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TESIS DOCTORAL

Control predictivo basado en modelos *fuzzy* de sistemas complejos. Aplicación al control y supervisión de procesos de depuración de aguas

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Universidad de Salamanca
Departamento de Informática y Automática
Facultad de Ciencias

Salamanca

Octubre, 2021

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A mis padres, Mª Victoria y Maximiano,
a mi mujer, Marta, a mis hijos, Pedrito y Martín,
a mi hermana y hermanos, Mª Victoria, Carlos, Nino y Maxi.

A las causas nobles.

Motivación: los avances científicos
deben beneficiar desinteresadamente
a toda la sociedad.

Tesis por compendio de artículos publicados

La presente Tesis Doctoral se presenta mediante la modalidad de compendio de artículos, estando integrada por tres artículos publicados en tres revistas diferentes, indexadas en *Journal Citation Reports* (JCR). A continuación se especifican las correspondientes citas bibliográficas y una referencia a los *índices de impacto JCR* (II-JCR) de cada revista (conocidos en la fecha de la publicación del artículo) y los correspondientes quartiles (Qi):

1. **Artículo nº1**
   Vallejo, P.M.[1] & Vega, P.[1] (2019). Analytical Fuzzy Predictive Control Applied to Wastewater Treatment Biological Processes. *Complexity*, *2019*, 5720185. https://doi.org/10.1155/2019/5720185
   II-JCR (JIF/JCI) y Qi (2017): 1.829/1.18; Q2/Q1 (Multidisciplinary Sciences)

2. **Artículo nº2**
   Vallejo, P.M.[1] & Vega, P.[1] (2021). Practical Computational Approach for the Stability Analysis of Fuzzy Model-Based Predictive Control of Substrate and Biomass in Activated Sludge Processes. *Processes*, *9*(3), 531. https://doi.org/10.3390/pr9030531
   II-JCR (JIF/JCI) y Qi (2019): 2.753/0.55; Q2/Q2 (Engineering, Chemical)

3. **Artículo nº3**
   Vallejo, P.M.[1] & Vega, P.[1] (2021). Integración de la estrategia FMBPC en una estructura de Control Predictivo en Lazo Cerrado. Aplicación al control de fangos activados. *Revista Iberoamericana de Automática e Informática Industrial (RIAI)*, *0*, 1-14. https://doi.org/10.4995/riai.2021.15793
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   Nota.- En el momento de la edición de esta Memoria, el DOI asignado al artículo nº3 aún no estaba activo (debido al poco tiempo transcurrido desde su asignación). Si aún estuviese inactivo en el momento de intentar acceder, se sugiere al lector que utilice uno de los siguientes enlaces de la revista RIAI (4º cuatrimestre de 2021):
   https://polipapers.upv.es/index.php/RIAI/article/view/15793/14338 (artículo nº3)
   https://polipapers.upv.es/index.php/RIAI/issue/view/761 (repositorio *preprint*)

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Autorización de la Directora de la Tesis

La **Dra. Pastora I. Vega Cruz**, catedrática del Dpto. de Informática y Automática (área de Ingeniería de Sistemas y Automática) de la Universidad de Salamanca

CERTIFICA

Que el presente trabajo titulado *Control predictivo basado en modelos fuzzy de sistemas complejos. Aplicación al control y supervisión de procesos de depuración de aguas* ha sido desarrollado bajo su supervisión por el doctorando **Pedro-Martín Vallejo LLamas** y constituye su Tesis Doctoral, en la modalidad de compendio de artículos, para optar al grado de **Doctor por la Universidad de Salamanca** (programa de Doctorado en en *Ingeniería Informática*).

Nota.- Se incluye la certificación en documento anexo (a continuación).

Salamanca, 27 de septiembre de 2021

Dra. Pastora Isabel Vega Cruz, con DNI 7833143X, quiere

HACER CONSTAR

que el trabajo titulado “CONTROL PREDICTIVO BASADO EN MODELOS FUZZY DE SISTEMAS COMPLEJOS. APLICACIÓN AL CONTROL Y SUPERVISIÓN DE PROCESOS DE DEPURACIÓN DE AGUAS” realizado por D. Pedro Vallejo Llamas ha sido realizado bajo su dirección y constituye la memoria de su trabajo de Tesis doctoral dentro del Programa de doctorado en Ingeniería Informática, y manifiesta su aceptación para que sea defendido en la modalidad de compendio de artículos.

Y para que conste firmo la presente en Salamanca a 27 de septiembre de 2021

Fdo. Pastora Isabel Vega Cruz

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La normativa de la Universidad de Salamanca para la presentación de la Tesis Doctoral mediante el formato de compendio de artículos establece que la memoria de tesis deberá incluir un informe de la Comisión Académica del Programa de Doctorado relativo a la presentación mediante esa modalidad. Tal informe será emitido tras la presentación de esta memoria, luego la inclusión en la misma podría hacerse con posterioridad, para la versión final imprimible de la memoria, insertando el documento a continuación de esta página, o bien constituir un anexo documental independiente posterior.

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Agradecimientos

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Resumen

El Control Predictivo basado en Modelos (MPC) es un caso particular de estrategia de control automático de procesos que abarca un conjunto de procedimientos cuyo denominador común es la utilización de un modelo de predicciones para determinar una ley de control óptima. El tipo de modelo elegido, los criterios de optimización y el procedimiento de deducción de la ley de control caracterizan cada una de las múltiples alternativas de MPC que existen. El control predictivo es una consolidada y, al mismo tiempo, prometedora estrategia de control con múltiples aplicaciones en el ámbito industrial y con numerosas líneas de investigación abiertas. Una de las modalidades de este tipo de control es el denominado Control Predictivo basado en Modelos Borrosos (FMBPC), que utiliza modelos cualitativos basados en reglas, globalmente no lineales, para representar el proceso a controlar. El control FMBPC está enmarcado en el subcampo del Control Predictivo No Lineal (NLMPC/NMPC) y al mismo tiempo pertenece también, parcialmente al menos, al campo del Control Inteligente (IC), debido a que utiliza una de las herramientas características de la inteligencia artificial, como es la lógica borrosa. En la Tesis Doctoral que aquí se presenta se considera una estrategia FMBPC cuyo modelo base es un modelo borroso, o *Fuzzy Model* (FM) en la literatura en inglés, de tipo *Takagi-Sugeno* (TS), obtenido mediante identificación a partir de series de datos numéricos de entrada-salida (que pueden ser datos estrictamente experimentales o adaptaciones de estos, generados en simulación). Esta característica dota a nuestra estrategia FMBPC de una interesante cualidad que aporta valor añadido dentro del campo del control NMPC, consistente en la útil información cualitativa implícita en el modelo borroso, consecuencia de la capacidad que tiene la identificación borrosa de capturar fielmente la dinámica de un sistema a partir de datos numéricos. Esta propiedad repercute directamente de forma positiva en la validez de las predicciones y supone, en última instancia, un incremento significativo del rendimiento o desempeño del algoritmo de control predictivo, en el caso de tratar con sistemas fuertemente no lineales, complejos o desconocidos. Esta es la razón por la que en esta tesis se propone la estrategia FMBPC como la idónea para abordar el control de un cierto tipo de procesos conocidos como *Procesos de Fangos Activados* (ASP), muy habituales como mecanismo de depuración biológica en Estaciones Depuradoras de Aguas Residuales (EDAR) (también conocidas en la literatura en inglés como *Wastewater Treatment Plants (WWTP)*). El interés de la propuesta es doble: por un lado, contribuir a ampliar las líneas de investigación en el campo del control predictivo no lineal y por otro, aportar una estrategia y una metodología que puedan ser útiles en la mejora de los procesos de depuración de aguas, cuya importancia en la salud pública y en

el cuidado del medio ambiente es creciente, cono así se refleja en las legislaciones medioambientales, cada vez más exigentes.

Una parte importante del esfuerzo investigador desarrollado en la presente tesis ha sido enfocado a la aplicación de la estrategia FMBPC propuesta al paradigmático caso de estudio elegido (procesos biológicos ASP en plantas depuradoras de aguas residuales). Dadas las características de estos procesos, principalmente su alta no linealidad, su complejidad intrínseca y su carácter multivariable, derivadas de su naturaleza biológica, las investigaciones realizadas pueden trascender más allá del mero ámbito del propio proceso. La implementación práctica se ha llevado a cabo mediante simulación y ello ha supuesto un importante reto, principalmente en dos aspectos: por un lado, el desarrollo del software necesario y por otro, la implementación de los cálculos matemáticos apropiados.

La investigación realizada puede descomponerse, de una manera esquemática, en las siguientes cuatro fases o etapas: a) identificación borrosa del proceso ASP a partir de datos numéricos de entrada-salida y conversión del modelo borroso obtenido en un modelo equivalente en el espacio de estados, discreto, lineal y variante en el tiempo (DLTV); b) determinación de una ley de control predictivo de tipo FMBPC, analítica y explícita, siguiendo los principios del denominado Control Predictivo Funcional (PFC); c) análisis de estabilidad local en lazo cerrado de la estrategia FMBPC propuesta; d) integración de esta estrategia dentro de la configuración de control predictivo conocida como *Paradigma de Lazo Cerrado* (CLP), también llamada *control predictivo en lazo cerrado*, con el objetivo de imponer restricciones de manera automática en la acción de control.

Los resultados obtenidos son satisfactorios, principalmente en lo que se refiere a la demostración de la utilidad de la estrategia FMBPC propuesta como una alternativa válida en el campo del control predictivo no lineal, para sistemas complejos o desconocidos, con dos ventajas destacables en relación con otras estrategias, a saber: por un lado, la útil información contenida en el modelo base de las predicciones, capturada durante el proceso de identificación borrosa previo a la aplicación de la estrategia y, por otro, la forma analítica y explícita de la ley de control deducida, que facilita tanto la implementación del algoritmo de control como las tareas de análisis (entre ellas, las de análisis estabilidad).

Palabras clave: Control predictivo basado en modelos, Modelado e identificación borrosa, Control inteligente, Control de sistemas con restricciones, Control multivariable, Plantas de tratamiento de aguas residuales, Procesos de fangos activados.

Abstract

Model-based Predictive Control (MPC) is a particular case of automatic process control strategy that encompasses a set of procedures whose common denominator is the use of a predictions model to determine an optimal control law. The type of model chosen, the optimization criteria and the control law deduction procedure characterize each of the multiple MPC alternatives that exist. Predictive control is a consolidated control strategy and, at the same time promising, with multiple applications in the industrial field and with numerous open research lines. One of the modalities of this type of control is the so-called Fuzzy Model-based Predictive Control (FMBPC), which uses qualitative models based on rules, globally non-linear, to represent the process to be controlled. The FMBPC control is framed in the subfield of Nonlinear Predictive Control (NLMPC/NMPC) and at the same time also belongs, partially at least, to the field of Intelligent Control (IC), because it uses one of the characteristic tools of artificial intelligence, as is the fuzzy logic. In the Doctoral Thesis presented here, a FMBPC strategy is considered whose base model is a *Fuzzy Model* (FM) of type *Takagi-Sugeno* (TS), obtained by identification from series of numerical input-output data (which can be strictly experimental data or adaptations of these, generated in simulation). This characteristic provides our FMBPC strategy an interesting quality that incorporates added value within the field of NMPC control, consisting of the useful qualitative information implicit in the fuzzy model, a consequence of the ability of the fuzzy identification to accurately capture the dynamics of a system from numerical data. This property has a direct positive impact on the validity of the predictions and ultimately represents a significant increase in the performance of the predictive control algorithm, in the case of dealing with strongly non-linear, complex or unknown systems. This is the reason why in this thesis the FMBPC strategy is proposed as the ideal one to address the control of a certain type of process known as *Activated Sludge Process* (ASP), very common as a biological purification mechanism in *Wastewater Treatment Plants (WWTP)*. The interest of the proposal is twofold: on the one hand, to contribute to broadening the lines of research in the field of non-linear predictive control and, on the other, to provide a strategy and a methodology that may be useful in improving the waters purification processes, whose importance in public health and in caring for the environment is growing, as is reflected in environmental legislation, increasingly demanding.

An important part of the research effort developed in this thesis has been focused on the application of the proposed FMBPC-strategy to the chosen paradigmatic case study (ASP biological processes in wastewater treatment plants). Given

the characteristics of these processes, mainly their high non-linearity, their intrinsic complexity and their multivariable character, derived from their biological nature, the investigations carried out can transcend beyond the process itself. The practical implementation has been carried out by simulation and this has been an important challenge, mainly in two aspects: on the one hand, the development of the necessary software and on the other, the implementation of the appropriate mathematical calculations.

The research carried out can be broken down, in a schematic way, into the following four phases or stages: a) fuzzy identification of the ASP process from numerical input-output data and conversion of the fuzzy model obtained into an equivalent model in the state space, discrete, linear and time-varying (DLTV); b) determination of an analytical and explicit FMBPC-type predictive control law, following the principles of the so-called Functional Predictive Control (PFC); c) closed-loop local stability analysis of the proposed FMBPC-strategy; d) integration of this strategy within the predictive control configuration known as *Closed Loop Paradigm* (CLP), also called *closed-loop predictive control*, with the aim of automatically imposing restrictions on the control action.

The results obtained are satisfactory, mainly with regard to the demonstration of the usefulness of the proposed FMBPC-strategy as a valid alternative in the field of non-linear predictive control, for complex or unknown systems, with two notable advantages in relation to other strategies, namely: on the one hand, the useful information contained in the base model of the predictions, captured during the fuzzy identification process prior to the application of the strategy and, on the other, the form analytical and explicit of the deduced control law, which facilitates both the implementation of the control algorithm and the analysis tasks (among them, stability analysis).

Keywords: Model-Based Predictive Control, Fuzzy Modeling and Identification, Intelligent Control, Control of Constrained Systems, Multivariable Control, Wastewater Treatment Plants, Activated Sludge Processes.

CONTENIDO

Capítulo 1

Introducción

La presente Tesis Doctoral se enmarca, en general, en el ámbito científico-técnico del Control Automático (CEA, 2009; Kuo, 2003; Ogata, 2003) y, en particular, en el campo del Control Predictivo basado en Modelos (*Model-based Predictive Control*) (MPC/MBPC) (Camacho & Bordons, 1999, 2004; Limón, 2002; Richalet, 1993; Rossiter, 2003) y dentro de él, en el subcampo del Control Predictivo basado en Modelos Borrosos (*Fuzzy Model-based Predictive Control*) (FMBPC) (Babuška, 1998; Boulkaibet et al., 2017; Bououden et al., 2013; Bououden et al., 2015; Mollov et al., 2004; Mollow, 2002; Roubos et al., 1999). Este tipo de control (FMBPC) puede considerarse también incluido (parcialmente) en el campo del Control Inteligente (*Intelligent Control*) (IC) (Jimenez & Al-Hadithi, 2014; White & Sofge, 1992), debido a la utilización de razonamiento cualitativo en el proceso de obtención (identificación) del modelo base borroso (*Fuzzy Identification*) (Babuška, 1998; Babuška et al., 1998). Las investigaciones llevadas a cabo con motivo de la presente tesis giran en torno al diseño, implementación (en simulación) y análisis de un caso particular de estrategia FMBPC, aplicada al control de cierto tipo de proceso biológico de depuración de aguas residuales, conocido como *Proceso de Fangos Activados* (*Activated Sludge Process*) (ASP) (Wahlberg, 2019), que será representado en los experimentos de simulación mediante su modelo matemático conocido como ASM1 (Henze et al., 1987), simplificado a efectos de estudio. Por tanto, la tesis se sitúa también en el ámbito de las disciplinas relacionadas con los procesos de depuración biológica de aguas y con las plantas de tratamiento en las que se llevan a cabo tales procesos, llamadas Estaciones Depuradoras de Aguas Residuales (EDAR) (también conocidas en la literatura en inglés como *Wastewater Treatment Plants (WWTP)*) (X. Chen et al., 2018; Pell & Wörman, 2011).

1.1. Planteamiento y antecedentes

La estrategia de control MPC clásico consiste, básicamente, en el uso de un modelo de predicciones para determinar las acciones de control necesarias en cada instante, imponiendo la minimización de una *función de coste* o *función objetivo* (que generalmente incluirá un término dependiente del error y otro dependiente de los esfuerzos de control, entre otras posibles alternativas). Esta estrategia incluye múltiples variantes o enfoques, dependiendo de varios factores, principalmente los tres siguientes: el tipo de modelo que se utilice para calcular las predicciones, el procedimiento para deducir la ley de control y la forma matemática final de esta ley. Una parte importante de las variantes más consolidadas de MPC toman como base un modelo lineal del proceso (MPC lineal) y determinan la ley de control mediante optimización (imponiendo el mínimo de la función de coste elegida). Otros métodos o estrategias de control predictivo toman como base de las predicciones modelos matemáticos no lineales, como modelos borrosos, también llamados difusos (*Fuzzy Model*) (FM) (Zadeh, 1990), modelos formalizados por medio de redes neuronales artificiales (*Artificial Neural Networks*) (ANN) (S. Chen & Billings, 1992), u otras alternativas, pero determinan la señal de control del mismo modo que en control MPC lineal tradicional, es decir, mediante algún procedimiento de optimización aplicado a la función de coste previamente definida. Haciendo uso de estas estrategias, pertenecientes al denominado Control Predictivo No Lineal (*Non-Linear Predictive Control*) (NLMPC/NMPC), generalmente puede conseguirse un mejor desempeño del algoritmo de control, en el supuesto de que el modelo represente más fielmente el proceso, pero normalmente se tendrá también como contrapartida una mayor complejidad en el proceso de modelado, en el proceso de deducción de la ley de control, en los procesos de cálculo y computación, o en todos ellos. En (Boulkaibet et al., 2017) puede consultarse una amplia y ordenada revisión bibliográfica de la evolución del control predictivo MPC, tanto lineal, como no lineal, incluyendo numerosas contribuciones y enfoques en este campo del control de procesos.

La estrategia de control predictivo FMBPC propuesta en la presente Tesis Doctoral se enmarca dentro de la categoría de las estrategias de control predictivo basadas en modelos no lineales borrosos. Pertenece por tanto a la categoría de control predictivo NMPC, pero con la particularidad de que la determinación de la ley de control no se realiza mediante un procedimiento clásico de optimización, sino que es resultado de aplicar el denominado *Principio de Equivalencia*, propio del paradigma de control predictivo conocido como Control Predictivo Funcional (*Predictive Functional Control*) (PFC) (Haber et al., 2016; Richalet, 1993; Richalet & O'Donovan, 2009; Škrjanc & Matko, 2000). Nuestra estrategia utiliza un tipo de algoritmo de control predictivo con varias características específicas diferentes a las de las estrategias clásicas de MPC. Destacamos, principalmente, las dos siguientes: la utilización de un modelo no lineal borroso como base de las predicciones y la deducción de la ley de control aplicando el mencionado *Principio de Equivalencia*.

La estrategia considerada tiene además otras características especiales, relacionadas con el procedimiento previo de identificación y modelado. Por un lado, el modelo base de predicciones es un modelo borroso, de tipo *Takagi-Sugeno* (TS) (Takagi & Sugeno, 1985), obtenido mediante identificación borrosa a partir de datos numéricos de entrada-salida, directamente relacionados con campañas reales llevadas a cabo en una depuradora de aguas residuales industrial (EDAR municipal de la localidad de Manresa, provincia de Barcelona, España; en la década de los 90) (Moreno, 1994), utilizando datos estrictamente experimentales o bien adaptaciones de ellos (generados mediante simulación). Y por otro lado, el modelo borroso obtenido es convertido, mediante el apropiado tratamiento matemático, en un modelo discreto equivalente en el espacio de estados, de tipo lineal y variante en el tiempo (*Discrete Linear Time-Varying*) (DLTV).

Como consecuencia de las características descritas, la estrategia propuesta tiene importantes ventajas en relación con otras estrategias de control predictivo, destacando las siguientes: una mayor capacidad para capturar de manera fiel la dinámica del proceso a controlar, a partir de datos numéricos de entrada-salida (capacidad reconocida a las técnicas de identificación borrosa) y una ley de control expresada en forma analítica y explícita, como resultado de la utilización del modelo DLTV deducido y de la aplicación del *Principio de Equivalencia*. La primera ventaja es especialmente útil en el caso de tratar con procesos altamente no lineales, complejos o desconocidos, pudiendo incrementarse el nivel de desempeño del algoritmo de control predictivo en relación con otras técnicas cuyo modelo base sea menos fiel. Y la segunda repercute positivamente en aspectos computacionales y de análisis, facilitando, tanto la implementación del algoritmo de control, como el análisis de estabilidad (u otros posibles).

Las investigaciones llevadas a cabo se han desarrollado en varias fases, las cuales describimos de manera resumida a continuación (en orden secuencial):

1. Identificación borrosa del proceso, modelado en el espacio de estados y deducción de una ley de control predictivo borroso de tipo FMBPC.

2. Análisis de estabilidad de la estrategia FMBPC mediante un enfoque práctico computacional.

3. Incorporación de un mecanismo de imposición de restricciones en la acción de

control, mediante la integración de la estrategia FMBPC en una estructura de control predictivo en modo dual denominada Paradigma de Lazo Cerrado (*Closed-Loop Paradigm*) (CLP o CLP-MPC), también llamada *control predictivo en lazo cerrado* (El Bahja, 2017; Rossiter, 2003).

En relación con los antecedentes de la presente tesis, desde un punto de vista general podríamos considerar como antecedente cualquier trabajo, general o aplicado, perteneciente al subcampo del control predictivo no lineal de tipo FMBPC. Sin embargo, bajo esta denominación caben diferentes arquitecturas y enfoques y por tanto resulta conveniente mencionar algunos trabajos previos relacionados más directamente (en distintos aspectos parciales) con las características específicas de la estrategia FMBPC propuesta en esta tesis.

Destacamos, en primer lugar, el siguiente grupo de referencias (ya citadas anteriormente), las cuales han sido una fuente relevante en el desarrollo inicial de nuestro trabajo: (Babuška, 1998; Mollov et al., 2004; Mollow, 2002; Roubos et al., 1999). La primera referencia corresponde a un libro donde se sientan las bases de la utilización del modelado borroso en sistemas de control, al tiempo que se desarrolla (complementariamente) una aplicación software para la identificación de modelos borrosos a partir de datos numéricos. Las otras tres referencias corresponden a diversos trabajos bastante interesantes sobre control FMBPC, siendo un referente muy significativo el trabajo correspondiente a la cuarta cita, puesto que en él se utiliza de forma específica la estrategia FMBPC con objetivos de control y es anterior en el tiempo a los otros dos trabajos, tomando además como base modelos borrosos de tipo TS, que son los elegidos también en la presente tesis. Las otras dos contribuciones (tercera y cuarta citas) son también dos antecedentes importantes, de una manera especial la tercera cita, al tratarse de un tesis doctoral, con el consiguiente volumen de trabajo que la sustenta.

En segundo lugar, consideramos un antecedente significativo el trabajo correspondiente a la siguiente referencia: (Blažič & Škrjanc, 2007). En este trabajo se presenta un algoritmo de control predictivo de tipo FMBPC, desarrollado en el espacio de estados y expresado en forma analítica y explícita, destacando sus autores que ello constituye una importante ventaja en comparación con los esquemas de control predictivo basados en optimización. La ley de control se deduce en este caso mediante la aplicación del *Principio de Equivalencia*, utilizado, como ya se mencionó anteriormente, en control PFC. Repetiremos también en este apartado las referencias ya citadas anteriormente relativas al control PFC: (Haber et al., 2016; Richalet, 1993; Richalet & O'Donovan, 2009; Škrjanc & Matko, 2000). La formalización en el espacio de estados y la posterior deducción de una ley de control analítica y explícita han sido también consideradas en el desarrollo de la estrategia presentada en esta tesis, pero con objetivos de aplicación al control un sistema con dinámica compleja, altamente no lineal y multivariable. Algunas otras contribuciones pertenecientes al ámbito del control FMBPC también consideradas, más recientes, son (entre otras) las siguientes (ya citadas anteriormente): (Boulkaibet et al., 2017; Bououden et al., 2013; Bououden et al., 2015).

En tercer lugar y en relación con el análisis de estabilidad de la estrategia FMBPC propuesta, las referencias que pueden ser consideradas como antecedentes son numerosas y en un amplio rango cronológico, incluyendo tanto las teorías más consolidadas sobre estabilidad, en general, como diversas variantes o enfoques de las mismas y gran cantidad de aplicaciones a casos de estudio concretos, en todos los campos y subcampos del control automático. En (Vallejo & Vega, 2021b) se detallan (en la sección introductoria) numerosas referencias al respecto. Citaremos únicamente algunas, empezando por una referencia especial, relativa a la paradigmática *Teoría de estabilidad de Lyapunov*, originariamente expuesta en idioma ruso en el año 1892, pero que fue publicada de nuevo en el año de su centenario (1992) y que puede ser consultada en (Lyapunov, 1992). Además, nos parecen muy relevantes (aunque hay muchas más) las tres referencias siguientes (mostradas en orden cronológico) relativas al análisis de estabilidad de sistemas borrosos, cuya dificultad es grande debido, entre otras razones, a la intrínseca no linealidad de este tipo de sistemas: la primera, (Tanaka & Sugeno, 1992), donde se discute tanto el análisis de estabilidad como el diseño de sistemas de control borrosos (*Fuzzy Control Systems*) (FCS); la segunda, (Al-Hadithi et al., 2007), donde se realiza (en una fecha más reciente) una revisión detallada del estado del arte del análisis de estabilidad de los sistemas borrosos (*Fuzzy Systems*) (FS) (hasta 2006); y la tercera, (Lendek et al., 2011), donde también se incluye una revisión de diversos trabajos relacionados con el análisis de estabilidad y el diseño de algoritmos de control, para sistemas no lineales representados por modelos borrosos de tipo TS, siendo el procedimiento general de estudio de la estabilidad utilizado en ellos, el método directo de Lyapunov.

En cuarto lugar, destacamos dos referencias relacionadas con la estructura CLP-MPC (anteriormente mencionada), la cual es utilizada en la presente tesis con el objetivo de imponer restricciones en la acción de control de la estrategia FMBPC propuesta. Las dos referencias son las siguientes (ambas ya citadas anteriormente): (El Bahja, 2017; Rossiter, 2003).

Y en quinto y último lugar de esta selección de referencias (que determinan, al menos parcialmente, los antecedentes más cercanos al trabajo abordado en la presente tesis), mencionaremos algunas citas relativas a las plantas de tratamiento de aguas residuales (WWTP) y a los procesos biológicos de depuración objeto de estudio (ASP), incluyendo modelado, identificación y control predictivo (MPC): (Babuška, 1998; Babuška et al., 1998; X. Chen et al., 2018; Francisco & Vega, 2006; Francisco et al., 2011; Henze et al., 1987; Moreno, 1994; Pell & Wörman, 2011).

Además de los trabajos citados en los párrafos anteriores, se han tenido en cuenta también otros muchos trabajos, los cuales están recogidos en las correspondientes bibliografías (Referencias) de los tres artículos que han sido publicados con motivo de la presente tesis (Vallejo & Vega, 2019, 2021a, 2021b).

1.2. Hipótesis de trabajo y objetivos

El Control Predictivo basado en Modelos (tanto MPC, como NLMPC), clásico, puede considerarse una de la estrategias de control más exitosas, tanto en el ámbito industrial como en el de la investigación científica en control automático, debido a las importantes ventajas que presenta (Camacho & Bordons, 2004; Limón, 2002), entre otras, las siguientes: la claridad y utilidad práctica de la formulación matemática del algoritmo de control (expresado en el dominio del tiempo, de manera flexible, abierta e intuitiva); su capacidad para tratar con sistemas de distintos tipos y características, tanto lineales como no lineales, monovariables y multivariables (con una estructura del algoritmo de control, básica, común para todos los casos); su especial característica de controlador óptimo (determinación de la ley de control mediante algún procedimiento de optimización) y la posibilidad de incluir restricciones en la síntesis del controlador de manera directa (en el marco de la formulación matemática del problema de optimización). Sin embargo, cuando el objetivo es el control de sistemas no lineales, y especialmente si estos presentan una alta no linealidad, el control predictivo clásico encuentra más dificultades para su implementación y eficacia. Por supuesto, se han desarrollado muchas alternativas válidas en el subcampo del control NLMPC, pero este ámbito el control predictivo admite aún bastante margen de mejora.

Una de las áreas de trabajo o aspectos relevantes en los que cabe investigar, con el propósito de intentar mejorar el rendimiento del control predictivo, es el modelado e identificación de procesos muy no lineales. El objetivo es obtener modelos que sean capaces de capturar las dinámicas no lineales lo más fielmente que sea posible. Para alcanzar ese objetivo, una de las herramientas científicas más adecuadas, con un amplio desarrollo y con un número de aplicaciones creciente (en muy diversos campos de la ciencia y la tecnología), es la denominada Lógica Borrosa o *razonamiento cualitativo (Fuzzy Logic)*, que constituye una subdisciplina de la Inteligencia Artificial (*Artificial Intelligence*) (AI). Esta disciplina, al igual que el control predictivo, puede considerarse también bastante exitosa en el campo del control automático, tanto en lo que se refiere a su utilización en la identificación de sistemas (*Fuzzy Identification*), como a su aplicación al control propiamente dicho (*Fuzzy Control*).

Teniendo en cuenta lo que acaba de ser expuesto, parece razonable pensar que la integración del control predictivo y de la lógica borrosa, utilizando esta en las tareas de modelado e identificación del proceso y diseñando un algoritmo de control predictivo basado en modelos borrosos (FMBPC), podría ser una buena alternativa dentro del campo del control NLMPC. En consonancia con ello, la principal hipótesis de trabajo de la presente tesis doctoral consiste en considerar que, en el marco de las estrategias de control predictivo NLMPC y para el caso de sistemas complejos altamente no lineales (o desconocidos), la utilización de modelos borrosos obtenidos mediante identificación borrosa, a partir de datos numéricos de entrada-salida, podría suponer una mejora importante en el rendimiento del controlador predictivo, o al menos en alguna o algunas de sus propiedades, debido a la previsible mejora en la capacidad de predicción del modelo base.

Por otra parte y en relación con el procedimiento de deducción de la ley de control, que en control predictivo clásico consiste en resolver un problema de optimización (minimización de la función objetivo), parece interesante intentar explorar otros métodos, que conduzcan a la obtención de una ley de control analítica y explícita, lo cuál podría suponer una mejora apreciable en dos aspectos importantes del control predictivo, a saber: el cálculo computacional asociado a la implementación del controlador y el análisis de estabilidad (entre otros). En esa línea de trabajo, podemos enunciar una segunda hipótesis, consistente en considerar que la formalización de los modelos borrosos, previamente identificados, en modelos equivalentes en el espacio de estados y la utilización del *Principio de Equivalencia* (propio del control predictivo PFC), podrían conducir a la obtención de una ley de control expresada en forma analítica y explícita, con las consiguientes mejoras para el controlador predictivo que acabamos de mencionar, incluso en casos de difícil tratamiento, como sistemas complejos, muy no lineales o multivariables (o incluso con varias de esas características).

Teniendo como norte la validación de las hipótesis de trabajo expuestas, los principales objetivos fijados para el desarrollo de la presente tesis doctoral son los siguientes:

- Determinar un procedimiento de identificación borrosa adecuado para obtener modelos borrosos fiables de procesos complejos, muy no lineales y/o multivariables, a partir de datos numéricos de entrada-salida.

- Aplicar algún método de conversión de modelos borrosos en modelos equivalentes en el espacio de estados, lineales y variantes en el tiempo (DLTV).

- Deducir una ley de control predictivo no lineal de tipo FMBPC, analítica y explícita, tomando como modelo base para las predicciones un modelo en el espacio de estados de tipo DLTV y aplicando el *Principio de Equivalencia* (propio del control predictivo PFC).

- Analizar la estabilidad de un sistema controlado con la ley FMBPC deducida y estudiar las condiciones que deberán satisfacerse para garantizar la estabilidad en lazo cerrado.

- Utilizar la estructura de control predictivo dual denominada *Paradigma de Lazo Cerrado* (CLP), también conocida como *control predictivo en lazo cerrado*, como posible mecanismo para imponer restricciones a la ley de control FMBPC deducida, integrando esta adecuadamente en la estructura CLP.

- Validar la estrategia FMBPC propuesta para el caso de un sistema con dinámica compleja, muy no lineal y multivariable, como son los procesos biológicos de depuración de aguas conocidos como fangos activados (ASP).

- Tomar como planta de referencia un modelo de una depuradora industrial, utilizando de forma útil registros de datos de entrada-salida procedentes de campañas reales de la depuradora.

1.3. Resumen de las principales contribuciones

Las contribuciones del trabajo desarrollado con motivo de la presente tesis aparecen reflejadas en las tres publicaciones incluidas en esta misma memoria. Sin embargo, parece conveniente anticiparse en esta sección introductoria y mostrar un resumen de tales aportaciones, para completar la visión panorámica de la tesis. Mostramos a continuación de forma resumida cuáles son las principales aportaciones:

- Sistematización de un procedimiento de identificación borrosa a partir de series de datos numéricos de entrada-salida, aplicable a cualquier sistema multivariable y especialmente adecuado para la identificación de sistemas complejos, muy no lineales o desconocidos. Está basado en métodos teóricos y en el uso de herramientas-software existentes, junto con el adecuado desarrollo de software. El procedimiento admite la incorporación (manual) de conocimiento experto previo, relativo a la dinámica del proceso, en la fase inicial de parametrización. El resultado de la identificación son modelos borrosos discretos de tipo TS expresados mediante reglas del tipo *si-entonces*.

- Identificación de un proceso de depuración biológica de fangos activados (ASP) a partir de datos de entrada-salida, recogidos en simulación en lazo abierto (estando el proceso representado por el modelo matemático no lineal estándar conocido como ASM1, simplificado a efectos de estudio). Para las entradas se tomaron como referencia valores típicos, extraídos de registros procedentes de campañas reales de una EDAR industrial (depuradora municipal de aguas residuales de la localidad de Manresa, en la provincia de Barcelona, España; década de los 90), utilizando directamente los datos recogidos o bien variaciones lógicas de los mismos. La realización de un gran número de pruebas con diferentes parámetros de identificación condujo a la obtención de modelos borrosos multivariables con altos índices de validación para las dos salidas consideradas, a saber: concentración de sustrato en el efluente y concentración de biomasa en el reactor.

- Formalización de los modelos borrosos discretos de tipo TS obtenidos mediante identificación, expresados mediante reglas del tipo *si-entonces*, en modelos equivalentes en el espacio de estados, discretos, lineales y variantes en el tiempo (modelos de tipo DLTV).

  Esta contribución tiene especial importancia, puesto que implica la posibilidad de que los modelos borrosos identificados puedan ser utilizados como modelos de predicciones en diferentes esquemas de control predictivo, ya consolidados, que trabajan con modelos en el espacio de estados.

- Deducción de una ley de control predictivo no lineal, de tipo FMBPC, analítica y explícita, utilizando como modelo de predicciones un modelo DLTV en el espacio de estados (equivalente al modelo *fuzzy* original) y aplicando el *Principio de Equivalencia*, propio del control predictivo PFC.

  Esta contribución es una de las más relevantes de esta tesis y tiene dos particularidades importantes a resaltar, en el campo del control predictivo no lineal, a saber: a) la generalización del *Principio de Equivalencia* (PFC) para un sistema no lineal y multivariable; b) la deducción de una ley de control predictivo mediante un procedimiento analítico directo, en lugar de hacerlo mediante un procedimiento de optimización, como es habitual en control predictivo clásico, con el resultado de la obtención de una ley de control analítica y explícita, con las consiguientes ventajas en el ámbito de la implementación de la acción de control (en el aspecto de la computación) y en el del análisis de la estrategia de control (análisis de estabilidad, entre otros posibles).

- Desarrollo de un procedimiento de análisis de estabilidad local en lazo cerrado para la estrategia FMBPC diseñada, con un enfoque computacional práctico. El procedimiento fue aplicado al caso de un proceso biológico de depuración de tipo ASP, comprobando su validez mediante test numéricos.

  El método desarrollado aborda la estabilidad en las cercanías de un punto de operación y se basa en la deducción de sendos modelos incrementales de tipo DLTI (discretos, lineales e invariantes en el tiempo), tanto para la planta en lazo abierto, como para la planta en lazo cerrado, válidos para estados suficientemente cercanos al punto de operación, cuya forma permite aplicar de manera directa los criterios prácticos de determinación de la estabilidad de Lyapunov, como el criterio de estabilidad asintótica (estabilidad interna), que establece como condición necesaria y suficiente para que un sistema sea asintóticamente estable, en el sentido de Lyapunov, que los autovalores de la matriz de estado estén estrictamente dentro del círculo unidad. Haciendo uso de tal criterio y mediante cálculo simbólico, se termina deduciendo la siguiente condición suficiente de estabilidad (local) en lazo cerrado: si la planta en lazo abierto es localmente asintóticamente estable (alrededor de algún punto de equilibrio), entonces para un *horizonte de coincidencia* suficientemente grande, la correspondiente planta en lazo cerrado también será localmente asintóticamente estable (alrededor de ese mismo punto de equilibrio).

- Diseño de una estrategia o arquitectura de control predictivo mixta, integrando el algoritmo FMBPC dentro de una estructura de control predictivo dual en lazo cerrado (basada en optimización) de tipo CLP, con el objetivo de imponer restricciones a la ley de control FMBPC. La estrategia mixta diseñada, además de servir para poder imponer restricciones a la ley FMBPC, constituye también en sí mismo una importante aportación en el ámbito del control predictivo, al colaborar entre sí dos estrategias de control predictivo diferentes: una basada en PFC (la estrategia FMBPC) y otra basada en optimización (la estructura CLP).

  La demostración de que la estrategia de control predictivo mixta diseñada cumple con su función (relativa a las restricciones) se basa en la identificación de la ley de control FMBPC con una ley de realimentación de estados del tipo $-kx$, forma a la que puede llegarse si se consideran los modelos DLTI mencionados en el anterior punto y mediante el adecuado desarrollo. Tomando esa ley como la ley de control base necesaria (durante las predicciones) para la estructura CLP, las restricciones que pueden imponerse en el marco de esta estructura se extenderán de manera directa a la ley FMBPC.

Por otra parte, teniendo en cuenta el análisis de estabilidad de la estructura CLP desarrollado en la literatura, según el cuál se garantizará la estabilidad en lazo cerrado si la ley de control base es una ley de control estabilizante, entonces la estructura de control predictivo mixta diseñada será estable en lazo cerrado, al ser la ley de control base -kx estabilizante.

- Validación de la estrategia FMBPC propuesta para el caso de un sistema con dinámica compleja, muy no lineal, multivariable y con fuertes perturbaciones en la entrada, como son los procesos biológicos de depuración de aguas conocidos como fangos activados (ASP).

- Desarrollo del software adecuado para la implementación, mediante simulación, de un sistema de control predictivo multivariable de tipo FMBPC aplicado a un proceso de fangos activados (ASP), representado mediante el modelo matemático no lineal estándar ASM1 (simplificado a efectos de estudio, como ya se indicó anteriormente), considerando una única entrada manipulable (el caudal de recirculación de fangos), dos perturbaciones de entrada con fuertes oscilaciones (el caudal de entrada y el sustrato de entrada) y dos salidas controladas (concentración de sustrato en el efluente y concentración de biomasa en el reactor).

  La realización de un gran número de experimentos de control predictivo, aplicando la estrategia FMBPC deducida al caso de estudio considerado (el proceso ASP), condujo a importantes conclusiones relativas al comportamiento del proceso de depuración de aguas, al rendimiento del controlador, a su estabilidad y al cumplimiento de restricciones en la acción de control, mediante las arquitecturas y configuraciones propuestas. Tales conclusiones han sido convenientemente descritas en cada una de las tres publicaciones científicas desarrolladas con motivo de la presente tesis.

1.4. Publicaciones: coherencia y relación entre los artículos desarrollados

Las investigaciones realizadas con motivo de la presente tesis se han ido desarrollado en varias etapas consecutivas (especificadas en el apartado 1.1), recogiéndose los correspondientes trabajos y resultados en tres publicaciones, también consecutivas, las cuáles constituyen el soporte principal de la tesis y de esta memoria. Hay una correspondencia natural entre tales etapas y los artículos publicados, según se indica a continuación:

- En (Vallejo & Vega, 2019) (artículo nº1) se recoge el trabajo y los resultados correspondientes a la primera fase (identificación borrosa, modelado equivalente en el espacio de estados y deducción de la ley FMBPC).
- En (Vallejo & Vega, 2021b) (artículo nº2) se publican las investigaciones correspondientes a la segunda fase (análisis de estabilidad mediante un enfoque computacional práctico).
- En (Vallejo & Vega, 2021a) se muestran las investigaciones correspondientes a la tercera fase (integración de la estrategia FMBPC en una estructura CLP-MPC con el objetivo de imponer restricciones a la ley de control FMBPC).

La coherencia de los artículos es absoluta, puesto que entre los tres cubren toda la materia objeto de investigación de esta tesis, hay continuidad entre cada artículo y el siguiente y los tres comparten los principales desarrollos. Por otra parte, cada artículo es consistente, considerado independientemente, y por tanto resulta lógico que cada uno de ellos constituya un capítulo diferenciado de esta memoria, tal como indicamos a continuación: el capítulo 2 corresponde al artículo nº1, el capítulo 3, al artículo nº2 y el capítulo 4, al artículo nº3.

Capítulo 2

Artículo nº1: Estrategia de control FMBPC

2.1. Título original del artículo

Analytical Fuzzy Predictive Control Applied to Wastewater Treatment Biological Processes.

2.2. Resumen en castellano

Título del artículo en castellano:

Control predictivo borroso analítico aplicado a procesos biológicos de tratamiento de aguas residuales.

En este artículo se propone una nueva ley de control predictivo borroso (*fuzzy*) (FMBPC) y se aplica con éxito a un proceso de tratamiento de aguas residuales. La ley de control propuesta permite expresar la señal de control de forma analítica, en cada periodo de muestreo, siendo una alternativa no lineal y difusa a otros controladores predictivos clásicos. Puede considerarse, además, que el controlador predictivo propuesto pertenece también, al menos parcialmente, al campo del control inteligente, considerando como tal la capacidad de la identificación borrosa de capturar de manera fiel la dinámica del proceso.

La estrategia de control propuesta utiliza como modelo de predicciones un modelo inicialmente borroso del proceso, obtenido mediante identificación borrosa a partir de datos de entrada-salida previamente recogidos en simulación. Sin embargo, antes de ser utilizado directamente, el modelo borroso es convertido en otro modelo equivalente en el espacio de estados, discreto, lineal y variante en el tiempo (DLTV), que se actualiza en cada instante de tiempo discreto, teniendo en cuenta los efectos de no linealidad debidos a la acción de las perturbaciones y a cambios en el punto de operación a lo largo del tiempo.

La deducción de ley de control se basa en la utilización del modelo DLTV equivalente como modelo efectivo de predicciones y en la aplicación del denominado *principio de equivalencia*, asociado a la estrategia de control predictivo llamada *control predictivo funcional* (PFC), imponiendo al mismo tiempo a las salidas el seguimiento de las denominadas *trayectorias de referencia*, previamente establecidas.

El trabajo aquí presentado muestra la aplicación, con éxito, de esta particular estrategia de control predictivo al caso de un proceso de depuración biológica de aguas residuales mediante lodos activados, en un entorno de simulación, teniendo en cuenta valores de las perturbaciones relacionados con registros realizados en campañas de una depuradora industrial real. Dicho proceso es multivariable, no lineal, variante en el tiempo y difícil de controlar, debido a su naturaleza biológica.

La ley de control propuesta podría usarse directamente dentro de un esquema MPC de modo dual para manejar restricciones, como una alternativa no lineal y difusa a la clásica ley de control de realimentación de estados.

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Research Article

Analytical Fuzzy Predictive Control Applied to Wastewater Treatment Biological Processes

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A novel control *fuzzy predictive control* law is proposed and successfully applied to a wastewater treatment process in this paper. The proposed control law allows us to evaluate the control signal in an analytical way, each sampling time being a nonlinear and fuzzy alternative to other classic *predictive controllers*. The control law is based on the formalization of the internal fuzzy predictive model of the process as linear time-varying state space equations that are updated every discrete time instant to take into account the nonlinearity effects due to disturbance action and changes in the operating point with time. The model is then used to evaluate the predictions, and, taking them as a starting point and considering them as a paradigm of the *predictive functional control* strategy, a control law, it is derived in an analytical and explicit way by imposing on the outputs of the follow-up of certain reference trajectories previously established. The work presented here addresses the application of this particular strategy of intelligent predictive control to the case of an activated sludge wastewater treatment process successfully in a simulation environment of a real plant taking into account real data for the disturbance records. Such a process is multivariable, nonlinear, time varying, and difficult to control due to its biological nature. The proposed control law can be straightforwardly used within a *dual-mode MPC* scheme to handle constraints, as a nonlinear and fuzzy alternative to the classic *state feedback* control law.

1. Introduction

The traditional strategy of *model-based predictive control* (MBPC or MPC) [1–3] consists basically of the use of a prediction model to determine the necessary control actions at each instant, imposing the minimization of a cost function (which will generally include a term dependent on the error and another dependent on the control efforts, among other possible scenarios). This strategy includes however multiple variants or approaches depending on various factors, mainly on the type of model that will be used to calculate the predictions and the mathematical algorithm used to determine the control law. There are various fundamental methods based on linear models of the process which determine the control variable by means of optimization (the minimization of the cost function chosen). Other methods consider nonlinear mathematical models (*fuzzy models* [4], models formalized by means of *artificial neuronal networks* [5], or other alternatives), but they also determine the control signal by means of optimization in the same way as the first ones. In [6], an orderly bibliographic review of the evolution of the linear and nonlinear MPC is carried out, detailing numerous contributions and approaches in this field of process control. Our study falls within the category of predictive control strategies based on nonlinear models and more specifically on fuzzy models: *fuzzy model-based predictive control* (FMBPC). For the internal model of the process, a fuzzy model of the *Takagi-Sugeno* (TS) [7] type was chosen in which the premises of the rules are diffuse logical expressions while the conclusions are linear numerical combinations of the consequents. *TS*-type fuzzy models are suitable for the identification and description of complex nonlinear processes from numerical input-output data and, if available, expert knowledge of the process. The parameters of our *TS* fuzzy model were deduced by identification as from input-output numerical data previously obtained in simulation. In [6, 8–15], several process control strategies can be seen, based on *Takagi-Sugeno* fuzzy models, some of them framed in the *FMBPC* field, with the identification made from available input-output data.

Concerning the calculation of the control variable, in our study, such a calculation is carried out following a methodology that could be considered as an extension of the so-called *predictive functional control* (PFC) [2, 16–18] (which was initially designed for linear systems) to nonlinear systems. The approach that we have chosen uses a fuzzy model of the process (for the calculation of the predictions) developed in the state space form and follows a *PFC* strategy for the derivation of a control law in an analytical explicit form, which can be an advantage in comparison with optimization-based control schemes [10, 18]. In relation to this issue, it is possible to say that some strategies or approaches of *MPC* could be considered as analytical *MPC* algorithms, such as *dynamic matrix control*, *generic model control*, or *predictor-corrector control* (in case of unconstrained linear *MPC* problems with quadratic cost functions, the control rules can be expressed in an analytical form and even for nonlinear *MPC* problems, if the model is linearized). But in the case of these algorithms, the analytical obtaining of a control law implies the statement and solution of a problem of optimization of a cost function. However, the *PFC* strategy offers us a method for the obtaining of the control law in which it is not necessary to solve an optimization problem, which is mathematically less complex. In addition, our approach deals with nonlinear models in a direct and practical way, by formalizing the fuzzy model in the form of time-varying state space equations (with coefficients that must be updated at each discrete time instant) and without this supposing a great increment in the complexity of the mathematical derivation of the control law.

The main contribution of our work to the *FMBPC* field is the application of this approach to a multivariable, with disturbances, strongly nonlinear, with a complex dynamic and of a biological nature case study (carried out by simulation), with the aim of a future generalization.

No parameters have been included to handle constraints in the proposed control law, that is, it is an unconstrained control law. But our control law can be used, as base law, within a *MBPC* scheme that is designed to satisfy previously fixed constraints (on the control action, on the control action increment, on the plant output, etc.). In particular, following the approach of the *MPC* schemes proposed by Rossiter [3] could be used (*OLP* or *CLP dual-mode MPC* schemes).

To obtain the control law by means of the procedure mentioned above, some relevant mathematical tasks must be carried out. It will be necessary to deduce the mathematical expression that relates the model outputs, for a certain prediction horizon, to the control variable, that is, the appropriate expression for calculating the predictions, making use of the fuzzy model of the process. In addition, it will also be necessary to specify the control objective for the outputs of the plant, which in our case will consist in the imposition of the tracking of the previously established reference trajectories. And, of course, we must also formalize mathematically the *model-based predictive control* characteristic or nature, by means of some kind of relationship between the model and the process (the plant). We will establish such a relationship using the incremental equivalence principle between the model and the plant (equivalence between the plant output-desired increment and the model output increment) [2, 16, 17]. Finally, combining appropriately all the relationships and equations mentioned, it will be necessary to find the control variable that guarantees the control objective established at each sampling instant. Due to the multivariable character of the process considered and the intrinsic mathematical complexity of the fuzzy models, in order to obtain the control variable in an analytical and explicit manner, we need to start from a (fuzzy) mathematical model expressed compactly and clearly. In addition to working with matrix expressions, therefore, in this study, we have formalized the expressions of the fuzzy model with a format similar to that of state equations, with the peculiarity that the coefficients of the various terms are not constant but depend on the instant of sampling. This is due to the fact that they depend on the instantaneous premise vector (more specifically on the levels of compliance with the various rules by the premise vector at each instant); it is therefore necessary for these coefficients to be updated (recalculated) during each period. This particular strategy of predictive control considered in this paper has previously been approached by other authors; to be precise, it is developed in [10] for a case study with a manipulated input and a single controlled output and without considering disturbances (previously, something similar was also developed in [18], for another case study). This work however approaches the case of a multivariable system and disturbances in the input are considered. To be precise, our system has three inputs: a manipulated input and two disturbances. It also has two outputs, both of which are controlled. Its multivariable character involves working with matrices, which leads to an increase in mathematical complexity. Moreover, due to the existence of two disturbances and a single manipulated input, it is assumed that it will not be easy to control the process, especially taking into account that the process considered is of a biological nature (significantly more unpredictable than many industrial physical-chemical processes). Tackling all these complexities is another of the contributions of this paper.

The present work has focused on the application of our fuzzy predictive control strategy to wastewater treatment biological processes. In our case study, the plant to be controlled was a *wastewater treatment plant* (WWTP) that originally had a relatively simple architecture (something that is clearly advantageous for the purposes of the study) and whose depuration method was the well-known *activated sludge process*. The entire study was implemented through simulation.

The control of the biological processes present in wastewater treatment plants is a rather complex problem, due to the nonlinear character of the corresponding physical-chemical reactions and also to the abrupt changes that can occur (sometimes unpredictably) in the influent input flow rate and in its degree of contamination (disturbances). The proposals that have been made to improve the control and operation of biological processes (generic or related to *WWTPs*) have been numerous and different, highlighting in our case those that include some variant of nonlinear model-based predictive control *(NLMPC)* and, especially, the proposals framed in the field of the fuzzy model-based predictive control (*FMBPC*). In [19–34], some of these contributions can be seen. Our work tries to be, precisely, a contribution more in this last area, proposing a strategy of predictive control based on a fuzzy model (obtained from numerical input-output data), but formalizing the law of control in an analytical and explicit way.

The paper is organized as follows. Section 2 describes the case study chosen: a municipal *WWTP* with a relatively simple architecture and with a biological purification method, the so-called *activated sludge*. The subsection related to the mathematical model is complemented by the Appendix D. In Section 3, the fuzzy modeling and identification procedure of the treatment plant is developed, starting only from input-output numerical data and also the subsequent formalization of the fuzzy model in an equivalent model in the state space (with time-dependent coefficients). In Section 4, we address the problem of control, in the *FMBPC* field, adopting a nonlinear *PFC* strategy based on a *fuzzy* model. In this section, all the mathematical tasks necessary to deduce an analytical and explicit control law for our case study are carried out (using Appendices A, B, and C). In Section 5, we explain the *FMBPC* scheme that will be implemented and we present and explain some experiments developed by simulation and their corresponding results. And, finally, Section 6 contains the conclusions of the work.

2. Case Study: Wastewater Treatment Processes

Our case study consists of applying the previously introduced the *FMBPC* strategy to a multivariable biological process, highly nonlinear and with complex dynamics, but with a simple architecture that can facilitate the extraction of useful conclusions. For this reason, a municipal wastewater treatment plant (WWTP) located in Manresa (in the province of Barcelona, Spain) [35], equipped with only the elements and processes necessary for the purification of basic organic water pollution, that is, for the purification of the substrate, was chosen. The study was carried out in a simulation environment, for obvious reasons of feasibility and availability.

2.1. Plant Description and Input-Output Configuration. The wastewater treatment plant chosen was originally designed to achieve a relatively simple purification process, without including a nitrification-denitrification process. So, the central objective of the overall process of purification in this plant was mainly the reduction of the concentration of the substrate, carried out through the so-called *activated sludge process*, of a biological nature. The plant had a single subprocess of substrate biological depuration (with six reactors working in parallel, arranged in two lines of three, which can be considered equivalent to one only for the purposes of mathematical modeling) and, following the reactors, a single decantation subprocess (with two settlers working in parallel, which can also be considered as one, for modeling purposes). The equivalent architecture of the considered plant corresponds to that shown in Figure 1 with a single aerated biological reactor followed by a secondary settler; the meaning of the main variables of the WWTP can be seen in Table 1.

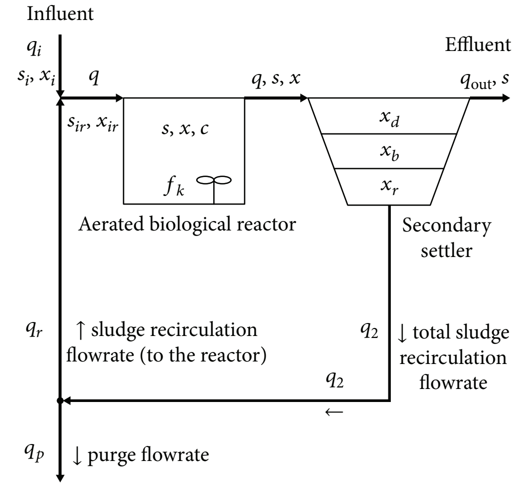

Figure 1: Wastewater treatment plant with activated sludge.

As already indicated in the introductory chapter, the purification technique used in this plant is consisting of the elimination of organic pollutants by means of *activated sludge* (a mixed culture of microorganisms in suspension in the aerated biological reactor), with the recirculation of this sludge being the main control action and the sludge recirculation flow rate to the reactor (q_r) being the only manipulated variable considered. The purification process that reduces the substrate concentration in the water is based on the interaction, through an aerobic reaction, between the microorganisms (the biomass) and the organic matter present in the water (the substrate). The microorganisms feed on the substrate (digest it), and consequently, its concentration decreases and therefore the contamination. This reaction takes place in the biological reactor (where the mix is supposed to be perfect) and requires a sufficient concentration of oxygen dissolved in the water. The concentration of oxygen (c) is one of the three variables involved in the basic mathematical models of the purification process and also needs to be controlled. However, in order to reduce the complexity of the case study, we will assume that the control of c will be guaranteed by the implementation of an adequate aeration system (a set of aeration turbines) and one independent control loop (with a PID algorithm for example), with an appropriate sampling period. After the biological reaction, the treated water is sent to a secondary settler, where the clean water and the activated sludge are separated. The clean water is sent to the outside of the plant and the activated sludge is recirculated and divided in two flows: one part of the sludge (q_p) is purged from the bottom of the settler, and the other one (q_r) is recirculated into the reactor with the objective of maintaining the population of the microorganisms.

In our case study, we will control only the substrate and the biomass (s, x). The input variables are the input flow rate (q_i), the input substrate concentration (s_i), and the sludge recirculation flow rate sent to the reactor (q_r). The first two

Table 1: Main variables of the WWTP with activated sludge.

Influent	Biological reactor	Secondary settler and recirculation flows	Effluent
q_i: input flow rate	q: reactor input-output flow rate	q_2: total sludge recirculation flow rate q_p: purge flow rate q_r: sludge recirculation flow rate (to the reactor)	q_{out}: output flow rate
s_i: input substrate concentration x_i: input biomass concentration	s_{ir}: substrate concentration in the input of the reactor x_{ir}: biomass concentration in the input of the reactor s: output substrate concentration x: output biomass concentration c: output oxygen concentration f_k: aeration factor	x_d: biomass concentration in the top layer of the settler x_b: biomass concentration in the middle layer of the settler x_r: biomass concentration in the lower layer of the settler	s: output substrate concentration

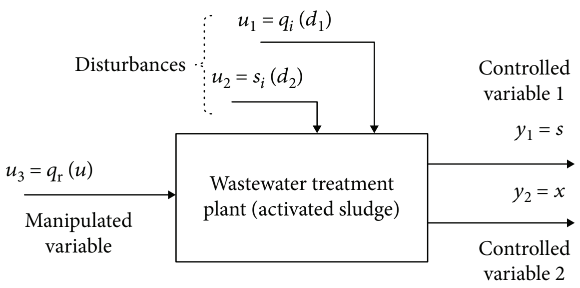

Figure 2: The multivariable nonlinear system considered.

are the system disturbances and the third is the manipulated variable. Figure 2 shows the block diagram of the multivariable nonlinear system considered, with the input and output variables involved and their role in the control system (in the case of the inputs, an alternative notation has been used, between brackets, more suitable and usual in control systems).

The information relating to the considered inputs and outputs of the plant is shown more detailed in Table 2 including for each of them and also their biological or physical-chemical significance.

The system chosen has two disturbances in the input, and the ultimate objective is to simultaneously control two output variables (coupled) by means of a single manipulated variable. In this study, we consider the wastewater treatment plant basically as a multivariable system case study with a complex dynamic and strongly nonlinear, the model of which (that could be unknown in a generic case) is identified from input-output numerical data.

2.2. Mathematical Model of the Wastewater Treatment Process. The mathematical model of the wastewater treatment process taken as reference is based on mass balances for the substrate, biomass, and oxygen, and it is founded on the classical *Monod and Maynard-Smith model* (with the assumption of a perfectly mixed tank reactor). This model could be considered a simplification of the standard model denominated as *activated sludge model no. 1*, which is better known by its initials, *ASM1* [36], but it is the corresponding model to the plant considered, a simple plant in its origin, but real.

The equations that constitute the model and the values of the different parameters of the Manresa *WWTP* [35] can be seen in [23] and in Appendix D (in English language). We will use this model, in substitution or representation of the plant, both in the identification process and in the simulation study of the proposed control strategy. But we will only use the equations of the substrate and the biomass, taking into account the aforementioned, in the sense of assigning oxygen control to an independent loop. This simplification does not however make the study any less interesting because it approaches a general problem of interest: the possible utility of a predictive control scheme based on fuzzy models to control strongly nonlinear multivariable systems, starting only from the information implicit in input-output sets of data.

3. Fuzzy Identification and State Space Modeling of a Wastewater Treatment Plant

One of the great advantages of using fuzzy models is their potential for describing nonlinear behaviour. This capacity depends on the suitability of the identification process. Fuzzy identification is therefore a data treatment procedure (with modelling objectives) of great significance both in the general sector of the modelling of nonlinear systems and in the more specific one of predictive control based on models. However, the identification of fuzzy models and their use within predictive control strategies can be studied and applied from many points of view. Our study focuses on the application to nonlinear multivariable systems of a particular fuzzy modelling methodology which consists of the following: expressing a fuzzy model (previously obtained by means of identification) in the form of state equations in such a way that it can be useful in calculating predictions and in the search for and obtaining of an explicit analytical expression for the predictive control law within the framework of a *fuzzy model-based predictive control* (FMBPC) system [14]. It is for this reason that in this article, we will concentrate more on the analytical

Table 2: Inputs and outputs of the wastewater treatment plant.

Inputs	Disturbances	$u_1\ (d_1)$	Input flow rate (influent)	q_i
		$u_2\ (d_2)$	Substrate concentration in the influent	s_i
	Manipulated variable	$u_3\ (u)$	Sludge recirculation flow rate	q_r
Outputs	Controlled variable 1	y_1	Substrate concentration in the effluent	s
	Controlled variable 2	y_2	Biomass concentration in the reactor	x

formalization of the fuzzy model that has already been identified than on the identification itself.

