O n the semi classical mass of $S^2$-kinks


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One-loop mass shifts to the classical masses of stable kinks arising in a massive non-linear $S^2$-sigma model are computed. Ultraviolet divergences are controlled using the heat-kernel/zeta function regularization method. A comparison between the results achieved from exact and high-temperature asymptotic heat traces is analyzed in depth.

PACS numbers: 11.15.Kc;11.27+d;11.10.Gh

I . I N T R O D U C T I O N

In a seminal paper, Olive and Witten $^1$ linked extended supersymmetric theories to BPS solitons by showing that the classical mass of these stable lumps agreed exactly with the central charge of the extended SUSY algebra. The subsequent issue concerning BPS saturation at one-loop (rather than tree) level has proved to be extremely subtle, prompting a remarkable amount of work over the last twelve years. See, e.g., $^2$ and References quoted therein to an in-depth report on these developments.

A new actor entered the stage when in $^3$ a Stony Brook/Witten group computed the one-loop mass shift of the supersymmetric CP$^2$-kink in a $N = (2,2)$ non-linear sigma model with twisted mass. Kinks of several types in massive non-linear sigma models were, however, discovered earlier, see $^4$, $^5$, $^6$, $^7$. In Reference $^8$, three of us found several families of non-topological kinks in another non-linear sigma model: we chose $S^2$ as the target space and considered the case when the mass of the pseudo-Nambu-Goldstone particles were different. The $O(2)$-symmetry of the equal mass case is explicitly broken to $Z_2 \times Z_2$ and the $SO(2)$-symmetry of topological kinks in the former system are deformation to the four families of non-topological kinks arising in the second system. The boundary of the moduli space of non-topological kinks in the latter is formed by a pair of topological kinks of different energy. The analysis of kink stability in the massive non-linear $S^2$-sigma model performed in $^9$ allowed us to calculate the one-loop mass shift for the topological kinks by using the Cahill-Comtet-Gauß-Laub's formula $^10$. These authors showed that the one-loop mass shift for static solitons can be read from the eigenvalues of the bound states of the kink second-order quantization operator and the threshold to the continuous spectrum when this operator is a transparent Schroedinger operator of the Posch-Teller type. This is the case of the topological kinks of the massive non-linear $S^2$-sigma model when a parallel frame to the kink orbits is chosen to refer to the "instantons."

The aim of this paper is to offer another route for computing the one-loop kink mass shift in order to unveil some of the intricacies hidden in this subtle problem. We shall follow the method developed in References $^11$ and $^12$ based on heat kernel/zeta function regularization of ultraviolet divergences. See also the lectures $^13$, where full details can be found. Because the spectrum of small kink excitations in our system can be extended analytically, we are able to give the exact answer for the mass shifts. We shall also show, however, how to reach approximate results using the coefficients of the heat kernel asymptotic expansion. The interest of this calculation is that a formula belonging to the class of formulas shown in $^12$ will be derived. The importance of this type of formulas lies in the fact that it can be applied to obtain the one-loop mass shifts of topological defects even when the spectrum of the second-order quantization operator is not known; for instance, in the case of two-component topological kinks: see $^11$, $^12$. Similar formulas work even for Abelian gauge theories in $(2+1)$-dimensions and thus the mass shifts of self-dual Nielsen-Olesen vortices and semi-local strings can be calculated approximately, see $^14$, $^15$, and $^16$.

To end this brief Introduction we emphasize that interesting calculations have recently appeared addressing one-loop kink mass corrections and kink melting at high temperatures in the sine-Gordon, CP$^1$, and $S^2$ models in a purely bosonic setting, see $^17$.

The organization of the paper is as follows: In Section x.II, we introduce the model and explain our conventions. In Section x.III, the perturbative sector as well as the mass renormalization procedure are discussed. Section x.IV is devoted to the analysis of the stable topological kinks in this system. The second-order kink quantization operator is obtained, placing special emphasis on its geometric properties. In Section x.V, the one-loop mass shift is computed using the heat kernel/zeta function regularization method. Section x.VI gives a comparison of the exact result obtained in x.V with the approximation reached from the high-temperature asymptotic expansion. Finally, a summary and outlook are offered whereas two Appendices containing some technical material are included.
II. THE (1+1)-DIMENSIONAL MASSIVE NON-LINEAR $S^2$-SIGMA MODEL

The action governing the dynamics of the non-linear $S^2$-sigma model and the constraint on the scalar fields are:

$$S[1; 2; 3] = \frac{1}{2} \int d\tau d\xi \left( \frac{\partial \phi}{\partial \xi} \right)^2 + \frac{1}{2} \left( \frac{\partial \phi}{\partial \tau} \right)^2 - \frac{1}{2} \left( \frac{\partial \phi}{\partial \xi} \right)^2$$

The scalar fields are thus maps, $\phi(t;\xi)$, from the (1+1)-dimensional space-time to a $S^2$-sphere of radius $R$, which is the target manifold of this non-linear sigma model. Our conventions for $R^{1+2}$ are as follows: $x = R^{1+2}, = 0$ with $x^0 = t^2 - x^2 = 0$ and $g = \text{diag}(1, 1)$. Then $x = g x^1 x^2$ and $g = \text{diag}(1, 1, 1)$.

The infrared asymptotics for massless particles in (1+1)-dimensional scalar field theories, see [16]. We therefore include the simplest potential energy density that would be generated by quantum fluctuations [21]:

$$V(1; 2; 3) = \frac{1}{2} \left( \frac{\partial \phi}{\partial \xi} \right)^2 + \frac{1}{2} \left( \frac{\partial \phi}{\partial \tau} \right)^2 - \frac{1}{2} \left( \frac{\partial \phi}{\partial \xi} \right)^2$$

where

$$V_S[1; 2] = \frac{1}{2} \left( \frac{\partial \phi}{\partial \xi} \right)^2 + \frac{1}{2} \left( \frac{\partial \phi}{\partial \tau} \right)^2 - \frac{1}{2} \left( \frac{\partial \phi}{\partial \xi} \right)^2$$

The masses of the pseudo-Nambu-Goldstone bosons are respectively:

$$m_1 = \frac{\partial \phi}{\partial \xi}, m_2 = \frac{\partial \phi}{\partial \tau}$$

1. Solving for $\phi(1), \phi(2), \phi(3)$, we get:

$$S = \frac{1}{2} \int d\tau d\xi \left( \frac{\partial \phi}{\partial \xi} \right)^2 + \frac{1}{2} \left( \frac{\partial \phi}{\partial \tau} \right)^2 - \frac{1}{2} \left( \frac{\partial \phi}{\partial \xi} \right)^2$$

2. Interactions, however, come from the geometry:

$$S = \frac{1}{2} \int d\tau d\xi \left( \frac{\partial \phi}{\partial \xi} \right)^2 + \frac{1}{2} \left( \frac{\partial \phi}{\partial \tau} \right)^2 - \frac{1}{2} \left( \frac{\partial \phi}{\partial \xi} \right)^2$$

and $\frac{1}{R^2}$ is a non-dimensional coupling constant.

