Kinks from Dynamical Systems: Domain Walls in a Deformed O(N) Linear Sigma Model

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Abstract

It is shown how a integrable mechanical system provides all the localized static solutions of a deformation of the linear O(N)-sigma model in two space-time dimensions. The proof is based on the Hamilton-Jacobi separability of the mechanical analogue system that follows when time-independent field configurations are being considered. In particular, we describe the properties of the different kinds of kinks in such a way that a hierarchical structure of solitary wave manifolds emerges for distinct N.

1 Introduction

We divide this Introduction into three parts: A. A brief history and the "state of the art". B. New developments and results to be presented in this work. C. Scenarios of possible physical applications.

A.

Kinks are solitary (non-dispersive) waves arising in several one-dimensional physical systems. Here, we shall focus on the relativistic theory of N-interacting scalar fields built on a space-time that is the (1+1)-dimensional Minkowski space $\mathbb{R}^{1,1}$. In this context, kinks are finite energy solutions of the Euler-Lagrange equations, such that the time-dependence is dictated by the Lorentz invariance: $\vec{\phi}_K(x,t) = \vec{\phi}_K\left(\frac{x-vt}{\sqrt{1-v^2}}\right)$. Thus, the search for kinks leads to the solving of a system of N-coupled non-linear ordinary differential equations and therefore becomes a very interesting problem in Mathematical Physics.

The study of topological defects began as an area of research in field theory by the mid-seventies; see [1]. It was immediately recognized that defects of the kink type are in one-to-one correspondence with the separatrix trajectories between the bounded and unbounded motion of a mechanical system, for which the motion equations precisely form the non-linear system of differential equations mentioned above. The equivalent mechanical system is also Lagrangian and thus automatically integrable if $N = 1$. For $N \geq 2$, complete integrability is generically non-guaranteed and the equivalence to a mechanical system is not useful. This circumstance has been emphasized by Rajaraman; see [2] pp. 23–24, and partially circumvented by himself: the trial orbit method allows one to guesstimate particular types of kink trajectories.

There are, nevertheless, theories with $N = 2$-coupled scalar fields such that the equivalent dynamical system is completely integrable. The prototype of this kind of system is the MSTB model: in [3] this was proposed in the context of the search for non-topological solitons with stability provided by a $U(1)$ internal symmetry. In Reference [4], the model was considered as a classical continuum approximation to a 1D crystal with a two-component order parameter and it was shown that the search for kinks in this system requires that a completely integrable dynamical system be addressed. Ito, in a seminal paper [5], showed the Hamilton-Jacobi separability of the system of non-linear differential equations. He found all the kink trajectories and explained a very peculiar kink energy sum rule.

Very rich manifolds of kinks were discovered in two $N = 2$ field theoretical models, close relatives to the MSTB system, in a recent research performed by the authors of the present work [6]. The investigation of kink properties in these models requires the analysis of the separatrix trajectories in two related dynamical systems which are type I and III respectively in the classification of Liouville bidimensional ($N = 2$) completely integrable systems, see [7].

In fact, on choosing between the four types of Liouville dynamical systems those that meet appropriate critical point structure, one builds an enormous list of related $N = 2$ field theoretical models exhibiting manifolds of kinks of growing complexity; see [8]. The rôle of these models can be understood by noticing that the MSTB system is a deformation of the $O(2)$-linear sigma model. Instead of spontaneous symmetry breaking of $O(2)$ by a degenerated S^1 vacuum manifold, the $O(2)$ symmetry group is explicitly broken to $\mathbb{Z}_2 \times \mathbb{Z}_2$ by a mass term; only invariance under $\phi_a \to (-1)^{\delta_{ab}}\phi_a$, for $b = 1, 2$, survives. From the point of view of quantum field theory, this deformation is very natural because in $(1+1)$-dimensions infrared divergences forbid the existence of Goldstone bosons, according to a theorem of Coleman [9]. Even if it is absent in the classical action, a mass term will be generated by quantum corrections.

