On the derivatives of generalized Gegenbauer polynomials

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Abstract

We prove some new formulae for the derivatives of the generalized Gegenbauer polynomials associated to the Lie algebra $A_2$.

As it is well known [1], the classical Gegenbauer polynomials $C_m(z)$ suffer, when differentiated in $z$, a shift in the parameter, namely

$$\frac{dP_m}{dz} = m P_m^{+1};$$
$$P_m(z) = \frac{m!}{(\frac{m}{2})_m} C_m(\frac{z}{2}); \quad (\frac{m}{2})_m = ( + 1)\cdots( + m 1)$$

The classical Gegenbauer polynomials are (up to a factor) the eigenfunctions of the simplest quantum Calogero-Sutherland Hamiltonian $[2],[3], [4], [5], [6], [7], [8], [9]$, that related to the Lie algebra $A_1$. It is the purpose of this note to show that the same shift in takes place in the derivatives of the generalized Gegenbauer polynomials $P_{m,n}(z_1;z_2)$ giving the quantum eigenfunctions of the Calogero-Sutherland system with Lie algebra $A_2$:

$$P_{m,n} = \binom{m}{n} P_{m,n}(z_1;z_2);$$
$$P_{m,n} = z_1^n z_2^m + \text{lower terms};$$
$$= (z_1^2 - 3z_2)@_{z_1}^2 + (z_2^2 - 3z_1)@_{z_2}^2 + (z_1 z_2 - 9)@_{z_1} @_{z_2} + (3 + 1)(z_1 @_{z_1} + z_2 @_{z_2})$$

$$\binom{m}{n} = m^2 + n^2 + mn + 3 (m + n);$$

see [3], [4],[5],[6],[7],[8]. Specifically, we will prove the following formulae:

$$\frac{\partial P_{m,n}}{\partial z_1} = m P_m^{+1} + A_{m,n}( ) P_{m,n}^{+1} + B_{m,n}( ) P_{m,n}^{+1}$$ (1)
$$\frac{\partial P_{m,n}}{\partial z_2} = n P_m^{+1} + A_{n,m}( ) P_{m,n}^{+1} + B_{n,m}( ) P_{m,n}^{+1}$$ (2)

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where

\[
A_{m,n}(x) = \frac{m \left((m+1)n(m+n+1)(m+n+1)\right)}{(m+1)(m+1)(m+n+1)(m+n+2)(m+n+2)}
\]

\[
B_{m,n}(x) = \frac{n(n+1)(m+n+1)}{(n+1)(n+1)}
\]

Consider (1). The proof of this formula proceeds by induction on the second quantum number. The generating function for the Jack polynomials \(P_{m,n}\) is known to be [3]

\[
(1 - z_1 t + z_2 t^2 - t^3) = \frac{x^t}{m!} P_{m,n} t^n.
\]

Differentiation of this expression shows the validity of (1) when \(n = 0\). On the other hand, we can use the recurrence relations for the generalized Gegenbauer polynomials [1] to express \(P_{m,n}\) in terms of polynomials with lower \(n\):

\[
P_{m,n} = z_2 P_{m,n-1} - \sum_{k=n}^{\min(m,n)} a_{m,n-1} P_{m,k} c_{n,k} P_{m+2,k-1}
\]

with

\[
a_{m,n}(x) = \frac{m(n+n+1)(n+1+2)(n+m+1+3)}{(n+1)(n+1)(n+m+2)(n+m+1+2)}
\]

\[
c_{n}(x) = \frac{n(n+1+2)}{(n+1)(n+1)}
\]

Differentiating (5) with respect to \(z_1\) under the assumption that (1) is valid when the second quantum number is lower than \(n\), and applying the recurrence relation (4) to get rid of the remaining \(z_2\) factors, we obtain:

\[
\frac{\partial P_{m,n}}{\partial z_1} = m P_{m+1,n} + 1
\]

\[
\left[ A_{m,n} 1() a_{m,n} 2() + ( + 1) A_{m,n} 1() a_{m,n} 1() \right] P_{m+1,n} 2
\]

\[
\left[ A_{m,n} 1() + m a_{m,n} 1() + ( + 1) (m 1) a_{m,n} 1() \right] P_{m+2,n} 1
\]

\[
\left[ B_{m,n} 1() c_{n,1} 1() + m c_{n,1} 1() + ( + 1) \right] P_{m+1,n} 2
\]

\[
\left[ B_{m,n} 1() c_{n,1} 3() + ( + 1) B_{m+1,n} 2() c_{n,1} 1() \right] P_{m+1,n} 4
\]

\[
\left[ a_{m,n} 1() B_{m,n} 1() + a_{m,n} 3() + ( + 1) B_{m,n} 1() \right]
\]

\[
+ A_{m,n} 1() c_{n,2} 1() + ( + 1) A_{m+1,n} 2() c_{n,1} 1() \right] P_{m+1,n} 3
\]

and by explicit use of (3) and (6), we nd:

\[
A_{m,n} 1() a_{m,n} 2() + ( + 1) A_{m,n} 1() a_{m,n} 1() = 0
\]

\[
B_{m,n} 1() c_{n,3} 1() + ( + 1) B_{m+1,n} 2() c_{n,1} 1() = 0
\]

\[
a_{m,n} 1() B_{m,n} 1() + a_{m,n} 3() + ( + 1) B_{m,n} 1() 
\]

\[
+ A_{m,n} 1() c_{n,2} 1() + ( + 1) A_{m+1,n} 2() c_{n,1} 1() = 0
\]
and

\[ A_{m,n}^{1} + m A_{m+1,n}^{1} = A_{m+1,n}^{1} + (m+1)A_{m,n}^{1} = B_{m,n}^{1} + m B_{m+1,n}^{1} = B_{m,n}^{1} + (m+1)C_{n+1}^{1} + m C_{n+1}^{1} = B_{m,n}^{1} \]

which establishes the desired result. The proof of (2) takes advantage of the two recurrence relation to (3), see [7], and is completely analogous. In conclusion we would like to mention that the approach of this note may be used also for the \( A_n \) case. We hope to return to this problem in the future.

Acknowledgments

We are grateful to Prof. M. Lorente for interesting discussions. One of the authors (A. M. P.) would like to express his gratitude to the Department of Physics of the University of Oviedo for the hospitality during his stay as a Visiting Professor.

References