

## The influence of super-horizon scales on cosmological observables generated during inflation

This content has been downloaded from IOPscience. Please scroll down to see the full text.

JCAP05(2004)008

(<http://iopscience.iop.org/1475-7516/2004/05/008>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 130.183.84.31

This content was downloaded on 18/11/2015 at 10:18

Please note that [terms and conditions apply](#).

# The influence of super-horizon scales on cosmological observables generated during inflation

Sabino Matarrese<sup>1</sup>, Marcello A Musso<sup>2</sup> and Antonio Riotto<sup>3</sup>

<sup>1</sup> Dipartimento di Fisica ‘G Galilei’, Università di Padova and INFN—Sezione di Padova, via F Marzolo 8, I-35131 Padova, Italy

<sup>2</sup> Dipartimento di Fisica Nucleare e Teorica, Università di Pavia and INFN—Sezione di Pavia, via U Bassi 6, I-27100, Pavia, Italy

<sup>3</sup> INFN—Sezione di Padova, via F Marzolo 8, I-35131 Padova, Italy  
E-mail: [matarrese@pd.infn.it](mailto:matarrese@pd.infn.it), [marcello.musso@pv.infn.it](mailto:marcello.musso@pv.infn.it) and [antonio.riotto@pd.infn.it](mailto:antonio.riotto@pd.infn.it)

Received 16 January 2004

Accepted 11 May 2004

Published 26 May 2004

Online at [stacks.iop.org/JCAP/2004/i=05/a=008](http://stacks.iop.org/JCAP/2004/i=05/a=008)

DOI: 10.1088/1475-7516/2004/05/008

**Abstract.** Using the techniques of out-of-equilibrium field theory, we study the influence on properties of cosmological perturbations generated during inflation on observable scales coming from fluctuations corresponding today to scales much bigger than the present Hubble radius. We write the effective action for the coarse grained inflaton perturbations, integrating out the sub-horizon modes, which manifest themselves as a coloured noise and lead to memory effects. Using the simple model of a scalar field with cubic self-interactions evolving in a fixed de Sitter background, we evaluate the two- and three-point correlation function on observable scales. Our basic procedure shows that perturbations do preserve some memory of the super-horizon scale dynamics, in the form of scale dependent imprints in the statistical moments. In particular, we find a *blue* tilt of the power spectrum on large scales, in agreement with the recent results of the WMAP collaboration which show a suppression of the lower multipoles in the cosmic microwave background anisotropies, and a substantial enhancement of the intrinsic non-Gaussianity on large scales.

**Keywords:** CMBR theory, cosmological perturbation theory, inflation

---

**Contents**

<b>1. Introduction</b>	<b>2</b>
<b>2. Effective super-horizon action</b>	<b>4</b>
<b>3. Spectra and two-point correlation functions</b>	<b>7</b>
<b>4. The interacting scalar field</b>	<b>10</b>
<b>5. Conditional probability distribution and moments</b>	<b>12</b>
<b>6. Discussion and conclusions</b>	<b>20</b>
<b>Appendix A: stochastic noise and the Langevin equation</b>	<b>21</b>
<b>Appendix B: dissipation</b>	<b>22</b>
<b>Appendix C: the noise correlation matrix</b>	<b>24</b>
<b>Appendix D: conditional probability</b>	<b>25</b>
<b>Appendix E: <math>D_{ijk}</math> coefficients for the Gaussian window</b>	<b>27</b>
<b>References</b>	<b>27</b>

---

**1. Introduction**

Stochastic inflation provides an efficient approach for studying inflationary dynamics and has become a very popular way to describe the growth of density perturbations on scales larger than the Hubble radius. In the first fundamental works [1]–[7], the inflaton field was split into a super-horizon and a sub-horizon part directly in the equation of motion. This splitting is operated in Fourier space through a window function, that separates high from low frequencies. The relevant variable is the long wavelength part, while the sub-horizon modes are collected in an effective noise term, playing the role of a classical perturbation to the super-horizon dynamics.

The resulting effective equation of motion is then a Langevin-like equation analogous to the one describing *Brownian motion*, where the deterministic evolution is influenced and modified by the stochasticity of the source, whose effects can be taken into account only as a statistical average over time. Indeed, in this formalism there is no knowledge of the exact form of the noise, only of its statistical properties.

A more general approach exploits the influence functional method [8, 9], and operates the frequency splitting at the action level getting rid of the high frequencies via a path integral over the sub-horizon part of the field. The effective action thus obtained contains some extra terms that can be interpreted as the coupling of the super-horizon field with a classical random noise source, whose configurations are statistically weighted by an appropriate functional probability distribution, becoming the origin of the stochastic character of the Langevin-like equation of motion.

The super-horizon degrees of freedom are then treated as a purely classical field, all the quantum fluctuations being collected in the classical noise term. This feature

is claimed not to be a simple computational trick, but an intrinsic characteristic of the system. Stochasticity is thus not only a clue to understanding the properties of inflation and the origin of the large scale structure in the universe, but also a way to explain the transition from a quantum to a classical world [10]. From a formal point of view, the quantum decoherence process in the stochastic inflation framework has been discussed in various works [11]–[15], where it was pointed out that the classicality of the coarse grained field (implicitly assumed in the first papers) is not necessarily assured, but is subject to some restrictive conditions.

Using standard techniques of stochastic processes [16, 17], the Langevin equation for the field expectation value leads to an evolution equation (the Fokker–Planck equation) for its probability distribution function. In the first works, the noise correlation time was assumed to be infinitesimally short, and the correlation function for different times can therefore be considered as being proportional to Dirac’s delta function  $\delta(t - t')$ , which sets its *white noise* properties. This assumption allows one to apply a well known formalism for the derivation of the Fokker–Planck equation and its solution. However, the characteristic of the correlation function strongly depends on the window function, whose choice is not a mere mathematical tool, but has several physical effects [18]. A white noise arises only as a consequence of a sharp momentum cut-off, whereas a smooth weighting avoids highly singular noise correlators and produces a *coloured noise*.

The choice of a coloured noise is interesting for at least two reasons. The first is that a sharp momentum-space cut-off seems rather unphysical, while a smooth weighting of the modes is much more likely. Actually, the most natural way to integrate out the small scale fluctuations is to average the field in configuration space, choosing an appropriate finite volume window function. In most cases, this choice results in a smooth weighting in Fourier space (thus producing a coloured noise), while the sharp momentum-space cut-off corresponds to a rather complicated infinite volume window function in configuration space. Moreover, it is possible to single out a wide class of window functions for which the shape of the coloured noise correlation is asymptotically the same [19]. A second reason may be the fact that a coloured noise could play, during inflation, an important role in producing intrinsically non-Gaussian density fluctuations as initial conditions for the subsequent evolution of the large scale structure of the universe [20].

In the simplest single-field slow-roll models of inflation, non-Gaussian features in scalar perturbations are produced by either the self-interaction of the inflaton field [21], which are however constrained to be very small by the slow-roll conditions, or by the back-reaction of field fluctuations on the background metric, whose amplitude is also strongly constrained by the slow-roll conditions [22]–[25]. It has been shown, however, that the most copious source of large scale non-Gaussianity is stored in the post-inflationary second-order evolution of perturbations, which sets in a universal level of non-Gaussianity for the gauge-invariant gravitational potential, which turns out to be of order unity [26, 27].

In this paper we point out that there is another source of intrinsic, and generally scale dependent non-Gaussianity in the fluctuation pattern, which originates from the cross-talk between super- and sub-horizon scale perturbation modes. On scales much larger than the Hubble radius, non-Gaussian features generally arise as a consequence of the non-linear multiplicative form of the Langevin equation, when back-reaction effects are accounted for [29, 28, 30]–[32]. However, as discussed in [29]–[35], these effects are not directly reflected in the statistical properties of cosmological perturbations on sub-horizon

scales. Indeed, in order to deal with fluctuations relative to our patch of the universe, one cannot simply perform statistical averages over the entire ensemble of possible states; rather one should allow for the observed smoothness of our universe on large scales. A possible, though approximate, way to take this constraint into account is to replace ensemble averages with averages over the conditional probability density functional so that fluctuations on sub-horizon scales assume a certain value, *given that* the inflaton field is spatially homogeneous at  $t = t_{60}$ , i.e. about 60 e-folds before the end of inflation (corresponding to a comoving scale slightly larger than the present Hubble radius). This is equivalent to setting, for the probability distribution of the fluctuations, the ‘initial’ condition  $P(\delta\varphi, t_{60}) = \delta(\delta\varphi - \delta\varphi_{60})$  [29]–[35, 22, 23]. Although this may happen (and in most models it does) well after the beginning of the accelerated expansion, if the noise driving the fluctuations is white their evolution is Markovian, implying that any notion of the previous history is erased. The probability distribution then behaves exactly as if inflation had started at that time and the level of the inflaton non-Gaussianity remains fully negligible. In contrast, a coloured noise has a non-vanishing correlation time: because of this fact the inflaton keeps a memory of what happened before the constraint, and its evolution ceases to be a Markov process. In this scenario, the probability distribution evolves in a different way, and also higher moments become important.

Since the solution of the Fokker–Planck equation with coloured noise carries several complications and is still a partly unknown matter, in this paper we followed a different approach, trying to perturbatively determine the probability distribution for the inflaton field directly solving the Langevin equation in a statistical way.

The plan of the paper is as follows. In section 2 we briefly describe the derivation of the stochastic equation of motion for the inflaton field averaged over super-horizon scales, using the influence functional method, and evaluate the dependence of the noise on the choice of window function. In section 3 we then choose a specific Gaussian shaped filter, obtaining the related (coloured) noise correlation functions, and the variance and power spectrum of the coarse grained fluctuation field. In section 4, after introducing a small non-linear (cubic) term in the potential, we evaluate the bispectrum and the third moment of the field. In section 5 we investigate the memory effects induced by this coloured noise and build up a formalism to quantitatively determine the relevance of the non-Gaussian features of the distribution and their sensitivity to the times before  $t_{60}$ . Finally, in section 6 we draw our conclusions. Some technical aspects of our calculation are reported in five appendices.

## 2. Effective super-horizon action

We consider a background de Sitter space–time, whose metric reads

$$ds^2 = dt^2 - a^2(t) d\mathbf{x}^2 = a^2(\eta)(d\eta^2 - d\mathbf{x}^2), \quad (2.1)$$

where  $a(t) = e^{Ht}$  is the scale factor (the Hubble parameter  $H \equiv \dot{a}/a$  is constant in time) and  $\eta$  is the conformal time defined by  $d\eta = dt/a$ , i.e.  $\eta = -[Ha(t)]^{-1}$ .

The action for the inflaton field is the ordinary action in curved space–time for a free scalar field with mass  $m$  is

$$S[\phi] = \int d^4x \sqrt{-g} \frac{1}{2} [g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2] \quad (2.2)$$

(where Greek letters label space–time indices), that with our choice of background de Sitter metric becomes

$$S[\phi] = \int d^4x a^3 \frac{1}{2} \left[ (\partial_t \phi)^2 - \frac{(\nabla \phi)^2}{a^2} - m^2 \phi^2 \right]. \quad (2.3)$$

The equation of motion for such a field is

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2 \phi}{a^2} + m^2 \phi = 0 \quad (2.4)$$

and the standard solution for the inflaton Fourier modes reads

$$\phi_{\mathbf{k}}(x) = \frac{H}{(2\pi)^{3/2}} \frac{\sqrt{\pi}}{2} \left( \frac{1}{aH} \right)^{3/2} H_{\nu}^{(1)} \left( \frac{k}{aH} \right) e^{i\mathbf{k}\cdot\mathbf{x}}, \quad (2.5)$$

where  $\nu^2 = \frac{9}{4} - \frac{m^2}{H^2}$  and  $H_{\nu}^{(1)}(x)$  are Hankel functions of the first kind. In the special case of a massless field,  $\nu = 3/2$  and

$$\phi_{\mathbf{k}}(x) = \frac{H}{(2\pi)^{3/2}} \frac{1}{\sqrt{2k}} \left( \eta - \frac{i}{k} \right) e^{i(-k\eta + \mathbf{k}\cdot\mathbf{x})}. \quad (2.6)$$

Let us now split the field  $\phi$  in two components, dividing the short wavelength normal modes (with wavelength  $a/k$  smaller than the horizon scale  $H^{-1}$ ) from the long wavelength ones. The short wavelength part is defined as

$$\phi_{>} = \int d\mathbf{k} W(|\mathbf{k}|, t) [\phi_{\mathbf{k}}(x) a_{\mathbf{k}} + \phi_{\mathbf{k}}^*(x) a_{\mathbf{k}}^{\dagger}], \quad (2.7)$$

where the window function  $W(k, t)$  projects out the long wavelength modes. The long wavelength part is then simply  $\phi_{<} = \phi - \phi_{>}$ . Substituting this field decomposition into the action (2.3) we obtain two distinct free actions for the two fields plus an interaction term:

$$S[\phi_{<}, \phi_{>}] = S[\phi_{<}] + S[\phi_{>}] + S_{\text{int}}[\phi_{<}, \phi_{>}]. \quad (2.8)$$

In order to obtain real *in-in* vacuum expectation values for the field  $\phi$  instead of the usual *in-out* transition amplitudes, the standard procedure is to work in the *closed time path* (CTP) formalism [36] of out-of-equilibrium field theory. Indeed, in ordinary quantum field theory, one deals with transition amplitudes in particle reactions and one may not study the dynamics of the system. This is because one needs the temporal evolution with definite initial conditions and not simply the transition amplitudes of particle reactions with fixed initial and final conditions. While ordinary quantum field theory yields quantum averages of operators evaluated with an *in* state and an *out* state, the CTP formalism yields quantum averages of operators evaluated in the *in* state without specifying the *out* state. In the CTP formalism [37] the time integration is made along a closed path going from the initial time to positive infinity and back to the initial time; the path integral on the field configurations is evaluated on this closed path, along which they need not assume the same values on the forward and backward branches of the time contour. This is equivalent to considering two fields  $\phi^+$  and  $\phi^-$ , with the constraint  $\phi^+(+\infty) = \phi^- (+\infty)$ , with the ordinary single-time integration on the real axis. These two fields have to be set equal to each other in the equation of motion. We thus end up with four different fields  $\phi_{>}^+$ ,  $\phi_{>}^-$ ,  $\phi_{<}^+$ ,  $\phi_{<}^-$  for small and large scales, and forward and backward times.