We interpret this as follows: in the parameter space of the $N = 2$ relativistic scalar field theories invariant under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ with generators mentioned above, and potential energy of the form

$$U(\vec{\phi}) = \frac{1}{2}\left(\alpha_1\phi_1^2 + \alpha_2\phi_2^2 + \frac{\beta_1}{2}\phi_1^4 + \beta_{12}\phi_1^2\phi_2^2 + \frac{\beta_2}{2}\phi_2^4\right) + C$$

there are at least two distinguished points. There is a choice of coupling constants such that there is explicit $O(2)$ symmetry, which is spontaneously broken. This is the linear $O(2)$-sigma model. The other interesting point is the MSTB model where the explicit $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry generated by $\phi_a \to (-1)^{\delta_{ab}}\phi_a$, for $b = 1, 2$, breaks spontaneously to the $\mathbb{Z}_2$ subgroup generated by $\phi_2 \to -\phi_2$. The key observation is that the renormalization group flow induced by quantum corrections in the parameter space avoids the $O(2)$-sigma system and instead leads to the MSTB model, which also offers a variety of kinks. All the other field theoretical models exhibiting an abundant supply of kinks also correspond to deformations of $O(2)$-symmetric systems with potential energies that depend on higher powers of ϕ_1 and ϕ_2, [6], [8].

There are strong analogies with the Zamolodchikov c-theorem, [10]: deformations in the space of (1+1)-dimensional field theories leading from conformal to integrable systems are the most interesting ones. We meet an analogous finite dimensional situation: replace the (infinite dimensional) conformal group by the $O(2)$ group and integrability of one system with infinite degrees of freedom by integrability of a bidimensional mechanical system.

B.

This paper is devoted to investigating the kink solitary waves of the deformation of the linear $O(N)$-sigma model that generalize the MSTB system to the case of N-interacting scalar fields. Non-linear waves in relativistic field theories with $N \geq 3$ scalar fields were sketchily described for the first time in Reference [11]. In this work, we offer a detailed analysis of this issue. The following points merit emphasis:

a) The dynamical system that encodes the solitary waves of the model as separatrix trajectories has N first integrals in involution and hence is completely integrable. Passing from Cartesian to Jacobi elliptic coordinates in the "internal" space, $\mathbb{R}^N$, the dynamical system becomes Hamilton-Jacobi separable. All the kink trajectories, and hence all the solitary waves, are then found by a special choice of the separation constants.

b) Deep insight into the structure of the kink manifold is gained by focusing on the $N = 3$ case. There are three kinds of kinks: 1. A two-parameter family of topological kinks with three non-null components that are "generic", i.e. they are not fixed under the action of the $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ group generated by $\phi_a \to (-1)^{\delta_{ab}}\phi_a$, for $b = 1, 2, 3$. 2. Four one-parameter families of "enveloping" non-topological kinks, also with three non-null components. The four families are related through the action of one $\mathbb{Z}_2 \times \mathbb{Z}_2$ sub-group and, together, form the envelop of the separatrix trajectories. 3. All the solitary waves of the $N = 2$ MSTB model appear "embedded" twice; once in each plane containing the two ground states. Different $\mathbb{Z}_2$ sub-groups leave these embedded kinks invariant.

c) The structure of the kink manifold of the $O(N)$ system with both explicit and spontaneous symmetry breaking repeats the patterns shown in the $N = 2$ (MSTB) model and its generalization for $N = 3$. There are also generic, enveloping and embedded kinks, although when N increases the complexity of the kink manifold also increases. For instance, the $N-1$ kink manifold is embedded $N-1$ times in the manifold of kinks of the deformed linear $O(N)$-sigma model.

d) In a remarkable system obtained from the generalized MSTB model by also allowing asymmetries in the quartic terms of the potential, only the embedded and enveloping topological kinks living on singular edges survive as solitary wave solutions. In this system, proposed in Reference [12] for the $N = 2$ case, the energy of all the above topological kinks is exactly the same. Together with vacuum degeneration, there is therefore kink degeneration, a phenomenon that deserves further analysis.

