|_{0,\Omega}$. Then, we have the first result simplifying from

$$\|\mathcal{D}_h\varphi\|_{0,\Omega}^2 \leq \frac{1}{(\beta^\star)^2} \|\boldsymbol{z}_h\|_{0,\Omega}^2 \leq \frac{1}{(\beta^\star)^2} \|\mathcal{D}_h\varphi\|_{0,\Omega} \|\varphi\|_{0,\Omega}.$$

Taking square roots in the following expression we obtain the second result,

$$\left\|\mathcal{D}_h^{\frac{1}{2}}\varphi\right\|_{0,\Omega}^2 = \left(\mathcal{D}_h^{\frac{1}{2}}\varphi, \mathcal{D}_h^{\frac{1}{2}}\varphi\right) = (\mathcal{D}_h\varphi, \varphi) \leq \|\mathcal{D}_h\varphi\|_{0,\Omega} \|\varphi\|_{0,\Omega} \leq \frac{1}{(\beta^\star)^2} \|\varphi\|_{0,\Omega}^2.$$

*Proof of lemma* 3. We have seen that $\left\|\mathcal{D}_h^{\frac{1}{2}}\varphi\right\|_{0,\Omega}^2 = \|\boldsymbol{z}_h\|_{0,\Omega}^2$ where, as being the solution of the auxiliary elliptic problem, it has $\|\boldsymbol{z}_h\|_{0,\Omega} \xrightarrow{h\to 0} \|\boldsymbol{z}\|_{0,\Omega}$. Then it suffices to show that $\|\boldsymbol{z}\|_{0,\Omega} = \|\varphi\|_{-1,\Omega}$. By the definition of the norm of $H^{-1}(\Omega)$, dual of $H_0^1(\Omega)$, we have $\forall\varphi \in L^2(\Omega) \subset H^{-1}(\Omega)$

$$\|\varphi\|_{-1,\Omega} = \sup_{w\in H_0^1(\Omega)} \frac{\langle\varphi, w\rangle_{H_0^1(\Omega)}}{\|w\|_{H_0^1(\Omega)}}.$$

But $\forall w \in H_0^1(\Omega)$ we have $\langle\varphi, w\rangle_{H_0^1(\Omega)} = (\varphi, w) = (\boldsymbol{grad}\ v, \boldsymbol{grad}\ w)$ where $-\Delta v = \varphi$ and $v|_\Gamma = 0$. Thus,

$$\|\varphi\|_{-1,\Omega} = \sup_{w\in H_0^1(\Omega)} \frac{(\boldsymbol{grad}\ v, \boldsymbol{grad}\ w)}{\|w\|_{H_0^1(\Omega)}} = \|\boldsymbol{grad}\ v\|_{0,\Omega} = \|\boldsymbol{z}\|_{0,\Omega}.$$

With these previous results we first establish a semidiscrete error estimate for (17) in the norm $\left\|\mathcal{D}_h^{\frac{1}{2}}(\cdot)\right\|_{0,\Omega}$, which is a discrete version of the dual norm $\|\cdot\|_{-1,\Omega}$ as Lemma 3 shown.

**theorem 4** *If $u$ is a solution of (12) and $u_h$ is a solution of (17), there exist two constants $\sigma$ and $C = C\left(u\left(0\right), u, \alpha, \mu, \beta^{\star}\right)$, independents of $h$, such that $\forall t \in (0, T)$*

$$\left\| \mathcal{D}_h^{\frac{1}{2}} \left(u - u_h\right)(t) \right\|_{0,\Omega} \leq C h^k e^{\sigma t}.$$

*Proof.* Subtract (12) and (17), subtract and add $\mathcal{D}_h u_t$, $\mathcal{D}_h f\left(u\left(t\right)\right)$ and $H\left(u_h\left(t\right)\right)$ then rearrange terms to get

$$\begin{aligned}\mathcal{D}_h \left(u - u_h\right)_t(t) + H\left(u\left(t\right)\right) - H\left(u_h\left(t\right)\right) = \left(\mathcal{D} - \mathcal{D}_h\right)\left(f\left(u\left(t\right)\right) - u_t\left(t\right)\right) + \\ + \mathcal{D}_h\left(f\left(u\left(t\right)\right) - f\left(u_h\left(t\right)\right)\right) - \left(I - \mathcal{P}_h\right) H\left(u_h\left(t\right)\right).\end{aligned}$$