The effective equation of motion for the *in-in* expectation value of the super-horizon fields  $\phi_{<}^{\pm} \equiv \varphi^{\pm}$  can be derived by integrating the action over the sub-horizon field variable  $\phi_{>}$ . The super-horizon effective action obtained in this way contains two extra terms in addition to the ordinary action (2.3), describing physical effects related to the horizon crossing of the various normal modes [8]. One of them contains both dissipation and non-local mass renormalization effects, while the other describes the influence on the super-horizon modes of the sub-horizon ones, whose quantum fluctuation can be treated as a stochastic noise. This second term is purely imaginary, and as such it cannot be interpreted as a standard action: indeed, it appears as the result of a statistical weighting over the configurations of the stochastic noise fields representing sub-horizon quantum fluctuations, which couple to  $\varphi$  in the effective action  $S_{\text{eff}}[\varphi^{\pm}]$ .

After some manipulations (see appendix A for details), introducing two real classical fields  $\xi_1$  e  $\xi_2$ , whose configurations are statistically weighted by the probability distribution functional  $\mathcal{P}[\xi_1(x), \xi_2(x)]$ , it is actually possible to write

$$e^{i\Gamma[\varphi^{\pm}]} \equiv \int \mathcal{D}\phi_{>}^{\pm} e^{i(S[\phi_{<}^{\pm}, \phi_{>}^{\pm}] - S[\phi_{<}^{\pm}, \phi_{>}^{-}])} = \langle e^{iS_{\text{eff}}[\varphi^{\pm}]} \rangle_S = \int \mathcal{D}\xi_1 \mathcal{D}\xi_2 \mathcal{P}[\xi_1, \xi_2] e^{iS_{\text{eff}}[\varphi^{\pm}]}, \quad (2.9)$$

where the effective action  $S_{\text{eff}}[\varphi^{\pm}]$  is (introducing  $\varphi^{\Delta} = \varphi^{+} - \varphi^{-}$ )

$$S_{\text{eff}} = S[\varphi^{+}] - S[\varphi^{-}] + \int d^4x a_i^3 H \left[ \left( \left( \frac{3}{2} - \nu \right) \varphi^{\Delta}(x) + \frac{\dot{\varphi}^{\Delta}(x)}{H} \right) \xi_1(x) + \varphi^{\Delta}(x) \xi_2(x) \right]. \quad (2.10)$$

The quantum noise on small scales acts then as a classical random source. Here,  $S$  is simply the free action (2.3), since the extra dissipation and mass renormalization terms are small and can be neglected in a first approach, as shown in appendix B.

The statistical weight of the two random fields  $\xi_1$  and  $\xi_2$  is Gaussian,

$$P[\xi_1, \xi_2] = \exp \left\{ -\frac{H^2}{2} \int d^4x d^4x' [\xi_1(x), \xi_2(x)] \mathbf{A}^{-1}(x, x') \begin{bmatrix} \xi_1(x') \\ \xi_2(x') \end{bmatrix} \right\}, \quad (2.11)$$

where  $\mathbf{A}^{-1}(x, x')$  is the functional inverse of the symmetric (under simultaneous exchange of discrete and continuous indices  $i, t \rightarrow j, t'$ ) kernel

$$\mathbf{A}(x, x') = \frac{H^6}{8\pi} \int \frac{dk}{k} \frac{\sin kr}{kr} k^5 (\eta\eta')^{5/2} W'(k\eta) W'(k\eta') \text{Re}[\mathbf{M}_{\nu}(k\eta, k\eta')], \quad (2.12)$$

with

$$\mathbf{M}_{\nu}(k\eta, k\eta') \equiv \begin{bmatrix} H_{\nu}^{(1)}(k|\eta|) H_{\nu}^{(1)*}(k|\eta'|) & k|\eta'| H_{\nu}^{(1)}(k|\eta|) H_{\nu-1}^{(1)*}(k|\eta'|) \\ k|\eta| H_{\nu-1}^{(1)}(k|\eta|) H_{\nu}^{(1)*}(k|\eta'|) & k^2 |\eta| |\eta'| H_{\nu-1}^{(1)}(k|\eta|) H_{\nu-1}^{(1)*}(k|\eta'|) \end{bmatrix}. \quad (2.13)$$

In the procedure we described, quantum fluctuations of the sub-horizon modes are collected via the path integral in a rapidly varying classical noise term coupled to the super-horizon part of the scalar field: these fluctuations can thus talk to the super-horizon modes and perturb the dynamics on scales larger than the Hubble radius during inflation.

The coupling of the two random noises  $\xi_1$  and  $\xi_2$  with the scalar field is slightly different from that in [8]; the reason is that the choice of a general window function  $W(k\eta)$  (which is not necessarily able to produce a sharp cut in the frequencies, but can have some spread around the horizon scale) can introduce a  $k$  dependence in the effective

field coupling to the noise, thereby spoiling the separation between super- and sub-horizon scales. Our different choice of basis avoids this problem. Another consequence of this formulation is that the correlation functions of the two noise fields are different from each other, and cross-correlations appear, in a similar way to in [18].

The effective equation of motion obtained from this action with the usual CTP method is then

$$0 = \left. \frac{\delta S_{\text{eff}}}{\delta \phi^\Delta} \right|_{\phi^\Delta=0} = \ddot{\varphi} + 3H\dot{\varphi} - \frac{\nabla^2 \varphi}{a^2} + m^2 \varphi - 3H\xi_1 - \dot{\xi}_1 + H\xi_2, \quad (2.14)$$

that is a Langevin-like stochastic equation, where the dynamics of the field  $\varphi$  is subjected to random ‘kicks’ given by the rapidly varying stochastic force  $\xi$ . We treat the effect of the random force as a perturbation of the classical dynamics and split the field  $\varphi$  into its mean, obeying the classical equation of motion (2.4), plus a fluctuation  $\delta\varphi$  that by definition vanishes when averaged over all noise configurations. In the massless case, the equation for the fluctuations becomes

$$\delta\ddot{\varphi} + 3H\delta\dot{\varphi} - \frac{\nabla^2 \delta\varphi}{a^2} = 3H\xi_1 + \dot{\xi}_1 - H\xi_2; \quad (2.15)$$

if we neglect the exponentially suppressed spatial gradients, the second time derivative  $\delta\ddot{\varphi}$  (assuming the validity of the slow-roll conditions) and the  $\dot{\xi}_1$  term, we finally get

$$\delta\dot{\varphi} = \xi_1 - \frac{\xi_2}{3} \equiv \xi. \quad (2.16)$$

Even in this simple form, a deterministic treatment of this equation is impossible, since we do not know the exact form of  $\xi_1$  and  $\xi_2$ . However, we can study this equation from a stochastic point of view [16, 17], in order to understand how the statistical properties of the Gaussian noise (that are completely characterized by the two-point correlation functions  $A_{ij}(x, x')$ ) determine the behaviour of  $\delta\varphi[\xi]$ , now treated as a stochastic variable itself. In other words, our goal will not be the exact determination of the evolution of the field  $\delta\varphi$ , but that of its probability distribution functional.

### 3. Spectra and two-point correlation functions

From the equation of motion (2.16), one can immediately compute the two-point correlation function for  $\delta\varphi$ , which reads

$$\langle \delta\varphi(x)\delta\varphi(x') \rangle = \int_{t_{60}}^t d\tilde{t} \int_{t_{60}}^{t'} d\tilde{t}' \langle \xi(\tilde{t}, \mathbf{x})\xi(\tilde{t}', \mathbf{x}') \rangle, \quad (3.1)$$

where the correlation function of the noise  $\xi$  is given by

$$\langle \xi(x)\xi(x') \rangle = \frac{1}{H^2} \left[ A_{11}(x, x') - \frac{A_{12}(x, x')}{3} - \frac{A_{21}(x, x')}{3} + \frac{A_{22}(x, x')}{9} \right], \quad (3.2)$$

and the exact form of the matrix elements  $A_{ij}$  depends through (2.12) on the choice of the window function  $W(k\eta)$ .

It is interesting to note that if we project the modes using the Heaviside step function  $W(k\eta) = \vartheta((k/aH) - \varepsilon)$ , in such a way that  $W'(k\eta) = \delta((k/aH) - \varepsilon) = aH\delta(k - \varepsilon aH)$ , the

noise correlator  $\langle \xi_1 \xi_1 \rangle$  gives the standard result obtained in the first stochastic inflation works [1]–[7]

$$\langle \xi_1(x) \xi_1(x') \rangle = \frac{H^3}{4\pi^2} \frac{\sin \epsilon a_t H r}{\epsilon a_t H r} (1 + \epsilon^2) \delta(t - t'), \quad (3.3)$$

while the other correlators are of order  $\mathcal{O}(\epsilon^2)$ . Therefore, in the small  $\epsilon$  limit, we get the standard result (for  $\mathbf{x} = \mathbf{x}'$  and  $t < t'$ )

$$\langle \delta\varphi(t) \delta\varphi(t') \rangle = \frac{H^3}{4\pi^2} (t - t_{60}). \quad (3.4)$$

Integrating out long wavelengths using the step function gives what in stochastic language is called a *Markov process*. However, this is not the most natural choice one can make. Actually, the smoothing of the field is generally performed in configuration space through a function  $w(r/R)$  that rapidly decays for distances much larger than  $R$ . In momentum space, this operation produces a weighting of the modes with the Fourier transform  $\tilde{w}(kR)$  that projects out the high frequencies. Since we are interested in the short wavelength part of the field, and our smoothing scale is the comoving Hubble length  $|\eta|$ , a natural choice of the momentum window function will then be  $W(k\eta) = 1 - \tilde{w}(k|\eta|)$ .

In a general case the choice of a more physical way to separate the modes gives a coloured noise term, which is unfortunately much more difficult to treat. Namely, if we smooth the field with a Gaussian filter

$$w(r/|\eta|) = e^{-(k^2\sigma^2)/(2\eta^2)}, \quad (3.5)$$

we get the window function

$$W(k\eta) = 1 - e^{-(k^2\eta^2)/(2\sigma^2)}, \quad (3.6)$$

also in this case we can compute the noise correlation function for  $\nu = 3/2$ , and in the limit  $r = |\mathbf{x} - \mathbf{x}'| \rightarrow 0$  we obtain (appendix C), setting  $\tau = t - t'$ ,

$$\langle \xi_i(t) \xi_j(t') \rangle_{r=0} = \frac{H^4}{4\pi^2} \frac{1}{\cosh^2 H\tau} \sum_{k=0}^{\infty} \mathcal{A}_{ij}^{(k)}(\tau) \frac{(-1)^k (k+1)(k+2)}{(2k-1)!!} \left( 2\sigma^2 \frac{\sinh^2 H\tau/2}{\cosh H\tau} \right)^k, \quad (3.7)$$

where

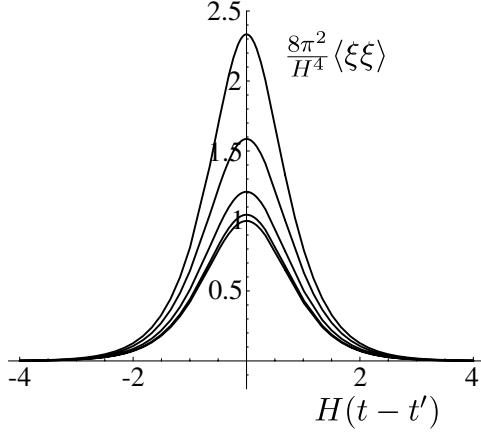
$$\mathcal{A}_{ij}^{(k)}(\tau) = \frac{1}{2} \left[ \begin{array}{cc} \frac{1-2k}{k+2} + \frac{\sigma^2}{\cosh H\tau} & \frac{\sigma^2 e^{H\tau}}{\cosh H\tau} \left( 1 + \frac{2k}{e^{H\tau} - 1} \right) \\ \frac{\sigma^2 e^{-H\tau}}{\cosh H\tau} \left( 1 + \frac{2k}{e^{-H\tau} - 1} \right) & \sigma^4 \frac{(k+3)}{\cosh^2 H\tau} \end{array} \right]. \quad (3.8)$$

For generic values of the parameter  $\sigma$ , the noise correlation function  $\langle \xi(t) \xi(t') \rangle$  can have a rather complicated functional form, and is plotted in figure 1. However, in the small  $\sigma$  limit, the leading term of the series (3.7) is that with  $k = 0$ . Therefore, all the  $A_{ij}$  vanish but  $A_{11}$ , and the noise correlation function becomes [19]

$$\langle \xi(t) \xi(t') \rangle = \frac{H^4}{8\pi^2} \frac{1}{\cosh^2 (H(t-t'))} + \mathcal{O}(\sigma^2), \quad (3.9)$$

which inserted in (3.1) and after a double time integration gives (to leading order in  $\sigma^2$ )

$$\langle \delta\varphi(t) \delta\varphi(t') \rangle = \frac{H^2}{8\pi^2} \ln \frac{\cosh(H(t-t_{60})) \cosh(H(t'-t_{60}))}{\cosh(H(t-t'))}, \quad (3.10)$$



**Figure 1.** Noise correlation functions  $\langle \xi(t)\xi(t') \rangle$  for  $r = 0$  and five values of  $\sigma$  varying from 1 (top line) to 0 (bottom line). The only non-vanishing function for  $\sigma = 0$  is  $\langle \xi_1 \xi_1 \rangle$ .

that for  $t' \gg t \gg t_{60}$  approaches the standard result (3.4). Conversely, for  $t = t' \rightarrow t_{60}$  one obtains

$$\langle \delta\varphi(t)^2 \rangle = \frac{H^4}{8\pi^2} \ln \cosh^2(H(t - t_{60})) \sim \frac{H^4}{8\pi^2} (t - t_{60})^2; \quad (3.11)$$

we thus find that soon after the initial condition is set the variance depends quadratically on time, unlike the white noise case (3.4), where the time dependence is linear. This behaviour, which comes from the fact that the Dirac delta function is divergent while a more physical correlation function is not, will have an important role in the following analysis.