C.

Solitary waves of the kind that we are to describe play an important rôle in condensed matter physics. Phase transitions characterized through order parameters of the vector type are understood in terms of the linear (or non linear) $O(N)$-sigma model. The order parameter is organized in the fundamental representation of $O(N)$ and the system becomes non-linear when this N-vector is forced to take its values in the coset space $M = O(N)/O(N-1)$. In $(1+1)$-dimensional space-time, kinks are accompanied by the fermion fractionization phenomenon [13]; this describes the continuous approximation to the bizarre behaviour of certain one-dimensional polymers such as poly-acetilene. When the spatial dimension is 3, as in the real world, kinks become domain walls which are thus related to theories involving spontaneous breaking of discrete symmetries. This happens in the hot Big Bang cosmology, where domain wall topological defects can be formed in a phase transition occurring in the expansion of the very early Universe; see [14]. More recently, domain walls have been characterized as BPS states of SUSY gluodynamics and the Wess-Zumino model, [15]. In all these cases there are sets of scalar fields, as in our system, that presents a variety of domain walls with different characteristics when seen from a 3-dimensional perspective.

In quantum field theory, the linear $O(N)$-sigma model describes systems with spontaneous symmetry breakdown to an $O(N-1)$ sub-group and $N-1$ Goldstone bosons in the particle spectrum. At the beginning of the sixties Gell-Mann and Levy analyzed low energy hadronic phenomenology by introducing an effective Lagrangian field theory of this type [16]. Besides becoming the central element of current algebra, linear sigma models also enter fundamental physics in the Higgs sector of gauge theories for elementary particle physics, see report [17] for a comprehensive review (of the perturbative sub-sector). For instance, the linear $O(4)$-sigma model corresponds to the Higgs sector of the electro-weak theory, while the $O(24) \times O(5)$ case provides the bosonic sector of the $SU(5)$ Grand Unified Theory.

Either considered on their own or forming part of Gauge theories, there are reasons to discuss deformations of the linear sigma model. In the phenomenological approach, pions are identified with the Goldstone bosons of the model; a deformation is then necessary to convert these massless excitations in pseudo-Goldstone particles, accounting for the pions light mass. Gauge theories are today found in the low energy limit of (fundamental) string theory. Even though deformations in the bosonic sector of gauge theories produced by small mass terms spoil renormalizability, the low energy features remain (almost) untouched and it is (almost) legitimate to trust them.

Here, we shall search for domain walls when these mild deformations are performed in the linear $O(N)$-sigma model. It is precisely in this kind of model where the cosmological problem of wall domination is avoided [18]. Moreover, the system has a rich manifold of topological and non-topological solitons, allowing for topological defects with "internal" structure and leading to the existence of defects inside defects, a situation that generalizes a proposal of Morris [19].

The organization of the paper is as follows: In Section §2 we discuss the particle spectrum of the deformed linear $O(N)$-sigma model as well as the manifold of the solitary wave solutions of the system. Section §3 is devoted to the $N = 3$ case, which is described in full detail. We describe the situation of the generalized MSTB model for any N in Section §4 and briefly discuss the phenomenon of kink degeneration in the Bazeia system. Finally, some conclusions are drawn and some new prospects opened in Section §5. An appendix on elliptic Jacobi coordinates is also offered.

2 Kinks in the deformed linear O(N)-sigma model

In a generic sense we understand "kinks" as the solitary waves of a relativistic (1+1)-dimensional scalar field theory. We shall stick to the standard definition of solitary waves; see [2] and [6]:

A solitary wave is a non-singular solution of the non-linear coupled field equations of finite energy such that their energy density has a space-time dependence of the form:

$$\varepsilon(\mathbf{x},t) = \varepsilon(\mathbf{x} - \mathbf{v}t)$$

where $\mathbf{v}$ is some velocity vector.