In our case study, the wastewater treatment plant was identified starting from a series of input-output numerical data previously obtained by open-loop simulation and subsequently treated with the fuzzy identification software tool known as *FMID* (*fuzzy model identification toolbox)* [37]. Some functions of the toolbox were (partially) adapted and some code complements were also included (in the *Matlab* and *Simulink* environment) so as to be able to make our calculations and carry out our experiments. This tool, which is based on clustering techniques by means of the *Gustafson-Kessel* algorithm, was developed by Babuška et al. as software support for the theories and techniques of fuzzy modelling and identification described in the book *Fuzzy Modelling for Control* [9].

3.1. Input-Output Data. The input-output numerical data of our system were obtained by means of simulation in open loop with the wastewater treatment plant represented by the differential equations corresponding to its traditional nonlinear mathematical model ([23], Appendix D). The identification process could also have been carried out with samples from the real system, but in this case, the predictive control tests would have been carried out with the real wastewater treatment plant, which is not usually possible.

The multivariable system to identify corresponding to our case study has 3 inputs and 2 outputs. Figure 3 shows another simplified block diagram of the wastewater treatment plant to emphasize the multivariable nature of the said system with multiple inputs and outputs (*MIMO system*).

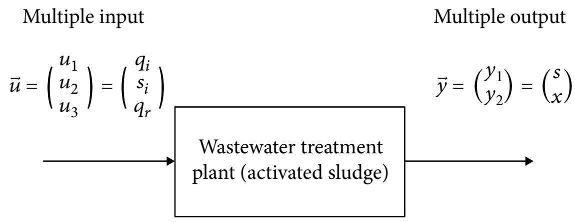

Figure 3: The MIMO system to identify (3 inputs and 2 outputs).

The inputs of the considered plant were the following: the incoming water flow rate (influent), $q_\text{i}(k)$; the substrate concentration in the influent (organic pollution of the incoming water), $s_\text{i}(k)$; and the activated sludge recirculation flow rate, $q_\text{r}(k)$. The first two are disturbances and the third is an input variable that can be used as a control variable (manipulated variable). And the two outputs taken into account are the substrate concentration in the outgoing water (effluent), $s(k)$, and the biomass concentration (microorganisms that feed on the substrate and therefore purify the water) in the reactor, $x(k)$.

The values chosen for the first two inputs (disturbances) were chosen by taking as a reference data from real experimental campaigns of an industrial wastewater treatment plant, to be precise, of the municipal wastewater treatment plant of Manresa, in the province of Barcelona, Spain (see the physical-chemical parameters of such a treatment plant in Table 12, in sub-Appendix D1). These data originated from tests and measures carried out at this plant on the occasion of the work on predictive control carried out at the time by Moreno [35].

For the control variable, pseudorandom value sequences were chosen, initially taking as a reference the values of the control variable used in the aforementioned campaigns of the industrial wastewater treatment plant and subsequently incorporating modifications both in variability and in the extreme values, aiming to cover different areas of operation, all this with the aim of capturing and extracting sufficient information on the dynamics of the wastewater treatment plant. For each combination of values of the input variables ($\{q_\text{i}(k), s_\text{i}(k), q_\text{r}(k)\}$), we determined by simulation the corresponding output values ($\{s(k), x(k)\}$), thus obtaining (for each test) a matrix of input-output data with 5 columns and as many rows as samples (thousands of samples), with each line consisting of 5 values (three inputs and two outputs). The sampling period of the data set recorded in the campaign was of the order of one hour. But the data was extended by performing a mathematical interpolation between each two samples, obtaining intermediate estimated data every 0.2 hours (i.e., 12 minutes).

Figures 4 and 5 represent the input-output data of one of the numerous identification tests carried out, which we will refer to as case A. Figure 4 is a graphic representation of the sequences of values of the inputs, and Figure 5 is a graphic representation of the sequences of values of the outputs, the latter having been obtained by means of simulation in open loop (being the wastewater treatment plant represented by the aforementioned standard nonlinear mathematical model). The variable associated with the abscissa axis of both graphs is the *sample number*, not the time, but the samples were ordered temporarily, as is logical.

3.2. Fuzzy Model: Type, Structure, and Parameters. The internal model that we will use for to the predictions, in our model-based predictive control frame, will be a *Takagi-Sugeno*-type discrete time fuzzy model [7]. These models are composed of a series of *if-then* rules, each of which represents a linear submodel corresponding to a certain subset or

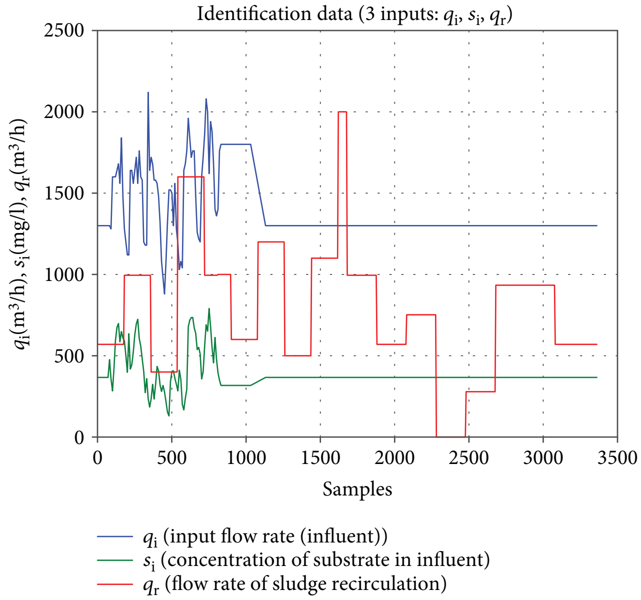

Figure 4: Identification data of case A (inputs).

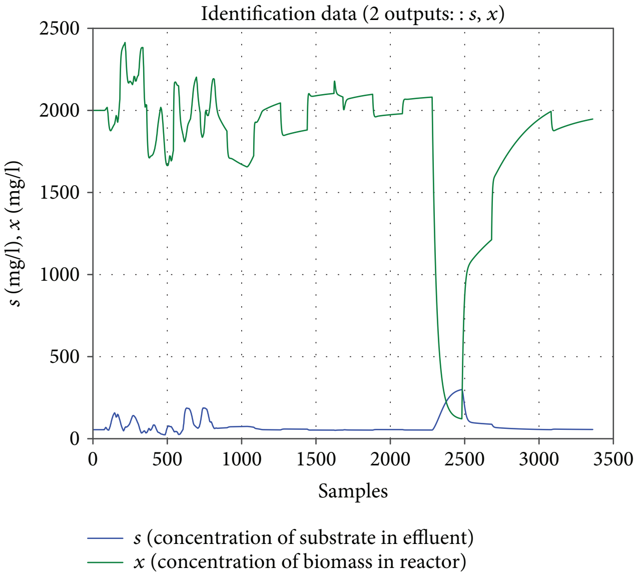

Figure 5: Identification data of case A (outputs).

$\mathbf{R}_j$: **if** (x_{a1} **is** A_{j1} **and** x_{a2} **is** A_{j2} **and**, … , **and** x_{ap} is A_{jp}) **then**:

$$y(k) = \phi_j(\mathbf{x}) = \alpha_{j1}x_1 + \alpha_{j2}x_2 + \cdots + \alpha_{jq}x_q + \delta_j$$

where:

$j = (1, 2, \ldots, \text{mr})$; mr: number of rules

$\mathbf{x}_a = (x_{a1}, x_{a2}, \ldots, x_{ap})$: antecedent vector

$\mathbf{x} = (x_1, x_2, \cdots, x_q)$: consequent vector

$(A_{j1}, A_{j2}, \ldots, A_{jp})$: fuzzy sets related to the components of the antecedent vector (rule $\mathbf{R}_j$)

Membership function associated to the fuzzy set A_{jh}:

$\theta_{A_{jh}} : \mathbb{R} \longrightarrow [0, 1]$ (smooth function)

$x_{ah} \longrightarrow \mu_{A_{jh}}(x_{ah})$ (membership grade of x_{ah} with respect to the fuzzy set A_{jh})

$(h = 1, \ldots, p)$

Algorithm 1: General form of the *Takagi-Sugeno*-type fuzzy models.

partition of the universe of fuzzy values of the antecedent vector (vector premise). The antecedent or premise of each of the rules is composed of several simple propositions connected by means of *and* logical operators. The simple propositions compare each of the components of the antecedent vector with a certain set or associated fuzzy *value* (characterised by its *membership function*). And the consequent or conclusion of each rule allocates to the output a linear combination of the variables that form the consequent vector, plus an independent term. Algorithm 1 shows the general form of this type of discrete time fuzzy models. The mathematical expression by means of which the numeric value that will allocate to the output is calculated is a function of the consequent vector $\mathbf{x}$ and has been represented in the said algorithm by $\phi_j(\mathbf{x})$.

The composition of both the antecedent vector and the consequent vector, as well as the number of rules, is two of the main aspects of the structure of a fuzzy TS model. The first is directly related to the dynamics of the process we want to model, that is, will depend on the input-output dynamic dependencies or relationship, and the number of rules will depend on the number of clusters observed or considered in the *product space* of the available input-output data set. Another characteristic aspect of the fuzzy model (less intuitive) is the type of membership functions associated with different sets or fuzzy values.

Before carrying out the identification process, we must choose what structure we will consider, that is, what decision we will take on the characteristics mentioned above. Such a choice could be based on hypothesis or be the consequence of expert knowledge of the process to be identified (if such empirical information is available) and implies the need to set various concrete numerical parameters, such as the specific number of rules and the order of discrete time recursive

Table 3: Structural parameters for fuzzy identification.

Parameter	Meaning	Description
c	Number of data clusters (for each output)	Vector with two integer components
$\mathbf{N}_y$	Output-output dynamic relationship	Matrix (2×2)
$\mathbf{N}_u$	Input-output dynamic relationship	Matrix (2×3)
$\mathbf{N}_d$	Input-output transport delays	Matrix (2×3)
T_s	Sample time	Real number

models that make up each rule. Subsequently, after the appropriate identification process, based on our case in input-output numerical data, we will obtain the specific numerical values of the coefficients of the different terms of the mathematical expression of the consequent of each of the rules, as well as the centers of clusters and constants relating to the membership functions.

3.3. Analytical Expression of the Fuzzy Model Output. To calculate the outputs of the considered fuzzy models (given by several rules), a numerical calculation method should be applied from among those contemplated in the fuzzy logic theory, adding or combining all the rules, with the appropriate weight to the consequent of each rule; for example (in our case), by means of the centroid method:

$$\tilde{y}_i = \frac{\sum_{j=1}^{\mathrm{mr}_i} \mu_{A_{j1}}(x_{a1})\, \mu_{A_{j2}}(x_{a2}) \cdots \mu_{A_{jp_i}}\left(x_{ap_i}\right) \phi_{ij}(\mathbf{x})}{\sum_{j=1}^{\mathrm{mr}_i} \mu_{A_{j1}}(x_{a1})\, \mu_{A_{j2}}(x_{a2}) \cdots \mu_{A_{jp_i}}\left(x_{ap_i}\right)}, \tag{1}$$

(i) $i = 1, 2$ (two outputs).

(ii) $j = 1, 2, \ldots, mr_i$.

(iii) mr_i is the number of rules of the output y_i.

(iv) p_i is the number of components of the antecedent vector corresponding to y_i.

(v) $\mu_{A_{jp_i}}(x_{ap_i})$ is the membership grade of x_{ap_i} with respect to the fuzzy set A_{jp_i}.

And defining for each output and rule the following membership functions (normalized) for all the antecedent vector

$$\beta_{ij}(\mathbf{x}_a) = \frac{\mu_{A_{j1}}(x_{a1})\, \mu_{A_{j2}}(x_{a2}) \cdots \mu_{A_{jp_i}}\left(x_{ap_i}\right)}{\sum_{j=1}^{\mathrm{mr}_i} \mu_{A_{j1}}(x_{a1})\, \mu_{A_{j2}}(x_{a2}) \cdots \mu_{A_{jp_i}}\left(x_{ap_i}\right)}, \tag{2}$$

the numerical expression of the output will remain as

$$\begin{aligned} \tilde{y}_i &= \sum_{j=1}^{\mathrm{mr}_i} \beta_{ij}(\mathbf{x}_a)\phi_{ij}(\mathbf{x}), \\ i &= 1, 2, \\ j &= 1, 2, \ldots, \mathrm{mr}_i, \end{aligned} \tag{3}$$

where mr_i is the number of rules of the output y_i.

3.4. Identification of the WWTP. The fuzzy identification process consists of searching, as from the input-output data map obtained, a fuzzy model to represent the behaviour of the system as faithfully as possible. As we said above, in order to do so, it is necessary to choose, previously, a certain type of fuzzy model with a specific structure and with determined dynamic characteristics and then to try to adjust unknown coefficients or parameters by means of an appropriate mathematical method. In our experiments, as has been mentioned in the introduction to this section, the whole of the identification process was carried out by using the *FMID* software tool [37], which is capable of *extracting* a fuzzy model of the *Takagi-Sugeno* type [7] for each of the outputs of the system to identify, as from the objective information implicit in the input-output data provided and according to the previous choice of certain parameters concerning the possible dynamics of the plant. Numerous identification tests were carried out with different input-output data set (corresponding to different campaigns) and with different choices for the values of various parameters (available in the tool) related to the dynamic characteristics of the recursive discrete model that will represent the plant. The result obtained in each of the tests carried out was a fuzzy model for each of the two outputs. Some of the experiments done are presented in [31, 32] and many others were subsequently carried out as the study and the research progressed. In this paper, we will refer to some of the case studied during the whole of the process, which we will refer to as case A, case B, case C, and case D.

The main dynamic structural parameters that need to be chosen to carry out the identification of the WWTP with the *FMID* software tool can be summarized in Table 3.

Next, in Table 4, we will specify the choice of parameters carried out in case D and its concrete meaning, as well as the relationship with the dynamics of the model.

The result of each identification test carried out consists of a fuzzy model of the *Takagi-Sugeno* type for each of the two outputs of the wastewater treatment plant. After the identification, all parameters of each fuzzy model are available in appropriate numerical structures and we did an adequate use of them for the automatic processing of information in our simulation study. The rules of the two fuzzy models that have been identified corresponding to case D of the WWTP identifications

Table 4: Structural parameters in case D.

Parameter	Value	Dynamic model
c	$[6 \quad 5]$	Out-$y_1(k) = s(k)$: 6 clusters/6 rules Out-$y_2(k) = x(k)$: 5 clusters/5 rules
$\mathbf{N}_y$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	[Row 1]: $y_1(k)$ depends on $y_1(k-1)$ and $y_2(k-1)$ [Row 2]: $y_2(k)$ depends on $y_2(k-1)$
$\mathbf{N}_u$	$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$	[Row 1]: $y_1(k)$ depends on $u_1(k-1)$, $u_2(k-1)$, $u_3(k-1)$ and $u_3(k-2)$ (the parameter value of the first row and the third column, equal to 2, implies two consecutive terms of u_3 (in discrete time)) [Row 2]: $y_2(k)$ depends on $u_1(k-1)$ and $u_3(k-1)$
$\mathbf{N}_d$	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$	[Row 1]: they are 1 transport delay for the three inputs (u_1, u_2, and u_3) with respect to $y_1(k)$ [Row 2]: they are 1 transport delay for the u_1 and u_3 inputs with respect to $y_2(k)$
T_s	0.2	Sample time (hours)

1. if ($y_1(k-1)$**is** A_{11} **and** $y_2(k-1)$ **is** A_{12} **and** $u_1(k-1)$ **is** A_{13} **and** $u_2(k-1)$ **is** A_{14} **and** $u_3(k-1)$ **is** A_{15} **and** $u_3(k-2)$ **is** A_{16}) **then**
$y_1(k) = (8.74)\cdot 10^{-1}\cdot y_1(k-1) - (2.72)\cdot 10^{-3}\cdot y_2(k-1) + (3.96)\cdot 10^{-3}\cdot u_1(k-1) + (2.24)\cdot 10^{-2}\cdot u_2(k-1) + (5.33)\cdot 10^{-3}\cdot u_3(k-1) - (5.98)\cdot 10^{-3}\cdot u_3(k-2) - (6.47)\cdot 10^{-1}$

2. if ($y_1(k-1)$ **is** A_{21} **and** $y_2(k-1)$ **is** A_{22} **and** $u_1(k-1)$ **is** A_{23} **and** $u_2(k-1)$ **is** A_{24} **and** $u_3(k-1)$ **is** A_{25} **and** $u_3(k-2)$ **is** A_{26}) **then**
$y_1(k) = (8.58)\cdot 10^{-1}\cdot y_1(k-1) - (3.87)\cdot 10^{-3}\cdot y_2(k-1) + (5.19)\cdot 10^{-3}\cdot u_1(k-1) + (2.31)\cdot 10^{-2}\cdot u_2(k-1) - (3.89)\cdot 10^{-3}\cdot u_3(k-1) + (3.69)\cdot 10^{-3}\cdot u_3(k-2) + (6.93)\cdot 10^{-1}$

3. …

4. …

5. …

6. if ($y_1(k-1)$ **is** A_{61} **and** $y_2(k-1)$ **is** A_{62} **and** $u_1(k-1)$ **is** A_{63} **and** $u_2(k-1)$ **is** A_{64} **and** $u_3(k-1)$ **is** A_{65} **and** $u_3(k-2)$ **is** A_{66}) **then**
$y_1(k) = (8.70)\cdot 10^{-1}\cdot y_1(k-1) - (6.54)\cdot 10^{-3}\cdot y_2(k-1) + (1.33)\cdot 10^{-2}\cdot u_1(k-1) + (3.59)\cdot 10^{-2}\cdot u_2(k-1) + (3.01)\cdot 10^{-2}\cdot u_3(k-1) - (3.04)\cdot 10^{-2}\cdot u_3(k-2) - (1.25)\cdot 10^{1}$

Algorithm 2: *Takagi-Sugeno* fuzzy model for substrate $y_1(k) = s(k)$.

1. if ($y_2(k-1)$ **is** $A^*{}_{11}$ **and** $u_1(k-1)$ **is** $A^*{}_{12}$ **and** $u_3(k-1)$ **is** $A^*{}_{13}$) **then**
$y_2(k) = (9.58)\cdot 10^{-1}\cdot y_2(k-1) - (2.22)\cdot 10^{-2}\cdot u_1(k-1) + (1.33)\cdot 10^{-2}\cdot u_3(k-1) + (1.02)\cdot 10^{2}$

2. if ($y_2(k-1)$ **is** $A^*{}_{21}$ **and** $u_1(k-1)$ **is** $A^*{}_{22}$ **and** $u_3(k-1)$ **is** $A^*{}_{23}$) **then**
$y_2(k) = (1.01)\cdot 10^{0}\cdot y_2(k-1) - (4.10)\cdot 10^{-2}\cdot u_1(k-1) + (3.54)\cdot 10^{-2}\cdot u_3(k-1) + (2.45)\cdot 10^{1}$

3. …

4. …

5. if ($y_2(k-1)$ **is** $A^*{}_{51}$ **and** $u_1(k-1)$ **is** $A^*{}_{52}$ **and** $u_3(k-1)$ **is** $A^*{}_{53}$) **then**
$y_2(k) = (9.35)\cdot 10^{-1}\cdot y_2(k-1) - (5.22)\cdot 10^{-2}\cdot u_1(k-1) - (3.96)\cdot 10^{-2}\cdot u_3(k-1) + (2.62)\cdot 10^{2}$

Algorithm 3: *Takagi-Sugeno* fuzzy model for biomass $y_2(k) = x(k)$.

carried out can be seen as follows: in Algorithm 2, the fuzzy model corresponding to output 1 and in Algorithm 3, the fuzzy model corresponding to output 2. The relationship between the parameters contained in Table 4 (except T_s) and the structure of the rules of the fuzzy model of the two outputs is quite clear. Thus,

Table 5: Consequent parameters for substrate $y_1(k) = s(k)$.

Rule	$y_1(k-1)$	$y_2(k-1)$	$u_1(k-1)$	$u_2(k-1)$	$u_3(k-1)$	$u_3(k-2)$	Offset
1	$8.74 \cdot 10^{-1}$	$-2.72 \cdot 10^{-3}$	…	…	…	$-5.98 \cdot 10^{-3}$	…
2	$8.58 \cdot 10^{-1}$	$-3.87 \cdot 10^{-3}$	…	…	…	$3.69 \cdot 10^{-3}$	…
3	…	…	…	…	…	…	…
4	…	…	…	…	…	…	…
5	$8.83 \cdot 10^{-1}$	$-3.15 \cdot 10^{-3}$	…	…	…	$5.72 \cdot 10^{-3}$	…
6	$8.70 \cdot 10^{-1}$	$-6.54 \cdot 10^{-3}$	…	…	…	$-3.04 \cdot 10^{-2}$	…

Table 6: Cluster centers for substrate $y_1(k) = s(k)$.

Rule	$y_1(k-1)$	$y_2(k-1)$	$u_1(k-1)$	$u_2(k-1)$	$u_3(k-1)$	$u_3(k-2)$
1	$5.71 \cdot 10^{1}$	$1.84 \cdot 10^{3}$	…	…	…	$8.29 \cdot 10^{2}$
2	$5.72 \cdot 10^{1}$	$1.99 \cdot 10^{3}$	…	…	…	$1.07 \cdot 10^{3}$
3	…	…	…	…	…	…
4	…	…	…	…	…	…
5	$1.06 \cdot 10^{2}$	$1.99 \cdot 10^{3}$	…	…	…	$7.90 \cdot 10^{2}$
6	$1.66 \cdot 10^{2}$	$1.90 \cdot 10^{3}$	…	…	…	$1.56 \cdot 10^{3}$

Table 7: Consequent parameters for biomass $y_2(k) = x(k)$.

Rule	$y_1(k-1)$	$u_1(k-1)$	$u_3(k-1)$	Offset
1	$9.58 \cdot 10^{-1}$	$-2.22 \cdot 10^{-2}$	$1.33 \cdot 10^{-2}$	$1.02 \cdot 10^{2}$
2	$1.01 \cdot 10^{0}$	$-4.10 \cdot 10^{-2}$	$3.54 \cdot 10^{-2}$	$2.45 \cdot 10^{1}$
3	…	…	…	…
4	…	…	…	…
5	$9.35 \cdot 10^{-1}$	$-5.22 \cdot 10^{-2}$	$-3.96 \cdot 10^{-2}$	$2.62 \cdot 10^{2}$

Table 8: Cluster centers for biomass $y_2(k) = x(k)$.

Rule	$y_2(k-1)$	$u_1(k-1)$	$u_3(k-1)$
1	$1.80 \cdot 10^{3}$	…	$5.68 \cdot 10^{2}$
2	$1.82 \cdot 10^{3}$	…	$4.81 \cdot 10^{2}$
3	…	…	…
4	…	…	…
5	$2.12 \cdot 10^{3}$	…	$9.96 \cdot 10^{2}$

the recursive dynamic models of $y_1(k)$ and $y_2(k)$ have the following structure:

$$y_1(k) = f_1(y_1(k-1), y_2(k-1), u_1(k-1), u_2(k-1), u_3(k-1), u_3(k-2)),$$

or alternatively,

$$y_1(k+1) = f_1(y_1(k), y_2(k), u_1(k), u_2(k), u_3(k), u_3(k-1)), \quad (4)$$

$$y_2(k) = f_2(y_2(k-1), u_1(k-1), u_3(k-1)),$$

or alternatively, (5)

$$y_2(k+1) = f_2(y_2(k), u_1(k), u_3(k)).$$

In Tables 5 and 6, the consequent parameters and the cluster centers, respectively, can be seen corresponding to output 1, and in Tables 7 and 8, the consequent parameters and the cluster centers, respectively, can be seen corresponding to output 2. All these numerical coefficients are results provided by the identification tool that uses the *Gustafson-Kessel* algorithm. The clustering techniques implicit in the tool are decisive when identifying the linear submodels.

The identification tool also provides the necessary numerical information relating to the fuzzy sets or values A_{ij}, with which the components of the antecedent vectors are compared. Such information must be searched in the analytical expressions of the *membership functions* corresponding to such sets, which will be given in a parametric form. Using these parameters automatically and developing the suitable software, we can represent in graphic form the membership functions of each set. As an example, we show below a graph (Figure 6) with the membership functions corresponding to the six fuzzy sets (one for each rule) associated with the $u_3(k-1) = q_r(k-1)$ variable, which is one of the components of the antecedent vector of the rules of the output $y_1(k)$, for the case A.

Observing the previous tables, we can see the composition of the antecedent and consequent vectors of each of the two outputs. The antecedent vector of output-1, $\mathbf{x}_{a|\text{out1}}$, coincides with the consequent vector of output 1, $\mathbf{x}_{|\text{out1}}$, and the same is true for the antecedent vector and consequent vector of output 2, $\mathbf{x}_{a|\text{out2}}$ and $\mathbf{x}_{|\text{out2}}$. Therefore, we will refer to the *antecedent-consequent vector* of output 1 and the *antecedent-consequent vector* of output 2. The composition of such vectors and the physical meaning of each component are specified below.

$$\mathbf{x}_a = \mathbf{x}_{|\text{out1}} = [y_1(k-1), y_2(k-1), u_1(k-1), u_2(k-1), u_3(k-1), u_3(k-2)], \quad (6)$$

$$\mathbf{x}_a = \mathbf{x}_{|\text{out2}} = [y_2(k-1), u_1(k-1), u_3(k-1)], \quad (7)$$

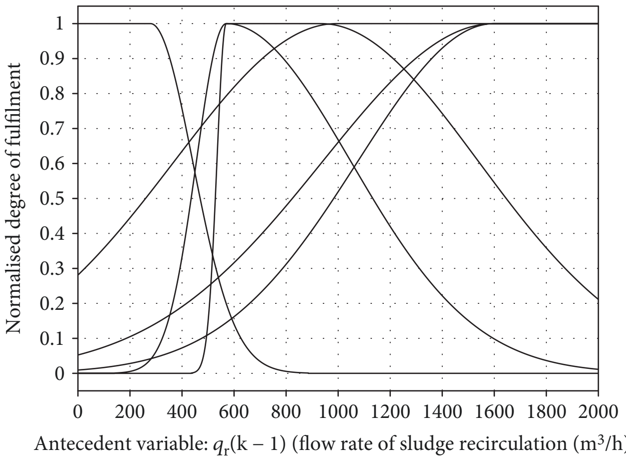

FIGURE 6: Membership functions of $u_3(k-1) = q_r(k-1)$.

where

$$\begin{aligned}
y_1(k-1) &= s(k-1)\text{; effluent substrate at } (k-1),\\
y_2(k-1) &= x(k-1)\text{; reactor biomass at } (k-1),\\
u_1(k-1) &= q_i(k-1)\text{; input flow rate at } (k-1),\\
u_2(k-1) &= s_i(k-1)\text{; influent substrate at } (k-1),\\
u_3(k-1) &= q_r(k-1)\text{; sludge recirculation at } (k-1),\\
u_3(k-2) &= q_r(k-2)\text{; sludge recirculation at } (k-2),\\
&(k-N) = (k-N) * T,\\
&N = 1, 2,\\
&T = \text{sampling period (discrete time system)}.
\end{aligned} \tag{8}$$

As can be seen by comparing (6) and (7), initially, the *antecedent-consequent vectors* of both outputs do not coincide. However, in this study, we decided to treat the mathematical expressions of the fuzzy models jointly for both outputs (using matrices), defining a single mixed *antecedent-consequent vector* formed by the union of the two *antecedent-consequent* vectors (considering, naturally, null factors where appropriate in the necessary mathematical developments). Such vector, *antecedent-consequent* common to both outputs, is as follows:

$$\begin{aligned}
\mathbf{x}_a = \mathbf{x} = [&y_1(k-1), y_2(k-1), u_1(k-1), u_2(k-1),\\
&u_3(k-1), u_3(k-2)],
\end{aligned} \tag{9}$$

where the involved variables are the same as those detailed in (8).

3.5. Validation of the Identified Fuzzy Models. An essential part of any identification process is the validation of the identified models. The *FMID* software tool [37] uses a specific mathematical procedure of validation and provides the results both in graphic form, comparing the *real* process output with the model *estimated* output and, numerically, by calculating and giving the validation index known as *VAF* (VAF: percentile variance accounted for between two signals), which is among those habitually used for giving validity to the identified models. In our study, the *real* process output was obtained by means of simulation (with the wastewater treatment plant represented by the nonlinear model mentioned in Section 2.2 and detailed in Appendix D, given in the form of differential equations) and the model *estimated* output was obtained by applying to the identified fuzzy model the same inputs as to the wastewater treatment plant simulated.

In our research process, numerous identifications were made with the same or different input-output data sets. Likewise, the use of the available input-output data was diverse, beginning with basic tests, using the same set of input-output data in the identification and validation of the model, and then carrying out more reliable tests in which a partition was made of the available input-output data, using a certain percentage of the data to identify the model and the rest to validate the model. On the other hand, and as we said at the beginning of Section 3.4, identifications with different dynamic characteristics of the recursive discrete model that will represent the plant were made. The presentation, analysis, and discussion of the results of all the identifications made, together with the study of the relationship between the used data or the choice of parameters of the structure of the fuzzy model and the goodness of the identification, are beyond the scope of this article. However, it is necessary to consider at least the four identification cases mentioned at the beginning of Section 3.4, cases A, B, C, and D, commenting in this section the graphic and numerical results of the validation process of the corresponding fuzzy models identified. We summarize in Table 9 the main characteristics of the four cases.

For case A and case B, dependence on $y_1(k)$ and $y_2(k)$ outputs with respect to $u_1(k-1) = q_i(k-1)$ input has not

Table 9: Parameters and use of input-output data in 4 cases.

Case	Dynamical Parameters of the recursive discrete model: $\mathbf{N}_y$	$\mathbf{N}_u$	$\mathbf{N}_d$	Identification in-out data versus validation in-out data: The same	Different (partition)
A	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	●	
B	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$		●
C	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$	●	
D	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$		●

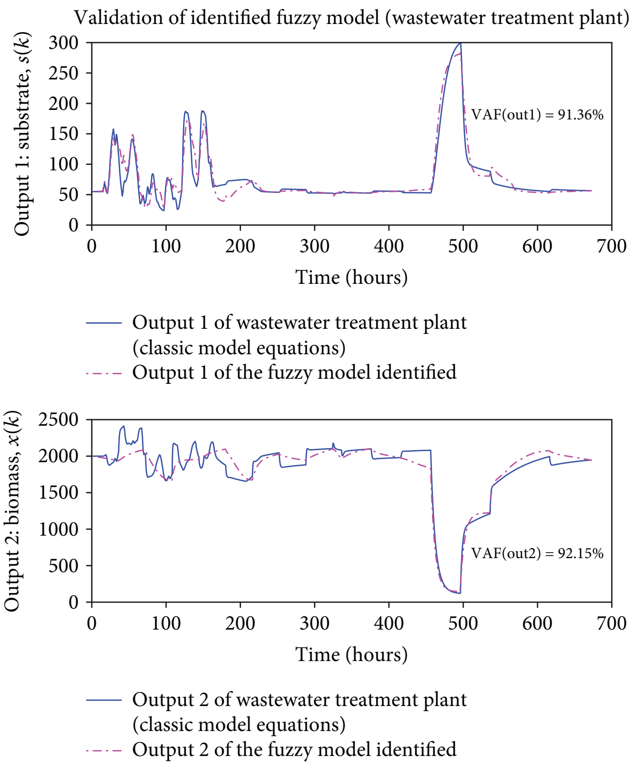

Figure 7: Validation of the identified fuzzy model corresponding to case A for output 1, $s(k)$, and output 2, $x(k)$.

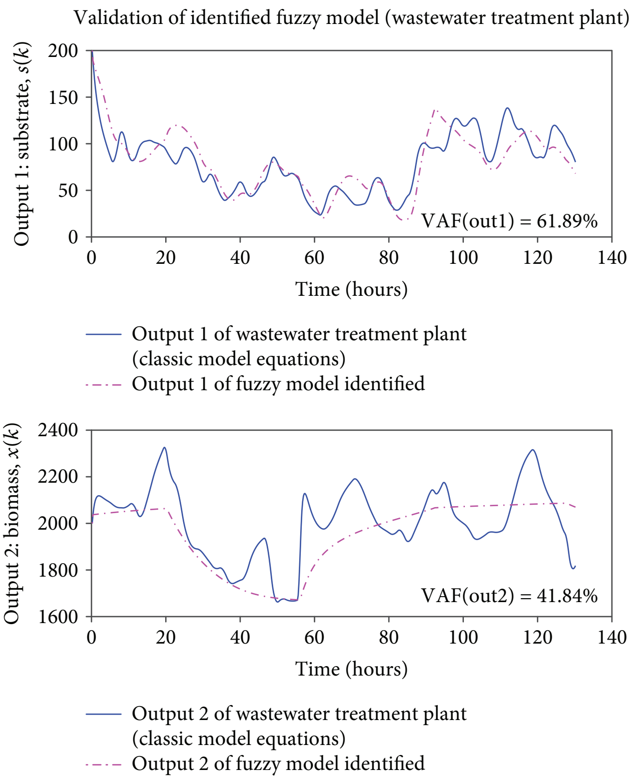

Figure 8: Validation of the identified fuzzy model corresponding to case B for output 1, $s(k)$, and output 2, $x(k)$.

been considered. But it has been considered in case C and case D. We are interested in testing the effectiveness of the control algorithm, both in cases where the model is the result of more complete identification (C and D) and in cases where the identification is incomplete or less precise (A and B).

3.5.1. Results of the Validation Process for Case A and Case B. The results of the validation corresponding to case A are shown in Figure 7. This figure contains two graphic representations (one for each output of the wastewater treatment plant under study) and compares the values of the *real* process output with those of the model *estimated* output (for the same set of values of the inputs), including in addition the *VAF* index, which measures the goodness of the identification for each of the two partial identifications.

In case B, we have the same structure of rules for the fuzzy model as in the case A and therefore the same components for the antecedent vector and the consequent vector, but with the difference that in this case, the numerical input-output data sequences used in the identification process were different from those used in the validation process (the available input-output data were partitioned). The results of the corresponding validation, for the two outputs of the wastewater treatment plant, can be seen in Figure 8.

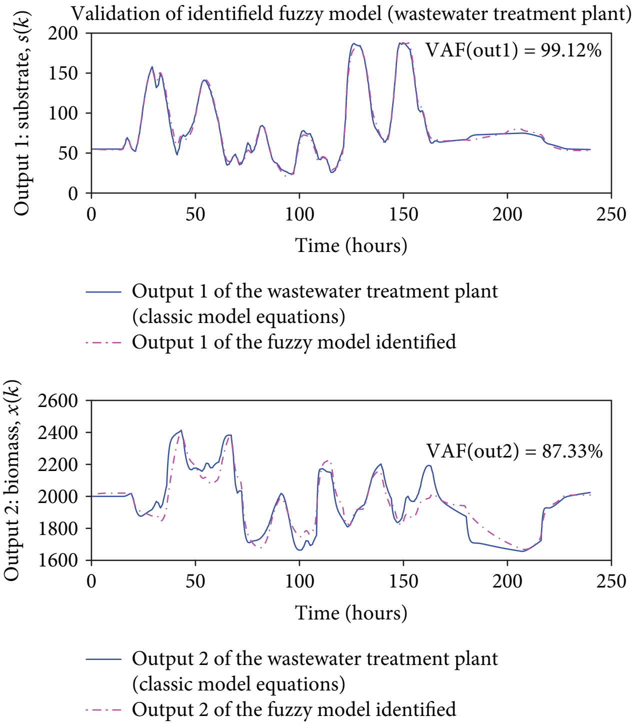

FIGURE 9: Validation of the identified fuzzy model corresponding to case C for output 1, $s(k)$, and output 2, $x(k)$.

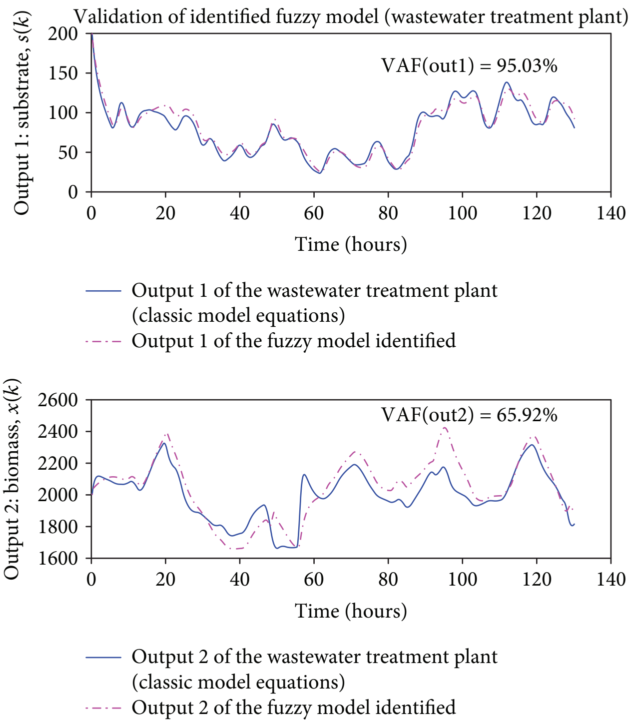

FIGURE 10: Validation of the identified fuzzy model corresponding to case D for output 1, $s(k)$, and output 2, $x(k)$.

In relation to the two identifications shown above, we can give a brief comparative analysis of both cases, taking into account the corresponding graphs and validation indexes. In case A (Figure 7), the adjustment is quite precise for the two outputs, with very high *VAF* indexes (over 92%), while in case B (Figure 8), the adjustment is markedly lower with *VAF* indexes around 62% (output 1) and 42% (output 2). The difference is due to the fact that in the second case, the input-output data used to identify the model were not the same as the input-output data used to validate the model, a situation in which a more difficult adjustment can be expected. In case B, however, although the *VAF* indexes are lower, the response of the fuzzy model follows quite well the evolution of output 1 of the wastewater treatment plant and acceptably that of output 2, the tendency of which was at least detected by the identification. On the other hand, one of the potential complementary objectives of this study is precisely the assessing of the efficiency of fuzzy predictive control algorithms, including with imperfect models, as indeed we have checked in the control experiments (FMBPC) carried out with both fuzzy models and which we will subsequently discuss.

3.5.2. Results of the Validation Process for Case C and Case D. In case C and case D, dependence on $y_1(k)$ and $y_2(k)$ outputs with respect to $u_1(k-1) = q_i(k-1)$ input has been considered. Therefore, in principle, these two cases are more precise and realistic when it comes to identifying the model of the plant and one would expect a greater utility of the models within a strategy of predictive control based on models. The dynamic structure of the model considered for the identification of the plant is the same in both cases, as can be seen in Table 8, but they differ (as in the previous two cases) in that, in case D, the numerical input-output data sequences used in the identification process were different from those used in the validation process (the available input-output data were partitioned). The results of the validation, for the two outputs of the wastewater treatment plant, corresponding to case C and case D, can be seen in Figures 9 and 10, respectively.

In relation to the other two identifications shown above, we can also give a brief comparative analysis of both cases, considering the corresponding graphs and validation indexes shown. In case C (Figure 9), the adjustment is quite precise for the two outputs, with very high *VAF* indexes (around 99% for output 1 and of 87% for output 2). In case D (Figure 10), the adjustment is also quite precise for output 1 (95%) and quite acceptable for output 2 (around 66%). In addition, the responses of the two fuzzy models follow quite well the evolution of the corresponding outputs of the wastewater treatment plant (output 1 and output 2), that is, the tendency in the two outputs was detected very well by the identification. Finally, it is relevant to emphasize that the goodness of the validation of the model identified in case D is significantly better than that of case B. The difference between both cases is the consideration, in case D, of the dependence of the two outputs with respect to $u_1(k-1) = q_i(k-1)$ input. In the simulation study presented in Section 5, four different cases of predictive control (FMBPC) applied to the *WWTP* have been considered, each of them corresponding to the use of one of the four identified models (case A, case B, case C, and case D). We are interested in analyzing the possible influence of the models on the effectiveness of the control algorithm, although a detailed study of this aspect is not possible to address it in this article.

3.6. State Space Modeling. By using suitable definitions and mathematical treatments, we can describe or formalize the rules of the fuzzy models of the *Takagi-Sugeno* type in the form of *state equations*. This involves the huge advantage of being able to treat the fuzzy models in an analytical manner and also to calculate the predictions analytically and ultimately being able to express the control law in an analytical and explicit manner. The application of this methodology to the case study which concerns us constitutes one of the main contributions of our study, which follows the line previously initiated by other authors [10]. The mathematical development necessary for our case study has previously been presented [31]. The discrete time fuzzy models in the *state space* were formalized jointly for the two outputs of the water treatment plant (with the necessary definitions), and the equations corresponding to them both were grouped together using matrices to obtain a single *state equation* and a single *output equation*. The result of the mathematical formalization was as follows:

$$\mathbf{z}_m(k+1) = \bar{\mathbf{A}}_m \mathbf{z}_m(k) + \bar{\mathbf{B}}_m \mathbf{u}_a(k) + \bar{\mathbf{R}}_m, \quad (10)$$

$$\mathbf{y}_m(k) = \bar{\mathbf{C}}_m \mathbf{z}_m(k), \quad (11)$$

where

(i) $\mathbf{z}_m(k)$ is the *extended-state vector*, integrated by the outputs and disturbances of the *WWTP*, at the kth instant

(ii) $\mathbf{z}_m(k+1)$ is the *extended-state vector*, at the $(k+1)$th instant

(iii) $\mathbf{y}_m(k)$ is the *output* of the *model in the state space*, integrated by the outputs of the *WWTP*, at the kth instant

(iv) $\mathbf{u}_a(k)$*input* of the *model in the state space*, integrated by the manipulated variable of the *WWTP*, at the *kth* instant and the previous one

(v) $\bar{\mathbf{A}}_m$, $\bar{\mathbf{B}}_m$, $\bar{\mathbf{R}}_m$, and $\bar{\mathbf{C}}_m$ are the *state matrices*

and the corresponding expressions are

$$\mathbf{z}_m(k) = \begin{pmatrix} y_1(k) \\ y_2(k) \\ d_1(k) \\ d_2(k) \end{pmatrix} = \begin{pmatrix} y_1(k) \\ y_2(k) \\ u_1(k) \\ u_2(k) \end{pmatrix} = \begin{pmatrix} s(k) \\ x(k) \\ q_{\mathrm{i}}(k) \\ s_{\mathrm{i}}(k) \end{pmatrix}, \quad (12)$$

$$\mathbf{u}_a(k) = \begin{pmatrix} u_3(k) \\ u_3(k-1) \end{pmatrix} = \begin{pmatrix} u(k) \\ u(k-1) \end{pmatrix} = \begin{pmatrix} q_{\mathrm{r}}(k) \\ q_{\mathrm{r}}(k-1) \end{pmatrix}, \quad (13)$$

$$\mathbf{y}_m(k) = \begin{pmatrix} y_1(k) \\ y_2(k) \end{pmatrix}, \quad (14)$$

and the *state matrices* are

$$\begin{aligned} \bar{\mathbf{A}}_m &= \sum_{j=1}^{\mathrm{mr}} \left(\boldsymbol{\beta}_j(\mathbf{x}_a) \mathbf{A}_{m_j} \right), \\ \bar{\mathbf{B}}_m &= \sum_{j=1}^{\mathrm{mr}} \left(\boldsymbol{\beta}_j(\mathbf{x}_a) \mathbf{B}_{m_j} \right), \\ \bar{\mathbf{C}}_m &= \sum_{j=1}^{\mathrm{mr}} \left(\boldsymbol{\beta}_{j_{12}}(\mathbf{x}_a) \mathbf{C}_{m_j} \right), \\ \bar{\mathbf{R}}_m &= \sum_{j=1}^{\mathrm{mr}} \left(\boldsymbol{\beta}_j(\mathbf{x}_a) \mathbf{R}_{m_j} \right), \end{aligned} \quad (15)$$

where

$$\begin{aligned} \boldsymbol{\beta}_j(\mathbf{x}_a) &= \begin{pmatrix} \beta_{1j}(\mathbf{x}_a) & 0 & 0 & 0 \\ 0 & \beta_{2j}(\mathbf{x}_a) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ \boldsymbol{\beta}_{j_{12}}(\mathbf{x}_a) &= \begin{pmatrix} \beta_{1j}(\mathbf{x}_a) & 0 \\ 0 & \beta_{2j}(\mathbf{x}_a) \end{pmatrix}, \\ \beta_{26}(\mathbf{x}_a) &= 0, \\ \mathrm{mr} &= \max \cdot (\mathrm{mr}_1, \mathrm{mr}_2), \end{aligned} \quad (16)$$

being that

$$\mathbf{A}_{m_j} = \begin{pmatrix} a_{j1} & a_{j2} & b_{j1} & b_{j2} \\ 0 & a_{j2}^* & b_{j1}^* & 0 \\ 0 & 0 & \dfrac{1}{\mathrm{mr}} & 0 \\ 0 & 0 & 0 & \dfrac{1}{\mathrm{mr}} \end{pmatrix},$$

$$\mathbf{B}_{m_j} = \begin{pmatrix} b_{j3} & b_{j4} \\ b_{j3}^* & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\mathbf{C}_{m_j} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\mathbf{R}_{m_j} = \begin{pmatrix} r_j \\ r_j^* \\ 0 \\ 0 \end{pmatrix}, \quad (17)$$

(i) $a_{j1}, a_{j2}, b_{j1}, b_{j2}, b_{j3}, b_{j4}, r_j$ are the coefficients of the antecedent vector and independent term in the jth rule of the output 1 fuzzy model of the *WWTP*.

(ii) $a_{j2}^*, b_{j1}^*, b_{j3}^*, r_j^*$ are the coefficients of the antecedent vector and independent term in the jth rule of the output 2 fuzzy model of the *WWTP* (with $a_{62}^* = 0$, $b_{61}^* = 0$, $b_{63}^* = 0$, and $r_6^* = 0$).

It is important and relevant to point out that the matrix coefficients of the state equations obtained, $\bar{\boldsymbol{A}}_m$, $\bar{\boldsymbol{B}}_m$, $\bar{\boldsymbol{R}}_m$, and $\bar{\boldsymbol{C}}_m$, depend on the antecedent vector $\mathbf{x}_a$ (through $\boldsymbol{\beta}_j(\mathbf{x}_a)$) and therefore also depend on the kth instant, because $\mathbf{x}_a$ depends on time. This assumes that concerning the necessary calculations to achieve the simulation, it will be necessary to update those coefficients in each iteration after having updated the antecedent vector $\mathbf{x}_a$ and subsequently $\boldsymbol{\beta}_j(\mathbf{x}_a)$. And as a conclusion to this process of formalization of the fuzzy models identified in the *state space*, we can summarize by saying that the behaviour of our nonlinear multivariable system, initially identified by fuzzy models, was finally represented by a state equation system with a linear shape, but with coefficients depending on time. In [10], two theoretical references are mentioned on the association of systems with nonlinear dynamics, with variant linear systems over time (Leith, D.J. and Leithead, W.E., 1998 and 1999).

4. The Control Law in Analytical Form

The next step of our *fuzzy predictive control* strategy consists of deducing an analytical and explicit control law, making use of the model in the form of *state equations*. As was mentioned in the introduction, the deduction method for the control law can be considered an extension of the so-called *predictive functional control* 1 [2, 16–18] for the multivariable case. To be able to use this method, the following will be necessary: on the one hand, to define both the desired behaviour of the *closed-loop* system and the *control goal* and, on the other hand, to formalize the relationship between the model and the plant that allows carrying out a *model-based predictive control* strategy. The desired behaviour will be defined by imposing on the plant outputs the follow-up of certain reference trajectories (one for each output), as faithfully as possible, and the control goal will consist on calculating the future control action so that the predicted plant output values coincide with the reference trajectory in at least one point. In our case, we will consider a single *coincidence point*. The difference between the *current time* (kth time instant) and the one corresponding to the *coincidence point*, measured on the number of *sampling periods*, is called *coincidence horizon* and it is usually represented by H. Therefore, starting from time instant k, the coincidence with each trajectory must occur at time instant $k + H$.

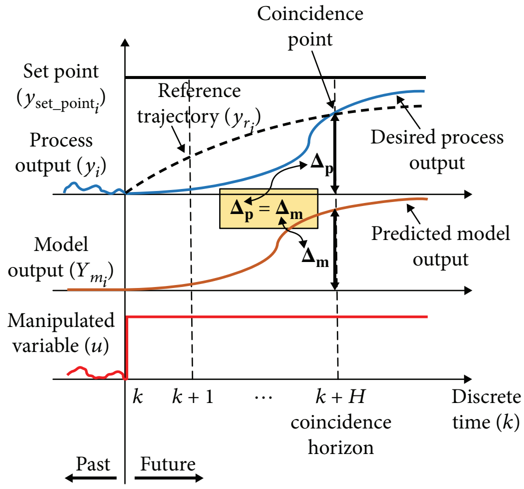

Figure 11: Predictive functional control-equivalence principle.

The relationship between the model and the plant will be established assuming the so-called *equivalence principle* between the plant and the model, used in *PFC*. This principle consists in supposing that, for each output of the plant, the *plant output increment*, between k and $k + H$, necessary to coincide with the reference trajectory, Δ_p, will be equal to the *predicted model output increment*, also between k and $k + H$, Δ_m. This principle is analytically expressed by the following expression:

$$\Delta_p = \Delta_m, \quad (18)$$

which must be satisfied for each output of the plant. If we consider the equations of both outputs simultaneously, we would have a vector equality: $\Delta_p = \Delta_m$, where Δ_p is usually the so-called *objective increment vector* and Δ_m is the *model output increment vector*. The idea of the *PFC strategy* and *equivalence principle* can be shown in graphical form (Figure 11).

4.1. Predictions Based on the Model. The control actions in the various existing *predictive control* strategies are not calculated in the same way in all cases; there are different methods and different formulas or algorithms that are applied. However, all these strategies share the use of some type of mathematical model to represent as faithfully as possible the process to be controlled (in our case, a fuzzy model formalized in the state space) and which serves to predict the future behaviour of the process and ultimately to determine the appropriate control action for compliance with the control objectives. We can therefore say that the mathematical basis of the predictive control strategies will be the general expression corresponding to $\mathbf{y}_m(k + H)$, that is, the expression of the output at the instant $(k + H)$, predicted by the model at the kth instant, with H being the prediction horizon.

The deduction of such an expression, from the state equations previously obtained, is fairly extensive in our case and has for this reason been developed in its totality in Appendix A with the objective of facilitating the follow-up of the article. The calculation of *Hstep ahead* prediction is done under the assumption of constant future manipulated variables, that is, $u(k) = u(k+1) = \cdots = u(k+H-1)$. The final expression deduced is the following (see (A.26), at the end of Appendix A):

$$\mathbf{y}_m(k+H) = \bar{\mathbf{C}}_m\left(\bar{\mathbf{A}}_m{}^H\mathbf{z}_m(k) + \left(\bar{\mathbf{A}}_m{}^{H-1}\bar{\mathbf{B}}_m + \left(\bar{\mathbf{A}}_m{}^{H-1} - \mathbf{I}\right)\cdot\left(\bar{\mathbf{A}}_m - \mathbf{I}\right)^{-1}\bar{\mathbf{B}}_m\mathbf{P}_{1010}\right)\mathbf{u}_a(k) + + \left(\bar{\mathbf{A}}_m{}^{H-1} + \left(\bar{\mathbf{A}}_m{}^{H-1} - \mathbf{I}\right)\left(\bar{\mathbf{A}}_m - \mathbf{I}\right)^{-1}\right)\bar{\mathbf{R}}_m\right), \quad (19)$$

where

$$H \in Z^+, H \geq 1\,; \exists\bar{\mathbf{A}}_m{}^{H-n} \text{ if } (H-n) \geq 1 \text{ (with } n \in Z^+),$$
$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{the order 2 identity matrix, and } \mathbf{P}_{1010} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}. \quad (20)$$

4.2. Deduction of the Control Law in an Analytical and Explicit Way. As from the expression of $\mathbf{y}_m(k+H)$, which allows the prediction at the kth instant of the evolution of the outputs of the model, H periods further on, we can search an analytical and explicit expression for the control law. This is precisely the main objective of our paper. We detail below the necessary mathematical process. In the first place, we will find $\mathbf{u}_a(k)$ in (19), obtaining an expression that will be a function of $\mathbf{y}_m(k+H)$ and we will subsequently extract, of the vector variable $\mathbf{u}_a(k)$ obtained, the scalar variable $u(k)$, which will also be a function of $\mathbf{y}_m(k+H)$. This first stage is developed in Section 4.2.1. Secondly, we will establish the desired behaviour of our system in a closed loop, imposing the follow-up by the outputs of the WWTP of certain reference trajectories, which will also condition to $\mathbf{y}_m(k+H)$. And as from the relations obtained, we will deduct the expression of the control action $u(k)$ which is necessary to comply with the control objective, which will be a function of the reference (set point) of the future outputs (at $(k+H)$). The second stage is developed in Section 4.2.2.

4.2.1. Manipulated Variable $u(k)$ as a Function of $\mathbf{y}_m(k+H)$. Leaving at (19) to one side of equality, the term in which $\mathbf{u}_a(k)$ is multiplying

$$\mathbf{y}_m(k+H) - \bar{\mathbf{C}}_m\bar{\mathbf{A}}_m{}^H\mathbf{z}_m(k) - \bar{\mathbf{C}}_m\left(\bar{\mathbf{A}}_m{}^{H-1} + \left(\bar{\mathbf{A}}_m{}^{H-1} - \mathbf{I}\right)\left(\bar{\mathbf{A}}_m - \mathbf{I}\right)^{-1}\right)\bar{\mathbf{R}}_m = \bar{\mathbf{C}}_m\left(\bar{\mathbf{A}}_m{}^{H-1}\bar{\mathbf{B}}_m + \left(\bar{\mathbf{A}}_m{}^{H-1} - \mathbf{I}\right)\left(\mathbf{A}_m - \mathbf{I}\right)^{-1}\bar{\mathbf{B}}_m\mathbf{P}_{1010}\right)\mathbf{u}_a(k), \quad (21)$$

and grouping together terms for simplification, defining matrix $\mathbf{M}_a$ as follows:

$$\mathbf{M}_a = \bar{\mathbf{C}}_m\left(\bar{\mathbf{A}}_m{}^{H-1}\mathbf{B}_m + \left(\bar{\mathbf{A}}_m{}^{H-1} - \mathbf{I}\right)\left(\bar{\mathbf{A}}_m - \mathbf{I}\right)^{-1}\bar{\mathbf{B}}_m\mathbf{P}_{1010}\right), \quad (22)$$

we will have (21) simplified, obtaining

$$\mathbf{y}_m(k+H) - \bar{\mathbf{C}}_m\bar{\mathbf{A}}_m{}^H\mathbf{z}_m(k) - \bar{\mathbf{C}}_m\left(\bar{\mathbf{A}}_m{}^{H-1} + \left(\bar{\mathbf{A}}_m{}^{H-1} - \mathbf{I}\right)\left(\bar{\mathbf{A}}_m - \mathbf{I}\right)^{-1}\right)\bar{\mathbf{R}}_m = \mathbf{M}_a\mathbf{u}_a(k), \quad (23)$$

and multiplying on the left, on both sides of equality, by the matrix $\mathbf{M}_a{}^{-1}$, we will have the following matrix expression:

$$\mathbf{u}_a(k) = \mathbf{M}_a{}^{-1}\left(\mathbf{y}_m(k+H) - \bar{\mathbf{C}}_m\bar{\mathbf{A}}_m{}^H\mathbf{z}_m(k) - \bar{\mathbf{C}}_m\left(\bar{\mathbf{A}}_m{}^{H-1} + \left(\bar{\mathbf{A}}_m{}^{H-1} - \mathbf{I}\right)\left(\bar{\mathbf{A}}_m - \mathbf{I}\right)^{-1}\right)\bar{\mathbf{R}}_m\right), \quad (24)$$

and replacing $\mathbf{u}_a(k)$ by its generic expression specified in (12), we will have

$$\begin{pmatrix} u(k) \\ u(k-1) \end{pmatrix} = \mathbf{M}_a{}^{-1}\left(\mathbf{y}_m(k+H) - \bar{\mathbf{C}}_m\bar{\mathbf{A}}_m{}^H\mathbf{z}_m(k) - \bar{\mathbf{C}}_m\cdot\left(\bar{\mathbf{A}}_m{}^{H-1} + \left(\bar{\mathbf{A}}_m{}^{H-1} - \mathbf{I}\right)\left(\bar{\mathbf{A}}_m - \mathbf{I}\right)^{-1}\right)\bar{\mathbf{R}}_m\right), \quad (25)$$

where $u(k-1)$ will be the value of the variable u at the previous sampling instant to the kth and should therefore have been conveniently memorized (or for the first sampling instant have a predetermined initial value). In short, $u(k-1)$ will be a data, a specific numerical value saved, while $u(k)$ will be the unknown quantity to find or determine.

If matrix (25), with $u(k-1)$ given, has a single numerical solution (or if it has none, assuming that it is possible to determine an approximate solution by computer), then we will be able to determine the numerical value of $u(k)$ by means of the following expression:

$$u(k) = \mathbf{P}_{10}\mathbf{u}_a(k), \quad (26)$$

being

$$\mathbf{P}_{10} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad (27)$$

and replacing $\mathbf{u}_a(k)$ by its expression given by (24), we would finally have

$$u(k) = \mathbf{P}_{10}\mathbf{M}_a{}^{-1}\left(\mathbf{y}_m(k+H) - \bar{\mathbf{C}}_m\bar{\mathbf{A}}_m{}^H\mathbf{z}_m(k) - \bar{\mathbf{C}}_m\cdot\left(\bar{\mathbf{A}}_m{}^{H-1} + \left(\bar{\mathbf{A}}_m{}^{H-1} - \mathbf{I}\right)\left(\bar{\mathbf{A}}_m - \mathbf{I}\right)^{-1}\right)\bar{\mathbf{R}}_m\right), \quad (28)$$

which is the expression that will allow the determining of the variable $u(k)$ which will be necessary at the kth sampling instant so that the model can attain, H periods later (i.e., at the $(k+H)$th instant), a certain vector *value*$\mathbf{y}_m(k+H)$.

4.2.2. Control Action $u(k)$ Necessary for the Follow-Up of the Output Reference Trajectories. The control action $u(k)$ which we search will be that guaranteeing compliance with the control objectives for the process output. Thus, following the same approach as in [10, 18], we will establish the desired behaviour of the *closed-loop* system imposing for the outputs the follow-up of discrete reference trajectories, which must gradually approach to the corresponding references. These trajectories constitute the *reference model* and will be given formally by means of the following recursive equations:

$$y_{r_i}(k+1) = a_{r_i} y_{r_i}(k) + b_{r_i} y_{\text{set\_point}_i}(k), \quad i = 1, 2 \text{ (number of outputs)}, \tag{29}$$

where $y_{\text{set\_point}_i}(k)$ is the reference or desired value of the ith output $y_i(k)$ at the kth instant, $y_{r_i}(k)$ and $y_{r_i}(k+1)$, the values in k and in $k+1$, respectively, of the reference trajectory, and a_{r_i} and b_{r_i} the parameters of the reference model, the values of which must be such so as to ensure that the gain of the reference model is the unit (for both outputs). For this to occur, as can be seen in Appendix B expression (B.5), the following mathematical relationship should be complied with

$$\left(1 - a_{r_i}\right)^{-1} b_{r_i} = 1, \quad i = 1, 2 \text{ (number of outputs)}. \tag{30}$$

And as we have introduced at the beginning of the section, to achieve the follow-up of the reference trajectories, the control goal fixed will consist on to calculate the future control action so that the predicted plant output values coincide with the reference trajectory (for each output of the plant) in a single *coincidence point*. Such a point will correspond to the instant $k+H$, where H is called the *coincidence horizon*.