In the unit natural system, $c = 1$, the dimensions of fields, masses and coupling constants are respectively:

$$[\phi] = 1 = [\xi], [\tau] = 1 = [\xi]$$

We define non-dimensional space-time coordinates and masses:

$$x^1 = \frac{2}{3}, x^2 = \frac{2}{3}, 0 < x^1 < 1$$

To write the action and the energy in terms of them:

$$S = \frac{1}{2} \int d\tau d\xi \left( \frac{\partial \phi}{\partial \xi} \right)^2 + \frac{1}{2} \left( \frac{\partial \phi}{\partial \tau} \right)^2 - \frac{1}{2} \left( \frac{\partial \phi}{\partial \xi} \right)^2$$

$$E = \frac{1}{2} \int d\tau d\xi \left( \frac{\partial \phi}{\partial \xi} \right)^2 + \frac{1}{2} \left( \frac{\partial \phi}{\partial \tau} \right)^2 - \frac{1}{2} \left( \frac{\partial \phi}{\partial \xi} \right)^2$$

There are two homogeneous minima of the action or vacua of our model: $x = y = z = 0$, the North and South Poles. Choice of one of the poles to quantize the system spontaneously breaks the $Z_2 x Z_2 x Z_2$ symmetry of the action [31]. There are two disconnected sectors $C = C_{kN} + C_{kS} + C_{kN} + C_{kS}$ labeled by the vacuum reached by each quantization at the two disconnected components of the boundary of the real line: $x = 1$. 

$$S[1; 2; 3] = \frac{1}{2} \int d\tau d\xi \left( \frac{\partial \phi}{\partial \xi} \right)^2 + \frac{1}{2} \left( \frac{\partial \phi}{\partial \tau} \right)^2 - \frac{1}{2} \left( \frac{\partial \phi}{\partial \xi} \right)^2$$
The field equations

\[ 2_1 \ + \ @ \ \frac{P_2}{R^2} \ = \ \frac{P_2}{R^2} \ + \ 1 \ = \ 0 \]

\[ 2_2 \ + \ @ \ \frac{P_2}{R^2} \ = \ \frac{P_2}{R^2} \ + \ 2 \ = \ 0 \]

.. become linear for small fluctuations, \( G(x) = v \ + \ G(x) \), around the vacuum:

\[ 2 G_1(t;x) + G_1(t;x) = 0 \ (G \ G) \ ; \ 2 G_2(t;x) + 2 G_2(t;x) = 0 \ (G \ G) \]  \hspace{1cm} (3)

We shall need the Feynman rules only for the four-valent vertices. Besides the two propagators for the (pseudo) Nambu-Goldstone bosons there are three vertices with four external legs. The derivatives appearing in the interactions induce dependence on them onenta in the weights. This also affects the sign and the combinatorial factors. Naturally, there are many more vertices in this model, but we list only the vertices that contribute to the self-energy of the Nambu-Goldstone bosons up to one-loop order.

### Table I: Propagators

<table>
<thead>
<tr>
<th>Particle</th>
<th>Field</th>
<th>Propagator</th>
<th>Diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nambu-Goldstone ( G_1(x) )</td>
<td>( k )</td>
<td>( k^2 )</td>
<td>( 1 \ + \ i )</td>
</tr>
<tr>
<td>Nambu-Goldstone ( G_2(x) )</td>
<td>( k )</td>
<td>( k^2 )</td>
<td>( 2 \ + \ i )</td>
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</tbody>
</table>

### Table II: Fourth-order vertices

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Weight</th>
<th>Vertex</th>
<th>Weight</th>
<th>Vertex</th>
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</thead>
<tbody>
<tr>
<td>( k )</td>
<td>( p )</td>
<td>( q )</td>
<td>( k )</td>
<td>( p )</td>
<td>( q )</td>
</tr>
</tbody>
</table>

A. Plane waves and vacuum energy

The general solution of the linearized field equations governing the small fluctuations of the Nambu-Goldstone fields is:

\[ G_1(x_0;x) = \frac{1}{2} \left( \frac{1}{i k_x} \right) \left( \frac{1}{2 \Gamma_1(k)} \right) a_1(k)e^{ik_0x_0 + ikx} + a_1(k)e^{-ik_0x_0 - ikx} \]

\[ G_2(x_0;x) = \frac{1}{2} \left( \frac{1}{i q_x} \right) \left( \frac{1}{2 \Gamma_2(q)} \right) a_2(q)e^{iq_0x_0 + iqx} + a_2(q)e^{-iq_0x_0 - iqx} \]
where \( k_0 = \gamma_1(k) = \frac{p}{k^2 + 1}, \gamma_0 = \gamma_2(q) = \frac{p}{q^2 + 2} \), and the dispersion relations \( k_0 \), \( k^2 \rightarrow 0, \gamma_0 \rightarrow 2 \) hold:

\[
K_0 \begin{pmatrix} e^{i\gamma x} & e^{i\gamma x} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \gamma_0 & 0 \\ 0 & \gamma_0 \end{pmatrix} ; \quad K_0 \begin{pmatrix} e^{i\gamma x} & 0 \\ 0 & e^{i\gamma x} \end{pmatrix} = \begin{pmatrix} \gamma_0 & 0 \\ 0 & \gamma_0 \end{pmatrix} .
\]

W e h ave c hosen a nor mal ization i nterval of non-dimen sional length \( L = L, I = \{ \frac{1}{2}, \frac{1}{2} \} \), and we im pose PBC on the plane waves so that: \( k = 2 \pi p, q = 2 \pi q \) with \( n_1, n_2 \in \mathbb{Z} \). Thus, \( K_0 \) acts on \( L^2 = L^2 \{ s \} \), and its spectral density at the \( 1 \) limit is:

\[
\gamma_0(k) = \frac{dk}{dx} \mid_{x=0} = \frac{1}{L} 1 0 0 1 .
\]

