Non-interference and continuity: impossibility results for the evaluation of infinite utility streams

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Abstract

In this work we analyse social welfare relations that satisfy various types of non-interference principles. Earlier contributions have established that (finitely) anonymous and strongly Paretian quasiorderings exist that capture such behaviors together with weak preference continuity and further consistency. Here we investigate a related problem: namely, the possibility of combining "standard" semicontinuity with efficiency in the presence of non-interference. We provide several impossibility results that prove that there is a generalised incompatibility between continuity and non-interference principles, both under ordinal and cardinal views of the problem.

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1 Introduction and motivation

Hammond's [6] characterization of the leximin ordering is based on anonymity, the strong Pareto axiom, and a principle now called Hammond Equity. It has been recently proven that Hammond Equity is equivalent to a liberal non-interference property called *Harm Principle* in the presence of anonymity and strong Pareto optimality (cf., Mariotti and Veneziani [11]). This seems fairly surprising since the Harm Principle does not embody any egalitarian consideration while Hammond Equity is a strongly egalitarian property. Extensions of the analysis to the case of the leximax criterion and also to the case of infinitely-lived societies appear in Lombardi and Veneziani [8,9]. In particular, preference continuities permit to characterize infinite extensions of the leximin criterion both on the basis of Hammond Equity (cf., Asheim and Tungodden [3], Bossert et al. [4]) and of adapted versions of the Harm Principle. Nevertheless, [9] shows that in the evaluation of infinite streams by orderings, anonymity, the strong Pareto axiom, and preference continuity properties are incompatible with full non-interference. Restricting ourselves to a finite economy, Mariotti and Veneziani [10] prove that a fully liberal non-interference view of the society leads to dictatorship if weak Pareto optimality is imposed.

We intend to explore the consequences of adding *standard* continuity properties to non-interference. Our main interest lies on the case of infinite utility streams but we also state some paralel implications for the case of finitely-lived societies. The results above inform us of trivial incompatibilities that derive from lack of continuity of the leximin/leximax criteria. We investigate more accurate reasons for the conflict among non-interference, optimality, the equal treatment of the generations and continuity in the evaluation of infinity utility streams. Then we elaborate on more general views of non-interference that scarcely provide some routes of escape to the generalised impossibilities that arise.

2 Notation and axioms

Let $\mathbf{X}$ denote a subset of $\mathbb{R}^{\mathbb{N}}$, that represents a domain of utility sequences or infinite-horizon utility streams. We adopt the usual notation for such utility streams: $\mathbf{x} = (x_1, ..., x_n, ...) \in \mathbf{X}$. By $(y)_{con}$ we mean the constant sequence $(y, y, ...)$, and $(x_1, ..., x_k, (y)_{con}) = (x_1, ..., x_k, y, y, ...)$ denotes an eventually constant sequence. We write $\mathbf{x} \geqslant \mathbf{y}$ if $x_i \geqslant y_i$ for each $i = 1, 2, ...$, and $\mathbf{x} \gg \mathbf{y}$ if $x_i > y_i$ for each $i = 1, 2, ...$ Also, $\mathbf{x} > \mathbf{y}$ means $\mathbf{x} \geqslant \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$.

Denote by ${}_1\mathbf{x}_{H-1} = (x_1, ..., x_{H-1})$ the H-head of $\mathbf{x} \in \mathbf{X}$, and denote by ${}_T\mathbf{x} = (x_T, x_{T+1}, ...)$ its T-tail, thus $\mathbf{x} = {}_1\mathbf{x} = ({}_1\mathbf{x}_{n-1}, {}_n\mathbf{x})$ for each $n \in \mathbb{N}$.

The notation above can be trivially adapted to domains of finite-length utility sequences. Also, a *social welfare relation* (SWR) is a binary relation $\succcurlyeq$ on $\mathbf{X} = \mathbb{R}^n$ with $n \in \mathbb{N} \cup \{+\infty\}$. Its asymmetric part is denoted by $\succ$, and its symmetric part is denoted by $\sim$. If $\succcurlyeq$ is an ordering (i.e., complete and transitive) then we call it a social welfare ordering or SWO. We are concerned with two sets of axioms of different nature on SWRs. We proceed to state and discuss them.