The control action at each kth instant, $u(k)$, should therefore be such that the plant outputs match the corresponding reference trajectories, H sampling periods later, such that $y_i(k+H)$ equals $y_{r_i}(k+H)$. Therefore, the desired *plant output increment* should be equal to $[y_{r_i}(k+H) - y_i(k)]$, for each ith output $(i = 1, 2)$, that is,

$$\boldsymbol{\Delta}_p\big|_i = y_{r_i}(k+H) - y_i(k), \quad i = 1, 2 \text{ (number of outputs)}. \tag{31}$$

On the other hand, we need now to do use of the relationship between the model and the plant established in the introduction of this section, consisting of adopting the so-called *equivalence principle* (following again the same approach as in [10, 18]), which is a way to introduce the main idea or mechanism implicit in the *model-based predictive control*. We must therefore consider equality (18), $\boldsymbol{\Delta}_p = \boldsymbol{\Delta}_m$, for each ith output $(i = 1, 2)$, where $\boldsymbol{\Delta}_m$ is the *predicted model output increment*, whose expression is as follows:

$$\boldsymbol{\Delta}_m\big|_i = y_{m_i}(k+H) - y_{m_i}(k), \quad i = 1, 2 \text{ (number of outputs)}, \tag{32}$$

and matching (31) and (32) (i.e., matching the desired *plant output increments* with the *predicted model output increments*: $\boldsymbol{\Delta}_p|_i = \boldsymbol{\Delta}_m|_i$), we will have

$$y_{r_i}(k+H) - y_i(k) = y_{m_i}(k+H) - y_{m_i}(k), \quad i = 1, 2 \text{ (number of outputs)}. \tag{33}$$

The control action at each kth instant, $u(k)$, should be such that equality (33) is satisfied for each of the two outputs. And considering together the two equalities implicit in (33), we will have the following single vector expression:

$$\mathbf{y}_r(k+H) - \mathbf{y}(k) = \mathbf{y}_m(k+H) - \mathbf{y}_m(k). \tag{34}$$

Finding now $\mathbf{y}_m(k+H)$ in (34), we will have that the predicted model output in the $(k+1)$th time instant should satisfy the following expression:

$$\mathbf{y}_m(k+H) = \mathbf{y}_r(k+H) - \mathbf{y}(k) + \mathbf{y}_m(k). \tag{35}$$

Replacing (28) $\mathbf{y}_m(k+H)$ by the second member of equality (35), which we have just obtained, we will finally have the expression that we were looking for $u(k)$, dependent on terms that can be determined using the plant model or the reference model or that can be measured at the kth instant. The final expression for $u(k)$ is the following, where $\boldsymbol{M}_a$ is given by (22) and $\mathbf{P}_{10}$ by (27) and where the matrix coefficients that intervene must be updated in each iteration of the implementation of the control algorithm (due to their dependence on the antecedent vector and therefore on time)

$$\begin{aligned} u(k) = \mathbf{P}_{10}\mathbf{M}_a^{-1}\Big(&\mathbf{y}_r(k+H) - \mathbf{y}(k) + \mathbf{y}_m(k) - \bar{\mathbf{C}}_m\bar{\mathbf{A}}_m^{\,H}\mathbf{z}_m(k) \\ &- \bar{\mathbf{C}}_m\Big(\bar{\mathbf{A}}_m^{\,H-1} + \Big(\bar{\mathbf{A}}_m^{\,H-1} - \mathbf{I}\Big)\left(\bar{\mathbf{A}}_m - \mathbf{I}\right)^{-1}\Big)\bar{\mathbf{R}}_m\Big) \\ &H \in Z^+, \quad H \geq 1, \end{aligned} \tag{36}$$

where the term $\mathbf{y}_r(k+H)$ would remain to be specified, which would be developed by induction. This development is shown in an abbreviated form together with the result obtained under the hypothesis of maintenance of the reference on the prediction horizon in Appendix C. The result is as follows (see (C.10) at the end of said appendix):

$$\begin{aligned}
\mathbf{y}_r(k+H) &= \mathbf{A}_{rH}\mathbf{y}_r(k) + (\mathbf{I} - \mathbf{A}_{rH})\mathbf{y}_{\text{set_point}}(k), \\
\mathbf{A}_{rH} &= \begin{pmatrix} a_{r_1}^H & 0 \\ 0 & a_{r_2}^H \end{pmatrix}, \\
\mathbf{I} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
H &\in Z^+, \\
H &\geq 1.
\end{aligned} \tag{37}$$

Formula (36), complemented with (37), constitutes the analytical and explicit expression of the control law $u(k)$ which would be needed, at each kth instant, to guarantee the desired behaviour of our multivariable closed-loop system, which is precisely what we were searching in this section.

5. FMBPC Applied to WWTP

The main objective of this work is the deduction of a *fuzzy predictive control* (*FMBPC*) law, expressed in an analytical and explicit way and to apply it to the control of wastewater treatment biological processes that are multivariable, highly nonlinear, time-varying, and complex processes. These processes are difficult to control due to their biological nature, and, in addition, our control system will use a single manipulated variable to control two output variables. Therefore, the possible obtaining of positive results would be quite interesting.

Next, we show the configuration of the implemented control system and the experiments carried out by means of simulation.

5.1. FMBPC Scheme. The implemented control strategy is in the field of *nonlinear predictive control* (*NLMPC*) and uses principles and methodology of *functional predictive control* (*PFC*) in the deduction of the control law. The basic mechanisms of operation of this strategy have been reflected in Figure 12.

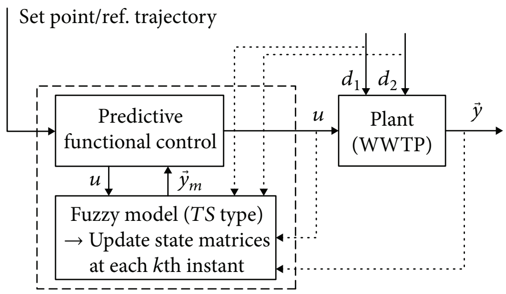

Figure 12: *FMBPC* scheme with an internal fuzzy model.

As can be seen, our *FMBPC* scheme uses an internal fuzzy model to carry out the predictions and uses *PFC* principles for the calculation of the control law. The controller needs the information related to the set point of the outputs and the corresponding reference trajectories. In addition, in each iteration (sampling period), state matrices must be updated, because they depend on the premise or antecedent vector (and ultimately also of time). Therefore, the controller must also have the instantaneous information of all the variables that constitute the antecedent vector.

The task sequence that must be executed for the implementation of our *FMBPC* strategy and that constitute the *control algorithm* is shown in Algorithm 4.

5.2. Simulation Study. The predictive control experiments were carried out by means of simulation in the *Matlab* and *Simulink* environment. Of all the developed software, the most important part was that corresponding to the implementation of the control algorithm, that is, the necessary code for the calculation in each iteration of the control variable, using the analytical and explicit expression obtained ((36) and (37)), including the necessary updating of the state matrices, which depend on time.

The study included numerous experiments carried out at different periods of time, some of which are presented in [31, 32]. Numerous tests were carried out with different fuzzy models, different input disturbances, and different output references (mainly between 45 (mg/l) and 65 (mg/l) for the substrate and between 700 (mg/l) and 2000 (mg/l) for the biomass). For each case, different tests with different values of H, the *coincidence horizon*, were carried out. The simulation time interval chosen was from 0 to 166 hours. For this study, four predictive control experiments were selected, based on fuzzy models that were the result of four identifications considered in Section 3 of this paper: case A, case B, case C, and case D. The dynamic characteristics of these models were summarized in Table 9. In addition to implementing for all cases our *FMBPC* scheme applied to the WWTP, in the last two cases (C and D), a *closed-loop control system* for the substrate was also implemented (in parallel), based on a previously tuned *PID* controller.

We will show the results of the selected experiments by means of several graphic representations. For each experiment, the temporal evolutions of the different variables involved are included: the two controlled variables, that is, the substrate concentration in effluent ($s(k)$) and the biomass concentration in the reactor ($x(k)$), and the control variable, that is, the activated sludge recirculation flow rate ($q_r(k)$), whose numerical values are precisely the result of the application of the control law obtained in Section 4 of this article. On the other hand, in some of the representations, the simultaneous temporal evolutions of the two perturbations considered have also been included: the input flow rate ($q_i(k)$) and the substrate concentration in the influent ($s_i(k)$). The reason is that it is relevant to show the high level of the disturbances that are being tried to compensate with the control action and, at the same time, to see the evolution of the controlled variables. That is, the objective is to relate the response of the controlled variables to the level and variability of the disturbances. In the case, the graph of the

Repeat at each kth instant (sampling period, T_s):

Step 1. Update the current premise vector. Measurement of all the necessary variables for the determination of the current premise or antecedent vector, $\mathbf{x}_a$:

$$y_1(k-1), y_2(k-1), u_1(k-1), u_2(k-1), u_3(k-1), u_3(k-2)$$

Step 2. Update the state matrices. The *TS* fuzzy model was formalized in the form of linear time-varying state space equations, and the state matrices depend on the current premise vector. Therefore, at each instant, the following parameters must be updated, using (14):

$$\bar{\mathbf{A}}_m, \bar{\mathbf{B}}_m, \bar{\mathbf{R}}_m, \text{ and } \bar{\mathbf{C}}_m$$

Step 3. Calculate the predicted model outputs. The use of a model to predict future behaviour is essential in MPC. By means of the appropriate mathematical treatment, the outputs predicted by the model can be calculated. In our case, using (18), we will get:

$$\mathbf{y}_m(k+H)$$

Step 4. Compute the current control action. By establishing the desired behaviour of our system through the previous definition of certain reference trajectories, which must be followed by the outputs of the plant, and applying principles of PFC, we can determine the expression of the necessary control action to achieve such behaviour. Thus, using (36), we will obtain:

$$u(k)$$

Step 5. Apply the calculated control action to WWTP.

Algorithm 4: Control algorithm for *FMBPC* implementation.

Table 10: Set points and H of experimental cases.

Case	Set point value: Substrate, s_{sp} (mg/l)	Set point value: Biomass, x_{sp} (mg/l)	Coincidence horizon $H \in \mathbb{Z}$
A	55	1837	6
B	55	750	3
C	55	1900	6
D	55	1837	180

temporal evolution of the substrate is even more relevant, because the control system manages to reduce the value of the substrate concentration to significantly lower levels. The disturbances considered in case A were different from those considered in case B and also different (both) to the perturbations of C and D cases (being the same in case C and in case D).

The set points of s and x, as well as the coincidence horizons considered in each of the four experiments are detailed in Table 10.

5.2.1. FMBPC Using Fuzzy Models of A and B Cases. The three graphic representations for the experiment corresponding to case A are shown in Figure 13 ($s(k)$), Figure 14 ($x(k)$), and Figure 15 ($q_r(k)$).

And the three graphic representations for the experiment corresponding to case B are shown in Figure 16 ($s(k)$), Figure 17 ($x(k)$), and Figure 18 ($q_r(k)$).

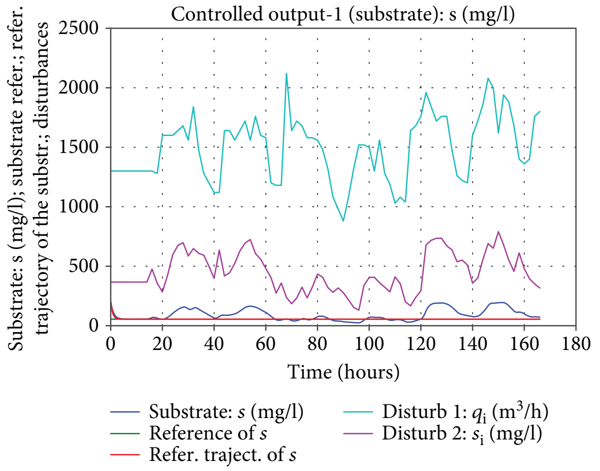

Figure 13: Substrate in the effluent and disturbances (case A).

The graphic representations corresponding to case A show that the control law obtained is able to maintain the levels of the substrate in effluent around the reference value (with acceptable deviations), despite the strong disturbances of the input flow rate and the pollution of the incoming water (the substrate concentration in the influent). At the same time, the concentration of biomass in the reactor is led to its reference value by following the preestablished reference trajectory. All of this is with a single manipulated variable. On the other hand, from the observation of the evolution of

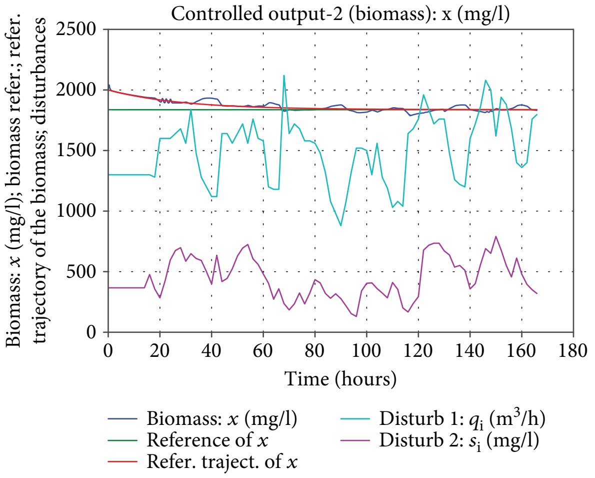

Figure 14: Biomass in the reactor and disturbances (case A).

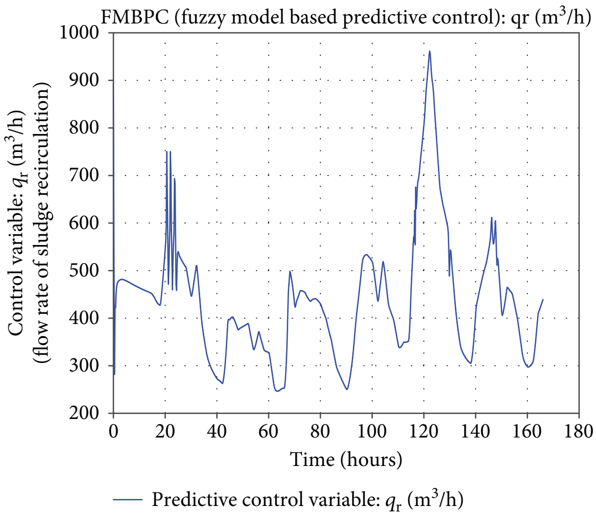

Figure 15: Activated sludge recirculation flow rate: the calculated predictive control variable (case A).

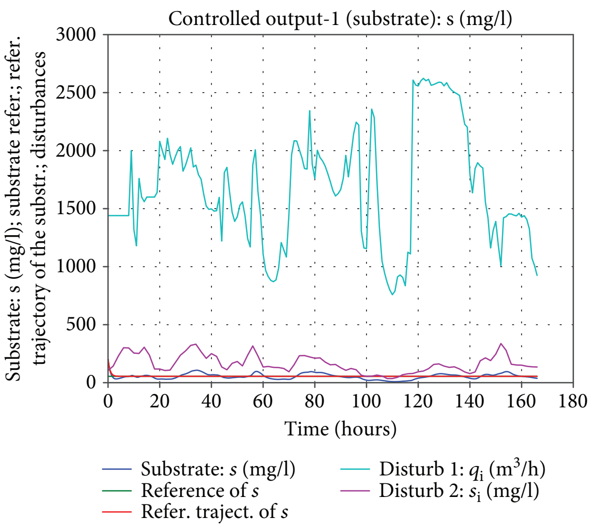

Figure 16: Substrate in the effluent and disturbances (case B).

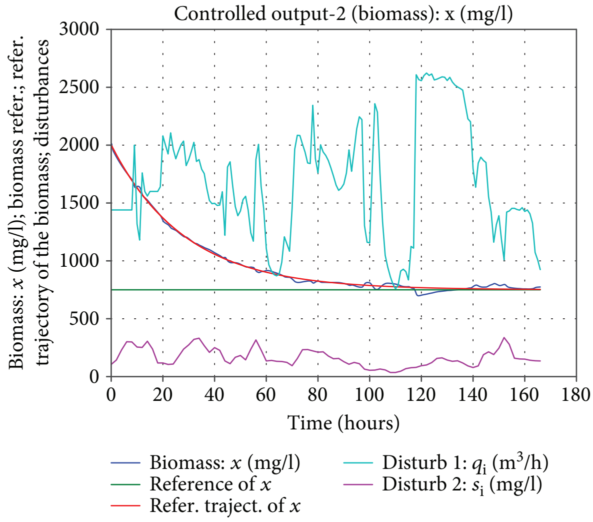

Figure 17: Biomass in the reactor and disturbances (case B).

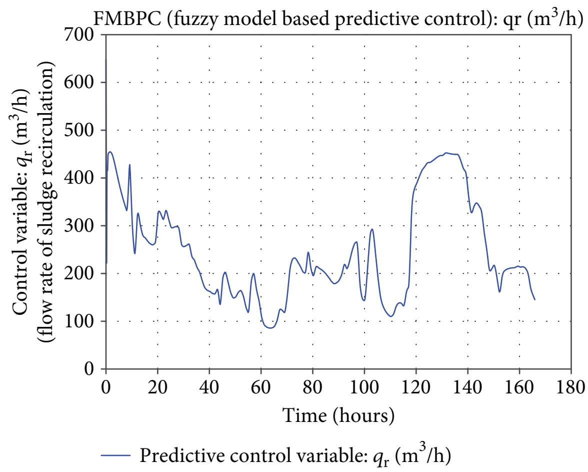

Figure 18: Activated sludge recirculation flow rate: the calculated predictive control variable (case B).

the control actions and the comparison with the disturbances and the outputs, we can say that the variations in the activated sludge recirculation flow rate, determined by the control law, timely counteract the output deviations due to the disturbances and by means of acceptable control efforts.

The graphic representations corresponding to case B show, as in case A, that the control law obtained is also able to maintain the levels of the substrate in effluent around the reference value, despite the disturbances of the input flow rate and the pollution of the incoming water (which are different from those of case A). At the same time, the concentration of biomass in the reactor is also led to its reference value (which is also different from that of case A), following the preestablished reference trajectory fairly faithfully (better even than in case A). All of this is also with a single manipulated variable. From the observation of the control actions, we can say that the activated sludge recirculation flow rate, calculated by means of the control law deduced, also

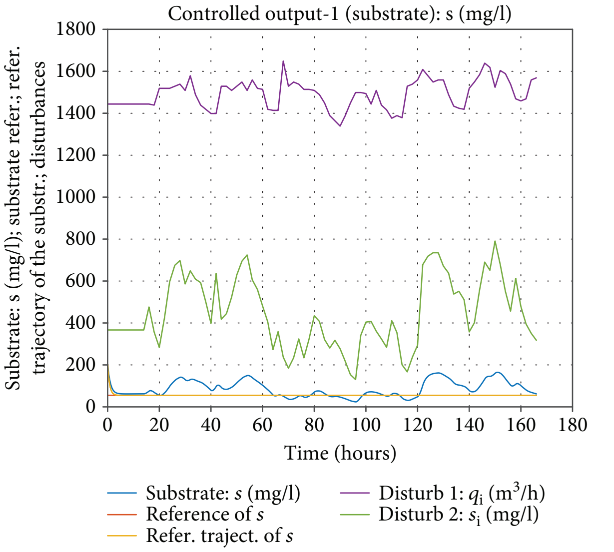

FIGURE 19: Effluent substrate by *FMBPC* and disturbances (case C).

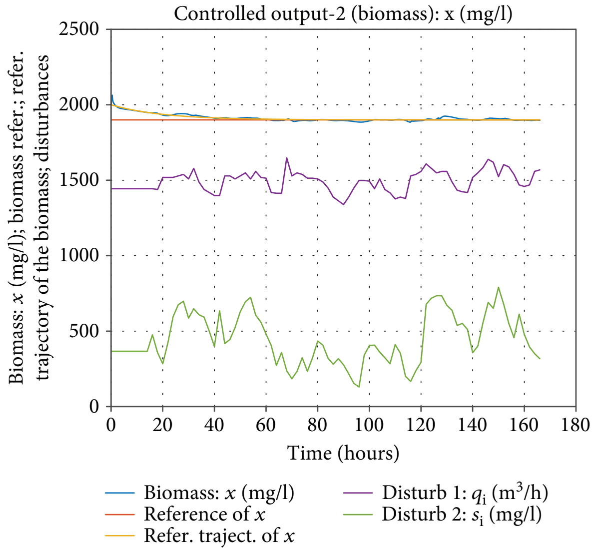

FIGURE 21: Reactor biomass by *FMBPC* and disturbances (case C).

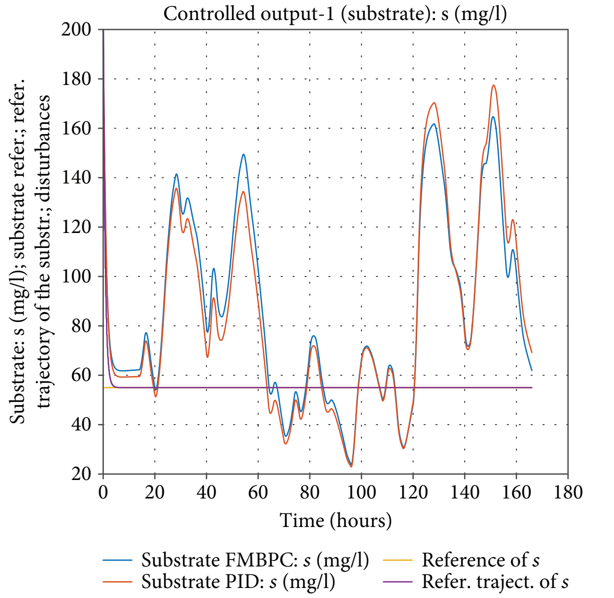

FIGURE 20: Effluent substrate by *FMBPC* and *PID* (case C).

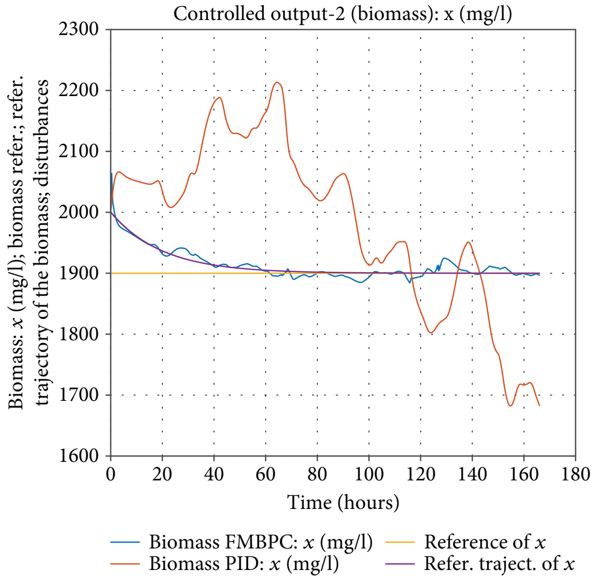

FIGURE 22: Reactor biomass by *FMBPC* and *PID* (case C).

compensates in this case, quite well, the output deviations due to the disturbances and also by means of acceptable control efforts. The maximum values of the control variable (as well as its range of variation) are lower than in case A, but also the pollution of the incoming water is much lower in this case than in case A.

In none of the cases shown, it was necessary to impose limits on the increase of the control action. However, in some cases, it is necessary to do so to avoid instability. The analysis of this problem requires further study. In relation to the prediction horizon, many tests were carried out with different prediction horizons, but herein, only the results corresponding to an H value, for each of the cases studied, have been shown. A specific study of the influence of the prediction horizon, which is a very important design parameter in predictive control, is needed.

5.2.2. FMBPC Using Fuzzy Models of C and D Cases. The dynamic characteristics of C and D cases are different from those of A and B cases (see Table 9). In addition, a previously tuned PID controller has been implemented with the objective of evaluating the *FMBPC* performance, not only individually but also comparatively. The PID tune parameters were $k_p = 1$ and $k_i = 0.1$ (with an offset or threshold for the control signal value equal to 775 m^3/h).

The results of the experiments corresponding to case C can be seen in the graphic representations shown in Figures 19–23. And the results of the experiments

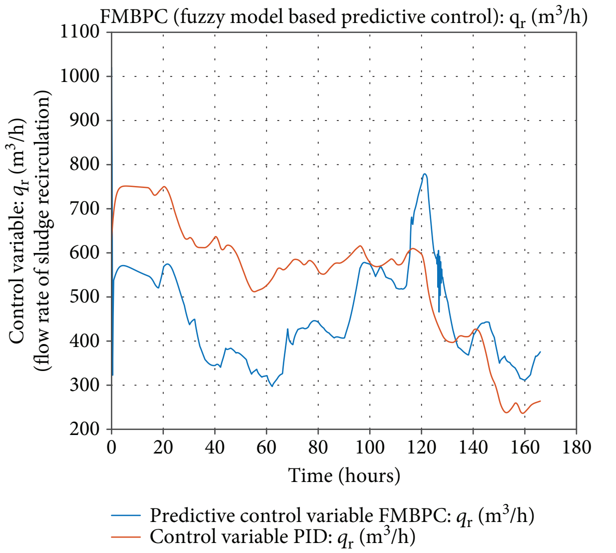

FIGURE 23: Control variables calculated by *FMBPC* and *PID* strategies (case C).

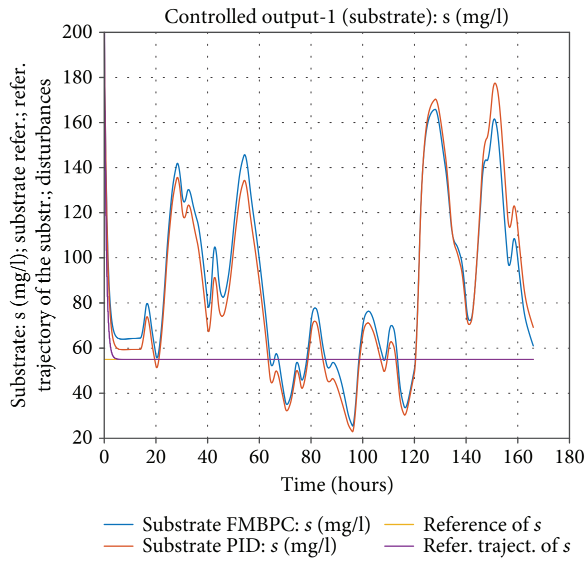

FIGURE 25: Effluent substrate by *FMBPC* and *PID* (case D).

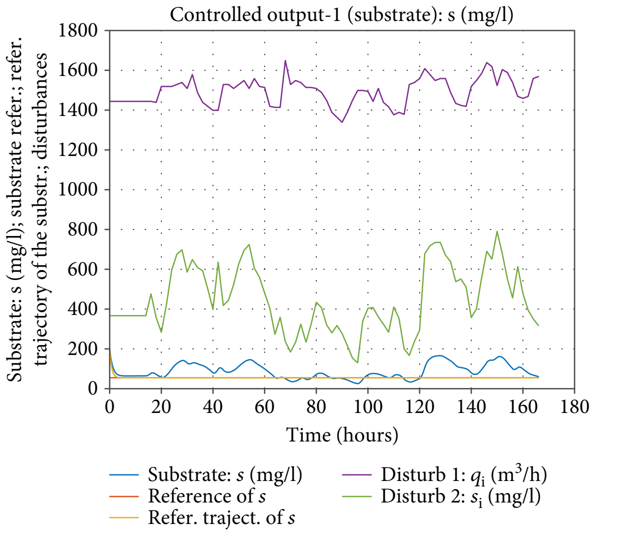

FIGURE 24: Effluent substrate by *FMBPC* and disturbances (case D).

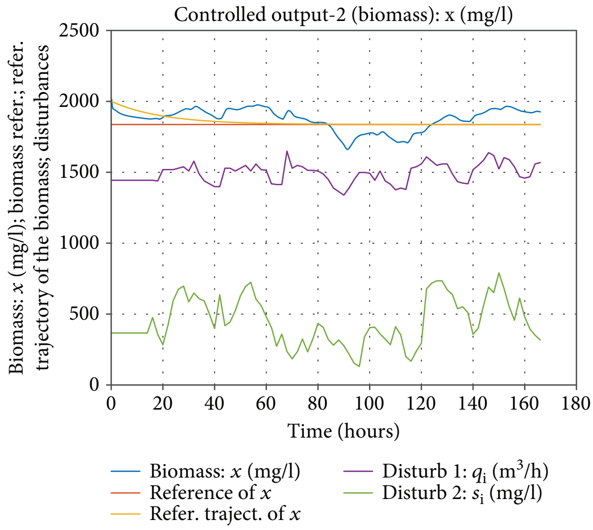

FIGURE 26: Reactor biomass by *FMBPC* and disturbances (case D).

corresponding to case D can be seen in the graphic representations shown in Figures 24–28. From the observation of such figures, we can summarize the results obtained in the experiments corresponding to C and D cases. In both cases (for the two models used), the substrate is controlled in an acceptable manner considering the strong and permanent disturbances in the input flow rate and in the substrate present in the effluent. In addition, the responses of the controlled substrate with our *FMBPC* strategy and with the *PID* strategy are also similar. The same does not occur with the biomass present in the reactor. On the one hand, the answer in case C is a little different from that in case D: tracking the reference trajectory, it is more precise in case C than in case D, probably due to the best *VAF* indexes in the first case, especially for biomass. However, the case D model is more realistic, due to the validation procedure and, therefore, it would be expected that in other areas of operation, further away from those of the identification, the case D model would respond better than the case C model. In any case, as we said at the beginning, this type of interpretation would require a broader study. The

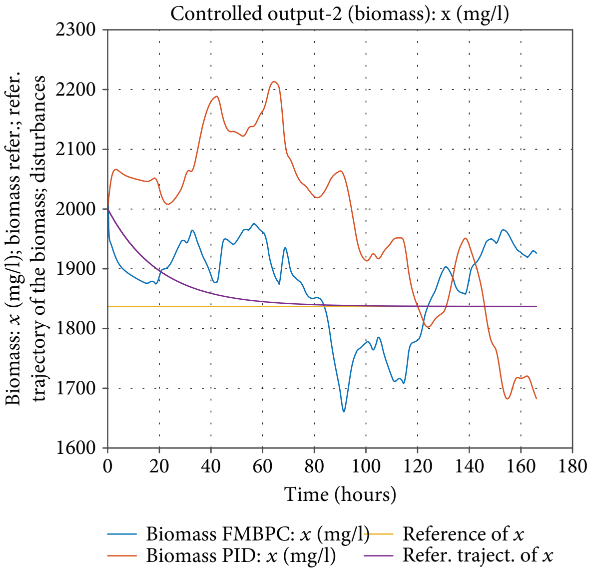

Figure 27: Reactor biomass by *FMBPC* and *PID* (case D).

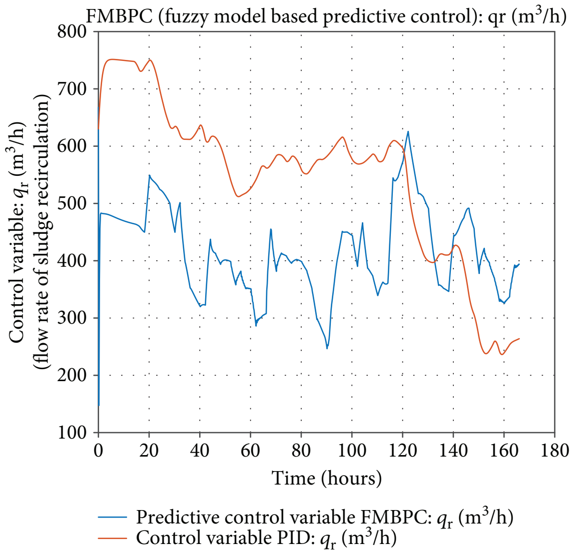

Figure 28: Control variables calculated by *FMBPC* and *PID* strategies (case D).

verification of the usefulness of the proposed *FMBPC* strategy has more weight in this work. Thus, with respect to the comparison of our *FMBPC* strategy and the *PID* strategy in the control of biomass, the differences are clear: by our *FMBPC* strategy, the biomass is acceptably controlled and simultaneously also the substrate, acting with a single manipulated variable, while the PID, acting with the same manipulated variable, manages to control the substrate, but at the cost of losing control of the biomass.

5.2.3. Performance Evaluation. The goal of the proposed control strategy is to follow the output references as closely as possible. To evaluate the degree of performance in the experiments carried out, the *integral square* (*or quadratic*) *error index*, known as the *ISE* index, was taken as a criterion. In the discrete case, the ISE corresponding to each output is the sum of the differences between the reference of the output (set point) and the values of the outputs, squared. The results obtained in the experiments corresponding to C and D cases can be seen in Table 11.

Analyzing the numerical results of Table 11, we can conclude that, for the two cases analyzed, the performance of the *FMBPC* strategy is a little better than the PID strategy, for the substrate, and rather better for biomass, something that was also evident after the qualitative analysis of the graphics shown. For biomass, our FMBPC strategy reduces 97.2% of the *ISE* of the PID in case C and 82% in case D.

6. Conclusions

In this article, a fuzzy predictive control law in an analytical and explicit way has been developed and has been applied to a multivariable, with disturbances, strongly nonlinear, with a complex dynamic and of a biological nature process. The process was an urban wastewater treatment plant (WWTP) with purification by activated sludge. The case study was developed in a simulation environment. The main conclusion of this study is the capacity of the proposed nonlinear MPC strategy to control a strongly nonlinear and multivariable system, in the presence of strong disturbances, even with adaptation to changes in the operating point with time. Our *FMBPC* strategy has been able to control two variables of a WWTP (substrate and biomass) simultaneously, making use of a single manipulated variable (sludge recirculation). The second important conclusion is that the performance of the proposed strategy improves that of a *PID* controller, in a very appreciable way for the case of biomass and with a similar performance for the substrate. The result of the evaluation made on the performance of each strategy (using the *ISE* index) is, in summary, that the proposed *FMBPC* approach reduces between 82% and 97.2% the *ISE* of the *PID* for the biomass variable (82% for one of the studied cases and 97.2% for the other).

Another line of future work could be the search for a variant of the proposed control algorithm that incorporates parameters (degrees of freedom), to be determined by means of optimization, with the aim of avoiding instabilities and improving the operation of the controlled plant. Likewise, the possibility, already mentioned in the introductory section of this article, that the proposed fuzzy control law can be straightforwardly used within a *dual-mode MPC* scheme to handle constraints if needed opens another interesting line of future work both in the field of nonlinear predictive control as well as in the field of intelligent control.

Table 11: Performance evaluation by the *ISE* index.

Case	Out 1 (substrate)		Out 2 (biomass)		Both outputs (sum)	
	FMBPC	PID	FMBPC	PID	FMBPC	PID
C	3.9824e + 05	4.0415e + 05	1.1959e + 05	4.3241e + 06	5.1783e + 05	4.7282e + 06
D	3.9619e + 05	4.0415e + 05	1.2063e + 06	6.7338e + 06	1.6025e + 06	7.1380e + 06

Appendix

A. Predictions Based on the Model: Deduction of the General Expression of the Outputs Predicted by the Model

The objective of this appendix is the obtaining of the general expression of the predictions of the wastewater treatment plant outputs, as from the fuzzy model identified and subsequently formalized in the state space. It is a case of deducing the general mathematical expression corresponding to $\mathbf{y}_m(k+H)$, that is, the expression of the output at the instant $(k+H)$, predicted by the model at the kth instant, with H being the prediction horizon. Making use of the model obtained, that is, of the state equations (10) and (11), we will develop the expression for a finite number of cases, giving to H consecutive numerical values, integers (beginning with $H=1$). Then we will reason by means of induction. We detail the deductive-inductive process below, beginning with the case $\underline{H=1}$.

$\underline{H=1}$ $(k+H=k+1)$

If we replace k with $(k+1)$ in state equation (11), we will have

$$\mathbf{y}_m(k+1)=\bar{\mathbf{C}}_m\mathbf{z}_m(k+1), \quad (A.1)$$

and then replacing $\mathbf{z}_m(k+1)$ with the expression specified in state equation (10), this will give

$$\mathbf{y}_m(k+1)=\bar{\mathbf{C}}_m\left(\bar{\mathbf{A}}_m\mathbf{z}_m(k)+\bar{\mathbf{B}}_m\mathbf{u}_a(k)+\bar{\mathbf{R}}_m\right). \quad (A.2)$$

$\underline{H=2}$ $(k+H=k+2)$

If we replace k with $(k+2)$ in state equation (10) and then we perform the change of variable (intermediate) $k'=k+1$ and then use twice consecutively state equation (10), we will have the following, under the hypothesis of the invariability of the coefficients of state equations for the predictions (from the kth instant onwards):

$$\begin{aligned}\mathbf{y}_m(k+2)&=\bar{\mathbf{C}}_m\mathbf{z}_m(k+2)\\&=\bar{\mathbf{C}}_m\mathbf{z}_m((k+1)+1)\\&=\bar{\mathbf{C}}_m\mathbf{z}_m\left(k'+1\right)\\&=\bar{\mathbf{C}}_m\mathbf{z}_m\left(\bar{\mathbf{A}}_m\mathbf{z}_m\left(k'\right)+\bar{\mathbf{B}}_m\mathbf{u}_a\left(k'\right)+\bar{\mathbf{R}}_m\right)\\&=\bar{\mathbf{C}}_m\mathbf{z}_m\left(\bar{\mathbf{A}}_m\mathbf{z}_m(k+1)+\bar{\mathbf{B}}_m\mathbf{u}_a(k+1)+\bar{\mathbf{R}}_m\right)\\&=\bar{\mathbf{C}}_m\mathbf{z}_m\left(\bar{\mathbf{A}}_m\left(\bar{\mathbf{A}}_m\mathbf{z}_m(k)+\bar{\mathbf{B}}_m\mathbf{u}_a(k)+\bar{\mathbf{R}}_m\right)+\right.\\&\qquad\left.+\bar{\mathbf{B}}_m\mathbf{u}_a(k+1)+\bar{\mathbf{R}}_m\right)\\&=\bar{\mathbf{C}}_m\mathbf{z}_m\left(\bar{\mathbf{A}}_m\mathbf{z}_m(k)+\bar{\mathbf{A}}_m\left(\bar{\mathbf{B}}_m\mathbf{u}_a(k)+\bar{\mathbf{R}}_m\right)+\right.\\&\qquad\left.+\bar{\mathbf{B}}_m\mathbf{u}_a(k+1)+\bar{\mathbf{R}}_m\right).\end{aligned} \quad (A.3)$$

Imposing now as a hypothesis that $u(k)$ is maintained constant during the prediction horizon, that is,

$$u(k)=u(k+1)=u(k+2)=\cdots, \quad (A.4)$$

it can easily be proved that

$$\mathbf{u}_a(k+1)=\mathbf{P}_{1010}\mathbf{u}_a(k), \quad (A.5)$$

where

$$\mathbf{P}_{1010}=\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}, \quad (A.6)$$

and now, making use of equality (A.4), the corresponding development to $\mathbf{y}_m(k+2)$ initiated in (A.3) will finally be

$$\begin{aligned}\mathbf{y}_m(k+2)=\bar{\mathbf{C}}_m&\left(\bar{\mathbf{A}}_m{}^2\mathbf{z}_m(k)+\bar{\mathbf{A}}_m\left(\bar{\mathbf{B}}_m\mathbf{u}_a(k)+\bar{\mathbf{R}}_m\right)\right.\\&\left.+\left(\bar{\mathbf{B}}_m\mathbf{P}_{1010}\right)\mathbf{u}_a(k)+\bar{\mathbf{R}}_m\right).\end{aligned} \quad (A.7)$$

$\underline{H=3}$ $(k+H=k+3)$

By replacing k with $(k+3)$ in state equation (11) and using successively state equation (10) and performing the necessary changes of the variable, we will have the following, also under the hypothesis of the invariability of the coefficients of state equations for the predictions (from the kth instant onwards):

$$\begin{aligned}\mathbf{y}_m(k+3)&=\bar{\mathbf{C}}_m\mathbf{z}_m(k+3)\\&=\bar{\mathbf{C}}_m\left(\bar{\mathbf{A}}_m\left(\bar{\mathbf{A}}_m{}^2\mathbf{z}_m(k)+\bar{\mathbf{A}}_m\left(\bar{\mathbf{B}}_m\mathbf{u}_a(k)+\bar{\mathbf{R}}_m\right)\right.\right.\\&\qquad\left.\left.+\left(\bar{\mathbf{B}}_m\mathbf{P}_{1010}\right)\mathbf{u}_a(k)+\bar{\mathbf{R}}_m\right)+\bar{\mathbf{B}}_m\mathbf{u}_a(k+2)+\bar{\mathbf{R}}_m\right).\end{aligned} \quad (A.8)$$

Considering now again the hypothesis that $u(k)$ is maintained constant during the prediction horizon, expressed in abbreviated form as (A.4) and using $\mathbf{P}_{1010}$ (the matrix specified in (A.6)), it can be proved that

$$\mathbf{u}_a(k+2)=\mathbf{u}_a(k+1)=\mathbf{P}_{1010}\mathbf{u}_a(k), \quad (A.9)$$

and taking into account this equality, the development corresponding to $\mathbf{y}_m(k+3)$ initiated in (A.8) will finally be as follows:

$$\begin{aligned}\mathbf{y}_m(k+3) = \bar{\mathbf{C}}_m \Big(& \bar{\mathbf{A}}_m{}^3 \mathbf{z}_m(k) + \bar{\mathbf{A}}_m{}^2 \left(\bar{\mathbf{B}}_m \mathbf{u}_a(k) + \bar{\mathbf{R}}_m\right) \\ & + \bar{\mathbf{A}}_m \left(\left(\bar{\mathbf{B}}_m \mathbf{P}_{1010}\right)\mathbf{u}_a(k) + \bar{\mathbf{R}}_m\right) \\ & + \left(\left(\bar{\mathbf{B}}_m \mathbf{P}_{1010}\right)\mathbf{u}_a(k) + \bar{\mathbf{R}}_m\right)\Big). \end{aligned} \tag{A.10}$$

<u>$H = 4$</u> $(k+H = k+4)$

Taking into account once again the hypothesis of the constant maintenance of $u(k)$ in the prediction horizon, (A.4) and using again the $\mathbf{P}_{1010}$ matrix, (A.6), it can be proved that

$$\mathbf{u}_a(k+3) = \mathbf{u}_a(k+2) = \mathbf{u}_a(k+1) = \mathbf{P}_{1010}\mathbf{u}_a(k). \tag{A.11}$$

Carrying out the necessary development, totally analogous to that of the previous cases and which we will omit in the interests of simplification, and using equality (A.11), the following final expression would be reached for the output predicted by the model for the instant $(k+4)$:

$$\begin{aligned}\mathbf{y}_m(k+4) = \bar{\mathbf{C}}_m \Big(& \bar{\mathbf{A}}_m{}^4 \mathbf{z}_m(k) + \bar{\mathbf{A}}_m{}^3 \left(\bar{\mathbf{B}}_m \mathbf{u}_a(k) + \bar{\mathbf{R}}_m\right) \\ & + \bar{\mathbf{A}}_m{}^2 \left(\left(\bar{\mathbf{B}}_m \mathbf{P}_{1010}\right)\mathbf{u}_a(k) + \bar{\mathbf{R}}_m\right) \\ & + \bar{\mathbf{A}}_m \left(\left(\bar{\mathbf{B}}_m \mathbf{P}_{1010}\right)\mathbf{u}_a(k) + \bar{\mathbf{R}}_m\right) \\ & + \left(\left(\bar{\mathbf{B}}_m \mathbf{P}_{1010}\right)\mathbf{u}_a(k) + \bar{\mathbf{R}}_m\right)\Big). \end{aligned} \tag{A.12}$$

Induction Hypothesis. Once the development corresponding to $\mathbf{y}_m(k+H)$ has been made for the first four values of H, we will assume that we can generalize the expressions obtained and apply them to the pth case. For the case $H = p$, therefore, the final expression would be

<u>$H = p$</u>; $p \in \mathbb{Z}^+$, $p \geq 4(k+H = k+p)$

$$\begin{aligned}\mathbf{y}_m(k+p) = \bar{\mathbf{C}}_m \Big(& \bar{\mathbf{A}}_m{}^p \mathbf{z}_m(k) + \bar{\mathbf{A}}_m{}^{p-1} \left(\bar{\mathbf{B}}_m \mathbf{u}_a(k) + \bar{\mathbf{R}}_m\right) \\ & + \bar{\mathbf{A}}_m{}^{p-2} \left(\left(\bar{\mathbf{B}}_m \mathbf{P}_{1010}\right)\mathbf{u}_a(k) + \bar{\mathbf{R}}_m\right) + \cdots \\ & + \bar{\mathbf{A}}_m \left(\left(\bar{\mathbf{B}}_m \mathbf{P}_{1010}\right)\mathbf{u}_a(k) + \bar{\mathbf{R}}_m\right) \\ & + \mathbf{I}\left(\left(\bar{\mathbf{B}}_m \mathbf{P}_{1010}\right)\mathbf{u}_a(k) + \bar{\mathbf{R}}_m\right)\Big), \end{aligned} \tag{A.13}$$

where $\mathbf{P}_{1010}$ is the matrix specified in (A.6) and $\mathbf{I}$ is the order 2 identity matrix, that is,

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{A.14}$$

Next, we will reason by means of induction. Assuming that the expression corresponding to case $H = p$ is correct, it will need to be shown that it is also correct for $H = p+1$, that is, that it would also be complied with in it if we replace p with $p+1$. In order to do so, we will use the state equations (in a similar way to what was done in the previous cases and with the same hypotheses) and (A.13), which is corresponding to case $H = p$.

As from state equation (11), making a trivial change of variable, we will have

$$\mathbf{y}_m(k+p) = \bar{\mathbf{C}}_m \mathbf{z}_m(k+p), \tag{A.15}$$

and comparing (A.13) and (A.15) and equalling the right-hand side of both equalities, we will have the following:

$$\begin{aligned}\bar{\mathbf{C}}_m \mathbf{z}_m(k+p) = \bar{\mathbf{C}}_m \Big(& \bar{\mathbf{A}}_m{}^p \mathbf{z}_m(k) + \bar{\mathbf{A}}_m{}^{p-1} \left(\bar{\mathbf{B}}_m \mathbf{u}_a(k) + \bar{\mathbf{R}}_m\right) \\ & + \bar{\mathbf{A}}_m{}^{p-2} \left(\left(\bar{\mathbf{B}}_m \mathbf{P}_{1010}\right)\mathbf{u}_a(k) + \bar{\mathbf{R}}_m\right) + \cdots \\ & + \bar{\mathbf{A}}_m \left(\left(\bar{\mathbf{B}}_m \mathbf{P}_{1010}\right)\mathbf{u}_a(k) + \bar{\mathbf{R}}_m\right) \\ & + \mathbf{I}\left(\left(\bar{\mathbf{B}}_m \mathbf{P}_{1010}\right)\mathbf{u}_a(k) + \bar{\mathbf{R}}_m\right)\Big). \end{aligned} \tag{A.16}$$

Taking into account again the state equations and also the previous result, we can now approach the development corresponding to case $H = p+1$. First, we will use state equation (11) and then we will make the intermediate change of variable $k_p = (k+p)$ and then we will use state equation (10) to develop $\mathbf{z}_m(k_p+1)$

<u>$H = p+1$</u>; $p \in \mathbb{Z}^+$, $p \geq 4(k+H = k+(p+1))$

$$\begin{aligned}\mathbf{y}_m(k+(p+1)) &= \bar{\mathbf{C}}_m \mathbf{z}_m(k+(p+1)) \\ &= \bar{\mathbf{C}}_m \mathbf{z}_m((k+p)+1) \\ &= \bar{\mathbf{C}}_m \mathbf{z}_m\left(k_p+1\right) \\ &= \bar{\mathbf{C}}_m \left(\bar{\mathbf{A}}_m \mathbf{z}_m\left(k_p\right) + \bar{\mathbf{B}}_m \mathbf{u}_a\left(k_p\right) + \bar{\mathbf{R}}_m\right) \\ &= \bar{\mathbf{C}}_m \left(\bar{\mathbf{A}}_m \mathbf{z}_m(k+p) + \bar{\mathbf{B}}_m \mathbf{u}_a(k+p) + \bar{\mathbf{R}}_m\right). \end{aligned} \tag{A.17}$$

On the other hand, considering again the hypothesis that $u(k)$ is maintained constant during the prediction horizon, expressed in (A.4), and with $\mathbf{P}_{1010}$ being the matrix specified in (A.6), the compliance with the following equality sequence can easily be proved:

$$\mathbf{u}_a(k+p) = \mathbf{u}_a(k+p-1) = \cdots = \mathbf{u}_a(k+2) = \mathbf{u}_a(k+1) = \mathbf{P}_{1010}\mathbf{u}_a(k), \tag{A.18}$$

and taking into account (A.16) and (A.18), replacing the terms which correspond with (A.17), and operating in an appropriate manner, we will obtain the development corresponding to $\mathbf{y}_m(k+(p+1))$, initiated in (A.17), as follows:

$$\begin{aligned}\mathbf{y}_m(k+(p+1)) = \bar{\mathbf{C}}_m \Big(&\bar{\mathbf{A}}_m{}^{p+1}\mathbf{z}_m(k) + \bar{\mathbf{A}}_m{}^{p}\left(\bar{\mathbf{B}}_m\mathbf{u}_a(k)+\bar{\mathbf{R}}_m\right) \\ &+ \bar{\mathbf{A}}_m{}^{p-1}\left(\left(\bar{\mathbf{B}}_m\mathbf{P}_{1010}\right)\mathbf{u}_a(k)+\bar{\mathbf{R}}_m\right) + \cdots \\ &+ \bar{\mathbf{A}}_m{}^{2}\left(\left(\bar{\mathbf{B}}_m\mathbf{P}_{1010}\right)\mathbf{u}_a(k)+\bar{\mathbf{R}}_m\right) \\ &+ \bar{\mathbf{A}}_m\left(\left(\bar{\mathbf{B}}_m\mathbf{P}_{1010}\right)\mathbf{u}_a(k)+\bar{\mathbf{R}}_m\right) \\ &+ \mathbf{I}\left(\left(\bar{\mathbf{B}}_m\mathbf{P}_{1010}\right)\mathbf{u}_a(k)+\bar{\mathbf{R}}_m\right)\Big).\end{aligned} \tag{A.19}$$

This last result compared with (A.13) shows that the predictions in case $H=p+1$ satisfy the same *formula* as those of case $H=p$. We can therefore conclude that the validity of (A.13) can be extended to $p\in\mathbb{Z}^+$, $p\geq 1$, and understanding that the factors $\bar{\mathbf{A}}_m{}^{p-n}$ which will appear (with $n\in\mathbb{Z}^+$), one must comply with $(p-n)\geq 1$. This expression may therefore be considered to be the *general formula* of the calculation of predictions at the kth instant, based on our state-space fuzzy model.

And now, once the demonstration has been concluded, we will formalize the mathematical expression of the calculation of predictions in a more general manner, representing the prediction horizon with H. In short, the general expression of $\mathbf{y}_m(k+H)$ obtained will be as follows (A.20):

$$\begin{aligned}\mathbf{y}_m(k+H) = \bar{\mathbf{C}}_m \Big(&\bar{\mathbf{A}}_m{}^{H}\mathbf{z}_m(k) + \bar{\mathbf{A}}_m{}^{H-1}\left(\bar{\mathbf{B}}_m\mathbf{u}_a(k)+\bar{\mathbf{R}}_m\right) + \\ &+ \bar{\mathbf{A}}_m{}^{H-2}\left(\left(\bar{\mathbf{B}}_m\mathbf{P}_{1010}\right)\mathbf{u}_a(k)+\bar{\mathbf{R}}_m\right) + \cdots \\ &+ \bar{\mathbf{A}}_m\left(\left(\bar{\mathbf{B}}_m\mathbf{P}_{1010}\right)\mathbf{u}_a(k)+\bar{\mathbf{R}}_m\right) + \\ &+ \mathbf{I}\left(\left(\bar{\mathbf{B}}_m\mathbf{P}_{1010}\right)\mathbf{u}_a(k)+\bar{\mathbf{R}}_m\right)\Big),\end{aligned}$$

where

$$H\in\mathbb{Z}^+,\quad H\geq 1,\quad \exists\bar{\mathbf{A}}_m{}^{H-n} \text{ if } (H-n)\geq 1 (n\in\mathbb{Z}^+). \tag{A.20}$$

Expression (A.20) may be presented in another way with the objective being the appearance as main factors of $\mathbf{z}_m(k)$, $\mathbf{u}_a(k)$, and $\bar{\mathbf{R}}_m$. Regrouping terms, therefore, we will have

$$\begin{aligned}\mathbf{y}_m(k+H) = \bar{\mathbf{C}}_m \Big(&\bar{\mathbf{A}}_m{}^{H}\mathbf{z}_m(k) + \Big(\bar{\mathbf{A}}_m{}^{H-1}\bar{\mathbf{B}}_m + \Big(\bar{\mathbf{A}}_m{}^{H-2} + \bar{\mathbf{A}}_m{}^{H-3} + \cdots \\ &+ \bar{\mathbf{A}}_m + \mathbf{I}\Big)\bar{\mathbf{B}}_m\mathbf{P}_{1010}\Big)\mathbf{u}_a(k) + \Big(\bar{\mathbf{A}}_m{}^{H-1} + \Big(\bar{\mathbf{A}}_m{}^{H-2} \\ &+ \bar{\mathbf{A}}_m{}^{H-3} + \cdots + \bar{\mathbf{A}}_m + \mathbf{I}\Big)\Big)\bar{\mathbf{R}}_m\Big),\end{aligned} \tag{A.21}$$

and making use of the following development of matrix algebra (for square matrixes)

$$\begin{aligned}&\frac{\left(\bar{\mathbf{A}}_m{}^{p}-\mathbf{I}\right)}{\left(\bar{\mathbf{A}}_m-\boldsymbol{I}\right)} = \bar{\mathbf{A}}_m{}^{p-1} + \bar{\mathbf{A}}_m{}^{p-2} + \cdots + \bar{\mathbf{A}}_m + \mathbf{I}, \\ &p\in\mathbb{Z}^+, p\geq 1, \\ &\exists\bar{\mathbf{A}}_m{}^{p-n}\ \textit{if}\ (p-n)\geq 1, \\ &\bar{\mathbf{A}}_m{}^{0} = \mathbf{I}(\text{with } \mathbf{I} \text{ being the order 2 identity matrix}),\end{aligned} \tag{A.22}$$

or what is the same

$$\left(\bar{\mathbf{A}}_m{}^{p}-\mathbf{I}\right)\left(\bar{\mathbf{A}}_m-\mathbf{I}\right)^{-1} = \bar{\mathbf{A}}_m{}^{p-1} + \bar{\mathbf{A}}_m{}^{p-2} + \cdots + \bar{\mathbf{A}}_m + \mathbf{I}, \tag{A.23}$$

then we will have (considering that $p=H-1$)

$$\bar{\mathbf{A}}_m{}^{H-2} + \bar{\mathbf{A}}_m{}^{H-3} + \cdots + \bar{\mathbf{A}}_m + \mathbf{I} = \left(\bar{\mathbf{A}}_m{}^{H-1}-\mathbf{I}\right)\left(\bar{\mathbf{A}}_m-\mathbf{I}\right)^{-1}, \tag{A.24}$$

and replacing (A.24) in (A.21) in the two terms where it appears to leave finally

$$\begin{aligned}\mathbf{y}_m(k+H) = \bar{\mathbf{C}}_m \Big(&\bar{\mathbf{A}}_m{}^{H}\mathbf{z}_m(k) + \Big(\bar{\mathbf{A}}_m{}^{H-1}\bar{\mathbf{B}}_m + \Big(\bar{\mathbf{A}}_m{}^{H-1}-\mathbf{I}\Big) \\ &\cdot\left(\bar{\mathbf{A}}_m-\mathbf{I}\right)^{-1}\bar{\mathbf{B}}_m\mathbf{P}_{1010}\Big)\mathbf{u}_a(k) + \\ &+ \Big(\bar{\mathbf{A}}_m{}^{H-1} + \Big(\bar{\mathbf{A}}_m{}^{H-1}-\mathbf{I}\Big)\left(\bar{\mathbf{A}}_m-\mathbf{I}\right)^{-1}\Big)\bar{\mathbf{R}}_m\Big),\end{aligned}$$

being that

$$\begin{aligned}&H\in\mathbb{Z}^+, H\geq 1, \exists\bar{\mathbf{A}}_m{}^{H-n} \text{ if } (H-n)\geq 1 (n\in\mathbb{Z}^+), \\ &\mathbf{I} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \text{the order 2 identity matrix and } \mathbf{P}_{1010} = \begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}.\end{aligned} \tag{A.25}$$

The final expression obtained (A.26) is the general analytical expression that must be used for calculating the predictions, that is, for the calculation at instant k of the outputs predicted by the fuzzy model for the instant $(k+H)$, with H being the prediction horizon. It can easily be proved that the *final formula* obtained for $\mathbf{y}_m(k+H)$ is also satisfied for each of the four particular cases developed in the induction process ($H=1,2,3,4$).

B. Relationship between the Parameters of the Reference Model

The reference model of each of the two outputs of our system is given by reference trajectories that should gradually approach the corresponding references (*set points*). The discrete equations of these trajectories, already introduced in (29), are as follows:

$$y_{r_i}(k+1) = a_{r_i} y_{r_i}(k) + b_{r_i} y_{\text{set\_point}_i}(k),\quad i = 1, 2\ (\text{number of outputs}). \tag{B.1}$$

As is logical, for the reference trajectories to follow their corresponding references, the reference model gain must be the unit. The imposition of this condition will cause a certain relationship between the parameters a_{r_i} and b_{r_i}, which is precisely what we wish to determine. In order to do so, firstly, we will calculate the *discrete transfer function inz* of the reference models and then we will impose the unit gain condition.

By applying the *ztransform* to equality (B.1), we will have

$$zY_{r_i}(z) = a_{r_i} Y_{r_i}(z) + b_{r_i} Y_{\text{set\_point}_i}(z),\quad i = 1, 2\ (\text{number of outputs}), \tag{B.2}$$

and grouping terms together and operating properly

$$\begin{aligned} &(z - a_{r_i}) Y_{r_i}(z) = b_{r_i} Y_{\text{set\_point}_i}(z),\\ &G_i(z) = \frac{Y_{r_i}(z)}{Y_{\text{set\_point}_i}(z)} = \frac{b_{r_i}}{(z - a_{r_i})},\\ &i = 1, 2\ (\text{number of outputs}) \end{aligned} \tag{B.3}$$

And now, making use of the expression of the discrete gain that derives from the application of the *final value theorem* and imposing the unit gain condition, we will have

$$\begin{aligned} &\text{Gain}_i = \lim_{z\to 1} G_i(z) = \lim_{z\to 1} \frac{b_{r_i}}{(z - a_{r_i})} = \frac{b_{r_i}}{(1 - a_{r_i})},\\ &\text{Gain}_i = \frac{b_{r_i}}{(1 - a_{r_i})} = 1,\\ &i = 1, 2\ (\text{number of outputs}), \end{aligned} \tag{B.4}$$

and in this way, we will finally obtain the relationship between the parameters of the reference model that we were searching (for each of the two outputs)

$$(1 - a_{r_i})^{-1} b_{r_i} = 1,\quad i = 1, 2\ (\text{number of outputs}) \tag{B.5}$$

C. Reference Trajectory on the Prediction Horizon

The mathematical model of the reference trajectories, introduced in (B.1) and in (29), is a discrete time recursive model that allows the calculation of the value of the output variable (of that model) at the instant $(k+1)$. The objective of this appendix is to show in an abbreviated manner the procedure for obtaining the expression corresponding to the output of the reference model at the instant $(k+H)$.

In order to determine the expression corresponding to $y_{r_i}(k+H)$, for $i = 1, 2$, we will reason by means of induction (omitting the indication of the values of $i (i = 1, 2)$, except at the beginning and at the end so as not to overload the development).