Multiply through by $u\left(t\right) - u_h\left(t\right)$ and integrate over $\Omega$ to get

$$\begin{aligned}&\left(\mathcal{D}_h \left(u - u_h\right)_t(t), u\left(t\right) - u_h\left(t\right)\right) + \\ &+ \left(H\left(u\left(t\right)\right) - H\left(u_h\left(t\right)\right), u\left(t\right) - u_h\left(t\right)\right) = \\ &= \left(\left(\mathcal{D} - \mathcal{D}_h\right)\left(f\left(u\left(t\right)\right) - u_t\left(t\right)\right), u\left(t\right) - u_h\left(t\right)\right) + \\ &+ \left(\mathcal{D}_h\left(f\left(u\left(t\right)\right) - f\left(u_h\left(t\right)\right)\right), u\left(t\right) - u_h\left(t\right)\right) - \\ &- \left(\left(I - \mathcal{P}_h\right) H\left(u_h\left(t\right)\right), u\left(t\right) - u_h\left(t\right)\right).\end{aligned} \tag{21}$$

Properties of $H$ give

$$\begin{aligned}&\left(H\left(u\left(t\right)\right) - H\left(u_h\left(t\right)\right), u\left(t\right) - u_h\left(t\right)\right) \geq \\ &\geq \tfrac{\alpha}{2} \left\|u\left(t\right) - u_h\left(t\right)\right\|_{0,\Omega}^2 + \tfrac{1}{2\mu} \left\|H\left(u\left(t\right)\right) - H\left(u_h\left(t\right)\right)\right\|_{0,\Omega}^2.\end{aligned} \tag{22}$$

From Lemma 1 for $\varphi = f\left(u\left(t\right)\right) - u_t\left(t\right)$, supposing sufficient regularity conditions, yields

$$\begin{aligned}&\left(\left(\mathcal{D} - \mathcal{D}_h\right)\left(f\left(u\left(t\right)\right) - u_t\left(t\right)\right), u\left(t\right) - u_h\left(t\right)\right) \leq \\ &\leq \left\|\left(\mathcal{D} - \mathcal{D}_h\right)\left(f\left(u\left(t\right)\right) - u_t\left(t\right)\right)\right\|_{0,\Omega} \left\|u\left(t\right) - u_h\left(t\right)\right\|_{0,\Omega} \leq \\ &\leq C h^k \left\|f\left(u\left(t\right)\right) - u_t\left(t\right)\right\|_{k-1,\Omega} \left\|u\left(t\right) - u_h\left(t\right)\right\|_{0,\Omega} \leq \\ &\leq C_1\left(u\right) h^k \left\|u\left(t\right) - u_h\left(t\right)\right\|_{0,\Omega} \leq \tfrac{C_1^2\left(u\left(t\right)\right)}{\alpha} h^{2k} + \tfrac{\alpha}{4} \left\|u\left(t\right) - u_h\left(t\right)\right\|_{0,\Omega}^2.\end{aligned} \tag{23}$$

Similarly, from Lemma 2 for $\varphi = f\left(u\left(t\right)\right) - f\left(u_h\left(t\right)\right)$ we have

$$\begin{aligned}&\left(\mathcal{D}_h\left(f\left(u\left(t\right)\right) - f\left(u_h\left(t\right)\right)\right), u\left(t\right) - u_h\left(t\right)\right) = \\ &= \left(\mathcal{D}_h^{\frac{1}{2}}\left(f\left(u\left(t\right)\right) - f\left(u_h\left(t\right)\right)\right), \mathcal{D}_h^{\frac{1}{2}}\left(u - u_h\right)(t)\right) \leq \\ &\leq \left\|\mathcal{D}_h^{\frac{1}{2}}\left(f\left(u\left(t\right)\right) - f\left(u_h\left(t\right)\right)\right)\right\|_{0,\Omega} \left\|\mathcal{D}_h^{\frac{1}{2}}\left(u - u_h\right)(t)\right\|_{0,\Omega} \leq \\ &\leq \tfrac{1}{\beta^{\star}} \left\|f\left(u\left(t\right)\right) - f\left(u_h\left(t\right)\right)\right\|_{0,\Omega} \left\|\mathcal{D}_h^{\frac{1}{2}}\left(u - u_h\right)(t)\right\|_{0,\Omega} \leq \\ &\leq \tfrac{L_f}{\beta^{\star}} \left\|u\left(t\right) - u_h\left(t\right)\right\|_{0,\Omega} \left\|\mathcal{D}_h^{\frac{1}{2}}\left(u - u_h\right)(t)\right\|_{0,\Omega} \leq \\ &\leq \tfrac{\alpha}{4} \left\|u\left(t\right) - u_h\left(t\right)\right\|_{0,\Omega}^2 + \tfrac{L_f^2}{\alpha\left(\beta^{\star}\right)^2} \left\|\mathcal{D}_h^{\frac{1}{2}}\left(u - u_h\right)(t)\right\|_{0,\Omega}^2.\end{aligned} \tag{24}$$