Given one N-component scalar field, which is a map from the $\mathbb{R}^{1,1}$ Minkowski space-time to $\mathbb{R}^N$, $\vec{\phi}(x,t) \equiv (\phi_1(x,t), \phi_2(x,t), \ldots, \phi_N(x,t))$, the dynamics of the system is governed by the action:

$$S = \int d^2y \left[\frac{1}{2} \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} - V(\vec{\phi}) \right]$$

Here, $\mu = 0,1$ are indices in the space-time and we shall use $a = 1,2,\ldots,N$ to label components of the field in the "internal" $\mathbb{R}^N$ space in such a way that $\vec{\phi} \cdot \vec{\phi} = \sum_{a=1}^{N} \phi_a \phi_a$. In $\mathbb{R}^{1,1}$ we choose the metric as $g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and the Einstein convention will be used throughout the paper only for the indices in $\mathbb{R}^{1,1}$. The potential energy density is:

$$V(\phi_1,\ldots,\phi_N) = \frac{\lambda^2}{4} \left(\vec{\phi} \cdot \vec{\phi} - \frac{m^2}{\lambda^2} \right)^2 + \sum_{a=1}^{N} \frac{\sigma_a^2}{4} \phi_a^2$$

where λ, m and σ_a are coupling constants of inverse length. The linear O(N)-sigma model corresponds to the case $\sigma_a = 0$, $\forall a$, which exhibits maximum O(N) symmetry. We shall focus on the deformation of this system, which is maximally non-isotropic in the harmonic terms, i.e. $\sigma_a \neq \sigma_b$, $\forall a \neq b$. Somehow, the deformation is natural from a quantum field theoretical vantage point as we shall explain later and, moreover, we shall stick to the range $\sigma_a^2 < m^2$, $\forall a$, in the parameter space because in this regime the structure of the kink manifold is richer.

Introducing non-dimensional variables $\phi_a \to \frac{m}{\lambda} \phi_a$, $y \to \frac{\sqrt{2}}{m} x$ and $\frac{\sigma_a^2}{m^2} \to \sigma_a^2$, we find our expression for the action to be:

$$S = \frac{m^2}{\lambda^2} \int d^2x \left[\frac{1}{2} \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} - V(\phi_1,\ldots,\phi_N) \right] \qquad (1)$$

$$V(\phi_1,\ldots,\phi_N) = \frac{1}{2} \left(\vec{\phi} \cdot \vec{\phi} - 1 \right)^2 + \sum_{a=1}^{N} \frac{1}{2} \sigma_a^2 \phi_a^2$$

2.1 Configuration space and particle spectrum

The Cauchy problem for the field equations

$$\Box \phi_a = -\frac{\partial V}{\partial \phi_a}; \quad a = 1,2,\ldots,N \qquad (2)$$

is fixed by choosing a “point” $\vec{\phi}(x;t_0)\in \mathrm{Maps}(\mathbb{R};\mathbb{R}^N)$ in the configuration space $\mathcal{C}$, and its “tangent”, $\dot{\vec{\phi}}(x;t_0)\in T_{\vec{\phi}}\mathrm{Maps}(\mathbb{R};\mathbb{R}^N)$, as initial conditions to solve the system (2) of non-linear PDE.

The configuration space itself is isomorphic to the space of finite energy static configurations; if $\vec{\phi}(x;t)=\vec{\phi}(x)$, $\mathcal{C}$ is the set of continuous maps $\vec{\phi}:\mathbb{R}\to\mathbb{R}^N$ ($\vec{\phi}\equiv(\phi_1,\dots,\phi_N)$) such that the (static) energy is finite:

$$E=\frac{m^3}{\lambda^2\sqrt{2}}\int dx\left\{\frac{1}{2}\frac{d\vec{\phi}}{dx}\cdot\frac{d\vec{\phi}}{dx}+V(\vec{\phi})\right\}<+\infty\;; \tag{3}$$

thus, $\mathcal{C}=\{\vec{\phi}(x)/E[\vec{\phi}]<+\infty\}$. $\vec{\phi}(x)\in\mathcal{C}$ only if $\vec{\phi}$ satisfies the asymptotic conditions:

