UNIVERSIDAD DE SALAMANCA

Facultad de Ciencias

Grado en Matemáticas

ISOMETRY GROUPS OF COSMOLOGICAL MODELS

TRABAJO FIN DE GRADO

Autor:
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Tutor:
Tomás Carlos Tejero Prieto

Curso 2023-2024

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Certificado de los tutores TFG Grado en Matemáticas

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HACE CONSTAR:

Que el trabajo titulado "*Grupos de Isometría de modelos cosmológicos*", que se presenta, ha sido realizado por D. Guillermo Rubio González, con DNI 70921213T y constituye la memoria del trabajo realizado para la superación de la asignatura Trabajo de Fin de Grado en Matemáticas en esta Universidad.

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Introduction

The study of the universe through the lens of mathematics and physics has led to deep and important insights into the nature of space, time, and gravity. The mathematical framework for understanding these concepts is provided by semi-Riemannian geometry, a branch of differential geometry.

Historically, the development of semi-Riemannian geometry has been tied to the formulation of Einstein's theory of General Relativity in the early 20th century. Riemannian manifolds and their curvature were first introduced non-rigorously by Bernhard Riemann in 1854. Pioneering mathematicians such as Hermann Weyl, who was the first to explicitly define the concept of a smooth manifold in 1913 [4, Page 37], extended the classical notions of curvature and parallel transport to accommodate the mathematics needed by relativistic physics. Their work gave rise to the mathematical tools used for describing the spacetime continuum, a four-dimensional manifold that unifies space and time into a single entity.

The first chapter of this work introduces the framework of semi-Riemannian geometry and related concepts, which are essential for the following chapters. This includes a detailed examination of smooth manifolds with metrics, the Levi-Civita connection, Lie groups and algebras, and curvature tensors. By building a solid understanding of these basic concepts, we set the stage for exploring the main objectives of this dissertation.

Within this subject, and in their application to wider physics, isometry groups play a crucial role. An isometry is a distance-preserving transformation, and the set of all such transformations forms a group under composition. The study of isometry groups allows us to understand the symmetries of a given spacetime, providing insights into its geometric and physical properties. Killing vector fields are infinitesimal generators of isometries. They are named after Wilhelm Killing, who made significant contributions to Lie theory, particularly on the topic of simple Lie groups, in the late 19th century. Killing vector fields satisfy a specific equation that ensures the metric remains invariant under the flow generated by the vector field. This invariance is deeply connected to the conservation laws in physics, through Noether's theorem.

The second chapter of this dissertation studies the formal properties of isometry groups, their Lie algebras, Killing vector fields, and their relationships. The main result of this chapter is theorem 2.17, which shows that the set of isometries on a semi-Riemannian manifold is a Lie group, whose (finite-dimensional) Lie algebra is the set of all complete

Killing fields on such manifold.

An useful concept to us in semi-Riemannian geometry is that of warped products. The notion of warped products extends the idea of a direct product of manifolds by allowing the metric to “change” along different directions, governed by a warping function. We will be able to obtain a rich class of metrics, which will be particularly useful in cosmological models, where one can represent the spacetime continuum as a warped product manifold.

The third chapter, then, introduces warped products and explores their geometric properties, allowing for their application to cosmology. The formalism of warped products provides a versatile tool for describing a variety of spacetimes, such as Robertson-Walker spacetimes, which model our universe as a relativistic perfect fluid.

Robertson-Walker spacetimes are a family of solutions to Einstein’s field equations that are modelled as warped product manifolds. These frameworks give rise to the Friedmann models, which are foundational in cosmology, forming the basis of the Big Bang theory and our understanding of the large-scale structure of the cosmos, and its homogeneity and isotropy.

The fourth chapter applies the mathematical machinery developed in the earlier chapters to the study of generalized Robertson-Walker spacetimes. By leveraging the concepts of Killing vector fields and warped products, we produce some important results in this context: Especifically, Theorems 4.13 and 4.6 and Corollary 4.12.

In conclusion, this work tries to describe and develop an abstract and rich mathematical theory with concrete physical applications, illustrating how mathematics form the basis for our understanding of the universe. Ultimately, however, this dissertation provides only a small glimpse on how mathematical methods are indispensable for unraveling the mysteries of the cosmos in our quest to understand our reality.

Chapter I

Semi-Riemannian Geometry

1.1 Semi-Riemannian Manifolds

Note: All manifolds considered in this work are supposed to be Haussdorf and second-countable.

We start by introducing some fundamental concepts and definitions that will be of use throughout the whole work, chiefly among them the *semi-Riemannian manifolds*:

Definition 1.1 (Smooth tensor fields)**.** A (p, q) *smooth tensor field* smoothly assigns to each point x of a smooth manifold M a (p, q) tensor on the tangent vector space T_xM. We will denote the set of smooth tensor fields on M as $\mathcal{T}_q^p(M)$.

Definition 1.2 (Semi-Riemannian Manifold)**.** A semi-Riemannian manifold is a smooth manifold endowed with a $(0, 2)$ smooth tensor field g, which meets the following conditions:

1. g_x is symmetric $\forall\ x \in\ M$.
2. g_x is nondegenerate $\forall\ x \in\ M$.
3. g_x has constant index $\forall\ x \in\ M$.

We will call g the metric tensor of the manifold.

Remark 1.3. We will denote the dimension of the manifold with the letter n, and the manifolds themselves with the letters M and N.

Definition 1.4. If the common index ν of g_x for each $x \in M$ is 0, $\nu = 0$, then M is called a *Riemannian manifold*. Equivalently, each g_p is a positive definite metric tensor.

Definition 1.5. If $\nu = 1$ and $n \geq 2$, M is a *Lorentz manifold*.

We continue giving the following:

Definition 1.6. A tangent vector v on T_xM for any $x \in M$ is called

- *Spacelike* if $g_x(v, v) > 0$

- *Lightlike* or *null* if $g_x(v, v) = 0$
- *Timelike* if $g_x(v, v) < 0$

It follows immediately that, if M is a riemannian manifold, every tangent vector is spacelike. This terminology is derived from relativity. As such, we call this categorisation the *causal character* of tangent vector.

1.1.1 The Tangent Bundle

It is well known that the tangent space at a point x of a smooth manifold M, T_xM, is a vector space. However, we will shortly see that they can be structured in a way such that they form a smooth manifold. We can consider their disjoint union, $\forall\, x \in M$. This will be the *tangent bundle* of M.

Definition 1.7 (The tangent bundle)**.** For any smooth manifold M, we define the *tanget bundle* of M as

$$TM := \bigsqcup_{x \in M} T_xM$$

We will write their elements as an ordered pair (x, v), where $x \in M$ and $v \in T_xM$.

Every tangent bundle has a projection $\tau_M : TM \to M$ that sends each (x, v) to x. As we have mentioned, one can consider the tangent bundle as a collection of vector spaces, but it is much more than that. The next proposition is taken from [10, Page 81, Lemma 4.1].

Proposition 1.8. *For any n-dimensional smooth manifold M, the tangent bundle TM has a natural topology and smooth structure that make it into a $2n$-dimensional smooth manifold. With this structure, $\pi : TM \to M$ is a smooth map.*

The coordinates system of the tangent bundle will be $(x_i, \dot{x}_i)$, where the x_i are a coordinate system on an open set of M and $\dot{x}_i$ are the coefficientes of the tangent vector in the base $\frac{\partial}{\partial x_i}$.

Remark 1.9. As TM has a structre of smooth manifold, we can consider *its* tangent bundle, $T(TM)$, usually called the *double* or *second* tangent bundle, which will be a $4n$-dimensional smooth manifold. We can repeat this process for as long as we wish.

1.2 Vector fields and their flows

The concept of a vector field is well known. It can be described simply as a $(1, 0)$ tensor field. In this section we endeavour to describe their “flows” and related concepts.

As an introduction and connection with the previous section, we can state that vector fields are sections of the projection of the tangent bundle onto the manifold. Simply consider a smooth vector field D on M. We can think of it as a map:

$$\begin{aligned} D : & M \longrightarrow TM \\ & x \longmapsto (x, D_x) \end{aligned}$$

These are evidently sections of the projection τ_M:

$$\tau_M \circ D = Id_M$$

Moreover, it is well known [10, Page 86, Lemma 4.6] that:

Proposition 1.10. *A vector field D is smooth if and only if it is smooth as a section of the tangent bundle.*

Let D be a vector field on the semi-Riemannian manifold (M, g). We continue now by defining

Definition 1.11 (Integral curves)**.** An *integral curve* of D is a smooth curve $\gamma : I \longrightarrow M$ such that

$$\gamma'(t) = D_{\gamma(t)} \tag{1.1}$$

We will call $x = \gamma(0)$ the "starting point" of the curve. Every integral curve can be reparametrized to change its starting point to any one point in the image of γ, thanks to the following proposition, proved in [10, Page 437, Lemma 17.2]:

Proposition 1.12. *Let D be a smooth vector field on a smooth manifold M. Let $I \subset \mathbb{R}$ be an open interval, and let $\gamma : I \longrightarrow M$ be an integral curve of D. Then, for any $a \in \mathbb{R}$, the curve*

$$\begin{aligned} \overline{\gamma} : & I + a \longrightarrow M \\ & t \longmapsto \gamma(t - a) \end{aligned}$$

where $I + a = \{t + a : \ t \in I\}$, is also an integral curve of D.

Thus the starting point of any integral curve can be interchanged with any other point in the image.

Remark 1.13. As can be seen in (1.1), finding integral curves boils down to solving a system of ordinary differential equations. If we fix the initial value ($\gamma(0) = x$), then such a curve exists and is unique in its domain, in the sense that, if any two curves share a part of their domains, they must be equal in precisely their common domain. We know this thanks to the famous ODE existence and uniqueness theorem [10, Theorem 17.9, Page 443].

By virtue of Remark 1.13, any maximal integral curve must be unique. Now, we can adress the main point of this section. Let X be a vector field on a semi-Riemannian manifold M. We define a new concept, global flows, and we will shortly see their relation to integral curves and fields.

Definition 1.14 (Global flows)**.** We can define a *global flow* on M, or a *one-parameter group action*, as a map

$$\theta : \mathbb{R} \times M \longrightarrow M$$

that meets the following conditions:

- $\theta(t, \theta(s, x)) = \theta(t + s, x)$
- $\theta(0, x) = x$

It is clear that this is a left group action of $\mathbb{R}$ on M. We will also consider only *smooth* global flows.

By fixing either t or x, we obtain two collections of maps, which we will denote with θ_t if we are fixing the first parameter or $\theta^{(x)}$ for the second. The latter is a curve on M with starting point x, and, equivalently, the orbit of x under the group action. This gives rise to

Definition 1.15 (Infinitesimal generator)**.** For each $x \in M$ we can define a tangent vector $D_x \in T_x M$ with

$$D_x = \left(\theta^{(x)}\right)'(0)$$

We will call this assignment $x \mapsto D_x$ the *infinitesimal generator* of θ.

Proposition 1.16. *The infinitesimal generator D of a smooth global flow θ is a smooth vector field on M, and each curve $\theta^{(x)}$ is an integral curve of D.*

Proof. To show that D is smooth, we can prove that Df is smooth for every $\mathcal{C}^\infty$ function f defined in an open subset of M.

$$D_x f = (\theta^{(x)})'(0)(f) = \frac{d}{dt} f\left(\theta^{(x)}(t)\right)\Big|_{t=0}$$

As f and θ are smooth by hypothesis, their composition will be smooth as well. We can conclude, then, by noting that the derivative of a smooth function is smooth.
We must now show that each $\theta^{(x)}$ is an integral curve of D. By the definitions given before, we need to show that

$$\left(\theta^{(x)}\right)'(t) = D_{\theta^{(x)}(t)} \ \forall \ x \in M, \ t \in \mathbb{R}$$

Let $x_0 \in M$ and $t_0 \in \mathbb{R}$ be any point in M and parameter in $\mathbb{R}$. As D is an infinitesimal generator of θ by hypothesis, we have that:

$$D_x = \left(\theta^{(x)}\right)'(0) \ \forall x \in M$$

Using the properties given in Definition 1.14, we can write that, for any $f \in C^\infty(\theta^{(x_0)}(t_0))$,

$$D_{\theta^{(x_0)}(t_0)} f = \left(\theta^{(\theta^{(x_0)})(t_0)}\right)'(0) f = \frac{d}{dt} f(\theta(t, \theta(t_0, x_0)))\Big|_{t=0} =$$

$$= \frac{d}{dt} f(\theta(t + t_0, x_0))\Big|_{t=0} = \left(\theta^{(x_0)}\right)'(t_0) f$$

As this equality holds for any $f \in C^\infty(\theta^{(x_0)}(t_0))$, we can conclude the proof. □

With this proposition we have seen that every smooth global flow gives rise to a infinitesimal generator, as defined before, whose integral curves are precisely the various $\theta^{(x)}$. We would like to be able to say that every smooth vector field is the infinitesimal generator of a global flow. However, this is only true for flows defined on an open subset of $\mathbb{R} \times M$ (so-called *local flows*), not necessarily on the whole set, as is the case with global flows. This is guaranteed by the *fundamental theorem on flows*, which will be explained shortly. For a detailed explanation of all of the above, see [10, Pages 440-442], which also states and proves the following theorem.

Theorem 1.17 (Fundamental Theorem on Flows)**.** *Let D be a smooth vector field on a smooth manifold M. There is a unique maximal (not necessarily global) smooth flow $\theta : \mathbb{R} \times M \supset \mathcal{D} \to M$ whose infinitesimal generator is D. This flow has the following properties:*

1. *For each $x \in M$, the curve $\theta^{(x)} : \mathcal{D}^{(x)} \to M$ is the unique maximal integral curve of D starting at x, where $\mathcal{D}^{(x)} = \{t \in \mathbb{R} : (t, x) \in \mathcal{D}\}$.*
2. *If $s \in \mathcal{D}^{(x)}$, then $\mathcal{D}^{(\theta(s,x))}$ is the interval $\mathcal{D}^{(x)} - s = \{t - s : t \in \mathcal{D}^x\}$.*
3. *For each $t \in \mathbb{R}$, the set $M_t = \{x \in M : (t, x) \in \mathcal{D}\}$ is open in M, and $\theta_t : M_t \to M_{-t}$ is a diffeomorphism with inverse θ_{-t}.*
4. *For each $(t, x) \in \mathcal{D}$, $(\theta_t)_* D_x = D_{\theta_t(x)}$.*

The vector fields that do give rise to a global flow will be important throughout this work. As such, we define them as being *complete*:

Definition 1.18 (Complete vector field)**.** A smooth vector field D is *complete* if it generates a smooth global flow θ.

Proposition 1.19. *The above condition is equivalent to requiring that every maximal integral curve of said vector fields is defined $\forall\ t \in \mathbb{R}$.*

Proof. If a smooth vector field D generates a global flow θ, it is immediate that every maximal integral must be defined on the entire real line: As the various $\theta^{(x)} : \mathbb{R} \longrightarrow M$ where x is any point in M, are all of the integral curves possible of D (as proved in the Proposition 1.16) with domain $\mathbb{R}$, we can conclude this part of the proof.

We must now prove that if every integral curve γ_x (where we use the $x \in M$ to denote the starting point of the curve) of D is defined in $\mathbb{R}$, then D generates a global flow. Indeed, we can construct a map:

$$\begin{aligned} \theta : \mathbb{R} \times M &\longrightarrow M \\ (t, x) &\mapsto \theta(t, x) = \gamma_x(t) \end{aligned}$$

We now only need to check that this map is actually a smooth global flow. Remember that we defined the starting point of a curve as $\gamma_x(0) = x$.

- $\theta(t, \theta(s, x)) = \gamma_{\theta(s,x)}(t) = \gamma_{\gamma_x(s)}(t) = \gamma_x(t + s) = \theta(t + s, x)$

- $\theta(0, x) = \gamma_x(0) = x$

As θ meets the two required conditions, it is a global flow. We can end the proof by checking that it is, indeed, smooth: The $\gamma_x(t)$ are smooth curves in t, and thanks to the well-known smooth dependence of ODEs on initial conditions‘(see [10, Theorem 17.9, Page 443]), we also know that the γ_x are smooth in x, as they are the solutions to a system of ordinary differential equations (Remark 1.13). □

We now state an important theorem related to complete vector fields, and the lemma necessary to prove it. Both the lemma and the theorem are taken from [10, Pages 446-447].

Lemma 1.20 (Escape Lemma)**.** *Let D be a smooth vector field on a smooth manifold M. If γ is an integral curve of D whose maximal domain is not all of $\mathbb{R}$, then the image of γ cannot lie in any compact subset of M.*

Theorem 1.21. *If M is a compact smooth manifold, then every smooth vector field on M is complete.*

1.3 The Levi-Civita connection

This concept will allow us to introduce a way to find the "derivative" of vector fields. We begin by defining what a connection is:

Definition 1.22 (Connection)**.** A connection on a smooth manifold M is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

such that

1. ∇ is $C^\infty(M)$-linear in the first factor
2. ∇ is a derivation in the second factor, to wit:
   (a) It is $\mathbb{R}$-linear
   (b) It follows Leibniz's rule

$$\nabla_D f D' = D(f)D' + f\nabla_D D' \ \forall \ D, D' \in \mathfrak{X}(M), \ \ f \in C^\infty(M)$$

Let D, D' be vector fields in a smooth manifold M. Then we can call $\nabla_D D'$ the *covariant derivative* of D' with respect to D for the connection ∇.

The following is the extremely important and well known theorem, taken from [13, Theorem 11, Page 61],

Theorem 1.23 (The Levi-Civita connection)**.** *Let $D_1, D_2, D_3 \in \mathfrak{X}(M)$. On a semi-Riemannian manifold (M, g) there is a unique connection ∇ such that*

1. *∇ is torsion-free, i.e. $[D_1, D_2] = \nabla_{D_1} D_2 - \nabla_{D_2} D_1$, where $[D_1, D_2]$ is the Lie bracket of D_1 and D_2, defined in 1.30.*

2. *∇ is compatible with the metric, i.e.* $D_3 g(D_1, D_2) = g(\nabla_{D_3} D_1, D_2) + g(D_1, \nabla_{D_3} D_2)$

Additionally, it verifies the Koszul formula.

$$\begin{aligned} g(\nabla_{D_1} D_2, D_3) = \frac{1}{2} [&D_1 g(D_2, D_3) + D_2 g(D_3, D_1) - D_3 g(D_1, D_2) \\ &- g([D_2, D_3], D_1) - g([D_1, D_3], D_2) - g([D_2, D_1], D_3)] \end{aligned} \tag{1.2}$$

where we have omitted the points "x" in each g and field for the sake of brevity.

We can now define the concept of the covariant derivate of a vector field thanks to the connection ∇.

Remark 1.24. We denote by $\mathfrak{X}(\gamma(t))$ the set of all smooth vector fields over γ, that is, all smooth fields tangent to M on the image of γ. It is a $C^\infty(I)$-module, where I is the interval of $\mathbb{R}$ on which γ is defined.

