Explicit computations of low lying eigenfunctions for the quantum trigonometric Calogero-Sutherland model related to the exceptional algebra $E_7$

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Abstract

In a previous paper [1] we have studied the characters and Clebsch-Gordan series for the exceptional Lie algebra $E_7$ by relating them to the quantum trigonometric Calogero-Sutherland Hamiltonian with coupling constant $= 1$. Now we extend that approach to the case of general.

1 Introduction

The Calogero-Sutherland models [2,3] related to the root systems of the simple Lie algebras $[4,5,6]$ have been deeply investigated during the last two decades. Originally introduced on purely theoretical grounds, this class of models have however found a number of relevant applications in such diverse fields as condensed matter physics, supersymmetric Yang-Mills theory or black-hole physics. On the mathematical side, an interesting feature of the quantum version of this kind of models is that their energy eigenfunctions provide a natural generalization of several types of hypergeometric functions to the multivariable case. For the potential $v(q) = (1)sin^2(q)$ and special values of the coupling constant, these eigenfunctions are related to some orthogonal functional systems of particular interest in the theory of Lie algebras and symmetric spaces: for $= 1$ we obtain the characters of the irreducible representations of the algebra, while for $= 0$ the corresponding monomial symmetric functions arise; other values of lead to zonal spherical functions in symmetric spaces associated to the Lie algebra; in particular, for $E_7$, $= 1/2$ gives these functions for the symmetric space $E_7$. The Calogero-Sutherland Hamiltonian appears in this way as a natural unified tool for the computation of all these objects.

The Calogero-Sutherland Hamiltonian associated to the root system of a simple Lie algebra can be written as a second-order differential operator whose variables are the characters of the fundamental representations of the algebra. As it was shown in the papers [6,9,10], and later in [11,12,13,14,15,16], this approach gives the possibility of developing some systematic procedures to solve the Schrodinger equation and determine in a portant properties of the eigenfunctions, such as recurrence relations or generating functions for some subsets of them. The approach has been used for classical algebras of $A_n$ and $D_n$ type, for the exceptional algebra $E_6$, and recently also for $E_7$ for the special value of the coupling constant for which the eigenfunctions are proportional to the characters of the irreducible representations of the algebra. The aim of this paper is to show how to generalize the treatment given in [13] to arbitrary values of the coupling constant and to extend some of the particular results found there to the general case.

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2 The Calogero-Sutherland Hamiltonian for E\(\gamma\) in Weyl-invariant variables

The trigonometric Calogero-Sutherland model related to the root system \(R\) of a simply-laced Lie algebra of rank \(r\) is the quantum system in an Euclidean space \(\mathbb{R}^r\) defined by the standard Hamiltonian operator

\[
H = \frac{1}{2} \sum_{j=1}^{r} X_j^2 + \sum_{\alpha \in R^+} (1) \sin^2 (\alpha q); \tag{1}
\]

where \(q = (q_j)\) is a cartesian coordinate system and \(p_j = \alpha \theta_{q_j} R\) is the set of the positive roots of the algebra, and \(\alpha\) is the coupling constant. The (non-normalized) ground state wave function is

\[
0(q) = \prod_{\alpha \in R^+} \sin (\alpha q); \tag{2}
\]

while the excited states are indexed by the highest weights \(\pi = \sum_{\alpha \in R^+} \pi^\alpha \) (\(R^+\) is the cone of dominant weights) of the irreducible representations of the algebra, that is, by the \(r\)-tuple of non-negative integers \(m = (m_1; \ldots; m_r)\). Looking for solutions \(m\) of the Schrödinger equation in the form

\[
m(q) = 0(q) m(q); \tag{3}
\]

we are led to the eigenvalue problem

\[
m = m(q) m(q); \tag{4}
\]

where \(m(q)\) is the linear differential operator

\[
\frac{1}{2} \sum_{j=1}^{r} X_j^2 + \sum_{\alpha \in R^+} \cot (\alpha q)(\alpha q); \tag{5}
\]

due to the Weyl symmetry of the Hamiltonian, to solve the eigenvalue problem \(m\) it is convenient to express the operator \(m\) in a set of independent Weyl-invariant variables such as \(z_k = \sum_{J=1}^{r} q_j\), the characters of the irreducible representations of the algebra. The operator \(m\) in the z-variables has the structure:

\[
m = X_{j,k} \theta_{z_k} \theta_{z_k} + \sum_{\alpha \in R^+} \alpha_j^\alpha \theta_{z_k} \theta_{z_k} + \theta_{z_k} \theta_{z_k}; \tag{6}
\]

but to \(X\) the coefficients by direct change of variables is very cumbersome. As explained in [4], a different procedure, based on the computation of the quadratic Clebsch-Gordan series and the second order characters of \(E\gamma\), is possible. In [4], we applied this procedure to compute \(a_{jk}(z)\) and \(b_{jk}(z)\) separately, thus finding the operator for the special case \( = 1\). The remaining task is to compute \(b_{jk}(z)\) and \(b_{jk}(z)\) separately.