Firstly we introduce some equity axioms of two different classes for a reflexive binary relation $\succcurlyeq$ on $\mathbf{X} = [0,1]^{\mathbb{N}}$, with asymmetric part $\succ$. They can be easily regarded as axioms on $[0,1]^n$ with $n \in \mathbb{N}$ too as is dutifully clarified along the exposition.

Anonymity is the usual "equal treatment of all generations" postulate à-la-Sidgwik and Diamond.

Axiom AN (*Anonymity*). Any finite permutation of a utility stream produces a utility stream with the same social utility

We now recall a consequentialist equity axiom that implements preference for egalitarian allocations of utilities among generations in various senses. Axiom HE below states that in case of a conflict between two generations, every other generation being as well off, the stream where the least favoured generation is better off must be weakly preferred.

Axiom HE (*Hammond Equity*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are such that $x_j > y_j > y_k > x_k$ for some $j, k \in \mathbb{N}$, and $x_t = y_t$ when $j \neq t \neq k$, then $\mathbf{y} \succcurlyeq \mathbf{x}$.

In a different vein, Mariotti and Veneziani [10,11] introduce non-interference conditions in the context of a finite society. Under additional requirements they are intimately related to HE (cf., Mariotti and Veneziani [11, p. 127]). Their infinite counterparts are deeply analyzed in Lombardi and Veneziani [8,9]. We proceed to recall these versions for infinite streams:

Axiom HP (*Harm Principle*). Suppose $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are eventually coincident and $\mathbf{x} \succ \mathbf{y}$. Consider two streams $\mathbf{x}', \mathbf{y}'$ such that: for some $i \in \mathbb{N}$, $j \neq i$ implies $x'_j = x_j$ and $y'_j = y_j$. If $x'_i < x_i$ and $y'_i < y_i$ then $x'_i > y'_i$ implies $\mathbf{x}' \succ \mathbf{y}'$.

In case that only $\mathbf{x}' \succcurlyeq \mathbf{y}'$ is ensured in the definition above, we speak of Weak Harm Principle. Both versions of the principle coincide for weakly dominant SWOs on $\mathbf{X} = [0,1]^{\mathbb{N}}$.

Axiom IBP (*Individual Benefit Principle*). Suppose $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are eventually coincident and $\mathbf{x} \succ \mathbf{y}$. Consider two streams $\mathbf{x}', \mathbf{y}'$ such that: for some $i \in \mathbb{N}$,

$j \neq i$ implies $x'_j = x_j$ and $y'_j = y_j$. If $x'_i > x_i$ and $y'_i > y_i$ then $x'_i > y'_i$ implies $\mathbf{x}' \succ \mathbf{y}'$.

In case that only $\mathbf{x}' \succcurlyeq \mathbf{y}'$ is ensured in the definition above, we speak of Weak Individual Benefit Principle. Both versions of the principle coincide for weakly dominant SWOs on $\mathbf{X} = [0,1]^{\mathbb{N}}$. The respective versions for finite-length streams are the same except in that the restriction of the thesis to eventually coincident vectors does not apply.

We intend to account for some kind of efficiency too. Various axioms capture the general principle that with respect to a given infinite utility stream, adequate changes must improve the social welfare if every generation is at least as well off after the change. The *Weak Dominance* axiom captures the following spirit: Improving the welfare of *exactly one* generation suffices to improve the social welfare. In turn, the *Weak Pareto* axiom requests that *all* generations increase their utility for the social welfare to improve. The *Strong Pareto* axiom imposes that if *at least one* generation increases its utility then the social welfare must improve thus Strong Pareto and Weak Dominance coincide over sets of finite-length vectors. Formally:

Axiom WD (*Weak Dominance*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and there is $j \in \mathbb{N}$ such that $x_j > y_j$, and $x_i = y_i$ for all $i \neq j$, then $\mathbf{x} \succ \mathbf{y}$.

