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Space-Time adaptive algorithm for the mixed parabolic problem

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Abstract In this paper we present an a-posteriori error estimator for the mixed formulation of a linear parabolic problem, used for designing an efficient adaptive algorithm. Our space-time discretization consists of lowest order Raviart-Thomas finite element over graded meshes and discontinuous Galerkin method with variable time step. Finally, several examples show that the proposed method is efficient and reliable.

1 Introduction

A-posteriori error estimates are an essential component in the design of reliable and efficient adaptive algorithms for the numerical solutions of PDEs. Mixed formulations can be suitable for certain problems, as they allow to directly approach certain solution derivatives.

In this paper we introduce an a-posteriori error estimation for the mixed formulation of a linear parabolic problem. We particularly obtain,

$$\left.\begin{array}{l}\|u-U\|_{L^\infty(0,T;L^2(\Omega))}\\ \|\mathbf{p}-\mathbf{P}\|_{L^2(0,T;H^{-1}(\mathrm{div},\Omega))}\end{array}\right\}\leq \mathcal{E}(u_0,\, f,\, T,\, \Omega;\, U,\, \mathbf{P},\, h,\, k),$$

where u is the scalar variable and $\mathbf{p}$ its gradient. In the following, we will use boldface type for vector-valued functions and capital letters for representing the numerical approximations obtained with the lowest order Raviart-Thomas finite

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element discretization and variable time step discontinuous Galerkin method. The estimator $\mathcal{E}$ is computed in terms of problem data u_0, f, Ω, T, computed solutions U and $\mathbf{P}$, mesh size h and time step k. Therefore, the a-posteriori estimators are computable quantities depending on the discrete solution and the data, which provide bounds of the error, and thus they are a tool for modifying meshes and time steps (adaptivity).

First we establish the a-posteriori error estimation using duality, a popular technique from linear PDEs, firstly used in the context of a-posteriori error estimation by K. Eriksson et al. [12]. Next, we summarize the keys of duality for the classical formulation of the linear parabolic problem. We consider the equation,

$$u_t - \Delta u = f.$$

Being U a numerical approximation of the solution, we define the residual $\mathcal{R}$,

$$\mathcal{R} := -U_t + \Delta U + f.$$

We denote $e := u - U$ the error function, then we can write,

$$e_t - \Delta e = \mathcal{R},$$

and, formally multiplying by φ and integrating by parts over $(0, T)$ we get,

$$(e(T), \varphi(T)) = (e(0), \varphi(0)) + \int_0^T (e, \varphi_T + \Delta\varphi) + \int_0^T (\mathcal{R}, \varphi). \tag{1}$$

Therefore, an error estimate follows by selecting in (1) φ as the solution of the backward dual problem,

$$\varphi_t + \Delta\varphi = 0 \quad \text{in } (0, T), \quad \varphi(T) = \varrho,$$

and using the stability properties of φ in terms of ϱ, for evaluating $\varphi(0)$ and $\mathcal{R}$.

The use of duality is a really useful technique for establishing error controls for linear PDEs (see [12,13,15]). However it has serious constraints when there is a strong non-linear term in the equation, because in general it is hard to obtain strong stability properties for the corresponding dual problem. Anyway this technique can be used in special circumstances [9,21] or if the non-linearity is moderate [14].

In this paper we deal with the linear problem, but the new feature here is the development of estimations for the mixed formulation. That is, we control the error for the scalar variable and also for its gradient. As in [21], we obtain from the residual equations the error representation formulas for $u - U$ and $\mathbf{p} - \mathbf{P}$. The evaluation of the residual in the corresponding norms and the stability properties of the associated dual problem allow us to conclude the estimations. The scalar error bound is an extension to the mixed formulation of the results developed in [12]. For the error estimation of $\mathbf{p} - \mathbf{P}$, we use the Helmholtz decomposition in $L^2(\Omega; \mathbb{R})$, as in [1] and [8] for the stationary case.

The estimator obtained is used to design a time-space adaptive algorithm. Several examples show that the proposed method is efficient and reliable. The numerical experiments have been designed with the finite element toolbox ALBERTA [23], extended with new designed tools for the lowest order Raviart-Thomas finite element and the a-posteriori error estimator developed in this paper.

This paper is organized as follows. In §2, we describe the mixed formulation for the linear parabolic problem. We introduce the dual problems we will use for the error estimation in §3. In §4, we obtain the error representation formulas. In §5, we introduce the fully discrete problem, which combines finite elements in space, Raviart-Thomas element of the lowest order, with the discontinuous Galerkin method with variable time step. In §6 we present the a-posteriori errors estimator summarized in Theorem 1 and Theorem 2, as well as the details of the proofs. The Adaptive Algorithm is described in §7. Finally, we conclude with some numerical experiments in §8.

2 Continuous problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polyhedral domain, let $T > 0$ be the end time, and set $Q_T := \Omega \times (0,T)$ and $\Gamma_T := \partial\Omega \times (0,T)$. Given sufficiently regular initial condition u_0 and source term $f(x,t)$, let $(u, \mathbf{p})$ be the solution of problem

$$u_t + \operatorname{div} \mathbf{p} = f \quad \text{in} \quad Q_T, \tag{2}$$

$$\mathbf{p} + \nabla u = 0 \quad \text{in} \quad Q_T, \tag{3}$$

$$\nabla u \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_T, \tag{4}$$

$$u(.,0) = u_0 \quad \text{in} \quad \Omega, \tag{5}$$

where $\boldsymbol{n}$ is the unit exterior normal vector to $\partial\Omega$.

In the following, $(\cdot,\cdot)$ denotes the corresponding inner product in $L^2(\Omega)$ or $L^2(\Omega;\mathbb{R}^2)$ and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between the corresponding spaces. We consider $H^1(\Omega)$, the standard Sobolev space of $L^2(\Omega)$ functions with weak derivatives in $L^2(\Omega)$. We also introduce the appropriate spaces for our problem,

$$H(\operatorname{div}, \Omega) := \left\{\mathbf{q} \in L^2\left(\Omega;\mathbb{R}^2\right) : \operatorname{div}\mathbf{q} \in L^2(\Omega)\right\},$$
$$H_0(\operatorname{div}, \Omega) := \{\mathbf{q} \in H(\operatorname{div}, \Omega) : \mathbf{q}\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega\}.$$

We denote by $H^{-1}(\operatorname{div}, \Omega)$ the dual space of $H_0(\operatorname{div}, \Omega)$. Now, we introduce the following operators,

$$\mathbf{curl}\, v = \begin{bmatrix} -\partial_{x_2} v \\ \partial_{x_1} v \end{bmatrix} \quad \text{and} \quad \operatorname{rot}\mathbf{q} = \partial_{x_2}\mathbf{q}_1 - \partial_{x_1}\mathbf{q}_2.$$

For writing the corresponding variational mixed-formulation of the linear parabolic problem, we use the bilinear operator $B(\cdot,\cdot;\cdot,\cdot)$ and the linear application $L(\cdot,\cdot)$ defined as,

$$B(u,\mathbf{p};v,\mathbf{q}) := (u(\cdot,0), v(\cdot,0)) + \int_0^T \left[(u_t, v) + (\operatorname{div}\mathbf{p}, v) + (\operatorname{div}\mathbf{q}, u) - (\mathbf{p},\mathbf{q})\right] dt,$$

$$L(v,\mathbf{q}) := (u_0, v(\cdot,0)) + \int_0^T (f,v)\, dt$$

Multiplying equations (2) and (5) by v, equation (3) by $\mathbf{q}$, adding them, using Green's formula and integrating over $(0, T)$, we obtain the following variational problem,

Continuous problem. *Find u and $\mathbf{p}$ such that*

$$u \in \mathcal{M} := H^1(0, T; L^2(\Omega)), \quad \mathbf{p} \in \mathcal{X} = L^2(0, T; H_0(\text{div}\,, \Omega)), \tag{6}$$

$$B(u, \mathbf{p}; v, \mathbf{q}) \;=\; L(v, \mathbf{q}) \quad \forall (v, \mathbf{q}) \in \mathcal{M} \times \mathcal{X}. \tag{7}$$

Existence and uniqueness for this problem are known [20].

If $(U, \mathbf{P})$ is an approximation of $(u, \mathbf{p})$, that is, $(U, \mathbf{P})$ belongs to the discrete spaces $\mathbb{M} \subset \mathcal{M}$, $\mathbb{X} \subset \mathcal{X}$ and verifies the equation,

$$B(U, \mathbf{P}; V, \mathbf{Q}) = L(V, \mathbf{Q}) \quad \forall (V, \mathbf{Q}) \in \mathbb{M} \times \mathbb{X}, \tag{8}$$

then, the residual of the parabolic equation, $\mathcal{R}(\cdot, \cdot)$ is defined in the following way,

$$\begin{aligned} \mathcal{R}(v, \mathbf{q}) = L(v, \mathbf{q}) - B(U, \mathbf{P}\,; v, \mathbf{q}) &= (u_0 - U(\cdot, 0), v(\cdot, 0)) \\ &+ \int_0^T (f - U_t - \text{div}\,\mathbf{P}\,, v)\, dt \\ &+ \int_0^T (\mathbf{P}\,, \mathbf{q}) - (U, \text{div}\,\mathbf{q})\, dt. \end{aligned} \tag{9}$$

Notice that each line corresponds to the residual of each equation of the original problem (2)–(5).