Definition 1.25 (Covariant derivative on curves)**.** Given a smooth curve γ on M, we define the *covariant derivative*:

$$\begin{aligned} \frac{D}{dt} : \mathfrak{X}(\gamma(t)) &\longrightarrow \mathfrak{X}(\gamma(t)) \\ V &\longmapsto \frac{DV}{dt} \end{aligned}$$

Proposition 1.26 (Existence and uniqueness of the covariant derivative)**.** *The covariant derivative exists and is unique, and it satisfies:*

1. $\frac{D}{dt}(V + V') = \frac{DV}{dt} + \frac{DV'}{dt} \ \forall \ V, V' \in \mathfrak{X}(\gamma(t))$
2. $\frac{D}{dt} hV = \frac{dh}{dt} V + h \frac{DV}{dt} \ \forall \ h \in C^\infty(I), V \in \mathfrak{X}(\gamma(t))$
3. *Given any* $D \in \mathfrak{X}(M)$, $\frac{D}{dt}(D \circ \gamma) = \nabla_{\dot{\gamma}} D$

Proof. As we can cover $\gamma(I)$ with a finite number of open sets with coordinate charts (as it is compact), the proof will be local, using coordinates. We will suppose that one such open set covers the entirety of the smooth curve, without loss of generality. We can write every $V \in \mathfrak{X}(\gamma(t))$ in the form:

$$V(t) := V_{\gamma(t)} = \sum_{i=1}^{n} v_i(t) \left(\frac{\partial}{\partial xi} \right)_{\gamma(t)}$$

Therefore,

$$\begin{aligned} \frac{DV}{dt} = \frac{D}{dt} \left(\sum_{i=1}^{n} v_i(t) \left(\frac{\partial}{\partial xi} \right)_{\gamma(t)} \right) &= \sum_{i=1}^{n} \left(\frac{d}{dt} v_i(t) \left(\frac{\partial}{\partial x_i} \right)_{\gamma(t)} + v_i(t) \frac{D}{dt} \left(\frac{\partial}{\partial x_i} \right)_{\gamma(t)} \right) \\ &= \sum_{i=1}^{n} \left(\frac{d}{dt} v_i(t) \left(\frac{\partial}{\partial x_i} \right)_{\gamma(t)} + v_i(t) \sum_{j=1}^{n} \sum_{k=1}^{n} \dot{\gamma}_j(t) \Gamma_{ji}^k \left(\frac{\partial}{\partial x_k} \right)_{\gamma(t)} \right) \end{aligned}$$

$$= \sum_{k=1} \left[\left(\frac{d}{dt} v_k(t) + \sum_{i,j=1}^{n} v_i(t)\dot{\gamma}_j(t)\Gamma_{ji}^k \right) \left(\frac{\partial}{\partial x_k} \right)_{\gamma(t)} \right]$$

where $\dot{\gamma}_j$ is the j-th component of the field $\dot{\gamma}$. It is obvious, then, that the result is a vector field over γ. Thus, as we can see, this operator exists and is unique. □

Similarly, using the connection as well, we can define the covariant differential of a vector field D:

Definition 1.27 (Covariant differential)**.** We define the *covariant differential* of D, denoted by ∇D, as a map

$$\begin{aligned} \nabla D : \mathfrak{X}(M) &\longrightarrow \mathfrak{X}(M) \\ D' &\longmapsto \nabla_{D'} D \end{aligned}$$

Remark 1.28. It follows immediately from this definition that ∇D is a (1,1) smooth tensor field on (M, g).

We can immediately extend the connection to tensor fields

Definition 1.29 (Covariant derivative of tensor fields)**.** Given a connection ∇, we define the *covariant derivative of tensor fields* with respect to $D \in \mathfrak{X}(M)$ as the unique tensor derivation

$$\nabla_D : \mathcal{T}_q^p(M) \longrightarrow \mathcal{T}_q^p(M)$$

where [13, Pages 44-45,64]

$$\nabla_D f = Df \ \forall \ f \in C^\infty(M)$$

and, consequently:

$$\nabla_D \omega(D') = D(\omega D') - \omega(\nabla_D D') \ \forall \ \omega \in \mathcal{T}_0^1(M), D' \in \mathfrak{X}(M)$$

Thus we have defined the covariant derivative of any (p, q) tensor field.

1.4 The Lie derivative

In this section we will introduce a concept vital to our work, with great applications in the topic of isometries in manifolds: The well known Lie derivative.

Definition 1.30. Let D_1, D_2 be any two vector fields in a smooth manifold M, and let $f \in C^\infty(M)$. Then, we can define an operator called the *Lie bracket*, and denoted with $[,]$, such that:

$$[D_1, D_2]f = D_1(D_2 f) - D_2(D_1 f)$$

It can be easily shown (such as in [10, Page 90, Lemma 4.12]) that the Lie bracket of any pair of smooth vector fields is itself a smooth vector field. Thus, this operator is well-defined.

Proposition 1.31 (Properties of the Lie bracket)**.** *The well known Lie bracket operation on vector fields is itself a type of bracket, as defined above, as it satisfies the three necessary conditions. Besides, we can also state one further property. Thus, it can be shown that the Lie bracket has the following properties:*

1. *Bilinearity in* $\mathbb{R}$.
2. *Antisymmetry*
3. *Jacobi Identity: For all* $D_1, D_2, D_3 \in \mathfrak{X}(M)$,

$$[D_1, [D_2, D_3]] + [D_2, [D_3, D_1]] + [D_3, [D_1, D_2]] = 0$$

4. *For all* $f, g \in C^\infty(M)$ *and* $D_1, D_2 \in \mathfrak{X}(M)$,

$$[fD_1, gD_2] = fg[D_1, D_2] + (fD_1 g)D_2 - (gD_2 f)D_1$$

Proof. Bilinearity and antisymmetry are obvious from the Definition 1.30. The proof of the Jacobi identity is a (using the definition of the Lie bracket) straightforward, if tedious, computation, so we won't write it here: One can see such computation in [10, Page 92, Proof of Lemma 4.15]. We now intend to show the last property. Let $h \in C^\infty(M)$, then, using the definition of the Lie bracket:

$$\begin{aligned}[fD_1, gD_2]h &= fD_1(gD_2 h) - gD_2(fD_1 h) \\ &= (fD_1 g)(D_2 h) + fgD_1(D_2 h) - (gD_2 f)(D_1 h) - fgD_2(D_1 h) \\ &= fg[D_1, D_2]h + (fD_1 g)(D_2 h) - (gD_2 f)(D_1 h)\end{aligned}$$

And thus,

$$[fD_1, gD_2] = fg[D_1, D_2] + (fD_1 g)D_2 - (gD_2 f)D_1$$

□

The proof of the following proposition has been taken from [13, Page 31, Proposition 1.58].

Proposition 1.32. *Let* $D, D' \in \mathfrak{X}(M)$ *and let* θ *be a local flow of* D *in a neighbourhood of* $x \in M$. *We will omit the points in the tangent maps to make the expressions more readable. Then,*

$$[D, D']_x = \lim_{t \to 0} \frac{1}{t} \left[(\theta_{-t})_* (D'_{\theta_t(x)}) - D'_x \right] \tag{1.3}$$

Proof. Let $F_x(t) := (\theta_{-t})_*(D'_{\theta_t(x)})$, so that the equation above becomes $[D, D']_x = F'_x(0)$. We will prove the proposition in each of 3 cases:

1. $D_x \neq 0$. We can choose a coordinate system in a neighbourhood of x such that $D = \partial_1$. Thus, the flow of D only changes the x_1 coordinate of points y near x, and we can write:

$$x_1(\theta_t(y)) = x_1(y) + t$$

$$x_i(\theta_t(y)) = x_i(y)\ \forall\ 2 \le i \le n$$

As such, $(\theta_t)_* = Id_*\ \forall\ t$. Let $D' = \sum_{i=1}^n D'_i \partial_i$. Then:

$$F_x(t) = \sum_{i=1}^n D'_i(\theta_t(x))(\theta_{-t})_* \left(\frac{\partial}{\partial x_i}\right)_{\theta_t(x)} = \sum_{i=1}^n D'_i(\theta_t(x)) \left(\frac{\partial}{\partial x_i}\right)_x$$

We can now find the derivative of F_x:

$$\begin{aligned} F'_x(0) &= \sum_{i=1}^n \frac{d}{dt} D'_i(\theta^{(x)}(t))\bigg|_{t=0} \left(\frac{\partial}{\partial x_i}\right)_x = \sum_{i=1}^n \left(\frac{\partial D'_i}{\partial x_1}\right)_x \left(\frac{\partial}{\partial x_i}\right)_x \\ &= \left[\frac{\partial}{\partial x_1}, D'\right]_x = [D, D']_x \end{aligned}$$

2. $D = 0$ in a neighbourhood of x. Then:
   - $[D, D']_x = 0$
   - Integral curves starting in the neighbourhood are constant, as $\gamma'(t) = D_{\gamma(t)} = 0$. Therefore, $\theta^{(y)} = Id\ \forall\ y \in \{\text{Neighbourhood of } x\}$. Thus F_x is constant and $F'_x(0) = 0$.

   Hence, $[D, D']_x = F'_x(0) = 0$.

3. $D_x = 0$, but there exists arbitrarily close points y to x such that $D_y \neq 0$. As, clearly, both $F'_x(0)$ and $[D, D']_x$ depend continuosly on x, we can follow the same procedure as in the first case by taking a new x to be one of these arbitrarily close points.

We have now covered and proved the proposition in all possible cases. Hence, we end the proof. □

Definition 1.33 (Lie derivative)**.** Given a smooth vector field $D \in \mathfrak{X}(M)$ and a smooth tensor field $T \in \mathcal{T}^p_q(M)$ on a semi-Riemannian manifold (M, g), we can define the *Lie derivative* of T in the direction D (or with respect to D), and is denoted by $\mathcal{L}_D T$, as the smooth (p, q) tensor field determined by the following expressions

1. For $f \in \mathcal{T}^0_0(M) = C^\infty(M)$, $\mathcal{L}_D f = Df$
2. For $D' \in \mathcal{T}^0_1(M) = \mathfrak{X}(M)$, $\mathcal{L}_D D' = [D, D']$
3. For $\omega \in \mathcal{T}^1_0(M)$, and $\forall\ D' \in \mathfrak{X}(M)$

$$(\mathcal{L}_D \omega)(D') = \mathcal{L}_D(\omega(D')) - \omega(\mathcal{L}_D D') = D(\omega(D')) - \omega([D, D'])$$

4. In general, for $T \in \mathcal{T}^p_q(M)$, and $\forall\ D_i \in \mathfrak{X}(M),\ \omega_j \in \mathcal{T}^1_0(M)$

$$\begin{aligned} (\mathcal{L}_D T)(D_1, \ldots, D_p, \omega_1, \ldots, \omega_q) &= D(T(D_1, \ldots, D_p, \omega_1, \ldots, \omega_q)) \\ &- \sum_{i=1}^p T(D_1, \ldots, \mathcal{L}_D D_i, \ldots, D_p, \omega_1, \ldots, \omega_q) \\ &- \sum_{j=1}^q T(D_1, \ldots, D_p, \omega_1, \ldots, \mathcal{L}_D \omega_j, \ldots, \omega_q) \end{aligned}$$

Definition 1.34 (Tensor derivation)**.** A *tensor derivation* $\mathscr{D}$ on a smooth manifold M is a set of $\mathbb{R}$-linear functions

$$\mathscr{D} : \mathcal{T}^p_q(M) \longrightarrow \mathcal{T}^p_q(M)$$

such that for any tensor fields $T_1, T_2 \in \mathcal{T}^p_q(M)$,

- $\mathscr{D}(T_1 \otimes T_2) = \mathscr{D}T_1 \otimes T_2 + T_1 \otimes \mathscr{D}T_2$
- $\mathscr{D}(CT_1) = C(\mathscr{D}T_1)$ for any contraction C.

The following property is stated in [7, Chapter 7].

Proposition 1.35. *The Lie derivative in the direction of any smooth vector field D, $\mathcal{L}_D$, is a tensor derivation.*

Proposition 1.36. *Let D be a smooth vector field and let $\tau \in \mathcal{T}^p_0$ be a smooth covariant tensor field on the semi-Riemannian manifold (M, g). Then,*

$$\mathcal{L}_D \tau = \lim_{t\to 0} \frac{1}{t}[(\theta_t)^*(\tau) - \tau] \tag{1.4}$$

where θ is the local flow defined by the field D. We will, again, omit the points in the tangent maps to make the expressions more readable.

Proof. Let X, Y be smooth vector fields on M. We will prove the proposition for $p = 2$ for simplicity, so we will rename $\tau \equiv g$. Then, the left-hand side of the formula becomes

$$\mathcal{L}_D g(X, Y) = D(g(X, Y)) - g([D, X], Y) - g(X, [D, Y])$$

We can also manipulate the right-hand side expression by fixing a point $x \in M$:

$$\begin{aligned}
&\lim_{t\to 0} \frac{1}{t}[g_{\theta_t(x)}((\theta_t)_*(X_x), (\theta_t)_*(Y_x)) - g_x(X_x, Y_x)] = \\
&= \lim_{t\to 0} \frac{1}{t}[g_{\theta(x,t)}((\theta_t)_*(X_x), (\theta_t)_*(Y_x)) - g_{\theta_t(x)}(X_{\theta_t(x)}, Y_{\theta_t(x)})] + \\
&+ \lim_{t\to 0} \frac{1}{t}[g_{\theta_t(x)}(X_{\theta_t(x)}, Y_{\theta_t(x)}) - g_x(X_x, Y_x)] = \\
&= L_1 + L_2
\end{aligned}$$

where we have renamed the first and second terms of the expression to L_1 and L_2, respectively. We know that $\theta^{(x)}$ is the maximal integral curve of X starting at point $x \in M$. Then,

$$L_2 = \frac{d}{dt}\left(g_{\theta^{(x)}(t)}\left((X)_{\theta^{(x)}(t)}, (Y)_{\theta^{(x)}(t)}\right)\right)\bigg|_{t=0} = \left(\theta^{(x)}\right)'(0)g(X, Y) = D_x g(X, Y)$$

now, using the identity:

$$g_x(v_1', v_2') - g_x(v_1, v_2) = g_x(v_1' - v_1, v_2) + g_x(v_1.v_2' - v_2)\ \forall x \in M,\ \forall\ v_1, v_2, v_1', v_2' \in T_x M$$

We get:

$$L_1 = \lim_{t\to 0} \frac{1}{t} \Big[g_{\theta(x,t)} \Big((\theta_t)_*(X_x) - X_{\theta_t(x)}, (\theta_t)_*(Y_x) \Big) + g_{\theta(x,t)} \Big(X_{\theta_t(x)}, (\theta_t)_*(Y_x) - Y_{\theta_t(x)} \Big) \Big]$$

Using Proposition 1.32, and the fact that $\theta_t^{-1} = \theta_{-t}$, the first term becomes:

$$\begin{aligned}
&\lim_{t\to 0} \frac{1}{t} \Big[g_{\theta(x,t)} \Big((\theta_t)_*(X_x) - X_{\theta_t(x)}, (\theta_t)_*(Y_x) \Big) \Big] \\
&= \lim_{t\to 0} \frac{1}{t} \Big[g_{\theta(x,t)} \Big((\theta_t)_* \Big(X_x - (\theta_{-t})_*(X_{\theta_t(x)}) \Big), (\theta_t)_*(Y_x) \Big) \Big] \\
&= -g_x \Big(\lim_{t\to 0} (\theta_t)_* \lim_{t\to 0} \frac{1}{t} \Big((\theta_{-t})_*(X_{\theta_t(x)}) - X_x \Big), \lim_{t\to 0} (\theta_t)_*(Y_x) \Big) \\
&= -g_x \left([D, X]_x, Y_x \right)
\end{aligned}$$

Similarly, the second term is

$$\lim_{t\to 0} \frac{1}{t} \Big[g_{\theta(x,t)} \Big(X_{\theta_t(x)}, (\theta_t)_*(Y_x) - Y_{\theta_t(x)} \Big) \Big] = -g_x \left(X_x, [D, Y]_x \right)$$

Thus, $L_1 + L_2 = \mathcal{L}_D g(X, Y)$. □

In fact, this proposition is completely equivalent to the Definition 1.33. In other words, we could have defined the Lie derivative with (1.4). Indeed, many books choose to define it using flows, instead of bothering with all of the formulas that we used in the definition. In this work we shall end the proof here, as anything more is outside the scope of the dissertation. If interested, one can see such explanation in, for example, [7, Pages 130-131].

Proposition 1.37. *The Lie derivative is $\mathbb{R}-$linear in both components, and, additionally, given any two vector fields $D, D' \in \mathfrak{X}(M)$ and any tensor field (of any contravariant and covariant indices p and q) $T \in \mathcal{T}^p_q(M)$,*

$$\mathcal{L}_{[D,D']} T = [\mathcal{L}_D, \mathcal{L}_{D'}] T$$

where $[\mathcal{L}_D, \mathcal{L}_{D'}] = \mathcal{L}_D \circ \mathcal{L}_{D'} - \mathcal{L}_{D'} \circ \mathcal{L}_D$, and $[D, D']$ is the well-known Lie bracket of D and D'.

Proof. It is quite easy to check that it is $\mathbb{R}$-linear. We proceed in the same manner as when we defined the Lie derivative to prove the equality, working on each case in the same order. Let $D, D' \in \mathfrak{X}(M)$:

1. For $f \in C^\infty(M)$:

$$\mathcal{L}_{[D,D']} f = [D, D'] f = D(D' f) - D'(D f) = \mathcal{L}_D(\mathcal{L}_{D'} f) - \mathcal{L}_{D'}(\mathcal{L}_D f) = [\mathcal{L}_D, \mathcal{L}_{D'}] f$$

2. For $D'' \in \mathfrak{X}(M)$, and using the Jacobi Identity of the Lie bracket:

$$\mathcal{L}_{[D,D']}D'' = [[D,D'],D''] = [D,[D',D'']] + [D',[D'',D]] =$$
$$= [D,[D',D''] - [D',[D,D'']] = \mathcal{L}_D(\mathcal{L}_{D'}D'') - \mathcal{L}_{D'}(\mathcal{L}_D D'') = [\mathcal{L}_D, \mathcal{L}_{D'}]D''$$

3. For $\omega \in \mathcal{T}_0^1(M)$ and $\forall\, D'' \in \mathfrak{X}(M)$:

$$\begin{aligned}
\mathcal{L}_{[D,D']}\omega(D'') &= [D,D'](\omega(D'')) - \omega([[D,D'],D'']) \\
[\mathcal{L}_D, \mathcal{L}_{D'}](\omega)(D'') &= D(\mathcal{L}_{D'}\omega(D'')) - \mathcal{L}_{D'}\omega([D',D'']) - D'(\mathcal{L}_D\omega(D'')) + \mathcal{L}_D\omega([D',D'']) \\
&= D(D'(\omega(D''))) - D(\omega([D',D''])) - D'(\omega([D,D''])) \\
&+ \omega([D',[D,D'']]) - D'(D(\omega(D''))) + D'(\omega([D,D''])) \\
&+ D(\omega([D',D''])) - \omega([D,[D',D'']]) \\
&= [D,D'](\omega(D'')) - \omega([D,[D',D'']] + [D',[D'',D]])
\end{aligned}$$

Using the Jacobi Identity in this last expression, we finally get:

$$\begin{aligned}
[\mathcal{L}_D, \mathcal{L}_{D'}](\omega)(D'') &= [D,D'](\omega(D'')) - \omega(-[D'',[D,D']]) \\
&= [D,D'](\omega(D'')) - \omega([[D,D'],D''])
\end{aligned}$$

4. Now, we first state the well known fact [6, Theorem 8.2.2.] that, for any n-dimensional vector space E, and any bases $\{e_1, \dots, e_n\}$ of E and $\{\omega_1, \dots, \omega_n\}$ of E^*, the set of tensors
$$\{e_{i_1}\otimes, \dots, \otimes e_{i_p} \otimes \omega_{j_1}\otimes, \dots, \otimes\omega_{j_q}\}$$
where $i_1, \dots i_p$ and $j_1, \dots j_q$ take all values from 1 to n, is a basis of the vector space of (p,q) tensors over E. Consequently, we can write that, for any $T \in \mathcal{T}_q^p(M)$:

$$T = \sum_{\substack{(i_1,\dots,i_p)\\(j_1,\dots,j_q)}} \lambda_{(j_1,\dots,j_q)}^{(i_1,\dots,i_p)} \frac{\partial}{\partial x_{i_1}}\otimes, \dots, \otimes\frac{\partial}{\partial x_{i_p}} \otimes dx_{j_1}\otimes, \dots, \otimes dx_{j_q}$$

Where $\lambda_{(j_1,\dots,j_q)}^{(i_1,\dots,i_p)} \in \mathbb{R}$. Therefore, using Proposition 1.35 and the previously proved cases for vector and covector fields,

$$\begin{aligned}
\mathcal{L}_{[D,D']}T = &\sum_{\substack{(i_1,\dots,i_p)\\(j_1,\dots,j_q)}} \lambda_{(j_1,\dots,j_q)}^{(i_1,\dots,i_p)} \mathcal{L}_{[D,D']}\left(\frac{\partial}{\partial x_{i_1}}\otimes, \dots, \otimes\frac{\partial}{\partial x_{i_p}} \otimes dx_{j_1}\otimes, \dots, \otimes dx_{j_q}\right) \\
&= \sum_{\substack{(i_1,\dots,i_p)\\(j_1,\dots,j_q)}} \lambda_{(j_1,\dots,j_q)}^{(i_1,\dots,i_p)} \left(\mathcal{L}_{[D,D']}\frac{\partial}{\partial x_{i_1}}\otimes, \dots, \otimes\frac{\partial}{\partial x_{i_p}} \otimes dx_{j_1}\otimes, \dots, \otimes dx_{j_q} + \dots\right. \\
&\left.\dots + \frac{\partial}{\partial x_{i_1}}\otimes, \dots, \otimes\frac{\partial}{\partial x_{i_p}} \otimes dx_{j_1}\otimes, \dots, \otimes\mathcal{L}_{[D,D']}dx_{j_q}\right) \\
&= \sum_{\substack{(i_1,\dots,i_p)\\(j_1,\dots,j_q)}} \lambda_{(j_1,\dots,j_q)}^{(i_1,\dots,i_p)} \left([\mathcal{L}_D, \mathcal{L}_{D'}]\frac{\partial}{\partial x_{i_1}}\otimes, \dots, \otimes\frac{\partial}{\partial x_{i_p}} \otimes dx_{j_1}\otimes, \dots, \otimes dx_{j_q} + \dots\right.
\end{aligned}$$

$$
\begin{aligned}
&\cdots + \frac{\partial}{\partial x_{i_1}} \otimes, \ldots, \otimes \frac{\partial}{\partial x_{i_p}} \otimes dx_{j_1} \otimes, \ldots, \otimes [\mathcal{L}_D, \mathcal{L}_{D'}] dx_{j_q} \Big) \\
&= \sum_{\substack{(i_1,\ldots,i_p)\\(j_1,\ldots,j_q)}} \lambda^{(i_1,\ldots,i_p)}_{(j_1,\ldots,j_q)} [\mathcal{L}_D, \mathcal{L}_{D'}] \left(\frac{\partial}{\partial x_{i_1}} \otimes, \ldots, \otimes \frac{\partial}{\partial x_{i_p}} \otimes dx_{j_1} \otimes, \ldots, \otimes dx_{j_q} \right) \\
&= [\mathcal{L}_D, \mathcal{L}_{D'}] \sum_{\substack{(i_1,\ldots,i_p)\\(j_1,\ldots,j_q)}} \lambda^{(i_1,\ldots,i_p)}_{(j_1,\ldots,j_q)} \left(\frac{\partial}{\partial x_{i_1}} \otimes, \ldots, \otimes \frac{\partial}{\partial x_{i_p}} \otimes dx_{j_1} \otimes, \ldots, \otimes dx_{j_q} \right) \\
&= [\mathcal{L}_D, \mathcal{L}_{D'}] T
\end{aligned}
$$

□

1.4.1 Interpretation

What is the geometric meaning of the Lie derivative of a tensor field with respect to a vector field D? In this section we will see that the Lie derivative of a covariant tensor field is zero if and only it is invariant under the flow of D.