Axiom WP (*Weak Pareto*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $\mathbf{x} \gg \mathbf{y}$, then $\mathbf{x} \succ \mathbf{y}$.

Axiom SP (*Strong Pareto*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $\mathbf{x} > \mathbf{y}$ then $\mathbf{x} \succ \mathbf{y}$.

Another relaxed form of Strong Pareto that is unrelated to either WP or WD is the uncontroversial Monotonicity.

Axiom M (*Monotonicity*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $\mathbf{x} \geqslant \mathbf{y}$ then $\mathbf{x} \succcurlyeq \mathbf{y}$.

Finally, we list some semicontinuity properties. Below we discuss how they adapt to the case $\mathbf{X} = \mathbb{R}^n$, $n \in \mathbb{N}$. For a reflexive binary relation $\succcurlyeq$ on $\mathbf{X} = \mathbb{R}^{\mathbb{N}}$, the following definitions apply:

Axiom RUSC (*Restricted upper semicontinuity with respect to the sup topology*). For each $\mathbf{x} \in \mathbf{X}$ eventually constant, $\{\mathbf{y} \in \mathbf{X} : \mathbf{y} \succcurlyeq \mathbf{x}\}$ is closed with respect to the sup topology.

Axiom RLSC (*Restricted lower semicontinuity with respect to the sup topology*). For each $\mathbf{x} \in \mathbf{X}$ eventually constant, $\{\mathbf{y} \in \mathbf{X} : \mathbf{x} \succcurlyeq \mathbf{y}\}$ is closed with respect to the sup topology.

In general, the sup topology is finer than the product topology but when $\mathbf{X} = \mathbb{R}^n$ with $n \in \mathbb{N}$, both topologies coincide with the Euclidean topology. Also in this context, RUSC/RLSC are the ordinary USC/LSC.

We say that P, an *asymmetric* binary relation, verifies Restricted A-Upper (resp., A-Lower) semicontinuity or RAUSC (resp., RALSC), when for each $\mathbf{x} \in \mathbf{X}$ eventually constant, $\{\mathbf{y} \in \mathbf{X} : \mathbf{x}P\mathbf{y}\}$ (resp., $\{\mathbf{y} \in \mathbf{X} : \mathbf{y}P\mathbf{x}\}$) is open with respect to the sup topology. In case that $\succcurlyeq$ is an ordering, $\succcurlyeq$ is RUSC (resp., RLSC) if and only if $\succ$ is RAUSC (resp., RALSC).

3 Impossibility results for semicontinuous relations

In this Section we first produce various impossibility results for social welfare relations with non-interference properties on $\mathbf{X} = [0, 1]^n$, a setting where SP and WD are the same property. Afterwards we show that they naturally translate into results on $\mathbf{X} = [0, 1]^{\mathbb{N}}$. A cardinal variant of the analysis completes this Section.

3.1 The case of a finite society

In this context it is known that AN, SP, and IBP (resp., either HE or HP) characterize the extensions of the leximax (resp., leximin): cf., Mariotti and Veneziani [10, Proposition 2, 3]. Let us recall the definitions of these orderings. For any $x \in \mathbb{R}^n$, let $\bar{x}$ denote the permutation of x whose components $\bar{x}_1, .., \bar{x}_n$ are ranked in ascending order. The leximin ordering $\succcurlyeq^{LM}$ is defined by: $x \succ^{LM} y$ if and only if either $\bar{x}_1 > \bar{y}_1$ or there exists $l > 1$ such that $\bar{x}_1 = \bar{y}_1$, ... , $\bar{x}_{l-1} = \bar{y}_{l-1}$, $\bar{x}_l > \bar{y}_l$. The leximax ordering $\succcurlyeq^{LX}$ is defined by: $x \succ^{LX} y$ if and only if either $\bar{x}_n > \bar{y}_n$ or there exists $l < n$ such that $\bar{x}_n = \bar{y}_n$, ... , $\bar{x}_{l+1} = \bar{y}_{l+1}$, $\bar{x}_l > \bar{y}_l$.