<u>$H = 1$</u> $(k + H = k + 1)$

For this first case, we will use the expression defining the reference trajectories, (B.1) and also (B.5), that is, the relationship between the parameters of the trajectories

$$\begin{aligned} &y_{r_i}(k+1) = a_{r_i} y_{r_i}(k) + b_{r_i} y_{\text{set\_point}_i}(k),\\ &b_{r_i} = (1 - a_{r_i}),\\ &i = 1, 2\ (\text{number of outputs}), \end{aligned} \tag{C.1}$$

and replacing b_{r_i} in the expression of the trajectory

$$y_{r_i}(k+1) = a_{r_i} y_{r_i}(k) + (1 - a_{r_i}) y_{\text{set\_point}_i}(k). \tag{C.2}$$

<u>$H = 2$</u> $(k + H = k + 2)$

We develop $(k+H)$ for $H = 2$, considering $k + 2 = (k + 1) + 1$ and considering the trivial change of variable $k' = k + 1$, which would leave $k + 2 = k' + 1$. Using again (B.1) that define the reference trajectories, but with k' instead of k, and making $b_{r_i} = (1 - a_{r_i})$, we will have

$$y_{r_i}(k+2) = y_{r_i}(k'+1) = a_{r_i} y_{r_i}(k') + (1 - a_{r_i}) y_{\text{set\_point}_i}(k'), \tag{C.3}$$

and reversing the change of variable $(k' = k + 1)$ and again using (B.1) to develop $y_{r_i}(k+1)$, we will have

$$\begin{aligned} y_{r_i}(k+2) &= a_{r_i} y_{r_i}(k+1) + (1 - a_{r_i}) y_{\text{set\_point}_i}(k+1)\\ &= a_{r_i}\left(a_{r_i} y_{r_i}(k) + (1 - a_{r_i}) y_{\text{set}_{\text{point}_i}}(k)\right)\\ &\quad + (1 - a_{r_i}) y_{\text{set}_{\text{point}_i}}(k+1)\\ &= a_{r_i}^2 y_{r_i}(k) + a_{r_i}(1 - a_{r_i}) y_{\text{set}_{\text{point}_i}}(k)\\ &\quad + (1 - a_{r_i}) y_{\text{set}_{\text{point}_i}}(k+1). \end{aligned} \tag{C.4}$$

Now considering the reference on the prediction horizon to be constant

$$y_{\text{set\_point}_i}(k) = y_{\text{set\_point}_i}(k+1) = \cdots = y_{\text{set\_point}_i}(k+H-1), \tag{C.5}$$

the development of $y_{r_i}(k+2)$, initiated in (C.4), will continue as follows:

$$y_{r_i}(k+2) = \cdots = a_{r_i}^2 y_{r_i}(k) + (1+a_{r_i})(1-a_{r_i}) y_{\text{set\_point}_i}(k) = a_{r_i}^2 y_{r_i}(k) + \left(1-a_{r_i}^2\right) y_{\text{set\_point}_i}(k). \quad \text{(C.6)}$$

Induction Hypothesis. The expression corresponding to the following cases would be obtained in a similar way, deducing (with the consideration of the maintaining of the constant reference with constant values on the prediction horizon) the following generic expression, which will be taken as the induction hypothesis:

$\underline{H=p}$; $p \in \mathbb{Z}^+, p \ge 1$

$$y_{r_i}(k+p) = a_{r_i}^p y_{r_i}(k) + \left(1-a_{r_i}^p\right) y_{\text{set\_point}_i}(k). \quad \text{(C.7)}$$

If we suppose the above formula to be correct, it can be proved easily that it is also complied with for $(p+1)$

$$y_{r_i}(k+(p+1)) = a_{r_i}^{(p+1)} y_{r_i}(k) + \left(1-a_{r_i}^{(p+1)}\right) y_{\text{set\_point}_i}(k), \quad \text{(C.8)}$$

and reasoning by induction, the general expression will be the following (considering the reference on the prediction horizon to be constant), for each of the two outputs:

$$y_{r_i}(k+H) = a_{r_i}^H y_{r_i}(k) + \left(1-a_{r_i}^H\right) y_{\text{set\_point}_i}(k) \; H \in \mathbb{Z}^+, \; H \ge 1 \, ; i = 1, 2 \text{ (number of outputs)}, \quad \text{(C.9)}$$

expressions that can be formalized jointly by means of the following (single) matrix expression:

$$\mathbf{y}_r(k+H) = \mathbf{A}_{rH}\mathbf{y}_r(k) + (I - \mathbf{A}_{rH})\mathbf{y}_{\text{set\_point}}(k)$$
$$\mathbf{A}_{rH} = \begin{pmatrix} a_{r_1}^H & 0 \\ 0 & a_{r_2}^H \end{pmatrix}; \; \boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \text{(C.10)}$$
$$H \in \mathbb{Z}^+, H \ge 1,$$

with $\mathbf{y}_r(k+H)$, $\mathbf{y}_r(k)$ e $\mathbf{y}_{\text{set\_point}}(k)$ being size-two column-vectors that group together the components corresponding to both outputs.

D. Mathematical Nonlinear Model of the Wastewater Treatment Process with Activated Sludge

The mathematical model of the wastewater treatment process taken as reference in this article is founded on the classical *Monod and Maynard-Smith model* (with the assumption of a perfectly mixed tank reactor) and it is obtained taking into account the corresponding mass balances of the substrate, biomass, and oxygen. The equations of the model, as well as its variables and parameters, are the following:

(i) Aerated biological reactor

$$\frac{dx}{dt} = \mu_{\max} y \frac{sx}{(K_s+s)} - K_d \frac{x^2}{s} - K_c x + \frac{q}{V}(x_{ir} - x)$$
$$\frac{ds}{dt} = -\mu_{\max} \frac{sx}{(K_s+s)} + f_{kd} K_d \frac{x^2}{s} + f_{kd} K_c x + \frac{q}{V}(s_{ir} - s)$$
$$\frac{dc}{dt} = K_{la} f_k (c_s - c) - K_{01}\mu_{\max} \frac{x^2}{(K_s+s)} - \frac{q}{V} c$$
$$x_{ir} = \frac{x_i q_i + x_r q_r}{q}$$
$$s_{ir} = \frac{s_i q_i + s q_r}{q} \quad \text{(D.1)}$$

(ii) Secondary settler (three layers with increasing concentrations of biomass)

$$A l_d \frac{dx_d}{dt} = q_{\text{out}} x_b - q_{\text{out}} x_d - A v_s(x_d)$$
$$A l_b \frac{dx_b}{dt} = qx - q_{\text{out}} x_b - q_2 x_b + A v_s(x_d) - A v_s(x_b)$$
$$A l_r \frac{dx_r}{dt} = q_2 x_b - q_2 x_r + A v_s(x_b)$$
$$v_s(x_d) = (nnr) x_d \exp((aar) x_d)$$
$$v_s(x_b) = (nnr) x_b \exp((aar) x_b) \quad \text{(D.2)}$$

(iii) Equations of equilibrium of the flow rates

$$q = q_i + q_r$$
$$q_{\text{out}} = q_i - q_p \quad \text{(D.3)}$$
$$q_2 = q_r + q_p$$

(iv) Variables and parameters (with the areas expressed in m^2, the volumes in m^3, the concentrations in mg/l, and the flow rates in m^3/h):

A:	Settler area
V:	Reactor volume

s_i: Input substrate concentration
x_i: Input biomass concentration
x: Biomass concentration in the reactor
s: Substrate concentration in the reactor
c: Oxygen concentration in the reactor
x_d: Biomass concentration in the top layer of the settler
x_b: Biomass concentration in the middle layer of the settler
x_r: Biomass concentration in the lower layer of the settler
l_d: Height of the top layer of the settler
l_b: Height of the middle layer of the settler
l_r: Height of the lower layer of the settler
q_i: Input flow rate (contaminated water/influent)
q: Reactor input-output flow rate
q_2: Total sludge recirculation flow rate
q_p: Purge flow rate (excess sludge)
q_r: Sludge recirculation flow rate (to the reactor)
q_{out}: Output flow rate (purified water/effluent)
x_{ir}: Biomass concentration in the input of the reactor
s_{ir}: Substrate concentration in the input of the reactor
$v_s(x_d)$: Sedimentation rate in the settler of the top layer with respect to the middle layer
$v_s(x_b)$: Sedimentation rate in the settler of the middle layer with respect to the lower layer
(nnr), (aar): Empirical coefficients for the calculation of the sedimentation rates
$\mu_{\max}$: Maximum specific rate of growth
y: Fraction of metabolized substrate that is converted into biomass
K_s: Saturation constant
K_d: Endogenous decomposition coefficient (mortality constant)
K_c: Cellular activity coefficient of the microorganisms
f_{kd}: Fraction of dead biomass that becomes a substrate
K_{la}: Oxygen mass overall transfer coefficient
f_k: Aeration factor
c_s: Dissolved oxygen saturation concentration
K_{01}: Equivalence coefficient between cell growth and oxygen consumption rate.

D.1. Parameters of the Manresa WWTP. The numerical values of the Manresa wastewater treatment plant (taken as reference) [35] are specified in Table 12.

Table 12: Parameters of the Manresa WWTP.

Plant dimensions and heights of the settler layers	*Value*
V (m^3)	7268
A (m^2)	2770.9
l_d (m)	2
l_b (m)	1.5
lr (m)	1
Concentrations and flow rates in the influent	*Value*
q_i (m^3/h)	1300
s_i (mg/l)	366.67
x_i (mg/l)	80
Kinetic and stoichiometric parameters	*Value*
$\mu_{\max}$ (h^{-1})	0.1824
y	0.5948
K_s (mg/l)	300
K_d (h^{-1})	$5e-5$
K_c (h^{-1})	$1.3333e-4$
f_{kd}	0.2
nnr	3.1563
aar	−0.00078567
K_{01}	$1e-4$
K_{la} (h^{-1})	0.7
c_s (mg/l)	8

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Capítulo 3

Artículo nº2: Análisis de estabilidad

3.1. Título original del artículo

Practical Computational Approach for the Stability Analysis of Fuzzy Model-Based Predictive Control of Substrate and Biomass in Activated Sludge Processes.

3.2. Resumen en castellano

Título del artículo en castellano:

Enfoque computacional práctico del análisis de estabilidad del control predictivo basado en modelos difusos de sustrato y biomasa en procesos de lodos activados.

Este trabajo presenta un procedimiento para abordar el análisis de estabilidad en lazo cerrado de una determinada variante de la estrategia denominada genéricamente Control Predictivo Basado en Modelos Difusos (FMBPC) (siendo el modelo difuso un modelo del tipo Takagi-Sugeno), aplicada al proceso de tratamiento de aguas residuales conocido como Proceso de Lodos Activados (ASP), con el objetivo de controlar simultáneamente la concentración de sustrato en el efluente (una de las principales variables que deben limitarse según las legislaciones ambientales) y la concentración de biomasa en el reactor. Este caso de estudio fue elegido tanto por su relevancia ambiental, como por las características especiales del proceso en sí mismo, que son de gran interés en el campo del control no lineal, como la fuerte no linealidad que presenta, su carácter multivariable y su dinámica compleja, consecuencia de su naturaleza biológica.

El análisis de estabilidad, tanto de los sistemas difusos (FS), como de las muy diversas estrategias existentes de control predictivo no lineal (NLMPC), es en general una tarea matemáticamente laboriosa y difícil de generalizar, especialmente para procesos con dinámicas complejas. Para intentar minimizar estas dificultades, en este artículo se ha puesto el foco en la simplificación matemática del problema, tanto en lo relativo a la expresión del modelo matemático del proceso, como en lo que se refiere a los procedimientos matemáticos del análisis de estabilidad.

En relación con el modelo matemático, se utilizó como modelo base de predicciones un modelo en el espacio de estados, discreto, lineal y variante en el tiempo (DLTV), equivalente al modelo difuso de partida (previamente identificado). Además, en un paso posterior, el modelo DLTV se aproximó a un modelo local (incremental), discreto, lineal e invariante en el tiempo (DLTI) (válido para un entorno local de un punto de operación previamente elegido).

En cuanto al análisis de estabilidad en sí mismo se refiere, se desarrolló un método computacional, basado en cálculo simbólico, que simplifica enormemente esta difícil tarea, en comparación con otros métodos existentes en la literatura. El uso del método propuesto proporciona conclusiones útiles (condiciones suficientes) relativas a la estabilidad en bucle cerrado de la estrategia FMBPC considerada, relacionando la estabilidad (local) en lazo cerrado con la estabilidad (local) en lazo abierto. El método fue validado aplicándolo al caso de estudio considerado (proceso ASP) y se realizaron test numéricos de comprobación.

Finalmente, se plantea la posibilidad de que el método pueda ser útil de manera más general, para otras estrategias similares, difusas y predictivas, y para otros procesos con dinámica compleja, diferentes al considerado.

3.3. Artículo nº 2: copia completa de la publicación

processes

Article

Practical Computational Approach for the Stability Analysis of Fuzzy Model-Based Predictive Control of Substrate and Biomass in Activated Sludge Processes

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Abstract: This paper presents a procedure for the closed-loop stability analysis of a certain variant of the strategy called Fuzzy Model-Based Predictive Control (FMBPC), with a model of the Takagi-Sugeno type, applied to the wastewater treatment process known as the Activated Sludge Process (ASP), with the aim of simultaneously controlling the substrate concentration in the effluent (one of the main variables that should be limited according to environmental legislations) and the biomass concentration in the reactor. This case study was chosen both for its environmental relevance and for special process characteristics that are of great interest in the field of nonlinear control, such as strong nonlinearity, multivariable nature, and its complex dynamics, a consequence of its biological nature. The stability analysis, both of fuzzy systems (FS) and the very diverse existing strategies of nonlinear predictive control (NLMPC), is in general a mathematically laborious task and difficult to generalize, especially for processes with complex dynamics. To try to minimize these difficulties, in this article, the focus was placed on the mathematical simplification of the problem, both with regard to the mathematical model of the process and the stability analysis procedures. Regarding the mathematical model, a state-space model of discrete linear time-varying (DLTV), equivalent to the starting fuzzy model (previously identified), was chosen as the base model. Furthermore, in a later step, the DLTV model was approximated to a local model of type discrete linear time-invariant (DLTI). As regards the stability analysis itself, a computational method was developed that greatly simplified this difficult task (in a local environment of an operating point), compared to other existing methods in the literature. The use of the proposed method provides useful conclusions for the closed-loop stability analysis of the considered FMBPC strategy, applied to an ASP process; at the same time, the possibility that the method may be useful in a more general way, for similar fuzzy and predictive strategies, and for other complex processes, was observed.

Keywords: stability analysis; fuzzy model-based predictive control; Takagi-Sugeno; multivariable control; wastewater treatment plant; activated sludge process; computational approach

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1. Introduction

In general, the control of non-linear industrial processes is not an easy task and it is further complicated in the case of processes of a biological nature, especially if the process is affected by major disturbances. One of the possible alternatives to approach the control of this type of systems could be the well-known and effective advanced control strategy called Model-Based Predictive Control (MBPC or MPC) [1–4]. However, this strategy is not associated with a single control algorithm, but rather it is a methodology or procedure to search for optimal control actions based on the predictions of the behavior of the process to be controlled, provided by some mathematical model thereof (in classic MPC, the search for the optimal control actions is carried out imposing the minimization of a certain cost function). Consequently, there are many types of predictive control, depending on the type of model chosen, the method of identifying the model, the procedure or algorithm to determine the control actions, and the control law itself that is finally adopted, among other aspects. Regarding the nonlinear nature of the processes to be controlled, for systems whose nonlinearity is not very high, if there is or is possible to obtain a model or several linearized sub models of the process that are sufficiently valid under certain hypotheses and restrictions, then the well-known and proven techniques of the so-called linear MPC (highly consolidated) could be enough. However, if the system is highly nonlinear, it is necessary to consider a predictive control strategy based on nonlinear models that more accurately represent the behavior of the process. This approach is framed (along with other variants) within the so-called nonlinear model predictive control (nonlinear MPC, or NLMPC) [5–9]. For highly nonlinear, complex, or unknown systems, one class of models that seems quite adequate for predictive control purposes are fuzzy models (FM), especially the Takagi-Sugeno (TS) type fuzzy models [10], which are very useful because the outputs of the sub models are given in the form of affine linear combinations. In addition, there are consolidated procedures for obtaining TS fuzzy models from series of numerical input-output data (collected by simulation or from real experiments). The fuzzy models obtained using data-based identification techniques have the enormous advantage of being able to directly capture the dynamics of such systems. In addition, this type of modeling is also able to deal with multivariable systems in a straightforward manner. In other words, within the framework of predictive control, the strategy known as Fuzzy Model-Based Predictive Control (FMBPC) and especially the variant that uses TS fuzzy models [11–17] can be a good alternative, in general, to approach the control of highly nonlinear and complex systems, among them, the systems of a biological nature; in particular, the so-called activated sludge process (ASP), which is a very common purification procedure in Wastewater Treatment Plants (WWTP) [18–20].

In this article, a certain FMBPC multivariable control strategy is analyzed [21–23], which is framed at the same time in the so-called Predictive Functional Control (PFC) [1,24,25]. In such a strategy, the considered prediction original model is a TS fuzzy model, obtained by identification from input-output data, later represented in the state-space by means of an equivalent model of linear time-varying (LTV) type. As for the control action, it is calculated (at each sampling instant) by means of an analytical and explicit expression (with time-dependent coefficients), deduced using the state-space LTV model and applying the so-called *equivalence principle* (used in PFC and related to the concepts of *reference trajectory* and *coincidence point*), rather than by minimizing a cost function. This control algorithm is applied to an ASP biological process, taken as a case study, with the aim of simultaneously controlling the substrate concentration in the effluent and the biomass concentration in the reactor, with the activated sludge recirculation flow-rate being the only control variable used. The main objective of this paper is to study some issues of the closed-loop stability analysis of the specific FMBPC strategy that has just been described, by a practical computational approach.

The stability analysis of nonlinear control systems is much more complex, in general, than the linear systems stability analysis. Moreover, there are many different approaches, depending on both the type of control law and the type of system to be controlled. The objective of this work is not to analyze all these possibilities, but to focus on the field of fuzzy predictive control, especially in control strategy and in the case study presented in [23]. However, it is important to mention some of the works that have addressed the stability of nonlinear systems—it was highlighted in works [26–32], among many others. In the case of the FMBPC strategy, the stability analysis may be even more complicated than other types of nonlinear control strategies, due, on the one hand to the particular characteristics of the nonlinear predictive control and on the other hand, to the structure of the Fuzzy Systems (FS) [11,33], initially in the form of rules, which are intrinsically very nonlinear, in general, and that logically does not facilitate the use of already consolidated classical procedures. The stability analysis of the FMBPC strategy is, moreover, more uncertain and diverse than other strategies, as a consequence of the great variety of types of fuzzy models and possible parameterizations or choices (antecedent vector and consequent vector, set of fuzzy values for each variable, form and parameters of the membership functions, number of rules, and others). Potentially, many alternatives could be explored to analyze the stability of this type of strategy, but there are two fields in which it seems more logical to focus on initially, and explore the lines of work on this topic: the field of NLMPC control systems and the field of FS systems. Each of the two fields has its particularities, but a large part of the stability analysis approaches developed in them share a theoretical framework: the so-called Lyapunov's stability theory [34,35], which could be considered as the main theoretical reference on stability. One of the most useful mathematical procedures provided by this theory is the so-called *direct method*, consisting of the search for certain characteristic functions, called *Lyapunov Functions*, which provide sufficient conditions for the internal stability, simple or asymptotic, of a system.

Regarding the first of the considered alternatives—the NLMPC control systems—it is known that stability has been an essential and recurring matter in the evolution of the MPC. From its origin in the late 1970s, one of the main problems or disadvantages of the MPC was that stability could not be guaranteed (especially in the presence of constraints) and solutions were not introduced until the end of the 1990s and early year 2000, when the conditions to guarantee stability were clearly established [36,37]. Throughout this period of time, numerous works were carried out on the stability of predictive control algorithms, with the greatest efforts being directed to the search and proposal of procedures that could guarantee closed-loop stability (predictive controllers with guaranteed stability). The use of restrictive terminal regions for states [38] and the consideration of infinite prediction horizons and inclusion of terminal penalty terms in the cost function [39–42] constitute some of the most important contributions and have been, at the same time, a reference for further lines of research in predictive controllers analysis and design, from then until now. In [3], a precise list of works on predictive controllers with guaranteed stability, carried out until 2002, is shown, including methods such as: MPC with terminal restriction of equality, MPC with terminal cost, MPC with terminal restriction of inequality, and MPC with terminal cost and terminal restriction. In addition, the author's proposals on stability and robustness of predictive controllers of nonlinear systems subject to constraints (with or without uncertainties) are also presented, based on the combination of Lyapunov's theory and invariant set theory. In [43], the advances in stability analysis of predictive control are also reviewed, within the framework of a general review or state of the art of predictive control. In [44], a brief but precise review of some methods on stability in predictive control existing at that time is included, specifically focused on the formulation of the MPC with guaranteed stability. In [45], a methodical and complete study of nonlinear model predictive control (theory and algorithms) is carried out, treating the stability aspects with an amplitude and intensity consistent with their importance. In addition, [9] reviews the state of the art of predictive control today, with special emphasis on the main objective of research works on predictive control, which is (as in the early 2000s) to ensure stability and robustness of the closed-loop system in a compatible way with the optimization approach. In all the referenced works, the use of Lyapunov's theory (direct method) tries to demonstrate that the cost function (the minimization of which allows us to calculate the optimal control signal) is a Lyapunov function. However, all the proposed closed-loop stabilization methods are not directly applicable to our FMBPC control strategy, since in it the control variable is not determined from a cost function. It is, therefore, necessary to analyze other alternatives.

Regarding the second of the alternatives considered—FS systems—it must be noted that the stability analysis of control systems in such a frame or field is not a trivial task, due, on the one hand, to the intrinsic nonlinear structure of this type of systems and, on the other, to the great diversity of modalities of use of fuzzy logic for control purposes. In relation to the latter and among other possible classifications, we could differentiate between controllers based on fuzzy plant models and controllers based on fuzzy rules (the plant models being fuzzy or not). Our strategy, FMBPC, fits the first type, but even if we focus only on that category, there can be, in principle, many possible ways to approach stability analysis, since there is no single method of stability analysis for FS systems. Many lines of research have been developed and numerous works with different approaches have been presented. Many of the works carried out have consisted of deducing sufficient (but not necessary) stability conditions. Furthermore, due to the great difficulty of generalization, many of the results have been developed, or tested, only for specific case studies. In short, there is a great diversity of proposals. Nevertheless, a large part of them share the theoretical framework of reference, which is, of course, Lyapunov's stability theory, widely used also by many other control strategies, as we have already commented previously. In [46], an exhaustive review of the state of the art of FS systems stability analysis is carried out (until 2006), highlighting its difficulty due to the non-linearity characteristic of this type of systems, and focusing the study on global and asymptotic stability. Different stability criteria and various techniques based on Lyapunov's stability theory are reviewed (considering quadratic, but also fuzzy Lyapunov functions) and a specific review of FS systems' stability using TS fuzzy models is made. Likewise, the approach of Linear Matrix Inequalities (LMIs) [47,48] is analyzed. In [49], there is also a review of several works and results related to stability analysis and control algorithms design for nonlinear systems represented by TS fuzzy models. In such works, the general procedure used to address stability and derive stabilization results is the Lyapunov's direct method. The stability conditions and the control design problem are expressed by means of constraints in the form of LMIs. Additionally, regarding the stability analysis, most of the methods studied in the mentioned review use a quadratic Lyapunov function (quadratic stability) and obtain, as a result of the demonstrations, only sufficient stability conditions (stability does not necessarily imply quadratic stability). With regard to stabilization methods, the review focuses mainly on those based on state feedback, although other alternative methods, such as those using output feedback, are also discussed. Naturally, in addition to these two great references that have just been described (where several criteria, techniques and approaches are reviewed and numerous authors are cited), there are many other works on the stability of FS systems. Thus, in the following list of works (not exhaustive, but extensive), we can see other contributions and approaches, some of them very significant: [11,50–82]. In addition, some more specific references, relative to control systems that use fuzzy models (among them, FMBPC control systems) and that include (to a greater or lesser degree) stability analysis, or stability-based design, may be, among others, the following: [11–13,22,83–93].

Our stability analysis could follow, in principle, any of the multiple lines of work or approaches mentioned in the two fields analyzed. However, regardless of the approach, FMBPC control systems' stability analysis is not an easy task. For this reason, our choice consisted in looking for a practical computational approach, which is less complex, developed for our specific case study, but with theoretical possibilities of generalization. Our approach consists, basically, of starting from the identified model of the system, a TS fuzzy discrete model, expressing it in the state space in discrete linear time-varying (DLTV) form, and subsequently obtaining an approximate local model of type discrete linear time-invariant (DLTI) for states located in the vicinity of a certain operating point, and applying the first Lyapunov's stability method or theorem, demonstrating its (local) fulfillment computationally. As has been already previously commented, the stability analysis of control systems based on fuzzy models, starting from the original structure of these in the form of rules, is very complicated. However, the use of equivalent LTV state-space models could dramatically simplify the stability analysis task. The stability theory for state-space linear systems is well established and there are numerous works, both for continuous-time and for discrete-time. In [94–98] can be found the basic theoretical foundations, and in [99,100], two specific works on stability analysis of LTV systems of continuous-time and discrete-time, respectively, can be seen. The work presented in [101] can also be consulted, where a theoretical review is made regarding the so-called convex systems based on Takagi-Sugeno, linear parameter varying (LPV), or quasi-LPV models. LPV systems can be considered a generalization of certain LTV systems and therefore, the stability analysis carried out in the said paper could be useful in some way. In any case, the work developed in [100] cited above would be more suitable for our modeling approach and could be a good reference to study the stability corresponding to our case study, after having converted the TS fuzzy model of the process to an equivalent DLTV model. However, and although some necessary and sufficient stability conditions are derived in that specific work, most of the conclusions of this type of study refer only to sufficient stability conditions. Furthermore, the mathematical complexity of the proposed methods in this type of work does not facilitate their use, in a simple and methodical way, in determining the stability of other different case studies. This justifies our attempt to solve the problem by making use of an equivalent local DLTI model (valid for states close to a certain steady state) and using a computational method to analyze stability.

In Section 2, the chosen case study, the ASP wastewater treatment biological process, is presented. Furthermore, the plant identification procedure, which produces a discrete TS fuzzy model, as a result, is revised and this model is converted into another equivalent in the state space, of DLTV type. In Section 3, a local state-space DLTI model is deduced and using such a model, open-loop stability is analyzed using the appropriate criteria for DLTI systems. Section 4 is dedicated to describing the control law and the closed-loop stability criteria are derived. In Section 5, using a symbolic computational approach and an induction process, the closed-loop stability conditions are obtained as a function of the controller parameters and open-loop dynamics of the selected case study. In Section 6, several numerical results are presented: first a stability test and then a series of FMBPC control experiments performed by simulation. Finally, in Section 7, the conclusions are presented, together with some future research options. Complementarily, Appendix A contains detailed numerical information on the membership functions associated with the identification of the fuzzy models of the ASP process and Appendix B includes a complementary mathematical development of Section 4.

2. The Activated Sludge Process: Takagi-Sugeno Fuzzy Modeling and Representation in State Space

In this article, we chose as case study the ASP biological purification procedure, commonly used in wastewater treatment plants. This process has a complex dynamic and is highly nonlinear due to its biological nature. The identification of a fuzzy model of the process from input-output data is a good choice to represent the behavior of this type of processes (as already noted in the Introduction). Initially, the resulting models of the identification procedure are Takagi-Sugeno type fuzzy models, but through the appropriate formalization they will be expressed finally as equivalent state space models, in the form of DLTV models. This section describes the process chosen as a case study, as well as the procedure for the fuzzy identification of the said process and the representation of the resulting TS model in state space, in the form of a DLTV model.

2.1. The Activated Sludge Process: Description and Classical Mathematical Model

In our case study, the basic structure for the WWTP (an aerated biological reactor followed by a secondary settler) was considered. The corresponding purification process consists, essentially, of the elimination of organic contaminants (substrate) through a culture of bacteria (biomass) that will feed precisely on that substrate. This biological process takes place in the aerated and stirred reactor and the treated water is sent to a secondary decanter in which the clean water separates from the resulting sludge. This sludge is sedimented at the bottom of the decanter and sent back to the reactor (sludge recirculation process). The complete cycle is called the activated sludge process. The mass balance of the reactions that take place in this process can be described using the classical Monod model. Based on this model, together with the hypothesis of a perfectly mixed reactor, in [23], a certain mathematical model is specified to represent the activated sludge process. The adopted model, which can be considered as a simplification of the well-known Activated Sludge Model No. 1 (ASM1) [102], consists of a set of differential and algebraic equations that relate the temporal evolution of the masses of the substrate, biomass, and oxygen. From the analysis of the mathematical model, either the complete or the simplified model, it can be deduced that it is highly non-linear, and this is precisely one of the main reasons why it is interesting to adopt it as a case study, in the context of a control problem. Another important reason why it is an attractive system as a case study is that it is a difficult system to control (due to its biological nature). In addition, it is multivariable.

Our case study has focused on a sub model of the adopted model (mentioned above), consisting of the equations that describe the interactions of five variables: three inputs (one manipulable variable and two disturbances) and two outputs (to be controlled both, simultaneously). Figure 1 shows schematically the different inputs and outputs of the biological process taken into account, as well as its characterization. The variables to be controlled simultaneously are two: the substrate concentration in the effluent, s, and the biomass concentration in the reactor, x. The inputs variables are three: one manipulated variable, the sludge recirculation flow-rate, q_r, and two disturbances, the wastewater flowrate in the input of the plant, q_i, and the substrate concentration in the input of the plant, s_i. This set of inputs and outputs, and its role, is more clearly detailed in Table 1.

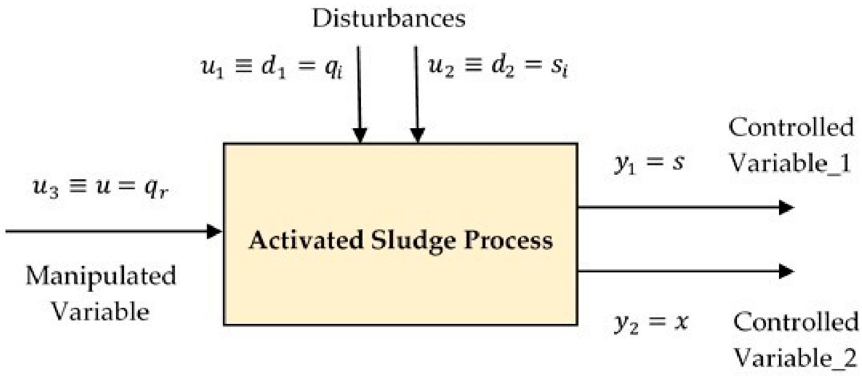


Figure 1. The activated sludge process (ASP) and the inputs and outputs taken into account (multi-variable nonlinear system).

Table 1. Inputs and outputs of the activated sludge process (ASP) and its role.

Inputs			Outputs	
Disturbances		Manipulated Variable	Controlled Variable_1	Controlled Variable_2
$u_1 \equiv d_1$	$u_2 \equiv d_2$	$u_3 \equiv u$	y_1	y_2
Input flow-rate (influent) q_i	Substrate concentration in the influent s_i	Sludge recirculation flow-rate q_r	Substrate concentration in the effluent s	Biomass concentration in the reactor x

In our work, the municipal WWTP of Manresa (Barcelona, Spain), operating since 1985 (with an evolution of the population of the urban area of the city from about 67,000 people in 1985 to about 76,000 people today), was considered as a reference plant. The dynamics of the ASP process of this treatment plant, in its initial period of operation, is acceptably well represented by the model and sub-model considered above and, therefore, it is coherent to make use of real data from such a plant for our studies. Thus, various series of input and output data recorded in such a period, in different industrial campaigns of this WWTP [103], were used (or taken into account for reference) to guide the fuzzy identification procedures of the simulated ASP process and also to adjust, in the control experiments implemented by means of simulation, the ranges of values of the process inputs. Both in the identification procedures and in the control experiments, a part of the numerical values used for the inputs correspond, or are related, to the data collected in the campaigns of the reference WWTP, as follows: for the disturbances, typical values (real or weighted), and for the manipulated variable, similar values to the real values at the different operating points.

2.2. Fuzzy-TS Identification of the ASP Process

The identification of the ASP process was carried out from input-output numerical datasets, with the inputs properly selected, taking as reference data recorded in industrial campaigns of the WWTP of Manresa. The outputs were obtained in simulation, using the mathematical model chosen for the treatment plant (specified in the previous subsection). The result of the identification carried out is a multivariable TS fuzzy model.

The tasks and procedures of the fuzzy identification were performed using the software tool known as FMID (Fuzzy Model Identification Toolbox) [104], complemented with some adaptations made within the framework of this work, programmed with Matlab® & Simulink®. The FMID tool was developed to support the fuzzy modeling and identification techniques described in [11]. The fuzzy identification functions of this tool allow the implementation of clustering techniques based on the Gustafson-Kessel algorithm. The numerical data of input-output provided to the FMID tool for the identification and validation of the process were composed as follows: for the input data, as we said in the previous subsection, typical values or similar to the real values of the WWTP of Manresa; as for the output data, they were obtained by simulation in open loop, with the mentioned inputs and with the plant represented by its continuous model in the form of differential equations. In the numerous identification tests carried out, the various available data series were generally divided into two subsets: identification data and validation data (although, for comparison purposes, in part of the tests, the same data set was used for both identification and validation). Our case study is a multivariable system (MIMO system), with three inputs and two outputs. This multivariable architecture can be seen schematically in Figure 2:

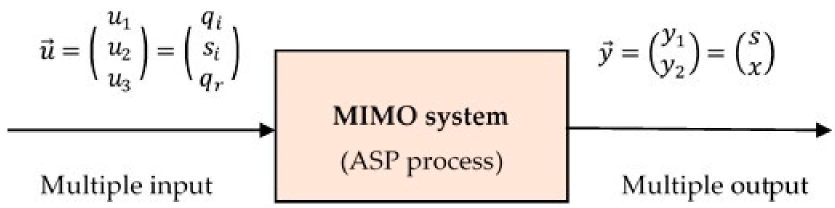


Figure 2. The multivariable architecture of the ASP process (3 inputs and 2 outputs).

Consistent with such architecture, the input-output datasets required for the FMID tool must be arranged as arrays with as many rows as samples and with five columns: three columns for the inputs and two columns for the outputs.

The identification tool provides TS fuzzy models (as well as mechanisms for their validation), expressed in the form of logical rules of the *if-then* type, that is: *if (antecedent) then (consequent)*, where the *consequent* is a linear combination of the components of the *consequent vector*, plus a constant term (*affine* function). In Equation (1), we show the structure of a generic rule, $\boldsymbol{R_j}$ (*j*-th rule), corresponding to a certain output, $y(k)$, of the identified process:

$$\begin{array}{c} \boldsymbol{R_j}: \quad \boldsymbol{if}\ \left(x_{a1}\ is\ A_{j1}\ and\ x_{a2}\ is\ A_{j2}\ and \cdots and\ x_{ap}\ is\ A_{jp}\right) \\ \boldsymbol{then} \\ y(k) = \phi_j(\boldsymbol{x}) = \alpha_{j1}x_1 + \alpha_{j2}x_2 + \cdots + \alpha_{jq}x_q + \delta_j \end{array} \tag{1}$$

where:

- $\boldsymbol{x_a}(k) = \left(x_{a1}, x_{a2}, \cdots, x_{ap}\right)$ is the antecedent vector
- $\boldsymbol{x}\ (k) = \left(x_1, x_2, \cdots, x_q\right)$ is the consequent vector
- A_{jp} is the fuzzy set (or *fuzzy value*) associated with the *p*-th component of the *antecedent vector* (for *j*-th rule); this fuzzy set will associate a certain membership function $\mu_{A_{jp}}(x) : \mathbb{R} \to [0,\ 1]$, which, applied to any value of the variable x_{ap}, will determine its membership grade with respect to the fuzzy set A_{jp}, that is: $\mu_{A_{jp}}\left(x_{ap}\right)$

(for each component of the antecedent vector, there will be as many membership functions as rules are)

- k is the k-th instant (discrete-time system)
- $\phi_j(\boldsymbol{x})$ is an affine function of $\boldsymbol{x}$
- α_{jq} is the coefficient, in the affine function, of the q-th component of the *consequent vector* (for j-th rule)
- δ_j is the independent term or offset (in the affine function)

Each rule of the TS fuzzy model is equivalent to a linear sub-model associated with a certain region of the input space of the model, in a greater degree than the sub models of the other rules. The output of each sub model is calculated by the corresponding affine function. In addition, the global output of the TS fuzzy model will be calculated by combining or aggregating the outputs of all the rules in a weighted way, according to the degree to which the antecedent vector fulfills or satisfies the premise or antecedent of each of the rules. This calculation will be called *global weighted output* and will be represented by $\widetilde{y}(k)$. In the fuzzy logic literature, there are several methods for combining or aggregating rules; among them is the well-known *centroid method*. Applying such a method to the fuzzy model represented in (1), for a general case with multiple outputs, the *weighted global output* can be expressed as shown in (2) (see [23]), where $\widetilde{y}_i(k)$ $(i = 1,\ 2,\ \ldots,\ l)$ is the expression corresponding to the global i-th output (having omitted, for the antecedent and consequent vectors, $\boldsymbol{x_a}$ and $\boldsymbol{x}$, respectively, and also for their respective components, the i index, i.e., indication of the dependence with respect to the i-th output, except in the dimension of the vectors (p_i and q_i, respectively), in order to simplify the notation; for the same reason, the express indication of the dependency of the two vectors with respect to k, has also been discarded):

$$\begin{gathered}\widetilde{y}_i(k) = \textstyle\sum_{j=1}^{mr_i} \beta_{ij}(\boldsymbol{x_a})\phi_{ij}(\boldsymbol{x}) \\ with: \\ \beta_{ij}(\boldsymbol{x_a}) = \frac{\mu_{A_{j1}}(x_{a1})\ \mu_{A_{j2}}(x_{a2})\ldots\ \mu_{A_{jp_i}}(x_{ap_i})}{\sum_{j=1}^{mr_i} \mu_{A_{j1}}(x_{a1})\ \mu_{A_{j2}}(x_{a2})\ldots\ \mu_{A_{jp_i}}(x_{ap_i})}\ ;\ \textstyle\sum_{j=1}^{mr_i} \beta_{ij}(\boldsymbol{x_a}) = 1 \\ \phi_{ij}(\boldsymbol{x}) = \alpha_{j1}x_1 + \alpha_{j2}x_2 + \cdots + \alpha_{jq}x_{q_i} + \delta_{ij}\end{gathered} \tag{2}$$

where:

- $\widetilde{y}_i(k)$ is the i-th weighted global output
- k *is* the k-th instant (discrete-time system)
- $i = 1,\ 2, \ldots,\ p$ (number of outputs)
- $j = 1,\ 2,\ \ldots,\ mr_i$ (number of rules of fuzzy model for output y_i)
- $\beta_{ij}(\boldsymbol{x_a})$ are the normalized membership functions of the *antecedent vector*, $\boldsymbol{x_a}$ (for j-th rule of i-th output)
- $\phi_{ij}(\boldsymbol{x})$ is the affine function of the *consequent vector*, $\boldsymbol{x}$ (for j-th rule of i-th output)

In the specific case of our ASP process, the output variables considered are two: $y_1(k)$, the substrate in the effluent, and $y_2(k)$, the biomass in the reactor. Therefore, each of the fuzzy identification tests carried out led to a fuzzy model constituted by two TS fuzzy sub-models (one TS model for each output). In the present work, the results obtained in two of the many identification tests carried out (see [23]) were selected, denoting the two obtained fuzzy models as FM_1 and FM_2, respectively. The criteria for their selection were two: on the one hand, the dynamic structure of the models and on the other, the identification validation index known as *VAF* (percentile *Variance Accounted For* between two signals). Thus, two different dynamic structures were chosen (see Dynamical Parameters in Table 2) to test the possible influence of this structural aspect on the performance of the control strategy, and from among the identified models, two models were chosen, one for each structure, with sufficiently high *VAF* indices. Specifying more, each of these fuzzy models is characterized by the choice (in the design phase) of different values of several identification parameters, which are shown below, in Table 2. Among them, N_y, N_u and N_d (Dynamical Parameters of the recursive discrete model) have special relevance in the identification process and its results, since they contain implicit information related to the dependencies of the process outputs at any k-th instant with respect to the different inputs and outputs in previous time instants (the legend of Table 2 explains the particular meaning for the model FM_1; the meaning is analogous for FM_2):

Table 2. Identification parameters of the FM_1 and FM_2 fuzzy models.

FM	Dynamical Parameters of the Recursive Discrete Model			Number of Data Clusters for Each Output		Identification in-out Data Versus Validation in-out Data		Sample Time T_s (hours)
	N_y	N_u	N_d	y_1	y_2	The Same	Differents (Partition)	
FM_1	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$	6	5	•		0.2
FM_2	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$				•	

where the N_y, N_u, and N_d parameters and its concrete values, for FM_1 (for FM_2 it would be analogous), mean the following:

- N_y: output-output dynamic relationship matrix (2 × 2) $N_y\ of\ FM_1(row\ 1)$ mean that $y_1(k)$ depends on $y_1(k-1)$ and $y_2(k-1)$ $N_y\ of\ FM_1(row\ 2)$ mean that $y_2(k)$ depends on $y_2(k-1)$
- N_u: input-output dynamic relationship matrix (2 × 3) $N_u\ of\ FM_1(row\ 1)$ mean that $y_1(k)$ depends on $u_1(k-1)$, $u_2(k-1)$, $u_3(k-1)$ and $u_3(k-2)$ $N_u\ of\ FM_1(row\ 2)$ mean that $y_2(k)$ depends on $u_1(k-1)$ and $u_3(k-1)$
- N_d: input-output transport delays matrix (2 × 3) $N_d\ of\ FM_1(row\ 1)$ mean that they is 1 transport delay for the three inputs (u_1, u_2 and u_3) with respect to $y_1(k)$ $N_d\ of\ FM_1(row\ 2)$ mean that they is 1 transport delay for the u_1 and u_3 inputs with respect to $y_2(k)$

From the parametric specifications contained in Table 2 and with a suitable selection of the available input and output data, the chosen identification tool (FMID) will determine the two fuzzy models of the ASP process (FM_1 and FM_2). Each of these models will be characterized by the rules corresponding to the two TS sub-models, making it necessary to know, for each rule, the fuzzy sets associated with each of the antecedent vector components (that is, the identifiers of those sets and their corresponding membership functions), and the affine function of the consequent, $\phi_{ij}(x)$ (or equivalently the numerical coefficients that multiply the components of the consequent vector, and the independent term or offset). The structure of the models identified by using the FMID tool, FM_1 and FM_2, their rules, and the corresponding fuzzy and numerical elements that have just been mentioned, are shown in detail in different tables. For the FM_1 model, Table 3 show the fuzzy sets of the antecedent and Table 4 the affine function of the consequent; and for the FM_2 model, Table 5 presents the fuzzy sets of the antecedent, and Table 6 shows the affine function of the consequent. In all cases, the input and output variables of the ASP process are the ones shown in Table 1. It can be observed that all these tables contain information corresponding to the two outputs of the ASP process (the two TS submodels), but a different notation has been used for each one. Thus, in the case of the FM_1 model, the rules are represented by por R_j, for $y_1(k)$, and R_j^*, for $y_2(k)$; as for the fuzzy sets of the antecedent vector, they are represented by A_{jp}, for $y_1(k)$, and A_{jp}^*, for $y_2(k)$. In the case of the FM_2 model, the established notation differences are analogous ($\overline{R}_j$ and $\overline{R}_j^*$, for the rules of $y_1(k)$ and $y_2(k)$, respectively, and $\overline{A}_{jp}$ and $\overline{A}_{jp}^*$, for the fuzzy sets of the antecedent vector of $y_1(k)$ e $y_2(k)$, respectively). Finally, precise and detailed information related to the membership functions $\mu_{A_{jp}}(x)$ defining each fuzzy set A_{jp} (using a general notation), has been specified, given its extension, in Appendix A (Tables A1 and A2), in parametric form. In our case, membership functions are piece-wise exponential type functions.

Table 3. Fuzzy sets of the antecedent in the rules of the identified FM_1 fuzzy model.

ASP Output	Rule	Components of the Antecedent Vector, x_a, and Corresponding Fuzzy Sets					
		$y_1(k-1)$	$y_2(k-1)$	$u_1(k-1)$	$u_2(k-1)$	$u_3(k-1)$	$u_3(k-2)$
$y_1(k) = s(k)$, substrate	R_1	A_{11}	A_{12}	A_{13}	A_{14}	A_{15}	A_{16}
	R_2	A_{21}	A_{22}	A_{23}	A_{24}	A_{25}	A_{26}
	R_3	A_{31}	A_{32}	A_{33}	A_{34}	A_{35}	A_{36}
	R_4	A_{41}	A_{42}	A_{43}	A_{44}	A_{45}	A_{46}
	R_5	A_{51}	A_{52}	A_{53}	A_{54}	A_{55}	A_{56}
	R_6	A_{61}	A_{62}	A_{63}	A_{64}	A_{65}	A_{66}
$y_2(k) = s(k)$, biomass	R_1^*	-	A_{11}^*	A_{12}^*	-	A_{13}^*	-
	R_2^*	-	A_{21}^*	A_{22}^*	-	A_{23}^*	-
	R_3^*	-	A_{31}^*	A_{32}^*	-	A_{33}^*	-
	R_4^*	-	A_{41}^*	A_{42}^*	-	A_{43}^*	-
	R_5^*	-	A_{51}^*	A_{52}^*	-	A_{53}^*	-

where:

- A_{uv} is the fuzzy set (or *fuzzy value*) corresponding to the v-th component of the antecedent vector, for the u-th rule, R_u, of the output $y_1(k)$ (substrate); the corresponding membership functions are of piece-wise exponential type.
- A_{wz}^* is the fuzzy set (or *fuzzy value*) corresponding to the z-th component of the antecedent vector, for the w-th rule, R_w^*, of the output $y_2(k)$ (biomass); the corresponding membership functions are of piece-wise exponential type.
- The membership functions corresponding to the fuzzy sets A_{uv} and A_{wz}^* are provided by FMID tool in parametric form; these parameters, as well as the mathematical expression of this type of membership functions, are detailed in Appendix A (Table A1).

Table 4. Coefficients of the consequent as per the rules of the identified FM_1 fuzzy model, and offset terms.

ASP Output	Rule	Components of the Consequent Vector, x, and Corresponding Coefficients						Offset
		$y_1(k-1)$	$y_2(k-1)$	$u_1(k-1)$	$u_2(k-1)$	$u_3(k-1)$	$u_3(k-2)$	
$y_1(k) = s(k)$, substrate	R_1	$(8.54) \times 10^{-1}$	$-(8.10) \times 10^{-3}$	$(4.09) \times 10^{-3}$	$(1.57) \times 10^{-2}$	$-(1.88) \times 10^{-2}$	$(1.91) \times 10^{-2}$	$(1.41) \times 10^{1}$
	R_2	$(5.85) \times 10^{-1}$	$(3.09) \times 10^{-2}$	$(3.37) \times 10^{-1}$	$-(1.30) \times 10^{0}$	0	0	0
	R_3	$(8.50) \times 10^{-1}$	$-(1.76) \times 10^{-2}$	$(1.11) \times 10^{-2}$	$(3.97) \times 10^{-2}$	$(1.43) \times 10^{-2}$	$-(1.62) \times 10^{-2}$	$(1.12) \times 10^{1}$
	R_4	$(8.90) \times 10^{-1}$	$-(2.66) \times 10^{-3}$	$(3.44) \times 10^{-3}$	$(3.01) \times 10^{-2}$	$(3.58) \times 10^{-4}$	$(2.34) \times 10^{-3}$	$-(5.16) \times 10^{0}$
	R_5	$(1.53) \times 10^{0}$	$(1.79) \times 10^{-2}$	$-(3.75) \times 10^{1}$	$-(3.80) \times 10^{2}$	$(3.14) \times 10^{2}$	0	0
	R_6	$(8.92) \times 10^{-1}$	$-(5.38) \times 10^{-3}$	$(1.66) \times 10^{-2}$	$(3.98) \times 10^{-2}$	$-(3.00) \times 10^{-2}$	$(2.72) \times 10^{-2}$	$-(2.55) \times 10^{1}$
$y_2(k) = x(k)$, biomass	R_1^*	-	$(9.93) \times 10^{-1}$	$-(2.08) \times 10^{-2}$	-	$(7.72) \times 10^{-3}$	-	$(4.37) \times 10^{1}$
	R_2^*	-	$(8.83) \times 10^{-1}$	$-(5.07) \times 10^{-2}$	-	$(1.24) \times 10^{-1}$	-	$(2.32) \times 10^{2}$
	R_3^*	-	$(7.95) \times 10^{-1}$	$-(1.32) \times 10^{-1}$	-	$(1.01) \times 10^{-1}$	-	$(4.57) \times 10^{2}$
	R_4^*	-	$(8.62) \times 10^{-1}$	$-(6.17) \times 10^{-2}$	-	$(2.48) \times 10^{-2}$	-	$(3.35) \times 10^{2}$
	R_5^*	-	$(9.57) \times 10^{-1}$	$-(4.07) \times 10^{-2}$	-	$(3.66) \times 10^{-2}$	-	$(1.20) \times 10^{2}$

Table 5. Fuzzy sets of the antecedent as per the rules of the identified FM_2 fuzzy model.

ASP Output	Rule	Components of the Antecedent Vector, x_a, and Corresponding Fuzzy Sets				
		$y_1(k-1)$	$y_2(k-1)$	$u_2(k-1)$	$u_3(k-1)$	$u_3(k-2)$
$y_1(k) = s(k)$, substrate	$\overline{R}_1$	$\overline{A}_{11}$	$\overline{A}_{12}$	$\overline{A}_{13}$	$\overline{A}_{14}$	$\overline{A}_{15}$
	$\overline{R}_2$	$\overline{A}_{21}$	$\overline{A}_{22}$	$\overline{A}_{23}$	$\overline{A}_{24}$	$\overline{A}_{25}$
	$\overline{R}_3$	$\overline{A}_{31}$	$\overline{A}_{32}$	$\overline{A}_{33}$	$\overline{A}_{34}$	$\overline{A}_{35}$
	$\overline{R}_4$	$\overline{A}_{41}$	$\overline{A}_{42}$	$\overline{A}_{43}$	$\overline{A}_{44}$	$\overline{A}_{45}$
	$\overline{R}_5$	$\overline{A}_{51}$	$\overline{A}_{52}$	$\overline{A}_{53}$	$\overline{A}_{54}$.	$\overline{A}_{55}$
	$\overline{R}_6$	$\overline{A}_{61}$	$\overline{A}_{62}$	$\overline{A}_{63}$	$\overline{A}_{64}$	$\overline{A}_{65}$
$y_2(k) = x(k)$, biomass	$\overline{R}^*_1$	-	$\overline{A}^*_{11}$	-	$\overline{A}^*_{12}$	-
	$\overline{R}^*_2$	-	$\overline{A}^*_{21}$	-	$\overline{A}^*_{22}$	-
	$\overline{R}^*_3$	-	$\overline{A}^*_{31}$	-	$\overline{A}^*_{32}$	-
	$\overline{R}^*_4$	-	$\overline{A}^*_{41}$	-	$\overline{A}^*_{42}$	-
	$\overline{R}^*_5$	-	$\overline{A}^*_{51}$	-	$\overline{A}^*_{52}$	-

where:

- $\overline{A}_{uv}$ is the fuzzy set (or *fuzzy value*) corresponding to the v-th component of the antecedent vector, for the u-th rule, $\overline{R}_u$, of the output $y_1(k)$ (substrate); the corresponding membership functions are of piece-wise exponential type.
- $\overline{A}^*_{wz}$ is the fuzzy set (or *fuzzy value*) corresponding to the z-th component of the antecedent vector, for the w-th rule, $\overline{R}^*_w$, of the output $y_2(k)$ (biomass); the corresponding membership functions are of piece-wise exponential type.
- The membership functions corresponding to the fuzzy sets $\overline{A}_{uv}$ and $\overline{A}^*_{wz}$ are provided by FMID tool in parametric form; these parameters, as well as the mathematical expression of this type of membership functions, are detailed in Appendix A (Table A2).

Table 6. Coefficients of the consequent in the rules of the identified FM_2 fuzzy model, and offset terms.

ASP Output	Rule	Components of the Consequent Vector, x, and Corresponding Coefficients					Offset
		$y_1(k-1)$	$y_2(k-1)$	$u_2(k-1)$	$u_3(k-1)$	$u_3(k-2)$	
$y_1(k) = s(k)$, substrate	$\overline{R}_1$	$-(2.47) \times 10^6$	$-(8.91) \times 10^4$	0	$(3.95) \times 10^5$	$-(6.26) \times 10^4$	0
	$\overline{R}_2$	$(9.53) \times 10^{-1}$	$-(4.07) \times 10^{-3}$	$(1.58) \times 10^{-2}$	$-(4.45) \times 10^{-3}$	$(4.60) \times 10^{-3}$	$(4.93) \times 10^0$
	$\overline{R}_3$	$(9.26) \times 10^{-1}$	$(4.13) \times 10^{-3}$	$-(1.39) \times 10^{-2}$	$-(1.84) \times 10^{-2}$	$(1.96) \times 10^{-2}$	$(1.61) \times 10^{-1}$
	$\overline{R}_4$	$(7.45) \times 10^{-1}$	$-(4.32) \times 10^{-2}$	$(8.83) \times 10^{-3}$	$(8.83) \times 10^{-3}$	$-(1.02) \times 10^{-2}$	$(1.01) \times 10^2$
	$\overline{R}_5$	$(5.70) \times 10^{-1}$	$-(5.68) \times 10^{-2}$	$(5.93) \times 10^{-2}$	$-(5.08) \times 10^{-1}$	$(5.07) \times 10^{-1}$	$(1.45) \times 10^2$
	$\overline{R}_6$	$(9.69) \times 10^{-1}$	$-(3.84) \times 10^{-3}$	$(2.57) \times 10^{-2}$	$(1.73) \times 10^{-3}$	$-(1.17) \times 10^{-2}$	0
$y_2(k) = x(k)$, biomass	$\overline{R}^*_1$	-	$(9.66) \times 10^{-1}$	-	$(1.32) \times 10^{-1}$	-	$(4.97) \times 10^0$
	$\overline{R}^*_2$	-	$(9.27) \times 10^{-1}$	-	$(3.73) \times 10^{-2}$	-	$(9.81) \times 10^1$
	$\overline{R}^*_3$	-	$(9.86) \times 10^{-1}$	-	$(3.79) \times 10^{-2}$	-	$(6.85) \times 10^0$
	$\overline{R}^*_4$	-	$(1.00) \times 10^0$	-	$(3.00) \times 10^{-2}$	-	$-(3.45) \times 10^1$
	$\overline{R}^*_5$	-	$(1.03) \times 10^0$	-	$-(1.21) \times 10^{-3}$	-	$-(7.36) \times 10^1$

The above tables specifically show the dynamic relationship between the outputs of the ASP process and the consequent vector $\boldsymbol{x}$, consistent with the configuration defined by the values of N_y, N_u, and N_d parameters shown in Table 2. Furthermore, it can also be observed in the above tables that, in the fuzzy TS models obtained by using the used identification tool, the antecedent vector is coincident with the original consequent vector, i.e., $\boldsymbol{x_a} = \boldsymbol{x}$. All these relationships or dependencies can also be functionally expressed as described below.

Thus, in the case of the FM_1 model, the observed dependencies for the output $y_1(k)$, and for the global output $\widetilde{y}_1(k)$, which is a weighted linear combination of the outputs of all rules (Equation (2)), as well as the components of the corresponding antecedent and consequent vectors, $\boldsymbol{x_a} = \boldsymbol{x}|_{(FM_1,\ out_1)}$, are the following (Equation (3)):

$$\begin{aligned} y_1(k)|_{(FM_1,\,R_j)} &= f_{1j}(y_1(k-1), y_2(k-1),\ u_1(k-1), u_2(k-1), u_3(k-1),\ u_3(k-2)) \\ \widetilde{y}_1(k)|_{FM_1} &= f_1(y_1(k-1), y_2(k-1),\ u_1(k-1), u_2(k-1), u_3(k-1),\ u_3(k-2)) \\ \boldsymbol{x_a} = \boldsymbol{x}|_{(FM_1,\ out_1)} &= (y_1(k-1), y_2(k-1), u_1(k-1), u_2(k-1), u_3(k-1),\ u_3(k-2)) \end{aligned} \tag{3}$$

Regarding the output $y_2(k)$ and the global output $\widetilde{y}_2(k)$, as well as the components of the corresponding antecedent and consequent vectors, $\boldsymbol{x_a} = \boldsymbol{x}|_{(FM_1,\ out_2)}$, the observed relationships are the following (Equation (4)):

$$\begin{aligned} y_2(k)|_{(FM_1,\ R_j^*)} &= f_{2j}(y_2(k-1),\ u_1(k-1), u_3(k-1)) \\ \widetilde{y}_2(k)|_{FM_1} &= f_2(y_2(k-1),\ u_1(k-1), u_3(k-1)) \\ \boldsymbol{x_a} = \boldsymbol{x}|_{(FM_1,\ out_2)} &= (y_2(k-1),\ u_1(k-1), u_3(k-1)) \end{aligned} \tag{4}$$

Note that from Equations (3) and (4) can be observed that the *consequent vector* corresponding to the $y_1(k)$ output is not exactly the same as the *consequent vector* corresponding to the $y_2(k)$ output, but the components of the first vector include the components of the second vector. We will consider then the set formed by the union of the components of both vectors and we will form a *consequent vector* common to both outputs, $\boldsymbol{x}|_{FM_1}$, to be able to mathematically deal together the fuzzy models of both outputs, within a matrix mathematical framework (by considering, logically, null factors where necessary in the numerical expressions related to output $y_2(k)$). Equation (5) summarizes the functional relationships of the two *weighted global outputs*, with respect to the common consequent vector, $\boldsymbol{x}|_{FM_1}$:

$$\begin{aligned} \widetilde{y}_1(k)|_{FM_1} &= f_{1c}(y_1(k-1), y_2(k-1),\ u_1(k-1), u_2(k-1), u_3(k-1),\ u_3(k-2)) \\ \widetilde{y}_2(k)|_{FM_1} &= f_{2c}(y_1(k-1), y_2(k-1),\ u_1(k-1), u_2(k-1), u_3(k-1),\ u_3(k-2)) \\ \boldsymbol{x}|_{FM_1} &= (y_1(k-1), y_2(k-1),\ u_1(k-1), u_2(k-1), u_3(k-1),\ u_3(k-2)) \\ & or\ (abbreviated): \\ \widetilde{y}_1(k)|_{FM_1} &= f_{1c}\left(\boldsymbol{x}|_{FM_1}\right) \\ \widetilde{y}_2(k)|_{FM_1} &= f_{2c}\left(\boldsymbol{x}|_{FM_1}\right) \end{aligned} \tag{5}$$

where:

- $f_{1c}(\cdot)$ and $f_{2c}(\cdot)$ are the trivial adaptations of $f_1(\cdot)$ and $f_2(\cdot)$, respectively, after the definition of the common *consequent vector*, $\boldsymbol{x}|_{FM_1}$
- $\boldsymbol{x}|_{FM_1}$ is the *consequent vector*, common to both outputs

For the FM_2 model, after a process of observation and deduction analogous to that of the FM_1 model, we observe the following dependency relationships for the two outputs of the ASP process, as well as the corresponding consequent vectors (Equations (6) and (7)):

$$\begin{aligned} y_1(k)|_{(FM_2,\,\overline{R}_j)} &= \overline{f}_{1j}(y_1(k-1), y_2(k-1),\ u_2(k-1), u_3(k-1),\ u_3(k-2)) \\ \widetilde{y}_1(k)|_{FM_2} &= \overline{f}_1(y_1(k-1), y_2(k-1),\ u_2(k-1), u_3(k-1),\ u_3(k-2)) \\ \boldsymbol{x_a} = \boldsymbol{x}|_{(FM_2,\ out_1)} &= (y_1(k-1), y_2(k-1), u_2(k-1), u_3(k-1),\ u_3(k-2)) \end{aligned} \tag{6}$$

$$\begin{aligned} y_2(k)|_{(FM_2,\ \overline{R}_j^*)} &= \overline{f}_{2j}(y_2(k-1),\ u_3(k-1)) \\ \widetilde{y}_2(k)|_{FM_2} &= \overline{f}_2(y_2(k-1),\ u_3(k-1)) \\ \boldsymbol{x_a} = \boldsymbol{x}|_{(FM_2,\ out_2)} &= (y_2(k-1),\ u_3(k-1)) \end{aligned} \tag{7}$$

Again, as in the case of the previous model, we observe that the consequent vectors of both outputs are not coincident, but given that the one corresponding to the output $y_1(k)$ includes among its components those of the one corresponding to the output $y_2(k)$, we can define a common consequent vector that is the union of both, $\boldsymbol{x}|_{FM_2}$(considering null factors in the numerical expressions related to output $y_2(k)$ where necessary). Equation (8) summarizes the functional relationships of the two *weighted global outputs*, with respect to the common consequent vector, $\boldsymbol{x}|_{FM_2}$:

$$\begin{gathered}
\widetilde{y}_1(k)|_{FM_2} = \overline{f}_{1c}(y_1(k-1), y_2(k-1),\ u_2(k-1), u_3(k-1),\ u_3(k-2)) \\
\widetilde{y}_2(k)|_{FM_2} = \overline{f}_{2c}(y_1(k-1), y_2(k-1),\ u_2(k-1), u_3(k-1),\ u_3(k-2)) \\
\boldsymbol{x}|_{FM_2} = (y_1(k-1),\ y_2(k-1), u_2(k-1),\ u_3(k-1),\ u_3(k-2)) \\
or\ (abbreviated): \\
\widetilde{y}_1(k)|_{FM_2} = \overline{f}_{1c}\left(\boldsymbol{x}|_{FM_2}\right) \\
\widetilde{y}_2(k)|_{FM_2} = \overline{f}_{2c}\left(\boldsymbol{x}|_{FM_2}\right)
\end{gathered} \tag{8}$$

where:

- $\overline{f}_{1c}(\cdot)$ and $\overline{f}_{2c}(\cdot)$ are the trivial adaptations of $\overline{f}_1(\cdot)$ and $\overline{f}_2(\cdot)$, respectively, after the definition of the common *consequent vector*, $\boldsymbol{x}|_{FM_2}$
- $\boldsymbol{x}|_{FM_2}$ is the *consequent vector*, common to both outputs

Generalization: Finally, and given that the components of the consequent vector $\boldsymbol{x}|_{FM_1}$ include those of the consequent vector $\boldsymbol{x}|_{FM_2}$, or equivalently, those of the latter are a subset of those of the first, we can consider the union of both as a consequent vector common to both models (in our case, it would be coincident with the first), representing it as $\boldsymbol{x}$. In this way, we can express the functional dependency relationships, for either of the two models, with respect to a single common consequent vector, $\boldsymbol{x}$, naturally considering null factors, where necessary, in the numerical expressions corresponding to the FM_2 model. The summarized general expressions, formally valid for both models, would be the following (Equation (9)):

$$\begin{gathered}
\widetilde{y}_1(k) = f_{af1}(\boldsymbol{x}) \\
\widetilde{y}_2(k) = f_{af2}(\boldsymbol{x}) \\
or,\ simplifying\ the\ notation \\
(denoting\ to\ the\ final\ weighted\ outputs\ in\ the\ same\ way \\
as\ to\ the\ individual\ rules\ outputs:\ \widetilde{y}_1(k) \equiv y_1(k)\ and\ \widetilde{y}_2(k) \equiv y_2(k)): \\
y_1(k) = f_{af1}(\boldsymbol{x}) \\
y_2(k) = f_{af2}(\boldsymbol{x}) \\
being: \\
\boldsymbol{x} = (y_1(k-1),\ y_2(k-1),\ u_1(k-1), u_2(k-1),\ u_3(k-1),\ u_3(k-2))
\end{gathered} \tag{9}$$

where:

- $f_{af1}(\cdot)$ and $f_{af2}(\cdot)$ are the affine functions corresponding to the outputs $\widetilde{y}_1(k)$ and $\widetilde{y}_2(k)$, respectively (the coefficients and offset term of these affine functions depend on the fuzzy model (FM_1 *or* FM_2))
- $\boldsymbol{x}$ is the *consequent vector*, common to both fuzzy models (and common to two outputs)

Remark: Both the outputs of the process, as well as the different antecedent and consequent vectors, have been expressed in the original form used by the FMID software tool, concerning temporal dependencies. Thus, the expressions corresponding to the outputs at the k-th time instant have been explicitly expressed. In order to specify the expressions of the outputs at the $(k+1)$-th time instant, as well as those of the corresponding antecedent and consequent vectors, we will simply have to substitute k by $(k+1)$ in the expressions we have shown previously. Thus, for example, in the case of the FM_1 model, the antecedent and consequent vectors corresponding to the outputs $y_1(k+1)$ and $y_2(k+1)$, as well as the fuzzy rules, in their generic form, would be as is detailed below (before adopting a common consequent vector):

$$FM_1 \; model \; (outputs \; at \; the \; (k+1)\text{-}th \; time \; instant).$$

$$\begin{array}{l} For \; the \; output \; y_1(k+1): \\ \qquad x_a = x|_{(FM_1, \;\; out_1)} = (y_1(k), y_2(k), u_1(k), u_2(k), u_3(k), \; u_3(k-1)) \\ \qquad j\text{-}th \; rule, \; \boldsymbol{R_j}: \\ \boldsymbol{if} \; (y_1(k) \; is \; A_{j1} \; and \; y_2(k) \; is \; A_{j2} \; and \; u_1(k) \; is \; A_{j3} \; and \; u_2(k) \; is \; A_{j4} \; and \; u_3(k) \; is \; A_{j5} \; and \; u_3(k-1) \; is \; A_{j6}) \\ \qquad \boldsymbol{then} \\ \qquad y_1(k+1) = \alpha_{j1}y_1(k) + \alpha_{j2}y_2(k) + \alpha_{j3}u_1(k) + \alpha_{j4}u_2(k) + \alpha_{j5}u_3(k) + \alpha_{j6}u_3(k-1) + \delta_j \\ \qquad j = 1, 2, \ldots, 6 \\ For \; the \; output \; y_2(k+1): \\ \qquad x_a = x|_{(FM_1, \;\; out_2)} = (y_2(k), u_1(k), u_3(k)) \\ \qquad j\text{-}th \; rule, \boldsymbol{R_j^*}: \\ \qquad \boldsymbol{if} \left(y_2(k) \; is \; A^*_{j1} \; and \; u_1(k) \; is \; A^*_{j2} \; and \; u_3(k) \; is \; A^*_{j3} \right) \\ \qquad \boldsymbol{then} \\ \qquad y_2(k+1) = \alpha^*_{j1}y_2(k) + \alpha^*_{j2}u_1(k) + \alpha^*_{j3}u_3(k) + \delta^*_j \\ \qquad j = 1, 2, \ldots, 5 \end{array} \tag{10}$$

2.3. State-Space DLTV Equivalent Model

Making use of the necessary mathematical basis (mainly, basic matrix calculation) and by an adequate treatment, the TS fuzzy models obtained can be transformed or expressed in the state space, in the form of discrete, linear, and time-varying models, that is, as DLTV type models.