To accomplish that task, a new piece of information is required: we need to know all the first-order symmetric monomials for \(E\gamma\) given as a function of the z-variables. To obtain them, we will rely on the expansions of the fundamental characters of \(E\gamma\) in terms of monomial functions computed by Loris and Sasakian [5]. In the notation of [5], these expansions are

\[
\begin{align*}
z_1 & = M_1 + 7; \\
z_2 & = M_2 + 6M_1; \\
z_3 & = M_3 + 5M_2 + 22M_1 + 77; \\
z_4 & = M_4 + 4M_3 + 15M_2 + 15M_1 + 45M_0 + 50M_1 + 145M_0 + 390M_1 + 980; \\
z_5 & = M_5 + 5M_4 + 21M_3 + 71M_2; \\
z_6 & = M_6 + 6M_5 + 27; \\
z_7 & = M_7;
\end{align*}
\]

To invert these formulas to compute the fundamental monomial functions, we have to proceed in increasing order of the height of the dominant weights associated to the characters. Once a first-order monomial function is known,
we compute the corresponding \( \ell^3 \) and \( \ell^3 \) following the procedure described in [1]. This makes it possible to use the part of the operator already known in each step to compute the second order monomial functions in advance, i.e. before they are needed to obtain the next fundamental monomial function. With this strategy, it is easy to nd

\[
M_1 = z_1^7; \\
M_2 = z_2^6 z_1; \\
M_3 = z_1^5 2z_2 + 8z_1 + 2; \\
M_4 = z_1^4 4z_2 z_3 + 9z_2^2 z_7 + 9z_1^3 + 9z_1^2 + 14z_3 + 39z_6 + 22z_1 + 18; \\
M_5 = z_5^5 5z_2^2 z_7 + 14z_2 + 15z_7; \\
M_6 = z_6 6z_1 + 15; \\
M_7 = z_7;
\]

and, therefore,

\[
\ell^3(z) + \ell^3(z) = 28 + 4z_1 + (28 + 68z_1) \\
\ell^3(z) + \ell^3(z) = 7z_2 + 24z_2 + (98z_2 + 24z_7) \\
\ell^3(z) + \ell^3(z) = 8 (56z_1 + 12z_3 + 20z_2 + (8 + 56z_1 + 132z_3 + 20z_6) \\
\ell^3(z) + \ell^3(z) = 72 + 72z_1 + 24z_2 + 24z_3 + 24z_4 + 16z_6 + 16z_7 + 24z_2 z_7 + 36z_7 + \\
(72 72z_1 + 24z_2 + 24z_3 + 192z_4 + 16z_6 + 16z_7 + 24z_2 z_7 + 36z_7) \\
\ell^3(z) + \ell^3(z) = 28z_2 + 15z_5 + 4z_7 + 20z_1 z_7 + (28z_2 + 150z_5 + 4z_7 + 20z_1 z_7) \\
\ell^3(z) + \ell^3(z) = 48 + 24z_1 + 8z_6 + (48 + 24z_1 + 104z_6) \\
\ell^3(z) + \ell^3(z) = 3z_1 + 54 z_7;
\]

This completes the computation of \( \ell^3 \). In the rest of the paper we present some results obtained through the use of this operator.

3 Some explicit results on the low lying eigenfunctions of the systems

In this section, we present some results on the first and second order polynomials and generalized quadratic Clebsch-Gordan series. Because some formulas are too long, we give the complete results, in a form suitable for use in Mathematica or Maple, in the adjoint files results31.txt (results35.txt), which are accessible through the "source" form at this document.

3.1 Second order monomial symmetric functions

Once we know \( \ell^3 \), we can compute its eigenfunctions by means of the iterative algorithm given in [1]. In particular, we can obtain the second order monomial symmetric functions for \( E_7 \) by simply taking \( \ell^3 = 0 \) in these algorithms. We present here the list of the second order monomial functions obtained in that way.