It is now trivial that AN, SP, HP, and IBP are incompatible properties for a SWO on $\mathbf{X} = [0, 1]^n$ when $n > 1$. This impossibility is avoided if AN is dropped and SP is relaxed to M plus WP (as any dictatorship by a generation proves). Relaxing SP to WP produces incompatibility too: Mariotti and Veneziani [10, Theorem 1] prove that WP, HP and IBP together entail dictatorship by a generation, which violates AN. As has been said, dropping either HP or IBP produces compatibility.

Since the extensions of the leximax (resp., leximin) do not verify lower (resp., upper) semicontinuity with respect to the sup topology, trivial impossibility consequences follow. [2] In this Subsection we clarify the extent of the conflict among non-interference arguments, Pareto optimality, and semicontinuity by proving that (a) AN plays no role in such incompatibilities, and (b) if a very mild reinforcement of M replaces WD then we still obtain conflicting axiomatics.

Regarding our first purpose, the following Propositions 1 and 2 are in order:

Proposition 1 *There is no* $\succcurlyeq$ *SWO on* $\boldsymbol{X} = [0,1]^2$ *satisfying IBP, WD, and LSC.*

Proof: We prove that the combination of properties in the statement conveys an absurd conclusion. Let us first show $(0,1) \succcurlyeq (\frac{1}{2}, 1-\frac{1}{i})$ for each $i = 2,3,...$. Suppose $(\frac{1}{2}, 1-\frac{1}{i_0}) \succ (0,1)$ for some $i_0 \in \{2,3,...\}$. An appeal to IBP yields $(1, 1-\frac{1}{i_0}) \succ (1-\frac{1}{m}, 1)$ for each $m = 2,3,...$. Now LSC entails $(1, 1-\frac{1}{i_0}) \succcurlyeq (1,1)$, contradicting WD.

With respect to the sup topology, $\{(\frac{1}{2}, 1-\frac{1}{i})\}_i$ converges to $(\frac{1}{2}, 1)$ thus LSC entails $(0,1) \succcurlyeq (\frac{1}{2}, 1)$, contradicting WD. □

Proposition 2 *There is no* $\succcurlyeq$ *SWO on* $\boldsymbol{X} = [0,1]^2$ *that verifies HP, WD, and USC.*

Proof: We prove that the combination of properties in the statement conveys an absurd conclusion. Let us first show $(\frac{1}{i}, \frac{1}{2}) \succcurlyeq (0,1)$ for each $i = 2,3,...$. Suppose $(0,1) \succ (\frac{1}{i_0}, \frac{1}{2})$ for some $i_0 \in \{2,3,...\}$. An appeal to HP yields $(0, \frac{1}{m}) \succ (\frac{1}{i_0}, 0)$ for each $m = 2,3,...$. Now USC entails $(0,0) \succcurlyeq (\frac{1}{i_0}, 0)$, contradicting WD.

With respect to the sup topology, $\{(\frac{1}{i}, \frac{1}{2})\}_i$ converges to $(0, \frac{1}{2})$ thus USC entails $(0, \frac{1}{2}) \succcurlyeq (0,1)$, contradicting WD. □

Regarding objective (b), we preliminarily explore the intimate relationship between the Harm Principle and Hammond Equity. This reveals another conflict between HP and RUSC in the presence of AN and a very mild reinforcement of M, which bears comparison with the conclusion in Proposition 2.

Mariotti and Veneziani [11, p. 127] state that when $\mathbf{X} = [0,1]^n$, $n \in \mathbb{N}$, HP

[2] Consider the case of the leximax. For each $i \in \mathbb{N}$ let $\mathbf{x}^{(i)} = (1-\frac{1}{i}, \frac{1}{2})$. With respect to the sup topology, $\mathbf{x}^{(i)}$ converges to $\mathbf{x} = (1, \frac{1}{2})$. However $(1,0) \succ^{LX} \mathbf{x}^{(i)}$ and $\mathbf{x} \succ^{LX} (1,0)$.
Now consider the case of the leximin. For each $i \in \mathbb{N}$ let $\mathbf{x}^{(i)} = (\frac{1}{i}, \frac{1}{2})$. With respect to the sup topology, $\mathbf{x}^{(i)}$ converges to $\mathbf{x} = (0, \frac{1}{2})$. However $\mathbf{x}^{(i)} \succ^{LM} (0,1)$ and $(0,1) \succ^{LM} \mathbf{x}$.

and HE are equivalent in the presence of AN and WD/SP. Proposition 3 below shows that relying on M alone permits to deduce the egalitarian HE from HP. The argument is exported to the case of infinitely-lived sociecties in subsection 3.2 below.