We will start from Equation (2), the mathematical expression that allows us to calculate the outputs of the process (at every k-th time instant) from the knowledge of the membership functions of the antecedent and of the numerical coefficients of the consequent, obtained as a result of the identification of fuzzy models (FM_1 or FM_2, in our case study). If we develop Equation (2) for a generic TS fuzzy model (of the type identified), for each of the two global outputs of the ASP process at the k-th time instant (that is, returning to the form provided by the software tool FMID), explicitly showing the affine function (as a linear combination of the components of the consequent vector, plus the independent term), we would obtain the following two equations (where, so as not to overload the notation, as in Equation (2), it has been omitted again the formal indication of the dependency of $\boldsymbol{x_a}$ with respect to the i index, that is, with respect to the i-th output, and also the dependency of $\boldsymbol{x_a}$ and $\boldsymbol{x}$ with respect to k):

$$\begin{array}{c} y_1(k) = \sum_{j=1}^{mr_1} \beta_{1j}(\boldsymbol{x_a})\phi_{1j}(\boldsymbol{x}) = \\ = \sum_{j=1}^{mr_1} \beta_{1j}(\boldsymbol{x_a})\big(\alpha_{j1}y_1(k-1) + \alpha_{j2}y_2(k-1) + \alpha_{j3}u_1(k-1) + \alpha_{j4}u_2(k-1) \\ +\alpha_{j5}u_3(k-1) + \alpha_{j6}u_3(k-2) + \delta_j\big) = \\ = \left(\sum_{j=1}^{mr_1} \beta_{1j}(\boldsymbol{x_a})\begin{pmatrix} \alpha_{j1} & \alpha_{j2} \end{pmatrix}\right)\begin{pmatrix} y_1(k-1) \\ y_2(k-1) \end{pmatrix} \\ +\left(\sum_{j=1}^{mr_1} \beta_{1j}(\boldsymbol{x_a})\begin{pmatrix} \alpha_{j3} & \alpha_{j4} \end{pmatrix}\right)\begin{pmatrix} u_1(k-1) \\ u_2(k-1) \end{pmatrix} \\ +\left(\sum_{j=1}^{mr_1} \beta_{1j}(\boldsymbol{x_a})\begin{pmatrix} \alpha_{j5} & \alpha_{j6} \end{pmatrix}\right)\begin{pmatrix} u_3(k-1) \\ u_3(k-2) \end{pmatrix} \\ +\left(\sum_{j=1}^{mr_1} \beta_{1j}(\boldsymbol{x_a})\delta_j\right) \\ being: \\ mr_1: \; number \; of \; rules \; of \; fuzzy \; model, \; for \; output \; y_1 \end{array} \tag{11}$$

$$
\begin{array}{c}
y_2(k) = \sum_{j=1}^{mr_2} \beta_{2j}(\boldsymbol{x_a})\phi_{2j}(\boldsymbol{x}) = \\
= \sum_{j=1}^{mr_2} \beta_{2j}(\boldsymbol{x_a})\Big(\alpha_{j1}^* y_1(k-1) + \alpha_{j2}^* y_2(k-1) + \alpha_{j3}^* u_1(k-1) + \alpha_{j4}^* u_2(k-1) \\
+\alpha_{j5}^* u_3(k-1) + \alpha_{j6}^* u_3(k-2) + \delta_j^*\Big) = \\
= \left(\sum_{j=1}^{mr_2} \beta_{2j}(\boldsymbol{x_a})\begin{pmatrix} \alpha_{j1}^* & \alpha_{j2}^* \end{pmatrix}\right)\begin{pmatrix} y_1(k-1) \\ y_2(k-1) \end{pmatrix} \\
+\left(\sum_{j=1}^{mr_2} \beta_{2j}(\boldsymbol{x_a})\begin{pmatrix} \alpha_{j3}^* & \alpha_{j4}^* \end{pmatrix}\right)\begin{pmatrix} u_1(k-1) \\ u_2(k-1) \end{pmatrix} \\
+\left(\sum_{j=1}^{mr_2} \beta_{2j}(\boldsymbol{x_a})\begin{pmatrix} \alpha_{j5}^* & \alpha_{j6}^* \end{pmatrix}\right)\begin{pmatrix} u_3(k-1) \\ u_3(k-2) \end{pmatrix} + \left(\sum_{j=1}^{mr_2} \beta_{2j}(\boldsymbol{x_a})\delta_j^*\right) \\
being: \\
mr_2:\ number\ of\ rules\ of\ fuzzy\ model,\ for\ output\ y_2
\end{array}
\tag{12}
$$

By introducing now suitable mathematical definitions (mr y $\boldsymbol{\beta_j}(\boldsymbol{x_a})$), we can group Equations (11) and (12) into a single matrix equation:

$$
\begin{array}{l}
\begin{pmatrix} y_1(k) \\ y_2(k) \end{pmatrix} = \left(\sum_{j=1}^{mr} \boldsymbol{\beta_j}(\boldsymbol{x_a})\begin{pmatrix} \alpha_{j1} & \alpha_{j2} \\ \alpha_{j1}^* & \alpha_{j2}^* \end{pmatrix}\right)\begin{pmatrix} y_1(k-1) \\ y_2(k-1) \end{pmatrix} + \left(\sum_{j=1}^{mr} \boldsymbol{\beta_j}(\boldsymbol{x_a})\begin{pmatrix} \alpha_{j3} & \alpha_{j4} \\ \alpha_{j3}^* & \alpha_{j4}^* \end{pmatrix}\right)\begin{pmatrix} u_1(k-1) \\ u_2(k-1) \end{pmatrix} \\
\quad + \left(\sum_{j=1}^{mr} \boldsymbol{\beta_j}(\boldsymbol{x_a})\begin{pmatrix} \alpha_{j5} & \alpha_{j6} \\ \alpha_{j5}^* & \alpha_{j6}^* \end{pmatrix}\right)\begin{pmatrix} u_3(k-1) \\ u_3(k-2) \end{pmatrix} + \left(\sum_{j=1}^{mr} \boldsymbol{\beta_j}(\boldsymbol{x_a})\begin{pmatrix} \delta_j \\ \delta_j^* \end{pmatrix}\right) \\
\qquad being: \\
\bullet\ mr = max.\,(mr_1, mr_2) = common\ number\ of\ rules\ (6,\ in\ our\ case\ study) \\
\bullet\ \boldsymbol{\beta_j}(\boldsymbol{x_a}) = \begin{pmatrix} \beta_{1j}(\boldsymbol{x_a}) & 0 \\ 0 & \beta_{2j}(\boldsymbol{x_a}) \end{pmatrix},\ where: \\
\quad \beta_{1j}(\boldsymbol{x_a})\ is\ the\ normalized\ membership\ function\ of\ the\ antecedent\ vector \\
\quad corresponding\ to\ the\ first\ output\ (y_1),\ for\ the\ j\text{-}th\ rule \\
\quad \beta_{2j}(\boldsymbol{x_a})\ is\ the\ normalized\ membership\ function\ of\ the\ antecedent\ vector \\
\quad corresponding\ to\ the\ second\ output\ (y_2),\ for\ the\ j\text{-}th\ rule \\
\bullet\ \beta_{26}(\boldsymbol{x_a}) = 0;\ \alpha_{6\mathrm{i}}^* \equiv 0\ (\mathrm{i} = 1, \ldots, 6),\ \delta_6^* \equiv 0(there\ are\ only\ 5\ rules\ for\ y_2)
\end{array}
\tag{13}
$$

and by substituting k for $(k+1)$ in Equation (13), that is, considering the outputs at the $(k+1)$-th time instant (see (10)), we will obtain the equivalent Equation (14) (having mr y $\boldsymbol{\beta_j}(\boldsymbol{x_a})$ the same or analogous meaning, respectively, as they have in Equation (13), and where $\boldsymbol{x_a}$ represents, in $\beta_{1j}(\boldsymbol{x_a})$, the corresponding antecedent vector to the output $y_1(k+1)$, and in $\beta_{2j}(\boldsymbol{x_a})$, $\boldsymbol{x_a}$ represents the corresponding antecedent vector to the output $y_2(k+1)$; all these meanings and the numerical peculiarities for $j = 6$ shown in (13) will be kept from now on):

$$
\begin{array}{l}
\begin{pmatrix} y_1(k+1) \\ y_2(k+1) \end{pmatrix} = \left(\sum_{j=1}^{mr} \boldsymbol{\beta_j}(\boldsymbol{x_a})\begin{pmatrix} \alpha_{j1} & \alpha_{j2} \\ \alpha_{j1}^* & \alpha_{j2}^* \end{pmatrix}\right)\begin{pmatrix} y_1(k) \\ y_2(k) \end{pmatrix} + \left(\sum_{j=1}^{mr} \boldsymbol{\beta_j}(\boldsymbol{x_a})\begin{pmatrix} \alpha_{j3} & \alpha_{j4} \\ \alpha_{j3}^* & \alpha_{j4}^* \end{pmatrix}\right)\begin{pmatrix} u_1(k) \\ u_2(k) \end{pmatrix} \\
\quad + \left(\sum_{j=1}^{mr} \boldsymbol{\beta_j}(\boldsymbol{x_a})\begin{pmatrix} \alpha_{j5} & \alpha_{j6} \\ \alpha_{j5}^* & \alpha_{j6}^* \end{pmatrix}\right)\begin{pmatrix} u_3(k) \\ u_3(k-1) \end{pmatrix} + \left(\sum_{j=1}^{mr} \boldsymbol{\beta_j}(\boldsymbol{x_a})\begin{pmatrix} \delta_j \\ \delta_j^* \end{pmatrix}\right)
\end{array}
\tag{14}
$$

Next, to improve the characterization of the different terms of Equation (14), we will introduce the following notation changes:

$$
\begin{array}{c}
\alpha_{j1} = a_{j1},\ \alpha_{j2} = a_{j2},\ \alpha_{j1}^* = a_{j1}^*,\ \alpha_{j2}^* = a_{j2}^* \\
\alpha_{j3} = d_{j1},\ \alpha_{j4} = d_{j2},\ \alpha_{j3}^* = d_{j1}^*,\ \alpha_{j4}^* = d_{j2}^* \\
\alpha_{j5} = b_{j1},\ \alpha_{j6} = b_{j2},\ \alpha_{j5}^* = b_{j1}^*,\ \alpha_{j6}^* = b_{j2}^* \\
\delta_j = r_j,\ \delta_j^* = r_j^*
\end{array}
\tag{15}
$$

Now, rearranging the terms of Equation (14) (exchanging the positions of the second and third terms) and applying the notation changes introduced in (15), the following equation will be obtained:

$$\begin{pmatrix} y_1(k+1) \\ y_2(k+1) \end{pmatrix} = \left(\sum_{j=1}^{mr} \boldsymbol{\beta_j}(\boldsymbol{x_a})\begin{pmatrix} a_{j1} & a_{j2} \\ a^*_{j1} & a^*_{j2} \end{pmatrix}\right)\begin{pmatrix} y_1(k) \\ y_2(k) \end{pmatrix} + \left(\sum_{j=1}^{mr} \boldsymbol{\beta_j}(\boldsymbol{x_a})\begin{pmatrix} b_{j1} & b_{j2} \\ b^*_{j1} & b^*_{j2} \end{pmatrix}\right)\begin{pmatrix} u_3(k) \\ u_3(k-1) \end{pmatrix}$$
$$+\left(\sum_{j=1}^{mr} \boldsymbol{\beta_j}(\boldsymbol{x_a})\begin{pmatrix} d_{j1} & d_{j2} \\ d^*_{j1} & d^*_{j2} \end{pmatrix}\right)\begin{pmatrix} u_1(k) \\ u_2(k) \end{pmatrix} + \left(\sum_{j=1}^{mr} \boldsymbol{\beta_j}(\boldsymbol{x_a})\begin{pmatrix} r_j \\ r^*_j \end{pmatrix}\right)$$
$$with:\ a^*_{6i} \equiv 0,\ b^*_{6i} \equiv 0,\ d^*_{6i} \equiv 0\ (\mathrm{i}=1,2),\ \mathrm{r}^*_6 \equiv 0\ (there\ are\ only\ 5\ rules\ for\ y_2) \quad (16)$$

As a further step, in the process of development and transformation of the original Equation (2), with integration of the two outputs, we will make the following vector and matrix definitions and formalizations (where the particularities corresponding to $j = 6$, indicated in (16), are maintained):

$$\boldsymbol{z_{mN}}(k) = \begin{pmatrix} y_1(k) \\ y_2(k) \end{pmatrix},\ \boldsymbol{z_{mN}}(k+1) = \begin{pmatrix} y_1(k+1) \\ y_2(k+1) \end{pmatrix}$$
$$\boldsymbol{u_a}(k) = \begin{pmatrix} u_3(k) \\ u_3(k-1) \end{pmatrix} \equiv \begin{pmatrix} u(k) \\ u(k-1) \end{pmatrix}$$
$$\boldsymbol{d}(k) = \begin{pmatrix} u_1(k) \\ u_2(k) \end{pmatrix} \equiv \begin{pmatrix} d_1(k) \\ d_2(k) \end{pmatrix} \quad (17)$$
$$\boldsymbol{A_{mN_j}} = \begin{pmatrix} a_{j1} & a_{j2} \\ a^*_{j1} & a^*_{j2} \end{pmatrix},\ \boldsymbol{B_{mN_j}} = \begin{pmatrix} b_{j1} & b_{j2} \\ b^*_{j1} & b^*_{j2} \end{pmatrix},\ \boldsymbol{D_{mN_j}} = \begin{pmatrix} d_{j1} & d_{j2} \\ d^*_{j1} & d^*_{j2} \end{pmatrix},\ \boldsymbol{R_{mN_j}} = \begin{pmatrix} r_j \\ r^*_j \end{pmatrix}$$

where:

- k represents $(k \cdot T)$ and T is the sampling period
- $\boldsymbol{z_{mN}}(k)$ is the definition of *state vector* at the *k-esimo* time instant (which groups the two outputs of the process at the k-th time instant)
- $\boldsymbol{u_a}(k)$ is the *input vector* or the *extended input vector* (with two components: the manipulated process variable at the k-th time instant and the manipulated variable at the$(k-1)$-th time instant, respectively)
- $\boldsymbol{d}(k)$ is the *disturbances vector* (which groups the two unmanipulated process variables at the k-th time instant)
- $\boldsymbol{y_{mN}}(k)$ is the *output vector* (that matches the *state vector* definition)

and where the different input and output process variables involved (whose physical meaning can be seen in Table 1) are the following:

- inputs: $u_1(k) = q_i(k)$, $u_2(k) = s_i(k)$, $u_3(k) = q_r(k)$
- outputs: $y_1(k) = s(k)$, $y_2(k) = x(k)$

Finally, by substituting in Equation (16) the definitions and formalizations made in (17), we will have the following *state equation*, where the time dependence of the antecedent vector, $\boldsymbol{x_a}$, ($\boldsymbol{x_a} = \boldsymbol{x_a}(k)$) has been shown explicitly due to its importance in the model characterization:

$$\begin{aligned} \boldsymbol{z_{mN}}(k+1) &= \left(\sum_{j=1}^{mr} \boldsymbol{\beta_j}(\boldsymbol{x_a}(k))\boldsymbol{A_{mN_j}}\right)\boldsymbol{z_{mN}}(k) + \left(\sum_{j=1}^{mr} \boldsymbol{\beta_j}(\boldsymbol{x_a}(k))\boldsymbol{B_{mN_j}}\right)\boldsymbol{u_a}(k) \\ &+\left(\sum_{j=1}^{mr} \boldsymbol{\beta_j}(\boldsymbol{x_a}(k))\boldsymbol{D_{mN_j}}\right)\boldsymbol{d}(k) + \left(\sum_{j=1}^{mr} \boldsymbol{\beta_j}(\boldsymbol{x_a}(k))\boldsymbol{R_{mN_j}}\right) \end{aligned} \quad (18)$$

On the other hand, taking into account that $\sum_{j=1}^{mr_i} \beta_{ij}(\boldsymbol{x_a}) = 1$, for any i (see Equation (2)), it is then true that $\sum_{j=1}^{mr} \beta_{1j}(\boldsymbol{x_a}) = 1$ and $\sum_{j=1}^{mr} \beta_{2j}(\boldsymbol{x_a}) = 1$ (since $mr = max.\ (mr_1, mr_2)$ and $\beta_{26}(\boldsymbol{x_a}) = 0$). Taking into account these last two equalities and grouping the outputs $y_1(k)$ and $y_2(k)$ in a single vector, it follows:

$$\begin{pmatrix} y_1(k) \\ y_2(k) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_1(k) \\ y_2(k) \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{mr} \beta_{1j}(\boldsymbol{x_a}) & 0 \\ 0 & \sum_{j=1}^{mr} \beta_{2j}(\boldsymbol{x_a}) \end{pmatrix}\begin{pmatrix} y_1(k) \\ y_2(k) \end{pmatrix} = \left(\sum_{j=1}^{mr}\begin{pmatrix} \beta_{1j}(\boldsymbol{x_a}) & 0 \\ 0 & \beta_{2j}(\boldsymbol{x_a}) \end{pmatrix}\right)\begin{pmatrix} y_1(k) \\ y_2(k) \end{pmatrix}$$
$$= \left(\sum_{j=1}^{mr}\begin{pmatrix} \beta_{1j}(\boldsymbol{x_a}) & 0 \\ 0 & \beta_{2j}(\boldsymbol{x_a}) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} y_1(k) \\ y_2(k) \end{pmatrix} = \left(\sum_{j=1}^{mr} \beta_j(\boldsymbol{x_a})\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} y_1(k) \\ y_2(k) \end{pmatrix} \quad (19)$$

and doing now the following vector and matrix definitions and formalizations:

$$\begin{array}{c} \boldsymbol{y_{mN}}(k) = \begin{pmatrix} y_1(k) \\ y_2(k) \end{pmatrix} \\ \boldsymbol{C_{mN_j}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \tag{20}$$

where:

- k represents $(k{\cdot}T)$ and T is the sampling period
- $\boldsymbol{y_{mN}}(k)$ is the *output vector* (which groups the two outputs of the process at the k-th time instant and matches the *state vector* definition)

and by substituting Equation (20) in Equation (19), we will have the following *output equation*, where the time dependence of the antecedent vector, $\boldsymbol{x_a}$, $(\boldsymbol{x_a} = \boldsymbol{x_a}(k))$ has been shown explicitly (as in the *state equation*) due to its importance in the model characterization:

$$\boldsymbol{y_{mN}}(k) = \left(\sum_{j=1}^{mr} \boldsymbol{\beta_j}(\boldsymbol{x_a}(k))\boldsymbol{C_{mN_j}}\right)\boldsymbol{z_{mN}}(k) \tag{21}$$

Next, we will represent by means of a single matrix each of the matrix sums of the different terms of Equations (18) and (21), according to the following definitions:

$$\begin{array}{c} \overline{\boldsymbol{A}}_{\boldsymbol{mN}} = \sum_{j=1}^{mr} \boldsymbol{\beta_j}(\boldsymbol{x_a}(k))A_{mN_j},\ \overline{\boldsymbol{B}}_{\boldsymbol{mN}} = \sum_{j=1}^{mr} \boldsymbol{\beta_j}(\boldsymbol{x_a}(k))\boldsymbol{B_{mN_j}} \\ \overline{\boldsymbol{D}}_{\boldsymbol{mN}} = \sum_{j=1}^{mr} \boldsymbol{\beta_j}(\boldsymbol{x_a}(k))\boldsymbol{D_{mN_j}},\ \overline{\boldsymbol{R}}_{\boldsymbol{mN}} = \sum_{j=1}^{mr} \boldsymbol{\beta_j}(\boldsymbol{x_a}(k))\boldsymbol{R_{mN_j}} \\ \overline{\boldsymbol{C}}_{\boldsymbol{mN}} = \sum_{j=1}^{mr} \boldsymbol{\beta_j}(\boldsymbol{x_a}(k))\boldsymbol{C_{mN_j}} \end{array} \tag{22}$$

The matrices defined in (22) can also be expressed in the following symbolic and generic way (which will be useful later to simplify the stability analysis), representing the elements of the matrices in a simplified way, taking into account the dynamic characteristics of each one of the models (see the identification parameters N_y, N_u and N_d, in Table 2), and by specifying its relationship with the original coefficients of the starting fuzzy models, FM_1 and FM_2:

$$\begin{array}{c} \overline{\boldsymbol{A}}_{\boldsymbol{mN}} = \begin{pmatrix} a & b \\ 0^{(1)} & f \end{pmatrix} (for\ FM_1\ and\ FM_2) \\ a = \left(\sum_{j=1}^{mr} \beta_{1j}(\boldsymbol{x_a})a_{j1}\right),\ b = \left(\sum_{j=1}^{mr} \beta_{1j}(\boldsymbol{x_a})a_{j2}\right),\ f = \left(\sum_{j=1}^{mr} \beta_{2j}(\boldsymbol{x_a})a_{j2}^{*}\right) \\ (1)\ for\ both\ fuzzy\ models,\ a_{j1}^{*} = 0,\ \forall j;\ then:\ \left(\sum_{j=1}^{mr} \beta_{2j}(\boldsymbol{x_a})a_{j1}^{*}\right) = 0 \\ \overline{\boldsymbol{B}}_{\boldsymbol{mN}} = \begin{pmatrix} m & n \\ p & 0^{(2)} \end{pmatrix} (for\ FM_1\ and\ FM_2) \\ m = \left(\sum_{j=1}^{mr} \beta_{1j}(\boldsymbol{x_a})b_{j1}\right),\ n = \left(\sum_{j=1}^{mr} \beta_{1j}(\boldsymbol{x_a})b_{j2}\right),\ p = \left(\sum_{j=1}^{mr} \beta_{2j}(\boldsymbol{x_a})b_{j1}^{*}\right) \\ (2)\ for\ both\ fuzzy\ models,\ b_{j2}^{*} = 0,\ \forall j;\ then:\ \left(\sum_{j=1}^{mr} \beta_{2j}(\boldsymbol{x_a})b_{j2}^{*}\right) = 0 \\ \overline{\boldsymbol{D}}_{\boldsymbol{mN}} = \begin{pmatrix} c & d \\ g & 0^{(3)} \end{pmatrix} (for\ FM_1;\ for\ FM_2:\ c = 0\ and\ g = 0\ \ (4)) \\ c = \left(\sum_{j=1}^{mr} \beta_{1j}(\boldsymbol{x_a})d_{j1}\right),\ d = \left(\sum_{j=1}^{mr} \beta_{1j}(\boldsymbol{x_a})d_{j2}\right),\ g = \left(\sum_{j=1}^{mr} \beta_{2j}(\boldsymbol{x_a})d_{j1}^{*}\right) \\ (3)\ for\ both\ fuzzy\ models,\ d_{j2}^{*} = 0,\ \forall j;\ then:\ \left(\sum_{j=1}^{mr} \beta_{2j}(\boldsymbol{x_a})d_{j2}^{*}\right) = 0 \\ (4)\ for\ FM_2:\ d_{j1} = 0,\ d_{j1}^{*} = 0,\ \ \forall j;\ then:\ \left(\sum_{j=1}^{mr} \beta_{1j}(\boldsymbol{x_a})d_{j1}\right) = 0\ and\ \left(\sum_{j=1}^{mr} \beta_{2j}(\boldsymbol{x_a})d_{j1}^{*}\right) = 0 \\ \overline{\boldsymbol{R}}_{\boldsymbol{mN}} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} (for\ FM_1\ and\ FM_2) \\ r_1 = \left(\sum_{j=1}^{mr} \beta_{1j}(\boldsymbol{x_a})r_j\right),\ r_2 = \left(\sum_{j=1}^{mr} \beta_{2j}(\boldsymbol{x_a})r_j^{*}\right) \\ \overline{\boldsymbol{C}}_{\boldsymbol{mN}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}(for\ FM_1\ and\ FM_2) \end{array} \tag{23}$$

Finally, and independently of the possible formal matrix representations shown in (23)), if we substitute the matrices defined in (22) in Equations (18) and (21), and we consider both equations together, we will obtain the final equivalent *state space model* (Equations (24) and (25)), which is of DLTV type, as we will reason below:

$$\boldsymbol{z_{mN}}(k+1) = \overline{\boldsymbol{A}}_{\boldsymbol{mN}}\boldsymbol{z_{mN}}(k) + \overline{\boldsymbol{B}}_{\boldsymbol{mN}}\boldsymbol{u_a}(k) + \overline{\boldsymbol{D}}_{\boldsymbol{mN}}\boldsymbol{d}(k) + \overline{\boldsymbol{R}}_{\boldsymbol{mN}} \tag{24}$$

$$\boldsymbol{y_{mN}}(k) = \overline{\boldsymbol{C}}_{\boldsymbol{mN}}\boldsymbol{z_{mN}}(k) \tag{25}$$

where:

- k represents $(k \cdot T)$ and T is the sampling period
- $\boldsymbol{z_{mN}}(k)$ is the state vector
- $\boldsymbol{u_a}(k)$ is the input vector (or the extended input vector)
- $\boldsymbol{d}(k)$ is the disturbances vector
- $\boldsymbol{y_{mN}}(k)$ is the output vector
- $\overline{\boldsymbol{A}}_{\boldsymbol{mN}}$, $\overline{\boldsymbol{B}}_{\boldsymbol{mN}}$, $\overline{\boldsymbol{D}}_{\boldsymbol{mN}}$, $\overline{\boldsymbol{R}}_{\boldsymbol{mN}}$, and $\overline{\boldsymbol{C}}_{\boldsymbol{mN}}$ are the system matrices

As can be seen in the equations of the obtained model, the *state* Equation (24) and the *output* Equation (25), the final equivalent model is a state-space discrete model, with input disturbances, and formally linear. However, observing Equation (22), it is immediate to deduce that the *system matrices* ($\overline{\boldsymbol{A}}_{\boldsymbol{mN}}$, $\overline{\boldsymbol{B}}_{\boldsymbol{mN}}$, $\overline{\boldsymbol{D}}_{\boldsymbol{mN}}$, $\overline{\boldsymbol{R}}_{\boldsymbol{mN}}$, and $\overline{\boldsymbol{C}}_{\boldsymbol{mN}}$) depend on time, since they depend on $\boldsymbol{\beta_j}(\boldsymbol{x_a}(k))$, and $\boldsymbol{x_a}(k)$ depends directly of k. That is: $\overline{\boldsymbol{A}}_{\boldsymbol{mN}} = \overline{\boldsymbol{A}}_{\boldsymbol{mN}}(\boldsymbol{x_a}(k)) = \overline{\boldsymbol{A}}_{\boldsymbol{mN}}(k)$, $\overline{\boldsymbol{B}}_{\boldsymbol{mN}} = \overline{\boldsymbol{B}}_{\boldsymbol{mN}}(\boldsymbol{x_a}(k)) = \overline{\boldsymbol{B}}_{\boldsymbol{mN}}(k)$, $\overline{\boldsymbol{D}}_{\boldsymbol{mN}} = \overline{\boldsymbol{D}}_{\boldsymbol{mN}}(\boldsymbol{x_a}(k)) = \overline{\boldsymbol{D}}_{\boldsymbol{mN}}(k)$, $\overline{\boldsymbol{R}}_{\boldsymbol{mN}} = \overline{\boldsymbol{R}}_{\boldsymbol{mN}}(\boldsymbol{x_a}(k)) = \overline{\boldsymbol{R}}_{\boldsymbol{mN}}(k)$, $\overline{\boldsymbol{C}}_{\boldsymbol{mN}} = \overline{\boldsymbol{C}}_{\boldsymbol{mN}}(\boldsymbol{x_a}(k)) = \overline{\boldsymbol{C}}_{\boldsymbol{mN}}(k)$. Therefore, the equivalent state-space model just described is time-varying and, more specifically, it is of DLTV type (which is what we want to demonstrate). Furthermore, as a global conclusion of the transformation process, we can highlight that the highly non-linear system under study (the ASP process), initially identified through fuzzy models, has finally been represented by a model in state space, with a linear formal appearance, although with varying coefficients over time.

Remark: The dependence of the *system matrices*, $\overline{\boldsymbol{A}}_{\boldsymbol{mN}}$, $\overline{\boldsymbol{B}}_{\boldsymbol{mN}}$, $\overline{\boldsymbol{D}}_{\boldsymbol{mN}}$, $\overline{\boldsymbol{R}}_{\boldsymbol{mN}}$, and $\overline{\boldsymbol{C}}_{\boldsymbol{mN}}$, with respect to the antecedent vector $\boldsymbol{x_a}(k)$, and therefore also with respect to time, implies that it is not possible to characterize the fuzzy models of the ASP process using a set of unique constant numerical matrices (as is the case of time-invariant linear systems), except for small environments around certain operating points or for small time intervals, in the case of sufficiently slow systems. On the other hand, as a consequence of this dependence of the *system matrices* with respect to $\boldsymbol{x_a}(k)$, in each iteration of the simulation experiments with the FMBPC control strategy, it is necessary to carry out the corresponding update sequence. That is, at the k-th time instant, both $\boldsymbol{x_a}(k)$ and $\boldsymbol{\beta_j}(\boldsymbol{x_a}(k))$ must be updated, as well as, finally, the *system matrices*, $\overline{\boldsymbol{A}}_{\boldsymbol{mN}}(k)$, $\overline{\boldsymbol{B}}_{\boldsymbol{mN}}(k)$, $\overline{\boldsymbol{D}}_{\boldsymbol{mN}}(k)$, $\overline{\boldsymbol{R}}_{\boldsymbol{mN}}(k)$, and $\overline{\boldsymbol{C}}_{\boldsymbol{mN}}(k)$.

State-space DLTV model of ASP process, in matrix form and with the elements of all matrices expressed in a generic symbolic way: Replacing in (24) and (25) the state, input, disturbances and output vectors by their corresponding expressions (Equations (17) and (20)) and the *system matrices* by their corresponding generic symbolic expressions (defined in (23)), we would have, for the specific case of our identified ASP process, the state-space DLTV model in matrix form and with the elements of the system matrices expressed in a generic way, as shown below (which will be useful, as we already said in the paragraph before Equation (23), to simplify the stability analysis of the system and its interpretation):

$$\overbrace{\begin{pmatrix} y_1(k+1) \\ y_2(k+1) \end{pmatrix}}^{\boldsymbol{z_{mN}}(k+1)} = \overbrace{\begin{pmatrix} a & b \\ 0 & f \end{pmatrix}}^{\overline{\boldsymbol{A}}_{\boldsymbol{mN}}} \overbrace{\begin{pmatrix} y_1(k) \\ y_2(k) \end{pmatrix}}^{\boldsymbol{z_{mN}}(k)} + \underbrace{\begin{pmatrix} m & n \\ p & 0 \end{pmatrix}}_{\overline{\boldsymbol{B}}_{\boldsymbol{mN}}} \underbrace{\begin{pmatrix} u(k) \\ u(k-1) \end{pmatrix}}_{\boldsymbol{u_a}(k)} + \underbrace{\begin{pmatrix} c & d \\ g & 0 \end{pmatrix}}_{\overline{\boldsymbol{D}}_{\boldsymbol{mN}}} \underbrace{\begin{pmatrix} d_1(k) \\ d_2(k) \end{pmatrix}}_{\boldsymbol{d}(k)} + \underbrace{\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}}_{\overline{\boldsymbol{R}}_{\boldsymbol{mN}}} \tag{26}$$

$$\overbrace{\begin{pmatrix} y_1(k) \\ y_2(k) \end{pmatrix}}^{\boldsymbol{y_{mN}}(k)} = \overbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}^{\overline{\boldsymbol{C}}_{\boldsymbol{mN}}} \overbrace{\begin{pmatrix} y_1(k) \\ y_2(k) \end{pmatrix}}^{\boldsymbol{z_{mN}}(k)} \tag{27}$$

this formalization being valid for both FM_1 and FM_2 (with $c = 0$ and $g = 0$, for FM_2).

3. Open-Loop Local Stability

We will now assume that for the model represented by Equations (24) and (25), there will be some *equilibrium point*, that is, we will assume that there will be some *steady state*, $\boldsymbol{z_{mNss}}$, for a certain input, $\boldsymbol{u_{ass}}$, for certain values of the disturbances, $\boldsymbol{d_{ss}}$, for a certain *antecedent vector* $\boldsymbol{x_{ass}}$ and consequently, for certain matrices $\overline{\boldsymbol{A}}_{\boldsymbol{mNss}}$, $\overline{\boldsymbol{B}}_{\boldsymbol{mNss}}$, $\overline{\boldsymbol{D}}_{\boldsymbol{mNss}}$, $\overline{\boldsymbol{R}}_{\boldsymbol{mNss}}$ and $\overline{\boldsymbol{C}}_{\boldsymbol{mNss}}$ and with a certain output, $\boldsymbol{y_{mNss}}$. Such a state must satisfy the steady state condition for a discrete-time system, i.e.:

$$\boldsymbol{z_{mN}}(k) = \boldsymbol{z_{mN}}(k+1) = \boldsymbol{z_{mNss}} \tag{28}$$

and on the other hand, it must also satisfy the equations of the model ((24) and (25)), that is:

$$\boldsymbol{z_{mNss}} = \overline{\boldsymbol{A}}_{\boldsymbol{mNss}}\boldsymbol{z_{mNss}} + \overline{\boldsymbol{B}}_{\boldsymbol{mNss}}\boldsymbol{u_{ass}} + \overline{\boldsymbol{D}}_{\boldsymbol{mNss}}\boldsymbol{d_{ss}} + \overline{\boldsymbol{R}}_{\boldsymbol{mNss}} \tag{29}$$

$$\boldsymbol{y_{mNss}} = \overline{\boldsymbol{C}}_{\boldsymbol{mNss}}\boldsymbol{z_{mNss}} \tag{30}$$

Let us now consider a generic state, $\boldsymbol{z_{mN}}(k)$, belonging to the model represented by (24) and (25) and suppose that the disturbances, $\boldsymbol{d}(k)$, are the same or very similar to those corresponding to the steady state, that is:

$$\boldsymbol{d}(k) - \boldsymbol{d_{ss}} \cong \boldsymbol{0} \tag{31}$$

and let us also suppose that the antecedent vector, $\boldsymbol{x_a}(k)$, is close enough to $\boldsymbol{x_{ass}}$ (in its corresponding space), to be able to consider that the system matrices are approximately equal to those of the steady state, that is:

$$\begin{array}{l} \overline{\boldsymbol{A}}_{\boldsymbol{mN}} \cong \overline{\boldsymbol{A}}_{\boldsymbol{mNss}} \\ \overline{\boldsymbol{B}}_{\boldsymbol{mN}} \cong \overline{\boldsymbol{B}}_{\boldsymbol{mNss}} \\ \overline{\boldsymbol{D}}_{\boldsymbol{mN}} \cong \overline{\boldsymbol{D}}_{\boldsymbol{mNss}} \\ \overline{\boldsymbol{R}}_{\boldsymbol{mN}} \cong \overline{\boldsymbol{R}}_{\boldsymbol{mNss}} \\ \overline{\boldsymbol{C}}_{\boldsymbol{mN}} \cong \overline{\boldsymbol{C}}_{\boldsymbol{mNss}} \end{array} \tag{32}$$

The assumption made, $\boldsymbol{x_a}(k)$ close to $\boldsymbol{x_{ass}}$, would imply the following, considering the more general case of the antecedent corresponding to the output $y_1(k+1)$ of the FM_1 model (for the others cases, it would be analogous): for this case, $\boldsymbol{x_a}(k) = (y_1(k), y_2(k), u_1(k), u_2(k), u_3(k),\ u_3(k-1))$, that can be also expressed as $\boldsymbol{x_a}(k) = \left(\boldsymbol{z_{mN}}^T(k), \boldsymbol{d}^T(k), \boldsymbol{u_a}^T(k)\right)$ (see (17)); consequently, the assumption $\boldsymbol{x_a}(k) \cong \boldsymbol{x_{ass}}$ would imply that all variables (state variables, disturbances and the extended input) should be close to the corresponding values associated to the considered stationary state, i.e.,: $\boldsymbol{z_{mN}}(k) \cong \boldsymbol{z_{mNss}}$, $\boldsymbol{d}(k) \cong \boldsymbol{d_{ss}}$ (hypothesis already considered in (31)), and $\boldsymbol{u_a}(k) \cong \boldsymbol{u_{ass}}$. We will assume the fulfillment of these conditions in our case study (for all cases) and will refer to them, in abbreviated form, simply as: *proximity condition* of state $\boldsymbol{z_{mN}}(k)$ with respect to the steady state $\boldsymbol{z_{mNss}}$.

Subtracting now the Equalities (24) and (29), on the one hand, and (25) and (30), on the other, we will obtain the equality relations that are shown in the following equations, by also considering some groupings of terms and some numerical considerations derived from the hypothesis mentioned above ($\boldsymbol{d}(k) - \boldsymbol{d_{ss}} \cong \boldsymbol{0}$ y $\overline{\boldsymbol{R}}_{\boldsymbol{mN}} - \overline{\boldsymbol{R}}_{\boldsymbol{mNss}} \cong \boldsymbol{0}$):

$$\underbrace{(\boldsymbol{z_{mN}}(k+1) - \boldsymbol{z_{mNss}})}_{\Delta \boldsymbol{z_{mN}}(k+1)} = \overline{\boldsymbol{A}}_{\boldsymbol{mN}}\underbrace{(\boldsymbol{z_{mN}}(k) - \boldsymbol{z_{mNss}})}_{\Delta \boldsymbol{z_{mN}}(k)} + \overline{\boldsymbol{B}}_{\boldsymbol{mN}}\underbrace{(\boldsymbol{u_a}(k) - \boldsymbol{u_{ass}})}_{\Delta \boldsymbol{u_a}(k)} + \overline{\boldsymbol{D}}_{\boldsymbol{mN}}\underbrace{(\boldsymbol{d}(k) - \boldsymbol{d_{ss}})}_{\cong \boldsymbol{0}} + \underbrace{(\overline{\boldsymbol{R}}_{\boldsymbol{mN}} - \overline{\boldsymbol{R}}_{\boldsymbol{mNss}})}_{\cong \boldsymbol{0}} \tag{33}$$

$$\underbrace{(\boldsymbol{y_{mN}}(k) - \boldsymbol{y_{mNss}})}_{\Delta \boldsymbol{y_{mN}}(\boldsymbol{k})} = \overline{\boldsymbol{C}}_{\boldsymbol{mN}} \underbrace{(\boldsymbol{z_{mN}}(k) - \boldsymbol{z_{mNss}})}_{\Delta \boldsymbol{z_{mN}}(\boldsymbol{k})} \tag{34}$$

The Equations (33) and (34) describe an *incremental model*, valid for small deviations with respect to a certain operating point, certain steady state taken as a reference (as just explained above), that can be simplified and expressed more compactly as follows (using the hypotheses and the complementary notation included in the same expressions (33) and (34)):

$$\Delta \boldsymbol{z_{mN}}(k+1) = \overline{\boldsymbol{A}}_{\boldsymbol{mN}} \Delta \boldsymbol{z_{mN}}(k) + \overline{\boldsymbol{B}}_{\boldsymbol{mN}} \Delta \boldsymbol{u_a}(k) \tag{35}$$

$$\Delta \boldsymbol{y_{mN}}(k) = \overline{\boldsymbol{C}}_{\boldsymbol{mN}} \Delta \boldsymbol{z_{mN}}(k) \tag{36}$$

and making the following change of notation:

$$\begin{array}{c} \Delta \boldsymbol{z_{mN}}(k+1) \equiv \boldsymbol{x_{inc}}(k+1) \\ \Delta \boldsymbol{z_{mN}}(k) \equiv \boldsymbol{x_{inc}}(k) \\ \Delta \boldsymbol{u_a}(k) \equiv \boldsymbol{u_{inc}}(k) \\ \Delta \boldsymbol{y_{mN}}(k) \equiv \boldsymbol{y_{inc}}(k) \end{array} \tag{37}$$

the incremental model will be finally expressed in the following standard form:

$$\boldsymbol{x_{inc}}(k+1) = \overline{\boldsymbol{A}}_{\boldsymbol{mN}} \boldsymbol{x_{inc}}(k) + \overline{\boldsymbol{B}}_{\boldsymbol{mN}} \boldsymbol{u_{inc}}(k) \tag{38}$$

$$\boldsymbol{y_{inc}}(k) = \overline{\boldsymbol{C}}_{\boldsymbol{mN}} \boldsymbol{x_{inc}}(k) \tag{39}$$

which constitutes a state-space local model, valid for states close to *zero state*, of DLTI type (considering the matrices $\overline{\boldsymbol{A}}_{\boldsymbol{mN}}$, $\overline{\boldsymbol{B}}_{\boldsymbol{mN}}$ and $\overline{\boldsymbol{C}}_{\boldsymbol{mN}}$ as constants, approximately equal to the corresponding matrices associated with the steady state, $\boldsymbol{z_{mNss}}$).

Open-loop local stability: The form of the approximate (local) model that we have just deduced allows us to directly apply the more known and accepted stability criterion (necessary and sufficient condition) for DLTI systems described in the state-space: a DLTI system is *asymptotically stable in the sense of Lyapunov* (internal stability) if and only the eigenvalues of the *state matrix* are strictly within the unit circle. Therefore, taking into account that the state matrix corresponding to our case, $\overline{\boldsymbol{A}}_{\boldsymbol{mN}}$, is triangular (see Equation (23)), the system described by Equations (38) and (39), which we will denote by S_{OL} below, will be stable if and only the absolute value of all the elements of the main diagonal of $\overline{\boldsymbol{A}}_{\boldsymbol{mN}}$ is strictly less than unity, that is:

$$\begin{array}{c} S_{OL} \ \ is\ asymptotically\ stable\ \ the\ eigenvalues\ of\ \overline{\boldsymbol{A}}_{\boldsymbol{mN}}\ are\ all\ strictly\ within\ the\ unit\ circle \\ \to\ for\ particular\ cases\ with\ triangular\ state\ matrix,\ \overline{\boldsymbol{A}}_{\boldsymbol{mN}} = \begin{pmatrix} a & b \\ 0 & f \end{pmatrix},\ we\ would\ have: \\ S_{OL}\ \text{is}\ asymptotically\ stable\ \ |a| < 1\ \wedge\ |f| < 1 \\ and\ in\ our\ case\ study\ (both\ for\ FM_1\ and\ FM_2),\ replacing\ the\ elements\ of\ the\ main\ diagonal\ of\ the \\ \overline{\boldsymbol{A}}_{\boldsymbol{mN}}\ matrix\ by\ their\ respective\ original\ expressions,\ based\ on\ the\ original\ coefficients \\ of\ the\ identified\ Fuzzy\ Models\ (see\ (23)),\ we\ would\ have: \\ S_{OL}\ is\ asymptotically\ stable\ \ |a| = \left| \sum_{j=1}^{mr} \beta_{1j}(\boldsymbol{x_a}) a_{j1} \right| < 1\ \wedge\ |f| = \left| \sum_{j=1}^{mr} \beta_{2j}(\boldsymbol{x_a}) a_{j2}^{*} \right| < 1 \end{array} \tag{40}$$

4. Closed-Loop Local Stability of ASP Process Controlled by FMBPC Law

The objective of this section is to analyze the closed-loop local stability of our ASP process, making use of a certain FMBPC control strategy. We will start from the equations of DLTI local model developed in Section 3 (Equations (38) and (39)) and will replace in them the FMBPC control law deduced in [23], thus obtaining the corresponding closed-loop model.

4.1. FMBPC Control Law

In [23], an analytical and explicit FMBPC control law was deduced following a method that could be considered an extension to the multivariable case of the well-known PFC strategy (already mentioned in the introductory section of this article), one of whose main bases is the so-called *equivalence principle*. In Figures 3 and 4, we summarize the FMBPC strategy implemented in such work and the mentioned principle, respectively:

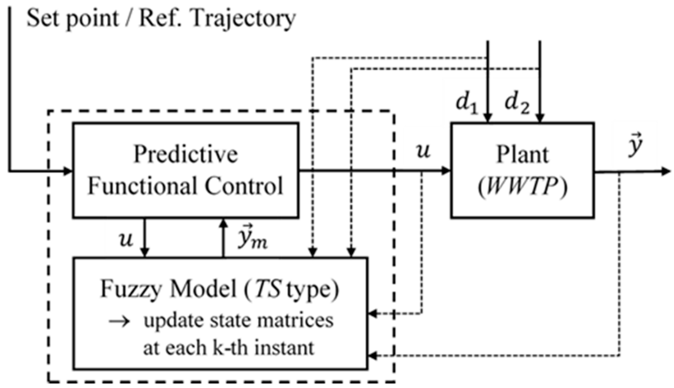


Figure 3. Scheme of the implemented FMBPC.

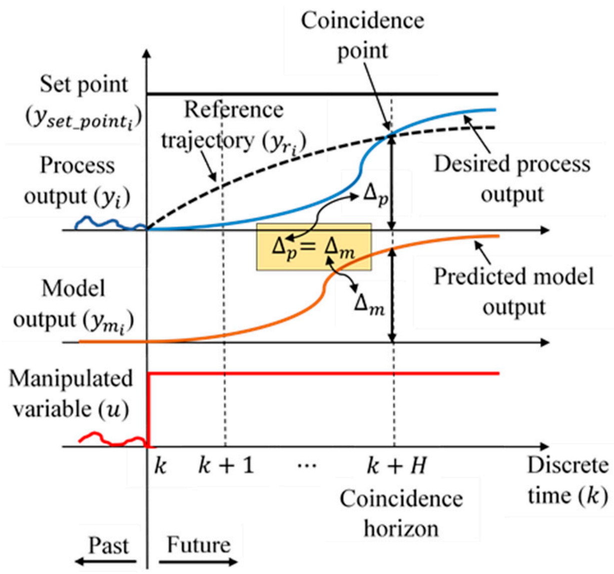


Figure 4. The equivalence principle in predictive functional control (PFC).

The control law was derived by imposing on the outputs of the prediction model the following certain reference trajectories, along the so-called coincidence horizon (H), and applying the equivalence principle to the end of that horizon. The result of the deductive process carried out in [23] was an analytical and explicit expression for the control action $u(k)$ (which is the sludge recirculation flow rate of the ASP process, i.e., $u(k) = q_r(k)$) and also, implicitly, the mathematical expression corresponding to the vector variable $\boldsymbol{u_a}(k)$ (reviewing the mathematical development and making some trivial considerations will suffice).

This control law is the one that has been chosen to be implemented and obtain the closed-loop control system considered in this work. The original form of the law obtained in [23] is a matrix algebraic expression, deduced from a certain global model of predictions in the state space, characterized by a certain *extended state* vector and certain *system matrices*. Consequently, the deduced control law is a function of those *system matrices* and of the aforementioned *extended state* vector. Therefore, in order to use such a law in the local model developed in Section 3 of this article (Equations (38) and (39)), must be first adapted. This local model derives from the global model in the state space obtained in Section 2.3 (Equations (24) and (25)), which is characterized by a *state vector* and *system matrices* different from those of the model used in [23]. However, the two state space global models considered, the one deduced in [23] and the one deduced in Section 2.3 of this article, both of type DLTV, are actually equivalent, since both are transformations of TS fuzzy models of the ASP process (previously identified). Due to this equivalence and also taking into account that the *extended state vector* of the first model includes the *state vector* of the second, it will be possible to transform the expression of the control law into an equivalent one, which is a function of the matrices of the system and of the state vector corresponding to the global model deduced in this article. On the other hand, and prior to this adaptation, the expression corresponding to the extended manipulated input variable, $\boldsymbol{u_a}(k)$, which is really implicit in the development done in [23], must be specified. The original expression of $\boldsymbol{u_a}(k)$ and its subsequent adaptation have been detailed in Appendix B of this article. The resulting final expression for this variable, already adapted and therefore suitable to be substituted in our local open-loop model, is the following (see details of the matrices involved in Appendix B):

$$\begin{gathered}\boldsymbol{u_a}(k) = \boldsymbol{M_{aN}}^{-1}(\boldsymbol{y_r}(k+H) - \boldsymbol{y}(k) + \boldsymbol{y_{mN}}(k) - \overline{\boldsymbol{C}}_{\boldsymbol{mN}}\overline{\boldsymbol{A}}_{\boldsymbol{mN}}{}^{H}\boldsymbol{z_{mN}}(k) - \overline{\boldsymbol{C}}_{\boldsymbol{mN}}\overline{\gamma}\boldsymbol{d}(k) - \overline{\boldsymbol{\lambda_{mN}}}\,) \\ being: \\ \boldsymbol{M_{aN}}^{-1},\ \overline{\boldsymbol{\lambda_{mN}}}:\ matrix\ functions\ of\ system\ matrices;\ \overline{\gamma}:\ submatrix\ of\ order\ 2; \\ H:\ coincidence\ horizon\ (PFC\ concept)\ [H \in \mathbb{Z}^{+},\ H \geq 1]\end{gathered} \tag{41}$$

4.2. Closed-Loop Local Stability Analysis

In the closed-loop stability study, we will also assume the existence of some *equilibrium point* for the open-loop plant model, given for Equations (24) and (25), and we will represent the corresponding *steady state* also by $\boldsymbol{z_{mNss}}$. We will consider that the objective of our control system is precisely to drive the system towards that state, whose output, $\boldsymbol{y_{mNss}}$, must therefore be the reference for the output of the closed-loop system. Furthermore, we will assume that the control strategy will achieve the system reaching this state and, consequently, the entry of the steady state input will be compatible with the control law. Finally, we will suppose that the model described in Equations (24) and (25) is a *perfect model* of the process that it represents, or what is the same, that for each k (k-th instant), the following will be verified: $\boldsymbol{y}(k) = \boldsymbol{y_{mN}}(k)$ and, therefore, $-\boldsymbol{y}(k) + \boldsymbol{y_{mN}}(k) = 0$.

Taking into account now the previous assumptions regarding the perfection of the model and the compatibility of the steady state input with the control law, we can first simplify the expression of the control law, $\boldsymbol{u_{aN}}(k)$ (Equation (41)), and secondly substitute in the resulting expression the values of the different variables and coefficients corresponding to the equilibrium point, obtaining the two following relationships:

$$\begin{gathered}\boldsymbol{u_a}(k) = \boldsymbol{M_{aN}}^{-1}(\boldsymbol{y_r}(k+H) - \overline{\boldsymbol{C}}_{\boldsymbol{mN}}\overline{\boldsymbol{A}}_{\boldsymbol{mN}}{}^{H}\boldsymbol{z_{mN}}(k) - \overline{\boldsymbol{C}}_{\boldsymbol{mN}}\overline{\gamma}d(k) - \overline{\boldsymbol{\lambda_m}}) \\ [H \in \mathbb{Z}^{+},\ H \geq 1]\end{gathered} \tag{42}$$

$$\begin{gathered}\boldsymbol{u_{ass}} = \boldsymbol{M_{aNss}}^{-1}(\boldsymbol{y_{mNss}} - \overline{\boldsymbol{C}}_{\boldsymbol{mNss}}\overline{\boldsymbol{A}}_{\boldsymbol{mNss}}{}^{H}\boldsymbol{z_{mNss}} - \overline{\boldsymbol{C}}_{\boldsymbol{mNss}}\overline{\gamma_{ss}}\boldsymbol{d_{ss}} - \overline{\boldsymbol{\lambda_{mss}}}) \\ [H \in \mathbb{Z}^{+},\ H \geq 1]\end{gathered} \tag{43}$$

The control law expressed in Equation (42) is the one that corresponds to the k-th instant and is a function of the generic state, $\boldsymbol{z_{mN}}(k)$, but also depends on the following variables and parameters: the coincidence horizon, H (which will have been previously chosen); the reference trajectory at $(k+H)$, $\boldsymbol{y_r}(k+H)$; the disturbances, $\boldsymbol{d}(k)$; and the system matrices, which depend on the antecedent vector, $\boldsymbol{x_a}(k)$. We will assume, as in the study of open-loop stability, that the disturbances will be the same or similar to those corresponding to the steady state ($\boldsymbol{d}(k) - \boldsymbol{d_{ss}} \cong \boldsymbol{0}$ or $\boldsymbol{d}(k) \cong \boldsymbol{d_{ss}}$; see Equation (31) and that the antecedent vector, $\boldsymbol{x_a}(k)$, will be close enough (in its corresponding space) to $\boldsymbol{x_{ass}}$, such that the system matrices (and also other matrices that appear in Equation (42) and that are a function of them) are approximately equal to those of the steady state (see Equation (32)). Furthermore, we will assume that what we called in Section 3 as *proximity condition* of state $\boldsymbol{z_{mN}}(k)$ with respect to the steady state $\boldsymbol{z_{mNss}}$, will be fulfilled (with its corresponding implications). Finally, we will assume that both the curvature of the reference trajectories, $\boldsymbol{y_r}(k)$ (function of the parameters a_{r_i} and b_{r_i}, $i = 1,\ 2$), and the value of H, will be adequate (H, large enough) to be able to consider that $\boldsymbol{y_r}(k+H)$ is sufficiently close to the output reference value, which is, as we said, the process output corresponding to the steady state, $\boldsymbol{y_{mNss}}$; that is, we will suppose that it is verified: $\boldsymbol{y_r}(k+H) \cong \boldsymbol{y_{mNss}}$. Considering all these assumptions, if we subtract expressions (42) and (43), we will have the following expression for the increase in the control action (with respect to that of the steady state) or incremental control law:

$$\begin{array}{l} \Delta \boldsymbol{u_a}(k) = -\left(\boldsymbol{M_{aN}}^{-1}\overline{\boldsymbol{C}}_{\boldsymbol{mN}}\overline{\boldsymbol{A}}_{\boldsymbol{mN}}{}^{H}\right)\Delta \boldsymbol{z_{mN}}(k) \\ \text{being :} \\ \quad \Delta \boldsymbol{z_{mN}}(k) = (\boldsymbol{z_{mN}}(k) - \boldsymbol{z_{mNss}}) \\ \text{and :} \\ \quad \boldsymbol{M_{aN}}^{-1} \cong \boldsymbol{M_{aNss}}^{-1} \\ \quad \overline{\boldsymbol{C}}_{\boldsymbol{mN}} \cong \overline{\boldsymbol{C}}_{\boldsymbol{mNss}} \\ \quad \overline{\boldsymbol{A}}_{\boldsymbol{mN}}{}^{H} \cong \overline{\boldsymbol{A}}_{\boldsymbol{mNss}}{}^{H} \end{array} \tag{44}$$

and using the change of notation introduced in Equation (37) (i.e., $\Delta \boldsymbol{z_{mN}}(k) \equiv \boldsymbol{x_{inc}}(k)$ and $\Delta \boldsymbol{u_a}(k) \equiv \boldsymbol{u_{inc}}(k)$), it will be as follows:

$$\boldsymbol{u_{inc}}(k) = -\left(\boldsymbol{M_{aN}}^{-1}\overline{\boldsymbol{C}}_{\boldsymbol{mN}}\overline{\boldsymbol{A}}_{\boldsymbol{mN}}{}^{H}\right)\boldsymbol{x_{inc}}(k) \tag{45}$$

Remark: It should be noted that, formally, the deduced incremental control law coincides with the known state-feedback control law ($\boldsymbol{u} = -\boldsymbol{K}\boldsymbol{x}$), although in our case it was obtained from a previously identified fuzzy model and under the PFC approach.