F rom the c lassical free (quad ratic) Ham iltonian

\[
H^{(2)} = \frac{1}{2} \int dx \left( \frac{\partial G_1}{\partial x_0} \frac{\partial G_1}{\partial x} + \frac{\partial G_1}{\partial x} \frac{\partial G_1}{\partial x_0} + \frac{\partial G_2}{\partial x_0} \frac{\partial G_2}{\partial x} \right) + G_1 + G_2 \Rightarrow G = \frac{x^2}{2} \sum_{k=1}^{N} \gamma_0(k)(a(k)a(k) + a(k)a(k));
\]

one passes via canonical quantization to the quantum free Hamiltonian:

\[
\hat{\gamma}_0^{(2)}(x) = \sum_{k=1}^{N} \gamma_0(k) \hat{a}_1^\dagger(k) \hat{a}_1(k) + \frac{1}{2} \hat{\gamma}_2(k) \hat{a}_2^\dagger(k) \hat{a}_2(k) + \frac{1}{2} ;
\]

The vacuum energy is:

\[
\hat{\gamma}(x) \hat{\gamma}(x) = 0; \quad \hat{E}_0 = \hbar \omega / \int_{0}^{L} \hat{\gamma}^{(2)}(x) \hat{\gamma}(x) = -\frac{X}{2} \gamma_1(k) + \frac{X}{2} \gamma_2(k) = \frac{N}{2} \gamma_0(k).
\]

B. O ne-loop mass renormalization counter-terms

T here are four ultraviolet divergent graphs in one-loop order of the \( \gamma \)-expansion contributing to the \( G_1(x) \) and \( G_2(x) \) Nambu-Goldstone bosons self-energies:

Self-energy of \( G_2 \)

\[
\frac{2i}{R^2} \int \frac{dk}{4} \left( \frac{1}{k^2 + 1} + \frac{1}{k^2 + 2} \right) = \frac{2i}{R^2} \int \frac{dk}{4} \left( \frac{1}{k^2 + 1} + \frac{1}{k^2 + 2} \right)
\]

Self-energy of \( G_1 \)

\[
\frac{2i}{R^2} \int \frac{dk}{4} \left( \frac{1}{k^2 + 1} + \frac{1}{k^2 + 2} \right) = \frac{2i}{R^2} \int \frac{dk}{4} \left( \frac{1}{k^2 + 1} + \frac{1}{k^2 + 2} \right)
\]
where we have computed the $k_0$ integrations by using the residue theorem. We only show this step explicitly in the computation of the self-energy of $G_i$ because it serves to point out how to regularize these divergent integrals by means of spectral zeta functions. The regularization just mentioned will be performed later in Section xV D.

The $p p$ factor becomes constant when the momentum is put "on shell" in the external legs, $p p = 1, p p = 2$. This process gives us the mass renormalization counter-terms. The Lagrangian density of counter-terms shown in Table III must be added to cancel the above divergences exactly. We also show the vertices generated at one-loop level.

<table>
<thead>
<tr>
<th>Diagram</th>
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<tbody>
<tr>
<td><img src="image" alt="Diagram" /></td>
<td>$\frac{21}{R^2}(I(1) + I(2))$</td>
</tr>
<tr>
<td><img src="image" alt="Diagram" /></td>
<td>$\frac{21}{R^2}(I(1) + I(2))$</td>
</tr>
</tbody>
</table>

Table III: One-loop counter-terms

$$L_{CPT} = \frac{1}{R^2} I(1) + I(2) \frac{2}{1} (\chi) + \frac{2}{2} (\chi)$$

IV. ISO THERMAL COORDINATES AND TOPOLOGICAL KINKS

In this Section we shall use the isothermal coordinates in the chart $S^2 \ f(0; 0; R) g$ obtained via stereographic projection from the South Pole:

$$x = \frac{1}{1 + \frac{R}{r}} = \frac{R}{R + \sqrt{3}} \frac{1}{R^2} \frac{1}{1 + \frac{2}{2} \frac{1}{2}} ; \quad y = \frac{2}{1 + \frac{R}{r}} = \frac{R}{R + \sqrt{3}} \frac{2}{R^2} \frac{2}{1 + \frac{2}{2} \frac{1}{2}} :$$

The geometric data of the sphere in this coordinate system are:

$$ds^2 = \frac{4R^4}{(R^2 + 1 + 1 + 2 + 2)^2} ; \quad g_{11}(1; 2) = g_{22}(1; 2) = \frac{4R^4}{(R^2 + 1 + 1 + 2 + 2)^2}$$

$$\frac{1}{11}(1; 2) = \frac{1}{12}(1; 2) = \frac{2}{12}(1; 2) = \frac{2}{11}(1; 2) = \frac{2}{21}(1; 2) = \frac{2}{22}(1; 2)$$

$$R_{122}^1(1; 2) = R_{212}^1(1; 2) = R_{212}^2(1; 2) = R_{221}^2(1; 2) = \frac{4R^4}{(R^2 + 1 + 1 + 2 + 2)^2} :$$

The kinetic and potential energy densities read:

$$T(1; 2) = \frac{2R^4}{(R^2 + 1 + 1 + 2 + 2)^2} \left( \partial_t \frac{1}{1} + \partial_t \frac{2}{2} \right)$$

$$V(1; 2) = \frac{2R^4}{(R^2 + 1 + 1 + 2 + 2)^2} \left( \partial_x \frac{1}{1} + \partial_x \frac{2}{2} \right)$$

From the action $S = \int d^3x \left[ T + V \right]$ one derives the eom equations:

$$2(\frac{1}{1} + \frac{1}{2}) \frac{1}{R^2 + 1 + 1 + 2 + 2} = 0 ;$$

which for static configurations reduce to:

$$\frac{d^2}{dx^2} \left( \frac{jk}{dx} \frac{1}{dx} + \frac{1}{1} + \frac{2}{2} \frac{1}{2} \frac{2}{2} \right) \frac{1}{R^2 + 1 + 1 + 2 + 2} = 0$$
A. Topological K kinks

We try the $^1 = 0$ orbit in (5) and reduce this ODE system to the single ODE:

$$\frac{d^2}{dx^2} \left[ \frac{2}{R^2 + \gamma^2} \frac{d^2}{dx^2} - \frac{2}{R^2 + \gamma^2} \right] x = 2 \frac{d^2}{dx^2}$$

(7)

$$\frac{1}{K}(x) = 0 \quad ; \quad \frac{2}{K}(x) = \text{Re} \left[ \kappa(x_0) \right]$$

(8)

are solutions of (7) of finite energy:

$$E[K] = \int Z_1 \frac{R^2}{\cosh^2 \left( \frac{x}{x_0} \right)} = 2 R^2$$

(9)

In (8), $x_0$ is an integration constant that sets the kink center. The kink field components in the original coordinates

$$\begin{align*}
\frac{1}{k}(x) &= 0 \\
\frac{2}{k}(x) &= \frac{R}{\cosh \left( \frac{x}{x_0} \right)} \\
\frac{3}{k}(x) &= R \tanh \left( \frac{x}{x_0} \right)
\end{align*}$$

are either kink-shaped, $\frac{1}{k}$, or bell-shaped, $\frac{3}{k}$. It is clear that the four solutions (8) belong to the topological sectors $C_{NS}$ or $C_{SN}$ of the configuration space. Lorentz invariance tells us that

$$\frac{1}{K}(x) = 0 \quad ; \quad \frac{2}{K}(x) = \text{Re} \left[ \frac{x}{x_0} \right]$$

(10)

are solitary wave solutions of the full field equations (5).