\[
\begin{align*}
M_{2000000} & = z_1^2 2z_3 2z_1^2 7 \\
M_{1100000} & = z_1 z_2 5z_5 + 3z_2 z_5 23z_7 \\
M_{0200000} & = z_2^2 2z_4 2z_2 z_7 2z_1^2 + 6z_2 + 14z_6 + 4z_1 + 12 \\
M_{1010000} & = z_1 z_3 3z_4 z_1 z_8 + 6z_2 z_7 + 3z_1^2 + 9z_1 + 9z_2 + 4z_6 + 20z_3 + 32 \\
M_{0110000} & = z_2 z_3 4z_2 z_5 + 5z_2 z_6 + 4z_1 z_7 4z_3 z_7 17z_2 z_5 16z_2 z_7 + 4z_5 + 13z_2 z_5 + 12z_5 + 7z_7 \\
M_{0020000} & = z_2^2 2z_1 z_4 + 2z_2 z_5 2z_1 z_8 7z_2^2 + 12z_2 z_7 2z_1^3 + 4z_1 z_3 + 2z_1 z_5 10z_4 + 2z_1 z_6 + 10z_1^2 + + 10z_1^2 16z_2 22z_6 + 16z_2 8 \\
M_{1001000} & = z_1 z_4 4z_2 z_4 4z_1 z_6 + 10z_2 z_4 + 9z_1 z_7 + 14z_2^2 34z_2 z_5 + 9z_1^2 21z_7^2 39z_2 z_3 21z_1 z_7^2 + + 66z_2 + 54z_1 z_6 + 23z_2 z_7 + 36z_2^2 22z_7^2 54z_3 24z_6 56z_1 24 \\
M_{0101000} & = z_2 z_4 3z_1 z_5 + 2z_1 z_2 z_6 2z_2 z_5 + 5z_1 z_3 z_5 + 5z_2 z_4 14z_4 z_7 19z_1 z_6 z_7 12z_2 z_5 + 15z_2 z_1
\end{align*}
\]
\( z^2 + 4z + 4 \)

This appears to be a polynomial equation. If you need further assistance or have any specific questions, feel free to ask!
### 3.2 Expansion of second order characters in monomial functions

As the name suggests, the orthogonal system of monomial symmetric functions is the simplest one among the different classes of symmetric polynomials associated to the Lie algebra $E_7$: each monomial symmetric function is nothing but the sum of all the monomials associated to one orbit of the Weyl group on the weight lattice. Now, we can easily expand other polynomials associated to the root system of $E_7$ in the basis of the monomial symmetric functions. In fact, the method is the same which we have described in [1] for the computation of Clebsch-Gordan series. In particular, the core consists in the decomposition of characters in monomial symmetric functions are interesting in that they give the multiplicities of the weights in the corresponding irreducible representations. As an example, we present such decomposition for all the second order characters.