$$\lim_{x\to\pm\infty}\frac{d\phi_a}{dx}=0\;;\qquad \lim_{x\to\pm\infty}\phi_a(x)=v_a\;;\ \forall a=1,\dots,N \tag{4}$$

where $\vec{v}\equiv(v_1,\dots,v_N)$ is a constant vector that belongs to the set $\mathcal{M}$ of vectors annihilating V. We assume, without loss of generality, the following ordering in the space of parameters: $\sigma_1=0<\sigma_2<\dots<\sigma_N<1$. $\mathcal{M}$ is thus formed by two vectors

$$\mathcal{M}=\left\{\vec{v}^{\,\pm}=(\pm1,0,\dots,0)\right\} \tag{5}$$

which are the absolute minima of V.

We refer to $\mathcal{M}$ as the vacuum manifold because in the quantum version of the theory points in $\mathcal{M}$ are the expectation values of the quantum field operators $\hat{\phi}_a$ at the ground states (“vacua”) of the system. The vacuum degeneration – i.e. the existence of more than one vector in $\mathcal{M}$ – is related to the breaking of symmetry. Besides two-dimensional Poincare invariance, there is a “internal” symmetry with respect to the discrete group $G=\mathbb{Z}_2\times\stackrel{N}{\dots}\times\mathbb{Z}_2=\mathbb{Z}_2^N$ generated by $\phi_a\to(-1)^{\delta_{ab}}\phi_a$, for $b=1,2,\dots,N$, $\forall a=1,\dots,N$. The vacuum manifold is the orbit of one element by the group action

$$\mathcal{M}=G/H_{\vec{v}}=\mathbb{Z}_2\;;\qquad H_{\vec{v}}\,\vec{v}=\vec{v}$$

$H_{\vec{v}}=\mathbb{Z}_2\times\stackrel{N-1}{\dots}\times\mathbb{Z}_2$ is the little group of the vacuum $\vec{v}$. The generators of $H_{\vec{v}}$ are the transformations $\phi_a\to(-1)^{\delta_{ab}}\phi_a$, for $b=1,2,\dots,N$, and $a=2,3,\dots,N$, so that $H_{\vec{v}}$ survives as a symmetry group when quantizing around $\vec{v}$. We can understand the internal parity group G as the discrete “gauge” symmetry: in (1+1)-dimensions no dynamical degrees of freedom related to gauge potentials appear.

Vectors in $\mathcal{M}$ are critical points of V satisfying $\left.\frac{\partial V}{\partial\phi_a}\right|_{\vec{\phi}=\vec{v}}=0$ and therefore constant solutions of the field equations (2). The plane wave expansion around $\vec{\phi}(x;t)=\vec{v}$

$$\phi_{v_a}(x;t)=v_a+\sum_k A_a(k)e^{i\omega t-ikx}$$

is a solution of (2) if the dispersion relation

$$\delta_{ab}\,\omega^2=\delta_{ab}k^2+M^2_{ab}(\vec{v})\;;\quad M^2_{ab}=\frac{\partial^2V}{\partial\phi_a\partial\phi_b}(\vec{v}) \tag{6}$$

holds.

In the quantum theory, these plane waves become the fundamental quanta with mass matrix $M^2_{ab}(\vec{v})$ and one reads the particle spectrum at a chosen critical point of V from (6). Because $\vec{v} \in \mathcal{M}$ are minima of V there are no negative eigenvalues of $M^2_{ab}(\vec{v})$ and the dependence on time of the plane waves around $\vec{v}$ is bounded: $e^{i\omega t}$. The choice of $\vec{v}$ as the starting point of the quantization procedure "spontaneously" breaks the symmetry $G = Z_2^N$ of the action to $H_{\vec{v}} = Z_2^{(N-1)}$, which is the remaining one that survives in the particle spectrum.