The power spectrum can be immediately derived by inserting the explicit form (2.12) of the  $A_{ij}$  into (3.1). The two integrations over time factor out, and performing the change of variables  $\tilde{t} \rightarrow x \equiv -k\tilde{\eta}$  and  $\tilde{t}' \rightarrow y \equiv -k\tilde{\eta}'$  we get

$$\langle \delta\varphi(x)\delta\varphi(x') \rangle = \int \frac{dk}{k} \frac{\sin kr}{kr} \frac{H^2}{4\pi^2} F(k\eta, k\eta'), \quad (3.12)$$

with

$$F(k\eta, k\eta') = \text{Re} \left[ \int_{k|\eta|}^{k|\eta_{60}|} dx (f_1(x) - f_2(x)) \int_{k|\eta'|}^{k|\eta_{60}|} dy (f_1(y) - f_2(y))^* \right], \quad (3.13)$$

and

$$f_1(x) = -\sqrt{\frac{\pi}{2}} x^{3/2} W'(x) H_\nu^{(1)}(x), \quad f_2(x) = -\sqrt{\frac{\pi}{2}} \frac{x^{5/2}}{3} W'(x) H_{\nu-1}^{(1)}(x). \quad (3.14)$$

For  $\nu = 3/2$  and using the Gaussian window (3.6), so that  $W'(x) = x \exp[-k^2\eta^2/2\sigma^2]/\sigma^2$ , the functions  $f_1$  and  $f_2$  are given by

$$f_1(x) = \frac{x(x+i)}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2} + ix\right), \quad f_2(x) = \frac{ix^3}{3\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2} + ix\right). \quad (3.15)$$

Setting  $t = t'$  in (3.12), we extract the power spectrum:

$$\mathcal{P}_{\delta\varphi} = \left(\frac{H}{2\pi}\right)^2 \left| \int_{k|\eta|}^{k|\eta_{60}|} dx \left( f_1(x) - f_2(x) \right) \right|^2. \quad (3.16)$$

For  $t \gg t_{60}$ , we can take into account scales that are much larger than the horizon at time  $t$  (such that  $-k\eta \ll 1$ ) but much smaller at  $t_{60}$  ( $-k\eta_{60} \gg 1$ ); for such scales we get

$$\mathcal{P}_{\delta\varphi} \simeq \left(\frac{H}{2\pi}\right)^2 \left| \int_0^\infty dx f_1(x) - \int_0^\infty dx f_2(x) \right|^2. \quad (3.17)$$

Studying the behaviour of the integrals of  $f_1$  and  $f_2$  for large and small values of the parameter  $\sigma$  we see that for  $\sigma \rightarrow 0$  we have  $\int_0^\infty dx f_1(x) \rightarrow i$  and  $\int_0^\infty dx f_2(x) \rightarrow 0$ , while for  $\sigma \rightarrow \infty$  both integrals tend to vanish. This means that for small  $\sigma$  the power spectrum on very large scales (such that  $k \ll \sigma aH$ ) approximates the standard result  $H^2/4\pi^2$ , while fluctuations do not appear at all for very large values of the parameter. This behaviour is due to the fact that for small  $\sigma$  the window function tends to unity, and larger and larger scales are included in the noise  $\xi$ , whose correlation function therefore reproduces the ordinary fluctuation behaviour (when all scales are taken into account). For increasing  $\sigma$ , instead, since the window function vanishes, the noise contains only smaller and smaller wavelengths, and is no longer able to influence super-horizon scales.

#### 4. The interacting scalar field

Let us now introduce a small self-interaction term in the potential for  $\varphi$ , in the form of a cubic term

$$V = \frac{\mu}{3!} \varphi^3, \quad (4.1)$$

where  $\mu/6H \ll 1$ . This toy model may be useful even for describing the generation of non-linearities in scalar fields other than the inflaton field and then transmitted to the latter, as described in [38, 39]. We expand to second order in the fluctuations the field  $\delta\varphi$  substituting  $\delta\varphi = \delta\varphi_1 + \delta\varphi_2$  in (2.16). Assuming that the small self-interaction term can only produce second-order effects (acting therefore only on  $\delta\varphi_2$ ), we can solve recursively the equation of motion for  $\delta\varphi_1$  and  $\delta\varphi_2$ . In the slow-roll approximation they become

$$\begin{aligned} 3H\delta\dot{\varphi}_1 &= 3H\xi_1 + \dot{\xi}_1 - H\xi_2, \\ 3H\delta\dot{\varphi}_2 + \frac{\mu}{2}\delta\varphi_1^2 &= 0. \end{aligned} \quad (4.2)$$

To first order in the field expansion, the three-point function decomposes into the sum of three terms:

$$\langle \delta\varphi_1(x_1)\delta\varphi_1(x_2)\delta\varphi_2(x_3) \rangle + \langle \delta\varphi_1(x_1)\delta\varphi_2(x_2)\delta\varphi_1(x_3) \rangle + \langle \delta\varphi_2(x_1)\delta\varphi_1(x_2)\delta\varphi_1(x_3) \rangle. \quad (4.3)$$

Since  $\delta\varphi_2$  depends quadratically on  $\delta\varphi_1$ , each of these first-order terms is actually constituted by the sum of three four-point functions of  $\delta\varphi_1$ , two connected and one disconnected. However, since in the perturbative expansion in  $\mu/6H$  of the equation of motion the originally vanishing mean value of  $\delta\varphi$  is shifted away from zero by  $\langle \delta\varphi_2 \rangle$ ,

The influence of super-horizon scales on cosmological observables generated during inflation

and the disconnected term is indeed proportional to this quantity, we can eliminate it just by requiring that the mean of the fluctuation vanishes. For this purpose, we set

$$\delta\varphi_2 = -\frac{\mu}{6H} \int_{t_{60}}^t dt' [\delta\varphi_1^2(t', \mathbf{x}) - \langle \delta\varphi_1^2(t', \mathbf{x}) \rangle]; \quad (4.4)$$

the second term, added in order to make the mean value vanish, cancels the disconnected contribution, while the connected ones are equal to each other. The first term of (4.3) namely becomes

$$-\frac{\mu}{3H} \int_{t_{60}}^{t_3} dt' \langle \delta\varphi_1(t_1, \mathbf{x}_1) \delta\varphi_1(t', \mathbf{x}_3) \rangle \langle \delta\varphi_1(t_2, \mathbf{x}_2) \delta\varphi_1(t', \mathbf{x}_3) \rangle, \quad (4.5)$$

that is (setting  $t_1 = t_2 = t_3 \equiv t$  and going to conformal time)

$$\frac{\mu}{3H^2} \int_{\eta_{60}}^{\eta} \frac{d\eta'}{\eta'} \frac{H^4}{4(2\pi)^6} \int \frac{d\mathbf{k}}{k^3} e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_3)} F(k\eta, k\eta') \int \frac{d\mathbf{k}'}{k'^3} e^{i\mathbf{k}'\cdot(\mathbf{x}_1-\mathbf{x}_3)} F(k'\eta, k'\eta'). \quad (4.6)$$

Going to Fourier space and extracting a factor of  $(2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)$ , we obtain the total bispectrum as the sum of the three contributions

$$B(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = B_{12} + \text{permutations}, \quad (4.7)$$

where

$$B_{12} \equiv \frac{\mu H^2}{12} \frac{1}{k_1^3 k_2^3} \int_{\eta_{60}}^{\eta} \frac{d\eta'}{\eta'} F(k_1\eta, k_1\eta') F(k_2\eta, k_2\eta'). \quad (4.8)$$

With a good approximation, we can write for the Gaussian window case

$$\begin{aligned} F(k\eta, k\eta') &\simeq \left( \exp\left(-\frac{k^2\eta^2}{2\sigma^2}\right) g(k\eta) - \exp\left(-\frac{k^2\eta_{60}^2}{2\sigma^2}\right) g(k\eta_{60}) \right) \\ &\times \left( \exp\left(-\frac{k^2\eta^2}{2\sigma^2}\right) g(k\eta) - \exp\left(-\frac{k^2\eta_{60}^2}{2\sigma^2}\right) g(k\eta_{60}) \right) + \mathcal{O}(\sigma^4), \end{aligned} \quad (4.9)$$

where

$$g(x) \equiv 1 + \frac{\sigma^2}{3} + \frac{1 + \sigma^2}{6} x^2 \quad (4.10)$$

and the contributions to the total bispectrum can be now calculated analytically. In the limit of super-horizon scales ( $-k_i\eta/\sigma \ll 1$ , but  $-k_i\eta_{60}/\sigma \gg 1$ ) and to order  $\mathcal{O}(\sigma^4)$  we obtain

$$B_{12} = \frac{\mu H^2}{24} \frac{1 + \sigma^2/3}{k_1^3 k_2^3} \left[ \ln \frac{(k_1^2 + k_2^2)\eta^2}{2\sigma^2} + \gamma - \frac{\sigma^2}{3} \right]. \quad (4.11)$$

If we take as window the step function, so that the derivative  $W'(x)$  is Dirac's delta function  $\delta(x - \varepsilon)$ , we get

$$F(k\eta, k\eta') = \left( 1 + \frac{\varepsilon^2}{3} + \frac{\varepsilon^4}{9} \right) \vartheta(\varepsilon - k|\eta|) \vartheta(\varepsilon - k|\eta'|) \vartheta(k|\eta_{60}| - \varepsilon), \quad (4.12)$$

and

$$B_{12} = \frac{\mu H^2}{12} \frac{1}{k_1^3 k_2^3} \left( 1 + \frac{\varepsilon^2}{3} + \frac{\varepsilon^4}{9} \right)^2 \ln \frac{k_{\max} |\eta|}{\varepsilon}, \quad (4.13)$$

with  $k_{\max}|\eta| < \varepsilon < k_{\min}|\eta_{60}|$ . This result is in good agreement with the one in [21], the main difference being that the relevant scale is not  $k_1 + k_2 + k_3$  but  $k_{\max}$ , which is due to our use of the slow-roll approximation, instead of the exact solution of the equation of motion.

For later convenience, let us also calculate here the skewness of our coarse grained inflaton field: from (4.3) and (4.5), with  $x_1 = x_2 = x_3$ , we have in the white noise case (when  $\langle \delta\varphi^2 \rangle \propto t - t_{60}$ )

$$\langle \delta\varphi^3(t, \mathbf{x}) \rangle = -\frac{\mu H^5}{3(2\pi)^4}(t - t_{60})^3, \quad (4.14)$$

while with our window function the result, in the small  $\sigma$  limit, is

$$\langle \delta\varphi^3(t, \mathbf{x}) \rangle = \frac{\mu H^2}{4(2\pi)^4} \int_{\eta_{60}}^{\eta} \frac{d\tilde{\eta}}{\tilde{\eta}} \left[ \ln \frac{(\eta^2 + \eta_{60}^2)(\tilde{\eta}^2 + \eta_{60}^2)}{(\eta^2 + \tilde{\eta}^2)2\eta_{60}^2} \right]^2; \quad (4.15)$$

for early times it can be approximated by

$$\langle \delta\varphi^3(t, \mathbf{x}) \rangle \simeq -\frac{\mu H^7}{12(2\pi)^4}(t - t_{60})^5, \quad (4.16)$$

while for  $t \gg t_{60}$  we asymptotically recover the ordinary result (4.14).

## 5. Conditional probability distribution and moments

In this section we derive an expression for the probability distribution of the field  $\delta\varphi$  at time  $t$  as a function of the stochastic variable  $\xi = \xi_1 - \xi_2/3$ . More precisely, we are interested in the conditional probability  $P_t(\delta\varphi|\delta\varphi_{60})$  that the stochastic variable  $\delta\varphi[\xi](t)$  assumes a specific value  $\delta\varphi$  at time  $t$  given that it assumed the value  $\delta\varphi_{60}$  at the earlier time  $t_{60}$ , i.e. 60 e-folds before inflation ends. In particular, we will consider  $\delta\varphi_{60} = 0$ , as argued in [33]–[35]. This is the crucial quantity that distinguishes Markovian processes from non-Markovian ones. A stochastic process is said to be Markovian if its conditional probability depends only on the value of the variable at the time when we impose the constraint, while in non-Markovian cases a memory of what happened at earlier times is kept.

It is a well known result of the theory of stochasticity [16, 17] that Langevin equations with white noise describe Markovian processes, while coloured noise is a typical cause of Markovianity breaking down. Thus, in our case the fluctuations of the inflaton field on the large scale behave like a Markovian process if we choose  $W(k\eta)$  to be a step function, that generates white noise, but become non-Markovian whenever a smoother separator is applied, as for example our Gaussian window function.

Our goal is to estimate the amount of non-Gaussianity produced in this latter case: in the standard scenario, non-Gaussian features are generally small, since they do not have enough time to develop after  $t_{60}$ , when the condition  $\delta\varphi_{60} = 0$  applies. However, if the constraint does not erase the memory of earlier times, non-Gaussianity could be significantly larger.