Proposition 3 *Let* $\succcurlyeq$ *be a SWO over* $\boldsymbol{X} = [0,1]^n$ *for some* $n \in \{2, 3, ...\}$. *If* $\succcurlyeq$ *verifies AN, HP and M then* $\succcurlyeq$ *verifies HE.*

Proof: Let $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ be such that $x_j > y_j > y_k > x_k$ for some $j, k \in \{1, 2, ..., n\}$, and $x_t = y_t$ when $j \neq t \neq k$. By contradiction, assume $\mathbf{x} \succ \mathbf{y}$. Due to AN we can fix $j = 1$, $k = 2$ thus ${}_3x = {}_3y$. Consider the vectors $\mathbf{x}' = (\frac{x_2+y_2}{2}, {}_2x)$ and $\mathbf{y}' = (x_2, y_2, {}_3x)$. They are obtained from $\mathbf{x}$ and $\mathbf{y}$ by reducing the endowment of the first generation to $\frac{x_2+y_2}{2}$ and x_2 respectively. Observe $\frac{x_2+y_2}{2} > x_2$.

An appeal to HP and then AN yields $\mathbf{x}' \succ \mathbf{y}' \sim (y_2, x_2, {}_3y)$. But this entails $\mathbf{x}' \succ (y_2, x_2, {}_3y) \succcurlyeq \mathbf{x}'$ due to M, a contradiction. □

Proposition 4 below yields the subsequent Corollary 1:

Proposition 4 *Let* $\succcurlyeq$ *be a SWR on* $\boldsymbol{X} = [0,1]^2$. *Suppose*

$$\exists\, \mathbf{x} \in \boldsymbol{X} \text{ such that } \mathbf{x} \succcurlyeq \mathbf{y} = (y_1,\, x_2) \text{ is false and } y_1 > x_1 > x_2 \tag{1}$$

Then $\succcurlyeq$ *does not verify HE and RUSC simultaneously.*[3]

Proof: By contradiction. Define $\mathbf{y}^{(n)}$ according to: $y_1^{(n)} = x_1$, $y_2^{(n)} = x_2 + \frac{1}{n}$. With respect to the sup topology, $\mathbf{y}^{(n)}$ converges to $\mathbf{x}$. For each $n > \frac{1}{x_1 - x_2}$, HE entails $\mathbf{y}^{(n)} \succcurlyeq \mathbf{y}$ because $y_1 > x_1 = y_1^{(n)} > x_2 + \frac{1}{n} = y_2^{(n)} > x_2$. This means $\mathbf{x} \succcurlyeq \mathbf{y}$ due to RUSC, contradicting the assumption. □

Corollary 1 *There is no* $\succcurlyeq$ *SWO on* $\boldsymbol{X} = [0,1]^2$ *that verifies AN, HP, RUSC and M plus condition* (1) *above.*

Proof: By Proposition 3, $\succcurlyeq$ verifies HE. Now Proposition 4 applies. □

Our last result in this regard replicates the incompatibility shown by Corollary 1 in terms of IBP. The reader is invited to mimick its proof in order to give a direct argument for Corollary 1 that circumvents Propositions 3 and 4.

Proposition 5 *There is no* $\succcurlyeq$ *SWO on* $\boldsymbol{X} = [0,1]^2$ *that verifies AN, IBP, RLSC and M plus the following condition* (2):

$$\exists\, \mathbf{x} \in \boldsymbol{X} \text{ such that there are } x_2 > x_1 > y_1 \text{ for which } \mathbf{x} \succ \mathbf{y} = (y_1,\, x_2) \tag{2}$$

[3] Observe that condition (1) holds under e.g., WD/SP.