In our model, we read the particle spectrum from:

$$M^2(\vec{v}) = \frac{m^2}{2}\begin{pmatrix} 4 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_N^2 \end{pmatrix} \tag{7}$$

Considering this system as a physical description of the continuum approximation to a one-dimensional crystal with an N-component order parameter, the particle spectrum describes a single phase with N phonon branches. We see explicitly how the symmetry group $G = Z_2^N$ is "broken" by the choice of the $\vec{v}$ vacuum to the $H_{\vec{v}} = Z_2^{(N-1)}$ subgroup: the N phonon branches have different masses or "energy gaps". From the point of view of particle physics we can say that there are no tachyons; only a pseudo-Goldstone particle becomes a Goldstone boson if the corresponding σ_a goes to zero.

It is interesting to see the model as a member of the family characterized by the potential energy densities:

$$V = \frac{1}{2}\left(\sum_{a,b=1}^{N} \alpha_{ab}\,\phi_a\phi_b - \gamma^2\right)^2 + \sum_{a,b=1}^{N} \frac{\beta_{ab}}{2}\,\phi_a\phi_b$$

where α_{ab}; β_{ab} and γ^2 are "bare" non-dimensional parameters. Ultraviolet divergences are controlled by normal ordering in the quantum theory, but the need arises to introduce a renormalization 'point' μ^2, and the dependence of the renormalized parameters on μ^2 is determined by the renormalization group equation. One special solution, a specific renormalization group flow, might lead to the "point":

$$\alpha^R_{ab}(\mu^2) = \delta_{ab}; \quad \beta^R_{ab}(\mu^2) = 0; \quad \gamma^R(\mu^2) = 1$$

in the space of quantum field theory models in the family. This point is the linear $O(N)$-sigma model which has $G = O(N)$ as the (continuous) symmetry group. The vacuum orbit is, however, $\mathcal{M} = O(N)/O(N-1) = S^{N-1}$, the $(N-1)$-dimensional sphere, and thus there is no unbroken symmetry left: there are $N-1$ massless particles. If the only modification of the renormalized parameters is to allow for non-zero values of $\beta^R_{ab}(\mu^2)$; $a = b = r+1, \ldots, N$ there are still $r-1$ Goldstone bosons.

Coleman [9] established that in $(1+1)$-dimensions the infrared asymptotics of the two-point Green functions of a quantum scalar field forbids poles at $\omega^2 = k^2$; there are no Goldstone bosons in $(1+1)$-dimensions. It is thus impossible to reach the $O(N)$-sigma model or its deformation with the $O(r)$ symmetry spontaneously broken to $O(r-1)$ in the renormalization group flow. The closest admissible points are the models characterized by:

$$\alpha^R_{ab}(\mu^2) = \delta_{ab}; \quad \gamma^R(\mu^2) = 1; \ \beta^R_{ab} = 0; a \neq b$$

$$\sigma^R_{11}(\sigma^2) = 0 < \sigma^R_{22}(\sigma^2) = \sigma_2^2 \le \cdots \le \sigma^R_{NN}(\sigma^2) = \sigma_N^2 < 1$$

In this paper we shall focus on the case of maximal explicit symmetry breaking; i.e. when strict inequalities in the parameter space occur. Nevertheless, we shall comment on the allowed situation characterized by

$$\sigma_1^2 = 0 < \sigma_2^2 = \cdots = \sigma_{r_1}^2 < \sigma_{r_1+1}^2 = \cdots = \sigma_{r_2}^2 < \cdots < \sigma_{r_k+1}^2 = \cdots = \sigma_N^2 < 1$$

when there is degeneration in the spectrum but no Goldstone bosons. Note that the generators of the $O(r_1-1)\times O(r_2-r_1)\times\cdots\times O(N-r_k)$ symmetry sub-group are in the little group of the vacuum. The symmetry group is $G = Z_2\times O(r_1-1)\times O(r_2-r_1)\times\cdots\times O(N-r_k)$, $H_{\vec{v}} = O(r_1-1)\times O(r_2-r_1)\times\cdots\times O(N-r_k)$ and the vacuum orbit is $M = G/H_{\vec{v}} = Z_2$.