Remark 1 In the previous formula, when the discrete solution U is not smooth, the time derivative is understood in the weak sense,

$$\int_0^T (U_t, v)\, dt \equiv (U(\cdot, T), v(\cdot, T)) - (U(\cdot, 0), v(\cdot, 0)) - \int_0^T (U, v_t)\, dt. \tag{10}$$

Remark 2 A non-homogeneous Neumann boundary condition in (4) would imply the use of either a non-conforming Galerkin scheme (see [6]) or a Lagrange multiplier approach (see [3]) .

3 Dual problems

In this section we motivate the duality and we use it to obtain the error representation formulas. We denote the errors by $e_u := u - U$ and $\mathbf{e_P} := \mathbf{p} - \mathbf{P}$. From equations (7) and (8) we have,

$$B(e_u, \mathbf{e_P}; v, \mathbf{q}) = \mathcal{R}(v, \mathbf{q}). \tag{11}$$

In the latter equation $(e_u, \mathbf{e_P})$ can be understood as a test function of a problem with solution $(v, \mathbf{q})$. This discussion motivates the following *dual problem*,

$$\begin{aligned}&\text{Find} \quad (v, \mathbf{q}) \in \mathcal{M} \times \mathcal{X} \text{such that,}\\ &B(w, \mathbf{s}; v, \mathbf{q}) = L^*(w, \mathbf{s}) \quad \forall (w, \mathbf{s}) \in \mathcal{M}_0 \times \mathcal{X},\end{aligned} \tag{12}$$

where the space $\mathcal{M}_0$ is defined as,

$$\mathcal{M}_0 := \{w \in \mathcal{M} : w(\cdot, 0) = 0\}.$$

The appropriate choice of the linear operator $L^*(\cdot,\cdot)$ allows us to obtain the error estimate. Throughout this paper, we will use two auxiliary problems, in order to obtain the estimations, with two different linear operators,

$$L^u(w, \mathbf{s}) = (\psi, w(\cdot, T)) \text{ representation of } \|e_u\|_{L^2(0,T;L^2(\Omega))},$$

$$L^{\mathbf{p}}(w, \mathbf{s}) = \int_0^T (\nabla\phi, \mathbf{s})\, dt \text{ representation of } \|\mathbf{e_P}\|_{L^2(0,T;H^{-1}(\mathrm{div},\Omega))}.$$

In the next section we select the appropriate functions ψ and $\nabla\phi$. For sufficient regularity assumptions, problem (12) with right terms $L^u(\cdot,\cdot)$ and $L^{\mathbf{p}}(\cdot,\cdot)$, respectively, can be written as,

– Problem $\mathcal{D}_u$

$$\begin{aligned} -v_t + \operatorname{div}\mathbf{q} &= 0 \quad \text{in} \quad Q_T,\\ \mathbf{q} + \nabla v &= 0 \quad \text{in} \quad Q_T,\\ \nabla v \cdot \boldsymbol{n} &= 0 \quad \text{on} \quad \Gamma_T,\\ v(., T) &= \psi \quad \text{in} \quad \Omega.\end{aligned}$$

– Problem $\mathcal{D}_{\mathbf{p}}$

$$\begin{aligned} -v_t + \operatorname{div}\mathbf{q} &= 0 \quad \text{in} \quad Q_T,\\ \mathbf{q} + \nabla v &= \nabla\phi \quad \text{in} \quad Q_T,\\ \nabla v \cdot \boldsymbol{n} &= 0 \quad \text{on} \quad \Gamma_T,\\ v(., T) &= 0 \quad \text{in} \quad \Omega.\end{aligned}$$

The stability properties of the previous problems are the key for establishing the error estimation. This question has been already studied in [12].

Lemma 1 **(D_u Stability.)** *Let v be the solution of D_u, then for all $t \in [0, T)$,*

$$\|v(t)\|_{L^2(\Omega)} \le \|\psi\|_{L^2(\Omega)}, \tag{13}$$

$$\int_t^T (T-s)\|\Delta v(s)\|^2_{L^2(\Omega)} ds \le \frac{1}{4}\|\psi\|^2_{L^2(\Omega)}, \tag{14}$$

$$\int_t^T (T-s)\|v_t(s)\|^2_{L^2(\Omega)} ds \le \frac{1}{4}\|\psi\|^2_{L^2(\Omega)}. \tag{15}$$

Lemma 2 **($D_\mathbf{p}$ Stability.)** *Let v be the solution of $D_\mathbf{p}$, with $\nabla\phi \cdot \boldsymbol{n} = 0$ and sufficiently smooth data, then for all $t \in [0, T)$,*

$$\|v(t)\|_{L^2(\Omega)} \leq \|\nabla\phi\|_{L^2(0,T;L^2(\Omega))}, \tag{16}$$
$$\|\nabla v\|_{L^2(0,T;L^2(\Omega))} \leq \|\nabla\phi\|_{L^2(0,T;L^2(\Omega))}, \tag{17}$$
$$\|v_t\|_{L^2(0,T;L^2(\Omega))} \leq \|\Delta\phi\|_{L^2(0,T;L^2(\Omega))}, \tag{18}$$
$$\|\Delta v\|_{L^2(0,T;L^2(\Omega))} \leq \|\Delta\phi\|_{L^2(0,T;L^2(\Omega))}. \tag{19}$$

4 Error representation formulas

The purpose of this section is to obtain formulas for the errors $(e_u, \mathbf{e_P})$.

To obtain the error formula for the scalar variable, let $(v, \mathbf{q})$ be the solution of problem $\mathcal{D}_u$, so that,

$$B(e_u, \mathbf{e_P}; v, \mathbf{q}) = (\psi, e_u(\cdot, T)),$$

then from (11) we obtain,

$$\|e_u(\cdot, T)\|_{L^2(\Omega)} = \sup_{\psi \in L^2(\Omega)} \frac{(e_u(\cdot, T), \psi)}{\|\psi\|_{L^2(\Omega)}} = \sup_{\psi \in L^2(\Omega)} \frac{|\mathcal{R}(v, \mathbf{q})|}{\|\psi\|_{L^2(\Omega)}}. \tag{20}$$

To obtain the error formula for $\mathbf{e_P}$, it should be noticed that any function $\boldsymbol{\kappa} \in L^2(\Omega; \mathbb{R}^2)$ (particularly in $H_0(\text{div}\,, \Omega)$) can be decomposed into,

$$\boldsymbol{\kappa} = \mathbf{curl}\,\mu + \nabla\phi, \tag{21}$$

where $\mu \in H_0^1(\Omega)$ and $\phi \in H^1(\Omega)/\mathbb{R}$ (see, for instance, [19]). Then, taking this into account, if $(v, \mathbf{q})$ is a solution of $D_\mathbf{p}$, we have,

$$B(e_u, \mathbf{e_P}; v, \mathbf{q}) = \int_0^T (\mathbf{e_P}, \nabla\phi)dt.$$

Using decomposition (21) we have,

$$\begin{aligned}
\|\mathbf{e_P}\|_{L^2(0,T;H^{-1}(\text{div},\Omega))} &= \sup_{\boldsymbol{\kappa} \in L^2(0,T;H_0(\text{div}\,,\Omega))} \frac{\langle \mathbf{e_P}, \boldsymbol{\kappa}\rangle}{\|\boldsymbol{\kappa}\|_{L^2(0,T;H(\text{div},\Omega))}} \\
&\leq \sup_{\boldsymbol{\kappa} \in L^2(0,T;H_0(\text{div}\,,\Omega))} \frac{\int_0^T (\mathbf{e_P}, \mathbf{curl}\,\mu)dt}{\|\boldsymbol{\kappa}\|_{L^2(0,T;H(\text{div},\Omega))}} \\
&\quad + \sup_{\boldsymbol{\kappa} \in L^2(0,T;H_0(\text{div}\,,\Omega))} \frac{\int_0^T (\mathbf{e_P}, \nabla\phi)dt}{\|\boldsymbol{\kappa}\|_{L^2(0,T;H(\text{div},\Omega))}}.
\end{aligned}$$

Finally, from (11) we obtain the following error formula,

$$\begin{aligned}
\|\mathbf{e_P}\|_{L^2(0,T;H^{-1}(\text{div},\Omega))} &\leq \sup_{\boldsymbol{\kappa} \in L^2(0,T;H_0(\text{div}\,,\Omega))} \frac{\int_0^T (\mathbf{e_P}, \mathbf{curl}\,\mu)dt}{\|\boldsymbol{\kappa}\|_{L^2(0,T;H(\text{div},\Omega))}} \\
&\quad + \sup_{\boldsymbol{\kappa} \in L^2(0,T;H_0(\text{div}\,,\Omega))} \frac{|\mathcal{R}(v, \mathbf{q})|}{\|\boldsymbol{\kappa}\|_{L^2(0,T;H(\text{div},\Omega))}}.
\end{aligned} \tag{22}$$

5 Discretization

We now introduce some useful notations and the fully discrete problem, which combines finite elements in space with the Galerkin discontinuous method in time.