B. Second-order fluctuation operator

Let us consider small kink fluctuations:

$$(x) = k(x) + \frac{1}{k}(x) \quad ; \quad (x) = (1(x); 2(x))$$

Here, $k(x) = (\frac{1}{k}(x); \frac{2}{k}(x))$ is the kink solution and $\frac{1}{k}(x) = \frac{1}{k}(x)^{\frac{1}{m}} + \frac{2}{k}(x)^{\frac{1}{m}} 2$ (TS2) are vector fields along the kink orbit—expressed in the orthonormal basis $\delta_i^j$; $j = 1, 2$ of TS2—giving the small fluctuations on the kink.

From the tangent vector field to the orbit $\frac{0}{k}(x) = \frac{d}{dx} \left[ \frac{1}{k}(x) \right] + \frac{d}{dx} \left[ \frac{2}{k}(x) \right]$, the covariant derivative, and the curvature tensor

$$\begin{align*}
\frac{r}{\kappa}(x) &= \frac{\partial}{\kappa}(x) + \frac{1}{jk}(x) \frac{\partial}{\kappa}(x) \frac{\partial}{\kappa} (x) \\
R(0; k) &= \frac{\partial}{\kappa}(x) R_{ijk}(x) \frac{\partial}{\kappa} (x) \frac{\partial}{\kappa} \frac{\partial}{\kappa} (x)
\end{align*}$$

we obtain the geodesic deviation operator:

$$\frac{D^2}{dx^2}(x) = R_{0k}(x) R_{0k}(x) + R(0; k)$$

We also need the Hessian of the mechanical potential $U(1; 2) = V(1; 2)$

$$\begin{align*}
\frac{r}{\text{grad} U(x)} &= \frac{\partial^2 U}{\partial x^2} (x) + \frac{k}{j}(x) \frac{\partial^2 U}{\partial x^2} (x) \\
&= \frac{\partial^2 U}{\partial x^2} (x) + \frac{k}{j}(x) \frac{\partial^2 U}{\partial x^2} (x)
\end{align*}$$

The second-order fluctuation operator around the kink $\kappa$ is:

$$(\kappa)(x) = \frac{D^2}{dx^2}(x) + R(0; k) + r \text{grad} U(x)$$

(11)
C. Small fluctuations on K kinks

Application to the K kink $K(x) = \left(\frac{1}{K}(x) = 0; \frac{2}{K}(x) = \text{Re}^x\right)$ gives:

$$
(K) = \frac{d^2}{dx^2} + 2 \left(1 + \tanh x\right) \frac{d^1}{dx} \left(1 + 2^2 \tanh x \frac{\theta}{\theta} + 2 \left(2 \tanh x \frac{\theta}{\theta} \right) = 0 \right)
$$

The second-order fluctuation operator in the orthonormal frame is a second-order differential operator that has first-order derivatives both in the direction of the kink orbit, $\frac{\theta}{\theta}$, and the orthogonal direction to the orbit $\frac{\theta}{\theta}$. Alternatively, we can use a parallel frame, $(x) = \left(\frac{1}{x} + \frac{2}{x}\right)$, along the K kink orbit:

$$
\frac{d}{dx} + \left(1 + \tanh x\right) \frac{\theta}{\theta} = 0 \quad \text{and} \quad \frac{d}{dx} + \left(1 + \tanh x\right) \frac{\theta}{\theta} = 0
$$

In this parallel frame the vectors of the basis $\left(\frac{\theta}{\theta}, \frac{\theta}{\theta}\right)$ point in the same directions as $\frac{\theta}{\theta}$, but their moduli vary along the kink orbit:

$$
h \left(\frac{1}{x} \frac{\theta}{\theta} + \frac{2}{x} \frac{\theta}{\theta} \right) = (1 + e^{2x})^2
$$

Writing the fluctuations in this frame, $(x) = \left\{\frac{1}{\theta} \frac{\theta}{\theta} + \frac{2}{\theta} \frac{\theta}{\theta}\right\}$, we find:

$$
(K) = \frac{d^2}{dx^2} + \left(1 + \frac{2}{\tanh x} \frac{\theta}{\theta} \right) \frac{d^1}{dx} + \frac{2}{\tanh x} \frac{\theta}{\theta} + \frac{2}{\tanh x} \frac{\theta}{\theta} = 0 \quad (13)
$$

In the parallel frame the second-order fluctuation operator is a transparent (reflection coefficient equal to zero) Posh-Teller Schrödinger operator both in the parallel and orthogonal directions to the kink orbit.

This analysis is deceptively simple: acting respectively on $\frac{1}{x} \frac{\theta}{\theta} = (1 + e^{2x})^2$ and $\frac{2}{x} \frac{\theta}{\theta} = (1 + e^{2x})^2$ the terms with first-order derivatives in $\frac{1}{x} \frac{\theta}{\theta}$ disappear and $(1 + e^{2x})^2$ factors out, leaving very well known Schrödinger operators acting respectively on $\frac{1}{x} \frac{\theta}{\theta}$ and $\frac{2}{x} \frac{\theta}{\theta}$. The key point is that the differential operators in $\frac{1}{x} \frac{\theta}{\theta}$ and $\frac{2}{x} \frac{\theta}{\theta}$ share the eigenvalues although their eigenfunctions differ by the $\frac{1}{x} \frac{\theta}{\theta}$ factors. The spectral functions associated to each of these factors are identical and it seems wise to use the best known form. What we have shown here is the geometric meaning of the $\frac{1}{x} \frac{\theta}{\theta}$ factors: they provide a parallel frame along the kink orbit.