Definition 1.38. We say that a tensor field T is invariant under a flow θ if

$$\theta_t^* \left(T_{\theta_t(x)} \right) = T_x$$

for all $(t, x) \in \{\text{domain of } \theta\}$. In the specific case that D is a complete vector field (and thus θ is a global flow), the above condition is equivalent to saying that

$$\theta_t^*(T) = T \;\forall\; t \in \mathbb{R}$$

The next lemma and proposition relate derivatives of t with Lie derivatives. They are taken from [10, Page 478, Lemma 18.15 and Proposition 18.16].

Lemma 1.39. *Let M be a smooth manifold, $D \in \mathfrak{X}(M)$, and let θ be the (local) flow of D. For any smooth covariant tensor field T and any (t_0, x) in the domain of θ,*

$$\left.\frac{d}{dt}\right|_{t=t_0} \theta_t^*(T_{\theta_t(x)}) = \theta_{t_0}^* \left((\mathcal{L}_D T)_{\theta_{t_0}(x)} \right)$$

Proposition 1.40 (Invariance of tensor fields)**.** *Let M be a smooth manifold and let $D \in \mathfrak{X}(M)$. A smooth covariant tensor field T is invariant under the flow of D if and only if $\mathcal{L}_D T = 0$.*

1.5 Lie Groups and Lie Algebras

We now introduce a new mathematical structure that will be essential: The Lie group. Lie groups are named after Sophus Lie, Norwegian mathematician (1842-1899). They have great applications in physics and differential equations. Lie algebras and Killing fields, which bear great importance too, are closely related to these structures.

Definition 1.41 (Lie groups)**.** A *Lie group* G is a smooth manifold that is also a group in the algebraic sense with smooth group operations, i.e., the maps

$$\begin{aligned} m : G \times G &\longrightarrow G \\ (g, g') &\longmapsto gg' \end{aligned}$$

$$\begin{aligned} I : G &\longrightarrow G \\ g &\longmapsto g^{-1} \end{aligned}$$

are both smooth.

We will denote with e the identity element of the Lie group.

Remark 1.42. It can be shown that the map I is smooth if m is smooth. Consequently, to prove that a smooth manifold with group operations is a Lie group, it is sufficient to prove that m is smooth.

Definition 1.43 (Left translation)**.** Let G be a Lie group. We define a left translation by $a \in G$ as

$$\begin{aligned} L_a : G &\longrightarrow G \\ g &\longmapsto ag \end{aligned}$$

We can just as well define a right translation, but we will use mainly left translations. Such maps are clearly diffeomorphims, as they have inverse ($L_{a^{-1}}$) and are smooth.

Definition 1.44 (φ-related)**.** Given a smooth map φ between two smooth manifolds M and N, $\varphi : M \to N$, we say that $D_1 \in \mathfrak{X}(M)$ and $D_2 \in \mathfrak{X}(M)$ are *φ-related* if $\varphi_* \circ D_1 = D_2 \circ \varphi$.

Definition 1.45 (Left-invariant fields)**.** A vector field D on G is *left-invariant* if for each $a \in G$, D is L_a-related to itself, that is,

$$(L_a)_{*,g}(D_g) = D_{ag} \ \forall \ g \in G \tag{1.5}$$

The proof of the following proposition is taken and adapted from [16, Page 85, Proposition 3.7(b)].

Proposition 1.46. *All left-invariant vector fields are smooth.*

Proof. Let $D \in \mathfrak{g}$, and let $f \in C^\infty(G)$. We need to show that $Df \in C^\infty(G)$. For each $g \in G$,

$$D_g(f) = (L_g)_{*,e}(D_e)(f) = D_e(f \circ L_g)$$

thus we need to prove that $g \mapsto D_e(f \circ L_g)$ is a C^∞ function on G. Let m denote the group multiplication as before, and let i_g^1 and i_g^2 be the inclusion maps of $g \in G \to G \times G$ in the first and second component, respectively:

$$i_g^1(g') = (g, g')$$

$$i_g^2(g') = (g', g)$$

Let D' be any smooth vector field on G such that $D'_e = D_e$. Then $(0, D')$ will be a smooth vector field on $G \times G$, and consequently $[(0, D')(f \circ m)] \circ i^1_e$ is a smooth function on G. Using [16, Page 52, Exercise 24(d)], we get:

$$\begin{aligned}[(0, D')(f \circ m)] \circ i^1_e(g) &= (0, D')_{g,e}(f \circ m) \\ &= 0_g(f \circ m \circ i^1_e) + D_e(f \circ m \circ i^2_g) \\ &= D_e(f \circ m \circ i^2_g) = D_e(f \circ L_g)\end{aligned}$$

Thus, we have shown that $D_e(f \circ L_g)$ is a smooth function on G, and we can end the proof. □

Proposition 1.47. *Let G be a Lie group and let $\mathfrak{g}$ be the set of left-invariant vector fields on G. Then $\mathfrak{g}$ is a finite-dimensional real vector space subspace of $\mathfrak{X}(G)$ and is isomorphic to T_eG. Consequently,* $\dim \mathfrak{g} = \dim T_eG = \dim G$.

Proof. That $\mathfrak{g}$ is a real vector space is immediate thanks to the linearity of $(L_a)_{*,x}$ $\forall x \in G$ and the fact that all vector fields are themselves a real vector space: Let $D, D' \in \mathfrak{g}$ and let $\lambda, \mu \in \mathbb{R}$, then

$$(L_a)_{*,x}((\lambda D + \mu D')_x) = \lambda (L_a)_{*,x}((D)_x) + \mu (L_a)_{*,x}((D')_x) =$$
$$= \lambda (D)_{ax} + \mu (D')_{ax} = (\lambda D + \mu D')_{ax} \ \forall x \in G$$

Consequently, a linear combination of elements of $\mathfrak{g}$ is an element of $\mathfrak{g}$ as well. We now need to prove that $\mathfrak{g}$ is indeed isomorphic to T_eG. We can do that by proving that the map between said spaces

$$\begin{aligned}\mathfrak{g} &\longrightarrow T_eG \\ D &\longmapsto D_e\end{aligned}$$

is a linear isomorphism:

- Linearity: It is immediate from the fact that the elements of $\mathfrak{g}$ are vector fields, like before.
- Injectivity: Let $D, D' \in \mathfrak{g}$ such that $D_e = D'_e$. Then,

  $$(L_a)_{*,e}(D_e) = (L_a)_{*,e}(D'_e) \ \forall a \in G \Rightarrow D_a = D'_a \ \forall a \in G \Rightarrow D = D'$$

  which proves that the map is injective.
- Surjectivity: Let $v \in T_eG$. We can define a vector field D in G such that $D_g = (L_g)_{*,e}(v)$ $\forall g \in G$. We know that it will indeed be a vector field because L_g is a diffeomorphism. This vector field is left invariant too: Let a, g be any points in G, then

  $$(L_a)_{*,g}(D_g) = ((L_a)_{*,g} \circ (L_g)_{*,e}(v)) = (L_a \circ L_g)_{*,e}(v) = (L_{ag})_{*,e}(v) = D_{ag}$$

  As the left translation by e is the identity map, $D_e = (L_e)_{*,e}(v) = Id_{*,e}(v) = v$. Thus, D both maps to v and is left-invariant. Consequently, it will necessarily be smooth too, thanks to the Proposition 1.46. Thus, we can conclude that $D \in \mathfrak{g}$, and, consequently, $D \mapsto D_e$ is surjective.

□

Definition 1.48 (Lie algebra). A *Lie algebra* over $\mathbb{R}$ is a real vector space $\mathfrak{g}$ equipped with a map called the *bracket*:

$$\begin{aligned} \mathfrak{g}\times\mathfrak{g} &\longrightarrow \mathfrak{g} \\ (D_1, D_2) &\longmapsto [D_1, D_2] \end{aligned}$$

that satisfies the following conditions $\forall\ D_1, D_2, D_3 \in \mathfrak{g}$:

(1) Bilinearity: For $a, b \in \mathbb{R}$,

$$\begin{aligned} [aD_1 + bD_2, D_3] &= a[D_1, D_3] + b[D_2, D_3] \\ [D_3, aD_1 + bD_2] &= a[D_3, D_1] + b[D_3, D_2] \end{aligned}$$

(2) Antisymmetry:

$$[D_1, D_2] = -[D_2, D_1]$$

(3) Jacobi Identity:

$$[D_1, [D_2, D_3]] + [D_2, [D_3, D_1]] + [D_3, [D_1, D_2]] = 0$$

Definition 1.49. Let S be a set of vectors of a Lie algebra $\mathfrak{g}$. We say that the Lie algebra $\mathfrak{h} \subseteq \mathfrak{g}$ is generated by S if it is the smallest Lie algebra that contains S.

Lemma 1.50. *Let $\varphi : M \to N$ be C^∞, where M and N are two smooth manifolds. Let $D_1, D_1' \in \mathfrak{X}(M)$ and let $D_2, D_2' \in \mathfrak{X}(N)$. Then, if D_1 is φ-related to D_2, and D_1' is φ-related to D_2', then $[D_1, D_2]$ is φ-related to $[D_1', D_2']$.*

Proof. Let $x \in M$, and let $f \in C^\infty(N)$. We must show that $\varphi_{*,x}([D_1, D_2]_x)(f) = [D_1', D_2']_{\varphi(x)}(f)$. Using the definition of the Lie bracket, we can compute both sides of the equation:

$$\begin{aligned} \varphi_{*,x}([D_1, D_2]_x)f &= \varphi_{*,x}(D_{1\,x}(D_2) - D_{2\,x}(D_1))(f) \\ &= D_{1\,x}(D_2(f \circ \varphi)) - D_{2\,x}(D_1(f \circ \varphi)) \\ &= D_{1\,x}(\varphi_*(D_2)(f)) - D_{2\,x}(\varphi_*(D_1)(f)) \\ &= D_{1\,x}(D_2'(f) \circ \varphi) - D_{2\,x}(D_1'(f) \circ \varphi) \\ &= \varphi_{*,x}(D_{1\,x})(D_2' f) - \varphi_{*,x}(D_{2\,x})(D_1' f) \\ &= D'_{1\,\varphi(x)}(D_2' f) - D'_{2\,\varphi(x)}(D_1' f) = [D_1', D_2']_{\varphi(x)} f \end{aligned}$$

□

Lemma 1.51. *The Lie bracket of two left-invariant vector fields is itself a left-invariant vector field.*

Proof. The proof is immediate thanks to the above lemma: Let $D_1, D_2 \in \mathfrak{g} \subset \mathfrak{X}(G)$, where G is the Lie group. From the definition of left-invariant fields 1.45, we know that, consequently, both D_1 and D_2 are L_a-related to themselves, $\forall\ a \in G$. Thus, by virtue of the aforementioned lemma, $[D_1, D_2]$ will be L_a-related to itself $\forall\ a \in G$. □

Proposition 1.52. *The vector space* $\mathfrak{g}$ *of all left-invariant vector fields of a Lie group* G *forms a finite-dimensional Lie algebra endowed with the Lie bracket operation on vector fields.*

Proof. This proof becomes straightforward thanks to the last two lemmas. We have already proved that the Lie bracket of vector fields satisfies the necessary properties stated in Definition 1.48. Therefore, we only need to check that $\mathfrak{g}$ is closed under this map. Let $D_1, D_2 \in \mathfrak{g}$ be two left-invariant vector fields. Is $[D_1, D_2]$ a left-invariant vector field as well? Using the last Lemma 1.51, we know that $[D_1, D_2]$ is indeed a left-invariant field too. □

Definition 1.53. The Lie algebra $\mathfrak{g}$ of left-invariant vector fields of a Lie group G is called the Lie algebra of the Lie group G.

1.5.1 Lie transformation groups

Definition 1.54 (Lie transformation group)**.** A *Lie transformation group* is a pair (G, M), where G is Lie group and M a smooth manifold, endowed with a smooth map, called the *left action*:

$$\begin{aligned} \Phi : G \times M &\longrightarrow M \\ (g,x) &\longmapsto \Phi(g,x) = \Phi_g(x) = \Phi_x(g) \end{aligned}$$

such that

- $\Phi_e = Id_M$
- $\Phi_{g_1 \cdot g_2} = \Phi_{g_1} \circ \Phi_{g_2}$

Remark 1.55. We could define Lie transformation groups through right actions as well, similary to left translations, as mentioned before.

Remark 1.56. As with left translations, it is also obvious that the partial maps Φ_g are diffeomorphisms, as they have inverse and are smooth. It is therefore immediate from this fact and the definition that there exists a group morphism

$$\begin{aligned} \hat{\Phi} : &G \longrightarrow \text{Diff}(M) \\ &g \longmapsto \hat{\Phi}(g) = \Phi_g \end{aligned}$$

where $\text{Diff}(M)$ denotes the set of all diffeomorphisms from M onto itself.

We now introduce an important theorem that will prove fundamental to our objectives in this chapter. It is taken from [11, Page 23, Theorem 2.40].

Theorem 1.57. *Let* G *be a group of diffeomorphisms of a smooth manifold* M *onto itself. Let* $\mathfrak{C}$ *be the set of all complete vector fields on* M *such that* $\theta_t \in G$*, where* θ *is the global flow generated by each complete vector field. If the set* $\mathfrak{C}$ *generates a finite-dimensional Lie algebra of vector fields on* M*, then* (G, M) *is a Lie transformation group and* $\mathfrak{C}$ *is the Lie algebra of* G*.*

1.6 Curvature

Definition 1.58 (Riemannian curvature tensor)**.** Let M be a semi-Riemannian manifold with Levi-Civita connection ∇. The (1,3) tensor field R given by

$$R_{D_1,D_2}D = \nabla_{[D_1,D_2]}D + \nabla_{D_2}(\nabla_{D_1}D) - \nabla_{D_1}(\nabla_{D_2}D)$$

is called the *Riemannian curvature tensor*.

Remark 1.59. Bear in mind that some sources define this tensor to be what we would call here $-R$.

It is immediate to prove that it is indeed a tensor field. We can now introduce the following properties, proved in [13, Proposition 36 and 37, Pages 75-76]:

Proposition 1.60 (Properties of the Riemannian curvature tensor)**.** *Let $u, v, w, r \in T_xM$. Then, the following equations hold:*

1. $R_{u,v} = -R_{u,v}$
2. $g_x(R_{u,v}w, r) = -g_x(R_{u,v}r, w)$
3. *First Bianchi identity:* $R_{u,v}w + R_{v,w}u + R_{w,u}v = 0$
4. $g_x(R_{u,v}w, r) = g_x(R_{w,r}u, v)$
5. *Second Bianchi identity:* $\nabla_w R(u,v) + \nabla_v R(w,u) + \nabla_u R(v,w) = 0$

Note that the second expression says that curvature operators are skew-adjoint.

Definition 1.61 (Sectional curvature)**.** Let R be the Riemannian curvature tensor as defined above. Given any two linearly independent tangent vectors to M at the same point $u, v \in T_xM$ such that $g_x(u,u)g_x(v,v) - g(u,v)^2 \neq 0$, the quantity

$$\kappa(u,v) = \frac{g_x(R_{u,v}u, v)}{g_x(u,u)g_x(v,v) - g(u,v)^2} \tag{1.6}$$

is called the sectional curvature κ of the tangent plane defined by u, v. It can be proven [13, Page 77] that it is independent of the choice of basis u, v of the tangent plane. Hence, κ can be considered a real-valued function on the set of tangent planes that satisfy $g_x(u,u)g_x(v,v) - g(u,v)^2 \neq 0$.

Remark 1.62. If M is a $n-$sphere S^n of radius a with the euclidean metric restricted to it, then its sectional curvature is constant and equal to $1/a^2$. Similarly, if M is a hyperbolic space H^n endowed with the metric

$$g = a^2 \frac{(a^2 - |x|^2)\sum_i (dx^i)^2 - (\sum_i x^i dx^i)^2}{(a^2 - \sum_i (x^i)^i)^2}$$

then its sectional curvature is $-1/a^2$ [9, Page 209].

The following theorem is stated in [2, Page 42] and [9, Page 202].

Theorem 1.63 (Schur's Lemma). *Let (M, g) be a connected semi-Riemannian manifold with $n \geq 3$. Then, if the sectional curvature κ depends only on each $x \in M$, the sectional curvature is constant.*

Definition 1.64 (Ricci curvature tensor). Let R be the Riemannian curvature tensor of M as defined above. The *Ricci curvature tensor*, denoted by Ric, of M is the contraction $C_3^1(R) \in \mathcal{T}_2^0(M)$. Locally, using an orthogonal frame field $\{E_1, \ldots, E_n\}$, there exists [13, Page 87] the following formula for computing the value of the Ricci curvature tensor $\forall\, D_1, D_2 \in \mathfrak{X}(M)$:

$$\mathrm{Ric}(D_1, D_2) = \sum_m g(R_{D_1,E_m} D_2, E_m) g(E_m, E_m) \tag{1.7}$$

It can also be calculated with the sectional curvature. Let $e_1, \ldots, e_m$ be a basis at $T_x M$ such that $u = e_1$. Then [13, Page 88],

$$\mathrm{Ric}_x(u, u) = g_x(u, u) \sum_m \kappa_x(u, e_m) \tag{1.8}$$

Definition 1.65 (Scalar curvature). The *scalar curvature* S_x of M for each $x \in M$ is the contraction of the Ricci curvature tensor $C(\mathrm{Ric})$. Thus, in coordinates, $S_x = g_x^{ij} R_{xij}$.

Definition 1.66 (Einstein manifold). Let (M, g) be a semi-Riemannian manifold. Then, M will be an *Einstein manifold* if there exists, for each $x \in M$ and each $u, v \in T_x M$, some real constant $\lambda \in \mathbb{R}$ such that:

$$\mathrm{Ric} = \lambda g \tag{1.9}$$

The following remarks and proposition are taken from [2, Page 44].

Remark 1.67.

1. If $n = \dim\ M = 1$, Ric $= 0$ and M is always Einstein.
2. If $n = \dim\ M = 2$, M is Einstein if and only if it has constant scalar or sectional curvature.
3. If $n = \dim\ M = 3$, M is Einstein if and only if it has constant sectional curvature.

