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A PARAMETRIC APPROACH TO ELECTORAL SYSTEMS

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1. Introduction

The study of electoral systems has developed into a rich and diverse field of research with important contributions coming from several disciplines including political science, philosophy, mathematics, and economics. Such studies, inter alia, address issues of classification, investigate the direct and indirect effects of different electoral systems, and evaluate existing and ideal type electoral systems against a large number of normatively or analytically relevant criteria. All these research questions are conditional on the understanding of the actual working of electoral systems. Consequently, the literature is rich in work highlighting how different systems translate votes into seats. While comparative studies typically calculate the effects of variation in the electoral systems' individual components (v. [3], [4], [6], [7], [8], [9]), country studies mostly demonstrate how these components interact to produce the electoral outcomes (v. [1], [5] and the country chapters in [2]). Common to most such endeavour is detailed description of institutional detail. Clearly, this is essential for understanding individual cases, illustrating the empirical range of various manifestations of the individual components of electoral systems, and reaching out to political practitioners. In this paper we take a different perspective at describing how electoral systems work that puts the emphasis on parsimony.

We approach algorithmically the electoral system of the organs determining the (central) Government (executive power). The bases of the electoral systems are contemplated, without regard to details (e.g., the representation of Trentino-Alto Adige in the Italian Senate or the subtleties of the three-tier apportionment of seats in Austria). In a general (pre-mathematical) sense, algorithms are precise procedures designed to solve a problem, be it locating a book in library or, as in our case, allocating a given number of parliamentary seats among the contenders. An algorithm has to satisfy several criteria including universality (it must work in all specific applications), definiteness (its steps must be clearly defined in their sequence and content), and finiteness (it must terminate after a finite number of steps and deliver a result).

By reducing electoral systems to an algorithmic scheme, it becomes clear how every particular system is determined by the choice of very few parameters. In this sense, our analysis is in the wake of the seminal contribution of [8], where the stylized facts of reality are profiled, and the famous three "electoral law variables" (ballot structure, electoral district size and electoral formula) are considered.

2. General framework

We consider a constituency with n seats. There is a set S of eligible candidates; we suppose that the number of candidates, $|S|$, is greater than or equal to the magnitude of the constituency: $|S| \geq n$.

Each ballot contains an ordered list of at most m candidates. Let $S^{(m)}$ be the set of (ordered) lists formed by at most m different elements of S; according to the relevant electoral system, perhaps not all these lists are admissible to be chosen by the voters. Let $T \subseteq S^{(m)}$ be the set of admissible lists that may be written on a ballot. For example, in a closed list system, only the lists proposed by the parties that have met the requirements for candidacy are in T.

The larger is T, the more information is provided by the voter. The maximum information is provided if $m = |S|$ and $T = S^{(m)}$; for this to be practicable, the magnitude of the constituency should be small. The minimum information corresponds to closed lists.

After the election, the set P of valid ballots is obtained. After counting, the final outcome of the election in the constituency is the set E of elected candidates.

We summarize the basic notation:

- n: number of seats of the constituency
- S: set of eligible candidates (we assume $|S| \geq n$)
- $T \subseteq S^{(m)}$, set of admissible contents of ballots
  (where $S^{(m)}$ is the set of (ordered) lists formed by at most m different elements of S)
- P: set of valid ballots.
- E: set of elected candidates.

The counting process is iterative; at the end of each iteration one candidate is either elected or disqualified. At the beginning E contains no candidate (i.e. $E = \emptyset$). During the implementation of the algorithm, E increases (as the candidates are elected) and S decreases (as the candidates are elected or disqualified). Thus if candidate k is elected, then it is added to E and eliminated from S:

$$E \leftarrow E \cup \{k\},\ S \leftarrow S \sim \{k\}$$

If candidate h is disqualified, it is eliminated from S:

$$S \leftarrow S \sim \{h\}$$

In each iteration only the first candidate in the ballot (after all the candidates already elected or disqualified are disregarded) is considered for election. Given an eligible candidate $i \in S$, we denote by $P_i \subseteq P$ the set of those ballots in which i is the top candidate among those still in S (i.e. among those not yet elected or disqualified).

Every ballot $j \in P$ has a weight ρ_j. This weight is initially 1, and then the weight decreases as candidates listed in the ballot are elected.

In each iteration we begin by counting the votes x_i of every candidate $i \in S$, i.e. the weighted number of ballots in which i is the top candidate (among those not yet elected or disqualified):

$$x_i \leftarrow \sum_{j \in P_i} \rho_j\ ,\ \text{for every } i \in S$$

3. Algorithms without disqualification

In the simplest case, no disqualification of candidates is considered. In each iteration simply the candidate k with the highest weighted number of ballots is elected:

$$k \leftarrow \arg\max \{x_i, i \in S\}$$

Recall that when candidate k is elected the weights of all the ballots in $j \in P_k$ decrease; let us call c_k the reduction factor, i.e.:

$$\rho_j \leftarrow c_k \rho_j \text{ , for every } j \in P_k$$

A scheme of a family of algorithms follows; when the three parameters n, T and c_k are fixed, a concrete algorithm is determined. Every time an algorithm is run, it is applied to a particular *instance* of the problem; the instance is determined by the two data S and P. The outcome of the algorithm is the set of elected candidates E.