Using the properties of the orthogonal projection $\mathcal{P}_h$ we have

$$((I-\mathcal{P}_h)\,H\,(u_h\,(t))\,,u\,(t)-u_h\,(t)) = ((I-\mathcal{P}_h)\,H\,(u_h\,(t))\,,u\,(t)-w_h)\ \ \forall w_h \in M_h.$$

As $\mathcal{P}_h H\,(u_h\,(t))$ is the element of $M_h$ that minimizes distance to $H\,(u_h\,(t))$, it follows that $\forall w_h \in M_h$

$$\begin{aligned}
&((I-\mathcal{P}_h)\,H\,(u_h\,(t))\,,u\,(t)-u_h\,(t)) \leq \\
&\leq \|(I-\mathcal{P}_h)\,H\,(u_h\,(t))\|_{0,\Omega}\,\|u\,(t)-w_h\|_{0,\Omega} \leq \\
&\leq \|H\,(u_h\,(t))-\psi_h\|_{0,\Omega}\,\|u\,(t)-w_h\|_{0,\Omega} \qquad\qquad \forall \psi_h \in M_h.
\end{aligned}$$

Particularly, choosing $\psi_h = \mathcal{P}_h H\,(u_h\,(t))$, adding and subtracting $H\,(u\,(t))$, and rearranging we get $\forall w_h \in M_h$

$$\begin{aligned}
&((I-\mathcal{P}_h)\,H\,(u_h\,(t))\,,u\,(t)-u_h\,(t)) \leq \\
&\leq \left(\|H\,(u_h\,(t))-H\,(u\,(t))\|_{0,\Omega} + \|H\,(u\,(t))-\mathcal{P}_h H\,(u_h\,(t))\|_{0,\Omega}\right)\|u\,(t)-w_h\|_{0,\Omega}\,,
\end{aligned}$$

and finally taking $w_h = \mathcal{P}_h u$,

$$\begin{aligned}
&((I-\mathcal{P}_h)\,H\,(u_h\,(t))\,,u\,(t)-u_h\,(t)) \leq \\
&\leq \left(\|H\,(u_h\,(t))-H\,(u\,(t))\|_{0,\Omega} + Ch^k\,\|H\,(u\,(t))\|_{k,\Omega}\right) Ch^k\,\|u\,(t)\|_{k,\Omega} \leq \\
&\leq C_2^2\,(u)\,h^{2k}\tfrac{\mu}{2} + \tfrac{1}{2\mu}\,\|H\,(u_h\,(t))-H\,(u\,(t))\|_{0,\Omega}^2 + C_3\,(u)\,h^{2k}
\end{aligned} \tag{25}$$

Combination of (22)-(25) yields $\forall t \in (0,T)$ with $C_4\,(u,\alpha,\mu) = \frac{C_1(u)}{\alpha} + \frac{C_2^2(u)\mu}{2} + C_3\,(u)$

$$\left(\mathcal{D}_h\,(u-u_h)_t\,(t)\,,u\,(t)-u_h\,(t)\right) \leq C_4 h^{2k} + \frac{L_f^2}{\alpha\,(\beta^\star)^2}\left\|\mathcal{D}_h^{\frac{1}{2}}\,(u-u_h)\,(t)\right\|_{0,\Omega}^2 \tag{26}$$

Now, integrating (26) over $(0,T)$, and taking into account that

$$\begin{aligned}
&\textstyle\int_0^t \left(\mathcal{D}_h\,(u-u_h)_s\,(s)\,,u\,(s)-u_h\,(s)\right) ds = \\
&= \textstyle\int_0^t \left(\mathcal{D}_h^{\frac{1}{2}}\,(u-u_h)_s\,(s)\,,\mathcal{D}_h^{\frac{1}{2}}\,(u-u_h)\,(s)\right) ds = \\
&= \tfrac{1}{2}\textstyle\int_0^t \frac{d}{ds}\left(\mathcal{D}_h^{\frac{1}{2}}\,(u-u_h)\,(s)\,,\mathcal{D}_h^{\frac{1}{2}}\,(u-u_h)\,(s)\right) ds = \\
&= \tfrac{1}{2}\left\|\mathcal{D}_h^{\frac{1}{2}}\,(u-u_h)\,(t)\right\|_{0,\Omega}^2 - \tfrac{1}{2}\left\|\mathcal{D}_h^{\frac{1}{2}}\,(u-u_h)\,(0)\right\|_{0,\Omega}^2,
\end{aligned} \tag{27}$$

we get

$$\begin{aligned}
\left\|\mathcal{D}_h^{\frac{1}{2}}\,(u-u_h)\,(t)\right\|_{0,\Omega}^2 \ \leq\ & \left\|\mathcal{D}_h^{\frac{1}{2}}\,(u-u_h)\,(0)\right\|_{0,\Omega}^2 + 2h^{2k}\int_0^T C_4\,(u,\alpha,\mu)\,ds + \\
& + \frac{2L_f^2}{\alpha\,(\beta^\star)^2}\int_0^t \left\|\mathcal{D}_h^{\frac{1}{2}}\,(u-u_h)\,(s)\right\|_{0,\Omega}^2 ds.
\end{aligned}$$