Once the expression of the incremental control law has been obtained (Equation (45)), the next step will be to substitute it in the state equation of the open-loop incremental DLTI model (Equation (38)) and adequately group terms, obtaining the following:

$$\begin{array}{c} \boldsymbol{x_{inc}}(k+1) = \overline{\boldsymbol{A}}_{\boldsymbol{mN}}\boldsymbol{x_{inc}}(k) - \overline{\boldsymbol{B}}_{\boldsymbol{mN}}\left(\boldsymbol{M_{aN}}^{-1}\overline{\boldsymbol{C}}_{\boldsymbol{mN}}\overline{\boldsymbol{A}}_{\boldsymbol{mN}}{}^{H}\right)\boldsymbol{x_{inc}}(k) \\ = \left(\overline{\boldsymbol{A}}_{\boldsymbol{mN}} - \overline{\boldsymbol{B}}_{\boldsymbol{mN}}\boldsymbol{M_{aN}}^{-1}\overline{\boldsymbol{C}}_{\boldsymbol{mN}}\overline{\boldsymbol{A}}_{\boldsymbol{mN}}{}^{H}\right)\boldsymbol{x_{inc}}(k) \\ where: \\ \overline{\boldsymbol{A}}_{\boldsymbol{mN}} \cong \overline{\boldsymbol{A}}_{\boldsymbol{mNss}},\ \overline{\boldsymbol{B}}_{\boldsymbol{mN}} \cong \overline{\boldsymbol{B}}_{\boldsymbol{mNss}},\ \boldsymbol{M_{aN}}^{-1} \cong \boldsymbol{M_{aNss}}^{-1},\ \overline{\boldsymbol{C}}_{\boldsymbol{mN}} \cong \overline{\boldsymbol{C}}_{\boldsymbol{mNss}},\ \overline{\boldsymbol{A}}_{\boldsymbol{mN}}{}^{H} \cong \overline{\boldsymbol{A}}_{\boldsymbol{mNss}}{}^{H} \end{array} \tag{46}$$

Now we will group in a single matrix the matrix expression that multiplies to $\boldsymbol{x_{inc}}(k)$ in Equation (46), which we will denote with $\overline{\boldsymbol{A}}_{\boldsymbol{mNCL}}$ and we will name as a *closed-loop state matrix* of the incremental model:

$$\overline{\boldsymbol{A}}_{\boldsymbol{mNCL}} = \left(\overline{\boldsymbol{A}}_{\boldsymbol{mN}} - \overline{\boldsymbol{B}}_{\boldsymbol{mN}}\boldsymbol{M_{aN}}^{-1}\overline{\boldsymbol{C}}_{\boldsymbol{mN}}\overline{\boldsymbol{A}}_{\boldsymbol{mN}}{}^{H}\right) \tag{47}$$

and, by analogy, we will also define the following matrix:

$$\overline{\boldsymbol{A}}_{\boldsymbol{mNCLss}} = \left(\overline{\boldsymbol{A}}_{\boldsymbol{mNss}} - \overline{\boldsymbol{B}}_{\boldsymbol{mNss}}\boldsymbol{M_{aNss}}^{-1}\overline{\boldsymbol{C}}_{\boldsymbol{mNss}}\overline{\boldsymbol{A}}_{\boldsymbol{mNss}}{}^{H}\right) \tag{48}$$

being the relationship between the matrices defined in Equations (47) and (48): $\overline{\boldsymbol{A}}_{\boldsymbol{mNCL}} \cong \overline{\boldsymbol{A}}_{\boldsymbol{mNCLss}}$, according to the *proximity specifications* between matrices summarized in Equation (46)). Substituting Equation (47) in Equation (46), we obtain Equation (49) (below) and repeating the output equation of the open-loop incremental DLTI model (Equation (39), in which $\boldsymbol{u_{inc}}(k)$ does not intervene), we have Equation (50) (below). Expressing both together (Equations (49) and (50)), we finally have the following closed-loop incremental DLTI model:

$$\boldsymbol{x_{inc}}(k+1) = \overline{\boldsymbol{A}}_{\boldsymbol{mNCL}}\boldsymbol{x_{inc}}(k) \tag{49}$$

$$\boldsymbol{y_{inc}}(k) = \overline{\boldsymbol{C}}_{\boldsymbol{mN}}\boldsymbol{x_{inc}}(k) \tag{50}$$

which constitutes a state-space local model, valid for states close to *zero state*, of DLTI type (considering the matrices $\overline{\boldsymbol{A}}_{\boldsymbol{mNCL}}$ and $\overline{\boldsymbol{C}}_{\boldsymbol{mN}}$ as constants, approximately equal to the corresponding matrices associated with the steady state, $\boldsymbol{z}_{\boldsymbol{mNss}}$, i.e.,: $\overline{\boldsymbol{A}}_{\boldsymbol{mNCL}} \cong \overline{\boldsymbol{A}}_{\boldsymbol{mNCLss}}$ and $\overline{\boldsymbol{C}}_{\boldsymbol{mN}} \cong \overline{C}_{\boldsymbol{mNss}}$).

Closed-loop local stability: The form of the obtained local DLTI model, given by Equations (49) and (50) and which we will denote by S_{CL} below, will allow us to apply the same stability criterion in the sense of *Lyapunov* (internal stability) that was applied in open-loop analysis. Such criterion will now be expressed as follows:

$$S_{CL} \textit{ is asymptotically stable the eigenvalues of } \overline{\boldsymbol{A}}_{\boldsymbol{mNCL}} \textit{ are all strictly within the unit circle} \tag{51}$$

that is, the asymptotic stability of S_{CL} will depend on the eigenvalues of $\overline{\boldsymbol{A}}_{\boldsymbol{mNCL}} = \overline{\boldsymbol{A}}_{\boldsymbol{mN}} - \overline{\boldsymbol{B}}_{\boldsymbol{mN}}\boldsymbol{M}_{\boldsymbol{aN}}{}^{-1}\overline{\boldsymbol{C}}_{\boldsymbol{mN}}\overline{\boldsymbol{A}}_{\boldsymbol{mN}}{}^{H}$. However, the analysis of the eigenvalues of this matrix expression is not as simple as in the case of the open-loop, both because of the presence of multiple operations between matrices and the dependence on H, and not in a simple way (H is the exponent of a power of matrix base). To try to solve this problem, a computational approach is proposed in the next section.

5. Practical Determination of Closed-Loop Local Stability: A Computational Approach

The purpose of this section is to analyze by computation the closed-loop local stability of our case study, carrying out the study through a direct numerical analysis, which can be considered as an alternative approach to the classical analytical analysis (which is not applicable in a direct way in the case of multivariable fuzzy systems).

The stability analysis will be carried out with the help of the so-called *symbolic calculation* considering generic expressions for the matrices involved in the previously deduced mathematical models of our system. We will use the software tool *Symbolic Math Toolbox*™ belonging to the *Matlab*® programming environment (*The MathWorks Inc.*, Natick, MA, USA).

The procedure consists of determining the position in the plane of the eigenvalues of the closed-loop state matrix, $\overline{\boldsymbol{A}}_{\boldsymbol{mNCL}}$ Equation (47), which depends on the coincidence horizon, H, for a succession of increasing values of this parameter, and afterwards the tendency when H tends to infinity is also determined by means of an inductive process. The position of these eigenvalues will be compared with that of the eigenvalues of the open-loop state matrix, $\overline{\boldsymbol{A}}_{\boldsymbol{mN}}$ (see Equations (22) and (23)), and conclusions will be drawn about the closed-loop stability, taking into account the stability criterion expressed in Equation (51).

Closed-loop stability analysis by means of symbolic computation: We will start from the following generic specifications for the matrices involved in the considered mathematical models, adding the *symbol* suffix to the subscript of the matrices names (consistent with the generic name for the system matrices introduced in Equation (23)):

$$\begin{gathered}\overline{\boldsymbol{A}}_{\boldsymbol{mN\_symb}} = \begin{pmatrix} a & b \\ 0 & f \end{pmatrix};\ \overline{\boldsymbol{B}}_{\boldsymbol{mN\_symb}} = \begin{pmatrix} m & n \\ p & 0 \end{pmatrix};\ \overline{\boldsymbol{D}}_{\boldsymbol{mN\_symb}} = \begin{pmatrix} c & d \\ g & 0 \end{pmatrix}; \\ \overline{\boldsymbol{R}}_{\boldsymbol{mN\_symb}} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix};\ \overline{\boldsymbol{C}}_{\boldsymbol{mN\_symb}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\end{gathered} \tag{52}$$

Of the above matrices, the most significant initially is the open-loop state matrix, $\overline{\boldsymbol{A}}_{\boldsymbol{mN\_symb}}$, whose eigenvalues (with *Matlab* code: $eig(\overline{\boldsymbol{A}}_{\boldsymbol{mN\_symb}})$) are the following:

$$\begin{matrix} a \\ f \end{matrix} \tag{53}$$

and therefore, the open-loop system will be *asymptotically stable* $\Leftrightarrow |a| < 1 \wedge |f| < 1$, as already established in Equation (40).

Next, we will assign increasing values to H and determine, for each value of H, the closed-loop state matrix and its eigenvalues, by means of symbolic calculation. The use of this type of calculation will allow us to carry out the necessary matrix operations (see the original expression of $\overline{\boldsymbol{A}}_{\boldsymbol{mNCL}}$, in Equation (47)) with matrices expressed in generic form, including sums, products, and powers with exponent H ($H \in \mathbb{Z}^+,\ H \geq 1$), even for very large H values, and later determine the eigenvalues of the matrix obtained.

The results obtained using the above cited software tool are (for some selected H values) the following (where $\overline{\boldsymbol{A}}_{\boldsymbol{mNCL\_symb\_Hn}}$ is the, generically expressed, closed-loop state matrix corresponding to case $H = n$ and f is the *second* of the eigenvalues shown in Equation (53), corresponding to the open-loop state matrix, $\overline{\boldsymbol{A}}_{\boldsymbol{mN\_symb}}$):

$\underline{H = 6}$

→ *Matlab* code: $eig\left(\overline{\boldsymbol{A}}_{\boldsymbol{mNCL\_symb\_H6}}\right)$

→ calculated eigenvalues:

$$\begin{array}{c} 0 \\ f \cdot \left(f^4 + f^3 + \cdots + f + 1\right) / \left(f^5 + f^4 + \cdots + 1\right) \end{array} \tag{54}$$

$\underline{H = 10}$

→ *Matlab* code: $eig\left(\overline{\boldsymbol{A}}_{\boldsymbol{mNCL\_symb\_H10}}\right)$

→ calculated eigenvalues:

$$\begin{array}{c} 0 \\ f \cdot \left(f^8 + f^7 + \cdots + f + 1\right) / \left(f^9 + f^8 + \cdots + 1\right) \end{array} \tag{55}$$

$\underline{H = 100}$

→ *Matlab* code: $eig\left(\overline{\boldsymbol{A}}_{\boldsymbol{mNCL\_symb\_H100}}\right)$

→ calculated eigenvalues:

$$\begin{array}{c} 0 \\ f \cdot \left(f^{98} + f^{97} + \cdots + f + 1\right) / \left(f^{99} + \cdots + 1\right) \end{array} \tag{56}$$

$\underline{H = 1000}$

→ *Matlab* code: $eig\left(\overline{\boldsymbol{A}}_{\boldsymbol{mNCL\_symb\_H1000}}\right)$

→ calculated eigenvalues:

$$\begin{array}{c} 0 \\ f \cdot \left(f^{998} + f^{997} + \cdots + f + 1\right) / \left(f^{999} + \cdots + 1\right) \end{array} \tag{57}$$

From observation of the results obtained and reasoning by induction, we can deduce, with respect to the eigenvalues of $\overline{\boldsymbol{A}}_{\boldsymbol{mNCL\_symb\_Hn}}$ (for all $H > 1$, H increasing), that the first is always 0 and that the second fits a sequence whose general term depends on H analytically, according to the following expression:

$$f \cdot \left(f^{H-2} + \cdots + f + 1\right) / \left(f^{H-1} + \cdots + 1\right) \tag{58}$$

That is, the eigenvalues of $\overline{\boldsymbol{A}}_{\boldsymbol{mNCL\_symb\_Hn}}$ are the following:

$$\begin{array}{c} eig_1 = 0 \\ eig_2 = f \cdot \left(\sum_{n=0}^{H-2} f^n\right) / \left(\sum_{n=0}^{H-1} f^n\right) \end{array} \tag{59}$$

where we can observe, for the eig_2, that the summations in the numerator and denominator satisfy (both) the shape of a geometric series (GS_m) of the following type:

$$GS_m = \sum_{n=0}^{m} k \cdot r^n \tag{60}$$

k being a nonzero constant coefficient, r the ratio of the series and m the upper level of the summation (in our case: $k = 1$, $r = f$ and $m = H - 2$ or $m = H - 1$).

The geometric series of the type shown in Equation (60), with $k \in \mathbb{R}$, $k \neq 0$ and ratio $r \in \mathbb{R}$, are convergent (when m tends to infinity) if and only $|r| < 1$ and, if this is verified, its sum will be: $k/(1 - r)$. That is:

$$if\ (k \neq 0\ \wedge\ |r| < 1) \Rightarrow \lim_{m \to \infty} (GS_m) = k \cdot \frac{1}{1 - r} \tag{61}$$

and therefore, in our case: if $|f| < 1$, both the numerator and the denominator of the expression corresponding to the eig_2 eigenvalue, in Equation (59), will tend to $1/(1 - f)$ when H tends to infinity. That is, we will have the following:

$$if\ \ |f| < 1 \Rightarrow \lim_{H \to \infty} (eig_2) = f \cdot \frac{\left(\frac{1}{1-f}\right)}{\left(\frac{1}{1-f}\right)} = f \tag{62}$$

Thus, for the limit case corresponding to H tending to infinity, we will finally have:

$\underline{H \to \infty}$
$\to$ deducted eigenvalues for $\overline{\boldsymbol{A}}_{\boldsymbol{mNCL\_symb\_H\infty}}$:

$$\begin{array}{c} eig_{1\infty} = 0 \\ eig_{2\infty} = f\ (if\ |f| < 1) \end{array} \tag{63}$$

Next, in Table 7, we will summarize the results obtained in the calculation of the eigenvalues corresponding to the closed-loop state matrix, together with the eigenvalues of the open-loop state matrix:

Table 7. Eigenvalues of the state matrices obtained by means of *symbolic computation*.

***eig*($\overline{A}_{mN}$) [*Open-Loop*]**		***H***	***eig*($\overline{A}_{mNCL}$)→ *H Dependent* [*Closed-Loop*]**	
***eig*$_1$**	***eig*$_2$**		***eig*$_1$**	***eig*$_2$**
a	f	6	0	$f \cdot (f^4 + \cdots + 1)/(f^5 + \cdots + 1)$
		10	0	$f \cdot (f^8 + \cdots + 1)/(f^9 + \cdots + 1)$
		100	0	$f \cdot (f^{98} + \cdots + 1)/(f^{99} + \cdots + 1)$
		1000	0	$f \cdot (f^{998} + \cdots + 1)/(f^{999} + \cdots + 1)$
		>1000	0	$f \cdot (f^{H-2} + \cdots + 1)/(f^{H-1} + \cdots + 1)$
		$\to \infty$	0	$f\ (if\ |f| < 1)$

The main characteristic of the trend that we can observe in Table 7 is that, for $H \to \infty$, the eigenvalues of the closed-loop state matrix, $\overline{\boldsymbol{A}}_{\boldsymbol{mNCL}}$, are: 0 and a *second* eigenvalue, f, which coincides with the *second* eigenvalue of the open-loop state matrix, $\overline{\boldsymbol{A}}_{\boldsymbol{mN}}$. Consequently, we can state the following conclusions:

Conclusion 1. *When the coincidence horizon tends towards infinity, $H \to \infty$, one of the two eigenvalues of the closed-loop state matrix is zero and the other coincides with one of the eigenvalues of the open-loop state matrix (the second eigenvalue of (53)) if the absolute value of this eigenvalue is less than 1.*

Conclusion 2. *If the open-loop plant is locally asymptotically stable (around some equilibrium point), then for a large coincidence horizon ($H \to \infty$), the corresponding closed-loop plant will also be locally asymptotically stable (around from that equilibrium point).*

Proof of the Conclusion 2. If the open-loop plant is *locally asymptotically stable*, then all the eigenvalues of the open-loop state matrix will be strictly within the unit circle Equation (40). On the other hand, if H tends towards infinity, the eigenvalues of the closed-loop state matrix will be (Conclusion 1): the zero (which is clearly within the unit circle) and a second eigenvalue, which will match some eigenvalue of the open-loop state matrix with absolute value less than 1, if it exists, and this happens, since its eigenvalues are all within the unit circle, as we just established in the starting hypothesis. Therefore, all the eigenvalues of the closed-loop state matrix will also be within the unit circle and, consequently, taking into account the criterion expressed in Equation (51), we can affirm that the closed-loop plant will also be *locally asymptotically stable*. □

6. Stability Test and FMBPC Experiments

This section is dedicated to the presentation of some practical and experimental tests carried out, showing both their description and their results. First, we carry out a stability test, consisting of the verification of the inductive process developed in the previous section, for a specific numerical example, that is, considering numerical values for the elements of the system matrices. Second, as an experimental test of the behavior of the closed loop control system, we analyze the evolution of the controlled variables in two of the multiple FMBPC simulation experiments carried out in the framework of the research corresponding to the present work.

6.1. Stability Test

To carry out this test, a specific case was chosen, determined by simulation in the vicinity of an equilibrium point, for a certain antecedent vector, and characterized by certain open-loop system matrices, with specific numerical values. From these matrices, we will determine, for each value of the coincidence horizon, H, the corresponding closed-loop state matrix, $\overline{\boldsymbol{A}}_{\boldsymbol{mNCL}}$ (according to expression shown in Equation (47)), expressed with specific numerical values as well. The numerical open-loop system matrices corresponding to our example are the following:

$$\begin{gathered}\overline{\boldsymbol{A}}_{\boldsymbol{mN}} = \begin{pmatrix} 0.8000 & 0.7700 \\ 0 & 0.8800 \end{pmatrix};\ \overline{\boldsymbol{B}}_{\boldsymbol{mN}} = \begin{pmatrix} 0.9000 & 0.5700 \\ 0.8100 & 0 \end{pmatrix}; \\ \overline{\boldsymbol{D}}_{\boldsymbol{mN}} = \begin{pmatrix} 0.2900 & 0.4800 \\ 0.2900 & 0 \end{pmatrix};\ \overline{\boldsymbol{R}}_{\boldsymbol{mN}} = \begin{pmatrix} 0.7900 \\ 0.8800 \end{pmatrix};\ \overline{\boldsymbol{C}}_{\boldsymbol{mN}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\end{gathered} \quad (64)$$

where the most significant matrix for our study is the open-loop state matrix, $\overline{\boldsymbol{A}}_{\boldsymbol{mN}}$, whose eigenvalues ($eig\left(\overline{\boldsymbol{A}}_{\boldsymbol{mN}}\right)$ in *Matlab* code) are:

$$\begin{matrix} 0.8000 \\ 0.8800 \end{matrix} \quad (65)$$

and therefore, since the two eigenvalues are strictly within the unit circle, in this case, the open-loop system would be *asymptotically stable* (see (40)).

Next, we will repeat the systematic calculation process performed in Section 5 (now with numbers), giving increasing values to H and determining, for each value of H ($H = n$), the corresponding closed-loop state matrix, $\overline{\boldsymbol{A}}_{\boldsymbol{mNCL\_Hn}}$ Equation (47), as well as its eigenvalues. The results obtained in the *Matlab*® environment are the following (where we have approximated the numbers using only four decimal places):

$\underline{H = 6}$

→ closed-loop state matrix:

$$\overline{A}_{mNCL\_H6} = \begin{pmatrix} 0.0000 & 0.8682 \\ 0 & 0.7759 \end{pmatrix} \tag{66}$$

→ calculated eigenvalues (Matlab code: $eig(\overline{A}_{mNCL\_H6})$):

$$\begin{matrix} 0.0000 \\ 0.7759 \end{matrix} \tag{67}$$

$\underline{H = 10}$

→ closed-loop state matrix:

$$\overline{A}_{mNCL\_H10} = \begin{pmatrix} 0.0000 & -3.6440 \\ 0 & 0.8336 \end{pmatrix} \tag{68}$$

→ calculated eigenvalues (Matlab code: $eig(\overline{A}_{mNCL\_H10})$):

$$\begin{matrix} 0.0000 \\ 0.8336 \end{matrix} \tag{69}$$

$\underline{H = 100}$

→ closed-loop state matrix:

$$\overline{A}_{mNCL\_H100} = \begin{pmatrix} 0.0000 & -51653.6594 \\ 0 & 0.8799 \end{pmatrix} \tag{70}$$

→ calculated eigenvalues (Matlab code: $eig(\overline{A}_{mNCL\_H100})$):

$$\begin{matrix} 0.0000 \\ 0.8799 \end{matrix} \tag{71}$$

$\underline{H = 1000}$

→ closed-loop state matrix:

$$\overline{A}_{mNCL\_H1000} = \begin{pmatrix} 0.0000 & * \\ 0 & 0.8800 \end{pmatrix} \tag{72}$$

$*$ *very large number (negative)*

→ calculated eigenvalues (Matlab code: $eig(\overline{A}_{mNCL\_H1000})$):

$$\begin{matrix} 0.0000 \\ 0.8800 \end{matrix} \tag{73}$$

$\underline{H > 1000}$

→ the eigenvalues of the closed-loop state matrix coincide with those obtained for $H = 1000$ (Equation (73)), repeating the same result as we continue increasing H

Next, in Table 8, we summarize all the results obtained:

Table 8. Eigenvalues of the state matrices obtained by means of direct computation.

***eig*($\overline{A}_{mN}$) [*Open-Loop*]**		***H***	***eig*($\overline{A}_{mNCL}$)→ *H Dependent* [*Closed-Loop*]**	
***eig*$_1$**	***eig*$_2$**		***eig*$_1$**	***eig*$_2$**
0.8000	0.8800	6	0.0000	0.7759
		10	0.0000	0.8336
		100	0.0000	0.8799
		1000	0.0000	0.8800
		>1000	0.0000	0.8800

From observation of previous calculations, summarized in Table 8, we can conclude that as H increases and tends towards large values, one of the eigenvalues of the closed-loop state matrix, $\overline{\boldsymbol{A}}_{\boldsymbol{mNCL}}$, is 0.0000, in all cases, and the other increases slowly, always with a value less than 1 (and greater than 0), until it reaches the value 0.8800, which coincides with the *second* eigenvalue (the greater) of the open-loop state matrix, $\overline{\boldsymbol{A}}_{\boldsymbol{mN}}$. For our particular numerical example, *Conclusion 1* established in Section 5 has been verified. On the other hand, in our case, we started from a plant that is locally asymptotically stable in open-loop and it has been found that, for sufficiently large values of H, the eigenvalues of the corresponding closed-loop state matrix are all strictly within the unit circle and therefore, according to criterion expressed in Equation (51), the corresponding closed-loop plant will also be locally asymptotically stable. That is, *Conclusion 2* established in Section 5 has also been verified for our particular numerical example.

6.2. FMBPC Control Experiments

Within the framework of this work, many control experiments were carried out applying the considered FMBPC control-law to a simulated ASP process (represented by a continuous model given by differential equations). The objective of these experiments was to test the behavior of our FMBPC control strategy, both for constant reference values and for variable reference values, in the neighborhood of certain operating points in terms of the control performance and the stability of the closed loop system. The simulations were performed using the *MATLAB & Simulink* software environment (*Matlab*®: programming environment of *The MathWorks, Inc.*). The controlled variables are the substrate concentration in the effluent and the biomass concentration in the reactor (s and x, respectively), the only manipulated variable being the sludge recirculation flow-rate (q_r).

The general control objective is to drive the system to an appropriate operating point (s_{ref}, x_{ref}) despite the strong variations in disturbance signals (the input flow-rate of contaminated water, q_i, and the organic substrate concentration in it, s_i). More precisely, the substrate concentration in the effluent must be kept at a certain reference value fulfilling legal regulations, while the biomass concentration approaches some convenient reference values according to biochemical criteria to guarantee a proper purification ability of the plant, avoiding certain inappropriate dynamic behaviors of the purification process (dependent on the work area) among other operational aspects.

In this section, a set of control experiments considering different changes in the biomass concentration reference while keeping the substrate concentration reference at a constant value, are presented. The action of strong disturbances is also considered. The experiments with the FMBPC strategy are structured in three cases: Case 1, Case 2, and Case 3. The performance of the FMBPC control algorithm will be qualitatively studied in terms of disturbances rejection and references tracking from the closed-loop simulation model. The stability of the closed-loop system, with the FMBPC control strategy, can be qualitatively analyzed if the design of the experiment is appropriate and, both the open-loop stability conditions of the plant and the conditions of the controller parameter H, are fulfilled. Consequently, tests to evaluate the performance and stability of the closed-loop control system were developed within an operating range in which the plant is locally open-loop stable and high-enough values of H are chosen.

Classical-PID control Tests: In the reference industrial plant [103], the control algorithm used to control the substrate concentration was the classical-PID algorithm. For this reason, it has been considered that it could be useful to simultaneously carry out tests with a monovariable classical PID control algorithm, for some of the FMBPC cases considered. Specifically, for Cases 1 and 2, two tests with the classical PID control algorithm were carried out: one, focusing the PID to substrate concentration control and the other, focusing the PID to biomass concentration control. The results of these PID tests, as well as a small qualitative study of the stability and performance of this control strategy applied to the control of the ASP process, are also included in this section.

Configuration of the FMBPC experiments and classical-PID control tests: For implementation of the FMBPC strategy, the identified fuzzy models FM_1 and FM_2, described in Section 2.2 and whose identification parameters, as well as their meaning, are shown in Table 2, were used. Three cases of multivariable FMBPC control are presented, with the substrate and biomass being the variables simultaneously controlled. In the first two cases, the same control experiment is also carried out with the classic PID methodology, in duplicate, considering that the controlled variable is either the substrate or the biomass. The information about the configuration, several characteristics, and parameters considered for these FMBPC and PID control experiments is shown in Table 9. In this table, we will refer to the three FMBPC control experiments as Case 1, Case 2, and Case 3 and to the PID control experiments as PID tests (1a and 1b, corresponding to Case 1, and 2a and 2b, corresponding to Case 2). This table contains (from left to right): an identifier (FM_1 or FM_2) for the original fuzzy model of the ASP process, chosen from several models, previously identified; the FMBPC control algorithm design parameters, $\left\{a_{r_j}\right\}_{j=1,2}$ and H, which are the outputs reference trajectories parameters and the coincidence horizon, respectively; the information regarding which are the controlled variables in the case of the FMBPC strategy (the substrate, s, and the biomass, x); an identifier (D_A or D_B) for the chosen input disturbances, which are certain sets of typical values, or logical variations thereof, of q_i and s_i variables that were measured in the real WWTP of reference; the simulation interval (0 to 166 h, in all cases); the reference values, s_{ref} and x_{ref}, of the controlled variables, s and x, respectively; the time instants scheduled for the changes in the biomass reference (*step time*); and, finally, the choice of the variable that will be the control target, that is, the variable controlled, for the implemented classical monovariable PID controller, considering two options: either the substrate (s) or the biomass (x), depending on the PID test.

The reference value s_{ref} is kept constant over time with the same value for all the tests. However, the reference value x_{ref} is changed throughout the entire simulation interval, going from a constant value to another in two different time instants for all the tests (two sequences were planned: one for the two sub-cases of *Case 1* and another for the two sub-cases of *Case 2* and for the *Case 3*). For the PID controller, the reference for the controlled variable (s or x) is the same (s_{ref} or x_{ref}, respectively) that considered for the FMBPC in the corresponding case.

The PID controller tuning was carried out in two phases: a first approximation was obtained by applying the *Ziegler-Nichols* method and later it was adjusted more finely by a trial-and-error procedure to improve the performance of the control system. The PID tuning parameters used were: $k_p = 1$ and $k_i = 0.1$ (in practice, a PI controller) and a bias value for the control signal equal to 775 m^3/h.

Table 9. Configuration of the FMBPC control experiments and classical PID tests.

FMBPC Strategy					Dist.	Simulation Parameters				Class. PID	
		FMBPC Parameters									
Case	FM	Refer. Traject. Param.	H	CV		Simul. interv. (h)	s_{ref} (mg/L)	x_{ref} (mg/L)	ST (h)	PID Test	CV
1	FM_1	$a_{r_1} = 0.76$ $a_{r_2} = 0.96$	6	$(s,\ x)$	D_A	0 *to* 166	55	1800 *to* 2200 2200 *to* 2000	30 80	1*a* 1*b*	(s) (x)
2	FM_2	$a_{r_1} = 0.76$ $a_{r_2} = 0.96$	3	$(s,\ x)$	D_B	0 *to* 166	55	1300 *to* 500 500 *to* 750	30 60	2*a* 2*b*	(s) (x)
3	FM_2	$a_{r_1} = 0.76$ $a_{r_2} = 0.96$	3	$(s,\ x)$	D_A	0 *to* 166	55	1300 *to* 500 500 *to* 750	30 60	–	–

where:

- FM: fuzzy model (FM_1 or FM_2)
- Refer. Traject. Param.: reference trajectories parameters (a_{r_1} and a_{r_2})
- H: coincidence horizon
- CV: controlled variables (s: substrate; x: biomass)
- Dist.: disturbances
- Simul. interv. (h): simulation interval (hours)
- s_{ref}, x_{ref}: substrate and biomass references (respectively)
- ST(h): step time (hours)
- Class. PID: classical PID controller (monovariable)

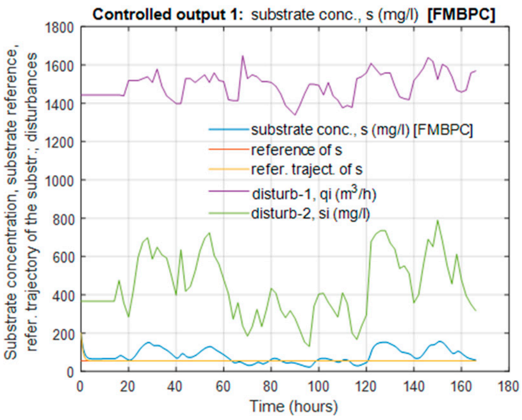
Results of the simulation: The results of the different experiments carried out are shown graphically. For all cases corresponding to FMBPC strategy (Cases 1, 2 and 3), the time evolution of each of the two controlled variables (substrate and biomass) is represented, separately, together with their corresponding references and reference trajectories. The two input disturbances are also included (in the graphs of the two controlled variables) to show the magnitude of the values that the controller must compensate. The time evolution of the control variable (sludge recirculation flow-rate) is also represented in a separate graphic. The objective of all the representations is to show the evolution of the controlled variables in terms of disturbance rejection and reference tracking.

Furthermore, for cases 1 and 2 and for each controlled variable, the temporal evolution corresponding to the FMBPC strategy and that corresponding to the classical monovariable PID control algorithm (in its two modalities or tests: substrate control and biomass control) have been represented jointly, although avoiding showing the input disturbances in these graphs, so as not to impair the observation of the controlled variables. The time evolutions of the control variables corresponding to both strategies have also been shown jointly, but in different graphs than those of the controlled variables.

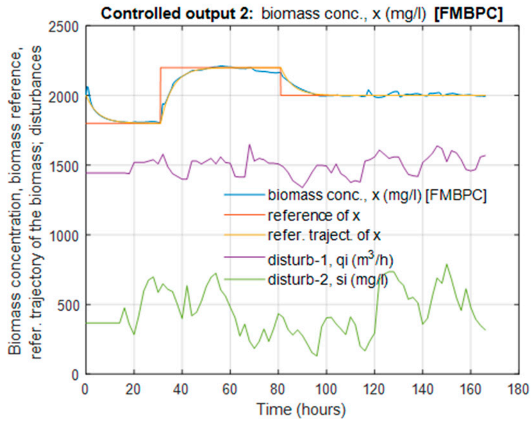
The graphical results obtained are shown below, organized according to the different cases and tests specified in Table 9:

6.2.1. FMBPC—Case 1 and PID-Tests (1a and 1b)

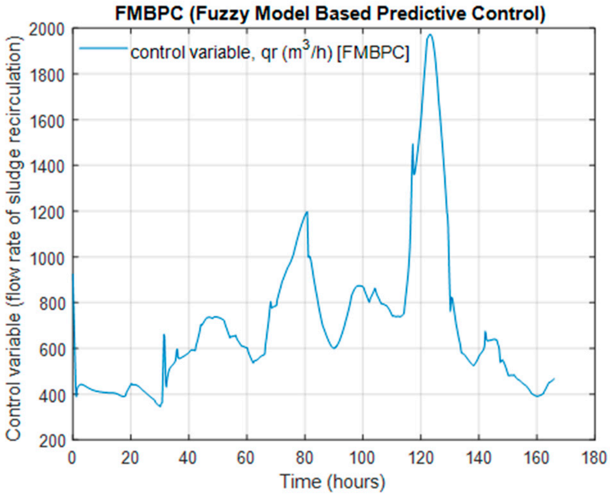
Case 1 is characterized by the use of the fuzzy model FM_1, the disturbance pattern D_A, and a certain sequence of changes in the biomass concentration reference signal, within an operation range between 1800 y 2200 (mg/L). The remainder parameters of this experiment are shown in Table 9. The output responses of the system controlled by the FMBPC strategy, as well as corresponding to the two tests performed with the PID (*PID-test 1a* and *PID-test 1b*), are presented below in Figures 5–7, respectively:



(**a**) Substrate concentration in the effluent and disturbances.

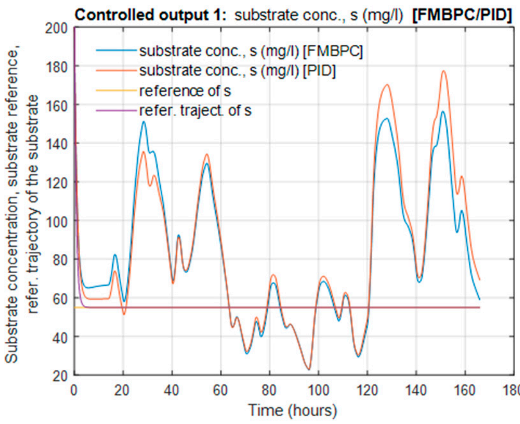


(**b**) Biomass concentration in the reactor and disturbances.

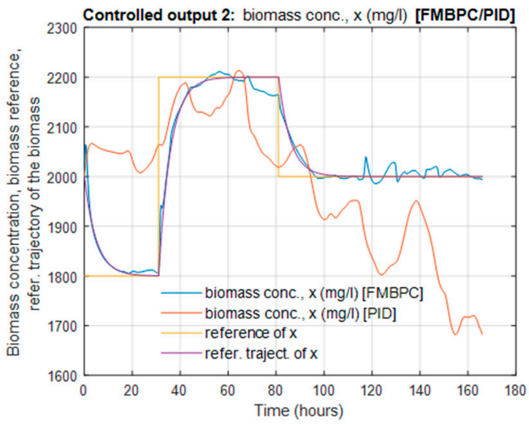


(**c**) Control variable calculated using the FMBPC algorithm.

Figure 5. FMBPC—Case 1.

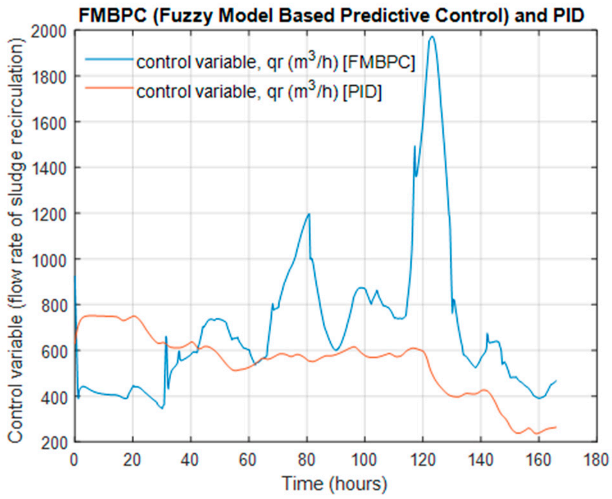


(**a**) Substrate concentration in the effluent.



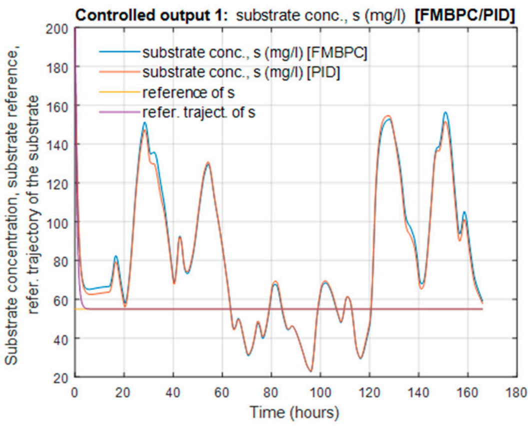
(**b**) Biomass concentration in the reactor.

Figure 6. *Cont.*

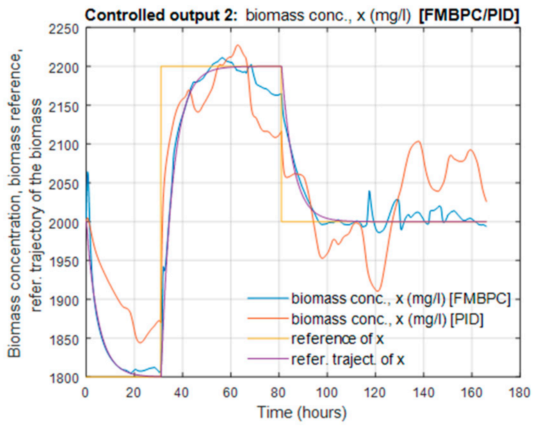


(**c**) Control variables calculated using both the FMBPC algorithm and the PID algorithm.

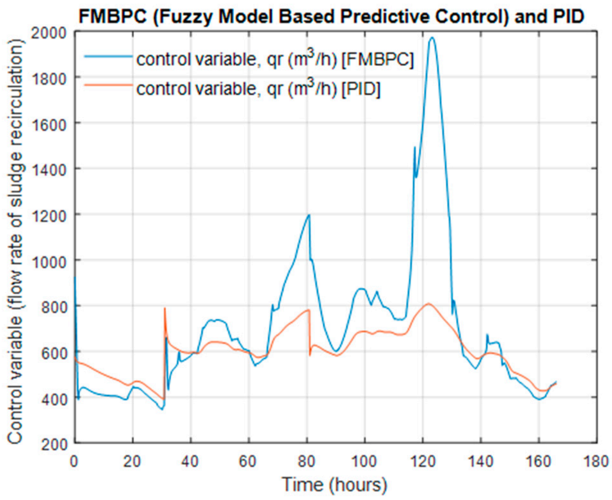
Figure 6. FMBPC—Case 1 and PID-test 1a.



(**a**) Substrate concentration in the effluent.



(**b**) Biomass concentration in the reactor.



(**c**) Control variables calculated using both the FMBPC algorithm and the PID algorithm.

Figure 7. FMBPC—Case 1 and PID-test 1b.

In Figure 5 shown above, the behavior of the ASP process controlled by using the FMBPC strategy is represented, for Case 1. From the observation of the graphic representations presented in this figure, we can conclude that the values of the substrate concentration along the time remain relatively close to their set points and, at the same time, the values of the biomass concentration follow quite closely the jump changes in the corresponding reference signal, following very precisely the predetermined reference trajectory. Furthermore, it can be noted that the control variable varies adequately to reject the disturbances, with moderate control efforts, except in certain cases, in which the two disturbances change simultaneously and abruptly. Moreover, stable behavior of the closed-loop system is observed, despite the strong changes in the biomass reference.

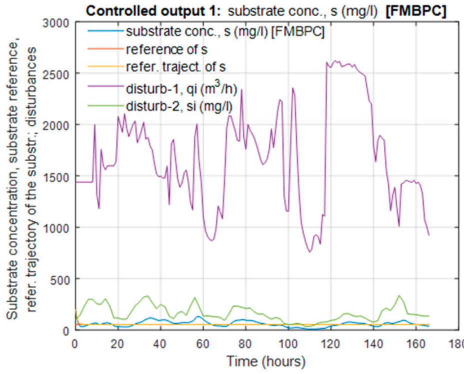
Figure 6 shows the time evolution of the two outputs of the ASP process, when it is controlled with two strategies: the FMBPC algorithm (Case 1) and the classic PID control designed, in this particular case, to control the substrate concentration. We can observe that the values of the substrate concentration produced when the PID algorithm is implemented are a little higher than the values of the substrate concentration produced by the FMBPC strategy, but in both cases, the dynamic response is similar. Note that, the tracking of the biomass reference is not guaranteed with this particular PID control while the FMBPC strategy (Figure 6b) allows us to do so. Specifically, it is observed that, during the last time-interval, the biomass values were significantly lower than the corresponding reference signal, what is not recommendable from an operational point of view in this type of plant. This can be avoided when implementing FMBPC because this strategy uses a model that simultaneously considers the dynamic behavior of both the substrate and the biomass with respect to the manipulated variable, as well as the interaction between both variables, and consequently is capable of adequately controlling the substrate, while maintaining the biomass at values prescribed by your reference signal (that is, it allows both variables to be controlled simultaneously).

In Figure 7, the responses of the ASP process controlled using two strategies are shown together in the same graphic representation. Particularly, the FMBPC algorithm (Case 1) is considered again, and a classic PID oriented, in this case, to the biomass concentration control (the PID changes its target variable from control). In this test, we can observe better PID behavior compared to the previous test (Figure 6). The values of the substrate concentration with the PID are very similar to those of the FMBPC, both with regard to the maximum values and the evolution over time. Regarding the values of the biomass concentration, the PID achieves much more acceptable levels over time than in the previous test, following quite well the selected changes in the reference signal, although the tracking capability is less than that obtained with the FMBPC strategy. By comparing the control actions of both strategies, we see again that the PID control actions are not able to make the necessary efforts to achieve the biomass concentration reference values properly in the face of high disturbances action (while maintaining the substrate concentration values around the reference values, acceptably). In the case of FMBPC, the biomass concentration accurately follows the reference trajectory. The multivariable control objective is successfully achieved with our strategy, even using a single manipulated variable, the sludge recirculation flow-rate.

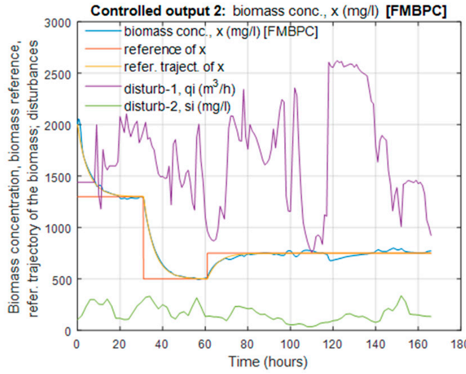
Note, in the case of the PID, that a better tuning (more aggressive) would allow better tracking of the biomass reference but causing greater variability in the values of the substrate concentration, with the risk of even exceeding values not allowed by environmental regulations. This is to be expected because the PID does not incorporate information on the interaction between substrate and biomass. Consequently, the tuning implemented for the PID was the result of the search for an equilibrium in the variations of both variables, placing the accent on the fulfillment of the substrate restrictions.

6.2.2. FMBPC—Case 2 and PID-Tests (2a and 2b)

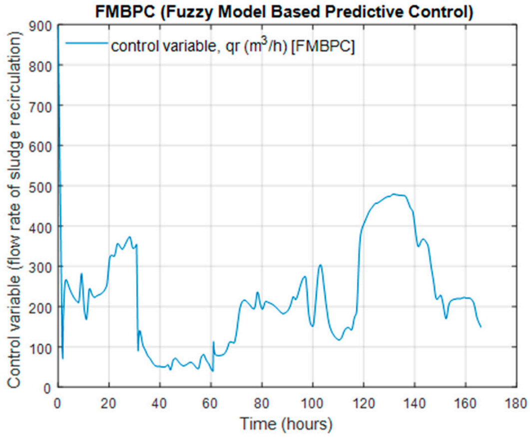
Case 2 is characterized by the use of the fuzzy model FM_2, the disturbance pattern D_B, and a particular sequence of changes in the reference of the biomass concentration, within an operation range between 500 × 1300 (mg/L). The remainder parameters of this experiment are shown in Table 9. The responses of the system controlled by the FMBPC strategy, as well as the corresponding to the two tests performed with the PID (*PID-test 2a* and *PID-test 2b*), are presented below in Figures 8–10, respectively:



(**a**) Substrate concentration in the effluent and disturbances.

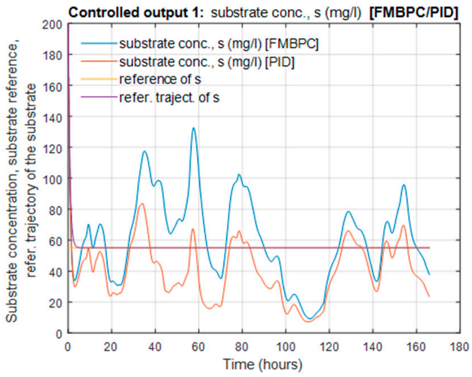


(**b**) Biomass concentration in the reactor and disturbances.

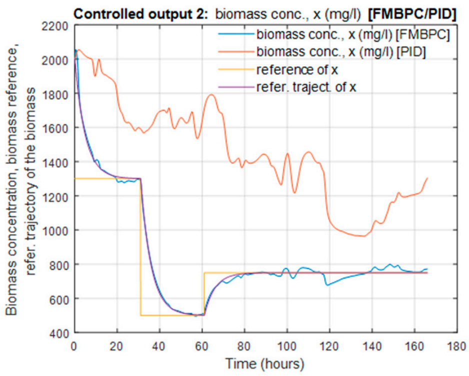


(**c**) Control variable calculated using the FMBPC algorithm.

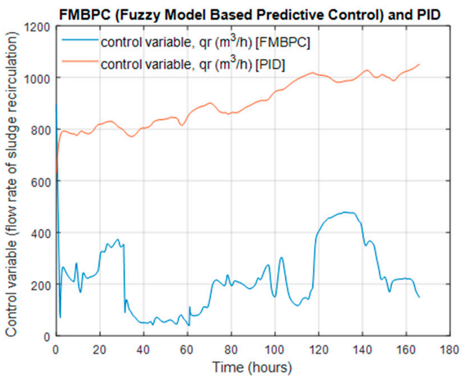
Figure 8. FMBPC—Case 2.



(**a**) Substrate concentration in the effluent.

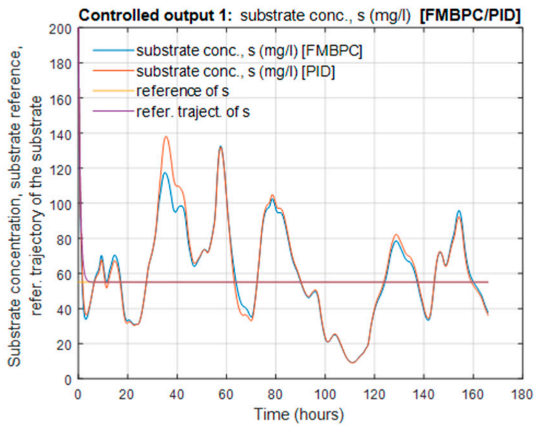


(**b**) Biomass concentration in the reactor.

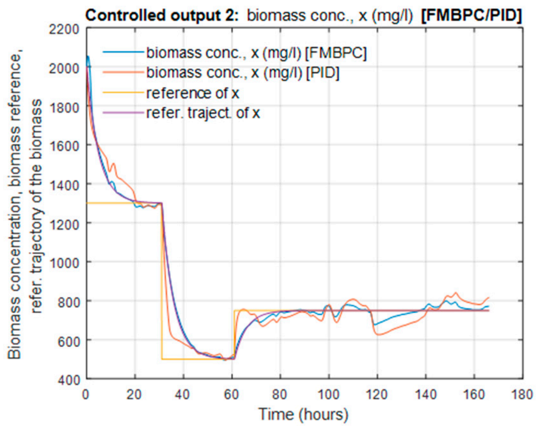


(**c**) Control variables calculated using both the FMBPC algorithm and the PID algorithm.

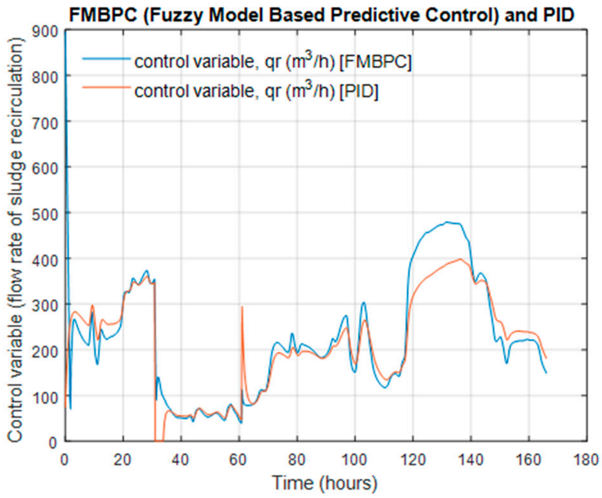
Figure 9. FMBPC—Case 2 and PID-test 2a.



(**a**) Substrate concentration in the effluent.



(**b**) Biomass concentration in the reactor.



(**c**) Control variables calculated using both the FMBPC algorithm and the PID algorithm.

Figure 10. FMBPC—Case 2 and PID-test 2b.

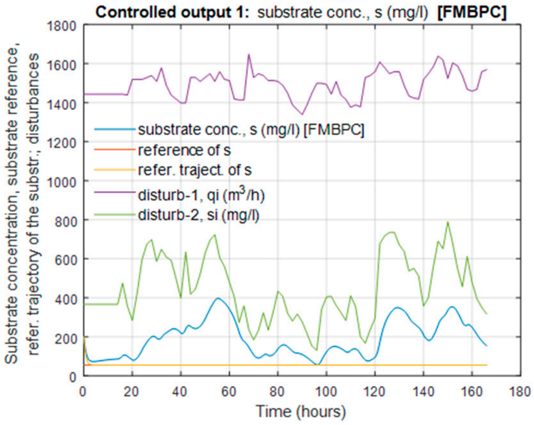
In Figure 8, shown above, the behavior of the ASP process controlled by using the FMBPC strategy is represented, for Case 2. From the observation of the time evolution of the two outputs represented in this figure, we can conclude that, as well as for Case 1, the values of the substrate concentration remain relatively close to its reference signal despite the disturbance action and, at the same time, the biomass concentration values follow quite fast the abrupt jumps in the corresponding reference variable, following very accurately the predefined reference trajectory. Moreover, it can be observed that the changes in the control variable and the control efforts to reject disturbances are softer than for Case 1 all the time. Note that, although the input flow variations are fairly large and, besides, are subjected to strong oscillations, the control system presents a very good disturbance rejection capability. This fact can also be observed by comparing the profile of the control signal time evolution and the input flow profile, particularly at times when the changes in the input flow are the largest ones. It must be pointed out that the control actions respond very quickly after each oscillation. It can also be observed that the closed-loop system keeps a stable behavior in the presence of strong changes in the biomass reference. A conclusion from this second case study, with important differences with respect to the first (in the fuzzy model, in the disturbances and in the operating zone), can be drawn: the FMBPC strategy also allows to obtain very good performance, in terms of disturbance rejection and reference tracking, while ensuring stable behavior of the closed-loop system.

In Figure 9, the responses of the ASP process provided by the two control strategies are shown together: the FMBPC (Case 2) and the classic PID oriented to control the substrate concentration, respectively. In this case, we observe that the values of the substrate concentration provided by the PID control algorithm are lower than the values of the substrate concentration corresponding to the FMBPC strategy. Now, if we look at the graphic showing the time evolution of the biomass concentration (Figure 9b), a tracking capability of its reference trajectory fairly optimal is observed when the FMBPC strategy is used, while with the PID algorithm, the biomass reference tracking cannot be ensured, as shown in the figure, and was expected. It must be noted at this point that the PID is a monovariable control law and, particularly for this test, it was designed with the aim of controlling just the substrate concentration. Nevertheless, it must be highlighted that the FMBPC strategy accomplishes the multivariable control objectives very well, keeping the two controlled variables, the substrate and the biomass concentrations, close to their reference values using just one manipulated variable. The reason why in this test the substrate values are slightly higher using the FMBPC than using PID control is clear and a logical consequence of having forced the biomass to follow too low values, so that the substrate cannot be reduced as much as it was expected. This fact can be sort out by setting higher values of the biomass reference as in Case 1. An important conclusion is drawn from this test: a compromised selection of the two reference variables is needed to keep the biomass at high-enough levels to ensure a good purification process of the water, while keeping the substrate at low-enough values according to its reference signal and legal specifications.

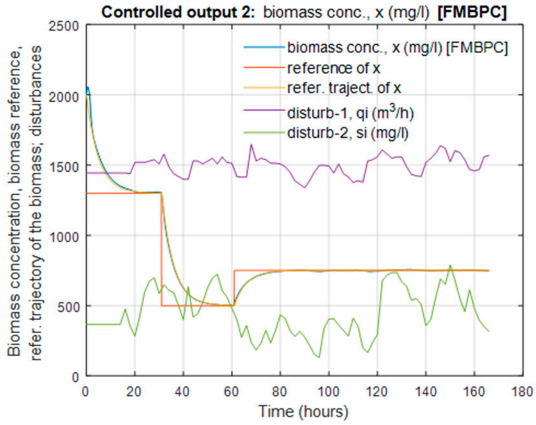
In Figure 10, the responses of the ASP process controlled using the two strategies are shown in the same graphic. Particularly, the FMBPC algorithm corresponding to Case 2 is implemented again but, this time, a classic PID oriented to biomass concentration control is considered (the PID changes its target variable from control). Regarding the time evolution of biomass concentration (Figure 10b), we can observe a better PID behavior compared to the previous test (Figure 9). Since the biomass is now the controlled variable for the PID control, we can observe that the closed-loop system has quite a good capability to follow the changes in the biomass reference. On the other hand, we can observe a similar reduction in substrate concentration for both strategies (Figure 10a), although a little more favorable to FMBPC. Finally, it is also interesting to observe the control actions of both strategies (Figure 10c). Roughly speaking, they are quite similar; nevertheless, some of the differences are interesting. Firstly, the PID controller presents a greater number of situations in which the control efforts are larger, which could be bad for the actuator. Secondly, the operation under the PID control presents saturation in the control signal, with values equal to zero during a period of several hours (in the time interval between 20 h and 40 h of simulation time), a circumstance that does not occur in the case of the FMBPC strategy.

6.2.3. FMBPC—Case 3

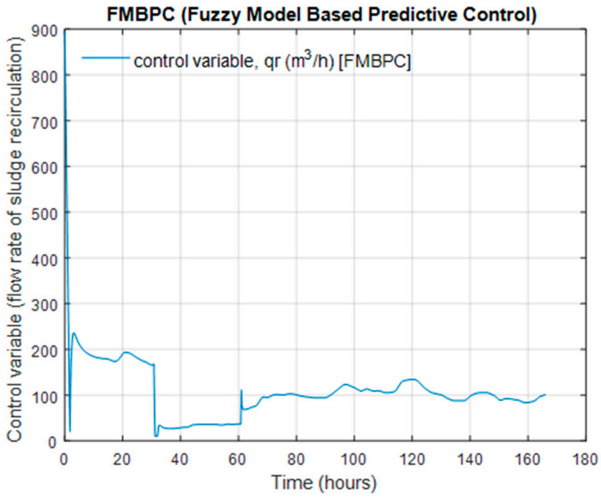
Case 3 is basically characterized by the use of the fuzzy model FM_2, the disturbance pattern D_A, and a particular sequence of changes in the reference of the biomass concentration, within an operation zone between 500 y 1300 (mg/L). The remainder parameters can be seen in Table 9. The configuration of this case is the same as Case 2, except for the disturbance pattern, which is coincident with that of Case 1. Precisely, the objective of the experiments in this case is to study the robustness of the FMBPC strategy by using a pattern of disturbances and operating conditions different from the ones used for the fuzzy model identification considered in the experiments. It must be taken into account that fuzzy models are identified with certain patterns of disturbances and in certain areas of operation and therefore, although the ranges of the values for different variables in the identification are wide, the validity of the models will be higher for such specific areas and will decrease when we move to remote ones. The responses of the system controlled by the FMBPC strategy are shown below, in Figure 11:



(**a**) Substrate concentration in the effluent and disturbances.



(**b**) Biomass concentration in the reactor and disturbances.



(**c**) Control variable calculated using the FMBPC algorithm.

Figure 11. FMBPC—Case 3.

Figure 11, shown above, shows the behavior of the ASP process controlled with the FMBPC strategy for Case 3. From observation of the figure, we can conclude that, as for Case 1 and Case 2, the substrate concentration values remain relatively close to the reference and, at the same time, the biomass concentration values also change accordingly the jumps in reference signal presents, following, especially in this case, an accurately predetermined reference trajectory. From a more detailed analysis of Figure 11, it can be noted that the substrate levels in Case 3 are higher than the corresponding ones in Case 2, due to the fact that the variations of the substrate in the inflow water for Case 3 are much greater than the corresponding ones for Case 2. Furthermore, although the control actions should compensate for the excess substrate by providing an adequate sludge recirculation flow-rate, as previously reasoned, this action is compromised by the requirement to follow certain low reference values for biomass (multivariable control). Regarding the control variable, it is observed that it acts adequately to reject the disturbances, as in Case 1 and Case 2, with moderate control efforts in general, although it requires greater efforts in some specific moments (initially and in the reference changes of biomass). Finally, we can observe that the sludge recirculation flows used to control the two variables are different (lower) in Case 3 than in Case 2. This is due to the differences between the patterns of disturbances in both cases.

Overall analysis of the results: From the observation of Figures 5, 8 and 11, we can conclude that our FMBPC closed-loop multivariable control system is stable in the face of reference changes, presenting a satisfactory reference tracking capability. In all cases studied, the FMBPC-strategy reacts adequately to steps in the biomass reference, in such a way that this variable follows its reference trajectory fairly closely and, at the same time, the substrate remains around its reference value, with acceptable deviations despite the strong disturbances present at the WWTP input. This behavior occurs for different patterns of disturbances (all with many oscillations), different steps sequences of the biomass reference and different operating points of the ASP process (the variation ranges of the references of output variables were: $s_{ref} = 55$ (mg/L), for the substrate, and $x_{ref} \in [500,\ 1300]$ (mg/L) or $x_{ref} \in [1800,\ 2200]$ (mg/L), for the biomass).

The other graphic representations (Figures 6, 7, 9 and 10) show the evolution of the different variables both for the FMBPC strategy and for the PID, simultaneously. Looking at these figures, we can deduce that both strategies are stable within the considered operating range. Regarding the performance of both control algorithms, the performance of the FMBPC algorithm is, in general, better than that of the monovariable PID algorithm, this difference being much greater in the cases in which the PID is oriented to substrate concentration control and not to the biomass concentration control. In these cases, the control of the substrate concentration with both strategies is similar, while the control of the biomass concentration with the FMBPC algorithm is much better than with the PID, as is logical, since the FMBPC strategy is multivariable and is oriented to simultaneously control both substrate and biomass. For the cases in which the PID is oriented to biomass concentration control, the differences are smaller, but the performance of the FMBPC algorithm continues to be better, mainly concerning biomass concentration control (in relation to substrate concentration control, the performances of both algorithms are closer). As mentioned above, the performance of the PID in relation to the tracking of the biomass reference could be improved, but at the cost of increasing variations in substrate concentration, with the consequent risk of not complying with environmental restrictions.

As a summary, we can highlight that the FMBPC multivariable advanced control strategy considered in this work would be competent to address the control of the substrate concentration in the effluent (reduction of contamination) in WWTP with ASP processes, in a stable way and with a performance similar or better to that of a classical PID, and simultaneously, also control the biomass concentration in the reactor, with high performance.

Remark: It is necessary to remember again here that the classical PID control algorithm was the technique implemented in the reference industrial plant [103], whose simulating model together with real data of the disturbance records are used in this work. For this reason, it has also been studied here. Being aware that the comparison with the FMBPC is not fair according to the different degrees of complexity of both approaches, the aim is just to show how the performance of the plant can be improved by using advanced control techniques and to encourage their use in a real environment.

Comparison with other more advanced PID techniques is also possible and recommendable. For instance, PID methods that consider a feedforward action to take into account the effect of measurable disturbances (see [105]) could be appropriate. In addition, comparison could be made with multivariable PID control or fuzzy PID control, and with other controllers using explicit control laws (as in the case of our FMBPC strategy). However, a controller benchmarking is out of the scope of this paper, although it could be very interesting for future research.