$$
\begin{align*}
200000 &= M_{200000} + M_{001000} + 4M_{000010} + 17M_{100000} + 63M_{000000} \\
110000 &= M_{110000} + 4M_{001000} + 16M_{100000} + 56M_{010000} + 171M_{000000} \\
020000 &= M_{020000} + M_{001000} + 3M_{100000} + 11M_{010000} + 10M_{200000} + 36M_{000002} + 34M_{001000} + \\
&+ 96M_{000010} + 248M_{100000} + 603M_{000000} \\
101000 &= M_{101000} + 2M_{001000} + 8M_{100000} + 24M_{010000} + 32M_{200000} + 64M_{000002} + 78M_{001000} + \\
&+ 208M_{000010} + 544M_{100000} + 1344M_{000000} \\
011000 &= M_{011000} + 3M_{100000} + 10M_{010000} + 10M_{200000} + 30M_{001000} + 90M_{110000} + 80M_{000011} + \\
&+ 231M_{000100} + 570M_{100000} + 1344M_{010000} + 3024M_{000001} \\
002000 &= M_{002000} + M_{101000} + 2M_{010000} + 3M_{200000} + 7M_{001000} + 19M_{110000} + 20M_{000020} + \\
&+ 46M_{000011} + 10M_{300000} + 49M_{020000} + 56M_{101000} + 104M_{100002} + 125M_{001100} + \\
&+ 291M_{100010} + 682M_{200000} + 638M_{010000} + 1338M_{000002} + 1402M_{010000} + 2908M_{000001} + \\
&+ 593M_{001000} + 1184M_{000000} \\
100100 &= M_{100100} + 3M_{010000} + 4M_{200000} + 10M_{010000} + 30M_{110000} + 25M_{000020} + 75M_{000010} + \\
&+ 15M_{300000} + 84M_{020000} + 90M_{110000} + 180M_{100002} + 213M_{001000} + 507M_{100000} + \\
&+ 1149M_{010000} + 1185M_{200000} + 2484M_{000002} + 2565M_{010000} + 5439M_{000010} + \\
&+ 11265M_{100000} + 22680M_{000000} \\
010100 &= M_{010100} + 2M_{001000} + 6M_{100000} + 20M_{020000} + 15M_{101000} + 15M_{000100} + 42M_{000011} + \\
&+ 96M_{100011} + 40M_{210000} + 114M_{011000} + 220M_{010002} + 256M_{100100} + 565M_{200000} + \\
&+ 480M_{000003} + 575M_{010010} + 1240M_{001000} + 2624M_{110000} + 2580M_{000011} + \\
&+ 5340M_{000100} + 10589M_{100000} + 20524M_{010000} + 38864M_{000001} \\
001100 &= M_{001100} + 2M_{100100} + 5M_{020000} + 6M_{101000} + 5M_{000020} + 14M_{001000} + 15M_{210000} + \\
&+ 33M_{100020} + 37M_{011000} + 83M_{100101} + 40M_{201000} + 180M_{200002} + 94M_{120000} + \\
&+ 100M_{020000} + 215M_{101000} + 180M_{010011} + 467M_{200010} + 456M_{010000} + 375M_{001000} + \\
&+ 958M_{001010} + 750M_{000012} + 1964M_{110000} + 1920M_{000020} + 3963M_{020000} + 3850M_{000011} + \\
&+ 1010M_{300000} + 4005M_{101000} + 7374M_{100002} + 7700M_{001000} + 16462M_{100000} + \\
&+ 27546M_{200000} + 27263M_{010001} + 94968M_{000002} + 50206M_{010000} + 90408M_{000010} + \\
&+ 160642M_{100000} + 281268M_{000000} \\
000200 &= M_{000200} + M_{011000} + 2M_{020000} + 2M_{100200} + 2M_{120000} + 5M_{300000} + 5M_{100100} + \\
&+ 11M_{111000} + 11M_{010100} + 27M_{010010} + 12M_{200020} + 25M_{220000} + 23M_{001020} + \\
&+ 23M_{200010} + 54M_{010101} + 25M_{102000} + 52M_{201000} + 109M_{110011} + 45M_{000030} + \\
&+ 64M_{021000} + 45M_{300002} + 210M_{000011} + 225M_{020002} + 210M_{101002} + 129M_{011000} + \\
&+ 258M_{110010} + 520M_{200010} + 408M_{001002} + 750M_{100002} + 105M_{300000} + 499M_{000020} + \\
&+ 501M_{101010} + 960M_{210000} + 968M_{001010} + 215M_{400000} + 1365M_{010003} + 1787M_{100020} + \\
&+ 1854M_{010001} + 3376M_{100010} + 1830M_{201000} + 3524M_{120000} + 6055M_{200002} + \\
&+ 3525M_{020000} + 6085M_{010011} + 10760M_{001002} + 6350M_{100100} + 11358M_{010010}
\end{align*}
$$
3.3 First order polynomials

The iterative methods given in [1] allow us to solve the Schrödinger equation [3] for general . The eigenfunctions are polynomials. In this and the next subsection, we present a partial list of such polynomials of first and second order.

\[
P_{100000}(z) = z_1 + \frac{7(1 + z)}{1 + 17}
\]

\[
P_{010000}(z) = z_2 + \frac{6(1 + z)}{1 + 11}
\]

\[
P_{001000}(z) = z_3 + \frac{5(1 + z)z_6}{1 + 7} + \frac{8(1 + z)(1 + 8)z_6}{(1 + 7)(1 + 8)} + \frac{2(1 + z)(1 + 159 + 136^2)}{(1 + 7)(1 + 8)(1 + 11)}
\]

\[
P_{000100}(z) = z_4 + \frac{4(1 + z)z_6}{1 + 5} + \frac{3(1 + z)^2(3 + 5)z_6z_7}{(1 + 5)^2} + \frac{9(1 + z)(1 + 22 + 5^2)z^2_7}{(1 + 5)^3(1 + 7)}
\]

\[
+ \frac{1}{(1 + 5)(1 + 7)(78 + 1255 + 3653^2) + 6295^2 + 3325^4}z_6
\]

\[
+ \frac{2(1 + z)(22 + 755 + 3477^2 + 11255^3 + 175^4)}{(1 + 5)^3(1 + 7)(2 + 11)}
\]

\[
+ \frac{2(1 + z)(18 + 365 + 8123^2 + 2045^3 + 2275^4)}{(1 + 5)^3(1 + 7)(2 + 11)}
\]

\[
P_{000010}(z) = z_5 + \frac{5(1 + z)z_6}{1 + 7} + \frac{7(1 + z)(4 + 7)}{(1 + 7)(2 + 13)} + \frac{5(1 + z)(6 + 137 + 56^2)}{(1 + 7)(1 + 8)(2 + 13)}
\]

\[
P_{000001}(z) = z_6 + \frac{6(1 + z)}{1 + 9} + \frac{15(1 + z)(1 + 5)}{(1 + 9)(1 + 13)}
\]

\[
P_{000000}(z) = z_7
\]