We denote by k_n the time step at the n-th step and set $t_n = \sum_{i=1}^{n} k_i$. Let N be the total number of time steps, that is, $t_N = T$. We associate to each time step $(t_{n-1}, t_n]$ the following three elements $(\mathcal{T}_n, \mathbb{M}_n, \mathbb{X}_n)$, where

- $\mathcal{T}_n = \{\mathcal{S}_n\}$ is a regular partition of Ω in triangles (see [11]) . Given a triangle $\mathcal{S} \in \mathcal{T}_n$, $h_{\mathcal{S}}$ stands for its diameter and h_n is the piecewise constant function over $\mathcal{T}_n$ which is h_S in $S \in \mathcal{T}_n$. We also denote by $\mathcal{B}_n$ the collection of interior interelement boundaries e of $\mathcal{T}_n$; h_e stands for the size of e. Mesh $\mathcal{T}_n$ is obtained from $\mathcal{T}_{n-1}$ by refining/coarsening.
- $\mathbb{M}_n \subset L^2(\Omega)$ is the usual space of piecewise constant finite element over $\mathcal{T}_n$.
- $\mathbb{X}_n \subset H_0(\text{div}\,, \Omega)$ is the lowest order Raviart-Thomas finite element (see [22]) with normal trace zero on the boundary.

Let $\mathcal{P}_\Omega^n : L^2(\Omega) \to \mathbb{M}_n$ be the usual $L^2(\Omega)$-projection operator over $\mathbb{M}_n$. The discrete initial condition $U^0 \in \mathbb{M}_0$ is defined as,

$$U^0 := \mathcal{P}_\Omega^0\, u_0. \tag{23}$$

Let J_t be the tangential jump of $\mathbf{P} \in \mathbb{X}$ along e,

$$J_{\boldsymbol{t}} := [\![\mathbf{P}]\!]_e \cdot \boldsymbol{t},$$

where $\boldsymbol{t}$ is the unit tangent vector to the edge e.

In the following, every function with superscript n stands for its value at time t_n, *i.e.* $g^n := g(\,.\,, t_n)$.

Discrete problem. *Given* $U^{n-1} \in \mathbb{M}_{n-1}$, then k_{n-1} and $\mathcal{T}_{n-1}$ are modified as described in section § 7 to get k_n and $\mathcal{T}_n$ and thereafter $U^n \in \mathbb{M}_n$, $\mathbf{P}^n \in \mathbb{X}_n$ computed according to,

$$\frac{1}{k_n}\big(U^n - \mathcal{P}_\Omega^n\, U^{n-1}, V\big) + \big(\text{div}\,\mathbf{P}^n, V\big) = \frac{1}{k_n}\int_{t_{n-1}}^{t_n} (f, V)dt \quad \forall V \in \mathbb{M}_n, \tag{24}$$

$$\big(\mathbf{P}^n, \mathbf{Q}\big) - \big(\text{div}\,\mathbf{Q}, U^n\big) = 0 \quad \forall \mathbf{Q} \in \mathbb{X}_n. \tag{25}$$

Remark 3 We can modify the previous scheme in order to avoid the coupled variables. The equivalent problem solved on each time step is,

$$\big(\mathbf{P}^n, \mathbf{Q}\big) + k_n\,\big(\text{div}\,\mathbf{P}^n, \text{div}\,\mathbf{Q}\big) = \int_{t_{n-1}}^{t_n} (f, \text{div}\,\mathbf{Q})\,dt + \big(\mathcal{P}_\Omega^n\, U^{n-1}, \text{div}\,\mathbf{Q}\big) \quad \forall \mathbf{Q} \in \mathbb{X}_n,$$

$$\big(U^n, V\big) = \int_{t_{n-1}}^{t_n} (f, V)dt - k_n\,\big(\text{div}\,\mathbf{P}^n, V\big) + \big(\mathcal{P}_\Omega^n\, U^{n-1}, V\big) \quad \forall V \in \mathbb{M}_n.$$

Notice the similarity of this scheme with an Augmented Lagrangian formulation (see [17]). Then the discrete problem (24)–(25) is well posedness because the matrix associated to its equivalent decoupled formulation is symmetric and positive definite for each time step.

Remark 4 The decoupling of (24)–(25) is possible thanks to the fact that the operator div is suprayective between the discrete spaces $\mathbb{M}_n$ and $\mathbb{X}_n$,

$$\operatorname{div} \mathbb{X}_n \subseteq \mathbb{M}_n \tag{26}$$

for the present choice of the Raviart Thomas space of order zero. In fact, this deocupling is also possible for higher order elements if the pair of spaces $(\mathbb{M}_n, \mathbb{X}_n)$ verify condition (26). In particular, all Raviart Thomas spaces verify this condition.

6 A-posteriori estimator

In this section we present a-posteriori error estimates of

$$\|e_u\|_{L^\infty(0,T;L^2(\Omega))}, \quad \|\mathbf{e_P}\|_{L^2(0,T;H^{-1}(\mathrm{div},\Omega))},$$

for the schemes developed bellow, summarized in Theorem 1 and Theorem 2.

Theorem 1 *Let* $(u, \mathbf{p})$ *be the solution of (6–7) and* $(U, \mathbf{P})$ *the solution of (24–25), if* Ω *is convex, for* $N \geq 1$,

$$\left\| e_u^N \right\|_{L^2(\Omega)} \leq \|e_u^0\|_{L^2(\Omega)} + \sum_{i=1}^{5} C_i \max_{1 \leq n \leq N} \mathcal{E}_i, \tag{27}$$

with,

$$\mathcal{E}_1 := \left\| U^n - \mathcal{P}_\Omega^n\, U^{n-1} \right\|_{L^2(\Omega)}, \tag{28}$$

$$\mathcal{E}_2 := \max \Big\{ \left\| U^{n-1} - \mathcal{P}_\Omega^n\, U^{n-1}) \right\|_{L^2(\Omega)}, \tag{29}$$

$$k_n^{-1} \left\| h_n (U^{n-1} - \mathcal{P}_\Omega^n\, U^{n-1}) \right\|_{L^2(\Omega)} \Big\}, \tag{30}$$

$$\mathcal{E}_3 := \|h_n \mathbf{P}\|_{L^2(\Omega)}, \tag{31}$$

$$\mathcal{E}_4 := \max_{t \in (t_{n-1}, t_n]} \|h_n (f - div\, \mathbf{P})\|_{L^2(\Omega)}, \tag{32}$$

$$\mathcal{E}_5 := \int_{t_{n-1}}^{t_n} \|f - div\, \mathbf{P}\|_{L^2(\Omega)} dt. \tag{33}$$

Remark 5 The constants of the previous estimation depend on the minimum mesh angle, on the interpolation constants and on the following logarithm factor,

$$\max_{1 \leq n \leq N} \left[\log \left(\frac{t_n}{k_n} \right) \right]^{1/2}.$$

Theorem 2 *Let* $(u, \mathbf{p})$ *be the solution of (6–7) and* $(U, \mathbf{P})$ *the solution of (24–25), if* Ω *is convex, for* $N \geq 1$*, we have,*

$$\|\mathbf{e_P}\|_{L^2(0,T;H^{-1}(div,\Omega))} \leq \left\|e_u^0\right\|_{L^2(\Omega)} + \sum_{i=6}^{11} C_i \mathcal{E}_i. \tag{34}$$

where,

$$\mathcal{E}_6 := \left(\sum_{n=1}^{N} k_n \sum_{e \in \mathcal{B}_n} h_e \|J_t\|^2_{L^2(e)}\right)^{1/2}, \tag{35}$$

$$\mathcal{E}_7 := \left(\sum_{n=1}^{N} k_n \left\|U^n - \mathcal{P}^n_\Omega\, U^{n-1}\right\|^2_{L^2(\Omega)}\right)^{1/2}, \tag{36}$$

$$\mathcal{E}_8 := \max\left\{\left(\sum_{n=1}^{N} k_n \left\|U^{n-1} - \mathcal{P}^n_\Omega\, U^{n-1}\right\|^2_{L^2(\Omega)}\right)^{1/2}, \right. \tag{37}$$

$$\left.\left(\sum_{n=1}^{N} k_n^{-1} \left\|h_n(U^{n-1} - \mathcal{P}^n_\Omega\, U^{n-1})\right\|^2_{L^2(\Omega)}\right)^{1/2}\right\}, \tag{38}$$

$$\mathcal{E}_9 := \left(\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|h_n(f - div\,\mathbf{P})\|^2_{L^2(\Omega)} dt\right)^{1/2}, \tag{39}$$

$$\mathcal{E}_{10} := \left[\sum_{n=1}^{N} \left(\int_{t_{n-1}}^{t_n} k_n^{1/2} \|f - div\,\mathbf{P}\|_{L^2(\Omega)} dt\right)^2\right]^{1/2}, \tag{40}$$

$$\mathcal{E}_{11} := \left(\sum_{n=1}^{N} k_n \|h_n \mathbf{P}\|^2_{L^2(\Omega)}\right)^{1/2}. \tag{41}$$

The proofs of Theorem 1 and Theorem 2 are obtained from the error representation formulas (20) and (22), respectively, by the evaluation of the residual in the corresponding norms and by the stability properties of the associated dual problem. Before proving these results, we introduce the interpolation operators we will use, and find a simpler expression for the residual.