PRELIMINARY SCHEME OF ALGORITHMS

STRUCTURAL PARAMETERS: n, T
COUNTING PARAMETER: c_k
DATA OF THE INSTANCE: S, P
OUTCOME: E

STEP 1 (inicialization)
$E \leftarrow \emptyset$;
$\rho_j \leftarrow 1$ for every $j \in P$
STEP 2 (counting)
$x_i \leftarrow \sum_{j \in P_i} \rho_j$ for every $i \in S$
STEP 3 (election)
$k \leftarrow \arg\max \{x_i, i \in S\}$;
$E \leftarrow E \cup \{k\}$; $S \leftarrow S \sim \{k\}$;
$\rho_j \leftarrow c_k \rho_j$ for every $j \in P_k$;
if $|E| < n$ **then** go to step 2 **else** END

In particular, the scheme above covers all the traditional list electoral systems, where voters may only vote for closed lists proposed by political parties. Two possibilities deserve special consideration:

- Factor c_k depends only on the position of candidate k in the original party list. The systems based on the *highest average* method (see, e.g., [4]) are obtained just setting the parameter c_k with the adequate values. For example, if candidate k is in the p-th position in his party list, in the D'Hondt system

$$c_k = \frac{p}{p+1}$$

and in the Sainte-Laguë system

$$c_k = \frac{2p-1}{2p+1}$$

- Factor c_k depends only on the (weighted) number of ballots x_k. The systems based on the *largest remainder* method result as particular cases setting the parameter c_k adequately. Each one of these systems is based on a quota q defined from the total number $|P|$ of valid ballots cast; so $q = |P|/(n+1)$ is the Droop quota and $q = |P|/n$ is the Hare quota. Now all the systems are obtained by taking

$$c_k = \frac{x_k - q}{x_k}$$

Note that the first possibility (positional c_k) requires party list voting, whereas the second possibility does not have this restriction.

In real world party list systems, often an *admissibility threshold* is imposed: only the ballots containing lists with a number of votes greater than this threshold are considered (e.g., 3% in Spain[1] (*Congreso de los Diputados*), 4% in Austria (*Nationalrat*) or 5% in Germany (*Bundestag*)). On the other hand, it is sometimes possible for the voters to intervene in the setting of the order of candidates in the party list through an additional voting process; this is the case in Germany, where only after considering the so-called "first vote" the order of the candidates in the lists of parties is established[2].

In the Italian case the algorithm has to be applied (at least) twice. The parties are grouped into *coalitions* (a party not belonging to an explicit coalition forms a coalition in itself). Both for the Chamber of Deputies and the Senate the voter chooses a closed party list. Let's consider the apportionment of 617 of the 630 seats of the Chamber of Deputies (12 seats for the Italian residents abroad and 1 seat for the Valle d'Aosta follow special rules). Firstly the algorithm is applied to the coalitions (Hare quota, with admissibility thresholds of 10% for multi-party coalitions and 4% for "single-party coalitions"). If, as a result, the winning coalition has obtained at least 340 seats (55% of 617), the first phase is finished; otherwise, 340 seats are allotted to the winning coalition and the remaining 277 seats are divided among the remaining coalitions as before. Once the seats are assigned to the coalitions, the algorithm is applied again to divide the seats of every multi-party coalition among the constituting parties (Hare quota, with an admissibility threshold of 2% for each party). The system is similar for the Senate, but now the constituency is no longer the whole nation, but every region.

4. Algorithms with disqualification

Now we introduce the possibility of disqualification of candidates. At the end of each iteration one candidate is either elected or disqualified. Let k be the candidate with the highest weighted number of ballots, x_k. If x_k is greater than the eligibility (lower) threshold l, then k is elected. Otherwise the candidate h with the lowest number of ballots is disqualified:

$$h \leftarrow \arg\min \{x_i, i \in S\}$$

[1]Considering the magnitude of the constituencies, this threshold has very limited significance in the Spanish system (and it can be effective only in the constituencies of Madrid and Barcelona).

[2]The first vote is applied to single-seat constituencies (first-past-the post). Those so elected for each party are now to be considered at the top of the corresponding party list (taking into account that the candidates for the single-seat constituencies may also be candidates of their parties in the party lists). Those elected with the first vote remain so even if they are not elected with the algorithm; the size of the Bundestag is increased accordingly ("Überhangmandate").

Apart from this simple idea, the resulting scheme of algorithms is like that of the last section. When the four parameters n, T, c_k and l are fixed, a concrete algorithm is determined.