3.4 Second order polynomials

\[
P_{200000}(z) = z_1^2 + \frac{2z_3}{1 + z} + \frac{10z}{1 + z}(1 + 4) + \frac{2(3 + 6 + 119^2 + 28^3)}{(1 + z)(1 + 4)(3 + 17)}
\]

\[
+ \frac{42 + 459 + 290^2 + 3205^3 + 196^4}{(1 + z)(1 + 4)(2 + 17)(2 + 17)}
\]

\[
P_{110000}(z) = z_1z_2 + \frac{5z_5}{1 + 4} + \frac{6z_5}{1 + z}(95 + 24^2)z_7 + \frac{28(1 + z)(26 + 11)}{(1 + 4)(2 + 11)(3 + 17)}
\]

\[
+ \frac{(1 + z)(138 + 365 + 9979^2 + 1176^3)}{(1 + 4)(1 + 7)(2 + 11)(3 + 17)}
\]
\[
P_{020000}(z) = z_2^2 + \frac{2z_4}{1 + (1 + 3)(1 + 3)} + \frac{8z_2z_6}{1 + (1 + 3)} + \frac{6(1 + 5)(1 + 4)(1 + 6^2)z_2z_7}{(1 + 3)(1 + 5)(1 + 5)} + \frac{2(1 + 23^2)z_5^2}{(1 + 1)(1 + 3)(1 + 5)} + 18(1 + 1)(1 + 13)(1 + 7^2 + 6^3)z_2^2 + 4(3 + 23 + 141^2 + 493^3 + 180^4)z_3^2 (1 + 1)(1 + 3)(1 + 3)(1 + 5)(1 + 5)(3 + 11) + \frac{2(42 + 537 + 2397^2 + 6715^3 + 14529^4 + 2380^5)z_6}{1 + (1 + 3)(1 + 4)(1 + 5)(2 + 11)(3 + 11)} + 4(1 + 1)(1 + 18 + 325 + 2143^2 + 2067^3 + 14045^4 + 22012^5) + (1 + 1)(1 + 3)(1 + 4)(1 + 5)(1 + 7)(2 + 11)(3 + 11) + 4(1 + 1)(1 + 18 + 325 + 2143^2 + 2067^3 + 14045^4 + 22012^5) + (1 + 1)(1 + 3)(1 + 4)(1 + 5)(2 + 11)(3 + 11) + 4(1 + 1)(1 + 18 + 325 + 2143^2 + 2067^3 + 14045^4 + 22012^5)
\]

\[
P_{100000}(z) = z_4z_3 + \frac{3z_4}{1 + 2} + \frac{2(2 + 3 + 10^2)z_2z_6}{(1 + 2)(2 + 7)} + \frac{6(1 + 1)(2 + 15)z_5z_7}{(1 + 2)(1 + 4)(2 + 7)} + (1 + 1)(1 + 2)(1 + 4)(2 + 7)(3 + 16)(4 + 17) + 2(1 + 1)(48 + 1190 + 6697^2 + 9260^3 + 2240^4)z_4^2 (1 + 2)(1 + 4)(2 + 7)(3 + 16)(4 + 17) + 12(1 + 1)(80 + 100 + 7708^2 + 27189^3 + 30716^4 + 12736^5)z_3 + (1 + 2)(1 + 4)(2 + 7)(2 + 11)(3 + 16)(4 + 17) + 4(1 + 1)(384 + 6676 + 12672^2 + 67253^3 + 75612^4 + 7616^5) + (1 + 2)(1 + 4)(2 + 7)(2 + 11)(3 + 16)(4 + 17) + 12(1 + 1)(80 + 100 + 7708^2 + 27189^3 + 30716^4 + 12736^5)z_3 + (1 + 2)(1 + 4)(2 + 7)(2 + 11)(3 + 16)(4 + 17) + 4(1 + 1)(384 + 6676 + 12672^2 + 67253^3 + 75612^4 + 7616^5) + (1 + 2)(1 + 4)(2 + 7)(2 + 11)(3 + 16)(4 + 17) + 12(1 + 1)(80 + 100 + 7708^2 + 27189^3 + 30716^4 + 12736^5)z_3
\]