6.1 Interpolation operators

Let $\mathcal{P}^n_\Omega$ (resp. $\mathcal{P}^n_t$) denote the L^2-projection over the piecewise constant functions over the mesh $\mathcal{T}_n$ (resp. constant functions on t in the interval $(t_{n-1}, t_n])$. We avoid the superscript n when it is redundant. It is well known, that if v is smooth we have,

$$\left\| v - \mathcal{P}_\Omega^n v \right\|_{L^2(\Omega)} \le c \|h_n \nabla v\|_{L^2(\Omega)}, \tag{42}$$

$$\max_{\xi \in (t_{n-1}, t_n]} |v(\xi) - \mathcal{P}_t^n v| \le \int_{t_{n-1}}^{t_n} |v_t| d\xi. \tag{43}$$

For vectorial functions, we define $\mathbf{\Pi}_t^n$ in the same way as for scalar functions, *i.e.* the L^2-projection over constant functions in the interval $(t_{n-1}, t_n]$. For the projection over $H_0(\text{div}\,, \Omega)$ we use the Raviart-Thomas interpolation operator $\mathbf{\Pi}_\Omega^n$, (see [7]),

$$\mathbf{\Pi}_\Omega^n : H(\text{div}\,, \Omega) \cap L^s\left(\Omega; \mathbb{R}^2\right) \longrightarrow \mathbb{X}_n, \quad \text{for } s > 2,$$

which verifies,

$$\langle (\mathbf{q} - \mathbf{\Pi}_\Omega \mathbf{q}) \cdot \boldsymbol{n}, 1 \rangle_e = 0, \quad \forall e \in \mathcal{B}_n, \tag{44}$$

where $\boldsymbol{n}$ is the normal vector to the edge e, moreover, if $\mathbf{q} \in H^1(\Omega; \mathbb{R}^2)$,

$$\|\mathbf{q} - \mathbf{\Pi}_\Omega \mathbf{q}\|_{L^2(S)} \le c h_S \|\nabla \mathbf{q}\|_{L^2(S)}. \tag{45}$$

We also use the Clément interpolation operator, then let $S_0^1(\mathcal{T}_n)$ be the space of piecewise linear functions over $\mathcal{T}_n$ that cancels out on the boundary. Let $\mathcal{C}_n \mu$ denote the Clément interpolation of function μ over $S_0^1(\mathcal{T}_n)$. Then we have the following error estimations ([10]),

$$\|\mu - \mathcal{C}_n \mu\|_{L^2(T)} \le c h_S \|\nabla \mu\|_{L^2(\Omega_T)}, \tag{46}$$

$$\|\mu - \mathcal{C}_n \mu\|_{L^2(e)} \le c h_e^{1/2} \|\nabla \mu\|_{L^2(\Omega_e)}, \tag{47}$$

where $\Omega_T = \{T^* \in \mathcal{T}_n : T \text{ and } T^* \text{have a common vertex}\}$ and $\Omega_e = \{T^* \in \mathcal{T}_n : T^* \text{ and } e \text{ have a common vertex}\}$. It is easy to prove that $\text{rot}\, \mathcal{C}_n \mu \in \mathbb{X}_n$.

6.2 Residual

We first express the residual (9) in a more appropriate way. From (10) taking into account that U is piecewise constant, integrating and adding by parts, we have,

$$\begin{aligned}
\int_0^{T=t_N} (U_t, v) dt &\equiv \left(U^N, v^N\right) - \left(U^0, v^0\right) - \int_0^T (U, v_t) dt \\
&= \left(U^N, v^N\right) - \left(U^0, v^0\right) - \sum_{n=1}^N \left(U^n, v^n - v^{n-1}\right) \\
&= \sum_{n=1}^N \left(U^n - U^{n-1}, v^{n-1}\right).
\end{aligned}$$

Substituting the former expression into (9), the residual $\mathcal{R}(v, \mathbf{q})$ becomes,

$$\mathcal{R}(v, \mathbf{q}) = \left(e_u^0, v^0\right) - \sum_{n=1}^{N} \left(U^n - U^{n-1}, v^{n-1}\right) + \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (f - \operatorname{div} \mathbf{P}\,, v) dt$$
$$+ \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \left[(\mathbf{P}\,, \mathbf{q}) - (\operatorname{div} \mathbf{q}, U)\right] dt.$$

Finally, adding to this expression equations (24)–(25) we obtain,

$$\begin{aligned}\mathcal{R}(v, \mathbf{q}) = & \left(e_u^0, v^0\right) + \sum_{n=1}^{N} \left(U^n - \mathcal{P}_\Omega^n\, U^{n-1}, V - v^{n-1}\right) \\ & + \sum_{n=1}^{N} \left(U^{n-1} - \mathcal{P}_\Omega^n\, U^{n-1}, v^{n-1}\right) \\ & + \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (f - \operatorname{div} \mathbf{P}, v - V)\, dt \\ & + \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \left[(\mathbf{P}, \mathbf{q} - \mathbf{Q}) - (\operatorname{div} (\mathbf{q} - \mathbf{Q}), U)\right] dt \\ = & \, \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V}. \end{aligned} \tag{48}$$

Now, we have the tools to prove Theorem 1 and Theorem 2.

6.3 Proof of Theorem 1

We will estimate each term of (48), and to conclude we will use these estimations in the error representation formula (20). In this proof we discuss in a different way the last temporal interval, in order to be able to apply Lemma 1.

Let $(v, \mathbf{q})$ be the solution of the dual problem $\mathcal{D}_u$ and select the following discrete functions, $V := \mathcal{P}_t^n\, \mathcal{P}_\Omega^n\, v \in \mathbb{M}_n$ and $\mathbf{Q} := \mathcal{P}_t^n\, \Pi_\Omega^n \mathbf{q} \in \mathbb{X}_n$. Using the Cauchy-Schwartz inequality in the first term of (48), we have,

$$|\mathrm{I}| \leq \left\|e_u^0\right\|_{L^2(\Omega)} \left\|v^0\right\|_{L^2(\Omega)}. \tag{49}$$

For the second term, notice that from the properties of the L^2-projection $\mathcal{P}_\Omega^n$, $(U^n - \mathcal{P}_\Omega^n\, U^{n-1}, v^{n-1} - \mathcal{P}_\Omega^n\, v^{n-1}) = 0$, therefore using Cauchy-Schwartz inequality and (43), we obtain,

$$
\begin{aligned}
|\mathrm{II}| &= \left| \sum_{n=1}^{N} \left(U^n - \mathcal{P}_\Omega^n\, U^{n-1}, V - \mathcal{P}_\Omega^n\, v^{n-1} \right) \right| \\
&\le \sum_{n=1}^{N-1} \left\| U^n - \mathcal{P}_\Omega^n\, U^{n-1} \right\|_{L^2(\Omega)} \int_{t_{n-1}}^{t_n} \|v_t\|_{L^2(\Omega)} dt \\
&\quad +2 \left\| U^N - \mathcal{P}_\Omega^N\, U^{N-1} \right\|_{L^2(\Omega)} \max_{t \in (t_{N-1}, t_N]} \|v\|_{L^2(\Omega)} \\
&\le \max_{1 \le n \le N} \left(\left\| U^n - \mathcal{P}_\Omega^n\, U^{n-1} \right\|_{L^2(\Omega)} \right) \\
&\quad \left(\int_0^{t_{N-1}} \|v_t\|_{L^2(\Omega)} dt + 2 \max_{t \in (t_{N-1}, t_N]} \|v\|_{L^2(\Omega)} \right). \qquad (50)
\end{aligned}
$$

The third term of (48), as $(\mathcal{P}_\Omega^n\, U^{n-1} - U^{n-1}, V) = 0$, we can write,

$$
\begin{aligned}
|\mathrm{III}| \le & \left| \sum_{n=1}^{N-1} \left(\mathcal{P}_\Omega^n\, U^{n-1} - U^{n-1}, \mathcal{P}_t^n\, v - v^{n-1} \right) \right| \\
&+ \left| \sum_{n=1}^{N-1} \left(\mathcal{P}_\Omega^n\, U^{n-1} - U^{n-1}, \mathcal{P}_t^n\, v - V \right) \right| \\
&+ \left| \left(\mathcal{P}_\Omega^N\, U^{N-1} - U^{N-1}, v^{N-1} \right) \right|,
\end{aligned}
$$

The first and the third terms of this sum can be estimated in the same way as in the previous case. For the second term, since functions in $\mathbb{M}_n$ are constant, using (42) we obtain,

$$
\begin{aligned}
&\left| \sum_{n=1}^{N-1} \left(\mathcal{P}_\Omega^n\, U^{n-1} - U^{n-1}, \mathcal{P}_t^n\, v - V \right) \right| \\
&\le \left| \sum_{n=1}^{N-1} k_n^{-1} \left(\mathcal{P}_\Omega^n\, U^{n-1} - U^{n-1}, \int_{t_n - 1}^{t_n} \left[v - \mathcal{P}_\Omega^n\, v \right] dt \right) \right| \\
&\le c \sum_{n=1}^{N-1} k_n^{-1} \left\| h_n \left(\mathcal{P}_\Omega^n\, U^{n-1} - U^{n-1} \right) \right\|_{L^2(\Omega)} \int_{t_{n-1}}^{t_n} \|\nabla v\|_{L^2(\Omega)} dt.
\end{aligned}
$$