7. Conclusions

In this paper, a computational approach of closed-loop stability analysis of a specific FMBPC control strategy, applied to an activated sludge biological process, was presented. The original model used for the predictions is a TS type fuzzy discrete model, previously identified and later formalized in an equivalent DLTV state-space model. Based on this model but considering a certain generic steady-state as the operating point, in this work we have deduced, for analysis purposes, a local incremental state-space model of the DLTI type, valid for states close enough to steady-state (that is, for small variations or increases with respect to the steady-state). The great advantage of working with models of that type is the possibility of applying the existing stability criteria for DLTI models, which are well defined and widely used, and demonstrating their compliance for the chosen case study. However, the demonstration procedures of compliance with such stability criteria for systems with complex dynamics, as is our case study, are often mathematically quite laborious and also difficult to generalize. For this reason, in the present work, it was decided to carry out stability analysis using a computational approach, alternative to the usual procedures existing in the literature.

The solution adopted consists of determining, first, the generic closed-loop state matrix (with algebraic variables) and its eigenvalues, as a function of the coincidence horizon H, by means of symbolic numerical calculation and an induction process (considering H, increasing). Second, deduce under what conditions the eigenvalues will be within the unit circle and consequently (stability criteria) when the closed-loop plant will be asymptotically stable and its relationship with the stability of the open-loop plant. The result is satisfactory, in the sense that by means of this procedure it has been possible to establish an important conclusion regarding closed-loop (local) stability, which is the following: if the open-loop plant is (locally) asymptotically stable, then for values large enough of H, the closed-loop plant will also be (locally) asymptotically stable.

As a final assessment, we consider that the work carried out constitutes an appreciable contribution in the field of stability analysis of FMBPC control systems, for two reasons. On the one hand, due to the simplicity of the method developed, in relation to other methods, most of them are more complicated. On the other hand, it has been considered a more complex case study and more difficult to approach than those chosen in previous similar works (SISO systems with not very complicated dynamics). In the present work, a useful conclusion regarding the stability of FMBPC control systems has been deduced, for a case study of a multivariable nature (MIMO system), highly nonlinear, with complex dynamics (biological system), and subject to strong disturbances.

As possible future research works, we propose the following: to generalize the proposed FMBPC methodology for any MIMO system and to integrate it within a *Closed-loop Predictive Control* scheme to manage input and output constraints, considering at the same time the stability analysis of the resulting system. This will allow the use of the proposed methodology for the control of more complex systems and for complete plant control (*Plantwide Control*).

Author Contributions: Design and approach of the general research line, P.M.V.L. and P.V.; research and development tasks, P.M.V.L.; control algorithm design, stability analysis approach and symbolic calculation, P.M.V.L.; software and simulations, P.M.V.L.; analysis and interpretation of simulations and results, P.M.V.L. and P.V.; writing—review and editing, P.M.V.L.; supervision, P.V. All authors have read and agreed to the published version of the manuscript.

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Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

Membership Functions

Table A1. Antecedent fuzzy sets and membership functions of the identified FM_1 fuzzy model.

Membership Functions of *piece-wise Exponential* Type:

$$\mu_{A_{jp}}(x) \text{ or } \mu_{A^*_{jp}}(x) = \begin{cases} exp(-(\frac{x-c_l}{2\omega_l})^2) & (if\ x<c_l) \\ exp(-(\frac{x-c_r}{2\omega_r})^2) & (if\ x>c_r) \\ 1 & (otherwise) \end{cases}$$

[Parameters: $c_l, c_r, \omega_l, \omega_r$]

ASP Process Outputs	Antecedent Vector Components $x_a(k)$	Rules R_j, R^*_j	Fuzzy Sets A_{jp}, A^*_{jp}	c_l	c_r	ω_l	ω_r
$y_1(k) = s(k)$, substrate	$y_1(k-1)$	R_1	A_{11}	−57.90	−57.90	66.51	100.82
		R_2	A_{21}	54.89	55.01	55.01	64.11
		R_3	A_{31}	−7.96	34.82	51.13	234.52
		R_4	A_{41}	−13.63	37.23	99.38	244.74
		R_5	A_{51}	68.76	73.27	74.27	83.19
		R_6	A_{61}	−32.93	144.60	269.57	269.57
	$y_2(k-1)$	R_1	A_{12}	182.44	2005.11	2057.60	3294.68
		R_2	A_{22}	1988.31	2000.00	2000.00	2011.60
		R_3	A_{32}	1614.02	1748.47	1798.43	2495.98
		R_4	A_{42}	1537.75	1783.24	1878.23	3373.81
		R_5	A_{52}	1276.69	1276.69	1697.58	1745.50
		R_6	A_{62}	1576.80	1813.15	2792.79	2792.79
	$u_1(k-1)$	R_1	A_{13}	260.02	260.02	1365.00	2870.08
		R_2	A_{23}	1285.78	1300.00	1300.00	1318.16
		R_3	A_{33}	−97.77	1520.00	1568.00	2702.66
		R_4	A_{43}	−1196.17	1800.00	1800.00	3032.13
		R_5	A_{53}	1756.45	1800.00	1800.00	1900.50
		R_6	A_{63}	23.29	1881.99	2739.95	2739.95
	$u_2(k-1)$	R_1	A_{14}	−199.86	−199.86	403.51	827.83
		R_2	A_{24}	363.10	366.67	366.67	381.52
		R_3	A_{34}	−75.60	230.14	296.95	1104.52
		R_4	A_{44}	−200.89	317.33	317.33	1591.24
		R_5	A_{54}	315.58	317.33	317.33	324.90
		R_6	A_{64}	−1.53	641.98	1120.67	1120.67
	$u_3(k-1)$	R_1	A_{15}	−843.67	1200.00	1200.00	4184.72
		R_2	A_{25}	−200.00	−200.00	570.00	598.83
		R_3	A_{35}	−153.92	1000.00	2200.00	2200.00
		R_4	A_{45}	−200.00	−200.00	600.00	2553.40
		R_5	A_{55}	548.75	600.00	600.00	1146.67
		R_6	A_{65}	−430.47	995.00	995.00	1008.62
	$u_3(k-2)$	R_1	A_{16}	−805.16	1199.96	2200.00	2200.00
		R_2	A_{26}	−200.00	−200.00	570.00	990.66
		R_3	A_{36}	−2372.29	995.00	2200.00	2200.00
		R_4	A_{46}	−13.53	600.00	995.00	4579.06
		R_5	A_{56}	549.66	600.00	600.00	1186.76
		R_6	A_{66}	−395.79	995.00	995.00	1469.02
$y_2(k) = x(k)$, biomass	$y_2(k-1)$	R^*_1	A^*_{11}	1276.69	1276.69	1704.53	2071.65
		R^*_2	A^*_{21}	1836.32	2000.00	2000.00	2062.48
		R^*_3	A^*_{31}	1670.72	2004.62	2020.39	2107.79
		R^*_4	A^*_{41}	1222.63	2089.77	2158.00	2246.92
		R^*_5	A^*_{51}	1586.15	2208.53	2792.79	2792.79
	$u_1(k-1)$	R^*_1	A^*_{12}	1050.97	1800.00	1800.00	1841.40
		R^*_2	A^*_{22}	260.02	260.02	1300.00	1879.38
		R^*_3	A^*_{32}	1285.84	1300.00	1310.01	1709.47
		R^*_4	A^*_{42}	−740.74	1743.98	1790.00	1813.00
		R^*_5	A^*_{52}	−239.52	1800.00	2739.95	2739.95
	$u_3(k-1)$	R^*_1	A^*_{13}	513.07	600.00	600.00	1283.64
		R^*_2	A^*_{23}	−200.00	−200.00	570.00	629.90
		R^*_3	A^*_{33}	799.28	1200.00	1200.00	2959.47
		R^*_4	A^*_{43}	1200.00	1200.00	2200.00	2200.00
		R^*_5	A^*_{53}	304.47	995.00	1000.00	1380.54

Table A2. Antecedent fuzzy sets and membership functions of the identified FM_2 fuzzy model.

ASP Process Outputs	Antecedent Vector Components $x_a(k)$	Rules $\overline{A}_j$, $\overline{A}_j^*$	Fuzzy Sets $\overline{A}_{jp}$, $\overline{A}_{jp}^*$	Membership Functions of *Piece-Wise Exponential* Type: $\mu_{\overline{A}_{jp}}(x)$ or $\mu_{\overline{A}_{jp}^*}(x)=\begin{cases} exp(-(\frac{x-c_l}{2\omega_l})^2) & (if\ x<c_l) \\ exp(-(\frac{x-c_r}{2\omega_r})^2) & (if\ x>c_r) \\ 1 & (otherwise) \end{cases}$ [parameters: c_l, c_r, ω_l, ω_r]			
				c_l	c_r	ω_l	ω_r
$y_1(k) = s(k)$, substrate	$y_1(k-1)$	$\overline{R}_1$	$\overline{A}_{11}$	40.44	52.75	57.79	75.11
		$\overline{R}_2$	$\overline{A}_{21}$	−113.86	−113.86	42.62	78.55
		$\overline{R}_3$	$\overline{A}_{31}$	−113.86	−113.86	74.76	399.32
		$\overline{R}_4$	$\overline{A}_{41}$	−80.17	139.90	175.53	395.52
		$\overline{R}_5$	$\overline{A}_{51}$	−113.86	−113.86	37.56	84.98
		$\overline{R}_6$	$\overline{A}_{61}$	−82.04	187.03	437.47	437.47
	$y_2(k-1)$	$\overline{R}_1$	$\overline{A}_{12}$	1316.27	1971.47	2087.49	2236.38
		$\overline{R}_2$	$\overline{A}_{22}$	1520.93	2087.49	2088.98	2102.61
		$\overline{R}_3$	$\overline{A}_{32}$	365.65	1695.29	3560.07	3560.07
		$\overline{R}_4$	$\overline{A}_{42}$	1867.75	1905.35	1910.25	2511.25
		$\overline{R}_5$	$\overline{A}_{52}$	1526.43	1781.70	1850.22	3748.13
		$\overline{R}_6$	$\overline{A}_{62}$	−1025.12	−1025.12	789.35	3530.74
	$u_2(k-1)$	$\overline{R}_1$	$\overline{A}_{13}$	358.63	366.67	366.67	480.10
		$\overline{R}_2$	$\overline{A}_{23}$	−199.86	−199.86	278.92	473.16
		$\overline{R}_3$	$\overline{A}_{33}$	−199.86	−199.86	329.66	2428.79
		$\overline{R}_4$	$\overline{A}_{43}$	−25.89	690.10	1120.67	1120.67
		$\overline{R}_5$	$\overline{A}_{53}$	−199.86	−199.86	231.77	578.89
		$\overline{R}_6$	$\overline{A}_{63}$	361.68	366.67	366.67	675.58
	$u_3(k-1)$	$\overline{R}_1$	$\overline{A}_{14}$	141.19	570.00	933.89	3196.54
		$\overline{R}_2$	$\overline{A}_{24}$	−360.98	1600.00	3000.00	3000.00
		$\overline{R}_3$	$\overline{A}_{34}$	−1341.60	995.00	3000.00	3000.00
		$\overline{R}_4$	$\overline{A}_{44}$	436.24	570.00	570.00	2359.54
		$\overline{R}_5$	$\overline{A}_{54}$	−866.17	1600.00	3000.00	3000.00
		$\overline{R}_6$	$\overline{A}_{64}$	−1000.00	−1000.00	278.89	885.23
	$u_3(k-2)$	$\overline{R}_1$	$\overline{A}_{15}$	101.12	570.00	933.89	3196.81
		$\overline{R}_2$	$\overline{A}_{25}$	−381.98	1600.00	3000.00	3000.00
		$\overline{R}_3$	$\overline{A}_{35}$	−1318.67	995.00	3000.00	3000.00
		$\overline{R}_4$	$\overline{A}_{45}$	435.38	570.00	570.00	2345.99
		$\overline{R}_5$	$\overline{A}_{55}$	−847.74	1600.00	3000.00	3000.00
		$\overline{R}_6$	$\overline{A}_{65}$	−1000.00	−1000.00	278.89	822.68
$y_2(k) = x(k)$, biomass	$y_2(k-1)$	$\overline{R}_1^*$	$\overline{A}_{11}^*$	−1025.12	−1025.12	260.29	3304.59
		$\overline{R}_2^*$	$\overline{A}_{21}^*$	70.17	1630.85	1689.19	2383.28
		$\overline{R}_3^*$	$\overline{A}_{31}^*$	1535.36	1858.85	1983.77	2286.34
		$\overline{R}_4^*$	$\overline{A}_{41}^*$	1330.50	2087.33	2097.72	2448.73
		$\overline{R}_5^*$	$\overline{A}_{51}^*$	1624.63	2101.30	3560.07	3560.07
	$u_3(k-1)$	$\overline{R}_1^*$	$\overline{A}_{12}^*$	−1000.00	−1000.00	278.89	580.19
		$\overline{R}_2^*$	$\overline{A}_{22}^*$	527.18	600.00	600.00	1341.93
		$\overline{R}_3^*$	$\overline{A}_{32}^*$	77.73	500.00	570.00	1355.74
		$\overline{R}_4^*$	$\overline{A}_{42}^*$	334.06	933.89	1100.00	3145.84
		$\overline{R}_5^*$	$\overline{A}_{52}^*$	374.09	1600.00	3000.00	3000.00

Appendix B

Adaptation of the Extended Manipulated Variable, $\boldsymbol{u_a}(k)$, to the Global Model in the State Space Obtained in Section 2.3 of This Article (Equations (24) and (25))

Previous observation: $\boldsymbol{u_a}(k)$ was derived in [23].

1. State-space DLTV global model (the predictions model) of [23].

$$\boldsymbol{z_m}(k+1) = \overline{\boldsymbol{A}}_m \boldsymbol{z_m}(k) + \overline{\boldsymbol{B}}_m \boldsymbol{u_a}(k) + \overline{\boldsymbol{R}}_m \quad \text{(A1)}$$

$$\boldsymbol{y_m}(k) = \overline{\boldsymbol{C}}_m \boldsymbol{z_m}(k) \quad \text{(A2)}$$

where:

- k represents $(k{\cdot}T)$ and T is the sampling period
- $\boldsymbol{z_m}(k)$ is the extended state vector
- $\boldsymbol{y_m}(k)$ is the output vector
- $\boldsymbol{u_a}(k)$ is the input vector (or the extended input vector)
- $\overline{\boldsymbol{A}}_{\boldsymbol{m}}(k)$, $\overline{\boldsymbol{B}}_{\boldsymbol{m}}(k)$, $\overline{\boldsymbol{R}}_{\boldsymbol{m}}(k)$ and $\overline{\boldsymbol{C}}_{\boldsymbol{m}}(k)$ are the system matrices

being:

$$\boldsymbol{z_m}(k) = \begin{pmatrix} y_1(k) \\ y_2(k) \\ u_1(k) \\ u_2(k) \end{pmatrix} = \begin{pmatrix} y_1(k) \\ y_2(k) \\ d_1(k) \\ d_2(k) \end{pmatrix} = \begin{pmatrix} s(k) \\ x(k) \\ q_i(k) \\ s_i(k) \end{pmatrix} \tag{A3}$$

$$\boldsymbol{y_m}(k) = \begin{pmatrix} y_1(k) \\ y_2(k) \end{pmatrix} = \begin{pmatrix} s(k) \\ x(k) \end{pmatrix} \tag{A4}$$

$$\boldsymbol{u_a}(k) = \begin{pmatrix} u_3(k) \\ u_3(k-1) \end{pmatrix} = \begin{pmatrix} u(k) \\ u(k-1) \end{pmatrix} = \begin{pmatrix} q_r(k) \\ q_r(k-1) \end{pmatrix} \tag{A5}$$

$$\begin{array}{c} \overline{\boldsymbol{A}}_{\boldsymbol{m}} = \sum_{\boldsymbol{j=1}}^{\boldsymbol{mr}}\left(\boldsymbol{\beta_j}(\boldsymbol{x_a})\boldsymbol{A_{m_j}}\right);\ \overline{\boldsymbol{B}}_{\boldsymbol{m}} = \sum_{\boldsymbol{j=1}}^{\boldsymbol{mr}}\left(\boldsymbol{\beta_j}(\boldsymbol{x_a})\boldsymbol{B_{m_j}}\right) \\ \overline{\boldsymbol{C}}_{\boldsymbol{m}} = \sum_{\boldsymbol{j=1}}^{\boldsymbol{mr}}\left(\boldsymbol{\beta_{j_{12}}}(\boldsymbol{x_a})\boldsymbol{C_{m_j}}\right);\ \overline{\boldsymbol{R}}_{\boldsymbol{m}} = \sum_{\boldsymbol{j=1}}^{\boldsymbol{mr}}\left(\boldsymbol{\beta_j}(\boldsymbol{x_a})\boldsymbol{R_{m_j}}\right) \end{array} \tag{A6}$$

$$\begin{array}{c} \boldsymbol{\beta_j}(\boldsymbol{x_a}) = \begin{pmatrix} \beta_{1j}(\boldsymbol{x_a}) & 0 & 0 & 0 \\ 0 & \beta_{2j}(\boldsymbol{x_a}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ \beta_{j_{12}}(\boldsymbol{x_a}) = \begin{pmatrix} \beta_{1j}(\boldsymbol{x_a}) & 0 \\ 0 & \beta_{2j}(\boldsymbol{x_a}) \end{pmatrix} \\ \beta_{26}(\boldsymbol{x_a}) = 0 \text{ (there are only 5 rules for } y_2) \\ \beta_{ij}(\boldsymbol{x_a}):\ \text{normalized membership functions of the antecedent } vector,\ \boldsymbol{x_a} \end{array} \tag{A7}$$

$$\boldsymbol{A_{m_j}} = \begin{pmatrix} a_{j1} & a_{j2} & b_{j1} & b_{j2} \\ 0 & a^*_{j2} & b^*_{j1} & 0 \\ 0 & 0 & \frac{1}{mr} & 0 \\ 0 & 0 & 0 & \frac{1}{mr} \end{pmatrix}; \boldsymbol{B_{m_j}} = \begin{pmatrix} b_{j3} & b_{j4} \\ b^*_{j3} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix};\ \boldsymbol{C_{m_j}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix};\ \boldsymbol{R_{m_j}} = \begin{pmatrix} r_j \\ r^*_j \\ 0 \\ 0 \end{pmatrix} \tag{A8}$$

being:

- $mr = max.\ (mr_1, mr_2)$: common number of rules
- $\{a_{ji}\}$, $\{b_{ji}\}$, and $\{r_j\}$, the coefficients of the *consequent vector*, and the independent term, respectively, in the j-th rule of the fuzzy model corresponding to output $y_1(k)$
- $\left\{a^*_{ji}\right\}$, $\left\{b^*_{ji}\right\}$, and $\left\{r^*_j\right\}$, the coefficients of the *consequent vector*, and the independent term, respectively, in the j-th rule of the fuzzy model corresponding to output $y_2(k)$ (with the following particularity: $a^*_{62} \equiv 0$, $b^*_{61} \equiv 0$ $b^*_{63} \equiv 0$, $r^*_6 \equiv 0$; (only 5 rules for y_2)

State-space global model of the ASP process, corresponding to [23], with the elements of the vectors and system matrices expressed in a generic way:

$$\overbrace{\begin{pmatrix} y_1(k+1) \\ y_2(k+1) \\ ---- \\ d_1(k+1) \\ d_2(k+1) \end{pmatrix}}^{z_m(k+1)} = \overbrace{\begin{pmatrix} a & b & | & c & d \\ 0 & f & | & g & 0 \\ - & - & - & - & - \\ 0 & 0 & | & 1 & 0 \\ 0 & 0 & | & 0 & 1 \end{pmatrix}}^{\overline{A}_m} \overbrace{\begin{pmatrix} y_1(k) \\ y_2(k) \\ --- \\ d_1(k) \\ d_2(k) \end{pmatrix}}^{z_m(k)} + \underbrace{\begin{pmatrix} m & n \\ p & 0 \\ - & - \\ 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\overline{B}_m} \underbrace{\begin{pmatrix} u(k) \\ u(k-1) \end{pmatrix}}_{u_a(k)} + \underbrace{\begin{pmatrix} r_1 \\ r_2 \\ - \\ 0 \\ 0 \end{pmatrix}}_{\overline{R}_m} \tag{A9}$$

$$\overbrace{\begin{pmatrix} y_1(k) \\ y_2(k) \end{pmatrix}}^{y_m(k)} = \overbrace{\begin{pmatrix} 1 & 0 & | & 0 & 0 \\ 0 & 1 & | & 0 & 0 \end{pmatrix}}^{\overline{C}_m} \overbrace{\begin{pmatrix} y_1(k) \\ y_2(k) \\ --- \\ d_1(k) \\ d_2(k) \end{pmatrix}}^{z_m(k)} \tag{A10}$$

2. State-space DLTV global model (the predictions model) of Section 2.3 of this article (Equations (24) y (25))

$$\boldsymbol{z_{mN}}(k+1) = \overline{\boldsymbol{A}}_{\boldsymbol{mN}}\boldsymbol{z_{mN}}(k) + \overline{\boldsymbol{B}}_{\boldsymbol{mN}}\boldsymbol{u_a}(k) + \overline{\boldsymbol{D}}_{\boldsymbol{mN}}\boldsymbol{d}(k) + \overline{\boldsymbol{R}}_{\boldsymbol{mN}} \tag{A11}$$

$$\boldsymbol{y_{mN}}(k) = \overline{\boldsymbol{C}}_{\boldsymbol{mN}}\boldsymbol{z_{mN}}(k) \tag{A12}$$

where the expressions corresponding to the state, extended input, disturbances and output vectors, as well as those of the system matrices, are described in Section 2.3 of this article (Equations (17), (20) and (22)).

State-space global model of the ASP process, corresponding to Section 2.3 of this article, with the elements of the vectors and system matrices expressed in a generic way:

$$\overbrace{\begin{pmatrix} y_1(k+1) \\ y_2(k+1) \end{pmatrix}}^{z_{mN}(k+1)} = \overbrace{\begin{pmatrix} a & b \\ 0 & f \end{pmatrix}}^{\overline{A}_{mN}} \overbrace{\begin{pmatrix} y_1(k) \\ y_2(k) \end{pmatrix}}^{z_{mN}(k)} + \underbrace{\begin{pmatrix} m & n \\ p & 0 \end{pmatrix}}_{\overline{B}_{mN}} \underbrace{\begin{pmatrix} u(k) \\ u(k-1) \end{pmatrix}}_{u_a(k)} + \underbrace{\begin{pmatrix} c & d \\ g & 0 \end{pmatrix}}_{\overline{D}_{mN}} \underbrace{\begin{pmatrix} d_1(k) \\ d_2(k) \end{pmatrix}}_{d(k)} + \underbrace{\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}}_{\overline{R}_{mN}} \tag{A13}$$

$$\overbrace{\begin{pmatrix} y_1(k) \\ y_2(k) \end{pmatrix}}^{y_{mN}(k)} = \overbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}^{\overline{C}_{mN}} \overbrace{\begin{pmatrix} y_1(k) \\ y_2(k) \end{pmatrix}}^{z_{mN}(k)} \tag{A14}$$

3. Extended manipulated variable, $\boldsymbol{u_a}(k)$, deduced in [23].

In [23], an analytical and explicit expression was deduced for the scalar control variable, $u(k)$, corresponding to the FMBPC control strategy whose stability is analyzed in the present article. In this mathematical process, the expression corresponding to the vector variable $\boldsymbol{u_a}(k)$ was also implicitly deduced (it is done with reviewing the corresponding mathematical development and making some trivial consideration). The expression of $\boldsymbol{u_a}(k)$, for the moment as a function of the state vector, other variables, and the matrices corresponding to the original model, is the following:

$$\begin{aligned} \boldsymbol{u_a}(k) = \boldsymbol{M_a}^{-1}\big(\boldsymbol{y_r}(k+H) - \boldsymbol{y}(k) + \boldsymbol{y_m}(k) - \overline{\boldsymbol{C}}_{\boldsymbol{m}}\overline{\boldsymbol{A}}_{\boldsymbol{m}}{}^{H}\boldsymbol{z_m}(k) \\ - \overline{\boldsymbol{C}}_{\boldsymbol{m}}\Big(\overline{\boldsymbol{A}}_{\boldsymbol{m}}{}^{H-1} + \big(\overline{\boldsymbol{A}}_{\boldsymbol{m}}{}^{H-1} - I\big)\big(\overline{\boldsymbol{A}}_{\boldsymbol{m}} - \boldsymbol{I}\big)^{-1}\Big)\overline{\boldsymbol{R}}_{\boldsymbol{m}} \big) \end{aligned}$$

being :

$$\begin{aligned} &\boldsymbol{u_a}(k) = \begin{pmatrix} u(k) \\ u(k-1) \end{pmatrix} \\ &\boldsymbol{M_a} = \overline{\boldsymbol{C}}_{\boldsymbol{m}}\Big(\overline{\boldsymbol{A}}_{\boldsymbol{m}}{}^{H-1}\,\overline{\boldsymbol{B}}_{\boldsymbol{m}} + \big(\overline{\boldsymbol{A}}_{\boldsymbol{m}}{}^{H-1} - \boldsymbol{I}\big)\big(\overline{\boldsymbol{A}}_{\boldsymbol{m}} - \boldsymbol{I}\big)^{-1}\overline{\boldsymbol{B}}_{\boldsymbol{m}}\boldsymbol{P_{1010}}\Big) \\ &\boldsymbol{P_{1010}} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \\ &\boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \quad \text{(A15)}$$

and the *reference model* for the output trajectories :

$$\begin{aligned} &\boldsymbol{y_r}(k+H) = \boldsymbol{A_{rH}}\boldsymbol{y_r}(k) + (I - \boldsymbol{A_{rH}})\boldsymbol{y_{set\_point}}(k) \\ &\boldsymbol{A_{rH}} = \begin{pmatrix} a_{r_1}^H & 0 \\ 0 & a_{r_2}^H \end{pmatrix} \\ &(1-a_{r_i})^{-1}b_{r_i} = 1;\ i = 1,\ 2; \\ &\qquad [H \in \mathbb{Z}^+,\ H \geq 1] \end{aligned}$$

where:

- $\boldsymbol{u_a}(k)$: the *input vector* or the *extended input vector* (see Equation (A5))
- $u(k)$: the *next* value for the control variable, computed at the k-th instant (the current instant); will be applied from the k-th instant. In the context of our case study (*activated sludge* processes), the control variable is, specifically, the *sludge recirculation flow-rate*: $u(k) = q_r(k)$
- $u(k-1)$: the control variable at the $(k-1)$-th instant (the previous sampling instant)
- H: coincidence horizon (PFC concept)
- $\boldsymbol{y_r}(k+H)$: output variables at the $(k+H)$-th instant, given by a certain, previously chosen, reference model for the output trajectories
- $\boldsymbol{y}(k)$: process output variables, measured at the k-th instant (the current instant)
- $\boldsymbol{y_m}(k)$: model output variables at the k-th instant, calculated using the model equations, with the current value of the control signal (see Equation (A4))
- $\boldsymbol{z_m}(k)$: state vector of the original model at the k-th instant (see Equation (A3))
- $\overline{\boldsymbol{A}}_{\boldsymbol{m}}$, $\overline{\boldsymbol{B}}_{\boldsymbol{m}}$, $\overline{\boldsymbol{R}}_{\boldsymbol{m}}$ and $\overline{\boldsymbol{C}}_{\boldsymbol{m}}$: time-varying *system matrices* (see Equations (A6)–(A8)). These matrices must be updated at each iteration of the control algorithm implementation because they depend on the *antecedent vector* $\boldsymbol{x_a}$, which depends on time ($\boldsymbol{x_a} = \boldsymbol{x_a}(k)$)

The expression of $\boldsymbol{u_a}(k)$ detailed in (A15) can be formally simplified by grouping in a single matrix the matrix relations included in the last term, resulting in the following:

$$\begin{aligned} &\boldsymbol{u_a}(k) = \boldsymbol{M_a}^{-1}\big(\boldsymbol{y_r}(k+H) - \boldsymbol{y}(k) + \boldsymbol{y_m}(k) - \overline{\boldsymbol{C}}_{\boldsymbol{m}}\overline{\boldsymbol{A}}_{\boldsymbol{m}}{}^{H}\boldsymbol{z_m}(k) - \overline{\boldsymbol{\lambda}_{\boldsymbol{m}}}\big) \\ &\text{being :} \\ &\overline{\boldsymbol{\lambda}_{\boldsymbol{m}}} = \overline{\boldsymbol{C}}_{\boldsymbol{m}}\Big(\overline{\boldsymbol{A}}_{\boldsymbol{m}}{}^{H-1} + \big(\overline{\boldsymbol{A}}_{\boldsymbol{m}}{}^{H-1} - \boldsymbol{I}\big)\big(\overline{\boldsymbol{A}}_{\boldsymbol{m}} - \boldsymbol{I}\big)^{-1}\Big)\overline{\boldsymbol{R}}_{\boldsymbol{m}} \\ &[H \in \mathbb{Z}^+,\ H \geq 1] \end{aligned} \quad \text{(A16)}$$

4. Adaptation of $\boldsymbol{u_a}(k)$ to the global model in the state space obtained in Section 2.3 of this article.

Next, it is necessary to relate both the state vector and the system matrices of the two considered global models: the original model and the model obtained in Section 2.3 of this article. Taking into account the composition of all the vector and matrix variables involved in the two state space models, the following relations can be easily deduced (see all the equations previously shown in this Appendix and especially Equations (A9), (A10), and (A13), (A14)):

$$\begin{gathered} \boldsymbol{z_m}(k) = \begin{pmatrix} \boldsymbol{z_{mN}}(k) \\ \boldsymbol{d}(k) \end{pmatrix} \\ \overline{\boldsymbol{A}}_{\boldsymbol{m}} = \begin{pmatrix} \overline{\boldsymbol{A}}_{\boldsymbol{mN}} & \overline{\boldsymbol{D}}_{\boldsymbol{mN}} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix};\ \overline{\boldsymbol{B}}_{\boldsymbol{m}} = \begin{pmatrix} \overline{\boldsymbol{B}}_{\boldsymbol{mN}} \\ \boldsymbol{0} \end{pmatrix};\ \overline{\boldsymbol{C}}_{\boldsymbol{m}} = \begin{pmatrix} \overline{\boldsymbol{C}}_{\boldsymbol{mN}} & \boldsymbol{0} \end{pmatrix};\ \overline{\boldsymbol{R}}_{\boldsymbol{m}} = \begin{pmatrix} \overline{\boldsymbol{R}}_{\boldsymbol{mN}} \\ \boldsymbol{0} \end{pmatrix} \\ \overline{\boldsymbol{A}}_{\boldsymbol{m}}{}^{\boldsymbol{H}} = \begin{pmatrix} \overline{\boldsymbol{A}}_{\boldsymbol{mN}}{}^{H} & \overline{\gamma} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix} \end{gathered} \tag{A17}$$

being:

- $\overline{\gamma}$: a submatrix of order 2
- $\boldsymbol{I}$: the order 2 identity matrix
- $\boldsymbol{0}$: the order 2 null matrix

Considering now the previous relations (Equation (A17)), then the penultimate term of the expression of $\boldsymbol{u_a}(k)$ in (A16), $\overline{\boldsymbol{C}}_{\boldsymbol{m}}\overline{\boldsymbol{A}}_{\boldsymbol{m}}{}^{H}\boldsymbol{z_m}(k)$, can be developed as follows:

$$\begin{aligned} \overline{\boldsymbol{C}}_{\boldsymbol{m}}\overline{\boldsymbol{A}}_{\boldsymbol{m}}{}^{H}\boldsymbol{z_m}(k) &= \begin{pmatrix} \overline{\boldsymbol{C}}_{\boldsymbol{mN}} & \boldsymbol{0} \end{pmatrix}\begin{pmatrix} \overline{\boldsymbol{A}}_{\boldsymbol{mN}}{}^{H} & \overline{\gamma} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix}\begin{pmatrix} \boldsymbol{z_{mN}}(k) \\ \boldsymbol{d}(k) \end{pmatrix} \\ &= \begin{pmatrix} \overline{\boldsymbol{C}}_{\boldsymbol{mN}}\overline{\boldsymbol{A}}_{\boldsymbol{mN}}{}^{H} & \overline{\boldsymbol{C}}_{\boldsymbol{mN}}\overline{\gamma} \end{pmatrix}\begin{pmatrix} \boldsymbol{z_{mN}}(k) \\ \boldsymbol{d}(k) \end{pmatrix} \\ &= \overline{\boldsymbol{C}}_{\boldsymbol{mN}}\overline{\boldsymbol{A}}_{\boldsymbol{mN}}{}^{H}\boldsymbol{z_{mN}}(k) + \overline{\boldsymbol{C}}_{\boldsymbol{mN}}\overline{\gamma}\boldsymbol{d}(k) \end{aligned} \tag{A18}$$

In a similar way, we could transform the expressions of two other matrices present in Equation (A16), $\boldsymbol{M_a}$ (specified in Equation (A15)) and $\overline{\boldsymbol{\lambda_m}}$ (specified in Equation (A16)), in terms of the matrices of the model of Section 2.3 of this article, making use of the relations included in Equation (A17). However, it is not necessary to detail these transformations and it is preferable to keep each of these expressions (matrix operations between system matrices) as a single compact matrix, adding a suffix $\boldsymbol{N}$ referring to the corresponding transformation. We will denote those adapted matrices as $\boldsymbol{M_{aN}}$ and $\overline{\boldsymbol{\lambda_{mN}}}$, respectively (and $\boldsymbol{M_{aN}}^{-1}$ for the inverse of $\boldsymbol{M_{aN}}$). On the other hand, we will adapt the corresponding notation to the output of the model: $\boldsymbol{y_m}(k) \equiv \boldsymbol{y_{mN}}(k)$. Finally, considering all these notation adaptations and the development made in (A18), and replacing all of it in Equation (A16), the expression of $\boldsymbol{u_a}(k)$, as a function of the vector and matrix variables corresponding to the model obtained in Section 2.3 of this article, will be the next:

$$\begin{gathered} \boldsymbol{u_a}(k) = \boldsymbol{M_{aN}}^{-1}\left(\boldsymbol{y_r}(k+H) - \boldsymbol{y}(k) + \boldsymbol{y_{mN}}(k) - \overline{\boldsymbol{C}}_{\boldsymbol{mN}}\overline{\boldsymbol{A}}_{\boldsymbol{mN}}{}^{H}\boldsymbol{z_{mN}}(k) - \overline{\boldsymbol{C}}_{\boldsymbol{mN}}\overline{\gamma}\boldsymbol{d}(k) - \overline{\boldsymbol{\lambda_{mN}}} \right) \\ being: \\ \boldsymbol{M_{aN}}^{-1}, \overline{\boldsymbol{\lambda_{mN}}}: \ matrix\ functions\ of\ system\ matrices;\ \overline{\gamma}: \ submatrix\ of\ order\ 2; \\ H: \ coincidence\ horizon\ (PFC\ concept)\ [H \in \mathbb{Z}^{+},\ H \geq 1] \end{gathered} \tag{A19}$$

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Capítulo 4

Artículo nº 3: FMBPC con restricciones

4.1. Título original del artículo

Integración de la estrategia FMBPC en una estructura de Control Predictivo en Lazo Cerrado. Aplicación al control de fangos activados

4.2. Artículo nº3: copia completa de la publicación

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Integración de la estrategia FMBPC en una estructura de Control Predictivo en Lazo Cerrado. Aplicación al control de fangos activados

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Resumen

En este trabajo se aborda la integración de dos métodos o estrategias de Control Predictivo basado en Modelos, a saber: Control Predictivo basado en Modelos Borrosos (FMBPC) y Control Predictivo en Lazo Cerrado (CLP-MPC). La primera de estas estrategias utiliza principios de Control Predictivo Funcional (PFC) y está enmarcada, al mismo tiempo, en el ámbito del Control Inteligente (IC). La integración tiene como principal objetivo proporcionar a la estrategia de control no lineal FMBPC un procedimiento de optimización que permita el manejo automático de restricciones en la variable de control. La solución propuesta consiste en hacer uso de una estructura complementaria de tipo CLP-MPC para determinar mediante optimización, en cada instante de muestreo, los valores óptimos de un cierto término aditivo, a sumar a la ley de control FMBPC, de tal modo que se satisfagan las restricciones. El modelo de predicciones y la ley de control base necesarios para realizar los cálculos en la estructura CLP-MPC son proporcionados por la estrategia FMBPC. La estrategia mixta FMBPC/CLP propuesta ha sido validada, en simulación, aplicándola al control de fangos activados en plantas de tratamiento de aguas residuales (EDAR), poniendo el foco en la imposición de restricciones a la acción de control. Los resultados obtenidos son satisfactorios, observando un buen rendimiento del algoritmo de control diseñado, al tiempo que se garantiza tanto la satisfacción de las restricciones, que era el principal objetivo, como la estabilidad del sistema en lazo cerrado.

Palabras clave: Control predictivo basado en modelo, Control borroso y sistemas borrosos en control, Técnicas de control inteligente, Control de sistemas con restricciones, Control multivariable, Control automático de sistemas de tratamiento de aguas.

Integration of the FMBPC strategy in a Closed-Loop Predictive Control structure. Application to the control of activated sludge.

Abstract

This work addresses the integration of two methods or strategies of Model-Based Predictive Control, namely: Fuzzy Model-Based Predictive Control (FMBPC) and Closed-Loop Predictive Control (CLP-MPC). The first of these strategies uses principles of Predictive Functional Control (PFC) and is framed, at the same time, in the field of Intelligent Control (IC). The main objective of the integration is to provide to the FMBPC nonlinear control strategy an optimization procedure that allows the automatic handling of constraints in the control variable. The proposed solution consists of making use of a complementary structure of the CLP-MPC type to determine by optimization, at each sampling instant, the optimal values of a certain additive term, to be added to the FMBPC control law, in such a way that they are satisfied the constraints. The prediction model and base control law necessary to perform the calculations on the CLP-MPC structure are provided by the FMBPC strategy. The proposed FMBPC/CLP mixed strategy has been validated, in simulation, applying it to the control of activated sludge processes in wastewater treatment plants (WWTP), focusing on the imposition of constraints on the control action. The results obtained are satisfactory, observing a good performance of the designed control algorithm, while guaranteeing both the satisfaction of the constraints, which was the main objective, and the stability of the closed-loop system.

Keywords: Model-based predictive control, Fuzzy control and fuzzy systems in control, Intelligent control techniques, Control of systems with restrictions, Multivariable control, Automatic control of water treatment systems.

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1. Introducción

La estrategia de control predictivo basado en modelos borrosos o, mediante su correspondiente expresión en inglés, Fuzzy Model-based Predictive Control (FMBPC) (Babuška, 1998a; Roubos et al., 1999; Mollov et al., 2004; Blažič et al., 2007; Bououden et al., 2015; Škrjanc et al., 2016; Boulkaibet et al., 2017; Vallejo et al., 2019, 2021), constituye una modalidad de control predictivo no lineal idónea para controlar de manera eficaz sistemas altamente no lineales o con incertidumbres o, en general, sistemas con dinámicas complejas, como por ejemplo los sistemas biológicos. Tal capacidad deriva de la utilización de un modelo de predicciones borroso (*fuzzy model*) (Zadeh, 1990), generalmente obtenido mediante identificación a partir de datos experimentales de entrada-salida. Si la identificación borrosa (Babuška, 1998b) está bien diseñada, será capaz de capturar la dinámica de la planta de manera fiel, lo cual resulta determinante en la fiabilidad de las predicciones y, en última instancia, en el rendimiento del controlador predictivo. Esta propiedad puede ser identificada como la componente o faceta de la estrategia de control FMBPC que la sitúa (parcialmente) también en el ámbito del control inteligente (IC), puesto que está relacionada con la utilización de razonamiento cualitativo, presente en el modelado borroso. Se trata, por tanto, de un controlador que pertenece, tanto a la familia de los controladores predictivos no lineales, como a la de los controladores inteligentes. En el contexto del presente trabajo, la estrategia FMBPC considerada se enmarca además en el ámbito del denominado control predictivo funcional o (en inglés) Predictive Functional Control (PFC), que utiliza el denominado *principio de equivalencia* (Richalet, 1993; Škrjanc et al., 2000; Richalet et al., 2009; Haber et al., 2016). El *principio de equivalencia* establece en esencia que, dado un cierto proceso y dado un modelo perfecto del mismo, el incremento en la salida del proceso en respuesta a un determinado incremento de la acción de control deberá ser equivalente al incremento en la salida del modelo para el mismo cambio en la acción de control. Esto es utilizado en PFC en el tratamiento matemático correspondiente a los denominados *puntos de coincidencia* de la trayectoria de referencia (véase también: Vallejo et al., 2019).

En (Vallejo et al., 2019, 2021), la estrategia FMBPC fue aplicada al control multivariable de procesos biológicos de depuración de aguas residuales mediante fangos activados, obteniendo buenos resultados. Sin embargo, en tales propuestas no fueron implementados procedimientos de manejo automático de restricciones en la acción de control, o en otras variables, sujetos a optimización (Maciejowski, 2002; Limón, 2002). La incorporación de un procedimiento de ese tipo para el manejo de las restricciones en la variable de control (en cada iteración de la implementación online del control predictivo) es precisamente una de las principales razones y motivaciones del presente trabajo.

La solución propuesta en el presente trabajo se basa en la combinación de la estrategia FMBPC con una determinada estructura de control predictivo basado en modelos (MPC) (Camacho et al., 1998), denominada *paradigma de lazo* cerrado o, conforme a su definición original en inglés, Closed-Loop Paradigm (CLP), conocida también como *control predictivo en lazo cerrado* (Rossiter, 2003). El control CLP es en realidad una modalidad del denominado *control predictivo en modo dual* o Dual-Mode Control (DM-MPC) (Michalska et al., 1993; Rossiter, 2003), que más que una estrategia propiamente dicha, es una manera particular de organizar el conjunto de las predicciones, dividiéndolo en dos horizontes o etapas, en cada una de las cuales se computan de manera diferente las acciones de control futuras. Esta configuración facilita el diseño de algoritmos de control MPC, con estabilidad garantizada y buen rendimiento. En el caso de la modalidad de control en modo dual de tipo CLP, a la que denominaremos en adelante CLP-MPC, en cada uno de los pasos del primer tramo de las predicciones (*modo 1*) se sustituye la variable de entrada manipulada por la expresión correspondiente a una ley de control estabilizante, previamente elegida, más un término complementario dependiente del tiempo y sujeto a optimización, denominado *perturbación* de la acción de control (al que representaremos en adelante como c_j, en referencia al paso j-ésimo, o como c, de forma genérica). En el segundo tramo (*modo 2*), las predicciones se computan sustituyendo la entrada manipulada únicamente por la expresión correspondiente a la ley de control estabilizante elegida. Los elementos de la secuencia de *perturbaciones* de la acción de control constituyen los grados de libertad del problema de optimización y sus valores óptimos se obtendrán mediante minimización de una determinada función de coste. El primer valor de la secuencia óptima obtenida será el elegido para la implementación de la acción de control *online*, en el periodo de muestreo correspondiente. La estructura CLP-MPC tiene las ventajas propias de la estructura DM-MPC (dado que es un caso particular de esta) y constituye un método útil para mejorar algunos aspectos importantes en procedimientos de optimización en tiempo real, ligados a estrategias de control predictivo MPC, como el acondicionamiento numérico y la carga computacional (reduciéndola), o el análisis de estabilidad o de robustez (Adetola et al., 2010; Marchetti et al., 2014).

Algunos trabajos relativamente recientes en los que se utilizan las estructuras de control predictivo DM-MPC o CLP-MPC pueden verse en: (Ramírez et al., 2014), (Shariati et al., 2015), (El Bahja et al., 2018a, 2018b). Y enmarcados en el ámbito del control predictivo distribuido, en: (Al-Gherwi et al., 2013), (Sorcia-Vázquez et al., 2015). La reducción de la carga computacional en los cálculos de optimización asociados a estas estructuras hace especialmente interesante su uso en control predictivo distribuido.

En relación con los métodos de diseño de las estructuras de control predictivo CLP-MPC, en (Rossiter, 2003) se propone utilizar como ley de control base estabilizante una ley de realimentación de estados estándar del tipo $u = -Kx$ y al mismo tiempo se sugiere la posibilidad de que dicha ley pueda ser sustituida por otra. En (El Bahja et al., 2018b) se propone una metodología que utiliza conjuntos invariantes poliédricos y se deduce una ley de control CLP-MPC de realimentación de estados, estabilizante y que asegura el cumplimento de las restricciones y de los requisitos de rendimiento del sistema operando en lazo cerrado. En (El Bahja et al., 2018a) se propone como ley de control base estabilizante de la estructura CLP-MPC una ley de Control Predictivo Generalizado no lineal (NLGPC), que será expresada de forma analítica, y se utiliza un modelo fenomenológico de la planta para el cálculo de las predicciones (formalizado como un modelo lineal con coeficientes dependientes del estado). Sin embargo, el uso de modelos fenomenológicos restringe la aplicabilidad de estas técnicas debido a la dificultad, tanto de la obtención, como del manejo de tal tipo de modelos. Como alternativa, se plantea la utilización de modelos basados en datos, más fáciles de obtener y de manejar.

Así, en el presente trabajo se utilizan modelos basados en datos numéricos de entrada-salida, obtenidos mediante técnicas de identificación borrosa, capaces de capturar fielmente la dinámica de la planta. La identificación borrosa produce como resultado, originariamente, modelos borrosos discretos de tipo *Takagi-Sugeno* (TS) (Takagi et al., 1985), pero después estos son convertidos en (Vallejo et al., 2019, 2021) en modelos equivalentes en el espacio de estados, lineales y variantes en el tiempo, es decir (mediante sus siglas en inglés) de tipo DLTV (discrete linear time-varying), a escala global, y en modelos lineales e invariantes en el tiempo, es decir (en inglés) de tipo DLTI (discrete linear time-invariant), a escala local. Concretamente, estos últimos serán los que se emplearán como modelos de predicciones dentro de la estructura CLP-MPC, correspondiendo un modelo DLTI diferente a cada período de muestreo.

La estrategia mixta de control predictivo presentada en este artículo consiste, en esencia, en considerar la estrategia FMBPC como la principal y utilizar de modo complementario la estructura CLP-MPC, en el marco de la cual se determinarán mediante optimización los valores óptimos de las *perturbaciones* de la acción de control (términos c), en cada período de muestreo. En el cálculo de predicciones correspondiente a la estructura CLP-MPC se utilizará como ley de control base estabilizante la ley correspondiente a la estrategia FMBPC, propuesta y desarrollada en (Vallejo et al., 2019, 2021), que puede considerarse matemáticamente equivalente a una ley de control del tipo $u = -Kx$, con K dependiente del tiempo. En relación con el procedimiento de optimización, este consistirá en minimizar una cierta función de coste, dependiente de los grados de libertad del problema, es decir, de las variables c (estrictamente, c_j), con sujeción al cumplimiento de las restricciones previamente establecidas, obteniéndose así la secuencia óptima de tales términos. El primer elemento de esa secuencia será sumado *online* al valor calculado mediante la ley FMBPC, obteniéndose así el valor efectivo de la acción de control, a aplicar a la planta en el instante de muestreo k-ésimo en curso (instante de muestreo *actual*), con la forma equivalente siguiente: $u = -Kx + c$. En relación con las restricciones, en nuestro caso se ha puesto el énfasis en la limitación de los valores de la acción de control.

Las principales características de la estrategia de control predictivo no lineal propuesta, a la que denominaremos FMBPC/CLP, son: la utilización de modelos borrosos no lineales, obtenidos a partir de datos numéricos de entrada-salida y la aplicación de una ley de control predictivo del tipo FMBPC, no lineal, analítica y explícita, que asegura la estabilidad local del sistema en lazo cerrado y que, con la ayuda de la estructura complementaria CLP-MPC, asegura también al mismo tiempo el cumplimiento de las restricciones.

En relación con las aportaciones de la estrategia FMBPC/CLP propuesta, podemos separarlas por ámbitos. En el ámbito de las estrategias FMBPC con leyes de control analíticas y explícitas, nuestra estrategia incorpora un mecanismo automático de imposición de restricciones. En el ámbito del control predictivo, una aportación importante es la integración de dos metodologías de control predictivo diferentes, como son el control predictivo PFC, utilizado en la estrategia FMBPC considerada y el control predictivo clásico, basado en la optimización de funciones de coste, que es el método utilizado en la estructura CLP-MPC para determinar las acciones de control óptimas. Pero además, la combinación de esas dos metodologías (FMBPC y CLP-MPC) produce, como resultado, una nueva estrategia de MPC no lineal, con ciertas características de control inteligente en la identificación del modelo y con una importante reducción de la carga computacional en el manejo de restricciones, en comparación con otras metodologías de MPC no lineal en lazo abierto. Y finalmente, en el ámbito de las estrategias basadas en la estructura CLP-MPC, la estrategia presentada utiliza modelos obtenidos mediante identificación borrosa, a partir de datos numéricos, en lugar de modelos fenomenológicos, con las consiguientes ventajas en cuanto a la obtención, fidelidad y manejo del modelo (como ya se indicó anteriormente). Así mismo, desde una perspectiva más global y genérica, la metodología desarrollada puede constituir un posible marco para el diseño de leyes de control de procesos no lineales, con múltiples entradas y salidas y restricciones en las entradas, dotado con capacidades explícitas de manejo de restricciones y de caracterización de las propiedades de estabilidad y desempeño del controlador, en la fase de diseño de este.

La estrategia de control desarrollada se ha validado aplicándola al control de procesos de fangos activados en estaciones depuradoras de aguas residuales, que es la planta que ha sido tomada como caso de estudio.

El resto del artículo está estructurado de la forma siguiente: en la sección 2 se resumen los fundamentos y la base matemática del control predictivo en modo dual (brevemente) y de la estructura CLP en particular; en la sección 3 se revisa la estrategia de control predictivo FMBPC considerada, enfocada al control de fangos activados; en la sección 4 se explican los fundamentos de la propuesta de integración de la estrategia FMBPC con una estructura de control predictivo CLP-MPC, con el objetivo de manejar las restricciones en la acción de control; en la sección 5 se describen los experimentos de simulación y se muestran y se analizan los resultados; y por último, en la sección 6 se detallan las conclusiones.

2. Control predictivo en lazo cerrado

La estructura CLP-MPC es, como se ha dicho, un caso particular de estructura de control predictivo en modo dual (DM-MPC). Resumimos a continuación los fundamentos y la base matemática necesarias (Rossiter, 2003; El Bahja, 2017).

Representación de la planta. Supondremos que la planta en lazo abierto podrá representarse mediante un modelo lineal en el espacio de estados, es decir:

$$\begin{gathered} \boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + B\boldsymbol{u}_k \\ \boldsymbol{y}_k = C\boldsymbol{x}_k \end{gathered} \tag{1}$$

siendo $\boldsymbol{x}_k$ el estado en el instante k-ésimo, $\boldsymbol{u}_k$ las entradas manipulables e $\boldsymbol{y}_k$ las salidas medibles de la planta.

Por otra parte, en general asumiremos como ley de control en lazo cerrado la conocida ley de realimentación de estados:

$$\boldsymbol{u}_k = -K\boldsymbol{x}_k \tag{2}$$

siendo K un vector o matriz con las dimensiones adecuadas y definiremos:

$$\Phi = (A - BK) \tag{3}$$

Control predictivo en modo dual. La estructura DM-MPC original, también llamada *paradigma en lazo abierto* (OLP), consiste en considerar el horizonte de predicción dividido en dos tramos diferentes o *modos* (*modo 1* y *modo 2*), con diferentes tipos de acción de control para cada uno: para los n_c primeros pasos (*modo 1*), correspondientes a estados supuestamente lejanos al punto de operación deseado, las acciones de control, $\boldsymbol{u}_{k+j|k}$ ($j = 0,1,\dots,n_c - 1$), se consideran libres, a determinar mediante optimización, y para el resto de pasos, a partir de $n_c + 1$ (*modo 2*), correspondientes a estados supuestamente más cercanos ya al punto de operación, las acciones de control vendrán dadas por una ley de control en lazo cerrado previamente fijada (muy habitualmente, del tipo $-K\boldsymbol{x}_{k+n_c+j|k}$, con $j = 0,1,\dots,n_y - n_c - 1$, para un horizonte de predicción de n_y pasos). La secuencia de acciones de control predichas para todo el horizonte de predicción (en el instante de muestreo k) será una matriz a la que denotaremos con $\boldsymbol{u}_{k(OLP\text{-}MPC)_\rightarrow}$. Las primeras n_c componentes de esta matriz (*modo 1*) constituyen los grados de libertad *o*, mediante sus siglas en inglés, *d.o.f. (degrees of freedom)* de esta configuración y son determinados mediante optimización, típicamente buscando el mínimo de una cierta función de coste, sin restricciones o con ellas. Esta configuración es tenida en cuenta únicamente para las predicciones. Para la implementación *online* en cada período de muestreo, se toma el primer valor de la secuencia óptima de acciones de control obtenida.

Estabilidad garantizada en modo dual. En (Rossiter, 2003) se analiza la estabilidad de sistemas de control predictivo en modo dual, estableciendo una conclusión importante: el uso de horizontes infinitos y la inclusión en las predicciones de la denominada *cola* (*tail*, en inglés) permiten demostrar que la función de coste asociada al problema de optimización, (a la que denotaremos con, $J_{k(DM)}$), es una función de *Lyapunov* y por tanto puede concluirse que la ley de control DM-MPC será estable, según la teoría de estabilidad de *Lyapunov* (Lyapunov, 1892, 1992).

Control predictivo en lazo cerrado. La modalidad del control predictivo en modo dual CLP-MPC consiste en considerar las acciones de control del *modo 1*, durante las predicciones, como suma de dos términos: un primer término correspondiente a una ley de control en lazo cerrado previamente fijada (típicamente, la misma que en el *modo 2*) y un segundo término, o término complementario, $\boldsymbol{c}_k$, con el rol de *grado de libertad*, sujeto a optimización, y cuyo objetivo sea el manejo de las restricciones. En el caso de que la ley de control preestablecida esté basada en una realimentación de estados, $-K\boldsymbol{x}$, el modelo de predicciones se expresará como se especifica (a continuación) en la ecuación (4). Los términos $\boldsymbol{c}_k$ son conocidos como *perturbations* (perturbaciones de la acción de control), lo cual no debe confundirse con el concepto estándar de perturbaciones en la entrada, o en el estado, nombrado en la literatura de control automático en inglés como *disturbances*). En el *modo 2* de la estructura CLP-MPC, las acciones de control predictivas serán las correspondientes a la ley de control en lazo cerrado previamente fijada, tal y como se muestra (a continuación) en la ecuación (5):

Modo 1 (CLP-MPC)

$$\begin{aligned} \boldsymbol{x}_{k+i+1|k} &= A\boldsymbol{x}_{k+i|k} + B\left(-K\boldsymbol{x}_{k+i|k} + \boldsymbol{c}_{k+i|k}\right) \\ &= \Phi\boldsymbol{x}_{k+i|k} + B\boldsymbol{c}_{k+i|k} \\ \boldsymbol{u}_{k+i|k} &= -K\boldsymbol{x}_{k+i|k} + \boldsymbol{c}_{k+i|k} \end{aligned} \tag{4}$$

siendo: $\boldsymbol{c}_{k+i|k}$: $d.o.f, \quad i = 0,1,2,\dots,n_c - 1$

Modo 2 (CLP-MPC)

$$\begin{aligned} \boldsymbol{x}_{k+i+1|k} &= A\boldsymbol{x}_{k+i|k} + B(-K\boldsymbol{x}_{k+i|k}) \\ &= \Phi\boldsymbol{x}_{k+i|k} \\ \boldsymbol{u}_{k+i|k} &= -K\boldsymbol{x}_{k+i|k} \end{aligned} \tag{5}$$

siendo: $i > n_c$

Y teniendo en cuenta ahora las ecuaciones (4) y (5) y mediante el oportuno desarrollo, la expresión matricial de la secuencia de acciones de control predichas en el instante de muestreo k correspondientes a la estructura CLP-MPC, a la que denotaremos con $\boldsymbol{u}_{k(CLP\text{-}MPC)_\rightarrow}$, quedará como sigue:

$$\boldsymbol{u}_{k(CLP\text{-}MPC)_\rightarrow} = \begin{bmatrix} -K\boldsymbol{x}_{k|k} + \boldsymbol{c}_{k|k} \\ -K\boldsymbol{x}_{k+1|k} + \boldsymbol{c}_{k+1|k} \\ \vdots \\ -K\boldsymbol{x}_{k+n_c-1|k} + \boldsymbol{c}_{k+n_c-1|k} \\ ---------- \\ -K\boldsymbol{x}_{k+n_c|k} \\ -K\Phi\boldsymbol{x}_{k+n_c|k} \\ \vdots \\ -K\Phi^{n_y-n_c-1}\boldsymbol{x}_{k+n_c|k} \end{bmatrix} \tag{6}$$

siendo los n_c términos complementarios del modo 1 los grados de libertad del problema de optimización, que pueden ser expresados en forma matricial de la forma siguiente:

$$\boldsymbol{d.o.f}_{k(CLP\text{-}MPC)_\rightarrow} = \boldsymbol{c}_{k(CLP\text{-}MPC)_\rightarrow} = \begin{bmatrix} \boldsymbol{c}_{k|k} \\ \boldsymbol{c}_{k+1|k} \\ \vdots \\ \boldsymbol{c}_{k+n_c-1|k} \end{bmatrix} \tag{7}$$

Haciendo uso de las ecuaciones (4), (5), (6) y (7) y mediante el apropiado desarrollo, es posible obtener las expresiones matriciales correspondientes al conjunto de las predicciones en todo el horizonte de predicción, que lógicamente dependerán de las variables *d.o.f*, $\boldsymbol{c}_{k(CLP\text{-}MPC)_\rightarrow}$. Así mismo, una vez que haya sido elegida la función de coste $J_{k(CLP)}$ y que hayan sido especificadas las restricciones, es posible también formalizar sus correspondientes expresiones en función de $\boldsymbol{c}_{k(CLP\text{-}MPC)_\rightarrow}$.

A partir de ello podrá expresarse el problema de optimización en función de las variables *d.o.f*, lo cual conducirá a la determinación de la acción de control óptima (o incremento óptimo) para cada instante k. A continuación se detallan las expresiones más relevantes consideradas y asumidas (comenzando por un cambio de notación con objetivos de simplificación).

Simplificación de la notación. Con el objetivo de mejorar la comprensión matemática, se utilizará la siguiente notación (donde se ha considerado también la secuencia de los estados $\boldsymbol{x}_k$ a lo largo del horizonte de predicción, $\boldsymbol{x}_{k(CLP\text{-}MPC)_{\rightarrow}}$):

$$\begin{aligned} \boldsymbol{u}_{k(CLP\text{-}MPC)_{\rightarrow}} &\equiv \boldsymbol{u}_{\rightarrow} \\ \boldsymbol{c}_{k(CLP\text{-}MPC)_{\rightarrow}} &\equiv \boldsymbol{c}_{\rightarrow} \\ \boldsymbol{x}_{k(CLP\text{-}MPC)_{\rightarrow}} &\equiv \boldsymbol{x}_{\rightarrow} \end{aligned} \tag{8}$$

Predicciones correspondientes a la estructura CLP-MPC. Las expresiones finales correspondientes al estado, $\boldsymbol{x}_{\rightarrow}$, y a las acciones de control, $\boldsymbol{u}_{\rightarrow}$, son mostradas a continuación (siendo A y B las matrices del modelo lineal dado por la ecuación (1) y Φ la expresión definida en (3)):

$$\boldsymbol{x}_{\rightarrow} = P_{cl}\boldsymbol{x}_k + H_c\boldsymbol{c}_{\rightarrow} \tag{9}$$

donde:

$$P_{cl} = \begin{bmatrix} \Phi \\ \Phi^2 \\ \Phi^3 \\ \vdots \end{bmatrix}, \quad H_c = \begin{bmatrix} B & 0 & 0 & \cdots \\ \Phi B & B & 0 & \cdots \\ \Phi^2 B & \Phi B & B & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$\boldsymbol{u}_{\rightarrow} = P_{clu}\boldsymbol{x}_k + H_{cu}\boldsymbol{c}_{\rightarrow} \tag{10}$$

donde:

$$P_{clu} = \begin{bmatrix} -K \\ -K\Phi \\ -K\Phi^2 \\ \vdots \end{bmatrix}$$

$$H_{cu} = \begin{bmatrix} B & 0 & 0 & \cdots \\ -KB & B & 0 & \cdots \\ -K\Phi B & -KB & B & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

En principio, las expresiones contenidas en las ecuaciones (9) y (10) serían sólo las correspondiente al modo 1 y por lo tanto las dimensiones de $\boldsymbol{x}_{\rightarrow}$ y de $\boldsymbol{u}_{\rightarrow}$, así como las de las matrices P_{cl}, H_c, P_{clu} y H_{cu}, dependerían del número de pasos del modo 1, es decir, de n_c. Sin embargo, es posible expresar las acciones de control del modo 2 de la misma forma que en el modo 1, es decir: $\boldsymbol{u}_{k+i|k} = -K\boldsymbol{x}_{k+i|k} + \boldsymbol{c}_{k+i|k}$, imponiendo $\boldsymbol{c}_{k+i|k} = 0$, para todo $i \geq n_c$. Esto es equivalente a extender $\boldsymbol{c}_{\rightarrow}$ a todo el horizonte de predicción, de tal manera que todas las componentes a partir de la n_c-ésima sean nulas. En tal supuesto, las ecuaciones (9) y (10) pueden representar a las predicciones en ambos modos. En la frontera de separación entre ambos modos, es decir, en el paso $(n_c + 1)$, la expresión del estado predicho, $\boldsymbol{x}_{k+n_c|k}$, puede deducirse de (9), obteniéndose lo siguiente:

$$\boldsymbol{x}_{k+n_c|k} = P_{cl2}\boldsymbol{x}_k + H_{c2}\boldsymbol{c}_{\rightarrow} \tag{11}$$

siendo P_{cl2} y H_{c2} las submatrices de P_{cl} y H_c, respectivamente, correspondientes a la fila $(n_c + 1)$-ésima de $\boldsymbol{x}_{\rightarrow}$. La razón por la que se ha detallado la expresión correspondiente al estado $\boldsymbol{x}_{k+n_c|k}$ (ecuación (11)) es que las acciones de control predichas para el modo 2 dependen todas de tal estado (véase la ecuación (6)).