\[
P_{011000}(z) = z_2z_3 + \frac{4z_2z_5}{1 + 3} + \frac{5(1 + 1)(2 + 3)z_2z_7}{(1 + 3)(2 + 7)} + \frac{6(1 + 7)z_2^2z_7}{(1 + 3)(1 + 5)} + (1 + 1)(123 + 821 + 1018^2 + 10196^3 + 1280^4)z_5 + (1 + 1)(1 + 4)(1 + 5)(2 + 7)(3 + 11) + 2(1 + 1)(1 + 3)(1 + 5)(2 + 7)(3 + 11) + 6(1 + 1)(13 + 156 + 2853^2 + 2376^3 + 27380^4 + 11328^5 + 1920^6)z_4 + (1 + 3)(1 + 4)(1 + 5)(2 + 7)(3 + 11) + 6(1 + 1)(12 + 595 + 2610^2 + 7325^3 + 2248^4 + 1360^5)z_2^2 + (1 + 1)(1 + 3)(1 + 4)(1 + 5)(2 + 7)(3 + 11) + 2(1 + 1)(42 + 3713 + 46855^2 + 49890^3 + 620062^4 + 178192^5 + 24480^6)z_2
\]

\[
P_{100010}(z) = z_1z_5 + \frac{5z_2z_6}{1 + 4} + \frac{5(1 + 1)z_2^2z_7}{1 + 7} + \frac{15(1 + 1)z_2z_7}{(1 + 4)^2(1 + 7)} + (1 + 1)(1 + 13)(1 + 4)(1 + 13)(1 + 7) + (1 + 1)(648 + 8119 + 26227^2 + 12296^3 + 7280^4)z_5 + (1 + 1)(1 + 4)(1 + 5)(1 + 13)(1 + 7) + (1 + 1)(1 + 4)(1 + 5)(1 + 13)(1 + 7) + (1 + 1)(1 + 4)(1 + 5)(1 + 13)(1 + 7) + (1 + 1)(1 + 4)(1 + 5)(1 + 13)(1 + 7)
\]
\[ \begin{align*}
P_{1000010}(z) &= z_2z_6 + \frac{6z_2z_7}{1 + 5} + \frac{9(1 + 7)z_7^2}{(1 + 5)(1 + 8)} + \frac{6(1 + 1)z_7^2}{1 + 9} + \frac{3(1 + 1)(7 + 55)z_7}{(1 + 5)2(1 + 9)} \\
&+ \frac{(18 + 1213 + 15375)^2 + 51579}{3} + \frac{15985}{3} + \frac{12600}{5}z_6 \\
&+ \frac{(1 + 5)(1 + 8)(1 + 9)(3 + 17)}{3(1 + 5)(1 + 8)(1 + 9)(2 + 13)(3 + 17)} + \frac{3(54 - 775 + 4734 + 107248) + 344272}{4} + \frac{252825}{5} + \frac{121400}{6}z_1 \\
&+ \frac{(1 + 5)2(1 + 8)(1 + 9)(2 + 13)(3 + 17)}{3(54 + 2459444 + 22702 + 391238)(4)} + \frac{115305}{5} + \frac{35000}{6} \\
\end{align*} \]

\[ \begin{align*}
P_{0100010}(z) &= z_2z_6 + \frac{5z_2z_7}{1 + 4} + \frac{(6 + 95 + 24)z_7}{(1 + 4)(2 + 11)} + \frac{6(1 + 1)(3 + 4)z_7}{(1 + 4)(2 + 9)} \\
&+ \frac{2(42 + 643 + 4519)^2 + 600}{3}z_5 \\
&+ \frac{(1 + 4)(2 + 9)(2 + 11)(3 + 13)}{2(1 + 1)(114 + 113 + 11592)^2 + 46218 + 4680(4)z_1z_7 \\
&+ \frac{(1 + 4)(1 + 5)(2 + 9)(2 + 11)(3 + 13)}{180 + 3864 + 20509 + 80848)^3 + 4515 + 16500(5)z_6 \\
&+ \frac{(1 + 4)(1 + 5)(2 + 9)(2 + 11)(3 + 13)}{2(1 + 1)(42 + 2347 + 38355)^2 + 27714 + 4500(4)z_7 \\
&+ \frac{(1 + 4)(1 + 5)(2 + 9)(2 + 11)(3 + 13)}{2(1 + 1)(7 + 36)z_2z_7 + \frac{2(1 + 1)(1 + 2)(1 + 3)(1 + 4)}{(1 + 1)(1 + 2)(1 + 3)(1 + 4)} + \frac{4(13 + 43 + 10 + 24)z_7z_6}{(1 + 1)(1 + 2)(1 + 3)(1 + 4)} \\
&+ \frac{4(18 + 213 + 140 + 100 + 1136 + 3 + 2787) + 2700}{3(1 + 1)(1 + 4)(2 + 9)(3 + 13)} \\
&+ \frac{4(18 + 285 + 949)^2 + 2675 + 5493 + 33950 + 1646 + 3000(7)z_1}{3(1 + 1)(1 + 2)(1 + 3)(1 + 4)(2 + 9)(2 + 13)(3 + 13)} \\
&+ \frac{(1 + 1)(204 + 4208 + 37487)^2 + 140165 + 24394 + 655613}{5} + \frac{66238 + 75000}{5} \\
&+ \frac{1 + 1)(4208 + 37487^2 + 140165 + 24394 + 655613}{5} + \frac{66238 + 75000}{5} \\
\end{align*} \]