Gronwall's Lemma provides

$$\left\|\mathcal{D}_h^{\frac{1}{2}}\,(u-u_h)\,(t)\right\|_{0,\Omega}^2 \leq \left(\left\|\mathcal{D}_h^{\frac{1}{2}}\,(u-u_h)\,(0)\right\|_{0,\Omega}^2 + 2h^{2k}\int_0^T C_4\,(u,\alpha,\mu)\,ds\right) e^{\frac{2L_f^2}{\alpha(\beta^\star)^2}t}.$$

Again from Lemma 2 with $u_h(0) = L_h^{(k-1)} u(0)$ we have,

$$\left\| \mathcal{D}_h^{\frac{1}{2}} (u - u_h)(0) \right\|_{0,\Omega} \leq \frac{1}{\beta^\star} \| u(0) - u_h(0) \|_{0,\Omega} \leq \frac{C_5(u(0))}{\beta^\star} h^k,$$

and finally,

$$\left\| \mathcal{D}_h^{\frac{1}{2}} (u - u_h)(t) \right\|_{0,\Omega}^2 \leq \left( \frac{C_5^2(u(0))}{(\beta^\star)^2} + 2\int_0^T C_4(u, \alpha, \mu)\, ds \right) h^{2k} \cdot e^{\frac{2L_f^2}{\alpha(\beta^\star)^2} t} \tag{28}$$

which provides the desired error estimates taking square roots with,

$$C(u, u(0), \alpha, \mu, \beta^\star) = \left( \frac{C_5^2(u(0))}{(\beta^\star)^2} + 2\int_0^T C_4(u, \alpha, \mu)\, ds \right)^{\frac{1}{2}},$$

and

$$\sigma = \frac{L_f^2}{\alpha(\beta^\star)^2}.$$

Small changes in the previous demonstration provides an error estimate in the norm $L^2(0, T; L^2(\Omega))$.

**theorem 5** *If $u$ is a solution of (12) and $u_h$ is a solution of (17), there exist a constant $C = C(u(0), u, \alpha, \beta^\star, L_f)$, independent of $h$, such that $\forall t \in (0, T)$*

$$\left( \int_0^t \| u(t) - u_h(t) \|_{0,\Omega}^2\, dt \right)^{\frac{1}{2}} \leq Ch^k.$$

*If $\mu < \infty$ then, there exist a constant $C = C(u(0), u, \alpha, \beta^\star, \mu, L_f)$, independent of $h$, such that $\forall t \in (0, T)$*

$$\left( \int_0^t \| H(u(t)) - H(u_h(t)) \|_{0,\Omega}^2\, dt \right)^{\frac{1}{2}} \leq Ch^k.$$

*Proof.* In order to keep the term $\| u(t) - u_h(t) \|_{0,\Omega}$, we can modify the expressions (23), (24) and (25) in the following way,

$$\begin{aligned} &((\mathcal{D} - \mathcal{D}_h)(f(u(t)) - u_t(t)), u(t) - u_h(t)) \leq \\ &\leq \tfrac{L_f^2+1}{\alpha} C_1^2(u) h^{2k} + \tfrac{\alpha}{4\left(L_f^2+1\right)} \| u(t) - u_h(t) \|_{0,\Omega}^2, \end{aligned}$$

with the aid of Lipschitz condition (3),

$$\begin{aligned} &(\mathcal{D}_h(f(u(t)) - f(u_h(t))), u(t) - u_h(t)) \leq \\ &\leq \tfrac{\alpha L_f^2}{4\left(L_f^2+1\right)} \| u(t) - u_h(t) \|_{0,\Omega}^2 + \tfrac{L_f^2+1}{2\alpha(\beta^\star)^2} \left\| \mathcal{D}_h^{\frac{1}{2}} (u - u_h)(t) \right\|_{0,\Omega}^2, \end{aligned}$$

and finally,

$$\begin{aligned}&\left(\left(I-\mathcal{P}_h\right)H\left(u_h\left(t\right)\right),u\left(t\right)-u_h\left(t\right)\right)\le\\&\le C_2^2\left(u\right)h^{2k}\mu+\tfrac{1}{4\mu}\left\|H\left(u_h\left(t\right)\right)-H\left(u\left(t\right)\right)\right\|_{0,\Omega}^2+C_3\left(u\right)h^{2k}.\end{aligned}$$