Proposition 1.68. *Let (M, g) be a semi-Riemannian manifold. If M has constant sectional curvature, then it is Einstein.*

Chapter II

Isometries

The concept of isometries will be fundamental in the following work. To that end, we have dedicated a whole chapter to them and their relation to the Killing fields, Lie groups and Lie algebras.

Definition 2.1 (Isometry)**.** Let (M, g) and (N, g') be semi-Riemannian manifolds. Then, a diffeomorphism $\psi : M \longrightarrow N$ is an isometry if $g(v, v') = g'(\psi_*(v), \psi_*(v')) \ \forall \ v, v' \in T_xM$. Equivalently, if it preserves the metric tensor: $\psi^*(g') = g$.

Proposition 2.2. *The following statements hold:*

- *The composition of isometries is an isometry.*
- *The inverse of an isometry is an isometry as well.*

Proof. Trivial. □

We also introduce the following proposition, which translates the well known concept of linear isometries in vector spaces to semi-Riemannian manifolds.

From the definition of isometry given before, it follows immediately that if ψ is an isometry between semi-Riemannian manifolds, then ψ_* is a linear isometry between its tangent spaces. As we saw before, an isometry between semi-Riemannian manifolds preserves the metric tensor. It can be inferred, then, that all related notions to the geometric structure (Riemannian curvature, covariant derivative...) remain invariant under an isometry. Thus, we can say that isometric manifolds are the *same*, in a sense.

Proposition 2.3. *The set of all isometries of a semi-Riemannian manifold (M, g) onto itself forms a group under the composition of mappings. We will denote such group as $I(M, g)$.*

Proof. As we know (thanks to Proposition 2.2) that the composition of isometries is an isometry, we only need to check that all elements of $I(M, g)$ satisfy the three group axioms:

1. Associativity: It is immediate, as the composition of mappings is always associative.

2. Identity element: The identity map Id is obviously an isometry, so it is an element of $I(M, g)$. As it is the identity map, then it follows that $\psi \circ Id = \psi$ and $Id \circ \psi = \psi$ $\forall\ \psi \in I(M, g)$.

3. Inverse element: Again, thanks to Proposition 2.2, we know that the inverse of an isometry exists and it is an isometry as well.

It follows, then, that $I(M, g)$ is a group. □

Definition 2.4 (Isometry group)**.** The set of all isometries of a semi-Riemannian manifold, $I(M, g)$, is called the *isometry group* of the manifold.

Remark 2.5. The importance of $I(M)$ to the geometry of M cannot be understated. Roughly speaking, the larger $I(M)$ is, the simpler M is. This can be understood intuitively as isometries represent the symmetries of the manifold. We will see shortly that we can apply the theory of Lie groups to study this group (as indeed it is a Lie group), if the group is non-trivial, i.e. it is larger than the identity map. In this case $I(M)$ is a geometric invariant of M, as important as the curvature and geodesics.

We can also define maps that are generalisations of the concept of isometry ("half-isometries"), as in that they preserve angles, but not lengths. These definitions are taken from [13, Page 92]:

Definition 2.6 (Conformal map)**.** Let (M, g) and (N, g') be semi-Riemannian manifolds. A smooth mapping $f : M \to N$ is *conformal* if $f^*(g') = hg$ for some function $h \in C^\infty(M)$ such that either $h > 0$ or $h < 0$.

The following is a special case of conformal maps:

Definition 2.7 (Homothety)**.** A diffeomorphism $f : M \to N$ is a *homothety of coefficient* c iff $f^*(g') = cg$, where $c \in \mathbb{R}$ is a constant.

Remark 2.8. A homothety with $c = 1$ is simply an isometry. If $c = -1$, we will call them *anti-isometries.*

2.1 Isometries and Killing vector fields

We now define a type of vector fields closely related to isometries: The Killing vector fields, also known simply as Killing fields, named after Wilhem Killing, German mathematician (1847-1923).

Definition 2.9 (Killing vector fields)**.** A smooth vector field K on a semi-Riemannian manifold (M, g) is a Killing vector field if the Lie derivative of the metric tensor with respect to K vanishes: $\mathcal{L}_K g = 0$

These fields can be understood as "infinitesimal" isometries. In the next propositions and following work we will see that this understanding is justified. The following proposition is taken from [13, Page 251, Proposition 23].

Proposition 2.10. *A vector field K on a semi-Riemannian manifold (M, g) is Killing if and only if $\theta_t : M \to M$ (where θ is the local flow generated by K) is an isometry for each t where the flow is properly defined.*

Proof. If each θ_t is an isometry, then $\theta_t^*(g) = g$ for all appropiate t. Therefore, by Proposition 1.36,

$$\mathcal{L}_K g = \lim_{t\to 0} \frac{1}{t}(\theta_t^*(g) - g) = 0$$

and K is Killing.

Conversely, if K is a Killing vector field, $\mathcal{L}_K g = 0$, we can consider its flow θ. If $(0, x)$ is in the domain of the flow, then so will (s, x) be for s sufficiently small thanks to the openness of the set M_t (see Theorem 1.17). Let $v, v' \in T_x M$. Now, by Proposition 1.36, like before, and remembering that $\theta_t \circ \theta_s = \theta_{t+s}$:

$$\lim_{t\to 0} \frac{1}{t} \left(g_x({\theta_{(t+s)}}_*(v), {\theta_{(t+s)}}_*(v')) - g_x({\theta_s}_*(v), {\theta_s}_*(v')) \right) = \left. \frac{d\psi}{dt} \right|_{t=s} = 0$$

Where ψ is the real-valued function $t \mapsto g({\theta_t}_*(v), {\theta_t}_*(v'))$ (recall that t needs to be in $\mathcal{D}^{(x)}$). It follows, then, that as the derivative of ψ is identically zero, the function ψ is constant. Therefore, $\psi(t) = \psi(0)$ for any t in its domain and we can write

$$g_x({\theta_t}_*(v), {\theta_t}_*(v')) = g_x(v, v') \ \forall\ x \in M,\ v, v' \in T_x M$$

□

We will also define another class of vector fields, related to conformal maps as Killing fields are related to isometries, to be used later in Chapters 3 and 4:

Definition 2.11 (Conformal Killing vector fields)**.** A *conformal Killing vector field* on a semi-Riemannian manifold (M, g) is a vector field C which satisfies

$$\mathcal{L}_C g = hg$$

for some $h \in C^\infty(M)$.

Remark 2.12. The (local) flow of a conformal Killing vector field defines conformal transformations, as opposed to standard Killing fields, whose flow is composed of isometries, as we have just seen in the prior proposition.

Proposition 2.13. *Let (M, g) be a semi-Riemannian manifold and let K be a smooth vector field on M. Then, the following statements are equivalent:*

1. *K is a Killing vector field.*
2. *$K(g(D_1, D_2)) = g([K, D_1], D_2) + g(D_1, [K, D_2]) \ \forall\ D_1, D_2 \in \mathfrak{X}(M)$.*
3. *$g(\nabla_{D_1} K, D_2) + g(D_1, \nabla_{D_2} K) = 0 \ \forall\ D_1, D_2 \in \mathfrak{X}(M)$.*

Proof.

1. $\iff$ 2. : From the definition of the Lie derivative 1.33, we know that $\forall\ D_1, D_2 \in \mathfrak{X}(M)$,

$$K \text{is Killing} \iff 0 = \mathcal{L}_K g(D_1, D_2) = K(g(D_1, D_2)) - g(\mathcal{L}_K D_1, D_2) - g(D_1, \mathcal{L}_K D_2) =$$

$$= K(g(D_1, D_2)) - g([K, D_1], D_2) - g(D_1, [K, D_2])$$

Thus,

$$K \text{is Killing} \iff K(g(D_1, D_2)) = g([K, D_1], D_2) + g(D_1, [K, D_2])$$

2. $\iff$ 3. : As the Levi-Civita connection is torsion-free and compatible with the metric (Theorem 1.23), we can write:

$$K(g(D_1, D_2)) = K(g(\nabla_K D_1, D_2) + g(D_1, \nabla_K D_2))$$

$$g([K, D_1], D_2) + g(D_1, [K, D_2]) = g(\nabla_K D_1 - \nabla_{D_1} K, D_2) + g(D_1, \nabla_K D_2 - \nabla_{D_2} K)$$

Consequently,

$$K(g(D_1, D_2)) = g([K, D_1], D_2) + g(D_1, [K, D_2]) \iff$$

$$\iff g(\nabla_{D_1} K, D_2) + g(D_1, \nabla_{D_2} K) = 0$$

□

We conclude this part by writing the following lemma, whose proof can be found in [11, Page 47, Lemma 4.14].

Lemma 2.14. *Let K be a Killing vector field on a connected semi-Riemannian manifold. If there exists a point $x \in M$ such that both $K_x = 0$ and $(\nabla K)_x = 0$, then $K = 0$.*

We can use this lemma to directly prove the following:

Proposition 2.15. *Let K and K' be two Killing fields on a semi-Riemannian manifold (M, g). If there exists a point $x \in M$ such that $K_x = K'_x$ and $(\nabla K)_x = (\nabla K')_x$, then $K = K'$.*

Proof. We can take the vector field $K - K'$, which is trivially a Killing field too thanks to the properties of the Lie derivative. Then, it follows that, as there exists a point $x \in M$ such that $(K - K')_x = 0$ and $(\nabla(K - K'))_x = 0$, we can use the aforementioned Lemma 2.14 to conclude that $K - K' = 0 \Rightarrow K = K'$. □

2.2 The Lie Algebra of Killing fields

Definition 2.16. Let (M, g) be a semi-Riemannian manifold. We will then denote by $i(M, g)$ the set of all Killing vector fields on (M, g).

Theorem 2.17. *Let (M, g) be a semi-Riemannian manifold. Then, $i(M, g)$ is a Lie algebra.*

Proof. Let $K_1, K_2 \in i(M, g)$ be two Killing vector fields. It is immediate from the properties of the Lie derivative (see Proposition 1.37) that

- $\lambda K_1 + \mu K_2 \in i(M, g) \ \forall \ \lambda, \mu \in \mathbb{R}$
- $[K_1, K_2] \in i(M, g)$

Therefore, $i(M, g)$ satisfies the properties of a Lie algebra, and is, consequently, a Lie algebra. □

The following lemma will be useful when proving the next theorem.

Lemma 2.18. *Let (M, g) be a connected semi-Riemannian manifold of dimension n. Let $x \in M$ be any point in M. The set of all linear operators A on T_xM that fulfill the condition $g_x(Av_1, v_2) = -g_x(v_1, Av_2)$, where $v_1, v_2 \in T_xM$, form a Lie algebra, which we will denote by $\mathfrak{o}(T_xM)$.*

Proof. The Lie bracket of this Lie algebra is the *conmutator*, i.e. given any two linear operators $A, B \in \mathfrak{o}(T_xM)$, $[A, B] = A \circ B - B \circ A$. It is easily shown that the Lie algebra is closed under the conmutator. Let $v_1, v_2 \in T_xM$ and let $A, B \in \mathfrak{o}(T_xM)$:

$$\begin{aligned} g_x([A, B]v_1, v_2) &= g_x(A(Bv_1), v_2) - g_x(B(Av_1), v_2) = -g_x(Bv_1, Av_2) + g_x(Av_1, Bv_2) \\ &= g_x(v_1, B(A(v_2))) - g_x(v_1, A(B(v_2))) = -g_x(v_1, [A, B]v_2) \end{aligned}$$

We now only need to check that the conmutator is, indeed, a bracket. Due to the properties of the composition of operators, it is trivial that it is bilinear and antisymmetric. A simple, if tedious, calculation proves that is also follows the Jacobi identity, which we shall not write due to its length. Thus, $\mathfrak{o}(T_xM)$ is a Lie algebra with the conmutator. □

Theorem 2.19. *Let (M, g) be a connected semi-Riemannian manifold of dimension n. Then, the dimension of its Lie algebra of Killing vector fields $i(M, g)$ is at most $n(n+1)/2$, i.e.*

$$\dim i(M, g) \leq \frac{n(n+1)}{2}$$

Proof. Let $x \in M$ be any (fixed) point in M. Thanks to the Lemma 2.18, we know that $\mathfrak{o}(T_xM)$ is a Lie algebra. It is well known that, given a fixed basis, there is a bijective correspondence between linear operators and matrices. Thus, it is clear that $\mathfrak{o}(T_xM)$ will be isomorphic to the set of skew-symmetric matrices with dimension n and index (of the inner product) ν, denoted as $\text{Skew}_\nu(n)$. Using an orthonomal basis, the matrix g_x becomes $G = \text{diag}(-1, \overset{\nu}{\ldots}, -1, 1, \overset{n-\nu}{\ldots}, 1)$. A simple calculation proves the aforementioned isomorphism. Given $v_1, v_2 \in T_xM$:

$$A \in \mathfrak{o}(T_xM) \iff (Av_1)^t G v_2 = -v_1^t G(Av_2) \iff v_1^t A^t G v_2 = -v_1^t G A v_2 \iff$$

$$\iff A^t G = -GA \iff A^t = -GAG \iff A \in \text{Skew}_\nu(n)$$

Consequently, $\mathfrak{o}(T_xM) \simeq \text{Skew}_\nu(n)$ as Lie algebras. It is well known that $\dim \text{Skew}_\nu(n) = \frac{n(n-1)}{2}$. Now, let us consider the map ϕ:

$$\phi : i(M, g) \longrightarrow T_xM \times \mathfrak{o}(T_xM)$$

$$K \longmapsto (K_x, (\nabla K)_x)$$

We need to show that $(\nabla K)_x \in \mathfrak{o}(T_xM)$ to prove that this map is well defined. Due to the Proposition 2.13(3.), for any $D_1, D_2 \in \mathfrak{X}(M)$:

$$g(\nabla_{D_1}K, D_2) + g(D_1, \nabla_{D_2}K) = 0$$

Thus, for any two $v_1, v_2 \in T_xM$,

$$g_x((\nabla K)_x v_1, v_2)) + g_x(v_1, (\nabla K)_x v_2) = 0 \iff (\nabla K)_x \in \mathfrak{o}(T_xM)$$

Now, thanks to the Proposition 2.15, we know that this map will be injective: Given $K, K' \in i(M,g)$ such that $(K_x, (\nabla K)_x) = (K'_x, (\nabla K')_x)$. Then, we can use the aforementioned proposition to conclude that $K = K'$. As a result:

$$\dim\, i(M,g) \leq \dim\, T_xM + \dim\, \mathfrak{o}(T_xM) = \dim\, T_xM + \dim\, \mathrm{Skew}_\nu(n) = n + \frac{n(n-1)}{2}$$

$$\implies \dim\, i(M,g) \leq \frac{n(n+1)}{2}$$

□

2.3 The Lie Group of isometries

Let us recall that we denoted as $I(M,g)$ the set of all isometries on (M,g) onto itself, and that we proved that it was a group (Proposition 2.3). Let $\mathfrak{ci}(M,g) = \{K \in i(M,g) : K \text{ is complete}\}$ be the set of all complete Killing vector fields on M. We will now try to use Theorem 1.57 to prove the final objective of this chapter: That $(I(M,g), M)$ is a Lie transformation group, and $\mathfrak{ci}(M,g)$ is the finite-dimensional Lie algebra of $I(M,g)$. To achieve this, we have to show that $(I(M,g), M)$ and $\mathfrak{ci}(M,g)$ fulfill the necessary hypotheses of Theorem 1.57:

Let G be a group of diffeomorphisms of a smooth manifold M onto itself. Let $\mathfrak{C}$ be the set of all complete vector fields on M such that $\theta_t \in G$, where θ is the global flow generated by each complete vector field. If the set $\mathfrak{C}$ generates a finite-dimensional Lie algebra of vector fields on M, then (G, M) is a Lie transformation group and $\mathfrak{C}$ is the Lie algebra of G.

We do this with the following two lemmas:

Lemma 2.20. *Let (M,g) be a semi-Riemannian manifold. The set of all complete Killing vector fields on M, $\mathfrak{ci}(M,g)$, is the set of all vector fields on M that generate global flows whose θ_t are isometries from M onto itself, which we will denote by $\mathfrak{C}$.*

Proof. Our goal is, then, to show that $\mathfrak{ci}(M,g) = \mathfrak{C}$. Proposition 2.10 tells us that a vector field is Killing if and only if its flows are isometries of M on their domains. If we consider complete vector fields, it is an inmediate corollary of the proposition that a complete vector field K is Killing if and only if its flows are isometries of M. Thus, we have proved that $\mathfrak{ci}(M,g) = \mathfrak{C}$. □

Lemma 2.21. *$\mathfrak{ci}(M,g)$ generates a finite-dimensional Lie algebra.*

Proof. Due to Theorem 2.19, this proof becomes almost trivial: We know that $i(M,g)$ is a finite-dimensional Lie algebra, with dimension at most $\frac{n(n+1)}{2}$. Let $\mathfrak{g}$ be the Lie algebra generated by $\mathfrak{ci}(M,g)$. As $\mathfrak{ci}(M,g) \subset i(M,g)$, by virtue of Theorem 2.17, we know that both $aK_1 + bK_2 \in \mathfrak{g}$ and $[K_1, K_2] \in \mathfrak{g}$ will be Killing fields (and thus, elements of $i(M,g)$) for all $K_1, K_2 \in \mathfrak{ci}(M,g)$ and $a, b \in \mathbb{R}$. Consequently, $\mathfrak{g}$ is a Lie subalgebra of $i(M,g)$, and therefore finite-dimensional too. □

Therefore, we can use the aforementioned Theorem 1.57 to write:

Theorem 2.22. *Let (M,g) be a semi-Riemannian manifold. Let $I(M,g)$ be the set of all isometries of M onto iself, and let $\mathfrak{ci}(M,g)$ be the set of all complete vector fields on (M,g). Then, $(I(M,g),M)$ is a Lie transformation group and $\mathfrak{ci}(M,g)$ is the finite-dimensional Lie algebra of the Lie group $I(M,g)$.*

Remark 2.23. An important consequence of the fact that $(I(M,g),M)$ is a Lie transformation group is that the map

$$\begin{aligned} I(M,g) \times M &\longrightarrow M \\ (\Phi, x) &\longmapsto \Phi(x) \end{aligned}$$

which is the naturally defined action of the Lie group $I(M,g)$ on M, is smooth.

Remark 2.24. Thanks to Proposition 1.47, we know that the Lie algebra $\mathfrak{g}$ of a Lie group G is isomorphic to T_eG. Thus, it is immediate that $\mathfrak{ci}(M,g)$ is isomorphic to $T_{Id}I(M,g)$. We can think of complete Killing fields, then, as elements of the group that are "infinitesimally close" to the identity. This explains the fact that Killing fields are usually called the "infinitesimal generators" of the isometries in physics. The exponential map, which lies outside the scope of this work, allows us to find the isometries using the infinitesimal generators.

Remark 2.25. It is important to note that not all isometries are generated as flows of complete Killing vector fields. This is only true for *smooth* isometries. There are other types of isometries: We call them "discrete" isometries (such as reflections, rotations in a square, or C-parity), as opposed to "continuous" isometries (for example, rotations in an isotropic manifold, or traslations in a homogeneous space), which are the smooth ones that we have been working with. Both bear great importance in physics and related areas.

Chapter III

Warped products

In this chapter we will first introduce the concept of "warped products", which is a particular type of product manifolds, and related propositions and lemmas, which will be crucial to our final objective. Next, we will finally begin writing our ultimate and most important objective: The application of all that we have explained to spacetime and its famous metric, the *Friedmann-Lemaître-Robertson-Walker metric*.