According to Bayes theorem, the conditional probability is obtained from the joint probability  $P_t(\delta\varphi, \delta\varphi_{60})$  that the random process  $\delta\varphi[\xi]$  assumes the values  $\delta\varphi_{60}$  at time  $t_{60}$  and  $\delta\varphi$  at time  $t$ , normalized with the probability  $P(\delta\varphi_{60})$  that  $\delta\varphi[\xi](t_{60}) = \delta\varphi_{60}$ . A simple

way to evaluate the joint probability (see e.g. [40]) is to average over all  $\xi$  configurations the product of two delta functions centred on these values:

$$\begin{aligned} P_t(\delta\varphi, \delta\varphi_{60}) &= \mathcal{N} \int \mathcal{D}\xi \delta(\delta\varphi[\xi](t) - \delta\varphi) \delta(\delta\varphi[\xi](t_{60}) - \delta\varphi_{60}) e^{-(1/2)\xi^T \mathbf{A}^{-1} \xi} \\ &= \int \frac{d\alpha_1}{(2\pi)} \frac{d\alpha_2}{(2\pi)} e^{-i(\alpha_1 \delta\varphi + \alpha_2 \delta\varphi_{60})} \mathcal{N} \int \mathcal{D}\xi e^{-(1/2)\xi^T \mathbf{A}^{-1} \xi + i\alpha_1 \delta\varphi[\xi](t) + i\alpha_2 \delta\varphi[\xi](t_{60})}, \end{aligned} \quad (5.1)$$

where the shorthand notation  $\xi^T \mathbf{A}^{-1} \xi$  stands for the double integration in the exponent of the Gaussian weight (2.11). The probability distribution  $P(\delta\varphi_{60})$  for  $\delta\varphi[\xi]$  at  $t_{60}$  can be obtained in the same way by averaging just one delta function:

$$\begin{aligned} P(\delta\varphi_{60}) &= \mathcal{N} \int \mathcal{D}\xi \delta(\delta\varphi[\xi](t_{60}) - \delta\varphi_{60}) e^{-(1/2)\xi^T \mathbf{A}^{-1} \xi} \\ &= \int \frac{d\alpha_2}{2\pi} e^{-i\alpha_2 \delta\varphi_{60}} \mathcal{N} \int \mathcal{D}\xi e^{-(1/2)\xi^T \mathbf{A}^{-1} \xi + i\alpha_2 \delta\varphi[\xi](t_{60})}. \end{aligned} \quad (5.2)$$

The equation of motion for  $\delta\varphi[\xi]$  is a first-order differential equation, and therefore for a given choice of the function  $\xi(t)$  the solution depends on the initial condition  $\delta\varphi_{\text{in}}$  at  $t = t_{\text{in}}$  defined to be time at the beginning of inflation.

Since  $\delta\varphi$  is a fluctuation (i.e. with vanishing mean value), we require that the solution vanishes at every time when averaged over all  $\xi$ s: the initial condition must then be  $\delta\varphi_{\text{in}} = 0$  for continuity.

If we could solve the equation of motion, we could then calculate the value of the fluctuation at  $t_{60}$ . However, because of the presence of the delta function, this value is constrained and the path integral runs only over the noise configurations that satisfy the condition on  $\delta\varphi[\xi](t_{60})$ . Moreover, the value  $\delta\varphi_{60}$  assumed at  $t_{60}$  constitutes a new initial condition for the equation of motion at later times, whose solution thus depends only on the noise configuration after  $t_{60}$ .

Writing the equation of motion in integral form and then solving perturbatively by iteration, to first order in  $\mu/6H$  we get

$$\begin{aligned} \delta\varphi[\xi](t_{60}) &= \int_{t_{\text{in}}}^{t_{60}} dt' \xi(t') - \frac{\mu}{6H} \int_{t_{\text{in}}}^{t_{60}} dt' \delta\varphi^2[\xi](t') \\ &\simeq \int_{t_{\text{in}}}^{t_{60}} dt' \xi(t') - \frac{\mu}{6H} \int_{t_{\text{in}}}^{t_{60}} dt' \left( \int_{t_{\text{in}}}^{t'} dt'' \xi(t'') \right)^2 \end{aligned} \quad (5.3)$$

that depends on the noise term configuration only up to  $t_{60}$ , and for later times

$$\begin{aligned} \delta\varphi[\xi](t) &= \delta\varphi_{60} + \int_{t_{60}}^t dt' \xi(t') - \frac{\mu}{6H} \int_{t_{60}}^t dt' \delta\varphi^2[\xi](t') \\ &\simeq \delta\varphi_{60} - \frac{\mu}{6H} \delta\varphi_{60}^2 (t - t_{60}) + \int_{t_{60}}^t dt' \xi(t') - \frac{\mu}{6H} \int_{t_{60}}^t dt' \left( \int_{t_{60}}^{t'} dt'' \xi(t'') \right)^2, \end{aligned} \quad (5.4)$$

where in the last equality we did not include the first-order correction to the linear term in  $\xi$  because its effect would just be a sub-leading contribution to the two-point correlation function.

As we already pointed out, the solution after  $t_{60}$  involves only integrals over later times: the only possibility for the fluctuation field to keep memory of earlier times is then

that this configuration itself is influenced by the configurations before the constraint, which is exactly what happens in the case of coloured noise.

We see that in these equations there are both linear and quadratic terms in  $\xi$ : the quadratic terms can be added to  $\xi^T \mathbf{A}^{-1} \xi$  of equation (5.1) in such a way as to obtain a modified integration kernel and perform the Gaussian integration over the  $\xi$ s. Skipping all the technicalities (see appendix D), after inverting perturbatively the new integration kernel we obtain for the conditional probability the following result:

$$P_t(\delta\varphi|\delta\varphi_{60}) = \frac{\int d\alpha_1 d\alpha_2 e^{-i\delta\varphi_i\alpha_i} e^{-(1/2)C_{ij}\alpha_i\alpha_j + i(\mu/6H)D_{ijk}\alpha_i\alpha_j\alpha_k}}{2\pi \int d\alpha_2 e^{-i\delta\varphi_2\alpha_2} e^{-(1/2)C_{22}\alpha_2^2 + i(\mu/6H)D_{222}\alpha_2^3}}, \quad (5.5)$$

where we adopted a convenient shorthand tensor notation, setting

$$\delta\varphi_1 = \delta\varphi - \delta\varphi_{60} + \frac{\mu}{6H}\delta\varphi_{60}^2(t - t_{60}), \quad \delta\varphi_2 = \delta\varphi_{60}, \quad (5.6)$$

$$C_{ij} = \int_{I_i} dt' \int_{I_j} dt'' \langle \xi(t') \xi(t'') \rangle, \quad (5.7)$$

$$D_{ijk} = \int_{I_j} d\tilde{t} \int_{I_i} dt' \int_{\tilde{I}_j} dt'' \langle \xi(t') \xi(t'') \rangle \int_{I_k} d\tau' \int_{\tilde{I}_j} d\tau'' \langle \xi(\tau') \xi(\tau'') \rangle \quad (5.8)$$

and  $I_1 = [t_{60}, t]$ ,  $I_2 = [t_{\text{in}}, t_{60}]$ ,  $\tilde{I}_1 = [t_{60}, \tilde{t}]$ ,  $\tilde{I}_2 = [t_{\text{in}}, \tilde{t}]$  for the integration supports.

For white noise processes, the only terms that survive are  $C_{11}$ ,  $C_{22}$ ,  $D_{111}$  and  $D_{222}$ , while those with mixed indices vanish. For instance, we have

$$C_{12} \propto \int_{t_{60}}^t dt' \int_{t_{\text{in}}}^{t_{60}} dt'' \delta(t' - t'') = 0, \quad (5.9)$$

$$D_{121} \propto \int_{t_{\text{in}}}^{t_{60}} d\tilde{t} \left( \int_{t_{60}}^t dt' \int_{t_{\text{in}}}^{\tilde{t}} dt'' \delta(t' - t'') \right)^2 = 0 \quad (5.10)$$

since the integration supports of the two-time variables are disjoint, and the same happens for all other mixed terms. In this case, thus, the two integrations over  $\alpha_1$  and  $\alpha_2$  in the numerator factor out, and the second one simplifies with the one in the denominator. We then get

$$\begin{aligned} P_t(\delta\varphi|\delta\varphi_{60}) &= \frac{1}{2\pi} \int d\alpha_1 e^{-i\delta\varphi_1\alpha_1} e^{-(1/2)C_{11}\alpha_1^2 + i(\mu/6H)D_{111}\alpha_1^3} \\ &= \frac{\exp[(\mu/6H)D_{111}(\partial^3/\partial\delta\varphi^3)] \exp[-(\delta\varphi_1^2/2C_{11})]}{\sqrt{2\pi C_{11}}}, \end{aligned} \quad (5.11)$$

i.e. a probability distribution function for a Markovian process, that does not keep any memory of what happened before the constraint (indeed, this function contains only integrations over times after  $t_{60}$ , since no index 2 appears). To zeroth order in  $\mu/6H$ , this distribution is Gaussian with mean  $\delta\varphi_{60}$  and variance  $C_{11}$ , which is exactly the standard case of equation (3.1).

In a case with coloured noise, the correlation function is no longer a delta function but has a finite width, and the coefficients with mixed indices no longer vanish even if the two-time integration supports are disjoint. In the general non-Markovian case no

The influence of super-horizon scales on cosmological observables generated during inflation

factorization is possible, and after integrating over  $\alpha_1$  and  $\alpha_2$  we can write

$$P_t(\delta\varphi|\delta\varphi_{60}) = \frac{1}{\sqrt{2\pi C_{11}(1-y)}} \times \frac{\exp[(\mu/6H)D_{ijk}(\partial^3/\partial\delta\varphi_i\partial\delta\varphi_j\partial\delta\varphi_k)] \exp[-(1/2)[C^{-1}]_{ij}\delta\varphi_i\delta\varphi_j]}{\exp[(\mu/6H)D_{222}(\partial^3/\partial\delta\varphi_2^3)] \exp[-(1/2)(\delta\varphi_2^2/C_{22})]}, \quad (5.12)$$

where  $C$  is the matrix with elements  $C_{ij}$  and

$$[C^{-1}]_{ij} = \frac{1}{1-y} \begin{bmatrix} 1 & -y \\ C_{11} & C_{12} \\ -y & 1 \\ C_{12} & C_{22} \end{bmatrix}, \quad y = \frac{C_{12}^2}{C_{11}C_{22}}. \quad (5.13)$$

It is immediate to check that from this general formula we can get as a particular case the Markovian formula by simply setting  $C_{12} = 0$  (i.e.  $y = 0$ ) and  $D_{ijk} = 0$  for  $\{ijk\} \neq \{111\}, \{222\}$ .

Expanding the derivation operators to first order in  $\mu/6H$ , after some algebra we get

$$P_t(\delta\varphi|\delta\varphi_{60}) = \frac{\exp\left[-\frac{1}{2} \frac{(\delta\varphi - \delta\varphi_{60}(1 + (C_{12}/C_{22}) - (\mu/6H)\delta\varphi_{60}(t - t_{60})))^2}{C_{11}(1-y)}\right]}{\sqrt{2\pi C_{11}(1-y)}} \times \left(1 + \frac{\mu}{6H} \left( \frac{D_{ijk}(\partial^3/\partial\delta\varphi_i\partial\delta\varphi_j\partial\delta\varphi_k)e^{-(1/2)[C^{-1}]_{ij}\delta\varphi_i\delta\varphi_j}}{e^{-(1/2)[C^{-1}]_{ij}\delta\varphi_i\delta\varphi_j}} - \frac{D_{222}(\partial^3/\partial\delta\varphi_2^3)e^{-(\delta\varphi_2^2/2C_{22})}}{e^{-(\delta\varphi_2^2/2C_{22})}} \right)\right). \quad (5.14)$$

To zeroth order in  $\mu/6H$  we then have again a Gaussian distribution, but now the mean is

$$\langle\delta\varphi\rangle|_{\text{NM}} = \delta\varphi_{60} \left(1 + \frac{C_{12}}{C_{22}}\right), \quad (5.15)$$

that is the sum of the constraint value plus an extra term due to non-Markovian memory effects, while the variance becomes

$$\langle\delta\varphi^2\rangle|_{\text{NM}} = C_{11}(1-y). \quad (5.16)$$

In order to calculate the skewness, we have to take into account all the first-order contributions. The full calculation gives

$$\langle\delta\varphi^3\rangle|_{\text{NM}} = \int_{-\infty}^{+\infty} d\delta\varphi \delta\varphi^3 P_t(\delta\varphi|\delta\varphi_{60})|_{\delta\varphi_{60}=0} = -\frac{\mu}{H} D_{111}(1+K), \quad (5.17)$$

where  $K$  is given by

$$K = -\frac{C_{12}}{C_{22}} \frac{2D_{112} + D_{121}}{D_{111}} - \frac{C_{11}}{C_{22}} \frac{1 - 3y}{2} \frac{2D_{122} + D_{212}}{D_{111}} + \frac{C_{11}C_{12}}{(C_{22})^2} \frac{3 - 5y}{2} \frac{D_{222}}{D_{111}}, \quad (5.18)$$

$y$  and  $K$  are then the terms monitoring the importance of memory effects induced by the coloured noise in the conditional probability distribution variance and skewness respectively. Whenever these coefficients are small (as for white noise cases, when they

identically vanish since  $C_{12} = 0$  and  $D_{112} = D_{121} = D_{122} = D_{212} = 0$ ) we get for the variance  $\langle \delta\varphi^2 \rangle|_M = C_{11}$  the ordinary result (3.1), while equation (5.17) reduces to

$$\langle \delta\varphi^3 \rangle|_M = -\frac{\mu}{H} \int_{t_{60}}^t d\tilde{t} \left( \int_{t_{60}}^{\tilde{t}} dt' \int_{t_{60}}^{\tilde{t}} dt'' \langle \xi(t') \xi(t'') \rangle \right)^2, \quad (5.19)$$

which is exactly what we obtain from equations (4.3)–(4.5) with  $x_1 = x_2 = x_3$ . If instead  $y$  and  $K$  are significantly different from zero the effect is big, and the procedure of neglecting times before  $t_{60}$ , as we did to derive for example equations (3.11) and (4.15), is not correct any more.

We now want to apply the formalism developed so far to the case of the Gaussian window (3.6). It is a generic feature of non-Markovian systems that, since memory effects appear through secular terms which depend on the whole time interval before the constraint, they increase as this interval increases. Conversely, much after the constraint these effects tend to be erased as a consequence of stochasticity. The results derived in section 3 should thus still hold for  $t - t_{60} \gg H^{-1}$ , while we expect the difference from the Markovian case to be maximal for  $t - t_{60} \ll H^{-1}$  and  $t_{60} - t_{\text{in}} \gg H^{-1}$ .