Then, $\forall t\in(0,T)$ yields for $C_5\left(u,\alpha,\mu,L_f\right)=\frac{L_f^2+1}{\alpha}C_1^2\left(u\right)+C_2^2\left(u\right)\mu$,

$$\begin{aligned}\left(\mathcal{D}_h\left(u-u_h\right)_t\left(t\right),u\left(t\right)-u_h\left(t\right)\right)+\tfrac{\alpha}{4}\left\|u\left(t\right)-u_h\left(t\right)\right\|_{0,\Omega}^2+&\\+\tfrac{1}{4\mu}\left\|H\left(u_h\left(t\right)\right)-H\left(u\left(t\right)\right)\right\|_{0,\Omega}^2\le&\\\le C_5\left(u,\alpha,\mu,L_f\right)h^{2k}+\tfrac{L_f^2}{\alpha(\beta^\star)^2}\left\|\mathcal{D}_h^{\frac{1}{2}}\left(u-u_h\right)\left(t\right)\right\|_{0,\Omega}^2&.\end{aligned}$$

Integrating over $(0,T)$, similarly to the previous proof, we get

$$\begin{aligned}\left\|\mathcal{D}_h^{\frac{1}{2}}\left(u-u_h\right)\left(t\right)\right\|_{0,\Omega}^2+\tfrac{\alpha}{2}\textstyle\int_0^t\left\|u\left(s\right)-u_h\left(s\right)\right\|_{0,\Omega}^2ds+&\\+\tfrac{1}{\mu}\textstyle\int_0^t\left\|H\left(u\left(s\right)\right)-H\left(u_h\left(s\right)\right)\right\|_{0,\Omega}^2ds\le&\\\le\left\|\mathcal{D}_h^{\frac{1}{2}}\left(u-u_h\right)\left(0\right)\right\|_{0,\Omega}^2+2h^{2k}\textstyle\int_0^T C_5^2\left(u,\alpha,\mu,L_f\right)ds+&\\+2\tfrac{L_f^2+1}{\alpha(\beta^\star)^2}\textstyle\int_0^t\left\|\mathcal{D}_h^{\frac{1}{2}}\left(u-u_h\right)\left(t\right)\right\|_{0,\Omega}^2dt.&\end{aligned}$$

Now, by Theorem 4,

$$\begin{aligned}&\left\|\mathcal{D}_h^{\frac{1}{2}}\left(u-u_h\right)\left(0\right)\right\|_{0,\Omega}^2\le C^2\left(u\left(0\right),u,\alpha,\beta^\star,\mu\right)h^{2k},\\&\textstyle\int_0^t\left\|\mathcal{D}_h^{\frac{1}{2}}\left(u-u_h\right)\left(t\right)\right\|_{0,\Omega}^2dt\le C^2\left(u\left(0\right),\alpha,u,\beta^\star,\mu\right)h^{2k}\frac{e^{2\sigma t}-1}{2\sigma},\end{aligned}$$

so, denoting $C=C^2\left(u\left(0\right),u,\beta^\star,\alpha\right)\left(1+\frac{L_f^2+1}{L_f^2}\left(e^{2\sigma T}-1\right)\right)+2\int_0^T C_5^2\left(u,\alpha,\mu,L_f\right)dt$, we have

$$\begin{aligned}\left\|\mathcal{D}_h^{\frac{1}{2}}\left(u-u_h\right)\left(t\right)\right\|_{0,\Omega}^2+\tfrac{\alpha}{2}\textstyle\int_0^t\left\|u\left(t\right)-u_h\left(t\right)\right\|_{0,\Omega}^2dt+&\\+\tfrac{1}{\mu}\textstyle\int_0^t\left\|H\left(u\left(t\right)\right)-H\left(u_h\left(t\right)\right)\right\|_{0,\Omega}^2dt\le Ch^{2k}&.\end{aligned}$$

Particularly,

$$\left(\int_0^t\left\|u\left(t\right)-u_h\left(t\right)\right\|_{0,\Omega}^2dt\right)^{\frac{1}{2}}\le\left(\frac{2}{\alpha}C\right)^{\frac{1}{2}}h^k,$$

$$\left(\int_0^t\left\|H\left(u\left(t\right)\right)-H\left(u_h\left(t\right)\right)\right\|_{0,\Omega}^2dt\right)^{\frac{1}{2}}\le\left(\mu C\right)^{\frac{1}{2}}h^k.$$