3.1 Product manifolds

As a warped product is a type of metric on a semi-Riemannian product manifold, we first explain the structure that any such manifold inherits from their two component manifolds. Given two semi-Riemannian manifolds M and N, we can consider their cartesian product $M \times N$, which will be, too, a semi-Riemannian manifold with the following metric tensor:

$$g = \pi^*(g_M) + \sigma^*(g_N)$$

Where π and σ are the projections of $M \times N$ onto M and N, respectively, and g_M, g_N the metric tensors on M and N. We must now show that this structure is well-defined and that, with it, $M \times N$ is actually a semi-Riemannian manifold:

Lemma 3.1 (Product manifolds). *Let M and N be smooth manifolds. Then, $M \times N$ is a smooth manifold.*

Proof. Suppose that $(U, \{x_i\})$ and $(V, \{y_i\})$ are coordinate charts on M and N, respectively. Then there is a coordinate system $\varphi : M \times N \to R^{n+m}$ on the open set $U \times V \subseteq M \times N$, called the *product coordinate system*:

$$\varphi(x, y) = (x_1(x), \ldots, x_n(x), y_1(y), \ldots, y_m(y))$$

Where $n = \dim M$ and $m = \dim N$. Using the atlases of M and N, we can use this coordinate system to construct an atlas on $M \times N$. □

Lemma 3.2 (Semi-Riemannian product manifolds). *Let M and N be semi-Riemannian manifolds with metric tensors g_M and g_N. Then g as written above is a metric tensor on $M \times N$, making it a semi-Riemannian product manifold.*

Proof. Let $(x, y) \in M \times N$, and let $v_1, v_2 \in T_{(x,y)}(M \times N)$, then

$$g(v_1, v_2) = (\pi^*(g_M) + \sigma^*(g_N))\,(v_1, v_2) = g_M(\pi_* v_1, \pi_* v_2) + g_N(\sigma_* v_1, \sigma_* v_2)$$

and g is obviously symmetric. To show non-degeneracy, suppose that there exists a $v \in T_{(x,y)}(M \times N)$ such that $g(v, v') = 0 \ \forall \ v' \in T_{(x,y)}(M \times N)$. Hence, $g_M(\pi_* v, \pi_* v') + g_N(\sigma_* v, \sigma_* v') = 0$. However, if we take in particular $v' = i_{M\,*}\, w$, where i_M is the inclusion map $M \to M \times N$ and $w \in T_x M$, then $\sigma_* v' = 0$ necessarily, and therefore

$$g_M(\pi_* v, \pi_* v') = 0 \ \forall \ v' \in T_{(x,y)}(M \times N) \implies g_M(\pi_* v, w') = 0 \ \forall \ w' \in T_x M \implies \pi_* v = 0$$

because g_M is a metric tensor. We can do the same with a $w \in T_y N$, i_N and g_N to get $\sigma_* v = 0$. We conclude, then, that $v = 0$ and g is non degenerate. Finally, as orthonormal bases on $T_x M$ and $T_y N$ can be combined to form an orthonormal basis on $T_{(x,y)}(M \times N)$, then g has constant index on every point of value ind M + ind N. □

Lemma 3.3. *Let π and σ be the canonical projections from $M \times N$ to M and N, respectively. Then, each element of the tangent space of the product manifold $M \times N$ at any point (x, y) has a unique expression in the product of the tangent spaces of M and N:*

$$v \in T_{(x,y)} M \times N \mapsto (\pi_{*,(x,y)} v, \sigma_{*,(x,y)} v) \in T_x M \times T_y N$$

In the sense that this map is a linear isomorphism. Consequently, $T_{(x,y)}(M \times N) \simeq T_x M \times T_y N$.

Proof. Let $i_{y'}$ and $i_{x'}$ for any $x' \in M$ and $y' \in N$ be the inclusions from M and N onto $M \times N$, such that $i_{y'}(x) = (x, y')$ and $i_{x'}(y) = (x', y) \ \forall \ x \in M, \ y \in N$. Let $(x, y) \in M \times N$. We will prove that the following map

$$\begin{aligned} \varphi_1 : T_{(x,y)}(M \times N) &\longrightarrow T_x M \times T_y N \\ v &\longmapsto (\pi_{*,(x,y)} v, \sigma_{*,(x,y)} v) \end{aligned}$$

is a linear isomorphism. It is obviously linear due to the linearity of π_* and σ_*. Now, consider the map

$$\begin{aligned} \varphi_2 : T_x M \times T_y N &\longrightarrow T_{(x,y)}(M \times N) \\ (v, w) &\longmapsto (i_{y'})_{*,x} v + (i_{x'})_{*,y} w \end{aligned}$$

It can be shown that $\varphi_1 \circ \varphi_2 = Id_{T_x M \times T_y N}$. Let $(v, w) \in T_x M \times T_y N$. Then:

$$(\varphi_1 \circ \varphi_2)(v, w) = \Big(\pi_{*,(x,y)}((i_{y'})_{*,x} v + (i_{x'})_{*,y} w), \sigma_{*,(x,y)}((i_{y'})_{*,x} v + (i_{x'})_{*,y} w)\Big)$$

Using that:

$$\pi \circ i_{y'} = Id_M, \ \sigma \circ i_{y'} = y', \ \pi \circ i_{x'} = x', \ \sigma \circ i_{x'} = Id_N$$

And that the tangent map of the composition of two or more maps is the composition of tangent maps, we can write:

$$(\varphi_1 \circ \varphi_2)(v, w) = ((Id_M)_{*,x} v, (Id_N)_{*,y} w) = (v, w)$$

Thus, as $\varphi_1 \circ \varphi_2 = Id_{T_x M \times T_y N}$, i.e., φ_2 is a section of φ_1, φ_1 is surjective. Now, as $\dim T_{(x,y)} M \times N = \dim M \times N = \dim M + \dim N$ and $\dim T_x M \times T_y N = \dim M + \dim N$, we can finally conclude that φ_1 is, indeed, an isomorphism. □

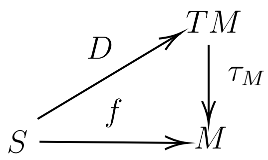
Theorem 3.4. *Let $M \times N$ be a smooth product manifold. Then, there is a diffeomorphism*

$$\begin{aligned} \varphi : T(M \times N) &\xrightarrow{\sim} TM \times TN \\ (x, y, v) &\longmapsto ((x, \pi_* v), (y, \sigma_* v)) \end{aligned}$$

and, aditionally,

$$\mathfrak{X}(M \times N) \simeq \mathfrak{X}_\pi(M) \times \mathfrak{X}_\sigma(N)$$

where $\mathfrak{X}_f(M)$ are the smooth maps D that make the following diagram commutative:



for any smooth manifold S and morphism f, where τ_M is the projection of the tangent bundle of M onto the manifold M.

Proof. To prove that such map is a diffeomorphism, we will make use of a corollary of the inverse function theorem for manifolds [10, Page 166]:

Suppose M and N are smooth manifolds of the same dimension, and $f : M \longrightarrow N$ is a local diffeomorphism. If f is bijective, it is a diffeomorphism.

Hence, we will work locally, using coordinate systems, to prove that it is indeed a local diffeomorphism. We also know that it is bijective thanks to Lemma 3.3, so, with that done, we will be able to conclude that it is a diffeomorphism.

Let (U, x_i) and (V, y_i) be coordinate charts on M and N. Then, the map φ can be written (locally) as:

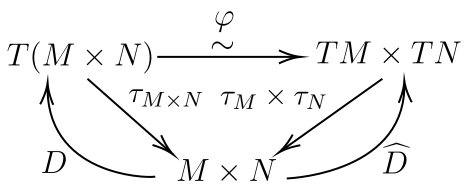
$$\begin{aligned} \hat{\varphi} \coloneqq \varphi|_{(U \times V \times \mathbb{R}^n \times \mathbb{R}^m)} : U \times V \times \mathbb{R}^n \times \mathbb{R}^m &\longrightarrow U \times \mathbb{R}^n \times V \times \mathbb{R}^m \\ (x, y, \lambda, \mu) &\longmapsto (x, \lambda, y, \mu) \end{aligned}$$

It is trivially bijective. We now have to show that both $\hat{\varphi}$ and their inverse are smooth. A map between manifolds is, by definition, smooth iff its expression in coordinates is smooth as a real-valued function. Let $\phi = (x_i)$ and $\psi = (y_i)$ be the coordinate charts of M and N on U and V. The coordinate charts of $\mathbb{R}^n$ and $\mathbb{R}^m$ are obviously the identities. Then, its expression in coordinates is:

$$\begin{aligned} (\phi, Id_{\mathbb{R}^n}, \psi, Id_{\mathbb{R}^m}) \circ \hat{\varphi} \circ (\phi, \psi, Id_{\mathbb{R}^n}, Id_{\mathbb{R}^m})^{-1}(x_1, \ldots, x_n, y_1, \ldots, y_m, \lambda, \mu) = \\ = (x_1, \ldots, x_n, \lambda, y_1, \ldots, y_m, \mu) \end{aligned}$$

This function is evidently C^∞. A similar reasoning concludes that the expression in coordinates of the inverse $\hat{\varphi}^{-1}$ is also smooth. Therefore, φ is a local diffeomorphism, and, through the inverse function theorem, we can come to the conclusion that φ is a diffeomorphism.

To finish the proof, we have to show that, esentially, every smooth vector field on $M \times N$ can be written as the sum of two smooth fields on M and N, and viceversa. Bear in mind that all smooth vector fields of a manifold are the smooth sections of the projection of the tangent bundle onto the manifold. Let $D \in \mathfrak{X}(M \times N)$. Consider the diagram:



We can then define the "vector field" $\widehat{D} = (\widehat{D}_M, \widehat{D}_Y)$ as the map that makes the diagram above commutative with D. Both $\widehat{D}_M$ and $\widehat{D}_N$ are smooth because they are a composition of smooth maps:

$$\widehat{D}_M = \pi_* \circ \varphi \circ D \qquad \widehat{D}_N = \sigma_* \circ \varphi \circ D$$

They also must be elements of $\mathfrak{X}_\pi(M)$ and $\mathfrak{X}_\sigma(N)$, respectively, so that they are well defined:

$$\tau_M \circ \widehat{D}_M = \pi \qquad \tau_N \circ \widehat{D}_N = \sigma$$

$$\tau_M(\widehat{D_M}(x, y)) = \tau_M(\varphi(D(x, y))) = \tau_M(x, \pi_* D_{x,y}) = x = \pi(x, y)$$

The argument for $\widehat{D}_N$ is completely equivalent. Evidently, for any $D_M \in \mathfrak{X}(M)$ or $D_N \in \mathfrak{X}(N)$, $D_M \circ \pi \in \mathfrak{X}_\pi(M)$ and $D_N \circ \sigma \in \mathfrak{X}_\sigma(N)$. Now we only need to check that this mapping

$$\begin{aligned} \alpha : \mathfrak{X}(M{\times}N) &\longrightarrow \mathfrak{X}_\pi(M) \times \mathfrak{X}_\sigma(N) \\ D &\longmapsto (\widehat{D}_M, \widehat{D}_N) \end{aligned}$$

is bijective. Let $(V_M, V_N) \in \mathfrak{X}_\pi(M) \times \mathfrak{X}_\sigma(N)$. We can define the inverse map:

$$\begin{aligned} \beta : \mathfrak{X}_\pi(M){\times}\mathfrak{X}_\sigma(N) &\longrightarrow \mathfrak{X}(M \times N) \\ (V_M, V_N) &\longmapsto \varphi^{-1} \circ (V_M, V_N) \end{aligned}$$

$\beta(V_M, V_N)$ is a smooth vector field on $M \times N$ as it is smooth (due to being a composition of smooth maps) section of $\tau_{M\times N}$. β is also indeed the inverse of α:

- $\alpha \circ \beta = Id_{\mathfrak{X}_\pi(M)\times\mathfrak{X}_\sigma(N)}$:

$$\alpha \circ \beta(V_M, V_N) = \varphi \circ \varphi^{-1} \circ (V_M, V_N) = (V_M, V_N)$$

- $\beta \circ \alpha = Id_{\mathfrak{X}(M\times N)}$:

$$\beta \circ \alpha(D) = \varphi^{-1} \circ \varphi \circ D = D$$

□

Definition 3.5 (Lift of a function)**.** Let $f \in C^\infty(M)$. Then, the *lift* of f to $M \times N$ is $\overline{f} = f \circ \pi$.

Definition 3.6 (Lift of a smooth endomorphism)**.** Let φ be a smooth endomorphism on F. Then, its *lift* to $B \times_f F$ is the map

$$\begin{aligned} \overline{\varphi} := Id_B \times \varphi : & B \times F \longrightarrow B \times F \\ & (x, y) \longmapsto (x, \varphi(y)) \end{aligned} \tag{3.1}$$

We can do the exact same procedure with endomorphisms on B.

Definition 3.7 (Lift of a vector field)**.** Let $D \in \mathfrak{X}(M)$. We define the *lift* of D to $M \times N$ as the vector field $\overline{D}$ such that $\forall\ (x, y) \in M \times N$:

$$D_x = \pi_{*,(x,y)}(\overline{D}_{(x,y)})$$

$$0_y = \sigma_{*,(x,y)}(\overline{D}_{(x,y)})$$

i.e. $\overline{D}$ is π-related to D and σ-related to 0. We know that each $\overline{D}_{(x,y)}$ will be unique thanks to the prior Lemma 3.3 and Theorem 3.4. Obviously, we can do the same with any $D' \in \mathfrak{X}(N)$. We will denote the set of all such lifts as $\mathfrak{L}(M)$ or $\mathfrak{L}(N)$.

Remark 3.8. Given the above definition, the lift of a vector field is, thanks to Theorem 3.4, essentially,

$$\overline{D}_{(x,y)} = (D_x, 0_y)\ \forall\ D \in \mathfrak{X}(M), (x, y) \in M \times N$$
$$\overline{V}_{(x,y)} = (0_x, V_y)\ \forall\ V \in \mathfrak{X}(N), (x, y) \in M \times N$$

We know that they will be smooth thanks to the aforementioned theorem.

Remark 3.9. We will use $\overline{D}$ to denote the lift of the vector field D on B or F to $B \times F$.

Remark 3.10. It is immediate from the metric tensor on the product manifold that the lift of vectors tangent to M are orthogonal to the lift of vectors tangent to N.

Proposition 3.11. *The lift $\overline{D}$ of $D \in \mathfrak{X}(M)$ to $M \times N$ is a smooth vector field.*

Proof. Using the product manifold coordinate system (see [13, Page 4] and the previous Lemma 3.1), where x_i and y_i are the coordinate functions of M and N respectively, the proof becomes easy. Let $D \in \mathfrak{X}(M)$, and let $n = \dim M$, $m = \dim N$. Then, $\forall\ (x, y) \in M \times N$,

$$D_x = \sum_{i=1}^{n} \lambda_i(x) \left.\frac{\partial}{\partial x_i}\right|_x$$

$$\overline{D} = \sum_{i=1}^{n} \overline{\lambda}_i(x, y) \left.\frac{\partial}{\partial x_i \circ \pi}\right|_{(x,y)} + \sum_{j=1}^{m} \mu_j(x, y) \left.\frac{\partial}{\partial y_j \circ \sigma}\right|_{(x,y)}$$

and we need to show that the $\overline{\lambda}_i$ and μ_j functions are smooth for all i, j. Now, we know that $\pi_*(\overline{D}) = D$, so:

$$D_x = \sum_{i=1}^{n} \overline{\lambda}_i(x, y) \pi_{*,(x,y)} \left(\left.\frac{\partial}{\partial x_i \circ \pi}\right|_{(x,y)} \right) = \sum_{i=1}^{n} \overline{\lambda}_i(x, y) \left.\frac{\partial}{\partial x_i}\right|_x$$

Thus we conclude that $\overline{\lambda}_i(x,y) = \lambda_i(x) \ \forall \ (x,y) \in M \times N$ and $\forall \ i = 1,\ldots,n$. Therefore, the $\overline{\lambda}_i$ functions are smooth, as $D \in \mathfrak{X}(M)$ by hypothesis. We also know that $\overline{D}$ is σ-related to the null vector field on N, and, like before, we get that $\mu_j \equiv 0 \ \forall \ j = 1,\ldots,m$, so they are smooth too. □

Lemma 3.12. *The Lie bracket of two vector fields, where one is tangent to the one manifold and the other to the other manifold, is the null vector field. That is to say, given any $(x,y) \in M \times N$, $D \in \mathfrak{X}(M)$ and $V \in \mathfrak{X}(N)$:*

$$[\overline{D}, \overline{V}] = 0$$

Proof. The proof is a straighforward computation using the definition of lifts: $\overline{D}$ is π-related to D and σ-related to 0 (viceversa with $\overline{V}$). Let $f \in C^\infty(M)$:

$$\begin{aligned}
\pi_*[\overline{D}, \overline{V}]_{(x,y)} f &= \overline{D}_{(x,y)}(\pi_* \overline{V} f) - \overline{V}_{(x,y)}(\pi_* \overline{D} f) \\
&= \overline{D}_{(x,y)}(0f) - \overline{V}_{(x,y)}((Df) \circ \pi) \\
&= -\pi_* \overline{V}_{(x,y)}(Df) = -0_x(Df) = 0
\end{aligned}$$

We can similarly conclude that $\sigma_*[\overline{D}, \overline{V}] = 0$. Thus, by virtue of Lemma 3.3, $[\overline{D}, \overline{V}]_{(x,y)} = 0_{(x,y)} \ \forall \ (x,y) \in M \times N \implies [\overline{D}, \overline{V}] = 0$.

□

Lemma 3.13. *The lift of the Lie bracket of two vector fields is the Lie bracket of the lifts:*

$$\overline{[D_1, D_2]} = [\overline{D_1}, \overline{D_2}] \ \forall \ D_1, D_2 \in \mathfrak{X}(M)$$
$$\overline{[V_1, V_2]} = [\overline{V_1}, \overline{V_2}] \ \forall \ V_1, V_2 \in \mathfrak{X}(N)$$

We will now state and prove an importat theorem for this chapter. It will allow us to more easily manipulate vectors

Proof. We will prove the first equation, as the other one is completely analogue. Due to the definition of lifts, we simply have to prove that $[\overline{D_1}, \overline{D_2}]$ is π-related to $[D_1, D_2]$ and σ-related to 0. A straightforward computation shows this to be the case. Let $f \in C^\infty(M)$ and let $(x,y) \in M \times N$:

$$\begin{aligned}
\pi_*[\overline{D_1}, \overline{D_2}]_{(x,y)} f &= (\overline{D_1})_{(x,y)}(\overline{D_2} f \circ \pi) - (\overline{D_2})_{(x,y)}(\overline{D_1} f \circ \pi) \\
&= (\overline{D_1})_{(x,y)}((D_2 \circ \pi) f) - (\overline{D_2})_{(x,y)}((D_1 \circ \pi) f)
\end{aligned}$$

As $(D_1 \circ \pi) f$ and $(D_2 \circ \pi) f$ are smooth functions that depend solely on the point x of M, and not on y,

$$\begin{aligned}
\pi_*[\overline{D_1}, \overline{D_2}]_{(x,y)} f &= (D_1)_x(D_2 f) - (D_2)_x(D_1 f) \\
&= ([D_1, D_2] \circ \pi)(x,y) f \ \forall \ (x,y) \in M \times N, f \in C^\infty(M) \implies \\
&\implies \pi_*[\overline{D_1}, \overline{D_2}] = [D_1, D_2] \circ \pi
\end{aligned}$$

□

3.1.1 Necessary notions on semi-Riemannian submanifolds

Now we provide an overview of the shape tensor (and related notions), a concept widely used in the geometry of semi-Riemannian submanifolds, which will evidently be very useful in the context of product manifolds and warped products. We will also fix the notation that will be used in this chapter.

Remark 3.14. In this section, we will mostly ignore the tangent maps of the inclusion map of a submanifold in a manifold to simplify the notation.

Remark 3.15. Given a submanifold $N \subseteq M$, we will denote by $\overline{\mathfrak{X}}(N)$ the set of all smooth vector fields D that smoothly assign to each point of N, $x \in N$, a vector tangent to M, $D_x \in T_xM$. It can be shown that it is a module over $C^\infty(N)$, and that $\mathfrak{X}(M)$ is a submodule of $\overline{\mathfrak{X}}(N)$.

Definition 3.16 (Shape tensor). The *shape tensor*, or *second fundamental form*, of a submanifold $N \subseteq M$ is a function $II : \mathfrak{X}(N) \times \mathfrak{X}(N) \to \mathfrak{X}(N)^\perp$ such that

$$II(D_1, D_2) = \text{nor } \nabla_{D_1} D_2$$

is $C^\infty(N)$-bilinear and symmetric, where:

- $\mathfrak{X}(N)^\perp$ is the set of all vector fields $D \in \overline{\mathfrak{X}}(N)$ such that D_x is normal to N $\forall x \in N$.
- nor is the orthogonal projection

  $$\text{nor}\colon\ T_xM \longrightarrow T_xN^\perp$$

  that sends each tangent vector of M to its normal component to N. Equivalently, we will define

  $$\tan\colon\ T_xM \longrightarrow T_xN$$

  as the orthogonal projection that send each tangent vector to its tangent component to N, such that each $v \in T_xM$ can be written as a sum $v = \tan v + \text{nor } v$.