Then, the third term of (48) is bounded by,

$$
\begin{aligned}
|\mathrm{III}| \le & \max_{1 \le n \le N} \left(\left\| U^{n-1} - \mathcal{P}_\Omega^n\, U^{n-1} \right\|_{L^2(\Omega)} \right) \left(\int_0^{t_{N-1}} \|v_t\|_{L^2(\Omega)} dt + 2 \max_{t \in (t_{N-1}, t_N]} \|v\|_{L^2(\Omega)} \right) \\
&+ c \max_{1 \le n \le N-1} \left(k_n^{-1} \left\| h_n (U^{n-1} - \mathcal{P}_\Omega^n\, U^{n-1}) \right\|_{L^2(\Omega)} \right) \left(\int_0^{t_{N-1}} \|\mathbf{q}\|_{L^2(\Omega)} dt \right). \qquad (51)
\end{aligned}
$$

A similar computation based on properties (42) and (43), allows us to estimate the fourth term of the residual (48), in the following way,

$$\begin{aligned}
|\mathrm{IV}| \le & \left| \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \big(f - \operatorname{div}\mathbf{P}\,,\, v - \mathcal{P}_\Omega^n\, v\big)\, dt \right| \\
& + \left| \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \big(f - \operatorname{div}\mathbf{P}\,,\, \mathcal{P}_\Omega^n\, v - V\big)\, dt \right| \\
\le & \left(\sum_{n=1}^{N-1} \int_{t_{n-1}}^{t_n} \|h_n(f - \operatorname{div}\mathbf{P}\,)\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} dt \right) \\
& + \left(\int_{t_{N-1}}^{t_N} \|f - \operatorname{div}\mathbf{P}\,\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} dt \right) \\
& + \left(\sum_{n=1}^{N} \max_{t\in(t_{n-1},t_n]} \left\|\mathcal{P}_\Omega^n\, v - V\right\|_{L^2(\Omega)} \int_{t_{n-1}}^{t_n} \|f - \operatorname{div}\mathbf{P}\,\|_{L^2(\Omega)} dt \right) \\
\le & \max_{1\le n\le N-1} \left(\max_{t\in(t_{n-1},t_n]} \|h_n(f - \operatorname{div}\mathbf{P}\,)\|_{L^2(\Omega)} \right) \int_0^{t_{N-1}} \|\mathbf{q}\|_{L^2(\Omega)} dt \\
& + \max_{1\le n\le N-1} \left(\int_{t_{n-1}}^{t_n} \|f - \operatorname{div}\mathbf{P}\,\|_{L^2(\Omega)} dt \right) \\
& \left(2 \max_{t\in(t_{N-1},t_N]} \|v\|_{L^2(\Omega)} + \int_0^{t_{N-1}} \|v_t\|_{L^2(\Omega)} dt \right). \qquad (52)
\end{aligned}$$

Finally, for the fifth term of (48), using (44) we have,

$$\begin{aligned}
|\mathrm{V}| \le & \left| \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \left[\big(\mathbf{P}\,,\, \mathbf{q} - \Pi_\Omega^n \mathbf{q}\big) - \big(\operatorname{div}\, \big(\mathbf{q} - \Pi_\Omega^n \mathbf{q}\big)\,,\, U\big) \right] dt \right| \\
& + \left| \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \left[\big(\mathbf{P}\,,\, \Pi_\Omega^n \mathbf{q} - \mathbf{Q}\big) - \big(\operatorname{div}\, \big(\Pi_\Omega^n \mathbf{q} - \mathbf{Q}\big)\,,\, U\big) \right] dt \right| \\
\le & \max_{1\le n\le N} \|h_n \mathbf{P}\|_{L^2(\Omega)} \left(\sum_{n=1}^{N} \left\| \int_{t_{n-1}}^{t_n} \nabla \mathbf{q}\; dt \right\|_{L^2(\Omega)} \right). \qquad (53)
\end{aligned}$$

Notice that due to the interpolation operator properties, all terms except the first one are null.

To conclude, we proceed as in [12]. Let $(v, \mathbf{q})$ be the solution of problem D_u, then,

$$\int_{t_{n-1}}^{t_n} \operatorname{div} \mathbf{q}\, dt = \int_{t_{n-1}}^{t_n} v_t\, dt = v^n - v^{n-1}. \tag{54}$$

Using the convexity of Ω, we deduce that $v \in H^2(\Omega)$ and

$$\|\nabla \mathbf{q}\|_{L^2(\Omega)} = |v|_{H^2(\Omega)} \leq \|\Delta v\|_{L^2(\Omega)} = \|\operatorname{div} \mathbf{q}\|_{L^2(\Omega)}. \tag{55}$$

Now, from the *inf-sup* condition we obtain,

$$\|\mathbf{q}\|_{L^2(\Omega)} \leq C \|\operatorname{div} \mathbf{q}\|_{L^2(\Omega)}. \tag{56}$$

On the other side, using the Cauchy-Schwartz inequality, we have,

$$\begin{aligned} \int_0^{t_{N-1}} \|z\|_{L^2(\Omega)} dt &\leq \left(\int_0^{t_{N-1}} (t_N - t)^{-1} dt \right)^{1/2} \left(\int_0^{t_{N-1}} (t_N - t) \|z\|^2_{L^2(\Omega)} dt \right)^{1/2} \\ &\leq \left(\log \frac{t_N}{k_N} \right)^{1/2} \left(\int_0^{t_{N-1}} (t_N - t) \|z\|^2_{L^2(\Omega)} dt \right)^{1/2}, \end{aligned} \tag{57}$$

with $z = v_t$ or $z = \operatorname{div} \mathbf{q}$.

Finally, Theorem 1 is an easy consequence of (20) combined with (50)–(51)–(52)–(53), together with (54)–(55)–(56)–(57) and stability properties (13)–(14)–(15) for problem D_u.

6.4 Proof of Theorem 2

We proceed in the same way as in the previous theorem, we estimate each term of the error representation formula (22).

Notice that from (7) and (25) we have $(\mathbf{e_P}, \mathbf{curl}\, \mathcal{C}_n \mu) = 0$. Then in the first term of (22) using this Galerkin orthogonality and integrating by parts, we obtain,

$$\begin{aligned} \int_0^{t_N} (\mathbf{e_P}, \mathbf{curl}\, \mu) dt &= \int_0^{t_N} (\mathbf{e_P}, \mathbf{curl}\, (\mu - \mathcal{C}\mu)) dt \\ &= \int_0^{t_N} \left[\sum_{S \in \mathcal{T}_n} (\operatorname{rot} \mathbf{P}, \mu - \mathcal{C}\mu)_S + \frac{1}{2} \sum_{e \in \partial T} \langle J_t, \mu - \mathcal{C}\mu \rangle_e \right\} dt. \end{aligned} \tag{58}$$

From property (47), taking into account that rot $\mathbf{P} \equiv 0$ and that $\mathbf{P}$ and J_t are constant on t in each time step, we have,

$$\left| \int_0^T (\mathbf{e_P}, \mathbf{curl}\,\mu) dt \right| \leq C \int_0^{t_N} \left(\sum_{e \in \mathcal{B}_n} h_e \| J_t \|_{0,e} \right)^{1/2} \| \nabla \mu \|_{L^2(\Omega)} \, dt$$
$$\leq C \left(\sum_{n=1}^{N} k_n \sum_{e \in \mathcal{B}_n} h_e \| J_t \|_{0,e} \right)^{1/2} \| \mathbf{curl}\,\mu \|_{L^2(0,T;L^2(\Omega))}. \tag{59}$$

We now estimate all terms of $\mathcal{R}(v, \mathbf{q})$, see equation (48). As in the former theorem, we consider $(v, \mathbf{q})$ the solution of problem $\mathcal{D}_\mathbf{p}$ and the discrete functions, $V := \pi_t \pi_\Omega v \in \mathbb{M}$ and $\mathbf{Q} := \mathbf{\Pi}_t \mathbf{\Pi}_\Omega \mathbf{q} \in \mathbb{X}$. The first term of (48) is bounded in the same way. For the second term of (48), using Cauchy-Schwartz, we obtain,

$$|\mathrm{II}| = \left| \sum_{n=1}^{N} \left(U^n - \mathcal{P}_\Omega^n U^{n-1}, V - \mathcal{P}_\Omega^n v^{n-1} \right) \right|$$
$$\leq \sum_{n=1}^{N} \left(\left\| U^n - \mathcal{P}_\Omega^n U^{n-1} \right\|_{L^2(\Omega)} \int_{t_{n-1}}^{t_n} \| v_t \|_{L^2(\Omega)} dt \right)$$
$$\leq \left(\sum_{n=1}^{N} k_n \left\| U^n - \mathcal{P}_\Omega^n U^{n-1} \right\|_{L^2(\Omega)}^2 \right)^{1/2} \left(\int_0^{t_N} \| v_t \|_{L^2(\Omega)}^2 dt \right)^{1/2}. \tag{60}$$

Similarly, for the third term of (48),

$$|\mathrm{III}| \leq \left| \sum_{n=1}^{N} \left(\mathcal{P}_\Omega^n U^{n-1} - U^{n-1}, \mathcal{P}_t^n v - v^{n-1} \right) \right|$$
$$+ \left| \sum_{n=1}^{N} \left(\mathcal{P}_\Omega^n U^{n-1} - U^{n-1}, \mathcal{P}_t^n v - V \right) \right|.$$