Proof: By contradiction. Define $\mathbf{x}^{(n)} \in \mathbf{X}$ according to: $x_1^{(n)} = x_1$, $x_2^{(n)} = x_2 - \frac{1}{n}$. With respect to the sup topology, $\mathbf{x}^{(n)}$ converges to $\mathbf{x}$. Thus there is n_0 such that $\mathbf{x}^{(n)} \succ \mathbf{y}$ when $n > n_0$, due to RLSC.

Select $m > n_0$ such that $x_2 - \frac{1}{2m} > y_1$. An appeal to AN and then IBP yields

$$(x_2 - \frac{1}{m}, x_2) \sim (x_2, x_2 - \frac{1}{m}) \succ (x_2 - \frac{1}{2m}, x_2)$$

because $x_1 < x_2$, $y_1 < x_2 - \frac{1}{2m}$, $x_2 - \frac{1}{2m} < x_2$ and $\mathbf{x}^{(m)} = (x_1, x_2 - \frac{1}{m}) \succ \mathbf{y} = (y_1, x_2)$. This conclusion contradicts M. □

3.2 The case of an infinitely-lived society

Most of the arguments in the preceding subsection carry forward to the case of infinite sequences of utilities. In this subsection we discuss the details.

In order to convert Propositions 1 and 2 into statements for infinitely-lived societies, it suffices to appeal to the following relaxed version of WD:

Axiom RWD (*Restricted Weak Dominance*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are eventually constant, and there is $j \in \mathbb{N}$ such that $x_j > y_j$, and $x_i = y_i$ for all $i \neq j$, then $\mathbf{x} \succ \mathbf{y}$.

By adding a (0_{con}) tail to the vectors in use along the proofs of Propositions 1 and 2 one gets:

Proposition 6 *There is no $\succcurlyeq$ SWO on $\boldsymbol{X} = [0,1]^{\mathbb{N}}$ that verifies IBP (res., HP), RWD, and RLSC (resp., RUSC).*

The reader can easily mimick the arguments in Propositions 3 and 5 to produce the following twin statement for infinitely-lived societies: [4]

Proposition 7 *Let $\succcurlyeq$ be a SWO over $\boldsymbol{X} = [0,1]^{\mathbb{N}}$. If $\succcurlyeq$ verifies AN, HP and either WD or M then $\succcurlyeq$ verifies HE.*

Proposition 8 *There is no anonymous $\succcurlyeq$ SWO on $\boldsymbol{X}$ satisfying either of the following sets of conditions:*

(a) HP, RUSC, and either RWD or M plus condition $(1')$ *below:*

$\exists\, \mathbf{x} \in \boldsymbol{X}$ eventually constant such that there are $y_1 > x_1 > x_2$ for which $\mathbf{x} \succ \mathbf{y} = (y_1, {}_2x)$ $\qquad (1')$, or

[4] Proposition 8 is reexplored in Subsection 3.3 below.

(b) IBP, RLSC, and either RWD or M plus condition (2′) *below:*

$\exists\, \mathbf{x} \in \boldsymbol{X}$ eventually constant such that there are $x_2 > x_1 > y_1$ for which $\mathbf{x} \succ \mathbf{y} = (y_1, {}_2x)$ (2′)

With respect to Proposition 6, Proposition 8 brings to light an incompatibility under another very mild version of Paretianity –in fact, an extremely weak strengthening of the unavoidable monotonicity– when AN is imposed.

3.3 Revisiting the analysis under a cardinal perspective

Now we check for possible changes in the analysis above when well-beings are universally comparable and cardinally measurable. We consider the following cardinal forms of the non-interference principles whose implications we have inspected thus far:

Axiom IEHP (*Individual Equal Harm Principle*). Suppose $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are eventually coincident and $\mathbf{x} \succ \mathbf{y}$. Consider two streams $\mathbf{x}', \mathbf{y}'$ such that: for some $i \in \mathbb{N}$, $j \neq i$ implies $x'_j = x_j$ and $y'_j = y_j$. If $x'_i = x_i - \varepsilon$ and $y'_i = y_i - \varepsilon$ for some $\varepsilon > 0$ then $\mathbf{x}' \succ \mathbf{y}'$.