Remark 3.17. The value of each tensor depends not on the entirety of each one-form or vector field, but on their values at each point solely [13, Proposition 2.2]. Thus, we could consider a map

$$II : T_xN \times T_xN \longrightarrow T_xN^\perp$$

that is $\mathbb{R}$-bilinear, well defined and totally equivalent to the previously defined shape tensor.

Definition 3.18 (Totally geodesic submanifold). A semi-Riemannian submanifold $N \subseteq M$ is *totally geodesic* provided its shape tensor vanishes: $II \equiv 0$.

Definition 3.19 (Totally umbilic submanifold). Given a semi-Riemannian submanifold $N \subseteq M$, a point $x \in N \subseteq M$ is *umbilic* if there is a vector $v \in T_xN^\perp$ such that

$$II(w, w') = g_x(w, w')v \ \forall\ w, w' \in T_xN$$

N is said to be *totally umbilic* if all of its points are umbilic.

3.2 Warped products

Let (B, g_B) and (F, g_F) be semi-Riemannian manifolds with dimensiones n and m respectively. We will denote their Levi-Civita connections as ∇^B and ∇^F, and their projections from $B \times F$ onto B and F as π and σ, respectively. Then, we can define a special type of metric tensors on $B \times F$:

Definition 3.20 (Warped product)**.** Let (B, g_B) and (F, g_F) be semi-Riemannian manifolds and let $f > 0$ be a smooth strictly positive function on B. The *warped product* $M = B \times_f F$ is the product manifold $B \times F$ endowed with the metric tensor

$$g = \pi^*(g_B) + (f \circ \pi)^2 \sigma^*(g_F)$$

More explicitly, given $v, v' \in T_{(x,y)} B \times F$,

$$g(v, v') = g_B(\pi_* v, \pi_* v') + f^2(x) g_F(\sigma_* v, \sigma_* v')$$

The exact same argument used to prove the Lemma 3.2 shows that g is indeed a metric tensor (Lemma 3.2):

Lemma 3.21. *Let (B, g_B) and (F, g_F) be semi-Riemannian manifolds, and let $f \in C^\infty(B)$ be a strictly positive smooth function on B. Then, $B \times_f F$ is a semi-Riemannian manifold.*

Proof. It is well known [13, Page 4] that the product manifold of two smooth manifolds, endowed with a product coordinate system, is a smooth manifold too. We will now prove that $g = \pi^*(g_B) + (f \circ \pi)\sigma^*(g_F)$ is a metric tensor, which would make $(B \times F, g)$ a semi-Riemannian manifold. To that end, we will check that $g_{(x,y)}$ is symmetric, nondegenerate, and has constant index $\forall\ (x, y) \in B \times F$. Symmetry and nondegeneracy are immediate using the exact same arguments as in Lemma 3.2. We can now combine orthonormal basis on B and an orthogonal basis on F such that $g_F(u, u) = 1/f(x)\ \forall\ u \in \{\text{basis}\}$ to form an orthonormal basis on $T_{(x,y)} B \times F$. In this basis, the matrix of $g_{(x,y)}$ is:

$$g_{(x,y)} = \operatorname{diag}(1, \overset{\dim B - \operatorname{ind} B}{\ldots}, 1, -1, \overset{\operatorname{ind} B}{\ldots}, -1, 1, \overset{\dim F - \operatorname{ind} F}{\ldots}, 1, -1, \overset{\operatorname{ind} F}{\ldots}, -1)$$

Thus, clearly g has constant index and $\operatorname{ind} B \times_f F = \operatorname{ind} B + \operatorname{ind} F$. □

We will call B the *base* of M, and F the *fiber*. In this chapter we will study the geometry of M in terms of B, F and the *warping function* f. It is easy to see that both the *fibers* $x \times F := \pi^{-1}(x)$ and the *leaves* $B \times y := \sigma^{-1}(y)$ are semi-Riemannian submanifolds of $B \times_f F$. In the same manner, we will call vector tangents to leaves *horizontal* and vectors tangents to fibers *vertical*.

We will denote by $\mathscr{H}$ the *horizontal* projection of $T_{(x,y)}(M)$ onto $T_{(x,y)}(B \times y)$, and by $\mathscr{V}$ the *vertical* projection onto $T_{(x,y)}(x \times F)$.

Remark 3.22.

1. For all $y \in F$, $\pi|_{(B \times y)}$ is an isometry onto B.

2. For all $x \in B$, $\sigma|_{(x \times F)}$ is a positive homothety onto F, with scale factor $1/f(x)$.

3. For all $(x, y) \in M$, the leaf $B \times y$ and the fiber $x \times F$ are orthogonal at (x, y).

Proposition 3.23. *Let $f \in C^\infty(B)$. Then, the gradient of the lift $f \circ \pi$ of f to M is the lift to M of the gradient of f.*

Proof. We need to show that $\text{grad}_g\ (f \circ \pi)$ is π-related to $\text{grad}_{g_B}\ f$. From the definition of the gradient, we know that

$$(\text{grad}_{g_B}\ f)_x = \sum_{j=1}^{n} \sum_{i=1}^{n} g^{ij} \left.\frac{\partial f}{\partial x_i}\right|_x \left.\frac{\partial}{\partial x_j}\right|_x$$

where $n = \dim\ B$ and g^{ij} is the element (i, j) of the inverse matrix G^{-1} of the metric tensor of M on x. Thus, given any $(x, y) \in B \times F$,

$$\begin{aligned}
\pi_{*,(x,y)} g \left(\text{grad}_g\ (f \circ \pi)\Big|_{(x,y)} \right) &= \sum_{j=1}^{n+m} \sum_{i=1}^{n+m} g^{ij} \left.\frac{\partial f \circ \pi}{\partial q_i}\right|_{(x,y)} \pi_{*,(x,y)} \left(\left.\frac{\partial}{\partial q_j}\right|_{(x,y)} \right) \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} g^{ij} \left.\frac{\partial f}{\partial x_i}\right|_x \left.\frac{\partial}{\partial x_j}\right|_x = \left(\text{grad}_g\ f\right)_{\pi(x,y)}
\end{aligned}$$

where $m = \dim\ F$, x_i are the coordinates of B and q_i are the product manifold coordinates, like those used in Proposition 3.11, such that if x_i and y_i are the coordinates of B and F, respectively, then $q_i = \{x_1 \circ \pi, \ldots, x_n \circ \pi, y_1 \circ \sigma, \ldots, y_m \circ \sigma\}$. ☐

The following proposition is proved in [13, Corollary 7.36, Page 207].

Proposition 3.24. *The leaves $B \times y$ of a warped product manifold are totally geodesic; the fibers $x \times F$ are totally umbilic.*

Proposition 3.25. *Given any vector fields $D_1, D_2, D \in \mathfrak{X}(B), V \in \mathfrak{X}(F)$, then*

$$g(\nabla_{\overline{D_1}} \overline{D_2}, \overline{V}) = 0 \tag{3.2}$$

$$g(\nabla_{\overline{D_1}} \overline{D_2}, \overline{D}) = g(\overline{\nabla^B_{D_1} D_2}, \overline{D}) \tag{3.3}$$

Proof. Bearing in mind that the Levi-Civita connection satisfies Koszul's formula (1.2), we can write:

$$\begin{aligned}
g(\nabla_{\overline{D_1}} \overline{D_2}, \overline{V}) = \frac{1}{2} \Big[&\overline{D_1} g(\overline{D_2}, \overline{V}) + \overline{D_2} g(\overline{V}, \overline{D_1}) - \overline{V} g(\overline{D_1}, \overline{D_2}) \\
&- g([\overline{D_2}, \overline{V}], \overline{D_1}) - g([\overline{D_1}, \overline{V}], \overline{D_2}) - g([\overline{D_2}, \overline{D_1}], \overline{V}) \Big]
\end{aligned}$$

As vectors tangent to leaves and fibers are orthogonal to each other, the first two terms cancel out. Using Lemmas 3.12 and 3.13, it can be shown that the last three terms are zero too. We now only have to show that $\overline{V} g(\overline{D_1}, \overline{D_2}) = 0$. We can do this by showing that $g(\overline{D_1}, \overline{D_2})$ is constant on the points $y \in F$:

$$g(\overline{D_1}, \overline{D_2})(x, y) = \pi^* g_B(\overline{D_1}, \overline{D_2})(x, y) + f(x)^2 \sigma^* g_F(\overline{D_1}, \overline{D_2})(x, y)$$

$$= g_B((D_1)_x, (D_2)_x) + f(x)^2 g_F(0_y, 0_y) = g_B((D_1)_x, (D_2)_x)$$

Hence $g(\nabla_{\overline{D_1}}\overline{D_2}, \overline{V}) = 0$. We continue with the second equation. We will, again, use Koszul's formula and Lemma 3.13:

$$\begin{aligned}
g(\overline{\nabla^B_{D_1} D_2}, \overline{D}) = g_B(\nabla^B_{D_1} D_2, D) &= \frac{1}{2}\left[D_1 g_B(D_2, D) + D_2 g_B(D, D_1) - D g_B(D_1, D_2)\right. \\
&\left. - g_B([D_2, D], D_1) - g_B([D_1, D], D_2) - g_B([D_2, D_1], D)\right] \\
&= \frac{1}{2}\Big[\overline{D_1} g(\overline{D_2}, \overline{D}) + \overline{D_2} g(\overline{D}, \overline{D_1}) - \overline{D} g(\overline{D_1}, \overline{D_2}) \\
&- g([\overline{D_2}, \overline{D}], \overline{D_1}) - g([\overline{D_1}, \overline{D}], \overline{D_2}) - g([\overline{D_2}, \overline{D_1}], \overline{D})\Big] \\
&= g(\nabla_{\overline{D_1}}\overline{D_2}, \overline{D})
\end{aligned}$$

□

Corollary 3.26. *For any vector fields $D_1, D_2 \in \mathfrak{X}(B)$, then*

$$\nabla_{\overline{D_1}}\overline{D_2} = \overline{\nabla^B_{D_1} D_2} \tag{3.4}$$

Proof. This equality is a direct consequence of the prior proposition, as it will be shortly seen. Let $Z \in \mathfrak{X}(B \times F)$ be any smooth vector field on M. We will prove that

$$g(\nabla_{\overline{D_1}}\overline{D_2}, Z) = g(\overline{\nabla^B_{D_1} D_2}, Z)$$

We can take the projections of this field Z onto B and F to compute the value of the product:

$$\begin{aligned}
g(\nabla_{\overline{D_1}}\overline{D_2}, Z) &= g_B(\pi_*\nabla_{\overline{D_1}}\overline{D_2}, \pi_* Z) + f^2 g_F(\sigma_*\nabla_{\overline{D_1}}\overline{D_2}, \sigma_* Z) = g_B(\nabla^B_{D_1} D_2, \pi_* Z) \\
&= g(\overline{\nabla^B_{D_1} D_2}, Z)\ \forall\ Z \in \mathfrak{X}(M) \implies \nabla_{\overline{D_1}}\overline{D_2} = \overline{\nabla^B_{D_1} D_2}
\end{aligned}$$

□

We can also prove a similar proposition to the above, but with the fibers:

Proposition 3.27. *Given any vector fields $D \in \mathfrak{X}(B), V_1, V_2, V \in \mathfrak{X}(F)$, then*

$$g(\nabla_{\overline{V_1}}\overline{V_2}, \overline{D}) = -g(\overline{V_2}, \nabla_{\overline{V_1}}\overline{D}) = -g(\overline{V_2}, \nabla_{\overline{D}}\overline{V_1}) \tag{3.5}$$

$$g(\nabla_{\overline{V_1}}\overline{V_2}, \overline{V}) = g(\overline{\nabla^F_{V_1} V_2}, \overline{V}) \tag{3.6}$$

Proof. We start the proof by showing the first expression. The first equality is immediate from the fact that ∇ is compatible with the metric g:

$$g(\nabla_{\overline{V_1}}\overline{V_2}, \overline{D}) = \overline{V_1} g(\overline{V_2}, \overline{D}) - g(\overline{V_2}, \nabla_{\overline{V_1}}\overline{D}) = -g(\overline{V_2}, \nabla_{\overline{V_1}}\overline{D})$$

Now, using Koszul's formula and Lemmas 3.12, 3.13,

$$g(\nabla_{\overline{V_1}}\overline{D}, \overline{V_2}) = \overline{D} g(\overline{V_1}, \overline{V_2}) = g(\nabla_{\overline{D}}\overline{V_1}, \overline{V_2})$$

The proof of the second equation is completely analogous to one done to show the second equality of Proposition 3.25. □

3.3 Killing fields on warped products

From now on, we will use $f = e^\theta$ to denote a positive function on B, and for any vector field D on the product manifold $B \times F$, we will denote the projections of D onto the leaves $B \times y$ and and fibers $x \times F$ as $D_B = (\pi_*(D), 0),\ D_F = (0, \sigma_*(D)) \implies D = D_B + D_F$. We already know that they are smooth thanks to Theorem 3.4. We will similary denote the pullback of any tensor field T on B or F onto $B \times F$ with the same symbol T if there is no likelihood of confusion. We will denote with $\mathcal{L}^B$ and $\mathcal{L}^F$ the Lie derivatives on $B \times y$ and $x \times F$, respectively.

Remark 3.28. In the following work, latin indices like a, b, c will run from 1 to n, while i, j, k will run from $n+1$ to $n+m$. Greek indices such as α, β, ν, μ will be used from 1 to $n+m$. We will also write $\mathcal{L}^B_{D_B} g_B := \mathcal{L}^B_{D_B|_{B\times y}} g_B$ (idem for F) in most cases for simplicity.

The following lemmas, which will be useful in this section, are taken from [15].

Lemma 3.29. *Let D be a vector field on $M = B \times_f F$. Then:*

$$\mathcal{L}_D g = \mathcal{L}_{D_B} g_B + D_B(f^2) g_F + f^2 \mathcal{L}_{D_F} g_F \tag{3.7}$$

where g is the warped product $g = \pi^*(g_B) + f^2\sigma^*(g_F)$, like before.

Proof. It is easy to first show that $\mathcal{L}_{D_F} g_B = \mathcal{L}_{D_B} g_F = 0$. We will prove the first equation. The proof for the second one is completely equivalent. Let $V, V' \in \mathfrak{X}(B \times_f F)$, then $\forall\ (x, y) \in B \times_f F$:

$$\begin{aligned}
\mathcal{L}_{D_F} g_B\ (V, V')_{(x,y)} &= (D_F)_{(x,y)}(g_B(V, V')) - (g_B)_{(x,y)}([D_F, V]_{(x,y)}, V'_{(x,y)}) \\
&- (g_B)_{(x,y)}(V_{(x,y)}, [D_F, V']_{(x,y)}) \\
&= i_{x\,*}\,\sigma_* D_{(x,y)}(g_B(\pi_* V, \pi_* V') \circ \pi) - (g_B)_x(\pi_*[D_F, V]_x, \pi_* V'_x) \\
&- (g_B)_x(\pi_* V_x, \pi_*[D_F, V]_x) \\
&= D_{(x,y)}(g_B(\pi_* V, \pi_* V') \circ \pi \circ i_x \circ \sigma) - (g_B)_x(0_x, \pi_* V'_x) \\
&- (g_B)_x(\pi_* V_x, 0_x) = 0
\end{aligned}$$

The lemma is immediate from this equalities. Using again the algebraic definition of the Lie derivative and the Proposition 1.37:

$$\begin{aligned}
\mathcal{L}_D g = \mathcal{L}_{D_B + D_F}\left(g_B + f^2 g_F\right) &= \mathcal{L}_{D_B} g_B + \mathcal{L}_{D_B + D_F} f^2 g_F \\
&= \mathcal{L}_{D_B} g_B + D_B(f^2) g_F + f^2 \mathcal{L}_{D_F} g_F
\end{aligned}$$

□

Remark 3.30. From now on we will not bother writing neither the points x, y in each vector field and/or metric tensor, nor the tangent maps of the inclusions, as the expressions get cluttered with notation and become difficult to read.

Lemma 3.31. *Locally, using coordinates (and the index notation convention stated before), for any smooth vector field $D \in \mathfrak{X}(B \times F)$:*

1. $\mathcal{L}_{D_B} g_B(\partial_\alpha, \partial_\beta) = \mathcal{L}^B_{D_B} g_B(\partial_a, \partial_b)$ *if* $\alpha, \beta = a, b$
2. $\mathcal{L}_{D_B} g_B(\partial_\alpha, \partial_\beta) = g_B(\partial_a, [\partial_i, D_B])$ *if* $\alpha = a,\ \beta = i$
3. $\mathcal{L}_{D_B} g_B(\partial_\alpha, \partial_\beta) = 0$ *if* $\alpha, \beta = i, j$

Similarly, for the fiber,

4. $\mathcal{L}_{D_F} g_F(\partial_\alpha, \partial_\beta) = \mathcal{L}^F_{D_F} g_F(\partial_i, \partial_j)$ *if* $\alpha, \beta = i, j$
5. $\mathcal{L}_{D_F} g_F(\partial_\alpha, \partial_\beta) = g_F(\partial_i, [\partial_a, D_F])$ *if* $\alpha = i,\ \beta = a$
6. $\mathcal{L}_{D_F} g_F(\partial_\alpha, \partial_\beta) = 0$ *if* $\alpha, \beta = a, b$

Proof. These equalities are easy to prove with the definition of the Lie derivative.

1. Let $\alpha, \beta = a, b \implies \pi_*\partial_\alpha = \partial_a,\ \pi_*\partial_\beta = \partial_b$:

$$\begin{aligned}\mathcal{L}_{D_B} g_B(\partial_\alpha, \partial_\beta) &= D_B g_B(\pi_*\partial_\alpha, \pi_*\partial_\beta) - g_B([D_B, \pi_*\partial_\alpha], \pi_*\partial_\beta) - g_B(\pi_*\partial_\alpha, [D_B, \pi_*\partial_\beta]) \\ &= D_B g_B(\partial_a, \partial_b) - g_B([D_B, \partial_a], \partial_b) - g_B(\partial_a, [D_B, \partial_b]) \\ &= \mathcal{L}^B_{D_B} g_B(\partial_a, \partial_b)\end{aligned}$$

2. Let $\alpha = a, \beta = i \implies \pi_*\partial_\alpha = \partial_a,\ \pi_*\partial_\beta = 0$:

$$\begin{aligned}\mathcal{L}_{D_B} g_B(\partial_\alpha, \partial_\beta) &= D_B g_B(\pi_*\partial_\alpha, \pi_*\partial_\beta) - g_B([D_B, \pi_*\partial_\alpha], \pi_*\partial_\beta) - g_B(\pi_*\partial_\alpha, [D_B, \pi_*\partial_\beta]) \\ &= -g_B(\partial_a, \pi_*[D_B, \partial_\beta])\end{aligned}$$

Let $f \in C^\infty(M)$. We compute the Lie bracket:

$$\begin{aligned}\pi_*[D_B, \partial_\beta] f &= D_B(\partial_\beta(f \circ \pi)) - \partial_\beta(D_B(f \circ \pi)) \\ &= -\partial_i(D_B f) = D_B(\partial_i(f)) - \partial_i(D_B f) \\ &= [D_B, \partial_i] f\end{aligned}$$

Therefore,

$$\mathcal{L}_{D_B} g_B(\partial_\alpha, \partial_\beta) = g_B(\partial_a, [\partial_i, D_B])$$

3. Let $\alpha, \beta = i, j \implies \pi_*\partial_\alpha = 0,\ \pi_*\partial_\beta = 0$:

$$\begin{aligned}\mathcal{L}_{D_B} g_B(\partial_\alpha, \partial_\beta) &= D_B g_B(\pi_*\partial_\alpha, \pi_*\partial_\beta) - g_B(\pi_*[D_B, \partial_\alpha], \pi_*\partial_\beta) - g_B(\pi_*\partial_\alpha, \pi_*[D_B, \partial_\beta]) \\ &= 0\end{aligned}$$

The proof for the second set of equations is exactly analogous to the one done for the base. □

Lemma 3.32. *Let D be any smooth vector field on $B \times F$. Then, $\forall\ D_1, D_2 \in \mathfrak{X}(M)$,*

$$\mathcal{L}_{D_B} g_B(D_1, D_2) = \mathcal{L}^B_{D_B} g_B(D_{1\,B}, D_{2\,B}) + g_B(D_1, [\sigma_* D_2, D_B]) + g_B([\sigma_* D_1, D_B], D_2) \quad (3.8)$$

$$\mathcal{L}_{D_F} g_F(D_1, D_2) = \mathcal{L}^F_{D_F} g_F(D_{1\,F}, D_{2\,F}) + g_F(D_1, [\pi_* D_2, D_F]) + g_F([\pi_* D_1, D_F], D_2) \quad (3.9)$$

Proof. They are both a direct consequence of the six equations proved in Lemma 3.31. □

With these lemmas done, we first consider a property of Killing vectors valid on an arbitrary submanifold. Then, we will apply it to warped products. This result is well known [13, Page 259, Exercise 7(a)].