\[ \begin{align*}
P_{0000020}(z) &= z_2^2 + \frac{2z_2z_7}{1 + 1} + \frac{2(1 + 1)z_4}{(1 + 1)(1 + 2)(1 + 3)(1 + 4)} + \frac{10z_7z_7^2}{(1 + 1)(1 + 4)} + \frac{4(13 + 43 + 10 + 24)z_7z_6}{(1 + 1)(1 + 2)(1 + 3)(1 + 4)} \\
&+ \frac{2(1 + 1)(1 + 2)(1 + 3)(1 + 4)}{(1 + 1)(1 + 2)(1 + 3)(1 + 4)} + \frac{2(1 + 1)(6 + 47 + 59)^3 + 88 + 3 + 48(4)z_7^2}{(1 + 1)(1 + 2)(1 + 3)(1 + 4)(2 + 9)} \\
&+ \frac{2(1 + 1)(1 + 2)(1 + 3)(1 + 4)}{(1 + 1)(1 + 2)(1 + 3)(1 + 4)} + \frac{4(6 + 29 + 36 + 199 + 3 + 60(4)z_7)}{(1 + 1)(1 + 2)(1 + 3)(1 + 4)(2 + 9)} \\
&+ \frac{4(18 + 213 + 140 + 100 + 1136 + 3 + 2787) + 2700}{3(1 + 1)(1 + 4)(2 + 9)(3 + 13)} \\
&+ \frac{4(18 + 285 + 949)^2 + 2675 + 5493 + 33950 + 1646 + 3000(7)z_1}{3(1 + 1)(1 + 2)(1 + 3)(1 + 4)(2 + 9)(2 + 13)(3 + 13)} \\
&+ \frac{(1 + 1)(204 + 4208 + 37487)^2 + 140165 + 24394 + 655613}{5} + \frac{66238 + 75000}{5} \\
&+ \frac{1 + 1)(4208 + 37487^2 + 140165 + 24394 + 655613}{5} + \frac{66238 + 75000}{5} \\
\end{align*} \]
3.5 Generalized quadratic C lebsch-Gordan series

For each , the product of polynomials can be decomposed as a linear combination of polynomials of the same . The terms entering in this decomposition are exactly the same as entering in the product of characters, i.e., in the corresponding C lebsch-Gordan series, while the coefficients are rational functions of . The method for computing these coefficients was explained in [11]. Here we give some of the quadratic C lebsch-Gordan series for general .

\[ \begin{align*}
\mathbf{P}_{0000001}(z) &= z_2 z_7 + \frac{4 z_4}{1 + 2} + \frac{4}{1 + 3} \left( \frac{5}{1 + 4} \right) z_1 z_2^2 + \frac{8}{1 + 7} (1 + 3) (1 + 4) (1 + 7) \\
&+ \frac{12}{1 + 12} (1 + 12) z_2^3 + \frac{1}{1 + 17} (1 + 3) (1 + 4) (1 + 7) \\
&+ \frac{327 + 3089}{1 + 13} + \frac{420}{1 + 17} (1 + 7) (1 + 9) (3 + 13) \\
&+ \frac{2 (1 + 15) (1 + 17) (1 + 9) (2 + 9) (3 + 13)}{(1 + 19) (1 + 2) (2 + 9)} z_2 \\
&+ \frac{2 (1 + 12) (1 + 15) (1 + 17) (1 + 9) (2 + 9) (2 + 13)}{(1 + 2) (1 + 15) (1 + 9)} z_7 \\
&+ \frac{2 (1 + 12) (1 + 15) (1 + 17) (1 + 9) (2 + 9) (2 + 13)}{(1 + 2) (1 + 15) (1 + 9) z_7} \end{align*} \]

\[ \begin{align*}
\mathbf{P}_{0000011}(z) &= z_6 z_7 + \frac{3 z_6}{1 + 2} + \frac{2}{1 + 4} \left( \frac{43 + 12}{1 + 5} \right) z_2 z_7 + \frac{7}{1 + 17} (1 + 3) (1 + 4) (1 + 5) (1 + 7) (1 + 9) (3 + 13) \\
&+ \frac{32 (1 + 2) (1 + 12) (1 + 15) (1 + 17) (1 + 9) (2 + 9) (2 + 13)}{(1 + 2) (1 + 15) (1 + 9)} z_7 \\
&+ \frac{2 (1 + 12) (1 + 15) (1 + 17) (1 + 9) (2 + 9) (2 + 13)}{(1 + 2) (1 + 15) (1 + 9) z_7} \end{align*} \]
\[
\begin{align*}
P_{0100000} \cdot P_{0000001} &= P_{0100000} + \frac{6}{1+5} P_{0010000} + \frac{16(1+2)(1+3)}{(1+6)(1+7)(1+11)} P_{0000100} \\
&\quad + \frac{32(1+2)(1+4)(1+7)}{(1+7)(1+8)(1+9)(1+11)} P_{1000000} \\
\end{align*}
\]