## 4. EXAMPLE

We consider the following simplified model for wildland fires (Asensio & Ferragut, 2002)where $u = u(t, \boldsymbol{x})$ is the nondimensional average temperature and $\beta = \beta(t, \boldsymbol{x})$ is the nondimensional mass fraction of unburned solid fuel on each $(t, \boldsymbol{x}) \in (0, T) \times \Omega$,

$$\begin{aligned} u_t - \Delta\left(\frac{\kappa}{4\varepsilon}(1+\varepsilon u)^4 - \frac{\kappa}{4\varepsilon}\right) &= s(u)^+ \beta e^{\frac{u}{1+\varepsilon u}} - \gamma u \\ \beta_t &= -s(u)^+ \frac{\varepsilon}{q} \beta e^{\frac{u}{1+\varepsilon u}} \end{aligned} \tag{29}$$

with $\varepsilon$ the nondimensional inverse of activation energy, $q$ the nondimensional reaction heat (Bebernes & Eberly, 1989), $\gamma$ the nondimensional natural convection coefficient and $\kappa$ the nondimensional inverse of thermal conductivity. The function,

$$s(u)^+ = \begin{cases} 1 & if\ u \geq u_{ref} \\ 0 & other \end{cases}$$

where $u_{ref}$ is the nondimensional change-phase temperature, represents the change from endothermic to exothermic phase which characterize wildland fires. The non-linearity of diffusive term is due to radiation, that is the principal heat transfer mechanism in this kind of combustion processes (Cox, 1995; Weber, 1989). The natural convection term represents the loss of heat due to the effect of gravity over the different densities caused from the distinct temperatures. This simplification let us to assume a two dimensional model. In this model, we do not consider the convective effect of wind and slope. We denote,

$$\begin{aligned} &H(u) = \tfrac{\kappa}{4\varepsilon}(1+\varepsilon u)^4 - \tfrac{\kappa}{4\varepsilon}, \\ &f(u, \beta) = s(u)^+ \beta e^{\frac{u}{1+\varepsilon u}} - \gamma u, \\ &g(u, \beta) = -s(u)^+ \tfrac{\varepsilon}{q} \beta e^{\frac{u}{1+\varepsilon u}}. \end{aligned}$$

Boundary and initial conditions are,

$$\begin{aligned} &H(u) = 0 && on\ (0, T) \times \partial\Omega, \\ &u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) && in\ \Omega, \\ &\beta(\boldsymbol{x}, 0) = \beta_0(\boldsymbol{x}) && in\ \Omega. \end{aligned}$$

Let $\Omega = [0, L_x] \times [0, L_y]$ the nondimensional domain and $\boldsymbol{\tau}_h$ a regular mesh of rectangles $\tau$ having sides parallel to the axis. We choose the finite subspaces of functions described before for $k = 1$, that is, $L_h^{(0)}$, the subspace of scalar functions constants over each rectangle, and $E_h^{(1)}$, the subspace of vectorial functions $\boldsymbol{q} = (q_1, q_2)$ such that over each rectangle are incomplete polynomials, $q_1$ is a polynomial of degree one in the first variable and zero in the second, and $q_2$ is a polynomial of degree zero in the first variable and one in the second. The basis functions for $L_h^{(0)}$are,

$$\varphi_k(\boldsymbol{x}) = \begin{cases} 1 & if\ \boldsymbol{x} \in \tau_k, \\ 0 & other, \end{cases}$$

for $k = 1, \ldots, L$, where $L = \dim L_h^{(0)}$ is the number of elements of the mesh. So the semidiscrete solution $u_h(t) = \sum_{k=1}^{L} u_k(t) \varphi_k(\boldsymbol{x})$ and $\beta_h(t) = \sum_{k=1}^{L} \beta_k(t) \varphi_k(\boldsymbol{x})$. We denote by $\boldsymbol{U}(t) = (u_1(t), \ldots, u_L(t))$ and $\boldsymbol{\beta}(t) = (u_1(t), \ldots, u_L(t))$ the coordinate vectors.

The basis functions for $E_h^{(1)}$ are $\boldsymbol{\phi}_m \in E_h^{(1)}$ for $m = 1, \ldots, E$ defined by,

$$\int_{l_n} \boldsymbol{\phi}_m \cdot \boldsymbol{\gamma}_n ds = \delta_{m,n}$$

where $l_n$ is the $n^{th}$ edge of the mesh and $\boldsymbol{\gamma}_n$ the unit normal to $\boldsymbol{\phi}_n$ with a prearranged orientation, and $E = \dim E_h^{(1)}$ is equal to the number of edges of the mesh. So $\boldsymbol{p}_h(t) = \sum_{m=1}^{E} p_m(t) \boldsymbol{\phi}_m(\boldsymbol{x})$ is the semidiscrete solution, and $\boldsymbol{P}(t) = (p_1(t), \ldots, p_E(t))$ is the coordinate vector.