D. The spectrum of small kink fluctuations

Changing from vector to matrix notation,

$\begin{pmatrix} \frac{\theta}{\theta} & \frac{\theta}{\theta} \\ \frac{\theta}{\theta} & \frac{\theta}{\theta} \end{pmatrix}$

we now use the differential operators from formula (13) to write the linearized di equation satisfied by the small kink fluctuations in the parallel frame:

$\frac{1}{x} \frac{\theta}{\theta} + \frac{2}{x} \frac{\theta}{\theta} K_1(t;\theta) = 0 \quad \frac{2}{x} \frac{\theta}{\theta} K_2(t;\theta) = 0$

Therefore, the eigenfunctions of the differential operator

$K = \begin{pmatrix} \frac{\theta}{\theta} & \frac{\theta}{\theta} \\ \frac{\theta}{\theta} & \frac{\theta}{\theta} \end{pmatrix}$

provide the general solution of the linearized equations via the separation ansatz: $K_1(t;\theta) = g_1(t) \frac{1}{x} \frac{\theta}{\theta}$, $K_2(t;\theta) = g_2(t) \frac{2}{x} \frac{\theta}{\theta}$. The eigenvalues and eigenfunctions of $K$ are shown in the following Table:
The spectrum of $K_{22}$ contains a bound state of zero eigenvalue—the translational mode—and a branch of the continuous spectrum, with the threshold at $m_{2}^{2}(0) = 2$. Spec$K_{11}$ also is formed by a bound state of positive eigenvalue and a branch of the continuous spectrum starting at $m_{1}^{2}(0) = 1$. Periodic boundary conditions in the $[\frac{1}{2}, \frac{1}{2}]$ interval require:

\[
q_{1+1}(q) = 2n_{1} \quad ; \quad k_{1+2}(k) = 2n_{2} \quad ; \quad n_{1}; n_{2} \in \mathbb{Z}
\]
such that the phase shifts and the induced spectral densities are:

\[
k_{1}(q) = \frac{1}{2} \left( 1 + \frac{d_{1}}{dq} (q) \right) \quad ; \quad k_{22}(k) = \frac{1}{2} \left( 1 + \frac{d_{2}}{dk} (k) \right)
\]

(15)

In sum, $K$ also acts in the Hilbert space $L^{2} = \bigoplus_{n=1}^{\infty} L^{2}(S^{1})$, and its spectral density in the limit of very large radius of the circle is:

\[
x(k) = \frac{dn}{dx} \left( \frac{d_{1}}{dq} (q) \right) = \frac{1}{2} \left( 1 + \frac{d_{2}}{dk} (k) \right) ;
\]

V. ONE-LOOP SHIFT TO THE CLASSICAL KINK MASSES IN THE MASSIVE NON-LINEAR S$^{2}$-SIGMA MODEL

A. Zero-point kink energy

The general solution of the linearized field equations governing the small kink fluctuations is:

\[
K_{1}(x_{0} ; x) = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} \left[ A_{1} \left( e^{i \pi_{1} x_{0}} \right) + A_{1} \left( e^{-i \pi_{1} x_{0}} \right) \right] f_{1} (x)
\]

\[
+ \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} \left[ A_{1} \left( e^{-i \pi_{2} x_{0}} \right) \right] f_{2} (x) + ...) + ...
\]

Note that the zero mode is not included because only contribute to quantum corrections at two-loop order. In the orthogonal complement to the kernel of $K_{22}$ in $L^{2} = \bigoplus_{n=1}^{\infty} L^{2}(S^{1})$, the eigenfunctions of the $K$ operator satisfying PBC form a complete orthonormal system. Therefore, the classical free Hamiltonian

\[
H^{(2)} = E + \frac{2}{2} \int_{x_{0}}^{x_{0}} \left( \frac{\partial^{2} A_{1}}{\partial x_{0}^{2}} \right) (x_{0}) + \frac{\partial A_{1}}{\partial x_{0}} (x_{0}) + \frac{\partial A_{1}}{\partial x_{0}} (x_{0}) + ... + ...
\]

\[
= E + \frac{2}{2} \int_{x_{0}}^{x_{0}} \left( A_{1} \right) \left( \frac{\partial^{2} A_{1}}{\partial x_{0}^{2}} \right) (x_{0}) + ... + ...
\]
can be written in terms of the normed modes of the system in the quadratic approximation. From this expression, one passes via canonical quantization, \[ H_{\text{kin}}(k) \hat{X}_{\nu}(q) = \sum_{k, \nu} \hat{A}_{k}^{\dagger} \hat{A}_{k} \hat{X}_{\nu}^{\dagger} \hat{X}_{\nu} \] to the quantum free Hamiltonian:

\[
H_{\text{kin}}^{(2)} = E + \frac{p^2}{2} A_{k}^{2} + \frac{1}{2} X^2 \sum_{k=1}^{N} (k) \hat{A}_{k}^{\dagger} (k) \hat{A}_{k} (k) + \frac{1}{2}
\]

The kink ground state is a coherent state annihilated by all the destruction operators:

\[
\hat{A}_{k} (k) \hat{\psi} \hat{I} = A_{k} \hat{\psi} \hat{I} = 0 \quad \text{for all } k \text{;} \\
\hat{A}_{k} (k) \hat{\psi} \hat{I} = 0 \quad \text{for all } k \text{.}
\]

The kink ground state energy is:

\[
E + E = \hbar \omega \hat{I} + (2) = 2 \hat{R}^2 + \frac{p^2}{2} A_{k}^{2} + \frac{1}{2} X^2 \sum_{k=1}^{N} (k) \hat{A}_{k}^{\dagger} (k) \hat{A}_{k} (k) + \frac{1}{2}
\]

(16)

B. Zeta function regularization and Casimir kink energy

Both \( T_{\text{L}} K_{0} \) and \( T_{\text{L}} K_{0}^{\dagger} \) are ultraviolet divergent quantities: one sum over an infinite number of eigenvalues, and a regularization-renormalization procedure must be performed to make sense of these formal expressions. We renormalize the zero-point kink energy by subtracting from it the vacuum energy to define the kink Casimir energy:

\[
4 E^{C} = 4 E^{C} - E_{0} = \frac{h}{2} T_{\text{L}} K^{\dagger} - T_{\text{L}} K_{0}^{\dagger}
\]

The subtraction of these two divergent quantities is regularized by using the associated generalized zeta functions, i.e., we temporarily assign to \( 4 E^{C} \) the finite value:

\[
4 E^{C}(s) = \frac{2}{2} T_{\text{L}} K^{s} - T_{\text{L}} K_{0}^{s} = \frac{2}{2} \left[ K(s) - K_{0}(s) \right]
\]

at a regular point of both \( K(s) \) and \( K_{0}(s) \). Here,

\[
K(s) = \sum_{\text{Spec} K} X^{s} \quad \text{and} \quad K_{0}(s) = \sum_{\text{Spec} K_{0}} X^{s}
\]

are the spectral zeta functions of \( K \) and \( K_{0} \), which are meromorphic functions of the complex variable \( s \). An auxiliary parameter with dimensions of inverse length is used to keep the physical dimensions right and we shall go to the physical limit \( E_{\infty} = \lim_{s \to 1} 4 E^{C}(s) \) at the end of the process.