\[
\begin{align*}
P_{0010000} \cdot P_{0000001} &= P_{0010000} + \frac{6}{1+5} P_{1100000} + \frac{20(1+10)}{(1+4)(1+7)(2+9)} P_{0000100} \\
&\quad + \frac{48(1+2)(1+3)(1+12)(2+17)}{(1+7)(1+8)(2+11)(3+16)} P_{1000001} \\
&\quad + \frac{252(1+1)(1+3)(1+4)(1+12)(1+14)}{(1+6)(1+7)(1+8)(2+11)(2+13)(3+17)} P_{0100000} \\
\end{align*}
\]

\[
\begin{align*}
P_{0000100} \cdot P_{0000001} &= P_{0000100} + \frac{4}{1+3} P_{0001000} + \frac{8(1+2)(1+9)}{(1+4)(1+5)(1+7)} P_{1000010} \\
&\quad + \frac{90(1+1)(1+3)(1+11)}{(1+5)(2+13)(3+13)} P_{0100001} \\
&\quad + \frac{40(1+1)(1+2)(2+7)(1+10)(1+11)}{(1+5)(2+9)(2+13)} P_{0010000} \\
&\quad + \frac{96(1+2)(1+3)(1+4)(2+7)(1+9)(1+12)(1+13)}{(1+5)(1+6)(1+7)(2+11)(2+13)(3+17)} P_{0000010} \\
\end{align*}
\]

\[
\begin{align*}
P_{0001000} \cdot P_{0000001} &= P_{0001000} + \frac{4}{1+3} P_{0010000} + \frac{12(1+1)(1+8)}{(1+3)(1+5)(2+7)} P_{1000100} \\
&\quad + \frac{60(1+1)(1+2)(1+6)(1+8)}{(1+4)(1+5)(2+7)(3+11)} P_{0101000} \\
&\quad + \frac{20(1+1)(1+2)(2+5)(1+8)(1+9)(3+16)}{(1+4)(1+5)(2+9)(3+15)} P_{0010001} \\
&\quad + \frac{30(1+1)(1+2)(1+3)(2+5)(1+8)(1+10)(3+17)}{(1+4)(1+5)(2+9)(3+13)} P_{1100000} \\
&\quad + \frac{40(1+1)(1+2)(1+3)(2+5)(1+8)(1+10)(3+10)(2+13)}{(1+4)(1+5)(2+9)(2+11)(3+13)(4+17)} P_{0000100} \\
\end{align*}
\]

\[
\begin{align*}
P_{0000100} \cdot P_{0000001} &= P_{0000100} + \frac{4}{1+3} P_{0001000} + \frac{8(1+2)(1+9)}{(1+4)(1+5)(1+7)} P_{1000100} \\
&\quad + \frac{90(1+1)(1+3)(1+11)}{(1+5)(2+13)(3+13)} P_{0100001} \\
&\quad + \frac{40(1+1)(1+2)(2+7)(1+10)(1+11)}{(1+5)(2+9)(2+13)} P_{0010000} \\
&\quad + \frac{96(1+2)(1+3)(1+4)(2+7)(1+9)(1+12)(1+13)}{(1+5)(1+6)(1+7)(2+11)(2+13)(3+17)} P_{0000100} \\
\end{align*}
\]

\[
\begin{align*}
P_{0000100} \cdot P_{0000001} &= P_{0000001} + \frac{4}{1+3} P_{0001000} + \frac{20(1+1)(1+10)}{(1+7)(1+9)(2+9)} P_{1000100} \\
&\quad + \frac{42(1+1)(1+4)(1+14)}{(1+5)(1+6)(1+9)(2+13)} P_{0100000} \\
&\quad + \frac{108(1+3)(1+4)(1+6)(1+14)(1+18)}{(1+8)(1+9)(1+11)(1+13)(2+13)(2+17)} P_{0000001} \\
\end{align*}
\]

\[
\begin{align*}
P_{0000001} \cdot P_{0000001} &= P_{0000001} + \frac{2}{1+1} P_{0000010} + \frac{12(1+4)}{(1+5)(1+9)} P_{1000000} \\
&\quad + \frac{56(1+4)(1+8)}{(1+9)(1+13)(1+17)} \
\end{align*}
\]
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