This axiom is not a direct descendant of the Harm Principle but it captures a related idea: If a stream is socially better than another one and the welfare of a given generation is reduced *by the same amount* in both distributions then the relative ranking of the resulting streams is the same as in their pre-reduction comparison. We believe that this is an appealing invariance behavior. It claims that an *equal* deprivation to one generation that does not affect any other generation should not be a cause for reconsidering the social judgement over distributions. The "individual" relative rankings that the distributions convey –x_i vs. y_i, x'_i vs. y'_i– is immaterial and only the fact that the penalisation is the same matters. A similar defense holds for the other side of the coin:

Axiom IEBP (*Individual Equal Benefit Principle*). Antecedent as in IEHP, thesis as follows: If $x'_i = x_i + \varepsilon$ and $y'_i = y_i + \varepsilon$ for some $\varepsilon > 0$ then $\mathbf{x}' \succ \mathbf{y}'$.

The respective versions for finite-length streams are the same except in that the restriction of the conclusion to eventually coincident vectors does not apply. We do not explore this context in depth because to the effect of comparing the ordinal and cardinal positions, we just need to observe that summing up the components is a WD/SP, AN, IEHP, IEBP, continuous with respect to the sup topology evaluation.

Let us therefore focus on infinitely-lived societies.

1) The reader can ellaborate on Proposition 3 in order to check that if a SWO is AN, IEHP, and either WD or M then it verifies the Weak Pigou-Dalton transfer principle (WPDT) as defined e.g., in Hara et al. [7, p. 185]. It is remarkable that in the presence of a procedural equity axiom like AN, cardinal non-interference implies a behavior that embodies a preference for egalitarian distribution of utilities among generations.

2) A possibility result emerges from Proposition 6 by replacing HP/IBP with their cardinal variants above. This reduces to check that discounted utilitarianism agrees with both IEHP and IEBP, as well as being SP, RUSC, and RLSC, and representable. By contrast with the case of general non-interference, utilitarianism can be reconciled with a cardinal approach to these principles.

3) It is less obvious that the conclusion in Proposition 8 does not vary if IEHP replaces HP in case (a), and IEBP replaces IBP in case (b). Proposition 10 below in particular leads to such consequence although in order to avoid distracting technicalities we state and prove a clear-cut statement and then comment upon other variations. Irrespective of the exact conclusion this shows that for infinitely-lived societies, the equal treatment of all generations is a cause for incompatibilities with cardinal non-interference principles under mild consistency, efficiency and continuity requirements. In order to state Propositions 9 and 10, the following properties are needed:

Axiom AEHP (*Anonymous Equal Harm Principle*). Suppose $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are eventually coincident and $\mathbf{x} \succ \mathbf{y}$. Consider two streams $\mathbf{x}', \mathbf{y}'$ such that: for some $i, k \in \mathbb{N}$, $j \neq i$ implies $x'_j = x_j$, and $j \neq k$ implies $y'_j = y_j$.
If $x'_i = x_i - \varepsilon$ and $y'_k = y_k - \varepsilon$ for some $\varepsilon > 0$ then $\mathbf{x}' \succ \mathbf{y}'$.

Axiom AEBP (*Anonymous Equal Benefit Principle*). Antecedent as in AEHP, thesis as follows: If $x'_i = x_i + \varepsilon$ and $y'_k = y_k + \varepsilon$ for some $\varepsilon > 0$ then $\mathbf{x}' \succ \mathbf{y}'$.

Observe that SWOs that satisfy AN and IEHP (resp., IEBP) verify AEHP (resp., AEBP) too. Therefore AEHP and AEBP capture consequences of the equal treatment of all generations together with cardinal non-interference under sufficiently consistent evaluations.