C. Partition and generalized zeta functions

Because analytical information about the spectrum of \( K \) is only available at the limit of large \( l \) (bound state energies, phase shifts and spectral densities) it is better to consider the partition or heat functions:

\[
T_{\text{L}} e^{K} = \sum_{l=1}^{Z_{l}} \frac{1}{2} dke^{(2k+1)} + \frac{1}{2} dke^{2(k+1)} = \frac{1}{2} \left( e^{p} + e^{-p} \right) ; \quad 2 R
\]

Note that here we have replaced \( k \) and \( q \) used in Section 3.1 by \( k \) and \( q \) for a better comparison between the spectra of \( K_{0} \) and \( K \). The PBC spectral density of \( K_{0} \) is thus obtained by replacing \( p \) by \( 0 \). The \( K \)-heat function is also expressed in terms of integrals over the continuous spectrum at the limit \( l=1 \) in these, rather than in infinite sums. The integrals, however, must be weighted with the PBC spectral densities:

\[
T_{\text{L}} e^{K_{0}} = \sum_{l=1}^{Z_{l}} \frac{1}{2} dke^{(2k+1)} + \frac{1}{2} dke^{2(k+1)} = \frac{1}{2} \left( e^{p} + e^{-p} \right) ; \quad 2 R
\]
The error and complementary error functions of \( \text{erf} \) and \( \text{erfc} \), a citious inverse temperature, arise and the asterisk means that we have not included the zero mode because zero modes do not enter the one-loop formula \( \text{Eq} \).

The generalized zeta functions are Mellin transforms of the heat functions:

\[
K_0(s) = \frac{1}{s}\int_0^1 \text{d} t t^{s-1} e^{-t K_0} = \frac{1}{2} \int_0^1 \text{d} k \left( \frac{k^2 + 1}{k^2 + 2} \right)^{s/2} + \frac{1}{2} \int_0^1 \text{d} k \left( \frac{k^2 + 1}{k^2 + 2} \right)^{s/2} = \frac{1}{4} (s \frac{1}{2}) \quad 1 + \frac{1}{\Gamma s 1}
\]

\[
K(s) = \frac{1}{s}\int_0^1 \text{d} t t^{s-1} e^{-t K} = K_0(s) + \text{polylog}_s(1, 2) \quad \frac{2}{s} (s \frac{1}{2}) \quad 1 + \frac{1}{\Gamma s 1}
\]

We indeed need meromorphic functions of \( s \) with poles and residues determined from the poles and residues of Euler (s) and Gaussian hypergeometric \( 2F_1(\alpha; \beta; \gamma; z) \) functions.[24]

In the Appendix I we show that the kink Casimir energy in the physical limit \( s = \frac{1}{2} \) is the divergent quantity:

\[
E^C = \lim_{s \to 0} \frac{2}{s} + 2 \text{ln} \frac{2}{s} + \text{ln} \frac{16}{(1 - s^2)} = 2 + \frac{2F_1(0; z)}{2} \left( \frac{z}{1} \right)^{1/2}
\]

where \( 2F_1(0; \gamma; \beta; \alpha; z) \) is the derivative of the Gaussian hypergeometric function with respect to the second argument.

\[
E^C = \sum_{n=1}^{\infty} \text{polylog}_s(1, 2) = \sum_{n=1}^{\infty} \frac{1}{2n}
\]

such that \( \text{Eq} \):

\[
E^M_R(s) = \frac{2}{s} + \frac{2}{2s} \left( \frac{z}{1} \right)^{s} \left( \frac{z}{1} \right)^{s} \left( \frac{z}{1} \right)^{s} \left( \frac{z}{1} \right)^{s}
\]

In the Appendix I it is proved that the physical limit \( s = \frac{1}{2} \) is also a pole of \( E^M_R(s) \):

\[
E^M_R = \frac{2}{s} \lim_{s \to 0} \frac{2}{s} + 2 \text{ln} \frac{2}{s} + 2 \text{ln} \frac{2}{s} = \frac{2}{s} \text{arcsec} \quad \frac{2}{2}
\]

The divergent terms in \( E^C \) \( \text{Eq} \) and \( E^M_R \) \( \text{Eq} \), as well as the \( s \)-dependent terms, cancel each other exactly and the one-loop kink mass shift is:

\[
E = \frac{2}{s} + \frac{2}{2s} \left( \frac{z}{1} \right)^{s} \left( \frac{z}{1} \right)^{s} \left( \frac{z}{1} \right)^{s} \left( \frac{z}{1} \right)^{s}
\]

In formula \( \text{Eq} \) we have also written the result found in our derivation a la Cahill-Cottingham, the quantum correction, see \( \text{Eq} \). The heat kernel/zeta function result is \( f(\cdot) \) whereas the CCH formula leads to \( g(\cdot) \), where

\[
f(\cdot) = 1 + \frac{1}{2} \left( \frac{z}{1} \right)^{s} \left( \frac{z}{1} \right)^{s} \left( \frac{z}{1} \right)^{s} \left( \frac{z}{1} \right)^{s} \right) \left( \frac{z}{1} \right)^{s} \left( \frac{z}{1} \right)^{s} \left( \frac{z}{1} \right)^{s} \left( \frac{z}{1} \right)^{s}
\]

\[
g(\cdot) = 2 \left( \frac{z}{1} \right)^{s} \left( \frac{z}{1} \right)^{s} \left( \frac{z}{1} \right)^{s} \left( \frac{z}{1} \right)^{s}
\]

Despite appearances, \( f(\cdot) \) and \( g(\cdot) \) are identical functions of \( z \) \( \text{Eq} \), as the Mathematica plots in the Figure 1 show. This is remarkable; there is no mention about the analytic identity between the functions \( f(\cdot) \) and \( g(\cdot) \) in the ample literature on special functions. Nevertheless, they trace identical curves as functions of \( z \).
Figure 1: Graphics of $f(\ )$ (left), $g(\ )$ (center), and $f(\ )$ and $g(\ )$ plotted together (right). In the Figure, is labeled as $s$ in the abscissa axis.