Lemma 3.33. *If K is a Killing vector field on a semi-Riemannian manifold (M, g), which at all the points of the submanifold $S \subseteq M$, is tangent to S, then $K|_S$ is a Killing vector field on S.*

Proposition 3.34. *Given a Killing vector field K on $B \times_f F$, if K lies on either the leaves or the fibers, then its restriction to a leaf or fiber is a Killing vector field. Additionally, if it lies on a leaf, then $K(f) = 0$.*

Proof. We can prove that K will indeed be a Killing vector field straight from the prior lemma. For the second assertion, making use of Lemma 3.29,

$$0 = \mathcal{L}_K g = \mathcal{L}_K g_B + K(f^2) g_F + f^2 \mathcal{L}_0 g_F$$

Now, from Lemmas 3.31, 3.32, the first term vanishes (this also proves the first assertion). As $\sigma^* g_F$ is a non-degenerate metric, necessarily $K(f^2) = 0 \implies K(f) = 0$. □

Remark 3.35. If a Killing vector field K is not tangent to a submanifold S then K_S will not, in general, be a Killing vector field on S.

The following lemma is stated and proved in [15]. It is needed to prove a proposition for warped products.

Lemma 3.36.

1. *If K is a Killing vector field on (M, g) and $S \subseteq M$ is a totally geodesic submanifold, then K_S is a Killing vector field on S.*

2. *If C is a conformal Killing vector field on (M, g) and $S \subseteq M$ is a totally umbilic submanifold, then C_S is a conformal Killing vector field on S.*

We have, therefore, as a direct consequence of the prior lemma and Proposition 3.24:

Proposition 3.37. *Let K be a Killing vector field on $B \times_f F$. Then, K_B is a Killing vector field on each leaf $B \times y$, and K_F is a conformal Killing vector field on each fiber $x \times F$.*

As we stated in a prior remark, K_B will not be a Killing vector field on $B \times F$, in general, nor will it satisfy $K(f) = 0$.

Proposition 3.38. *Let $D \in \mathfrak{X}(B)$ and $V \in \mathfrak{X}(F)$ be smooth vector fields on B and F, respectively. Then,*

1. *$\overline{D}$ is Killing if and only if D is Killing on B and $D(f) = 0$.*

2. *$\overline{V}$ is Killing if and only if V is Killing on F.*

Proof.

- $\overline{D}$ is Killing $\implies$ D is Killing on B and $D(f) = 0$, and $\overline{V}$ is Killing $\implies$ V is Killing on F.

  They are both a direct corollary of Proposition 3.34.

- D is Killing on B and $D(f) = 0 \implies \overline{D}$ is Killing

  We can use Lemma 3.29 to write:

$$\mathcal{L}_{\overline{D}} g = \mathcal{L}_D g_B + D(f^2) g_F + f^2 \mathcal{L}_0 g_F = \mathcal{L}_D g_B$$

  Now, through Lemma 3.32:

$$\begin{aligned} \mathcal{L}_D g_B(D_1, D_2) &= \mathcal{L}^B_D g_B(D_{1\,B}, D_{2\,B}) + g_B(D_1, [\sigma_* D_2, D]) + g_B([\sigma_* D_1, D], D_2) \\ &= 0(D_{1\,B}, D_{2\,B}) + g_B(D_1, 0) + g_B(0, D_2) = 0 \implies \\ \implies \mathcal{L}_{\overline{D}} g = 0 & \end{aligned}$$

  Hence, $\overline{D}$ is a Killing vector field on B.

- V is Killing on $F \implies \overline{V}$ is Killing.

  Using Lemma 3.29, just as before,

$$\mathcal{L}_{\overline{V}} g = \mathcal{L}_0 g_B + 0(f^2) g_F + f^2 \mathcal{L}_V g_F = f^2 \mathcal{L}_V g_F S$$

  and we can similarly conclude with Lemma 3.32.

□

The following proposition (without the proof) is stated as a remark in [15].

Proposition 3.39. *The following statements hold:*

1. *Let $D \in \mathfrak{X}(B)$. Then, $\overline{D}$ is properly conformal Killing on $B \times_f F$ if and only if D is properly conformal Killing on B and $D(\theta) = h/2$, where h is the conformal factor:*

$$\mathcal{L}_{\overline{D}} g = hg$$

2. *Let $V \in \mathfrak{X}(F)$. Then, $\overline{V}$ is **not** properly conformal Killing on the warped product.*

Proof.

1. If D is properly conformal on B and $D(\theta) = h/2$, $0 \neq h \in C^\infty(B)$, it is immediate from Lemmas 3.29 and 3.31 with a few computations that $\overline{D}$ will be properly conformal:

$$\mathcal{L}_{\overline{D}} g(\partial_\alpha, \partial_\beta) = \mathcal{L}_D g_B(\partial_\alpha, \partial_\beta) + D(f^2) g_F(\partial_\alpha, \partial_\beta) + f^2 \mathcal{L}_0 g_F(\partial_\alpha, \partial_\beta)$$

$$= \begin{cases} (h \circ \pi) g_B(\partial_a, \partial_b) \text{ if } \alpha, \beta = a, b \\ 0 \text{ if } \alpha, \beta = a, i \\ f^2 (h \circ \pi) g_F(\partial_i, \partial_j) \text{ if } \alpha, \beta = i, j \end{cases}$$

Thus, for any $D_1, D_2 \in \mathfrak{X}(B \times F)$,

$$\mathcal{L}_{\overline{D}} g(D_1, D_2) = (h \circ \pi) \left(g_B(D_{1\,B}, D_{2\,B}) + f^2 g_F(D_{1\,F}, D_{2\,F}) \right) = (h \circ \pi) g(D_1, D_2) \implies$$
$$\implies \mathcal{L}_{\overline{D}} g = (h \circ \pi) g$$

Conversely, if we assume that $\mathcal{L}_{\overline{D}} g = hg$, $0 \neq h \in C^\infty(B \times F)$, doing the same computations as before, we deduce that, for any $D_1, D_2 \in \mathfrak{X}(B \times F)$:

$$\begin{cases} hg(\partial_\alpha, \partial_\beta) = \mathcal{L}^B_D g_B(\partial_a, \partial_b) \text{ if } \alpha, \beta = a, b \\ hg(\partial_\alpha, \partial_\beta) = 2 f^2 D(\theta) g_F(\partial_i, \partial_j) \text{ if } \alpha, \beta = i, j \end{cases} \implies$$
$$\implies \begin{cases} h|_{B \times y} g_B(\partial_a, \partial_b) = \mathcal{L}^B_D g_B(\partial_a, \partial_b) \implies h g_B = \mathcal{L}^B_D g_B \\ h f^2 g_F(\partial_i, \partial_j) = 2 f^2 D(\theta) g_F(\partial_i, \partial_j) \implies D(\theta) = h/2 \end{cases}$$

Additionally, this means that h must be constant on the fibers, so everything works as it should.

2. Suppose that $\overline{V}$ is properly conformal Killing on $B \times_f F$, i.e., $\mathcal{L}_{\overline{V}} g = hg$, $0 \neq h \in C^\infty(B \times F)$. Then, from Lemma 3.29,

$$hg = f^2 \mathcal{L}_V g_F$$

and now, using Lemma 3.31,

$$hg(\partial_a, \partial_b) = h g_B(\partial_a, \partial_b) = 0 \ \forall\ a, b$$

which is impossible, as $h \neq 0$ and g_B is non degenerate. Thus, $\overline{V}$ can't be properly conformal Killing on the warped product metric.

□

3.3.1 Killing fields in local coordinates

Note: In this subsection we will be doing a long computation involving tensors, so we will use Einstein's summation convention and index notation to shorten the expressions.

We will now try to compute a expression for the Killing vector fields of the warped product in coordinates. Therefore, we will work locally, in the open set where such a coordinate system is valid. Let $D \in \mathfrak{X}(M)$. Remember that $D = (D_B, D_F)$. Using Lemma 3.29,

$$\mathcal{L}_D g = \mathcal{L}_{D_B} g_B + D_B(f^2) g_F + f^2 \mathcal{L}_{D_F} g_F \tag{3.10}$$

In coordinates, we can write [6] g as

$$g = (g_B)_{ab}(x) dx^a \otimes dx^b + f(x)^2 (g_F)_{ij}(y) dy^i \otimes dy^j \tag{3.11}$$

and thus, using the properties of the Lie derivative (Propositions 1.35 and 1.37),

$$\begin{aligned}
\mathcal{L}_D g &= \mathcal{L}_{D_B}(g_{ab}(x)dx^a \otimes dx^b) + D_B(f(x)^2)(g_{ij}(y)dy^i \otimes dy^j) \\
&+ f(x)^2 \mathcal{L}_{D_F}(g_{ij}(y)dy^i \otimes dy^j) \\
&= D_B g_{ab}(x)dx^a \otimes dx^b + g_{ij}(x)\left(\mathcal{L}_{D_B}dx^a \otimes dx^b + dx^a \otimes \mathcal{L}_{D_B}dx^b\right) \\
&+ D_B(f(x)^2)(g_{ij}(y)dy^i \otimes dy^j) + f(x)^2\left[D_F(g_{ij}(y))dy^i \otimes dy^j\right. \\
&\left.+ g_{ij}(y)\left(\mathcal{L}_{D_F}dy^i \otimes dy^j + dy^i \otimes \mathcal{L}_{D_F}dy^j\right)\right] \\
&= D_B g_{ab}(x)dx^a \otimes dx^b + g_{ab}(x)\left[(\partial_c D_B(x^a)dx^c + \partial_k D_B(x^a)dy^k) \otimes dx^b\right. \\
&\left.+ dx^a \otimes (\partial_c D_B(x^b)dx^c + \partial_k D_B(x^b)dy^k)\right] \\
&+ D_B(f(x)^2)(g_{ij}(y)dy^i \otimes dy^j) + f(x)^2 D_F(g_{ij}(y))dy^i \otimes dy^j \\
&+ f(x)^2 g_{ij}(y)\left[(\partial_c D_F(y^i)dx^c + \partial_k D_F(y^i)dy^k) \otimes dy^j\right. \\
&\left.+ dy^i \otimes (\partial_c D_F(y^j)dx^c + \partial_k D_F(y^j)dy^k)\right] = 0
\end{aligned}$$

Therefore, we can write this equation as a system of 3 equations:

$$\begin{cases}
D_B g_{ab}(x)dx^a \otimes dx^b + g_{ab}(x)\left[(\partial_c D_B(x^a)dx^c \otimes dx^b + dx^a \otimes \partial_c D_B(x^b)dx^c\right] = 0 \\
D_B(f(x)^2)(g_{ij}(y)dy^i \otimes dy^j) + f(x)^2 D_F(g_{ij}(y))dy^i \otimes dy^j \\
+ f(x)^2 g_{ij}(y)\left[\partial_k D_F(y^i)dy^k \otimes dy^j + dy^k \otimes dy^i \otimes \partial_k D_F(y^i)dy^j\right] = 0 \\
g_{ab}(x)\left[\partial_k D_B(x^a)dy^k \otimes dx^b + dx^a \otimes \partial_k D_B(x^b)dy^k\right] \\
+ f(x)^2 g_{ij}(y)\left[\partial_c D_F(y^i)dx^c \otimes dy^j + dy^i \otimes \partial_c D_F(y^j)dx^c\right] = 0
\end{cases} \tag{3.12}$$

This system is equivalent to:

$$\begin{cases}
\mathcal{L}^B_{D_B} g_B = 0 \\
\mathcal{L}^F_{D_F} g_F = -\dfrac{D_B(f(x)^2)}{f(x)^2} g_F \\
g_{ab}(x)\left[\partial_k D_B(x^a)dy^k \otimes dx^b + dx^a \otimes \partial_k D_B(x^b)dy^k\right] \\
+ f(x)^2 g_{ij}(y)\left[\partial_c D_F(y^i)dx^c \otimes dy^j + dy^i \otimes \partial_c D_F(y^j)dx^c\right] = 0
\end{cases} \tag{3.13}$$

We will make use of this result in Chapter 4.

Chapter IV

Cosmological spacetimes

Modern cosmology derives its results from the so-called ”Cosmological Principle,” which is essentially the assumption that, at appropriate scales, the universe is homogeneous and isotropic. These hypotheses lie at the base of the Standard Cosmological Model, also known as the ΛCDM model. The mathematical framework for this model is provided by Robertson-Walker spacetimes, that are solutions to Einstein's field equations that assume homogeneity and isotropy on large scales.

However, as our observational capabilities have improved, it has become increasingly evident that there may not be sufficient empirical evidence to regard the Cosmological Principle as completely valid. For example, extremely large-scale structures, such as the Sloan Great Wall, suggest the presence of inhomogeneities on scales that might challenge the traditional assumptions. Additionally, certain surveys, such as the one done by the Wilkinson Microwave Anisotropy Probe, suggest that there are a great number of anisotropies in the Cosmic Microwave Background.

Given these potential discrepancies, one wonders about the possibility of other cosmological models that do not rely on the strict homogeneity and isotropy assumed by the Cosmological Principle. In this chapter we endeavour to explore one of these alternative models: Generalized Robertson-Walker spacetimes, which do not require the sectional curvature of the fiber (space) to be constant. Exploring these alternatives not only tests the robustness of the Standard model but also deepens our understanding of the fundamental properties of the universe and the validity of the assumptions that have guided cosmology for decades.

Definition 4.1 (Robertson-Walker spacetimes)**.** A *Robertson-Walker spacetime* (from now on shortened to RW spacetime) is a warped product spacetime (see Definition 3.20) where the base is an open interval I of $\mathbb{R}$, with its usual metric reversed, i.e. $g_B \equiv -dt^2$, the fiber is a three-dimensional connected Riemannian manifold (F, g_F) of constant sectional curvature $\kappa_F = -1, 0, 1$ and the warping function f is any positive function $f > 0$ on I.

Definition 4.2 (Generalized Robertson-Walker spacetimes)**.** A *generalized Robertson-Walker spacetime* (from now on shortened to GRW spacetime) is a warped product spacetime where the base is an open interval I of $\mathbb{R}$, with its usual metric reversed, the fiber is a m-dimensional connected Riemannian manifold (F, g_F) and the warping function f is any positive function $f > 0$ on I.

In summary, a GRW spacetime is a product manifold $I \times F$ endowed with the metric tensor:

$$g = \pi^*(-dt^2) + f^2(t)\sigma^*(g_F)$$

We will denote such spacetimes usually with $M = I \times_f F$, and its dimension with dim $M = n = m + 1$. It is also clear, then, that a GRW spacetime *generalizes* Robertson-Walker spacetimes: A RW spacetimes is a GRW spacetime that requires $m = 3$ and κ_F to be constant and equal to -1,0 or 1.

Remark 4.3. The isotropy condition given by the Cosmological Principle implies in Robertson-Walker spacetimes the requirement for the sectional curvature of the fiber to be constant. This can be shown using Schur's lemma (Theorem 1.63), such as in [13, Page 342, Proposition 6].

4.1 Killing vector fields on GRW spacetimes

Definition 4.4 (Trivial Killing vector fields)**.** We will call a Killing vector field on a GLRW spacetime M *trivial* if it comes from the base or the fiber through Proposition 3.38, or if it is a linear combination of such fields.

We can make use of the equations (3.13) to write the expressions in local coordinates of the non-trivial Killing vector fields of these spacetimes. First, we compute the Killing fields on the base, i.e. on the time interval $(I, -dt^2)$. Let $D \in \mathfrak{X}(I)$:

$$\mathcal{L}_D g_B = -\left(dD(t) \otimes dt + dt \otimes dD(t)\right) = -2\frac{dD(t)}{dt} dt \otimes dt = 0 \implies$$
$$\implies \frac{d}{dt}D(t) = 0 \implies D(t) is constant \tag{4.1}$$

Thus, the Killing vector fields on the base must be of the form

$$D(t) = a\left.\frac{\partial}{\partial t}\right|_t \tag{4.2}$$

where $a \in \mathbb{R}$. We can use this to simplify solving the aforementioned system of equations. We have esentially already solved the first equation, and we get that, given any $K \in \mathfrak{X}(M)$, for it to be Killing it must satisfy that, by the first equation of (3.13):

$$K_B = a(y)\frac{\partial}{\partial t} \tag{4.3}$$

Now, using the second equation:

$$\mathcal{L}_{K_F} g_F = -\frac{a(y)2f(t)\dot{f}(t)}{f(t)^2} = -2a(y)\frac{\dot{f}(t)}{f(t)}g_F = -2a(y)H(t)g_F \tag{4.4}$$

Remark 4.5. In physics, $H(t) \coloneqq \frac{\dot{f}(t)}{f(t)}$ is usually called the *Hubble parameter*, and the warping function f represents the *scale factor*.

And, lastly, we have the final equation:

$$\begin{aligned} &-\left[\frac{\partial a(y)}{\partial y^k}dy^k \otimes dt + dt \otimes \frac{\partial a(y)}{\partial y^k}dy^k\right] \\ &+ f(t)^2 g_{ij}(y)\left[\frac{\partial K_F(y^i)}{\partial t}dt \otimes dy^j + dy^i \otimes \frac{\partial K_F(y^j)}{\partial t}dt\right] = 0 \end{aligned} \quad \Longrightarrow \tag{4.5}$$

$$\Longrightarrow \begin{aligned} &-\left[da \otimes dt + dt \otimes da\right] \\ &+ f(t)^2 g_{ij}(y)\left[\frac{\partial K_F(y^i)}{\partial t}dt \otimes dy^j + dy^i \otimes \frac{\partial K_F(y^j)}{\partial t}dt\right] = 0 \end{aligned} \tag{4.6}$$

Define

$$\begin{aligned} \mathbf{T} \coloneqq &-\left[da \otimes dt + dt \otimes da\right] \\ &+ f(t)^2 g_{ij}(y)\left[\frac{\partial K_F(y^i)}{\partial t}dt \otimes dy^j + dy^i \otimes \frac{\partial K_F(y^j)}{\partial t}dt\right] \end{aligned} \tag{4.7}$$

We shall compute the value of $\mathbf{T}$ on the vectors of the basis. These equations give no new information, as they are identically zero:

$$\mathbf{T}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = 0 \tag{4.8}$$

$$\mathbf{T}\left(\frac{\partial}{\partial y^k}, \frac{\partial}{\partial y^k}\right) = 0 \ \forall \ k \tag{4.9}$$

We calculate the rest:

$$\mathbf{T}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial y^k}\right) = -\frac{\partial a(y)}{\partial y^k} + f(t)^2 g_{ij}(y)\frac{\partial K_F(y^i)}{\partial t}\delta^j_k = 0 \Longrightarrow$$

$$\begin{aligned} &\Longrightarrow \frac{\partial a}{\partial y^k} = f(t)^2 g_{ij}(y)\frac{\partial K_F(y^i)}{\partial t}\delta^j_k \Longrightarrow \\ &\Longrightarrow \frac{\partial a}{\partial y^k} = f(t)^2 g_{ik}(y)\frac{\partial K_F(y^i)}{\partial t} \end{aligned} \tag{4.10}$$

$$\mathbf{T}\left(\frac{\partial}{\partial y^k}, \frac{\partial}{\partial t}\right) = -\frac{\partial a(y)}{\partial y^k} + f(t)^2 g_{ij}(y)\frac{\partial K_F(y^j)}{\partial t}\delta^i_k = 0$$

$$\begin{aligned} &\Longrightarrow \frac{\partial a}{\partial y^k} = f(t)^2 g_{ij}(y)\frac{\partial K_F(y^j)}{\partial t}\delta^i_k \Longrightarrow \\ &\Longrightarrow \frac{\partial a}{\partial y^k} = f(t)^2 g_{kj}(y)\frac{\partial K_F(y^j)}{\partial t} \end{aligned} \tag{4.11}$$

As g is symmetric, both (4.10) and (4.11) are the same equation. We can try to write these equations in an intrinsic way, without coordinates:

$$\begin{cases} \text{grad}_F\ a = \dfrac{\partial a}{\partial y^k} g^{kl}(y) \dfrac{\partial}{\partial y^l} \\ \dfrac{\partial a}{\partial y^j} = f(t)^2 g_{ij}(y) \dfrac{\partial K_F(y^i)}{\partial t} \end{cases} \implies \text{grad}_F\ a = f(t)^2 g_{ik}(y) \frac{\partial K_F(y^i)}{\partial t} g^{kl}(y) \partial_l$$

$$\implies \text{grad}_F\ a = f(t)^2 \frac{\partial K_F(y^l)}{\partial t} \partial_l = f(t)^2 \frac{\partial K_F}{\partial t} \implies \frac{\partial K_F}{\partial t} = \frac{1}{f(t)^2} \text{grad}_F\ a, \tag{4.12}$$

where g^{nm} represents the (n, m) component of the inverse of the metric g. We can solve this equation to conclude that K_F must be of the form:

$$K_F = G(t)\ \text{grad}_F\ a + E(y) \tag{4.13}$$

where $G(t) \coloneqq \int_0^t 1/f(s)^2 ds$ and $E(y)$ is any vector field on the fibers dependent only on the points $y \in F$ and not on $t \in I$.