In this limit, we can compare the white noise correlation functions (3.4) and (4.14) with the ones just derived. For the two-point function, while in the Markovian case we had  $\langle \delta\varphi^2 \rangle \propto H(t - t_{60})$ , now, since the noise correlation never diverges and  $C_{11}$  contains a double integration over time, for  $t \rightarrow t_{60}$  we get  $\langle \delta\varphi^2 \rangle \propto H^2(t - t_{60})^2$ , and the ratio of the two vanishes. Precisely, for a value of the parameter  $\sigma$  not too close to 1, at the  $C_{ij}$  read

$$C_{11} \simeq \left( \frac{H}{2\pi} \right)^2 \ln \frac{\eta^2 + \eta_{60}^2}{2\eta\eta_{60}} \sim \frac{H^4}{4\pi^2} \frac{(t - t_{60})^2}{2}, \quad (5.20)$$

$$C_{12} = C_{21} \simeq \left( \frac{H}{2\pi} \right)^2 \frac{1}{2} \ln \frac{2\eta_{60}^2(\eta^2 + \eta_{\text{in}}^2)}{(\eta^2 + \eta_{60}^2)(\eta_{60}^2 + \eta_{\text{in}}^2)} \sim \frac{H^3}{4\pi^2} \frac{(t - t_{60})}{2}, \quad (5.21)$$

$$C_{22} \simeq \left( \frac{H}{2\pi} \right)^2 \ln \frac{\eta_{60}^2 + \eta_{\text{in}}^2}{2\eta_{60}\eta_{\text{in}}} \sim \frac{H^2}{4\pi^2} [H(t_{60} - t_{\text{in}}) - \ln 2]. \quad (5.22)$$

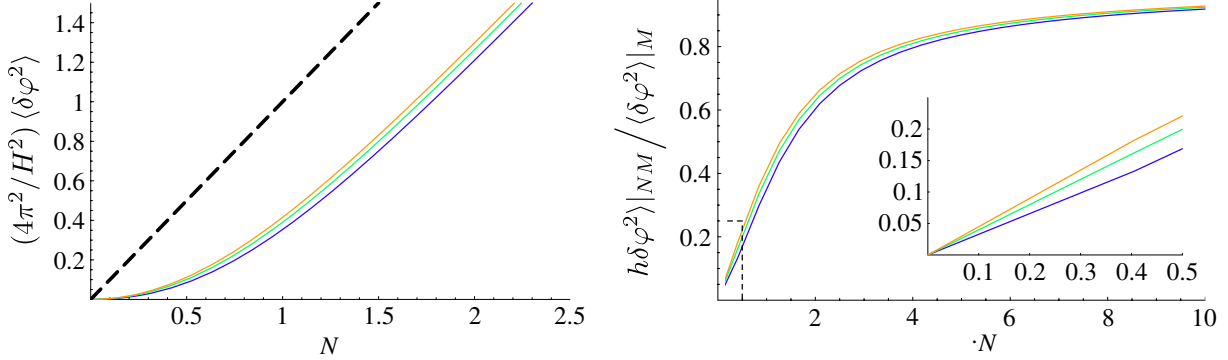
We thus get for the  $y$  parameter

$$y = \frac{C_{12}^2}{C_{11}C_{22}} \sim \frac{1}{2} \frac{1}{H(t_{60} - t_{\text{in}})}, \quad (5.23)$$

and the variance becomes

$$\langle \delta\varphi^2(t) \rangle|_{\text{NM}} = \frac{H^4}{8\pi^2} \left( 1 - \frac{1}{2H(t_{60} - t_{\text{in}})} \right) (t - t_{60})^2. \quad (5.24)$$

This expression shows two distinct effects. The first is the fact that the variance is an increasing function of the total amount of e-foldings of inflation before the constraint, and since  $y$  tends to vanish (though not very fast) for  $t_{60} - t_{\text{in}} \gg H^{-1}$  its dependence saturates to the value of the unconditional variance (3.11). This is a physical consequence of the fact that, even though highly oscillating noise configurations are statistically suppressed due to the non-vanishing correlation time, if there is much time after the beginning of inflation where the noise about  $t_{60}$  is almost uncorrelated with its initial value, while if the starting time is closer to  $t_{60}$  there cannot be very large fluctuations of the noise  $\xi$ , and also the variance of  $\delta\varphi$  gets smaller. A second effect is the quadratic time dependence soon after



**Figure 2.** Left panel: a plot of the variance  $\langle \delta\varphi^2(t) \rangle$  as a function of  $\Delta N = H(t - t_{60})$ , in the non-Markovian (thin continuous curves) and Markovian (thick dashed line) cases. Right panel: a plot versus  $\Delta N$  of the ratio  $\langle \delta\varphi^2(t) \rangle|_{\text{NM}} / \langle \delta\varphi^2(t) \rangle|_{\text{M}}$  between the non-Markovian (NM) and Markovian (M) cases. The inset contains a magnification of the region enclosed in the dashed box, showing in detail the behaviour for small  $\Delta N$  (i.e. close to  $t_{60}$ ). In all plots, different continuous curves represent values of  $H(t_{60} - t_{\text{in}})$  corresponding to 1.5, 2.5 and 5 (from bottom to top). The conditional variance is at any time after  $t_{60}$  an increasing function of the number of e-folds before  $t_{60}$ , converging to the value of the unconditional variance (3.11).

the constraint, versus the linear dependence of the standard Markovian case (3.4). This behaviour, already seen in equation (3.11), does not have anything to do with memory effects, but is merely due to the local shape of the noise correlation function, which is never divergent. Actually, we still have that

$$\frac{\langle \delta\varphi^2(t) \rangle|_{\text{NM}}}{\langle \delta\varphi^2(t) \rangle|_{\text{M}}} \sim \frac{\Delta N}{2}, \quad (5.25)$$

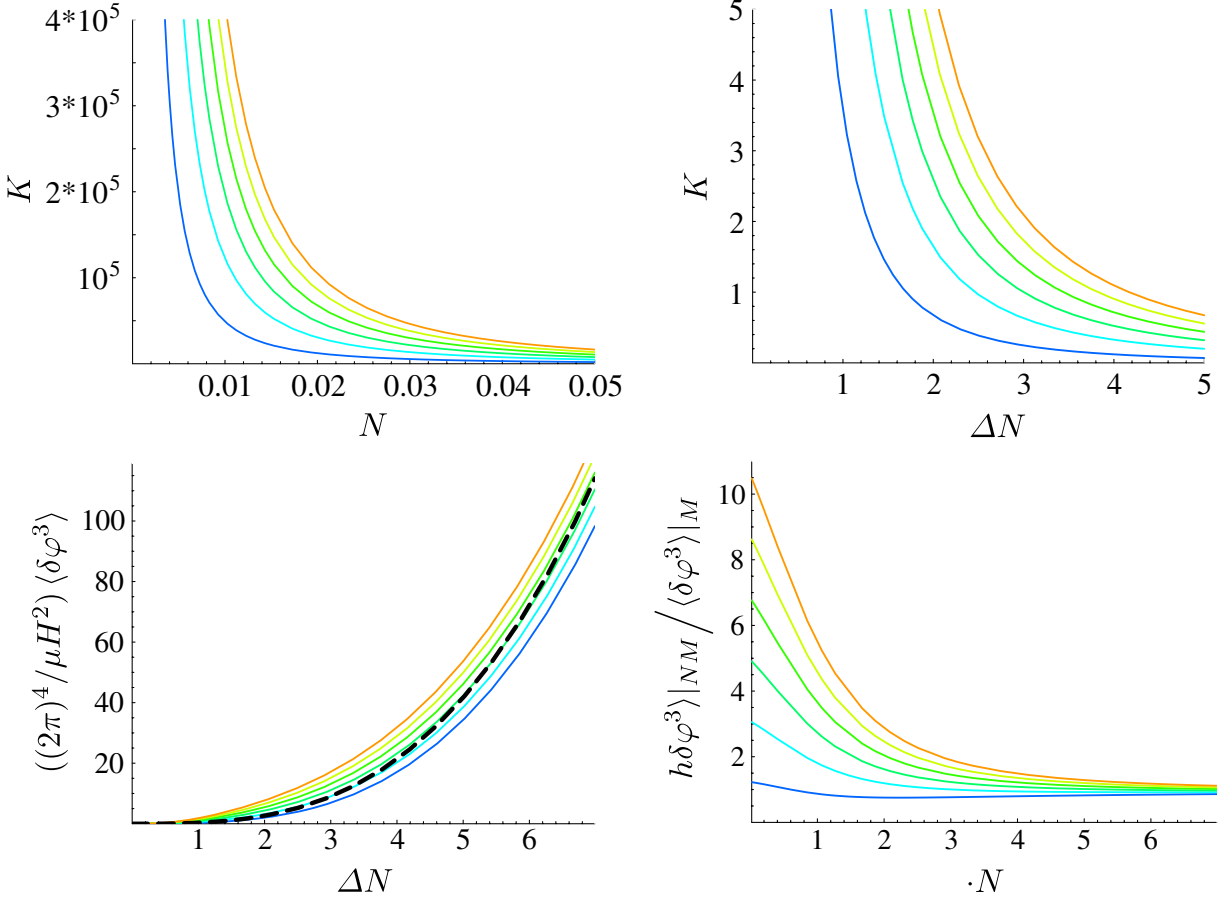
where we have defined  $\Delta N = H(t - t_{60}) = 60 - N$ , with  $N$  the number of e-folds until the end of inflation (see the inset in figure 2). This result shows that the power spectrum of perturbations on large scales is naturally bluer than the standard one. This basic conclusion is in qualitative agreement with the recent WMAP results which show a suppression of the lower multipoles in the cosmic microwave background (CMB) anisotropies.

While memory effects do not clearly show up in the variance, for the third moment we obtain, since  $K$  is dominated by the  $D_{222}$  term (see the explicit computation of all the  $D_{ijk}$  coefficients for this choice of the window in appendix E),

$$\langle \delta\varphi^3(t) \rangle|_{\text{NM}} \simeq -\frac{\mu}{H} \frac{3C_{11}C_{12}}{2C_{22}^2} D_{222} \sim -\frac{\mu H^6(t_{60} - t_{\text{in}})}{8(2\pi)^4} (t - t_{60})^3, \quad (5.26)$$

and therefore (recalling (4.14))

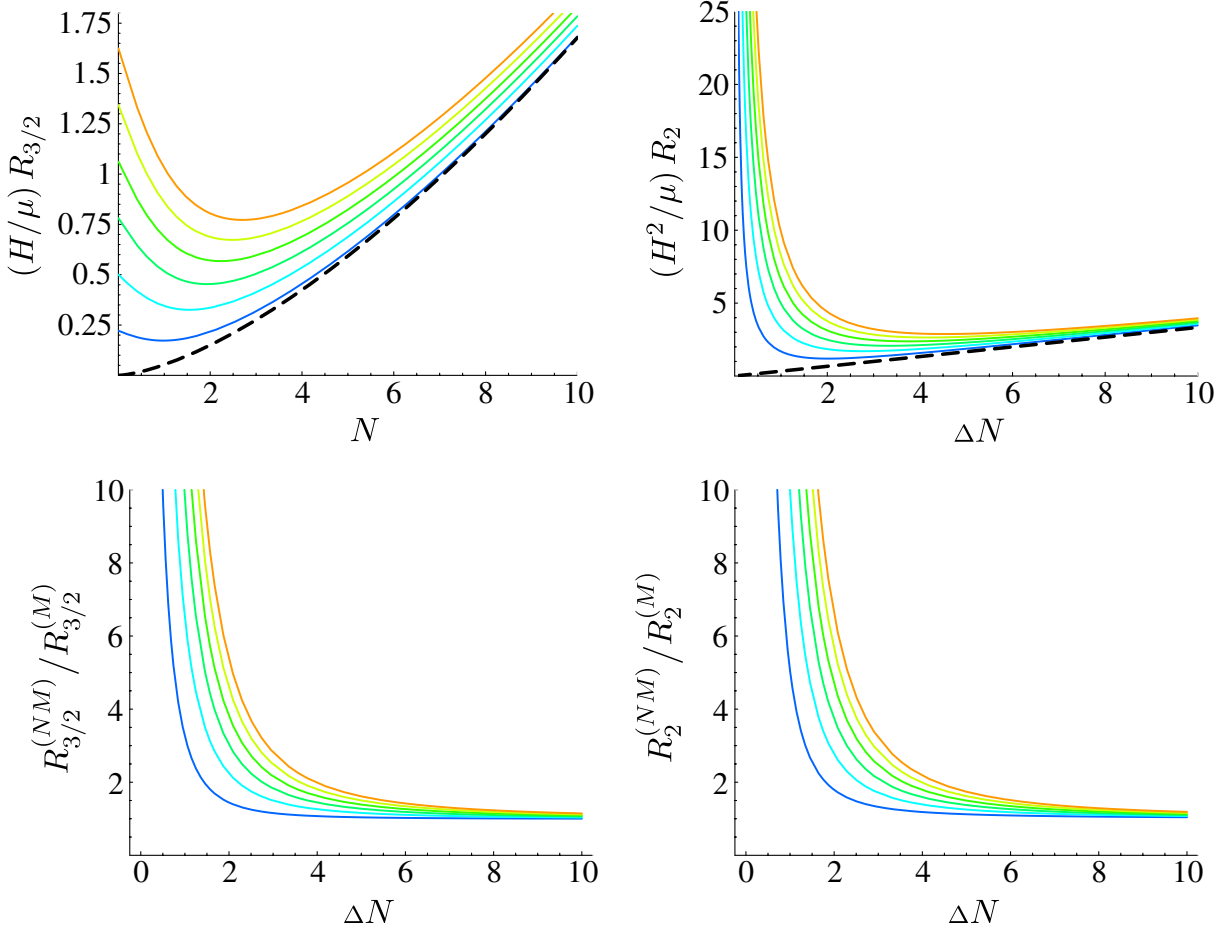
$$\frac{\langle \delta\varphi^3(t) \rangle|_{\text{NM}}}{\langle \delta\varphi^3(t) \rangle|_{\text{M}}} \sim \frac{3}{8} H(t_{60} - t_{\text{in}}), \quad (5.27)$$



**Figure 3.** Top panels: plots of the  $K$  coefficient for the Gaussian window versus  $\Delta N$ , showing memory effects in the third moment  $\langle \delta\varphi^3(t) \rangle$ , over two different timescales. Bottom panels: dependences on  $\Delta N$  of the third moment of  $\delta\varphi$  (left panel), in the non-Markovian (thin continuous curves) and Markovian (thick dashed curve) cases; the behaviour of the ratio  $\langle \delta\varphi^3(t) \rangle|_{NM}/\langle \delta\varphi^3(t) \rangle|_M$  (right panel) between the non-Markovian (NM) and Markovian (M) cases. Different curves correspond to values of  $H(t_{60} - t_{in})$  from 5 (bottom curve) to 30 (top curve).

showing that for wavelengths slightly smaller than the present horizon, the third moment can be considerably larger than is usually assumed, depending on the previous duration of inflation. However, at later times (for  $\Delta N \gg 1$ ) both the variance and the third moment converge to the standard Markovian behaviour. The precise value of all these quantities for generic times is shown in figures 2 and 3.