The semidiscrete approximation of (29) can be reduced, using these bases functions, to the following system of ODE's,

$$\begin{aligned} &\boldsymbol{IU}'(t) - \boldsymbol{BP}(t) = \boldsymbol{F}(t), \\ &\boldsymbol{AP}(t) + \boldsymbol{B}^t \boldsymbol{H}(t) = 0, \\ &\boldsymbol{I\beta}'(t) = \boldsymbol{G}(t), \\ &\boldsymbol{U}(0) = \boldsymbol{U}_0, \\ &\boldsymbol{\beta}(0) = \boldsymbol{\beta}_0, \end{aligned} \tag{30}$$

where,

$$\begin{array}{ll} \boldsymbol{B}_{m,j} = \int_\Omega div \boldsymbol{\phi}_m \varphi_j d\boldsymbol{x}, & \boldsymbol{A}_{m,n} = \int_\Omega \boldsymbol{\phi}_m \cdot \boldsymbol{\phi}_n d\boldsymbol{x}, \\ \boldsymbol{I}_{i,j} = \int_\Omega \varphi_i \varphi_j d\boldsymbol{x}, & (\boldsymbol{H}(t))_j = \int_\Omega H(u_h) \varphi_j d\boldsymbol{x}, \\ (\boldsymbol{F}(t))_j = \int_\Omega f(u_h, \beta_h) \varphi_j d\boldsymbol{x}, & (\boldsymbol{G}(t))_j = \int_\Omega g(u_h, \beta_h) \varphi_j d\boldsymbol{x}, \\ (\boldsymbol{U}_0)_i = \int_\Omega u_0 \varphi_i d\boldsymbol{x}, & (\boldsymbol{\beta}_0)_i = \int_\Omega \beta_0 \varphi_i d\boldsymbol{x}. \end{array}$$

The symmetric positive-definite matrix $\boldsymbol{A}$ with a suitable numeration of elements and edges is tridiagonal. Solving the corresponding tridiagonal linear system, the ODE's system to solve is,

$$\begin{aligned} &\boldsymbol{IU}'(t) + \boldsymbol{BA}^{-1} \boldsymbol{B}^t \boldsymbol{H}(t) = \boldsymbol{F}(t), \\ &\boldsymbol{I\beta}'(t) = \boldsymbol{G}(t), \\ &\boldsymbol{U}(0) = \boldsymbol{U}_0, \\ &\boldsymbol{\beta}(0) = \boldsymbol{\beta}_0, \end{aligned}$$

which has a unique solution $\forall t > 0$ because the globally Lipschitz conditions that $\boldsymbol{F}$ and $\boldsymbol{G}$ verifies.

Computational cost are low enough to allow high order temporal schemes, although low order temporal schemes give good results. A forward Euler method, keeping implicit the linear terms, for the semidiscrete problem (30), involves for each time step $n \geq 1$,

- to solve the linear system,

$$AP^{n-1} = -B^t H^{n-1}$$

- to compute for each $k = 1, \ldots, L$,

$$u_k^n = \frac{u_k^{n-1} + \frac{\Delta t}{area(\tau_k)}(BP^{n-1})_k + \Delta t \beta_k^{n-1} e^{\frac{u_k^{n-1}}{1+\epsilon u_k^{n-1}}}}{1 + \Delta t \alpha},$$

- to compute for each $k = 1, \ldots, L$,

$$\beta_k^n = \frac{\beta_k^{n-1}}{1 + \Delta t \frac{\epsilon}{q} e^{\frac{u_k^{n-1}}{1+\epsilon u_k^{n-1}}}}.$$

Fig. 1 represents contour plots of the approximate nondimensional average temperature and mass fraction of solid fuel for the following values of parameters, $\varepsilon = 0.03$, $q = 2.5$, $\alpha = 0.01$, $\kappa = 2.5$ and $u_{ref} = 15$. The domain is a square of side 100, the mesh has 1600 regular rectangles of sides parallel to the axes ($h = 2.5$) and time step is $5 \times 10^{-5}$. Initial fuel is random between 0 and 1, and there is an initial fire focus in the centre of the domain. We render instants $t = 0.1$, $t = 0.5$ and $t = 0.9$.