Now, if we take the equations (4.13) and (4.4), we get, using the properties of the Lie derivative:

$$G(t)\mathcal{L}_{(\text{grad}_F\ a)} g_F + \mathcal{L}_E g_F = -2a(y)H(t)g_F \tag{4.14}$$

We can take the Lie derivative with respect to ∂_t of both sides of this equation:

$$\dot{G}(t)\mathcal{L}_{(\text{grad}_F\ (a))} g_F = -2a(y)\dot{H}(t)g_F \tag{4.15}$$

Since as $\dot{G}(t) = 1/f(t)^2 =\neq 0\ \forall\ t$, we can divide by it and take again the Lie derivative with respect to ∂_t to get:

$$\frac{d}{dt}\left(\frac{\dot{H}(t)}{\dot{G}(t)}\right) = 0 \iff f\ddot{f} - \dot{f}^2 = C \tag{4.16}$$

where C is a constant. We shall bear in mind this equation, as we will use it extensively in the following section. To conclude, we will now find out the form of the field E. Thanks to equation (4.15), using (4.16), we can write:

$$\mathcal{L}_{(\text{grad}_F\ a)} g_F = -2Ca(y)g_F \tag{4.17}$$

We will now proceed by working through the possible different cases.

1. $C = 0 \implies \begin{cases} \dot{H}(t) = 0 \\ \mathcal{L}_E g_F = -2a(y)H(t)g_F \end{cases}$

   (a) $H(t) = 0 \implies \mathcal{L}_E g_F = 0 \implies E \in i(F, g_F)$ i.e. E is a Killing vector on F, and thus, a trivial Killing vector on M. Hence,

$$K_F = G(t)\ \text{grad}_F(a) + V,\ V \in i(F, g_F) \tag{4.18}$$

(b) $H(t) = H_0 \neq 0 \in \mathbb{R} \implies \mathcal{L}_E g_F = -2a(y)H_0 g_F$. If we take another solution to (4.12), we can write $\mathcal{L}_{E'} g_F = -2a(y)H_0 g_F$. Therefore, $E - E' \in i(F, g_F)$, and then,

$$K_F = G(t)\ \mathrm{grad}_F(a) + E + V \tag{4.19}$$

where E is a conformal Killing vector on F with factor $-2a(y)H_0$ and $V \in i(F, g_F)$ is a Killing vector field on F.

2. $C \neq 0$. We now have $\begin{cases} H(t) = CG(t) + H_0,\ H_0 \in \mathbb{R} \\ \mathcal{L}_{\mathrm{grad}_F\ (a)} g_F = -2Ca(y)g_F \end{cases}$

Therefore, using (4.14) and (4.17),

$$\begin{aligned} G(t)\mathcal{L}_{\mathrm{grad}_F\ (a)} g_F + \mathcal{L}_E g_F &= -2a(y)H(t)g_F \implies \\ G(t)(-2Ca(y)g_F + 2Ca(y)g_F) + \mathcal{L}_E g_F &= -2Aa(y)g_F \implies \\ \mathcal{L}_E g_F &= -2Aa(y)g_F \end{aligned} \tag{4.20}$$

and now, with (4.17) again, using the fact that: $C \neq 0$,

$$\mathcal{L}_{\frac{(\mathrm{grad}_F\ a)}{C}} g_F = -2a(y)g_F \implies \mathcal{L}_{A\frac{(\mathrm{grad}_F\ a)}{C}} g_F = -2Aa(y)g_F \tag{4.21}$$

We can conclude, with the two prior results, that:

$$\mathcal{L}_{E - A\frac{\mathrm{grad}_F\ (a)}{C}} = 0 \implies E - A\frac{(\mathrm{grad}_F\ (a))}{C} \in i(F, g_F) \tag{4.22}$$

And, consequently:

$$K_F = \frac{H(t)}{C}\mathrm{grad}_F\ (a) + V,\ V \in i(F, g_F) \tag{4.23}$$

Hence, if a vector field $K \in \mathfrak{X}(M)$ on a spacetime is Killing, it must satisfy that

$$\begin{cases} K_B = a(y)\dfrac{\partial}{\partial t} \\ \mathcal{L}_{\mathrm{grad}_F(a)} g_F = -2Ca(y)g_F \\ K_F \text{ must be of the forms (4.18), (4.19) and (4.23),} \\ \text{depending on the values of } C \text{ and } H(t). \end{cases} \tag{4.24}$$

We can summarize these results in the following theorem:

Theorem 4.6. *Let (M, g) be a GRW spacetime with warping function f. If M admits a non-trivial Killing vector field, then f satisfies the differential equation*

$$f\ddot{f} - \dot{f}^2 = C$$

for some constant $C \in \mathbb{R}$, and, additionally:

1. *If $C = 0$ and $H(t) = 0$,*

$$K = a(y)\frac{\partial}{\partial t} + \left(\int_{t_0}^{t} \frac{1}{f(s)^2} ds\right) \mathrm{grad}_F(a) + V \tag{4.25}$$

2. If $C = 0$ and $H(t) = H_0 \neq 0$,

$$K = a(y)\frac{\partial}{\partial t} + \left(\int_{t_0}^{t} \frac{1}{f(s)^2} ds\right) \mathrm{grad}_F(a) + E + V \tag{4.26}$$

3. If $C \neq 0$,

$$K = a(y)\frac{\partial}{\partial t} + \frac{H(t)}{C}\mathrm{grad}_F(a) + V \tag{4.27}$$

with V a Killing field on F, E a conformal Killing field on F with factor $-2a(y)H_0$ and $a \in C^\infty(F)$ satisfies the equation

$$\mathcal{L}_{grad_F(a)} g_F = -2Ca(y)g_F \tag{4.28}$$

or, equivalently [15],

$$\mathrm{Hess}_{g_F} a = -Ca(y)g_F \tag{4.29}$$

4.2 Einstein spacetimes and their curvature

We can now further expand on Theorem 4.6 with the following lemma:

Lemma 4.7. *Let f be a function without zeros. Consider the differential equations:*

$$f\ddot{f} - \dot{f}^2 = C \tag{4.30}$$

$$\frac{\ddot{f}}{f} = C' \tag{4.31}$$

If f satisfies (4.31)*, then it also satisfies* (4.30) *for an appropiate C, and viceversa for C'.*

Proof. Suppose that f satisfies (4.31). We will prove that the derivative of $f\ddot{f} - \dot{f}^2$ is zero:

$$\frac{d}{dt}\left(f\ddot{f} - \dot{f}^2\right) = \frac{d}{dt}\left(C'f^2 - \dot{f}^2\right) = 2C'f\dot{f} - 2\dot{f}\ddot{f} = 2C'f\dot{f} - 2C'f\dot{f} = 0$$
$$\implies f\ddot{f} - \dot{f}^2 = C$$

for some constant C. We will now suppose that f satisfies (4.30), and we will proceed in the same manner:

$$\frac{d}{dt}\left(\frac{\ddot{f}}{f}\right) = \frac{d}{dt}\left(\frac{C + \dot{f}^2}{f^2}\right) = \frac{2f^2\dot{f}\ddot{f} - (C + \dot{f}^2)(2f\dot{f})}{f^4}$$
$$= 2f\dot{f}\left(\frac{f\ddot{f} - (C + \dot{f}^2)}{f^4}\right) = 2f\dot{f}\,(C - C) = 0$$
$$\implies \frac{\ddot{f}}{f} = C'$$

for some constant C'. □

Using this result, we will show that any spacetime that has a non-trivial Killing field is an Einstein manifold (Definition 1.66) if and only if the space is Einstein with certain Ricci curvatures. We will then need the following equations for the Ricci curvature of a GLFRW spacetime (given $m > 1$), calculated with the help of [13, Page 211, Corollary 43].

$$\mathrm{Ric}(\overline{\partial_t}, \overline{\partial_t}) = -m\ddot{f}/f \tag{4.32}$$

$$\mathrm{Ric}(\overline{\partial_t}, \overline{D_1}) = \mathrm{Ric}(\overline{D_1}, \overline{\partial_t}) = 0 \tag{4.33}$$

$$\mathrm{Ric}(\overline{D_1}, \overline{D_2}) = \mathrm{Ric}_F(D_1, D_2) + (\ddot{f}f + (m-1)\dot{f}^2)g_F(D_1, D_2) \tag{4.34}$$

for all $D_1, D_2 \in \mathfrak{X}(F)$. We can now prove one of the main results of this chapter:

Theorem 4.8. *Let (M, g) be a GRW spacetime with $m = dim\ F > 1$. If M has a non-trivial Killing vector field, then M is Einstein with* $\mathrm{Ric} = mC' \cdot g$ *if and only if F is Einstein with* $\mathrm{Ric}_F = (m-1)C \cdot g_F$.

Proof. If M has a non-trivial Killing vector field, then the warping function f satisfies the equations (4.30) and (4.31), thanks to the prior Lemma 4.7. Let $D_1, D_2 \in \mathfrak{X}(F)$. Suppose that F is Einstein with $\mathrm{Ric}_F = (m-1)C \cdot g_F$. Using the three equations written above,

$$\begin{aligned}
\mathrm{Ric}(\overline{\partial_t}, \overline{\partial_t}) &= -m\ddot{f}/f = -mC' = mC'g(\overline{\partial_t}, \overline{\partial_t}) \\
\mathrm{Ric}(\overline{\partial_t}, \overline{D_1}) &= 0 = mC'g(\overline{\partial_t}, \overline{D_1}) \\
\mathrm{Ric}(\overline{D_1}, \overline{D_2}) &= \mathrm{Ric}_F(D_1, D_2) + (\ddot{f}f + (m-1)\dot{f}^2)g_F(D_1, D_2) \\
&= (m-1)Cg_F(D_1, D_2) + (\ddot{f}f + (m-1)(\ddot{f}f - C))g_F(D_1, D_2) \\
&= m\ddot{f}fg_F(D_1, D_2) = mC'f^2g_F(D_1, D_2) = mC'g(\overline{D_1}, \overline{D_2})
\end{aligned}$$

Now, as we previously proved that every vector field on M can be written as a sum of a smooth field on I and on F, we have shown that M is Einstein with proportionality factor mC'. Another simple computation shows the reciprocal statement: Let M be Einstein with $\mathrm{Ric} = mC' \cdot g$. Making use, again, of (4.34) we get:

$$\begin{aligned}
\mathrm{Ric}(\overline{D_1}, \overline{D_2}) &= mC'g(\overline{D_1}, \overline{D_2}) = mC'f^2g_F(D_1, D_2) \\
&= \mathrm{Ric}_F(D_1, D_2) + (\ddot{f}f + (m-1)\dot{f}^2)g_F(D_1, D_2) \\
\implies \mathrm{Ric}_F(D_1, D_2) &= (mC' - \ddot{f}f - (m-1)(\ddot{f}f - C))g_F(D_1, D_2) \\
\implies \mathrm{Ric}_F(D_1, D_2) &= (m-1)Cg_F(D_1, D_2)
\end{aligned}$$

□

We will now handle the case $m = 1$ separately.

Proposition 4.9. *Let (M, g) be a GRW spacetime with $m = 1$. Then, M is Einstein with $Ric = mC' \cdot g$ if and only if M has a non-trivial Killing vector field.*

Proof. Suppose that $m = 1$. From Remark 1.67, we know that M is Einstein with $n = 2$ if and only if the sectional curvature is constant with $\kappa = C'$. Hence, we will prove that the sectional curvature is constant if and only if M admits a non-trivial Killing vector field.

We now begin by computing the Christoffel symbols of the Levi-Civita connection. Since F is a connected one-dimensional manifold, then it is diffeomorphic either to the real line, or to the circle [12, Page 55, Theorem]. Therefore, there exists $D \in \mathfrak{X}(F)$ that is a basis of $\mathfrak{X}(F)$. Hence, $\mathfrak{X}(B) = \langle \partial_t \rangle$ and $\mathfrak{X}(F) = \langle D \rangle$. We now want to compute the sectional curvature, which is given by

$$\begin{aligned}\kappa &= \frac{g(R_{\partial_t,D}\partial_t, D)}{g(\partial_t,\partial_t)g(D,D) - g(\partial_t,D)^2} = \frac{g(R_{\partial_t,D}\partial_t, D)}{-f(t)^2 g_F(D,D)} = \frac{-g([\nabla_{\partial_t},\nabla_D] - \nabla_{[\partial_t,D]}(\partial_t), D)}{-f(t)^2 g_F(D,D)} \\ &= \frac{g(\nabla_{\partial_t}\nabla_D\partial_t - \nabla_D\nabla_{\partial_t}\partial_t, D)}{f(t)^2 g_F(D,D)} = \frac{g(\nabla_{\partial_t}(\nabla_D\partial_t), D)}{f(t)^2 g_F(D,D)} = \frac{g(\nabla_{\partial_t}(\nabla_{\partial_t}D), D)}{f(t)^2 g_F(D,D)}\end{aligned}$$

where we have used that ∂_t and D are orthogonal, that $[\partial_t, D] = 0$, and that $\nabla_{\partial_t} D = \nabla_D \partial_t$ since $\mathrm{Tor}_\nabla = 0$. Therefore, we need to compute $\nabla_{\partial_t} D$. By the Koszul formula, we have:

$$2g(\nabla_{\partial_t}D, \partial_t) = D(g(\partial_t,\partial_t)) = D(-1) = 0$$

In a similar way:

$$2g(\nabla_{\partial_t}D, D) = \partial_t(g(D,D)) = \partial_t(f(t)^2 g_F(D,D)) = 2f(t)\dot{f}(t)g_F(D,D)$$

Therefore, we get:

$$\nabla_{\partial_t}D = \frac{g(\nabla_{\partial_t}D, D)}{g(D,D)}D = \frac{f(t)\dot{f}(t)g_F(D,D)}{f(t)^2 g_F(D,D)}D = H(t)D$$

Hence:

$$\nabla_{\partial_t}(\nabla_{\partial_t}D) = \nabla_{\partial_t}(H(t)D) = \dot{H}(t)D + H(t)\nabla_{\partial_t}D = (\dot{H}(t) + H(t)^2)D$$

Thus, the sectional curvatures turns out to be:

$$\kappa = \frac{g(\nabla_{\partial_t}(\nabla_{\partial_t}D), D)}{f(t)^2 g_F(D,D)} = \dot{H}(t) + H(t)^2 = \frac{\ddot{f}}{f} = C'$$

And, using Lemma 4.7, we conclude the proof. □

Remark 4.10. An easily proven property of an Einstein manifold is that its scalar curvature will be constant.

Remark 4.11. Einstein manifolds are relevant because they imply that, for dim > 2, the metric is a solution to the Einstein field equations of a vacuum with cosmological constant, and viceversa:

$$\mathrm{Ric} - \frac{1}{2}Sg + \Lambda g = 0 \iff \mathrm{Ric} = \frac{2\Lambda}{\dim - 2}g \tag{4.35}$$

Furthermore, if we consider the specific case of our own universe, where $m = 3$, and the metric tensor satisfies the Einstein field equations, then Theorem 4.8 becomes even more relevant, as we will shortly see in the following corollary.

Corollary 4.12. *A GRW spacetime with three-dimensional fiber that satisfies the Einstein field equations for a vacuum with cosmological constant and admits a non-trivial Killing vector field, is necessarily RW spacetime.*

Proof. From the previous remark, we know that if we calculate the metric of our spacetime with the Einstein vacuum field equations with cosmological constant, then the spacetime will be Einstein. If one then finds a non-trivial symmetry of spacetime (i.e. no temporal translations or spatial symmetries that only exist in certain instants), then we can conclude, through Theorem 4.8, that F will be Einstein, and thanks to Remark 1.67, that the sectional curvature K_F will be constant. □

We can now find all possible warping functions by letting C and C' vary freely, using both differential equations which are, again:

$$\ddot{f}f - \dot{f}^2 = C$$

$$\frac{\ddot{f}}{f} = C'$$

The second equation has solutions of the form:

$$f(t) = A_1 e^{\sqrt{C'}} + A_2 e^{-\sqrt{C'}} \quad A_1, A_2 \in \mathbb{R} \tag{4.36}$$

Substituting this function in the first equation we find that the constants A_1 and A_2 must satisfy the relation:

$$C = 4A_1A_2C' \tag{4.37}$$

We can proceed manipulating these constants in the warping function f given before using this equation. Recall that we require $f(t) > 0$. The results are now presented in the following theorem:

Theorem 4.13. *Let M be a GRW spacetime. If M has a non-trivial Killing vector field, then the warping function f must be one of the six possibilities in Table 4.1.*

1	$C' > 0$	$C > 0$	$f(t) = \sqrt{\frac{C}{C'}}\cosh(\sqrt{C'}t + A)$
2	$C' > 0$	$C = 0$	$f(t) = e^{\sqrt{C'}t+A}$
3	$C' > 0$	$C < 0$	$f(t) = \sqrt{-\frac{C}{C'}}\sinh(\sqrt{C'}t + A)$
	$C' = 0$	$C > 0$	$C \leq 0 \implies \nexists f$
4	$C' = 0$	$C = 0$	$f(t) = e^A$
5	$C' = 0$	$C < 0$	$f(t) = \sqrt{-C}t + A$
	$C' < 0$	$C > 0$	$f \in \mathbb{C} \implies \nexists f$
	$C' < 0$	$C = 0$	$f \in \mathbb{C} \implies \nexists f$
6	$C' < 0$	$C < 0$	$f(t) = \sqrt{\frac{C}{C'}}\cos(\sqrt{C'}t + A)$

Table 4.1: Possible warping functions for a GRW spacetime with dim $F > 1$. $A \in \mathbb{R}$ constant.

We now finish this work by using Theorems 4.6 and 4.13 to show several interesting remarks.

Remark 4.14. As we said in Remark 4.5, the warping function f represents, in cosmology, the *scale factor* of the universe. As such, this theorem is very potent: Given can find any one non-trivial symmetry in a cosmological model, then the scale factor must be on of the six possibilities given in the table.

Remark 4.15. The constants C and C' are closely related to the sectional and scalar curvature, as it can be easily shown that, if M is Einstein with $\lambda = mC'$ and has constant sectional curvature,

$$S = n(n-1)C' \tag{4.38}$$

$$\kappa = C' \tag{4.39}$$

likewise with F,

$$S_F = m(m-1)C \tag{4.40}$$

$$\kappa_F = C \tag{4.41}$$

We can now simply reintroduce each warping function in the differential equations (4.30) and (4.31) to find the value of the constants C and C' where possible, and using the above equations, find the scalar and sectional curvature of each spacetime.

Corollary 4.16. *Let (M, g) be a GRW spacetime. Suppose that we are in the conditions of Theorem 4.6. Then, the only possible scalar and sectional curvatures of M for each warping function in table 4.1 are:*

1. $C' = 1 \implies S = n(n-1);\ \kappa = 1$
2. $C' = 1 \implies S = n(n-1);\ \kappa = 1$
3. $C' = 1 \implies S = n(n-1);\ \kappa = 1$
4. $C' = 0 \implies S = 0;\ \kappa = 0$
5. $C' = 0 \implies S = 0;\ \kappa = 0$
6. $C' = -1 \implies S = -n(n-1);\ \kappa = -1$

Remark 4.17. Note that for $n \geq 4$, if M is Einstein then, in general, the sectional curvature will not be constant. The results for K in the above corollary are only applicable when K is constant.

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