MAIN SCHEME OF ALGORITHMS

STRUCTURAL PARAMETERS: n, T
COUNTING PARAMETERS: l, c_k
DATA OF THE INSTANCE: S, P
OUTCOME: E

STEP 1 (inicialization)
$E \leftarrow \emptyset$;
$\rho_j \leftarrow 1$ for every $j \in P$
STEP 2 (counting)
$x_i \leftarrow \sum_{j \in P_i} \rho_j$ for every $i \in S$
STEP 3 (election)
$k \leftarrow \arg\max \{x_i, i \in S\}$;
if $x_k > l$
then $E \leftarrow E \cup \{k\}$; $S \leftarrow S \sim \{k\}$;
$\rho_j \leftarrow c_k \rho_j$ for every $j \in P_k$;
if $|E| < n$**then** go to step 2 **else** END
else go to step 4
STEP 4 (disqualification)
$h \leftarrow \arg\min \{x_i, i \in S\}$;
$S \leftarrow S \sim \{h\}$;
go to step 2

Note that the Preliminary Scheme of Algorithms is a particular case with $l = -\infty$. The Main Scheme of Algorithms covers also the traditional preferential electoral systems.

We consider, e.g., the Irish system in some detail. Here the voter provides the maximum of information: $m = |S|$ and $T = S^{(m)}$ (certainly the voters may refrain from listing all the candidates). The magnitude n of the constituencies is 3, 4 or 5. Now $l = |P|/(n+1)$, the Droop quota. In every iteration the election of a candidate is attempted; if no candidate can be elected in this iteration, one candidate is disqualified. If k is elected, the weight of the ballots in the set P_k (formed by the ballots in which k is the top candidate among those still in S) is decreased by the factor $c_k = \frac{x_k - l}{x_k}$. Alternatively, without using weights, a random sample (of the corresponding size) of ballots of P_k is taken, and the rest of the ballots of P_k is discarded. Either simple sampling or stratified sampling (with proportionate allocation) can be applied; in the latter case the strata are given by the candidate (in S) following k in the ballots. In the long run, weighting (called "Gregory method") and sampling are equivalent; in practice stratified sampling is used in Ireland.

Consider the following table, with $n = 3$ (the data are taken from [4]). In the first iteration, no candidate can be elected and O'Riordan is disqualified ($l = 8352$). Now his name plays no further part in all ballots, and thus the 3,796 ballots where he is in the top position are "transferred" according to the second preferences shown on them (in 287 ballots only the name of O'Riordan comes up;

these are "non-transferable ballots"). In the second iteration, Crowley is elected. Now $c_{CROWLEY} = \frac{166}{8518}$, and the algorithm applies the weight $\frac{166}{8518}$ to all ballots where Crowley is the first preference; alternatively, as in the table, 166 of these ballots are selected with stratified sampling (the apportionment for the 4 strata is given by the distribution of the 8,518 Crowley's ballots: 5,644 for Creed, 564 for Moynihan, 1,232 for Roche , and 1,078 not transferable), and the remaining 8,352 are put aside of the counting. Creed is elected in the third iteration, and Moynihan in the fourth one.

	1st PREF.	TRANSF.		TRANSF.	
Creed	7,037	+1,292	8,349	+110	8,469
Crowley	7,431	+1,087	8,518	-166	
Moynihan	7,777	+566	8,343	+11	8,354
O'Riordan	3,796	-3,796			
Roche	7,343	+564	7,907	+24	7,931
NON-TRANSF.		+287	287	+21	308
TOTAL	33,404				

In fact, when a candidate is elected a variant of the procedure above is used in Ireland. Stratified sampling is not implemented in P_k, but in a subset $V \subseteq P_k$. Thus when Crowley is elected, V is formed by those ballots transferred to Crowley after the disqualification of O'Riordan, excluding those having no preference after the names of O'Riordan and Crowley (non-transferable). The distribution of these 1,087 ballots is: 783 for Creed, 72 for Moynihan, 145 for Roche , and 87 not transferable. The proportions of Creed, Moynihan and Roche of the 1,087-87=1,000 ballots provide now the apportionment for the 3 strata, as shown in the table below.

	1st PREF.	TRANSF.		TRANSF.	
Creed	7,037	+1,292	8,349	+130	8,479
Crowley	7,431	+1,087	8,518	-166	8,352
Moynihan	7,777	+566	8,343	+12	8,355
O'Riordan	3,796	-3,796			
Roche	7,343	+564	7,907	+24	7,931
NON-TRANSF.		+287	287	+0	287
TOTAL	33,404				

When in the algorithm election and disqualification are carried out, there are usually non-transferable ballots. For simplicity, we have supposed that there are always enough transferable ballots for all seats to be allocated with the algorithm.

In a two-rounds electoral process, the algorithm has to be applied obviously twice. This is the case in France (*Assemblée Nationale*). The reader will find it trivial to carry out the relevant adjustments.

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