VI. HIGH-TEMPERATURE ASYMPTOTIC EXPANSION

The exact heat or partition function can be written in the form:

$$T_n = \text{tr} \left( T_r^2 e^{K_1} \right)^n,$$

$$0 \quad T_r^2 e^{K_2} = \frac{1}{4} + e^2 \text{Erf} \left( \frac{P}{2} \right) \text{tr} e^{0 \ e^2};$$

where $\text{tr}^m$ means trace in the matrix sense. There is an alternative way of computing this quantity by means of a high-temperature asymptotic expansion. Although we have the exact formula in our system, we shall also perform the approximate calculation, which is only one possible in other systems in order to gain control of this second approach in this favorable case.

In the Appendix II it is shown how the coefficients of the power expansion of the K-heat trace

$$T_n = \frac{1}{4} \sum_{n=0}^{\infty} c_n(K) n \text{tr} e^{0 \ e^2};$$

(20)

the Seeley coefficients $c_n(K)$, are obtained through integration of the Seeley densities over the whole line. The densities satisfy recurrence relations tantamount to the heat kernel equation starting from a general potential $U(x)$. In our problem we must solve the recurrence relations between these densities for the potential $U(x) = \frac{2}{\cos^2 x}$, essentially the same potential as for the sine-Gordon kink, see [11]. We list these coefficients up to the twentieth order in Table IV:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$c_n(K) - 2^n \ 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
</tr>
<tr>
<td>4</td>
<td>0.5</td>
</tr>
<tr>
<td>5</td>
<td>0.5</td>
</tr>
<tr>
<td>6</td>
<td>0.5</td>
</tr>
<tr>
<td>7</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Table IV: Seeley Coefficients

Write now the spectral zeta functions in the form:

$$k(s) = \kappa_0(s;b) + B_2 k_2(s;b)$$

$$= \frac{1}{(s) \ 4 \ 	ext{tr} d_{\frac{1}{2}} + \text{tr} d_{\frac{1}{2}} + \text{tr} d_{\frac{1}{2}} + \text{tr} d_{\frac{1}{2}} \ 0 \ e^2};$$

$$= \frac{1}{(s) \ 4 \ 	ext{tr} [s \ \frac{1}{2}; b] + \text{tr} [s \ \frac{1}{2}; b] + \text{tr} [s \ \frac{1}{2}; b]};$$

$$= \frac{1}{(s) \ 4 \ 	ext{tr} [s \ \frac{1}{2}; b] + \text{tr} [s \ \frac{1}{2}; b]};$$
The incomplete Euler Gamma functions \( \Gamma(z) \) are meromorphic functions of \( z \) whereas \( B_K(z;b) \) and \( B_K(z) \) are entire functions of \( s \). The splitting point of the Mellin transform is usually taken at \( b = 1 \). We leave \( b \) as a free parameter for reasons to be explained later.

Neglecting the entire parts, the zero-point energy renormalization

\[
\tilde{K}(s;b) = \frac{1}{2} \frac{\Gamma}{4} (s) c_n(K) \left[ s + n \frac{1}{2} b \right] 0
\]

gets rid of the \( c_0(K) \) term. The contribution of \( c_1(K) \)

\[
4 E_{(1)}^{c} = \frac{1}{2} \frac{\Gamma}{4} (s) \lim_{s \to 0} \frac{1}{2} \frac{\Gamma}{2} (s) \left[ s + n \frac{1}{2} b \right] 0
\]

is exactly canceled by the mass renormalization counter-term \( s \):

\[
4 E_{(1)}^{m} = \frac{1}{2} \frac{\Gamma}{4} (s) \lim_{s \to 0} \frac{1}{2} \frac{\Gamma}{2} (s) \left[ s + n \frac{1}{2} b \right] 0
\]

We must now subtract the contribution of the zero mode:

\[
\tilde{K}(s;b) = \tilde{K}(s;b) \frac{1}{2} \frac{\Gamma}{4} (s) \lim_{s \to 0} \frac{1}{2} \frac{\Gamma}{2} (s) \left[ s + n \frac{1}{2} b \right] 0
\]

Finally, the high-temperature one-loop correction to the \( K \) kink energy is:

\[
4 E(b) = \frac{1}{2} \frac{\Gamma}{4} (s) \lim_{s \to 0} \frac{1}{2} \frac{\Gamma}{2} (s) \lim_{s \to 0} \frac{1}{2} \frac{\Gamma}{2} (s) \left[ n + 1 \frac{1}{2} b \right] 0
\]

In practice, truncation of the series is also necessary:

\[
4 E(b;N_0) = 1 \frac{\Gamma}{4} (s) \frac{\Gamma}{4} (s) \frac{\Gamma}{4} (s) \lim_{s \to 0} \frac{1}{2} \frac{\Gamma}{2} (s) \left[ n + 1 \frac{1}{2} b \right] 0
\]

Using formula (21) to calculate the one-loop kink mass shift, we admit an error of:

\[
4 E(b;N_0) = 1 \frac{\Gamma}{4} (s) \frac{\Gamma}{4} (s) \frac{\Gamma}{4} (s) \lim_{s \to 0} \frac{1}{2} \frac{\Gamma}{2} (s) \left[ n + 1 \frac{1}{2} b \right] 0
\]

We offer a Figure where formula (21) has been applied for \( N_0 = 20 \) and several values of \( b \). The very good precision of the asymptotic formula was achieved by adapting the parameter \( b \) to the value of \( b \). For instance, we have taken \( b = 1000 \) for \( b = 0.1 \), \( b = 50 \) for \( b = 0.5 \), \( b = 20 \) for \( b = 0.7 \), \( b = 10 \) for \( b = 0.9 \), and \( b = 10 \) for \( b = 1 \). Physically, this means that the lighter the particle mass \( (m^2) \) is, the longer the integration interval in the Mellin transform must be taken to minimize the error produced by the neglected entire parts. In practice, we have chosen \( b \) in each case at the frontier near the point \( 0 \) for \( \left( b, 1 \right) \), where the asymptotic formula of the \( K \) heat trace departs from its exact value.
VII. CONCLUSIONS AND FURTHER COMMENTS

In sum, we may draw the following conclusions:

1. We have obtained the one-loop mass shift to the classical mass of the stable topological kink that exists in a massive anisotropic non-linear $S^2$-sigma model.

2. In the isotropic case, $\mu = 1$, our result agrees with the answer provided by other authors: the one-loop correction is twice (in modulus) the correction for the sine-Gordon kink, see [3] and [15].

3. Our procedure is based on the heat kernel/zeta function regularization method. The result is identical to the answer achieved by means of the Cahiell-Comtet-Glauber formula.

This is a remarkable fact: the CCH formula takes into account only the bound state eigenvalues and the thresholds to the two branches of the continuous spectrum of the Schrodinger operators that govern the small fluctuations. It is essentially finite. Our computation involves in finite renormalizations. The criterion chosen to set finite renormalizations-no modification of the particle mass at the one-loop level, equivalent to the vanishing tadpole criterion in linear sigma models-does exactly the job.