The first term of this expression is bounded by,

$$\left| \sum_{n=1}^{N} \left(\mathcal{P}_\Omega^n U^{n-1} - U^{n-1}, \mathcal{P}_t^n v - v^{n-1} \right) \right|$$
$$\leq \left(\sum_{n=1}^{N} k_n \left\| \mathcal{P}_\Omega^n U^{n-1} - U^{n-1} \right\|_{L^2(\Omega)}^2 \right)^{1/2} \left(\int_0^{t_N} \| v_t \|_{L^2(\Omega)}^2 dt \right)^{1/2}. \tag{61}$$

and the second, using property (42), is bounded by,

$$\left|\sum_{n=1}^{N}\left(\mathcal{P}_{\Omega}^{n}\, U^{n-1}-U^{n-1}, \mathcal{P}_{t}^{n}\, v-V\right)\right|$$
$$\leq\left(\sum_{n=1}^{N} k_{n}^{-1}\left\|h_{n}(\mathcal{P}_{\Omega}^{n}\, U^{n-1}-U^{n-1})\right\|_{L^{2}(\Omega)}^{2}\right)^{1/2}\left(\int_{0}^{t_N}\|\nabla v\|_{L^{2}(\Omega)}^{2} dt\right)^{1/2}. \tag{62}$$

The fourth term of (48) can be separated into two parts,

$$|\mathrm{IV}| \leq\left|\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n}\left(f-\operatorname{div} \mathbf{P}\,, v-\mathcal{P}_{\Omega}^{n}\, v\right)\right|+\left|\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n}\left(f-\operatorname{div} \mathbf{P}\,, \mathcal{P}_{\Omega}^{n}\, v-V\right)\right|.$$

Notice that using property (42), the first adding term of the last expression can be bounded as follows,

$$\left|\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n}(f-\operatorname{div} \mathbf{P}\,, v-\mathcal{P}_{\Omega}^{n}\, v)\, dt\right|$$
$$\leq\left(\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n}\|h_{n}(f-\operatorname{div} \mathbf{P}\,)\|_{L^{2}(\Omega)}^{2}\, dt\right)^{1/2}\|\nabla v\|_{L^{2}(0, T ; L^{2}(\Omega))}. \tag{63}$$

and using (43), the second adding term can be bounded as follow,

$$\left|\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n}\left(f-\operatorname{div} \mathbf{P}\,, \mathcal{P}_{\Omega}^{n}\, v-V\right) dt\right|$$
$$\leq \sum_{n=1}^{N}\left(\max_{t \in(t_{n-1}, t_n]}\left\|v-\mathcal{P}_{t}^{n}\, v\right\|_{L^{2}(\Omega)} \int_{t_{n-1}}^{t_n}\|f-\operatorname{div} \mathbf{P}\,\|_{L^{2}(\Omega)} dt\right)$$
$$\leq\left[\sum_{n=1}^{N}\left(\int_{t_{n-1}}^{t_n} k_{n}^{1/2}\|f-\operatorname{div} \mathbf{P}\,\|_{L^{2}(\Omega)} dt\right)^{2}\right]^{1/2}\|v_{t}\|_{L^{2}(0, T ; L^{2}(\Omega))}. \tag{64}$$

Finally, we proceed with the last term of (48) as in (53),

$$|\mathrm{IV}| \leq\left|\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n}(\mathbf{P}, \mathbf{q}-\mathbf{\Pi}_{\Omega} \mathbf{q}) dt\right|$$
$$\leq\left(\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n}\|h_{n} \mathbf{P}\|_{L^{2}(\Omega)}^{2}\right)^{1/2}\left(\int_{0}^{T_N}\|\nabla \mathbf{q}\|_{L^{2}(\Omega)}^{2}\right)^{1/2}. \tag{65}$$

Notice that for this last inequality we have used that $\mathbf{q} \in H^1(\Omega; \mathbb{R}^2)$. In fact, if $(v, \mathbf{q})$ is the solution of problem $D_{\mathbf{p}}$, with ϕ verifying (21), we have $\mathbf{q} = \nabla(v - \phi)$. Moreover, ϕ is the solution of,

$$\Delta\phi = \operatorname{div} \boldsymbol{\kappa} \quad \text{in } \Omega,$$
$$\nabla\phi \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma.$$

Then, as $\boldsymbol{\kappa} \in H_0(\operatorname{div}, \Omega)$ and Ω is convex, from the regularity theory (see [19]) it follows that $\mathbf{q} \in H^1(\Omega; \mathbb{R}^2)$.

Then from (22), using (59)–(60)–(61)–(62)–(63)–(64)–(65) and the stability properties (16)–(17)–(18)–(19) of problem $\mathcal{D}_{\mathbf{p}}$, we conclude,

$$\|\mathbf{e_P}\|_{L^2(0,T;H^{-1}(\operatorname{div},\Omega))} = \sup_{\kappa \in L^2(0,T;H_0(\operatorname{div},\Omega))} \frac{\langle \mathbf{e_P}, \boldsymbol{\kappa}\rangle}{\|\boldsymbol{\kappa}\|_{L^2(0,T;H(\operatorname{div},\Omega))}} \tag{66}$$

$$\leq \left\|e_u^0\right\|_{L^2(\Omega)} + \sum_{i=6}^{11} C_i \mathcal{E}_i, \tag{67}$$

where the constants C_i depend on the minimum mesh angle and the interpolation constants.

7 Adaptive algorithm

In this section we propose two adaptive strategies based on the former estimates. The goal is to determine the time steps and the meshes in order to obtain a uniform error distribution. As usual, in parabolic problems we distinguish four kinds of terms in the a-posteriori estimate (see [21]): initial error, space discretization error, coarsening error and time discretization error,

$$\mathcal{E}_n \leq \mathcal{E}_0 + \mathcal{E}_{h,n} + \mathcal{E}_{c,n} + \mathcal{E}_{k,n}.$$

We also select the parameters Γ_0, Γ_h and Γ_k, verifying,

$$\Gamma_0 + \Gamma_h + \Gamma_k \leq 1,$$

in a way that given a total error tolerance tol, the adaptive algorithm tries to select time steps and meshes verifying for all n,

$$\mathcal{E}_n \sim tol, \qquad \mathcal{E}_0 \sim \Gamma_0\, tol, \quad \mathcal{E}_{h,n} + \mathcal{E}_{c,n} \sim \Gamma_h\, tol, \quad \mathcal{E}_{k,n} \sim \Gamma_k\, tol.$$

The adjustment of the time step size has been done iteratively: the algorithm begins with an initial time step k_0; given $\theta_1 \in (0, 1)$ and $\theta_2 \in (0, \theta_1)$, if $\mathcal{E}_{k,n} > \theta_1 \Gamma_k\, tol$, then the time step diminishes by a factor $\delta_1 \in (0, 1)$; on the contrary, if $\mathcal{E}_k < \theta_2 \Gamma_k\, tol$, the time step increases by a factor $\delta_2 > 1$.

For the space refinement we use an error equidistribution strategy (see [12]). Given $\theta \sim 1$ and $\theta_c < 1$ and using the following error indicators,

$$\mathcal{E}_{h,n} = \left(\sum_{S \in \mathcal{T}_n} \mathcal{E}_{h,n}^2(S)\right)^{1/2}, \quad \mathcal{E}_{c,n} = \left(\sum_{S \in \mathcal{T}_n} \mathcal{E}_{c,n}^2(S)\right)^{1/2},$$

in each time step we refine the elements verifying,

$$\mathcal{E}_{h,n}(S) > \theta \frac{\Gamma_h \ tol}{\mathcal{N}_n^{1/2}},$$

and we mark to coarse the elements verifying,

$$\mathcal{E}_{h,n}(S) + \mathcal{E}_{c,n}(S) \le \theta_c \frac{\Gamma_h \ tol}{\mathcal{N}_n^{1/2}},$$

where $\mathcal{N}_n$ denotes the degrees of freedom of the n-th mesh.

We use the implicit adaptive strategy of type A described in [4], that means, for each time step we start from the previous step mesh and repeat the process,

$$\mathsf{SOLVE} \to \mathsf{ESTIMATE} \to \mathsf{REFINE/COARSEN},$$

until the estimated error is below the tolerance.

In the following table we summarized the Adaptive Algorithm, [23],

Space and Time Adaptive Algorithm

Given parameters $tol, \delta_1 \in (0,1), \delta_2 > 1, \theta_1 \in (0,1)$, and $\theta_2 \in (0, \theta_1)$, the discrete solution U_n on the triangulation $\mathcal{T}_n$, the time t_n and the time step size k_n.
$\mathcal{T}_{n+1} := \mathcal{T}_n$.
$k_{n+1} := k_n$.
$t_{n+1} := t_n + k_{n+1}$.
Solve the Discrete Problem for $(U_{n+1}, \mathbf{P}_{n+1})$ on the mesh $\mathcal{T}_{n+1}$.
Compute the error estimators.
While $\mathcal{E}_{k,n+1} > \theta_1 \Gamma_k \ tol$.
  $k_{n+1} := \delta_1 k_n$.
  $t_{n+1} := t_n + k_{n+1}$.
  Solve the Discrete Problem for $(U_{n+1}, \mathbf{P}_{n+1})$ on $\mathcal{T}_{n+1}$.
  Compute the error estimators.
End while.
Do
  Mark elements for refinement or coarsening using $\mathcal{E}_{h,n}$ and $\mathcal{E}_{c,n}$.
  If elements are marked then
    Adapt mesh $\mathcal{T}_{n+1}$.
    Solve the Discrete Problem for $(U_{n+1} \mathbf{P}_{n+1})$ on $\mathcal{T}_{n+1}$.
    Compute the error estimators.
  End if.
  While $\mathcal{E}_{k,n+1} > \theta_1 \Gamma_k \ tol$
    $k_{n+1} := \delta_1 k_n$.
    $t_{n+1} := t_n + k_{n+1}$.
    Solve the Discrete Problem for $(U_{n+1}, \mathbf{P}_{n+1})$ on $\mathcal{T}_{n+1}$.
    Compute the error estimators.
  End while
While $\mathcal{E}_{h,n+1} + \mathcal{E}_{c,n+1} > tol$
If $\mathcal{E}_{k,n+1} \le \theta_2 \Gamma_k \ tol$
  $k_{n+1} := \delta_2 k_n$.
End if.