## 4. CONCLUDING REMARKS

The nonlinear reaction-diffusion equations studied in this paper has applications in several areas. We use a mixed finite element method as opposed to standard Galerkin finite elements, due to their inherent conservation properties and the fact that the methods provide flux approximations as part of the formulation. The demonstration technique of the expected semidiscrete error estimates is based on the idea of obtaining first an error estimate in the norm $\left\| \mathcal{D}_h^{\frac{1}{2}} (\cdot) \right\|_{0,\Omega}$, and shown that it is a discrete version of the dual norm $\|\cdot\|_{-1,\Omega}$. This technique also could be applied to other spatial discretization like classical finite elements or finite differences for which lemma 1 verifies. The computational example presented as an application, although is too simple to allow comparisons with real physics, it provides a platform for a more sophisticated wildland fire model. The mathematical analysis asses that the mathematical model is well-posed.

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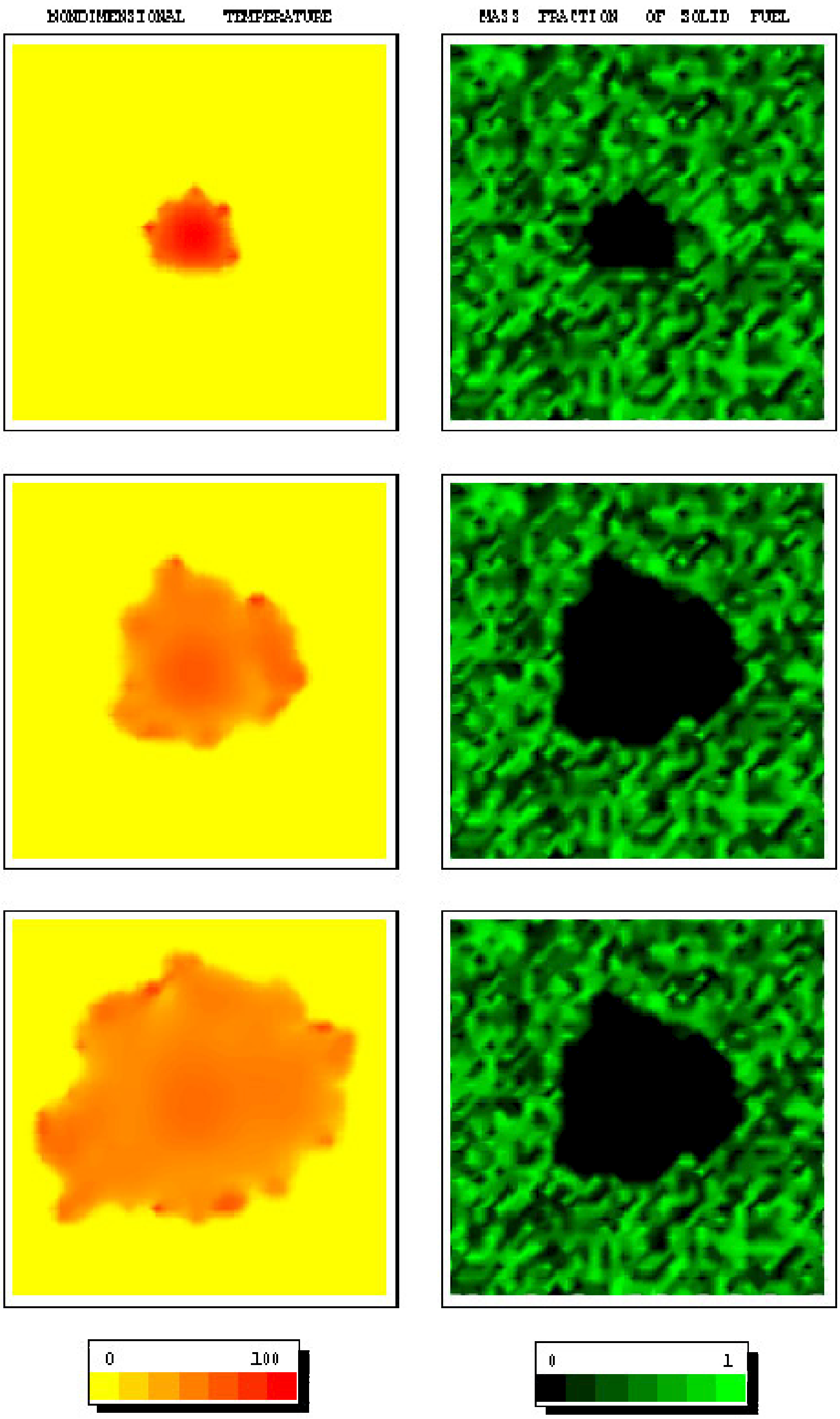


Figure 1: Nondimensional temperature and mass fraction of solid fuel.