4. We have also derived a high-temperature approximated formula for the mass shift, relying on the heat kernel asymptotic expansion. We stress that we have in proved a form of weakness of our method. The approximation to the exact result was poor for light masses -non-dimensional mass $\mu < 1$- in the model studied in [15]. We have achieved a very good approximation in this paper even for light particles by enlarging the integration interval of the Mellin transform and considering an optimum number of Seeley coefficient. We believe that this is a general procedure, working also in models where the exact generalized zeta function is not available.

As a final comment, we look forward to addressing the quantization procedure for: (a) Multi-solitons and breather modes of this model. (b) Stable topological kinks that may arise in other massive non-linear sigma models or different potentials, e.g., quartic, and/or different target manifolds, e.g., $S^3$.

VIII. ACKNOWLEDGEMENTS

We thank the Spanish Ministerio de Educacion y Ciencia and Junta de Castilla y Leon for partial financial support under grants FIS2006-09417, GR 224, and SA 034A 08. JM G thanks the ESF Research Network CASIMIR for providing excellent opportunities on the Casimir effect and related topics like topological defect fluctuations.

APPENDIX I: Kink Casimir energy and mass renormalization near the pole

The Casimir kink energy is, see Section x. V C:

$$E^C = \lim_{s!} E^C(z) = \lim_{s!} \frac{2^z}{z^2} \frac{2^z}{z^2} \frac{(s + \frac{1}{2})}{(s)} \frac{(1)^{s+z} \frac{\Gamma(\frac{1}{2} + \frac{1}{2})}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\right)}}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\right)} \frac{1}{s^z \Gamma(\frac{1}{2} + \frac{1}{2})} \frac{1}{(\frac{1}{2} + \frac{1}{2})^{2z}}$$

$$= \lim_{s!} \frac{2^z}{z^2} \frac{2^z}{z^2} \frac{(s + \frac{1}{2})}{(s)} \frac{(1)^{s+z} \frac{\Gamma(\frac{1}{2} + \frac{1}{2})}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\right)}}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\right)} \frac{1}{s^z \Gamma(\frac{1}{2} + \frac{1}{2})} \frac{1}{(\frac{1}{2} + \frac{1}{2})^{2z}}$$
but $s = \frac{1}{2}$ is a pole of $E^C(s)$. To find the residue, we expand this function in the neighborhood of the pole by using the following results

$$\frac{2}{\Gamma(1/2)} {}_2F_1\left[\frac{1}{2}, n; \frac{3}{2}; \frac{1}{2} \right] = \frac{2}{\Gamma(1/2)} \frac{2^n (1/2)}{\Gamma(n + 1)} = \frac{2}{\Gamma(1/2)} \frac{2^n (1/2)}{\Gamma(n + 1)} .$$

where $\text{er}_{\nu}(x)$ is the derivative of the Gauss hypergeometric function with respect to the second argument and we made use of the fact that $\text{er}_{\nu}(1) = 1.$

The physical limit $s = \frac{1}{2}$ is also a pole of $E^{-R}(s)$, see Section VI.D:

$$E^{-R} = \lim_{s \to \frac{1}{2}} \frac{2}{\Gamma(1/2)} \frac{2^n (1/2)}{\Gamma(n + 1)} = \frac{2}{\Gamma(1/2)} \frac{2^n (1/2)}{\Gamma(n + 1)} .$$

APPENDIX II: The heat kernel expansion

Consider the $K_0$- and $K_3$-heat kernels:

$$\begin{align*}
\frac{\partial}{\partial t} + K_0 \quad & K_0(x; y; t) = 0 ; \quad K_0(x; y; 0) = (x, y) \\
\frac{\partial}{\partial t} + K_3 \quad & K_3(x; y; t) = 0 ; \quad K_3(x; y; 0) = (x, y) .
\end{align*}$$

which provide an alternative way of writing the $K_0$- and $K_3$-heat traces:

$$\begin{align*}
\text{Tr}_{L^2} e^{-K_0} = \lim_{t \to \frac{1}{2}} \int \frac{1}{2} dx K_0(x; x; t) ; \\
\text{Tr}_{L^2} e^{-K_3} = \lim_{t \to \frac{1}{2}} \int \frac{1}{2} dx K_3(x; x; t) .
\end{align*}$$

Note that the form of the $K_3$-heat equation $\frac{\partial}{\partial t} = \frac{\partial}{\partial t} + K_0 \quad U(x) \quad K_K(x; y; t) = 0$, suggests a solution based on the $K_0$-heat kernel: $K_K(x; y; t) = C_K(x; y; t) U(x) C_K(x; y; t)$. The density $C_K(x; y; t)$ satisfies the in time temperature condition $C_K(x; y; 0) = \delta_0$. With the transfer equation:

$$\frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \frac{\partial^2}{\partial x^2} C_K(x; y; t) = U(x) C_K(x; y; t) .$$

Next we seek a power series solution, $C_K(x; y; t) = \sum_{n=0}^{\infty} c_n(x; y; t) \delta^n$, of $\frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \frac{\partial^2}{\partial x^2} C_K(x; y; t)$, which becomes tantamount to the recurrence relations:

$$nc_n(x; y; t) + (x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}) C(x; y; t) = \frac{\partial}{\partial x} (x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}) C(x; y; t) .$$

In fact, only the densities at coincident points $x = y$ on the line are needed. We introduce the notation $(k) c_n(x) = \lim_{x \to y} \frac{\partial^k}{\partial x^k} (x; y)$ to write the recurrence relations for the Seeley densities (and their derivatives) in the abbreviated form:

$$\begin{align*}
(k) c_n(x) = \frac{1}{n + k} 4^{(k - 2)} c_{n - 1}(x) \frac{\partial^k}{\partial x^k} U(x) c_{n - 1}(x) ;
\end{align*}$$

The (Seeley) coefficients $c_n(K)$ are the integrals over the line of the densities $c_n(x; x)$, i.e., $c_n(K) = \int_{-1}^{1} dx c_n(x; x).$

Without loss of generality, we choose the parameters such that: $\frac{1}{2} > \frac{s}{2} > 0$. Strictly speaking, Mellin transforms are defined in their fundamental strips, respectively $\text{Re} s > 1 - 2$, $\text{Re} s > 0$ in our problem. In the spirit of zeta function regularization, we extend the results of the Mellin transforms to the whole complex $s$-plane by analytic continuation.

The differential operators $K_{211}$ and $K_{222}$ are defined in Page 4.