Finalmente, cabe observar y resaltar que las expresiones de las predicciones dadas por (9) y (10), es decir, las correspondientes a los estados $\boldsymbol{x}_{\rightarrow}$ y a las acciones de control $\boldsymbol{u}_{\rightarrow}$, tienen, ambas, forma de *combinación lineal* o *función afín* de $\boldsymbol{x}_k$ y $\boldsymbol{c}_{\rightarrow}$, lo cual facilita la computación.

Función de coste $J_{k(CLP)}$. Asumiremos como función de coste la expresión siguiente (habitual en control predictivo):

$$\begin{aligned} J_{k(CLP)} &= \sum_{i=0}^{\infty}(\boldsymbol{x}_{k+i+1}^T Q\boldsymbol{x}_{k+i+1} + \boldsymbol{u}_{k+i}^T R\boldsymbol{u}_{k+i}) \\ &= \sum_{i=0}^{n_c-1}(\boldsymbol{x}_{k+i+1}^T Q\boldsymbol{x}_{k+i+1} + \boldsymbol{u}_{k+i}^T R\boldsymbol{u}_{k+i}) \\ &\quad + \sum_{i=n_c}^{\infty}(\boldsymbol{x}_{k+i+1}^T Q\boldsymbol{x}_{k+i+1} + \boldsymbol{u}_{k+i}^T R\boldsymbol{u}_{k+i}) \end{aligned} \tag{12}$$

donde los parámetros de ponderación Q y R son matrices definidas positivas y constituyen parámetros de sintonía en las estrategias de control predictivo. Llevando ahora a cabo el oportuno desarrollo y las consideraciones necesarias, la expresión (12) puede ser formalizada de la forma siguiente:

$$J_{k(CLP)} = \boldsymbol{c}_{\rightarrow}^T S_c\boldsymbol{c}_{\rightarrow} + 2\boldsymbol{c}_{\rightarrow}^T S_{cx}\boldsymbol{x}_{\boldsymbol{k}} + k \tag{13}$$

donde:

$$\begin{aligned} S_c &= H_c^T diag(Q)H_c + H_{cu}^T diag(R)H_{cu} \\ &\quad + H_{c2}^T PH_{c2} \\ S_{cx} &= H_c^T diag(Q)P_{cl} + H_{cu}^T diag(R)P_{clu} \\ &\quad + H_{c2}^T PP_{cl2} \end{aligned}$$

siendo posible ignorar el término k (el tercer sumando de la expresión de $J_{k(CLP)}$), dado que es independiente de $\boldsymbol{c}_{\rightarrow}$ y que la optimización se planteará respecto de $\boldsymbol{c}_{\rightarrow}$. Por tanto, podemos tomar como función $J_{k(CLP)}$ la expresión siguiente (equivalente a (13), en cuanto al procedimiento de optimización se refiere):

$$J_{k(CLP)} = \boldsymbol{c}_{\rightarrow}^T S_c\boldsymbol{c}_{\rightarrow} + 2\boldsymbol{c}_{\rightarrow}^T S_{cx}\boldsymbol{x}_{\boldsymbol{k}} \tag{14}$$

siendo S_c y S_{cx} las mismas expresiones que las especificadas arriba (en la ecuación (13)).

Como puede observarse en (14), la función de coste es cuadrática respecto de $\boldsymbol{c}_{\rightarrow}$, es decir, respecto de la variable matricial que contiene los grados de libertad del problema de optimización.

Formalización del problema de optimización y determinación de la ley de control correspondiente a la estructura CLP-MPC. De las dos posibilidades que cabría considerar, sin restricciones y con restricciones, el caso apropiado en el contexto del presente trabajo es el segundo. Las restricciones más habituales son las que afectan a los valores de las entradas manipulables y/o a sus incrementos, pero también puede haber restricciones en las salidas e incluso en los estados. En cualquier caso, en (Rossiter, 2003) se demuestra que todas las restricciones pueden expresarse de modo conjunto mediante una inecuación matricial de la forma genérica siguiente, donde L integra todos los límites o cotas:

$$M\boldsymbol{c}_{\rightarrow} + N\boldsymbol{x}_k - L \leq 0 \tag{15}$$

es decir, que las restricciones pueden formalizarse mediante una expresión afín respecto de las variables $\boldsymbol{c}_{\rightarrow}$ (*d.o.f*) y $\boldsymbol{x}_k$ (estado *actual*), con L (cotas) como término independiente.

Teniendo en cuenta ahora la expresión de la función de coste, mostrada en la ecuación (14) y la de las restricciones, mostrada en la ecuación (15), la formulación del problema de optimización para el caso con restricciones quedará de la forma siguiente:

$$\begin{gathered}\min_{\boldsymbol{c}_{\rightarrow}} J_{k(CLP)} = (\boldsymbol{c}_{\rightarrow}^T S_c \boldsymbol{c}_{\rightarrow} + 2\boldsymbol{c}_{\rightarrow}^T S_{cx}\boldsymbol{x_k}) \\ s.t. \\ M\boldsymbol{c}_{\rightarrow} + N\boldsymbol{x}_k - L \leq 0\end{gathered} \tag{16}$$

La solución de (16) nos dará la $\boldsymbol{c}_{\rightarrow}$ óptima para el caso con restricciones, $\boldsymbol{c}_{optR\rightarrow}$, y el primer elemento de esa secuencia, al que denotaremos como $\boldsymbol{c}_{kR}$, será el término complementario que habrá que sumar a la ley de control base para obtener la acción de control predictivo CLP-MPC, con restricciones, correspondiente al instante k, es decir:

$$\boldsymbol{u}_{kR} = -K\boldsymbol{x}_k + \boldsymbol{c}_{kR} \tag{17}$$

3. Estrategia de control predictivo FMBPC aplicada al control de fangos activados

La estrategia FMBPC abarca distintas alternativas, dependiendo de los principios y métodos aplicados para obtener la ley de control. La ley FMBPC tomada como referencia en el presente trabajo se dedujo (como se indicó en la Introducción) aplicando el denominado *principio de equivalencia*, propio del control PFC, obteniéndose como resultado una ley de control analítica y explícita. La forma original de esta ley fue deducida en (Vallejo et al., 2019), particularizada para el caso de un proceso de depuración de aguas residuales mediante fangos activados. Este tipo de proceso biológico ha sido elegido también como caso de estudio en nuestro trabajo y se describe brevemente a continuación, tanto el propio proceso, como el modelado matemático del mismo.

3.1 Proceso biológico de depuración de aguas residuales mediante fangos activados: identificación y modelado

El caso de estudio considerado consiste en un proceso de depuración biológica de aguas residuales mediante fangos activados que se caracteriza principalmente por lo siguiente: tiene una dinámica compleja debido a su carácter biológico, es fuertemente no lineal y es multivariable. A efectos de reducción de la complejidad, para un mejor tratamiento, se ha considerado un modelo simplificado, pero manteniendo en lo esencial las interesantes características de estudio que acabamos de enunciar.

En (Vallejo et al., 2019) puede verse una descripción detallada de esta planta y de su dinámica. A continuación, en la Figura 1, se representa de modo esquemático tal proceso, incluyéndose las principales variables físico-químicas relacionadas. Desde el punto de vista de nuestro sistema de control, esta planta es vista como un sistema multivariable, con tres entradas y dos salidas: una entrada manipulable, dos perturbaciones en la entrada y dos salidas controladas. Esta configuración de entrada-salida es mostrada en la Figura 2, donde se incluye también cuáles son las variables implicadas, su rol en el marco del sistema de control y su significado físico:

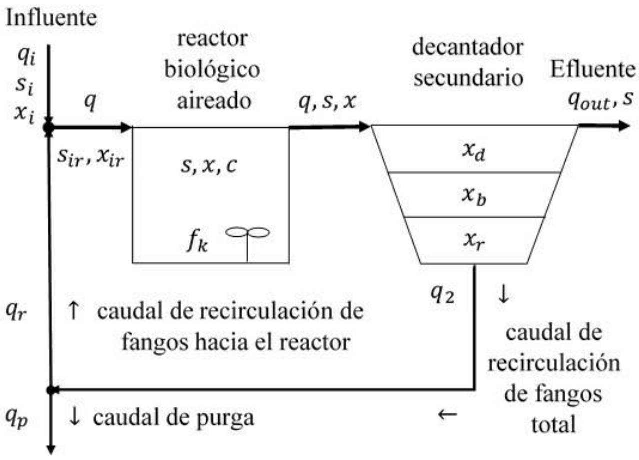


Figura 1: Proceso biológico de fangos activados simplificado.

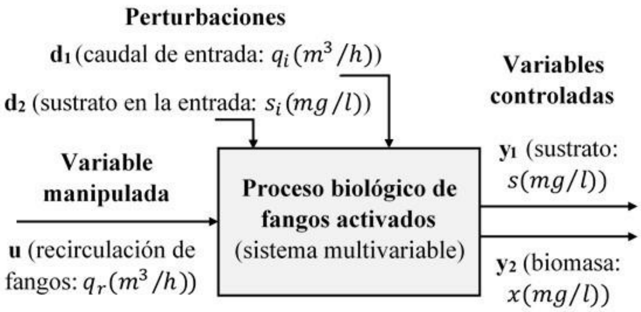


Figura 2: Sistema multivariable: tres entradas y dos salidas.

Modelo matemático fisicoquímico. Nuestro trabajo se ha llevado a cabo en simulación, representando la planta mediante un modelo matemático continuo, constituido por ecuaciones diferenciales que describen los balances de masa del sustrato, la biomasa y el oxígeno (en el reactor y en el decantador). El modelo utilizado puede consultarse en (Francisco et al., 2006; Vallejo et al., 2019) y consiste en una simplificación del modelo standard conocido como *Activated Sludge Model No. 1* (*ASM1*) (Henze et. al. ,1987).

Identificación borrosa. La estrategia FMBPC considerada en este trabajo utiliza como modelo base para para llevar a cabo las predicciones un modelo borroso de la planta, de tipo TS. Este modelo se obtiene mediante identificación borrosa, en nuestro caso a partir de series de datos de entrada-salida de la planta obtenidos mediante simulación en lazo abierto (aunque podrían proceder también de experimentos reales). Las series de datos temporales de entrada utilizados, agrupados en matrices de tres columnas (dos para las perturbaciones y una para la variable manipulada), contienen valores típicos de una planta real de tamaño medio que fue tomada como referencia. Esta planta es la depuradora municipal de la ciudad de Manresa (Barcelona, Spain), estudiada en su etapa inicial, en la década de los años 1990. En esas fechas se registraron series de datos reales de entrada-salidas correspondientes a diferentes campañas llevadas a cabo, siendo utilizados esos datos en trabajos de control predictivo de procesos de fangos activados (Moreno, 1994). En el presente trabajo también se han tomado como referencia esas colecciones de datos, utilizando los datos de entrada, o valores cercanos a los mismos, en los experimentos de simulación en lazo abierto. Como resultado de estas simulaciones se obtuvieron los correspondientes datos de salida.

A partir de los datos numéricos de entrada-salida obtenidos en simulación se desarrollaron diversos procesos de identificación borrosa, cada uno de los cuáles produjo como resultado un modelo borroso discreto de la planta en forma de reglas cualitativas del tipo si-entonces, con diferentes estructuras, valores de los parámetros de identificación y grados de validación (véase: Vallejo et al., 2019, 2021). En estos modelos, cada una de las reglas representa un clúster o submodelo, cuya validez será mayor cuanto mayor sea el *grado de pertenencia* de cada una de las componentes del *vector antecedente* instantáneo respecto de los correspondientes conjuntos borrosos. El vector antecedente está constituido por un cierto conjunto de variables que, en mayor o menor grado, influyen en la salida descrita de la planta. La identificación fue realizada mediante software desarrollado en el entorno *Matlab® & Simulink®* (The MathWorks, Inc., Natick, Massachusetts, USA), haciendo uso de la herramienta de software denominada FMID *(Fuzzy Model Identification Toolbox)* (Babuška, 1998b), adaptada convenientemente para su correcto funcionamiento en nuestro caso. La herramienta FMID está basada en técnicas de *clusterización* por medio del algoritmo de Gustafson-Kessel y fue desarrollada como soporte del libro *Fuzzy Modelling for Control* (Babuška, 1998a).

Modelo equivalente en el espacio de estados. Los modelos del proceso de fangos activados identificados por la herramienta FMID, de tipo borroso y discreto, fueron transformados en (Vallejo et al., 2019) en modelos discretos equivalentes en el espacio de estados, lineales, pero con matrices dependientes del tiempo, es decir, modelos de tipo DLTV. Mostramos a continuación la forma del modelo equivalente:

$$\begin{gathered}\boldsymbol{z_m}(k+1) = \boldsymbol{\bar{A}_m}\boldsymbol{z_m}(k) + \boldsymbol{\bar{B}_m}\boldsymbol{u_a}(k) + \boldsymbol{\bar{R}_m} \\ \boldsymbol{y_m}(k) = \boldsymbol{\bar{C}_m}\boldsymbol{z_m}(k)\end{gathered} \tag{18}$$

donde:

- $\boldsymbol{z_m}(k)$: *vector de estado extendido*, integrado por las dos salidas y las dos perturbaciones
- $\boldsymbol{u_a}(k) = \binom{u(k)}{u(k-1)}$: *vector de entrada extendido*, integrado por la variable manipulable en los instantes actual, $u(k)$ y anterior, $u(k-1)$
- $\boldsymbol{y_m}(k)$: *vector de salida*, integrado por las dos salidas controladas
- $\boldsymbol{\bar{A}}_m(k), \boldsymbol{\bar{B}}_m(k), \boldsymbol{\bar{R}}_m(k)$ y $\boldsymbol{\bar{C}}_m(k)$: *matrices del sistema en el espacio de estados*

3.2 Ley de control FMBPC

Haciendo uso del modelo que acaba de ser especificado y utilizando el *principio de equivalencia* (generalizado para sistemas multivariables), en (Vallejo et al., 2019) se dedujo una ley de control predictivo FMBPC, analítica y explícita, para el mismo caso de estudio que estamos considerando, es decir, para el proceso de fangos activados. La ley de control deducida es la siguiente, donde $u(k)$ es la entrada manipulable $q_r(k)$ (se omiten, por simplicidad, las expresiones matemáticas correspondientes al resto de variables y coeficientes):

$$\begin{aligned} u(k) = \boldsymbol{P_{10}}\boldsymbol{M_a}^{-1}(\, &\boldsymbol{y_r}(k+H) - \boldsymbol{y}(k) + \boldsymbol{y_m}(k) \\ &- \boldsymbol{\bar{C}_m}\boldsymbol{\bar{A}_m}^{H}\boldsymbol{z_m}(k) \\ -\boldsymbol{\bar{C}_m}(\boldsymbol{\bar{A}_m}^{\boldsymbol{H-1}} &+ (\boldsymbol{\bar{A}_m}^{\boldsymbol{H-1}} - \boldsymbol{I})(\,\boldsymbol{\bar{A}_m} - \boldsymbol{I})^{-\boldsymbol{1}})\boldsymbol{\bar{R}_m}\,) \end{aligned} \tag{19}$$

Observando la expresión anterior podemos concluir que $u(k)$ depende de las matrices $\boldsymbol{\bar{A}}_m, \boldsymbol{\bar{B}}_m, \boldsymbol{\bar{R}}_m$ y $\boldsymbol{\bar{C}}_m$, es decir, del modelo equivalente DLTV mostrado en la ecuación (18), de las *trayectorias de referencia* fijadas para las salidas ($\boldsymbol{y_r}$) y del parámetro de control predictivo *H (horizonte de coincidencia)*.

La ley de control FMBPC mostrada en la ecuación (19) será tomada como referencia en el presente trabajo, pero la expresión utilizada no será exactamente la misma, sino una expresión equivalente, justificada y detallada en la sección siguiente.

4. Integración de la estrategia FMBPC en una estructura de control predictivo CLP-MPC para el manejo de restricciones

El principal objetivo del presente trabajo consiste en integrar la estrategia FMBPC en un esquema de control predictivo CLP-MPC, como mecanismo para el manejo de restricciones, poniendo el foco en este caso en las restricciones de la acción de control. En la sección 2 de este artículo se expusieron los fundamentos del control predictivo en modo dual, en general, y para la estructura CLP-MPC en particular. La hipótesis de partida de los desarrollos mostrados es la existencia de un modelo lineal en el espacio de estados, representativo de la planta objeto de control, que pueda ser utilizado como *modelo de predicciones*. Así mismo, será necesaria una *ley de control base*, estabilizante, habitualmente una ley de realimentación de estados, que será utilizada en todo el horizonte de predicción de la forma siguiente: en el modo1, acompañada de un término aditivo sujeto a optimización y en el modo 2, sin ese término (o considerando que es nulo siempre). Es necesario, por tanto, elegir para nuestro caso ambas componentes de la estructura CLP-MPC.

4.1 Modelo de predicciones y ley de control base

Modelo de predicciones. El modelo del proceso de fangos activados detallado en la ecuación (18) fue reformulado en (Vallejo et al., 2021), obteniéndose otro modelo equivalente (también de tipo DLTV) y a partir de él, tomando como referencia un punto de equilibrio de la planta en lazo abierto, previamente elegido, se dedujo un *modelo incremental local* para el mismo proceso. El modelo deducido es un modelo en el espacio de estados, de tipo DLTI, válido para estados próximos al estado estacionario de referencia y tiene la forma siguiente:

$$\begin{gathered} \boldsymbol{x_{inc}}(k+1) = \boldsymbol{\bar{A}_{mN}} \boldsymbol{x_{inc}}(k) + \boldsymbol{\bar{B}_{mN}} \boldsymbol{u_{inc}}(k) \\ \boldsymbol{y_{inc}}(k) = \boldsymbol{\bar{C}_{mN}} \boldsymbol{x_{inc}}(k) \end{gathered} \qquad (20)$$

donde:

$\boldsymbol{\bar{A}_{mN}}$, $\boldsymbol{\bar{B}_{mN}}$ y $\boldsymbol{\bar{C}_{mN}}$ son las matrices correspondientes al estado estacionario

$\boldsymbol{x_{inc}}(k+1)$, $\boldsymbol{x_{inc}}(k)$, $\boldsymbol{u_{inc}}(k)$, $\boldsymbol{y_{inc}}(k)$: variables incrementales con respecto al punto de equilibrio

El modelo DLTI mostrado en la ecuación (20) es el que se utilizará como *modelo de predicciones* para representar al proceso de fangos activados en la estructura CLP-MPC. En la implementación *online* de la estrategia mixta FMBPC/CLP, para cada instante de muestreo k habrá un modelo diferente, dependiente de k, pero el modelo que se transferirá a la estructura CLP-MPC se mantendrá constante en ella durante las predicciones.

Ley de control base FMBPC. Partiendo de la expresión original de la ley de control FMBPC (ecuación (19)) y teniendo en cuenta la relación entre los diferentes modelos DLTV deducidos en (Vallejo et al., 2019, 2021) a los que hemos hecho referencia anteriormente, en (Vallejo et al., 2021) se demuestra que la ley de control FMBPC es equivalente, para el modelo incremental local mostrado en (20), a la expresión siguiente:

$$\boldsymbol{u_{inc}}(k) = -\left(\boldsymbol{M_{aN}}^{-1} \boldsymbol{\bar{C}_{mN}} \boldsymbol{\bar{A}_{mN}}^{\boldsymbol{H}}\right) \boldsymbol{x_{inc}}(k) \qquad (21)$$

donde $\boldsymbol{M_{aN}}$ es una expresión matricial que se obtiene a partir de $\boldsymbol{M_a}$ (matriz implícita en la ecuación (19)).

La ley de control obtenida es compatible con la conocida ley de realimentación de estados $\boldsymbol{u} = -\boldsymbol{Kx}$, es decir, la expresión mostrada en la ecuación (21) puede ser formalizada de manera análoga, como vemos a continuación:

$$\boldsymbol{u_{inc}}(k) = -\boldsymbol{K} \boldsymbol{x_{inc}}(k) \qquad (22)$$

donde: $\boldsymbol{K} = \left(\boldsymbol{M_{aN}}^{-1} \boldsymbol{\bar{C}_{mN}} \boldsymbol{\bar{A}_{mN}}^{\boldsymbol{H}}\right)$

En la implementación *online* de la estrategia mixta FMBPC/CLP, para cada instante de muestreo k habrá una $\boldsymbol{K}$ diferente, dependiente de k, pero esa $\boldsymbol{K}$ se mantendrá constante en el cálculo de predicciones de la estructura CLP-MPC correspondiente a ese instante de muestreo.

En (Vallejo et al., 2021) se demuestra que la ley de control mostrada en (21) y (22) garantiza la estabilidad (local) del sistema de control en lazo cerrado y, por tanto, podemos adoptarla para la estructura CLP-MPC como la ley de control base estabilizante, en los cálculos de las predicciones.

En la siguiente sección se utilizará una notación alternativa para las variables implicadas en esta ley de control, incorporando, por un lado, información relativa a su origen, denotando: $\boldsymbol{u_{inc}}(k) \equiv u_{inc.pred|FMBPC}$, y simplificando, por otro lado, la notación referente al estado: $\boldsymbol{x_{inc}}(k) \equiv \boldsymbol{x}(k)$. Con esos cambios, la ecuación (22) queda así:

$$u_{inc.pred|FMBPC} = -\boldsymbol{Kx}(k) \qquad (23)$$

4.2 Implementación de la estrategia mixta FMBPC/CLP

Tomando el modelo (incremental) mostrado en (20) como modelo de predicciones y la ley FMBPC mostrada en (23) como ley de control base del esquema de predicciones CLP-MPC, podremos hacer uso de las expresiones matemáticas mostradas en la sección 2 para una estructura genérica CLP-MPC. Así, el problema de imposición de restricciones a la ley de control FMBPC será equivalente a resolver un problema de optimización como el expresado en la ecuación (16) (con las particularidades matemáticas necesarias para adaptarlo a nuestro caso de estudio), cuya solución, en cada instante k, consistirá en una secuencia óptima de *perturbaciones* de la ley de control a lo largo del horizonte de predicción, $\boldsymbol{c}_{optR\rightarrow}$. El primer elemento de esa secuencia, $\boldsymbol{c}_{kR}$, será la *perturbación* óptima que deberá sumarse *online* a la ley de control base, obteniendo la acción de control óptima que garantizará el cumplimiento de las restricciones: $\boldsymbol{u}_{kR} = -K\boldsymbol{x}_k + \boldsymbol{c}_{kR}$ (ecuación (17)). Por otra parte, dado que el modelo usado en la estructura CLP-MPC es un modelo incremental, a continuación habrá que sumarle a $\boldsymbol{u}_{kR}$ la acción de control correspondiente al estado estacionario, que denotaremos con u_{ss}. En la implementación *online* de la estrategia mixta FMBPC/CLP este proceso se repetirá sistemáticamente cada período de muestreo, obteniendo en cada iteración un valor óptimo de la acción de control, dependiente de k. Este procedimiento puede formalizarse matemáticamente mediante la siguiente secuencia de expresiones (ecuaciones (24) a (27)):

$$u_{opt}(k) = u_{inc.pred|FMBPC} + c_{opt}(k) \qquad (24)$$

donde:

$u_{opt}(k)$: acción de control óptima en k (correspondiente al modelo incremental de predicciones), calculada mediante la estructura CLP-MPC (en las Figuras 4c, 4d, 4e y 5c, 5d y 5e: $u_{opt}(k) \equiv u_{mod.pred}(k)$)

$u_{inc.pred|FMBPC}$: ley de control base

$c_{opt}(k)$: *perturbación* óptima de la acción de control en k, calculada mediante la estructura CLP-MPC (en las Figuras 4c, 4d, 4e y 5c, 5d y 5e: $c_{opt}(k) \equiv c(k)$)

que puede expresarse también de la forma siguiente (utilizando la ecuación (23)):

$$u_{opt}(k) = -\boldsymbol{Kx}(k) + c_{opt}(k) \qquad (25)$$

donde:

$-\boldsymbol{Kx}(k)$: ley de control base FMPPC

$\boldsymbol{x}(k)$: *vector de estado* correspondiente al modelo incremental de predicciones y sumándole a $u_{opt}(k)$, expresado según la ecuación (24), la acción de control correspondiente al estado estacionario de referencia, u_{ss} y reordenando, obtendremos la acción de control global, $u_{FMBPC/CLP}(k)$:

$$\begin{aligned} u_{FMBPC/CLP}(k) &= u_{opt}(k) + u_{ss} \\ &= \left(u_{inc.pred|FMBPC} + c_{opt}(k)\right) + u_{ss} \\ &= \left(u_{inc.pred|FMBPC} + u_{ss}\right) + c_{opt}(k) \end{aligned} \tag{26}$$

que puede ser expresada finalmente, de forma más sencilla y compacta, como sigue:

$$u_{FMBPC/CLP}(k) = u_{pred|FMBPC} + c_{opt}(k) \tag{27}$$

donde: $u_{pred|FMBPC} = \left(u_{inc.pred|FMBPC} + u_{ss}\right)$

La ecuación (27) representa la expresión final de la implementación *online*, en cada instante k, de la ley de control correspondiente a la estrategia mixta FMBPC/CLP. En la práctica es equivalente a calcular la acción de control predictiva FMBPC con el modelo global y sumarle $c_{opt}(k)$, si asumimos que, para estados cercanos al estado estacionario de referencia, el modelo global es equivalente al modelo local incremental. La ley de control FMBPC/CLP obtenida está optimizada para el cumplimiento de las restricciones, que en nuestro caso de estudio se han concretado en las variaciones de la acción de control, en forma de cotas, máximas y mínimas. Por último, en relación con la estabilidad de la estrategia, asumiremos la conclusión establecida al respecto en (Rossiter, 2003) respecto a la estructura CLP-MPC, que establece que si la ley de control base en lazo cerrado es estabilizante, la estrategia de control predictivo en lazo cerrado será estable.

El esquema de implementación *online* de la estrategia FMBPC/CLP propuesta en el presente artículo se muestra en la Figura 3 y mediante el Algoritmo 1 se describe y explica de manera ordenada el procedimiento de implementación seguido. Para cada instante k se calcula, por un lado la acción de control predictivo $u_{pred|FMBPC}(k)$, y por otro, haciendo uso de la estructura CLP-MPC, la perturbación de control óptima $c_{opt}(k)$ que garantiza la satisfacción de las restricciones. Este segundo cálculo se realiza mediante la estructura CLP-MPC, a partir de la información transmitida por la estructura FMBPC, relativa a las matrices del modelo de predicciones, $\overline{\boldsymbol{A}}_{\boldsymbol{mN}}(k)$, $\overline{\boldsymbol{B}}_{\boldsymbol{mN}}(k)$, $\overline{\boldsymbol{C}}_{\boldsymbol{mN}}(k)$ y al *factor* $\boldsymbol{K}(k)$ de la ley de control base, todo ello actualizado para el instante de muestreo actual, k. El cálculo de predicciones y el procedimiento de optimización son realizados dentro de la estructura CLP-MPC utilizando un modelo lineal, pero el procedimiento en su conjunto es no lineal, dado que en el transcurso del tiempo de implementación, en cada iteración, se actualizan $\overline{\boldsymbol{A}}_{\boldsymbol{mN}}$, $\overline{\boldsymbol{B}}_{\boldsymbol{mN}}$, $\overline{\boldsymbol{C}}_{\boldsymbol{mN}}$ y $\boldsymbol{K}$. Tenemos por tanto que la estrategia mixta de control predictivo descrita es, globalmente, no lineal.

Algoritmo 1. Implementación de la estrategia FMBPC/CLP

1. Elección del modelo borroso FM (identificado a partir de series de datos de entrada/salida).
2. Elección de patrones para las perturbaciones de entrada (conforme a series de datos experimentales).
3. Elección de referencias y trayectorias de referencia.
4. Inicialización de variables: punto de inicio, vector de estado, vector antecedente.
5. Elección de parámetros de la estrategia de control FMBPC y de la estructura CLP. Principales parámetros: horizonte de coincidencia/FMBPC (H); número de pasos del modo 1 de la estructura CLP (n_c); cotas superior e inferior para la variable manipulada del *modelo de predicciones* de la estructura CLP ($u_{máx}$ y $u_{mín}$).
6. Actualización de los modelos DLTV/DLTI en el instante de tiempo *actual*, k.
7. Cálculo/actualización de la $\boldsymbol{K}(k)$ correspondiente a la ley de control base ($-\boldsymbol{Kx}$).
8. Cálculo de la variable de control FMBPC correspondiente al instante de tiempo *actual*: $u_{pred}(k)$.
9. Envío a la estructura CLP de las matrices del *modelo de predicciones*, $\overline{\boldsymbol{A}}_{\boldsymbol{mN}}(k)$, $\overline{\boldsymbol{B}}_{\boldsymbol{mN}}(k)$ y $\overline{\boldsymbol{C}}_{\boldsymbol{mN}}(k)$ y del *factor* $\boldsymbol{K}(k)$ de la ley de control base.
10. Cálculo de la perturbación de la acción de control, $c(k)$, mediante optimización dentro de la estructura CLP.
11. Cálculo de la variable de control de la estrategia mixta FMBPC/CLP en k : $u_{FMBPC/CLP}(k) = u_{pred}(k) + c(k)$. Aplicación de la variable de control $u_{FMBPC/CLP}(k) \equiv u(k)$ al proceso de fangos activados.
12. Si $t < t_{simul}$, volver al paso 6; si $t = t_{simul}$, terminar.

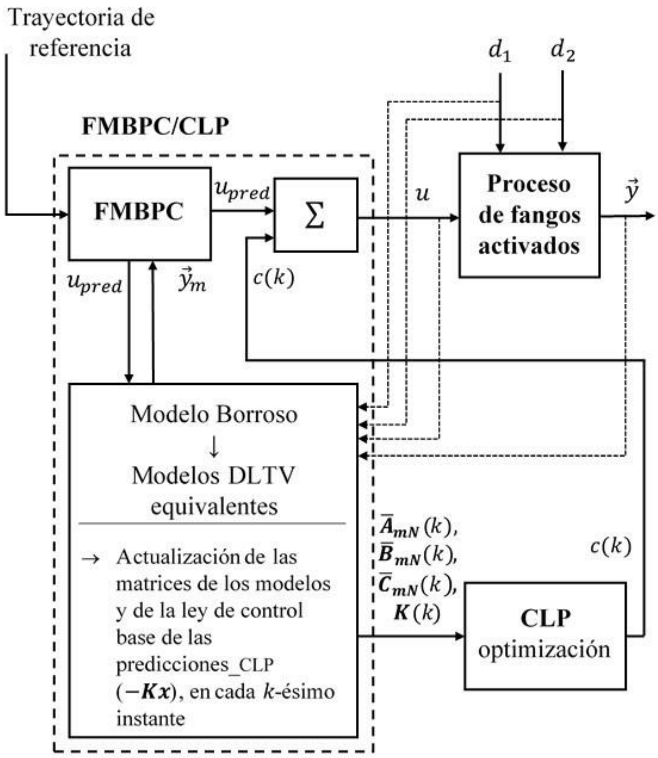


Figura 3. Estrategia mixta FMBPC/CLP: esquema de implementación.

5. Experimentos: simulación y resultados

La estrategia FMBPC/CLP propuesta fue aplicada en simulación al control multivariable de un proceso de fangos activados. La implementación se llevó a cabo utilizando el mismo entorno de software que el utilizado en la identificación, es decir, el entorno *Matlab® & Simulink®*. Las variables objeto de control simultáneo en los experimentos realizados fueron dos: la concentración de sustrato en el efluente y la concentración de biomasa en el reactor. Desde el punto de vista medioambiental, la principal variable a controlar es la concentración de sustrato, pero el control de la biomasa también tiene su importancia desde el punto de vista operativo en las plantas de tratamiento de aguas residuales. Además, el control simultáneo de esas dos variables es planteado como caso de estudio de una estrategia de control predictivo borroso multivariable. Por tanto, nos interesa por igual el control de ambas variables. En los experimentos realizados, la referencia de la concentración de sustrato fue mantenida en un valor constante a lo largo del tiempo, un valor bajo, supuestamente compatible con el máximo previsto por la legislación medioambiental, mientras que la referencia de la concentración de biomasa fue variable (constante por tramos, pero con dos saltos en dos momentos diferentes del tiempo).

Se muestra a continuación, en la Tabla 1, la configuración de dos de los muchos experimentos realizados, a los que hemos denominado caso 1 y caso 2, respectivamente, caracterizados por tener dos secuencias diferentes para la referencia de la concentración de biomasa y, en consecuencia, dos zonas de operación diferentes. Las referencias de la concentración de sustrato se mantuvieron constantes en el tiempo y con el mismo valor en ambos casos. Para cada uno de los dos casos se consideraron tres subcasos, correspondientes a tres pares de valores diferentes de las cotas mínima y máxima de la acción de control del modelo local de predicciones. El punto de inicio de los experimentos realizados (s_0 , x_0), fue el mismo en ambos casos: $s_0 = 200\,(mg/l)$ y $x_0 = 2000\,(mg/l)$. En la Tabla 1 se detallan los valores o referencias de los diferentes parámetros, tanto de la estrategia FMBPC, como de la estructura CLP-MPC, así como los parámetros de simulación. En relación con la estrategia FMBPC cabe resaltar la importancia del modelo borroso de predicciones (FM). Sin embargo, los detalles del modelo elegido (FM_1) no pueden ser incluidos en la Tabla, debido a las limitaciones de espacio del artículo. Pueden verse sus características en (Vallejo et al., 2021). Y en relación con los parámetros de sintonía Q y R, en los experimentos correspondientes a los resultados mostrados se eligió un valor unitario para R, es decir $R = 1$, y para Q se consideró la relación matemática $Q = C'C$, obteniéndose en nuestro caso: $Q = C'C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, lo cual puede ser expresado formalmente también como sigue: $Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$, con $Q_1 = 1$ y $Q_2 = 1$ (se probaron algunas otras parametrizaciones, pero sin variaciones significativas).

Tabla 1: Configuración de los experimentos de control realizados utilizando la estrategia mixta FMBPC/CLP

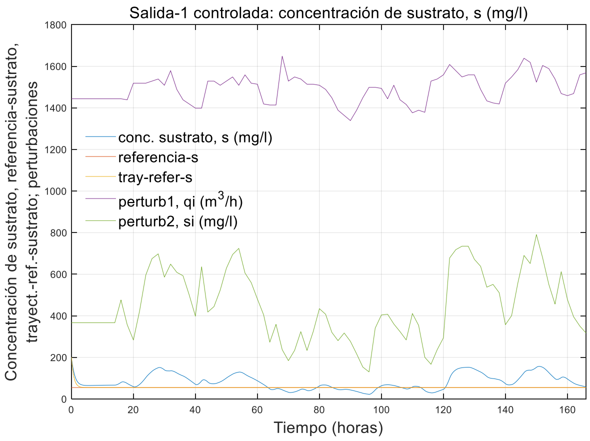
Caso		VC	Pert.	Estrategia FMBPC				Estructura CLP-MPC						Parámetros de simulación			
				FM	a_{r_1}	a_{r_2}	H	Q_1	Q_2	R	n_c	$u_{mín}$	$u_{máx}$	$I.sim.$	s_{ref}	$x_{ref}\,(saltos)$	$t.salto$
1	$1a$	s , x	D_A	FM_1	0.76	0.96	6	1	1	1	42	-200	200	0 a 166	55	1800 a 2200 2200 a 2000	31 81
	$1b$											-100	100				
	$1c$											-5	5				
2	$2a$	s , x	D_A	FM_1	0.76	0.96	6	1	1	1	42	-200	200	0 a 166	55	2000 a 1900 1900 a 2000	31 81
	$2b$											-100	100				
	$2c$											-5	5				

donde:

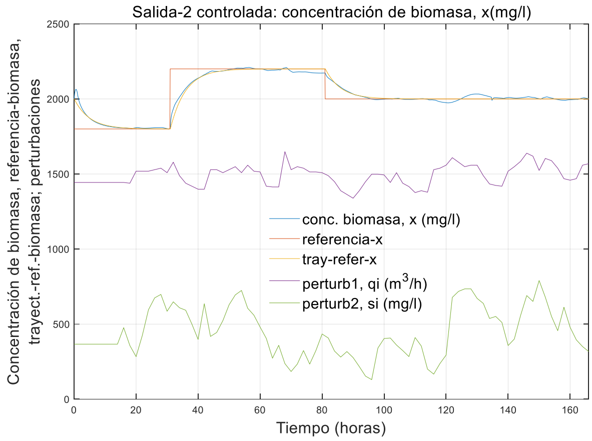
VC: variables controladas (s, x); punto de inicio de los experimentos: $s_0 = 200\ (mg/l)$, $x_0 = 2000\ (mg/l)$
Pert.: perturbaciones de entrada ($q_i(m^3/h)$ y $s_i(mg/l)$); patrón elegido: D_A (véase: Vallejo et al., 2021)
FM: modelo borroso (*fuzzy model*); FM_1: FM elegido (véase: Vallejo et al., 2021)
a_{r_1}, a_{r_2}: parámetros de la trayectoria de referencia (véase: Vallejo et al., 2021)
H: horizonte de coincidencia
Q_1, Q_2, R: parámetros de sintonía correspondientes a la función de coste $J_{k(CLP)}$ de la estructura CLP-MPC
n_c: número de pasos del *modo 1* en la estructura CLP
$u_{mín}$, $u_{máx}$: cotas mínima y máxima, respectivamente, de la acción de control ($q_r(m^3/h)$) del modelo de predicciones
$I.sim.(h.)$: intervalo de simulación, en horas
$s_{ref}\ (mg/l)$: referencia de la concentración de sustrato en el efluente
$x_{ref}(mg/l)$: referencia de la concentración de biomasa en el reactor, con dos saltos en dos instantes de tiempo diferentes
$t.salto$: tiempos de ocurrencia de los saltos en la referencia de la concentración de biomasa, en horas

Para cada uno de los dos casos presentados se hicieron tres pruebas (subcasos) considerando tres parejas de valores diferentes para las cotas $u_{mín}$ y $u_{máx}$ de la variable manipulable del modelo de predicciones (las mismas parejas de valores para ambos casos). Los resultados de las pruebas muestran que las secuencias de valores de las *perturbaciones*

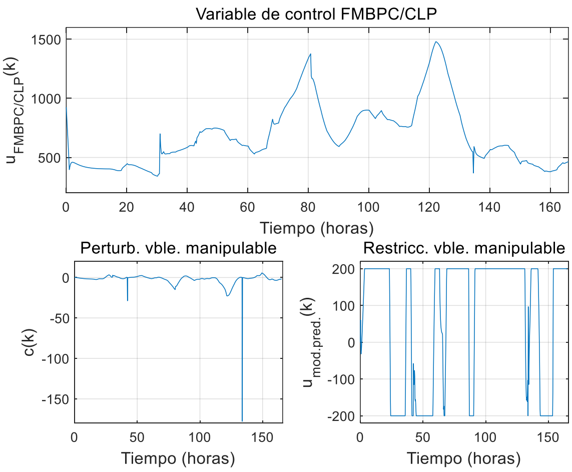
$c(k)$ obtenidas (calculadas como óptimas para asegurar las restricciones) dependen del caso y, para cada caso, dependen de los valores de las cotas. En lo que se refiere a la acción de control final de la estrategia mixta FMBPC/CLP, $u_{FMBPC/CLP}(k)$, se observó que la secuencia de acciones dependía sobre todo del caso, aunque para cada subcaso presentase algunos matices. Por ello y para no alargar excesivamente el artículo, sólo se mostrará, para cada caso, la $u_{FMBPC/CLP}(k)$ correspondiente a una de las tres pruebas (en el análisis posterior se justificará la elección del subcaso). Los resultados obtenidos, expresados gráficamente, se han agrupado en la Figura 4 (caso 1) y en la Figura 5 (caso 2):



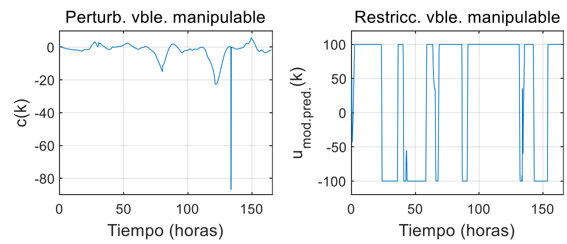
(**a**) Concentración de sustrato en el efluente y perturbaciones.



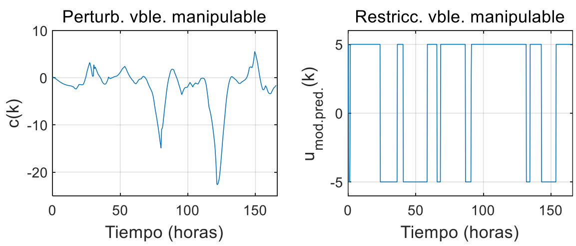
(**b**) Concentración de biomasa en el reactor y perturbaciones.



(c) Variable de control global $u_{\text{FMBPC/CLP}}(k)$, perturbaciones $c(k)$ y restricciones en $u_{mod.pred}(k)$ (mod. incremental) entre -200 y 200 (caso 1a).

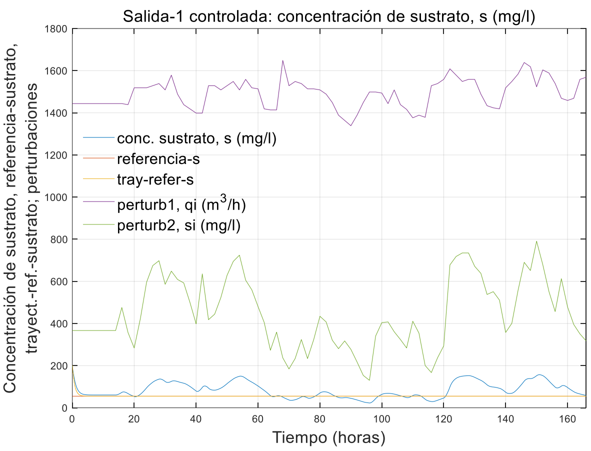


(d) Perturbaciones $c(k)$ y restricciones en $u_{mod.pred}(k)$ (mod. incremental) entre -100 y 100 (caso 1b).

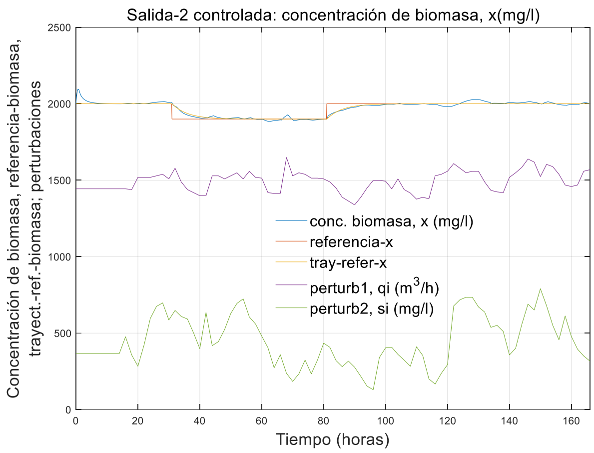


(e) Perturbaciones $c(k)$ y restricciones en $u_{mod.pred}(k)$ (mod. incremental) entre -5 y 5 (caso 1c).

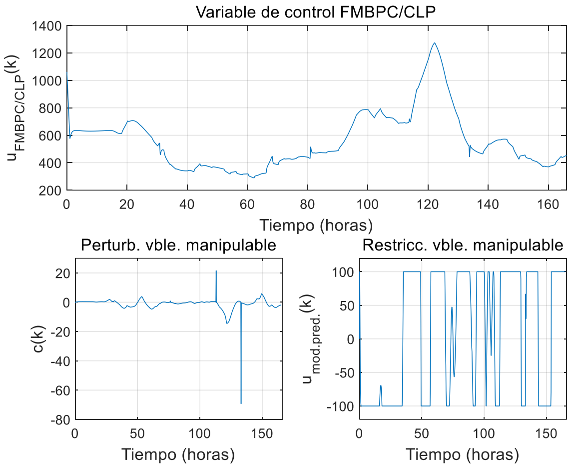
Figura 4. Experimentos FMBPC/CLP—Caso 1 (1a, 1b y 1c).



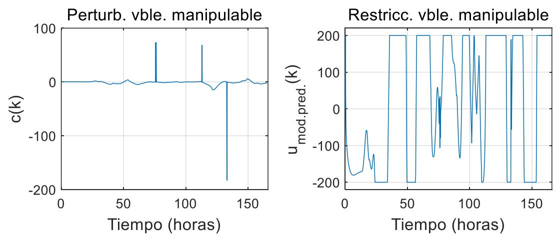
(a) Concentración de sustrato en el efluente y perturbaciones.



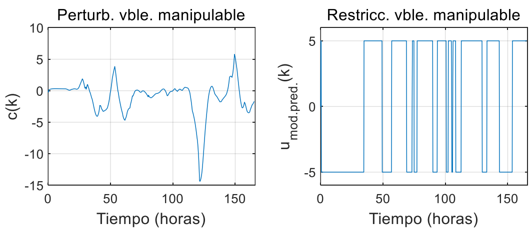
(b) Concentración de biomasa en el reactor y perturbaciones.



(c) Variable de control global $u_{\mathrm{FMBPC/CLP}}(k)$, perturbaciones $c(k)$ y restricciones en $u_{mod.pred}(k)$ (mod. incremental) entre -100 y 100 (caso 2b).



(d) Perturbaciones $c(k)$ y restricciones en $u_{mod.pred}(k)$ (mod. incremental) entre -200 y 200 (caso 2a).



(e) Perturbaciones $c(k)$ y restricciones en $u_{mod.pred}(k)$ (mod. incremental) entre -5 y 5 (caso 2c).

Figura 5. Experimentos FMBPC/CLP—Caso 2 (2b, 2a y 2c).

Análisis de los resultados. La primera consideración que procede hacer es que el mecanismo propuesto para la imposición de restricciones en la acción de control FMBPC, integrando esta estrategia en una estructura CLP-MPC, es realizable y eficaz. En los 6 subcasos presentados vemos que la perturbación óptima $c(k)$ calculada mediante este mecanismo actúa adecuadamente para que las acciones de control del modelo de predicciones satisfagan las restricciones. Por otra parte, se observa que las secuencias de valores $c(k)$ son diferentes para cada caso y subcaso, como cabría esperar. Así mismo, se observa claramente también que cuanto más grande sea el intervalo de restricciones de las acciones de control, menos se saturará en los extremos la acción de control. Este grado de saturación resulta lógico en el contexto de los experimentos considerados, puesto que las perturbaciones de entrada (q_i y s_i) son grandes en magnitud y con grandes oscilaciones, de tal forma que el algoritmo de control principal, es decir, el correspondiente a la estrategia FMBPC, trabaja para que las salidas sigan sus trayectorias de referencia a pesar de las perturbaciones, es decir, trata de rechazar las amplias perturbaciones incrementando la acción de control todo lo que sea necesario. Pero claro, cuando la orden del algoritmo tienda a incrementar la acción de control por encima de las cotas, la acción de control se saturará como consecuencia de la actuación del mecanismo de imposición de restricciones implementado. Y esta saturación será menor cuanto mayor sea el margen de actuación del algoritmo principal. Sin embargo, un margen muy amplio para las variaciones de la acción de control puede tener como consecuencia una acción global con cambios bruscos, es decir, no gradual. Estas observaciones nos conducen al razonamiento de que, desde el punto de vista operativo, parece conveniente buscar un equilibrio o compromiso entre las posibles ventajas derivadas de la imposición de un intervalo pequeño para las acciones de control (menor consumo energético, menor esfuerzo de los actuadores o acción de control global más suave o gradual) y las asociadas a una mayor grado de libertad de las acciones de control (capacidad para alcanzar el seguimiento de las trayectorias de referencia en presencia de fuertes perturbaciones).

Por otra parte, además del rechazo de las perturbaciones, otro objetivo habitual de cualquier sistema de control es, naturalmente, el seguimiento de la referencia. En nuestro caso, este objetivo resulta implícito de forma evidente en el propio diseño de los experimentos, los cuales incluyen dos saltos en la referencia de la concentración de la biomasa, diferentes para cada uno de los dos casos. La magnitud de los saltos del caso 2 es menor que la de los saltos del caso 1 y por tanto cabría esperar para el caso 2 una pequeña reducción de la necesidad de variabilidad de la acción de control para responder adecuadamente a esos saltos, lo cual parece corroborarse observando que mientras que en el caso 1 el intervalo de restricciones necesario para que comience a reducirse el grado de saturación de la acción de control es el intervalo $(-200, 200)$, en el caso 2 es posible tener una reducción similar de la saturación para restricciones de la acción de control en el intervalo $(-100, 100)$.

Por último, en relación con la acción de control final de la estrategia mixta FMBPC/CLP, $u_{FMBPC/CLP}(k)$, a partir de la observación de los resultados correspondientes a ambos casos, cabe deducir dos cosas. Por un lado, que la acción de control final de la estrategia propuesta es capaz de conducir a las dos salidas a sus valores de referencia, simultáneamente, con un seguimiento muy aceptable de las correspondientes trayectorias, para distintas zonas de operación, en presencia de fuertes perturbaciones en la entrada y de forma gradual y sin grandes esfuerzos. Por otro lado, cabe también comentar que, para cada uno de los dos casos, la acción de control final de los distintos subcasos es similar, aunque lógicamente presenta algunos matices derivados del margen de actuación permitido a la acción de control (mediante las restricciones). Solo se ha mostrado la acción de control final de uno de los subcasos (para abreviar la longitud del artículo), siendo el subcaso elegido el primero que permite una cierta reducción de la saturación, según vamos ampliado el margen de actuación.

6. Conclusiones

La principal conclusión que podemos extraer del trabajo desarrollado es que se ha conseguido integrar una estrategia de control predictivo borroso de tipo FMBPC, basado en control PFC, en una estructura de control predictivo en modo dual de tipo CLP-MPC, con el objetivo de dotar a la primera de un mecanismo de imposición de restricciones en la acción de control. Este objetivo ha sido alcanzado tomando la ley de control FMBPC analítica y explícita como ley de control base del modo 1 de la estructura CLP-MPC. Los experimentos con nuestro caso de estudio llevados a cabo en simulación corroboran que el mecanismo propuesto es posible y válido. Cabe resaltar la dificultad de la integración de ambas metodologías y la escasez de trabajos relativos a la estructura CLP-MPC.

Otra conclusión importante es que también es posible y útil diseñar técnicas de control avanzado basadas en la confluencia de técnicas de control inteligente con metodologías de control predictivo, en este caso dos metodologías diferentes y con restricciones. La parte inteligente de la estrategia mixta propuesta reside en el tipo de modelo utilizado como base de la estrategia FMBPC, que es un modelo borroso de tipo TS. La utilización de modelos borrosos, es decir de modelos cualitativos, obtenidos mediante identificación a partir de datos numéricos de entrada y salida, permite capturar la dinámica de la planta de forma muy fiel, incluso para sistemas complejos, muy cambiantes o desconocidos. Esta propiedad mejorará, lógicamente, la fidelidad de las predicciones y por tanto, en última instancia, mejorará el rendimiento del controlador predictivo.

Como conclusión final, resaltamos también el buen rendimiento de la estrategia mixta FMBPC/CLP, aplicada al control multivariable de procesos de fangos activados, un caso de estudio con un interés especial por su alta no linealidad y por la complejidad de la dinámica de la planta, debido a su carácter biológico.

En cuanto a posibles trabajos futuros, nos parece que sería interesante continuar y profundizar la investigación en los aspectos siguientes: a) influencia de los parámetros de cada una de las dos estrategias, FMBPC y CLP-MPC, en el comportamiento de las variables controladas, en el cumplimiento de las restricciones y en las características de la acción de control; b) aplicación de la estrategia propuesta a un caso de estudio con más restricciones, como restricciones en los incrementos de la acción de control (esfuerzos de control) u otros; c) utilización de modelos de predicciones alternativos en la estructura CLP-MPC, como modelos extendidos o aumentados, con los incrementos de la acción de control y/o con las perturbaciones; d) utilización de la estrategia propuesta en control MPC distribuido.

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Capítulo 5

Conclusiones

Las investigaciones y los trabajos llevados a cabo con motivo de la presente tesis doctoral nos llevan a enunciar las siguientes conclusiones:

- La primera conclusión importante es que las dos principales hipótesis de trabajo de partida han sido claramente verificadas. En relación con la primera hipótesis (la hipótesis principal), relativa a la potencialidad de la utilización en control predictivo de modelos borrosos obtenidos mediante identificación borrosa (a partir de datos numéricos de entrada-salida), creemos que queda comprobada con el trabajo descrito en la publicación nº1, donde se demuestra que la estrategia FMBPC propuesta, basada en modelos borrosos de la planta objeto de estudio (proceso ASP), es capaz de controlar, de manera eficaz y con un buen rendimiento del controlador (evaluado mediante el índice *ISE*), un sistema fuertemente no lineal y multivariable, en presencia de fuertes perturbaciones, comportándose incluso de manera robusta frente a cambios en el punto de operación a lo largo del tiempo. Y en relación con la segunda hipótesis, referente a la previsible utilidad de la formalización de los modelos borrosos en modelos equivalentes en el espacio de estados y la posterior aplicación del *Principio de Equivalencia* (propio del control predictivo PFC) para deducir la ley de control, también queda demostrado en la publicación nº 1 que tal formalización es posible, incluso para sistemas altamente no lineales y multivariables, y que, utilizando el modelo equivalente y aplicando el *Principio de Equivalencia*, se deduce una ley de control predictivo, expresada en forma analítica y explícita, lo cual facilita tanto la implementación del algoritmo de control (los cálculos computacionales), como su posterior análisis, entre otros el de estabilidad. Y todo esto es posible incluso para sistemas complejos o desconocidos, partiendo de series numéricas de datos de entrada-salida suficientemente completas y aplicando métodos de identificación borrosa que proporcionen modelos con suficiente grado de validez.

- La segunda conclusión importante se refiere al análisis de estabilidad de la estrategia FMBPC propuesta. En el trabajo descrito en la publicación nº2 se presentó un procedimiento de estudio de la estabilidad diferente a la mayoría de los procedimientos habituales en control predictivo (y en sistemas *Fuzzy*), más práctico, basado, por un lado, en la deducción de modelos incrementales de tipo DLTI (válidos en las cercanías de algún punto de equilibrio) y, por otro, en la utilización de cálculo simbólico. La gran ventaja de trabajar con modelos DLTI es la posibilidad de aplicar los criterios de estabilidad existentes para esos modelos, bien definidos y ampliamente utilizados, como, entre otros, los criterios derivados de la teoría de estabilidad de Lyapunov. Sin embargo, los procedimientos de demostración del cumplimiento de tales criterios de estabilidad para sistemas con dinámicas complejas, como es nuestro caso de estudio, son a menudo matemáticamente bastante laboriosos y también difíciles de generalizar. Por tal motivo, en el presente trabajo se decidió realizar el análisis de estabilidad mediante un enfoque computacional, basado en cálculo simbólico, alternativo a los procedimientos habituales existentes en la literatura. El resultado es bastante satisfactorio, en el sentido de que mediante este procedimiento se ha podido establecer una conclusión importante respecto a la estabilidad en lazo cerrado (local), que es la siguiente: si la planta en lazo abierto es (localmente) asintóticamente estable, entonces para valores suficientemente grandes del *horizonte de coincidencia*, *H*, la planta en lazo cerrado también será (localmente) asintóticamente estable. Cabe decir, finalmente, que consideramos que el trabajo realizado constituye una contribución relevante, o al menos apreciable, en el campo del análisis de estabilidad de los sistemas de control de FMBPC, por dos razones: por un lado, debido a la sencillez del método desarrollado en relación con otros métodos (la mayoría de ellos, más complicados) y, por otro, porque se ha considerado un caso de estudio más complejo y más difícil de abordar (sistema MIMO, altamente no lineal y con dinámica compleja, debido a su naturaleza biológica) que los elegidos en trabajos similares anteriores (sistemas SISO con dinámicas no muy complicadas).

- Y la tercera conclusión importante se extrae del trabajo expuesto en la publicación nº3, donde se describe lo que se ha denominado como estrategia mixta FMBPC/CLP, consistente en la integración de una estrategia de control predictivo borroso de tipo FMBPC (basado en control PFC) dentro de una estructura de control predictivo en modo dual de tipo CLP-MPC (basado en optimización), con el objetivo de dotar a la primera de un mecanismo de imposición de restricciones en la ley de control. La conclusión es que es posible llevar a cabo la integración FMBPC/CLP y que mediante el adecuado diseño es posible cumplir el objetivo de imponer restricciones a la ley FMBPC y además con estabilidad en lazo cerrado (en base a las propiedades de la estructura CLP-MPC descritas en la literatura). Concretamente, este objetivo ha sido alcanzado tomando la ley de control FMBPC, analítica y explícita, como ley de control base del *modo 1* de la estructura dual CLP-MPC y utilizando los modelos incrementales DLTI asociados a la estrategia FMBPC como modelos de predicciones. Los experimentos correspondientes a nuestro caso de estudio, llevados a cabo en simulación, corroboran que el mecanismo propuesto es posible y válido. Cabe resaltar la dificultad de la integración de ambas metodologías y la escasez de trabajos relativos a la estructura CLP-MPC.

- Otra conclusión de nuestras investigaciones, más global, es la siguiente: es posible y útil diseñar técnicas de control avanzado mediante la confluencia de técnicas de control inteligente con metodologías de control predictivo (en nuestro caso, dos metodologías diferentes). La parte inteligente de la estrategia mixta FMBPC/CLP propuesta en el trabajo nº3 reside en el tipo de modelo utilizado como base de la estrategia FMBPC, que es un modelo borroso de tipo TS. La utilización de modelos borrosos, es decir de modelos cualitativos, obtenidos mediante identificación a partir de datos numéricos de entrada y salida, permite capturar la dinámica de la planta de forma muy fiel, incluso para sistemas complejos, muy cambiantes o desconocidos. Esta propiedad mejorará, lógicamente, la fidelidad de las predicciones y por tanto, en última instancia, mejorará el rendimiento del controlador predictivo.

- Así mismo y como conclusión de carácter global también, pero más específica, destacamos el buen rendimiento, estabilidad y cumplimiento de restricciones (en la acción de control), de la estrategia FMBPC (sola o combinada con la estructura CLP-MPC), aplicada al control multivariable de procesos de fangos activados (ASP), un caso de estudio de especial interés debido a su alta no linealidad y a la complejidad de la dinámica de la planta (como consecuencia de su naturaleza biológica).

- En cuanto a posibles trabajos o líneas de investigación futuras, nos parece que sería interesante continuar y profundizar la investigación en los aspectos siguientes (entre otros): a) generalización de las estrategias propuestas, FMBPC y FMBPC/CLP, así como de las correspondientes metodologías, para cualquier sistema MIMO, no lineal y con dinámica compleja; b) profundización del estudio de la influencia de los parámetros de identificación borrosa en el rendimiento del control predictivo de tipo FMBPC y en consecuencia en la sintonía del mismo; c) investigar la posibilidad de incluir conocimiento experto de manera automática en la identificación borrosa del modelo base; d) extensión de las restricciones impuestas (mediante la estructura mixta FMBPC/CLP) a otras variables (como los esfuerzos de control y las salidas, además de la acción de control); e) extensión de la estrategia mixta FMBPC/CLP al campo del control MPC distribuido.

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