Proposition 9 *There is no asymmetric SWR on* $\boldsymbol{X}$ *that verifies AEHP (resp., AEBP), RAUSC (resp., RALSC), and RWD.*

Proof: We proceed by contradiction. Suppose $\succ$ is asymmetric and verifies AEBP, RALSC, and RWD. Using RWD we select $\mathbf{x} = (x_1, (x_2)_{con}) \succ (y_1, (x_2)_{con}) = \mathbf{y}$ with $x_2 > x_1 > y_1$.

For each $n > \frac{1}{x_2}$, define $\mathbf{x}^{(n)} \in \mathbf{X}$ according to: $x_1^{(n)} = x_1$, $x_k^{(n)} = x_2 - \frac{1}{n}$ for

$k = 2, 3, ...$ With respect to the sup topology, $\mathbf{x}^{(n)}$ converges to $\mathbf{x}$. Thus there is $n_0 > \frac{1}{x_2}$ such that $\mathbf{x}^{(n)} \succ \mathbf{y}$ when $n > n_0$, due to RALSC. We fix $m > n_0$ with the additional property $m > \frac{1}{2(1-(x_1-y_1))}$. Because $2m - 2m(x_1 - y_1) > 1$ there is $T \in \mathbb{N} \cap (\, 2m(x_1 - y_1), 2m\,)$ thus $1 > \frac{T}{2m} > x_1 - y_1$.

Because $\mathbf{x}^{(m)} \succ \mathbf{y}$, an appeal to AEBP yields

$$(x_1, x_2 - \frac{1}{m} + \frac{1}{2m}, (x_2 - \frac{1}{m})_{con}) = (x_1, x_2 - \frac{1}{2m}, (x_2 - \frac{1}{m})_{con}) \succ (y_1 + \frac{1}{2m}, (x_2)_{con})$$

From the latter expression we obtain

$$(x_1, x_2 - \frac{1}{2m}, x_2 - \frac{1}{m} + \frac{1}{2m}, (x_2 - \frac{1}{m})_{con}) \succ (y_1 + \frac{1}{2m} + \frac{1}{2m}, (x_2)_{con})$$

by appealing to AEBP again. An iterative argument leads to

$$(x_1, x_2 - \frac{1}{2m}, \overset{T}{\ldots\ldots}, x_2 - \frac{1}{2m}, (x_2 - \frac{1}{m})_{con}) \succ (y_1 + \frac{T}{2m}, (x_2)_{con})$$

RWD yields $(y_1 + \frac{T}{2m}, (x_2)_{con}) \succ (x_1, x_2 - \frac{1}{2m}, \overset{T}{\ldots\ldots}, x_2 - \frac{1}{2m}, (x_2 - \frac{1}{m})_{con})$, which contradicts asymmetry of $\succ$.

The other instance of the statement is proven by a simple modification of the argument above. □

With a tedious modification of the previous technique it is direct to prove the following variation that generalizes Proposition 8:

Proposition 10 *There is no SWO on* $\boldsymbol{X}$ *satisfying either of:*

(a) AEHP, RUSC, and either RWD or M plus condition $(1')$*, or*

(b) AEBP, RLSC, and either RWD or M plus condition $(2')$*.*

Remark 1 *The results in this Subsection can be confronted with Asheim and Tungodden [3, Prop. 4, 5]. They show that a property related to the conjunction of IEHP and IEBP, namely 2-Generation Unit Comparability or 2UC, permits to characterize utilitarian overtaking criteria. In particular they prove the existence of reflexive and transitive relations with SP, AN, 2UC and two respective forms of preference continuity. It is simple to check that overtaking not only verifies SP, AN, and Weak Preference Continuity (cf., [3, Prop. 5]), but also IEHP and IEBP. This speaks for the strong restrictions that weak semicontinuity axioms –in the usual sense– impose to anonymous equal harm/benefit behaviors.*

Remark 2 *In the context of a finite society, Mariotti and Veneziani [10] state a property in line with the conjunction of adapted versions of IEHP and IEBP,* *namely, Uniform Additive Non-Interference. They prove that SWOs that verify SP, AN, and Uniform Additive Non-Interference only deviate from the utilitarian ordering in comparisons between indifferent elements for the utilitarian rule.*

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