The amount of non-Gaussianity can be evaluated from the ratio of the three-point function and the variance to the appropriate power. It is useful to consider both the quantities  $R_{3/2} \equiv \langle \delta\varphi^3(t) \rangle / \langle \delta\varphi^2(t) \rangle^{3/2}$  and  $R_2 \equiv \langle \delta\varphi^3(t) \rangle / \langle \delta\varphi^2(t) \rangle^2$  (the second one is proportional to the non-Gaussianity strength parameter  $f_{NL}$ ; see e.g. [24, 25]) as a function of  $H(t - t_{60})$  and  $H(t_{60} - t_{in})$ . In this case too, as we see in figure 4, the effect increases



**Figure 4.** Top: the behaviour of  $R_{3/2}$  (left panel) and  $R_2$  (right panel) as a function of  $\Delta N$ , for the non-Markovian (thin continuous curves) and Markovian (thick dashed curve) cases. Bottom: the behaviour of the ratios  $R_{3/2}^{(NM)}/R_{3/2}^{(M)}$  (left panel) and  $R_2^{(NM)}/R_2^{(M)}$  (right panel) of the two cases. In each plot, different continuous curves represent values of  $H(t_{60} - t_{\text{in}})$  varying from 5 (bottom curve) to 30 (top curve).

as the time  $t$  gets closer to  $t_{60}$  and further from  $t_{\text{in}}$ . Analytically, in this limit we have

$$R_{3/2}^{(\text{NM})} \sim \frac{\mu}{H} \frac{H(t_{60} - t_{\text{in}})}{4\sqrt{2}\pi}, \quad (5.28)$$

$$R_2^{(\text{NM})} \sim \frac{\mu}{H^2} \frac{H(t_{60} - t_{\text{in}})}{2\Delta N}, \quad (5.29)$$

(where the superscript NM stands for non-Markovian) while for the white noise case

$$R_{3/2}^{(\text{M})} = \frac{\mu}{H} \frac{(\Delta N)^{3/2}}{6\pi}, \quad (5.30)$$

$$R_2^{(\text{M})} = \frac{\mu}{H^2} \frac{\Delta N}{3}, \quad (5.31)$$

(where the superscript M stands for Markovian). In the Markovian case both quantities vanish, but in non-Markovian ones they are finite or even divergent for  $\Delta N \rightarrow 0$ , because of the quadratic (instead of linear) time dependence of the variance, and are growing functions of  $H(t_{60} - t_{\text{in}})$ , due both to the increase of the third moment and the decrease of the variance with the total number of e-folds before  $t_{60}$ . Non-Gaussianity can then be larger by *orders of magnitude* in our scheme.

## 6. Discussion and conclusions

In this paper we used the stochastic inflation approach to address an important theoretical issue, namely that of estimating the possible influence of super-horizon perturbation modes on the statistics of cosmological perturbations on observable sub-horizon scales. To this end we first modified the standard scheme in order to allow for the cross-talk of perturbations on super- and sub-horizon scales. Such an effect is indeed totally absent in the traditional stochastic inflation dynamics; this is an artifact of the sharp  $k$ -space filter adopted in the coarse graining procedure, which is easily removed by adopting a smoother filter function, leading to a coloured—rather than white—noise source in the Langevin-like equation which governs the evolution of the coarse grained inflaton field. This modification implies that the evolution of the coarse grained inflaton field behaves as a non-Markovian stochastic process, in contrast to the standard case. Perturbations relevant to our smooth local patch of the universe are then consistently defined by constraining the inflaton field to be homogeneous at a conventional time  $t_{60}$  (i.e. about 60 e-folds before inflation ends), corresponding to the horizon crossing of a scale slightly larger than the present Hubble radius. As a consequence of its non-Markovianity, the coarse grained inflaton field preserves some memory of its dynamics prior to the constraint, which means that sub-horizon scale perturbations have some knowledge of the state of the universe on super-horizon scales.

Endowed with this *extended* stochastic inflation scheme we are finally able to calculate the conditional second- and third-order moments of inflaton perturbations on observable scales. We perform our calculations in the case of a simple model where a scalar field with a small cubic self-interaction term evolves in a fixed de Sitter background. Our most important findings are:

- The variance of inflaton fluctuations grows quadratically with time around  $t_{60}$ , and is therefore smaller than in the standard Markovian case, which is linearly dependent on time; this is equivalent to saying that the power spectrum of density perturbations gets bluer on very large scales, without invoking any ad hoc new physical input. This generic feature looks very intriguing, since the CMB anisotropy power on the largest angular scales observed by WMAP [41] appears to be lower than the one predicted by the standard model of cosmology with almost scale-invariant primordial perturbations arising from a period of inflation.
- The skewness of inflaton fluctuations is larger than the standard case at times around  $t_{60}$ , which is equivalent to an enhancement of the non-Gaussianity level on large scales, for a given value of the self-coupling strength. The possible presence of large scale non-Gaussianity is an important prediction to be confronted with current observational limits [42]. Let us also mention that some recent analyses have reported evidence for a positive detection of non-Gaussianity in the WMAP one-year data [43]–[47].

In spite of the simplicity of the model considered, we can conclude that the cross-talk between super- and sub-horizon scale perturbations is an important effect which would deserve an accurate treatment. The scheme presented here should be considered only as a first step in this direction, which needs to be substantially improved and extended in various ways. First of all, one should consistently include metric perturbations in the inflaton dynamics, going beyond our fixed de Sitter background treatment. Second, our choice of a Gaussian filter can be somewhat arbitrary, and it would be important to understand how much the results depend on its choice. The conclusions reached in [19], according to which there is a wide class of filters whose noise correlators show the same asymptotic behaviour  $\langle \xi(t)\xi(t') \rangle \propto e^{-2H|t-t'|}$  found in this paper (see equation (3.9)), are however quite encouraging in this respect. Actually, since the window function is only a technical device, we would expect our results to be independent of its choice, at least qualitatively. The white noise exception is not significant, being merely a consequence of the ‘bad’ choice of smoothing in configuration space.

The existence of memory effects means that the detailed dynamics of inflation plays a role in the specific form assumed by observable quantities. For instance, in our model one finds a residual dependence on the time when inflation started,  $t_{\text{in}}$ . Nonetheless, it is quite likely that this dependence saturates to a universal value determined by the general asymptotic behaviour, if the overall number of inflation e-folds is very large, as is usually the case. More difficult is to avoid a dependence on the precise time  $t_{60}$  at which we put the homogeneity constraint, and on the specific form of the constraint, which we introduced here through a delta function in field configuration space (see, however, [48] for a discussion of alternative approaches to the constraint). These are important issues which will certainly deserve further investigation.

## Appendix A: stochastic noise and the Langevin equation

The effective equation of motion for the *in-in* expectation value of the super-horizon field  $\varphi_{<}$  is obtained via a path integral over the sub-horizon field  $\varphi_{>}$ :

$$e^{i\Gamma[\phi_{<}^{\pm}]} = e^{i[S_{<}^{+} - S_{<}^{-}]} \int \mathcal{D}\phi_{>}^{\pm} \exp\left(i \int dx \left[ \frac{1}{2}\phi_{>}^{+}(x)\Lambda(x)\phi_{>}^{+}(x) - \frac{1}{2}\phi_{>}^{-}(x)\Lambda(x)\phi_{>}^{-}(x) + \phi_{<}^{+}(x)\Lambda(x)\phi_{>}^{+}(x) - \phi_{<}^{-}(x)\Lambda(x)\phi_{>}^{-}(x) \right]\right), \quad (\text{A.1})$$

where  $\Lambda$  is the integration kernel of the action for a free scalar field, given by

$$\Lambda(x) = -a_t^3 \left( \partial_t^2 + 3H\partial_t - \frac{\nabla^2}{a^2} + m^2 \right). \quad (\text{A.2})$$

After some manipulations [8], setting  $\phi_{<}^{\pm} = \varphi^{\pm}$ ,  $\varphi^{\Delta} = \phi_{<}^{+} - \phi_{<}^{-}$  and  $\varphi^C = (1/2)(\phi_{<}^{+} + \phi_{<}^{-})$ , the effective action reads

$$\Gamma[\varphi^{\pm}] = S[\varphi^{+}] - S[\varphi^{-}] + \frac{i}{2} \int d^4x d^4x' \varphi^{\Delta}(x) \text{Re}[\Pi(x, x')] \varphi^{\Delta}(x') - 2 \int d^4x d^4x' \theta(t - t') \varphi^{\Delta}(x) \text{Im}[\Pi(x, x')] \varphi^C(x'), \quad (\text{A.3})$$

where

$$\Pi(x, x') \equiv \int d\mathbf{k} a_t^3 \hat{P}_t \phi_{\mathbf{k}}(x) a_{t'}^3 \hat{P}_{t'} \phi_{\mathbf{k}}^*(x'), \quad \hat{P}_t = \ddot{W} + 3H\dot{W} + 2\dot{W}\partial_t \quad (\text{A.4})$$

and the normal modes  $\phi_{\mathbf{k}}$  of the field are given by (2.5).

Thanks to the useful relations  $a_t^3 \hat{P}_t = \partial_t(a_t^3 \dot{W}_t(k)) + 2a_t^3 \dot{W}_t(k) \partial_t$  and

$$\dot{\phi}_{\mathbf{k}} = -H \left( \frac{3}{2} - \nu + \frac{k}{aH} \frac{H_{\nu-1}^{(1)}(k/aH)}{H_{\nu}^{(1)}(k/aH)} \right) \phi_{\mathbf{k}} \equiv q_{\nu}(k\eta) \phi_{\mathbf{k}}, \quad (\text{A.5})$$

after integrating by parts, the imaginary term of the effective action (A.3) reads

$$\begin{aligned} & \frac{i}{2} \int d^4x d^4x' a_t^3 a_{t'}^3 \text{Re} \int d\mathbf{k} [\varphi^{\Delta}(x) q_{\nu}(k\eta) - \dot{\varphi}^{\Delta}(x)] \dot{W}_t(k) \Phi_{\mathbf{k}}(x) \\ & \times \dot{W}_{t'}(k) \Phi_{\mathbf{k}}^*(x') [\varphi^{\Delta}(x') q_{\nu}^*(k\eta') - \dot{\varphi}^{\Delta}(x')]. \end{aligned} \quad (\text{A.6})$$

Assuming  $W_t(k) = W(k\eta)$ , then  $\dot{W}_t = (k/a_t)W'(k\eta)$ , where the prime denotes differentiation with respect to the argument of  $W$ , and with this substitution we obtain

$$\dot{W}_t(k) \Phi_{\mathbf{k}}(x) \dot{W}_{t'}(k) \Phi_{\mathbf{k}}^*(x') = \frac{k^2 e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{32\pi^2 H} W'(k\eta) W'(k\eta') \frac{H_{\nu}^{(1)}(k/a_t H) H_{\nu}^{(1)*}(k/a_{t'} H)}{(a_t a_{t'})^{5/2}}; \quad (\text{A.7})$$

we now set  $\psi^{\Delta}(x) = ((3/2) - \nu)\phi_{\geq}^{\Delta}(x) + \dot{\phi}_{\geq}^{\Delta}(x)/H$ , and from the explicit form (A.5) of  $q_{\nu}$  (after evaluating the integral over angles) we can write the integral over  $d\mathbf{k}$  in a matrix form as

$$[\psi^{\Delta}(x), \varphi^{\Delta}(x)] \frac{H}{8\pi} \int dk k^4 \frac{\sin kr}{kr} \frac{W'(k\eta) W'(k\eta')}{(a_t a_{t'})^{5/2}} \text{Re}[\mathbf{M}_{\nu}(k\eta, k\eta')] \begin{bmatrix} \psi^{\Delta}(x') \\ \varphi^{\Delta}(x') \end{bmatrix}, \quad (\text{A.8})$$

with the matrix  $M_{\nu}^{i,j}(k\eta, k\eta')$  given by (2.13). This matrix is Hermitian under the simultaneous exchange of discrete and continuous indices  $i, t \rightarrow j, t'$ , and  $\text{Re}[\mathbf{M}_{\nu}]$  is therefore symmetric. Equation (A.6) can thus be rewritten in the bilinear and symmetric form

$$\frac{i}{2} \int d^4x d^4x' a_t^3 a_{t'}^3 [\psi^{\Delta}(x), \varphi^{\Delta}(x)] \mathbf{A}(x, x') \begin{bmatrix} \psi^{\Delta}(x') \\ \varphi^{\Delta}(x') \end{bmatrix}, \quad (\text{A.9})$$

where  $\mathbf{A}(x, x')$  is the correlation matrix (2.12); introducing two classical random fields  $\xi_1$  and  $\xi_2$  with the statistical weight (2.11), for the imaginary part of the effective action (A.3) we finally get

$$e^{-(1/2) \int d^4x d^4x' \phi_{\geq}^{\Delta}(x) \text{Re}[\Pi(x, x')] \phi_{\geq}^{\Delta}(x')} = \int \mathcal{D}\xi_1 \mathcal{D}\xi_2 P[\xi_1, \xi_2] e^{i \int d^4x a_t^3 H [\psi^{\Delta}(x) \xi_1(x) + \varphi^{\Delta}(x) \xi_2(x)]}, \quad (\text{A.10})$$

from which we immediately obtain (2.9).