8 Numerical examples

For the numerical examples we have selected the usual values for the parameters described before (see [23]),

$$\begin{aligned}&\Gamma_0 = 0.1, \quad \Gamma_h = 0.4, \quad \Gamma_k = 0.4,\\ &\theta_1 = 1.0, \quad \theta_2 = 0.3, \quad \delta_1 = \sqrt{2}, \quad \delta_2 = 1/\sqrt{2},\\ &\theta = 0.9 \quad \theta_c = 0.1.\end{aligned}$$

We consider,

$$u(x, y, t) = \sin\left(\frac{\pi t}{2}\right) \exp\left\{-20\| \left(x-\frac{1}{2}, y-\frac{1}{2}\right) - \frac{3}{4}\left(\cos\left(\frac{\pi t}{2}\right), \sin\left(\frac{\pi t}{2}\right)\right) \|^2\right\} \tag{68}$$

solution of the parabolic problem (6)–(7), with $\Omega = [0, 3]^2$ and $t \in [0, 4]$.

We solve the previous problem using every one of the estimators. To evaluate the quality of the error estimators we will compute two indicators,

- *Effectiveness index*, which gives the main ratio between the discretization error and the error estimated, *i.e.* we define,

$$C_u := \overline{e_u/\mathcal{E}^u}, \quad C_{\mathbf{p}} := \overline{\mathbf{e_p}/\mathcal{E}^{\mathbf{p}}},$$

  where the bar denotes the average value with respect to time. For a good estimator this quantity must be a constant independent of the mesh and the time step used. Although our theory does not include a proof of efficiency, the numerical examples show a good behaviour and provide numerical evidences of the efficiency of the estimators.
- *Correlation coefficient index* between the discretization error and the estimate, *i.e.*

$$\rho_u := \frac{\operatorname{cov}(\mathcal{E}^u, e_u)}{\sigma_{\mathcal{E}^u}\sigma_{e_u}}, \quad \rho_{\mathbf{p}} := \frac{\operatorname{cov}(\mathcal{E}^{\mathbf{p}}, \mathbf{e_p})}{\sigma_{\mathcal{E}^{\mathbf{p}}}\sigma_{\mathbf{e_p}}},$$

  where $\operatorname{cov}(\cdot, \cdot)$ denotes the covariance and σ_* denotes the standard deviation. For a good error estimator, the correlation coefficient index must be close to the ideal value 1.

8.1 Estimate e_u.

From Theorem 1 we define,

$$\begin{aligned}\mathcal{E}^u_{h,n}(S)^2 &:= C_3^2\|h_S\mathbf{P}\|^2_{L^2(S)} + C_4^2 \max_{t\in[t_{n-1},t_n)} \|h_S(f - \operatorname{div}\mathbf{P})\|^2_{L^2(S)},\\ \mathcal{E}^u_{c,n}(S)^2 &:= \max\left\{\|U^{n-1} - \mathcal{P}^n_\Omega\, U^{n-1}\|^2_{L^2(S)}, k_n^{-1}\|h_n\left(U^{n-1} - \mathcal{P}^n_\Omega\, U^{n-1}\right)\|^2_{L^2(S)}\right\}.\end{aligned}$$

So we obtain the following expressions for the error indicators,

$$\mathcal{E}^u_{h,n} := \left(\sum_{S \in \mathcal{T}_n} \mathcal{E}_{h,n}(S)^2 \right)^{1/2}, \quad \mathcal{E}^u_{c,n} := \left(\sum_{S \in \mathcal{T}_n} \mathcal{E}_{c,n}(S)^2 \right)^{1/2},$$
$$\mathcal{E}^u_{k,n} := C_1 \mathcal{E}_1 + C_5 \mathcal{E}_5.$$

We solve the test problem using the previous indicators for several tolerances: $tol = 1.0, 0.5$ and 0.25. In Table 1 we summarize the indicators of the estimator e_u. Notice that the effectiveness index is almost constant, independent of the prescribed tolerance, and the correlation coefficient is close to one in all cases. In Figure 1 we plot the estimate and the error progress in time, where we can appreciate the good correlation between both variables: error and estimator. The error discretization is computed in the L^2-norm in space for each time step. In Figure 2 we plot the progress of the degrees of freedom in time for the meshes used.

We also show several meshes and solutions at time $t = 0.90, 2.10$ and 3.50 in Figure 3. Notice that the Adaptive Algorithm localizes the region (depending on time) where the solution is not null and decides to place more grid points in this zone.

8.2 Estimate $\mathbf{e_P}$.

We can not directly use the results of Theorem 2 for the adaptive strategy because the time stock. As in [21], we define new indicators E_i verifying,

$$\mathcal{E}_i \le E_i, \quad E_i := \max_{1 \le n \le N} E_{i,n}, \quad i = 6, \ldots, 11.$$

Table 1 Estimate e_u. Estimate-Error correlation for several tolerances

tol	C_u	ρ_u
1.0	0.040	0.94
0.5	0.042	0.95
0.25	0.045	0.90

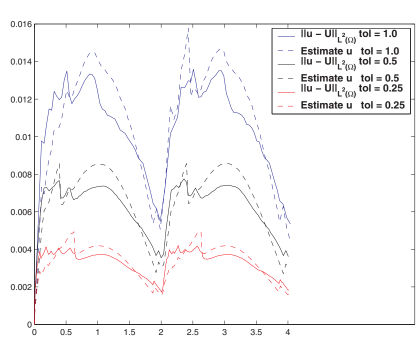

Fig. 1 Estimate e_u. Estimate and error progress in time. The scale is different to each variable

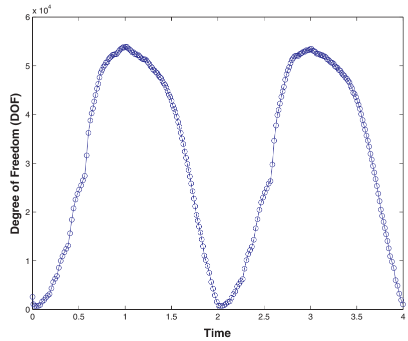

Fig. 2 Estimate e_u. Degrees of freedom for the meshes used versus time

To be precise, we define,

$$E_{6,n} := \left(\sum_{S \in \mathcal{T}_n} E_{6,n}^2(S) \right)^{1/2} := \left(\sum_{S \in \mathcal{T}_n} \sum_{e \in \mathcal{B}_n} T h_e \| J_{\boldsymbol{t}} \|_{L^2(e)}^2 \right)^{1/2},$$

$$E_{7,n} := \left(T \left\| U^n - \mathcal{P}_\Omega^n \, U^{n-1} \right\|_{L^2(\Omega)}^2 \right)^{1/2},$$

$$E_{8,n} := \left(\sum_{S \in \mathcal{T}_n} E_{8,n}^2(S) \right)^{1/2}$$

$$:= \max \left\{ \left(\sum_{S \in \mathcal{T}_n} T \| U^{n-1} - \mathcal{P}_\Omega^n \, U^{n-1} \|_{L^2(S)}^2 \right)^{1/2} \right.$$

$$\left. \left(\sum_{S \in \mathcal{T}_n} T k_n^{-2} \| h_n (U^{n-1} - \mathcal{P}_\Omega^n \, U^{n-1}) \|_{L^2(S)}^2 \right)^{1/2} \right\},$$

$$E_{9,n} := \left(\sum_{S \in \mathcal{T}_n} E_{9,n}^2(S) \right)^{1/2} := \left(\sum_{S \in \mathcal{T}_n} T \max_{t \in [t_{n-1}, t_n)} \| f - \operatorname{div} \mathbf{P} \|_{L^2(S)}^2 \right)^{1/2},$$

$$E_{10,n} := \left(T k_n^2 \max_{t \in [t_{n-1}, t_n)} \| f - \operatorname{div} \mathbf{P} \|_{L^2(\Omega)}^2 \right)^{1/2},$$

$$E_{11,n} := \left(\sum_{S \in \mathcal{T}_n} E_{11,n}^2(S) \right)^{1/2} := \left(\sum_{S \in \mathcal{T}_n} T \| h_S \mathbf{P} \|_{L^2(S)}^2 \right)^{1/2}.$$