## Appendix B: dissipation

In the Langevin theory of Brownian motion, the stochastic force arising as a consequence of the collisions with the particles of the thermal bath is usually split into two parts, one rapidly varying and stochastically distributed and another one proportional to the particle velocity, which plays the role of a friction term. The latter contribution describes

how fast the system reaches thermal equilibrium from an out-of-equilibrium configuration, making the mean velocity vanish exponentially and driving the mean kinetic energy to the equipartition value. The proportionality coefficient of this friction term (and therefore the characteristic relaxation time needed for the system to reach equilibrium and for the friction to become negligible) can be derived by integrating over time the correlation function of the rapidly varying part of the stochastic force. This relation between the macroscopic out-of-equilibrium behaviour and the microscopic equilibrium distribution is known as the *fluctuation-dissipation* theorem.

We then expect in our situation too a dissipation term of the type  $\alpha H \dot{\varphi}$  to show up in the equation of motion, with the coefficient  $\alpha$  related in some way to the correlation functions of the noise  $\xi$ . We also expect this effect to be negligible after a certain time.

In section 2 we disregarded the real term that appears in the action with the path integral over the sub-horizon degrees of freedom (the  $\text{Im}[\Pi]$  term in (A.3)). Had we kept this term, after integrating by parts with respect to  $t'$  we would have obtained in the equation of motion the two extra contributions

$$\int d^4x' a_{t'}^3 \vartheta(t-t') \zeta(x, x') \dot{\varphi}(x') + \int d^4x' a_{t'}^3 \delta m^2(x, x') \varphi(x'), \quad (\text{B.1})$$

corresponding to non-local effects of dissipation and mass renormalization respectively, with the integration kernels given by

$$\zeta(x, x') = 2 \text{Im} \int d\mathbf{k} \hat{P}_t \phi_{\mathbf{k}}(x) \dot{W}_{t'} \phi_{\mathbf{k}}^*(x'), \quad (\text{B.2})$$

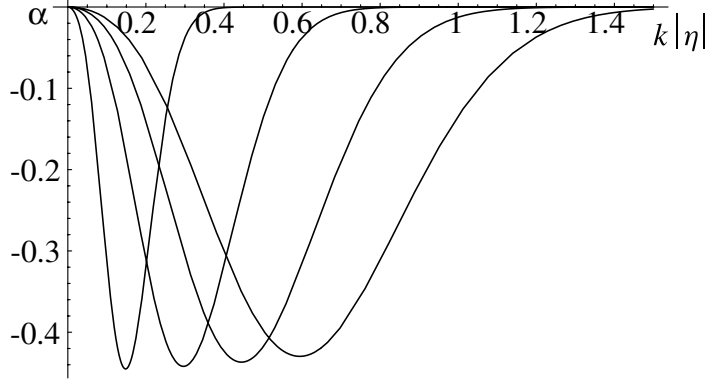
$$\begin{aligned} \delta m^2(x, x') = & -4\delta(t-t') \text{Im} \int d\mathbf{k} \dot{W}_t \dot{\phi}_{\mathbf{k}}(x) \dot{W}_{t'} \phi_{\mathbf{k}}^*(x') \\ & -2\vartheta(t-t') \text{Im} \int d\mathbf{k} \hat{P}_t \phi_{\mathbf{k}}(x) \dot{W}_{t'} \dot{\phi}_{\mathbf{k}}^*(x'), \end{aligned} \quad (\text{B.3})$$

where in  $\delta m^2$  a second term proportional to  $\delta(t-t') \phi_{\mathbf{k}}(x) \phi_{\mathbf{k}}^*(x')$  was dropped out since it gives a purely real result.

We will concentrate on the first term of (B.1), showing that it is negligible compared to the usual friction term  $3H \dot{\varphi}_{\mathbf{k}}$  related to the Hubble expansion. Since it has no space dependence but  $e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}' )}$ , going to Fourier space we have

$$\begin{aligned} \int_{-\infty}^t dt' a_{t'}^3 \frac{H^5}{k^3} \frac{k^4 \eta^2 \eta'^2}{\sigma^4} \text{Im} \left[ \exp\left(-\frac{(k\eta)^2}{2\sigma^2} - ik\eta\right) \exp\left(-\frac{(k\eta')^2}{2\sigma^2} + ik\eta'\right) \right. \\ \left. \times \left( (1 + ik\eta) \left(1 + \frac{k^2 \eta^2}{\sigma^2}\right) - 2k^2 \eta^2 \right) (1 - ik\eta') \right] \dot{\varphi}_{\mathbf{k}}(t'). \end{aligned} \quad (\text{B.4})$$

The field  $\varphi$  basically contains super-horizon modes, which are slowly varying with respect to the characteristic correlation time of the sub-horizon fluctuations. We can then assume that  $\dot{\varphi}_{\mathbf{k}}(t')$  is constant where the rest of the integrand is significantly different from zero, and take it out from the integration. We thus obtain a dissipation term  $\alpha_{\mathbf{k}} H \dot{\varphi}_{\mathbf{k}}(t)$  for the  $k$  mode, with a time dependent friction coefficient  $\alpha_{\mathbf{k}} = \alpha(k\eta, \sigma)$ . Changing variables, it



**Figure 5.** The friction coefficient  $\alpha(k\eta, \sigma)$  governing the fluctuation-induced dissipation effect for the  $k$  mode, for different values of  $\sigma$  (from left to right,  $\sigma = 0.125, 0.25, 0.375, 0.5$ ). The sign of the coefficient is always negative (meaning energy gain), and all functions are peaked roughly around the cut-off scale of the mode, for  $k|\eta| \sim \sigma$ .

becomes

$$\alpha(k\eta, \sigma) = \frac{k^2\eta^2}{\sigma^4} \text{Im} \left[ \exp\left(-\frac{(k\eta)^2}{2\sigma^2} - ik\eta\right) \left( (1 + ik\eta) \left(1 + \frac{k^2\eta^2}{\sigma^2}\right) - 2k^2\eta^2 \right) \times \int_{-\infty}^{k\eta} dx \exp\left(-\frac{x^2}{2\sigma^2} + ix\right) \frac{1 - ix}{x^2} \right]; \quad (\text{B.5})$$

this fluctuation-induced dissipation effect can then be neglected when  $|\alpha(k\eta, \sigma)| \ll 3$ . For small values of  $k|\eta|$  (sufficiently after the horizon crossing of each mode), this inequality reduces to

$$\frac{k^2\eta^2}{\sigma^2} \left( \frac{1}{3} - \frac{\sigma^2}{3 \cdot 5} + \frac{\sigma^4}{3 \cdot 5 \cdot 7} + \dots \right) \ll 3, \quad (\text{B.6})$$

and is always satisfied for  $k|\eta| \lesssim \sigma \leq 1$ .

The complete behaviour of the friction coefficient  $\alpha$  is shown in figure 5, where we see that for each mode the dissipative behaviour reaches its maximum for  $k|\eta| \sim \sigma$ , when the mode crosses the effective horizon  $(\sigma H)^{-1}$ ; the spatial non-locality of the dissipation term in (B.1) is a consequence of this effectiveness of the friction coefficient only about the cut-off scale for each mode [8]. We also note that the sign is always negative: this ‘anti-dissipation’ means for  $\varphi$  a global energy gain, instead of a loss, and is due to the continuous incoming of modes through the cut-off into the super-horizon field. Anyway, this effect can be neglected on a first approach, since  $\alpha/3$  is always much less than unity.

### Appendix C: the noise correlation matrix

The noise correlation matrix  $A_{ij}(x, x')$  can be calculated in the Gaussian window case (3.6). Since the relation

$$\sin(kr) \text{Re}[\mathbf{M}_\nu] = \frac{1}{2} \text{Im}[\mathbf{M}_\nu(e^{ikr} - e^{-ikr})] \quad (\text{C.1})$$

holds, equation (2.12) then becomes (setting  $\delta = \eta' - \eta$  and  $R \equiv \sqrt{\eta^2 + \eta'^2}$ )

$$\begin{aligned}
 \mathbf{A}(x, x') &= \frac{H^6}{8\pi^2 r} (\eta\eta')^3 \text{Im} \int_0^\infty dk k^5 \frac{e^{-(\eta^2 + \eta'^2/2\sigma^2)k^2}}{\sigma^4} \\
 &\quad \times \begin{bmatrix} \frac{(k\eta - i)(k\eta' + i)}{\eta\eta'k^3} & \eta' \frac{1 + ik\eta}{k\eta} \\ \frac{1 - ik\eta'}{\eta \frac{k\eta'}{k\eta'}} & \eta\eta'k \end{bmatrix} (e^{i(\delta+r)k} - e^{i(\delta-r)k}) \\
 &= \frac{H^6(\eta\eta')^3}{8\pi^2 r \sigma^4} [\partial] \frac{\partial^2}{\partial \delta^2} \text{Im} \int_0^\infty dk e^{-(\eta^2 + \eta'^2/2\sigma^2)k^2} (e^{i(\delta+r)k} - e^{i(\delta-r)k}) \\
 &= \frac{\sqrt{2}H^6(\eta\eta')^3}{8\pi^2 R r \sigma^3} [\partial] \frac{\partial^2}{\partial \delta^2} \left[ \exp(-(\delta+r)^2\sigma^2/2R^2) \int_0^{(\delta+r)\sigma/\sqrt{2}R} dy e^{y^2} \right. \\
 &\quad \left. - \exp(-(\delta-r)^2\sigma^2/2R^2) \int_0^{(\delta-r)\sigma/\sqrt{2}R} dy e^{y^2} \right], \tag{C.2}
 \end{aligned}$$

where we have defined the derivative matrix

$$[\partial] = \begin{bmatrix} \frac{\partial^2}{\partial \delta^2} + \frac{1}{\eta\eta'} \left( \delta \frac{\partial}{\partial \delta} - 1 \right) & \frac{\eta'}{\eta} \left( 1 + \eta \frac{\partial}{\partial \delta} \right) \frac{\partial^2}{\partial \delta^2} \\ \frac{\eta}{\eta'} \left( 1 - \eta' \frac{\partial}{\partial \delta} \right) \frac{\partial^2}{\partial \delta^2} & -\eta\eta' \frac{\partial^4}{\partial \delta^4} \end{bmatrix}. \tag{C.3}$$

For  $r \rightarrow 0$  we have

$$\begin{aligned}
 \mathbf{A}(t, t')_{r=0} &= -\frac{\sqrt{2}H^6(\eta\eta')^3}{4\pi^2 R^2 \sigma^2} [\partial] \frac{\partial^2}{\partial \delta^2} \left[ \frac{\delta\sigma}{R} \exp(-\delta^2\sigma^2/2R^2) \int_0^{\delta\sigma/\sqrt{2}R} dy e^{y^2} \right] \\
 &= -\frac{H^6}{4\pi^2} \frac{(\eta\eta')^3}{R^4} [\partial] 2 \sum_{k=0}^\infty \frac{(-1)^k (k+1)}{(2k-1)!!} \left( \frac{\delta\sigma}{R} \right)^{2k} \\
 &= \frac{H^6}{4\pi^2} \left( \frac{2\eta\eta'}{\eta^2 + \eta'^2} \right)^2 \sum_{k=0}^\infty \mathcal{A}^{(k)}(t-t') \frac{(-1)^k (k+1)(k+2)}{(2k-1)!!} \left( \frac{(\eta' - \eta)^2}{\eta^2 + \eta'^2} \sigma^2 \right)^k, \tag{C.4}
 \end{aligned}$$

( $\mathcal{A}^{(k)}(t-t')$  is given in (3.8)), and since in de Sitter space  $\eta = -e^{-Ht}/H$  we get (3.7).

## Appendix D: conditional probability

As a shorthand notation, to avoid writing all the integration symbols, we define the inner product of two functions  $f(x)$  and  $g(x)$  as

$$f^T g = \int_{-\infty}^{+\infty} d^4x f(x)g(x); \tag{D.1}$$

in this notation the equations of motion (5.3) and (5.4) become

$$\delta\varphi[\xi](t) = \delta\varphi_{60} - \frac{\mu}{6H} \delta\varphi_{60}^2(t-t_{60}) + J_1^T \xi - \frac{\mu}{6H} \xi^T \mathbf{B}_1 \xi, \tag{D.2}$$

$$\delta\varphi[\xi](t_{60}) = J_2^T \xi - \frac{\mu}{6H} \xi^T \mathbf{B}_2 \xi, \tag{D.3}$$

The influence of super-horizon scales on cosmological observables generated during inflation

with (the function  $\vartheta(t < t' < t'')$  is defined to be 1 if the inequality is true, 0 elsewhere)

$$J_1(x') = \vartheta(t_{60} < t' < t)\delta(\mathbf{x} - \mathbf{x}'), \quad J_2(x') = \vartheta(t_{\text{in}} < t' < t_{60})\delta(\mathbf{x} - \mathbf{x}'), \quad (\text{D.4})$$

$$\mathbf{B}_1(x', x'') = \delta(\mathbf{x} - \mathbf{x}')\delta(\mathbf{x} - \mathbf{x}'') \int_{t_{60}}^t d\tilde{t} \vartheta(t_{60} < t' < \tilde{t})\vartheta(t_{\text{in}} < t'' < \tilde{t}), \quad (\text{D.5})$$

$$\mathbf{B}_2(t', t'') = \delta(\mathbf{x} - \mathbf{x}')\delta(\mathbf{x} - \mathbf{x}'') \int_{t_{\text{in}}}^{t_{60}} d\tilde{t} \vartheta(t_{\text{in}} < t' < \tilde{t})\vartheta(t_{\text{in}} < t'' < \tilde{t}). \quad (\text{D.6})$$

Since we want a probability distribution for connected correlation functions, as discussed in (4.4) we subtract the mean of the perturbative solution, which in this notation reads

$$\langle \delta\varphi[\xi](t) \rangle = -\frac{\mu}{6H} \text{Tr}[\mathbf{A}\mathbf{B}_1], \quad \langle \delta\varphi[\xi](t_{60}) \rangle = -\frac{\mu}{6H} \text{Tr}[\mathbf{A}\mathbf{B}_2]. \quad (\text{D.7})$$