Then, the error indicators used in the adaptive strategy are given by,

$$\mathcal{E}_{h,n}^{\mathbf{p}} := \left(\sum_{S \in \mathcal{T}_n} C_6^2 E_{6,n}^2(S) + C_9^2 E_{9,n}^2(S) + C_{11}^2 E_{11,n}^2(S) \right)^{1/2},$$

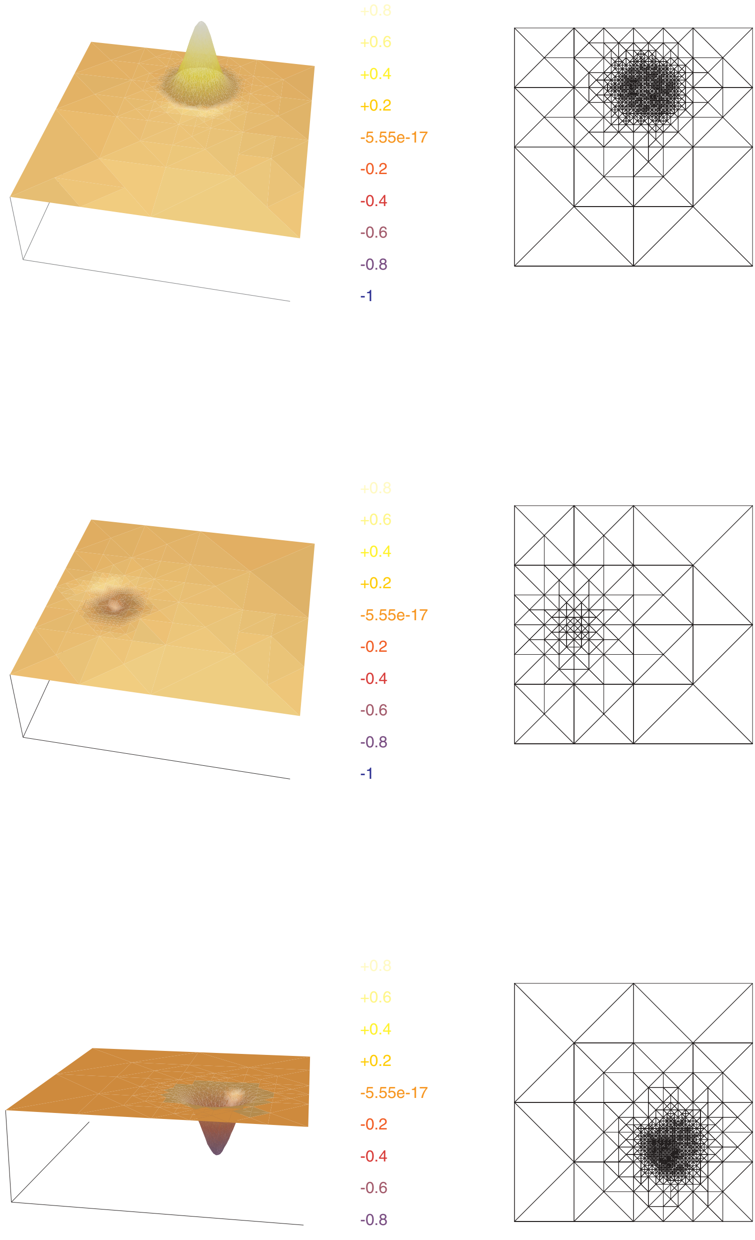

Fig. 3 Estimate e_u. Meshes and solutions at times $t = 0.90$, 2.10 and 3.50

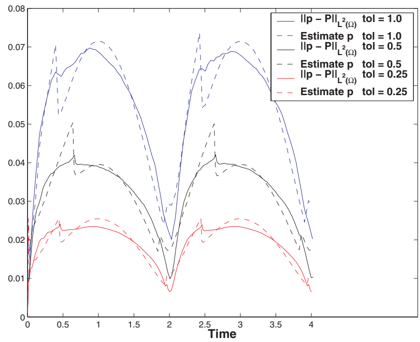

Fig. 4 Estimate $\mathbf{e_p}$. Estimate and error progress in time. The scale is different to each variable

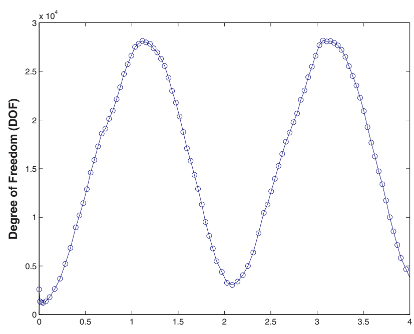

Fig. 5 Estimate $\mathbf{e_p}$. Degrees of freedom for the meshes used versus time

$$\mathcal{E}^{\mathbf{p}}_{c,n} := \left(\sum_{S \in \mathcal{T}_n} C_8^2 E_{8,n}^2(S) \right)^{1/2},$$

$$\mathcal{E}^{\mathbf{p}}_{k,n} := C_7 E_{7,n} + C_{10} E_{10,n}.$$

We repeat the same experiment with these error indicators, and we now study the adaptivity for the variable **p**. In Table 2 we can also see the behavior of the effectiveness index and the correlation coefficient. The discretization error is computed in the L^2-norm in space for each time step. Observe that this norm is stronger than the norm used in Theorem 2. Maybe, this is the reason why the effectiveness index seems to grow as the tolerance decreases.

Table 2 Estimate $\mathbf{e_p}$. Estimate-Error correlation for several tolerances

tol	$C_{\mathbf{p}}$	$\rho_{\mathbf{p}}$
1.0	0.18	0.96
0.5	0.14	0.92
0.25	0.24	0.92

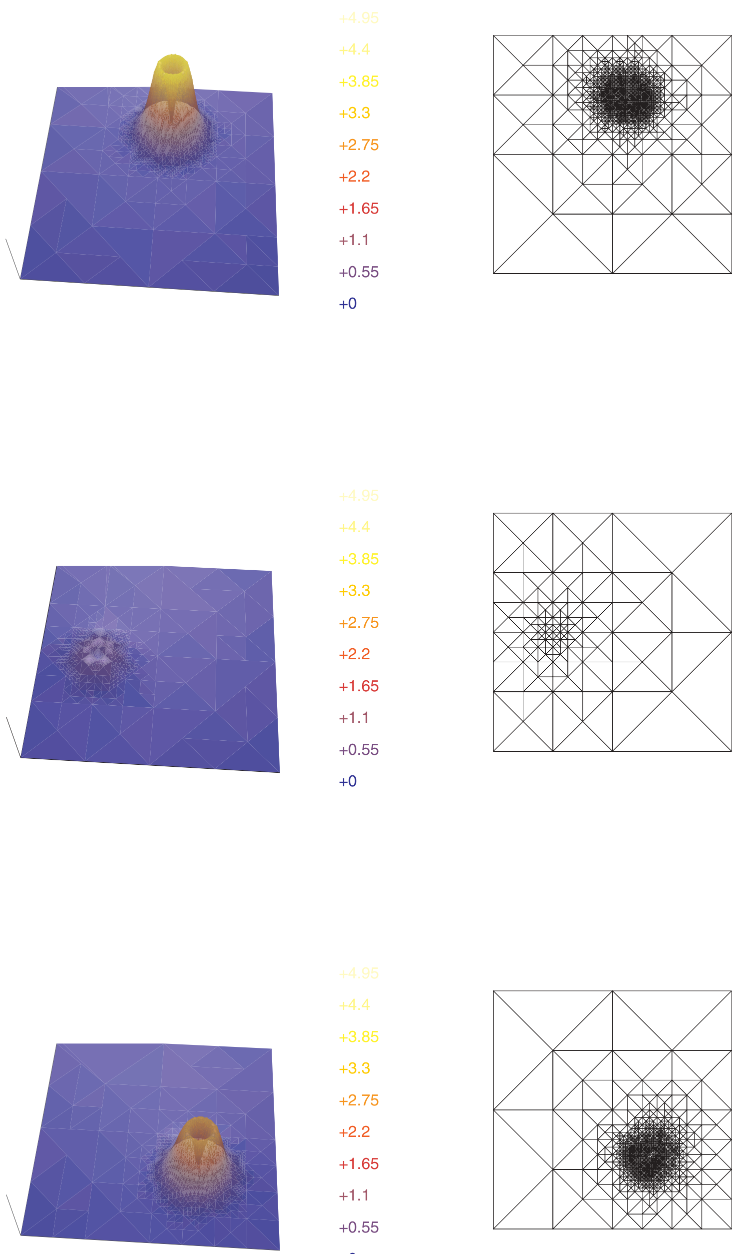

Fig. 6 Estimate $\mathbf{e_P}$. Meshes and norms of $\mathbf{P}$ at times $t = 0.90$, 2.10 and 3.50

The estimate and the error progress in time are plotted in Figure 4. Figure 5 shows the progress in time of the degrees of freedom for the meshes used when the adaptivity is done with the estimator $\mathbf{e_P}$.

Finally, the meshes and the norms of **P** are plotted at time $t = 0.90, 2.10$ and 3.50 in Figure 6.

9 Conclusions

The use of duality allows us to establish two a-posteriori error estimators for the mixed formulation of linear parabolic problems. These estimations are the essential component in the design of a reliable and efficient algorithm, as we can notice in the numerical examples. In future researchs we will try to extend this study to non-linear problems combining the duality ([12,13,9,21]) with the advantages provided by the mixed methods for dealing with this kind of problems ([2,5,16,18]).

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