If again we set  $\delta\varphi_1 \equiv \delta\varphi - \delta\varphi_{60} + \frac{\mu}{6H}\delta\varphi_{60}^2(t - t_{60})$  and  $\delta\varphi_2 \equiv \delta\varphi_{60}$ , we get for the joint probability (5.1)

$$\begin{aligned} P(\delta\varphi, \delta\varphi_{60}) &= \int \frac{d\alpha_1 d\alpha_2}{(2\pi)^2} \exp\left(i\left(\frac{\mu}{6H} \text{Tr}[\mathbf{A}\mathbf{B}_i] - \delta\varphi_i\right)\alpha_i\right) \mathcal{N} \\ &\times \int \mathcal{D}\xi \exp\left(-\frac{1}{2}\xi^T \mathbf{A}^{-1} \left(\mathbf{1} + i\frac{\mu}{3H}\alpha_k \mathbf{A}\mathbf{B}_k\right) \xi + i\alpha_i J_i^T \xi\right) \\ &= \int \frac{d\alpha_1 d\alpha_2}{(2\pi)^2} \exp\left(i\left(\frac{\mu}{6H} \text{Tr}[\mathbf{A}\mathbf{B}_i] - \delta\varphi_i\right)\alpha_i\right) \mathcal{N}' \\ &\times \exp\left(-\frac{1}{2}\alpha_i J_i^T \left(\mathbf{1} + i\frac{\mu}{3H}\alpha_k \mathbf{A}\mathbf{B}_k\right)^{-1} \mathbf{A} J_j \alpha_j\right), \end{aligned} \quad (\text{D.8})$$

where the new normalization  $\mathcal{N}'$  reads

$$\begin{aligned} \mathcal{N}' &= \frac{\int \mathcal{D}\xi e^{-(1/2)\xi^T \mathbf{A}^{-1} [\mathbf{1} + (i\mu/3H)\alpha_k \mathbf{A}\mathbf{B}_k] \xi}}{\int \mathcal{D}\xi e^{-(1/2)\xi^T \mathbf{A}^{-1} \xi}} \\ &= \text{Det} \left[ \mathbf{1} + \frac{i\mu}{3H} \alpha_k \mathbf{A}\mathbf{B}_k \right]^{-1/2} \equiv e^{-(1/2)\text{Tr} \ln [\mathbf{1} + \frac{i\mu}{3H} \alpha_k \mathbf{A}\mathbf{B}_k]}. \end{aligned} \quad (\text{D.9})$$

To first order in  $\mu/6H$ , the normalization  $\mathcal{N}'$  and the mean value of the field cancel out, and we have

$$P(\delta\varphi, \delta\varphi_{60}) = \int \frac{d\alpha_1 d\alpha_2}{(2\pi)^2} \exp(-i\delta\varphi_i \alpha_i) \exp\left(-\frac{1}{2} J_i^T \mathbf{A} J_j \alpha_i \alpha_j + i\frac{\mu}{6H} J_i^T \mathbf{A}\mathbf{B}_j \mathbf{A} J_k \alpha_i \alpha_j \alpha_k\right); \quad (\text{D.10})$$

normalizing the previous result for the joint probability with the probability distribution (5.2) for  $\delta\varphi_{60}$ , that becomes

$$P(\delta\varphi_{60}) = \int \frac{d\alpha_2}{2\pi} \exp(-i\delta\varphi_2 \alpha_2) \exp\left(-\frac{1}{2} J_2^T \mathbf{A} J_2 \alpha_2^2 + i\frac{\mu}{6H} J_2^T \mathbf{A}\mathbf{B}_2 \mathbf{A} J_2 \alpha_2^3\right), \quad (\text{D.11})$$

and making the notation uniform with the one adopted in section 5 (with  $J_i^T \mathbf{A} J_j = C_{ij}$  and  $J_i^T \mathbf{A}\mathbf{B}_j \mathbf{A} J_k = D_{ijk}$ ), we obtain the conditional probability distribution (5.5).

## Appendix E: $D_{ijk}$ coefficients for the Gaussian window

We perform here the explicit calculation (in the small  $\sigma$  limit) of the  $D_{ijk}$  coefficients appearing in the conditional probability of section 5, expanding in the limit  $t \rightarrow t_{60}$  and  $t_{60} - t_{\text{in}} \gg H^{-1}$ :

$$D_{111} = - \left( \frac{H}{2\pi} \right)^4 \frac{1}{4H} \int_{\eta_{60}}^{\eta} \frac{d\tilde{\eta}}{\tilde{\eta}} \left[ \ln \frac{(\eta^2 + \eta_{60}^2)(\tilde{\eta}^2 + \eta_{60}^2)}{(\eta^2 + \tilde{\eta}^2)2\eta_{60}^2} \right]^2 \simeq \frac{H^8}{12} \frac{1}{(2\pi)^4} (t - t_{60})^5, \quad (\text{E.1})$$

$$\begin{aligned} D_{112} = D_{211} &= - \left( \frac{H}{2\pi} \right)^4 \frac{1}{4H} \int_{\eta_{60}}^{\eta} \frac{d\tilde{\eta}}{\tilde{\eta}} \ln \frac{(\eta^2 + \eta_{60}^2)(\tilde{\eta}^2 + \eta_{60}^2)}{(\eta^2 + \tilde{\eta}^2)2\eta_{60}^2} \ln \frac{2\eta_{60}^2(\tilde{\eta}^2 + \eta_{\text{in}}^2)}{(\eta^2 + \tilde{\eta}^2)(\eta_{60}^2 + \eta_{\text{in}}^2)} \\ &\simeq \frac{H^7}{12} \frac{1}{(2\pi)^4} (t - t_{60})^4, \end{aligned} \quad (\text{E.2})$$

$$\begin{aligned} D_{122} = D_{221} &= - \left( \frac{H}{2\pi} \right)^4 \frac{1}{4H} \int_{\eta_{\text{in}}}^{\eta_{60}} \frac{d\tilde{\eta}}{\tilde{\eta}} \ln \frac{(\eta_{60}^2 + \eta_{\text{in}}^2)(\tilde{\eta}^2 + \eta_{\text{in}}^2)}{(\eta_{60}^2 + \tilde{\eta}^2)2\eta_{\text{in}}^2} \ln \frac{(\eta^2 + \eta_{\text{in}}^2)(\eta_{60}^2 + \tilde{\eta}^2)}{(\eta^2 + \tilde{\eta}^2)(\eta_{60}^2 + \eta_{\text{in}}^2)} \\ &\simeq \frac{H^4}{4} \frac{1}{(2\pi)^4} (t - t_{60})(t_{60} - t_{\text{in}}) \ln 2, \end{aligned} \quad (\text{E.3})$$

$$D_{121} = \frac{H^3}{8} \left( \frac{1}{2\pi} \right)^4 \int_b^1 \frac{dy}{y} \left[ \ln \frac{(1+ab)(y+b)}{(y+ab)(1+b)} \right]^2 \simeq \frac{H^3}{4} \frac{1}{(2\pi)^4} (t - t_{60})^2 \left( \frac{\ln 2}{2} - 1 \right), \quad (\text{E.4})$$

$$D_{212} = \frac{H^3}{8} \left( \frac{1}{2\pi} \right)^4 \int_b^1 \frac{dy}{y} \left[ \ln \frac{2(1+by)}{(1+y)(1+b)} \right]^2 \simeq \frac{H^6}{12} \left( \frac{1}{2\pi} \right)^4 (t - t_{60}), \quad (\text{E.5})$$

$$\begin{aligned} D_{222} &= \frac{H^3}{8} \left( \frac{1}{2\pi} \right)^4 \int_b^1 \frac{dy}{y} \left[ \ln \frac{(1+y)(1+b)}{2(y+b)} \right]^2 \\ &\simeq \frac{H^3}{(2\pi)^4} \left[ \frac{H^3(t_{60} - t_{\text{in}})^3}{3} - \frac{\ln 2}{2} H^2(t_{60} - t_{\text{in}})^2 + \left( \ln^2 2 - \frac{\pi^2}{6} \right) \frac{H(t_{60} - t_{\text{in}})}{4} \right]. \end{aligned} \quad (\text{E.6})$$

## References

- [1] Starobinsky A A, 1986 *Field Theory, Quantum Gravity and Strings* ed H J De Vega and N Sanchez, p 107
- [2] Rey S J, 1987 *Nucl. Phys. B* **284** 706 [SPIRES]
- [3] Goncharov A S, Linde A D and Mukhanov V F, 1987 *Int. J. Mod. Phys. A* **2** 561 [SPIRES]
- [4] Nakao K I, Nambu Y and Sasaki M, 1988 *Prog. Theor. Phys.* **80** 1041 [SPIRES]
- [5] Nambu Y and Sasaki M, 1989 *Phys. Lett. B* **219** 240 [SPIRES]
- [6] Nambu Y, 1989 *Prog. Theor. Phys.* **81** 1037 [SPIRES]
- [7] Linde A D, Linde D A and Mezhlumian A, 1994 *Phys. Rev. D* **49** 1783 [SPIRES]
- [8] Morikawa M, 1990 *Phys. Rev. D* **42** 1027 [SPIRES]
- [9] Hu B L and Sinha S, 1995 *Phys. Rev. D* **51** 1587 [SPIRES]
- [10] Polarski D and Starobinsky A A, 1996 *Class. Quantum Grav.* **13** 377 [SPIRES]
- [11] Habib S, 1992 *Phys. Rev. D* **46** 2408 [SPIRES]
- [12] Calzetta E and Hu B L, 1995 *Phys. Rev. D* **52** 6770 [SPIRES]
- [13] Bellini M, Casini H, Montemayor R and Sisterna P, 1996 *Phys. Rev. D* **54** 7172 [SPIRES]
- [14] Matacz A, 1997 *Phys. Rev. D* **55** 1860 [SPIRES]
- [15] Kiefer C, Lesgourgues J, Polarski D and Starobinsky A A, 1998 *Class. Quantum Grav.* **15** L67 [SPIRES]
- [16] Risken H, 1984 *The Fokker-Planck Equation: Methods of Solution and Applications* (Berlin: Springer)
- [17] Chandrasekhar S, 1943 *Rev. Mod. Phys.* **15** 1 [SPIRES]
- [18] Casini H, Montemayor R and Sisterna P, 1999 *Phys. Rev. D* **59** 063512 [SPIRES]
- [19] Winitzki S and Vilenkin A, 2000 *Phys. Rev. D* **61** 084008 [SPIRES]

- [20] Hu B L, Paz J P and Zhang Y H, 1992 *Chateau du Pont d'Oye 1992, Proceedings, The Origin of Structure in the Universe* ed E Gunzig and P Nardone, pp 227–51 [[gr-qc/9512049](#)]
- [21] Falk T, Rangarajan R and Srednicki M, 1993 *Astrophys. J.* **403** L1 [[SPIRES](#)]
- [22] Gangui A, Lucchin F, Matarrese S and Mollerach S, 1994 *Astrophys. J.* **430** 447 [[SPIRES](#)]
- [23] Gupta S, Berera A, Heavens A F and Matarrese S, 2003 *Phys. Rev. D* **66** 043510 [[SPIRES](#)]
- [24] Acquaviva V, Bartolo N, Matarrese S and Riotto A, 2003 *Nucl. Phys. B* **667** 119 [[SPIRES](#)]
- [25] Maldacena J, 2003 *J. High Energy Phys.* [JHEP05\(2003\)013](#) [[SPIRES](#)]
- [26] Bartolo N, Matarrese S and Riotto A, 2003 2004 *J. High Energy Phys.* [JHEP04\(2004\)006](#) [[SPIRES](#)]
- [27] Bartolo N, Matarrese S and Riotto A, 2004 *J. Cosmol. Astropart. Phys.* [JCAP01\(2004\)003](#) [[SPIRES](#)]
- [28] Matarrese S, Ortolan A and Lucchin F, 1989 *Phys. Rev. D* **40** 290 [[SPIRES](#)]
- [29] Salopek D S and Bond J R, 1991 *Phys. Rev. D* **43** 1005 [[SPIRES](#)]
- [30] Yi I, Vishniac E T and Mineshige S, 1991 *Phys. Rev. D* **43** 362 [[SPIRES](#)]
- [31] Yi I and Vishniac E T, 1992 *Phys. Rev. D* **45** 3441 [[SPIRES](#)]
- [32] Yi I and Vishniac E T, 1993 *Astrophys. J. Suppl.* **86** 333
- [33] Hodges H M, 1990 *Phys. Rev. Lett.* **64** 1080 [[SPIRES](#)]
- [34] Mollerach S, Matarrese S, Ortolan A and Lucchin F, 1991 *Phys. Rev. D* **44** 1670 [[SPIRES](#)]
- [35] Kofman L, Blumenthal G R, Hodges H and Primack J R, 1991 *ASP Conf. Ser.* **15** 339
- [36] Schwinger J, 1961 *J. Math. Phys.* **2** 407 [[SPIRES](#)]
- [37] Jordan R D, 1987 *Phys. Rev. D* **33** 444 [[SPIRES](#)]
- [38] Bartolo N, Matarrese S and Riotto A, 2001 *Phys. Rev. D* **64** 083514 [[SPIRES](#)]
- [39] Bartolo N, Matarrese S and Riotto A, 2002 *Phys. Rev. D* **65** 103505 [[SPIRES](#)]
- [40] Matarrese S, Verde L, Jimenez R, 2000 *Astrophys. J.* **541** 10 [[SPIRES](#)]
- [41] Spergel D N *et al*, 2003 *Astrophys. J. Suppl.* **148** 175
- [42] Komatsu E *et al*, 2003 *Astrophys. J. Suppl.* **148** 119
- [43] Chiang L Y, Naselsky P D, Verkhodanov O V and Way M J, 2003 *Astrophys. J.* **590** L65 [[SPIRES](#)]
- [44] Park C G, 2004 *Mon. Not. R. Astron. Soc.* **349** 313
- [45] Eriksen H K, Hansen F K, Banday A J, Gorski K M and Lilje P B, 2004 *Astrophys. J.* **605** 14 [[SPIRES](#)]
- [46] Copi C J, Huterer D and Starkman G D, 2003 *Preprint* [astro-ph/0310511](#)
- [47] Vielva P, Martinez-Gonzalez E, Barreiro R B, Sanz J L and Cayon L, 2003 *Preprint* [astro-ph/0310273](#)
- [48] Matarrese S, 1990 *6th Moriond Astrophysics Mtg: The Early Universe and Cosmic Structures*, ed J M Alimi *et al* p 21