2.2 Configuration space topology: kinks and dynamical systems

The configuration space of the model is the union of topologically disconnected sectors: $C = \bigsqcup_{\alpha;\beta=1}^{2} C_{\alpha\beta}$; thus, $\pi_0(C) = Z_2\times Z_2$ and $|\pi_0(C)| = 4$ are respectively the zeroth-order homotopy group of C and its order. This comes from the asymptotic conditions (4) and the continuity of the time evolution. There are topological charges defined for each configuration in C as:

$$Q_a^T = \frac{1}{2}\int_{-\infty}^{\infty} dx \frac{d\phi_a}{dx} = \frac{1}{2}(\phi_a(+\infty;t) - \phi_a(-\infty;t))$$

It should be noted that Q_a^T is independent of t, $\forall a$, and in our system equal to zero if $a\ge 2$. Therefore the four sectors $C_{\alpha\beta}$ are labelled by the values $\alpha;\beta$ of the fields at infinity compatible with finite energy and Q_1^T determines the homotopy class in $\pi_0(C) = Z_2\times Z_2$.

The critical points of E are time-independent finite-energy solutions of the field equations. If they are not spatially homogeneous, the critical points correspond to solitary waves that are therefore related to the topological structure of C. Besides complying with (4), solitary waves satisfy the system of ordinary differential equations:

$$\frac{d^2 q_a}{dx^2} = \frac{\partial V}{\partial q_a} \tag{8}$$

Recall that $\phi_a(x;t) = q_a(x)$. Solving the system (8) is tantamount to finding the solutions of the Lagrangian dynamical system in which $x = \tau$ plays the rôle of time, the "particle" position is determined by $q_a(\tau)$, and the potential energy of the particle is $U(\vec{q}) = -V(\vec{q})$. From this perspective the static field energy E is seen as the particle action:

$$E = J = \int d\tau \left\{ \frac{1}{2}\frac{d\vec{q}}{d\tau}\cdot\frac{d\vec{q}}{d\tau} - U(\vec{q}) \right\} \tag{9}$$

Trajectories that behaves asymptotically in the τ-time as ruled by (4) have a finite action, J, in the mechanical problem and are in one-to-one correspondence with solitary waves/kinks that have energy $E = J$ in the field theoretical system.

The mechanical analogy is very helpful when one is dealing with a real scalar field theory because, then, a first integral is all that we need to find all the solutions. Vector scalar fields of N components lead to N-dimensional dynamical systems which are seldom solvable. Magyari and Thomas [4] realized that the two-dimensional dynamical system arising in connection with the MSTB model is a completely integrable one in the Liouville sense; there are two first integrals in involution. Moreover, Ito [5] has shown that the mechanical system is Hamilton-Jacobi separable, finding all the trajectories and hence all the kinks of the MSTB model. In a recent publication [6], we have developed this procedure for two $N = 2$ models with interesting features: the first system is a deformation of the (1+1)-dimensional scalar field theory, where the potential energy density is the Chern-Simons-Higgs potential arising in self-dual planar gauge theories. The second one is a deformation of the linear $O(2)$-sigma model, which is different from the MSTB model.

To extend this method of finding kinks to the linear $O(N)$-sigma model, $N \geq 3$, deformed in such a way that the $O(N)$ symmetry is explicitly broken to $G = Z_2^{\,N}$, we start from the "particle" action:

$$J = \int d\tau \left\{ \frac{1}{2}\frac{d\vec{q}}{d\tau}\cdot\frac{d\vec{q}}{d\tau} + \frac{1}{2}(\vec{q}\cdot\vec{q} - 1)^2 + \frac{1}{2}\sum_{a=1}^{N}\sigma_a^2 q_a^2 \right\} = \int d\tau \, L(\vec{q},\dot{\vec{q}})$$

The particle motion equations are:

$$\frac{d^2 q_a}{d\tau^2} = 2q_a(\vec{q}\cdot\vec{q} - 1) + \sigma_a^2 q_a \, ; \quad \forall a = 1,\ldots,N \tag{10}$$

which are mathematically identical to the field equations for static configurations. Finite action trajectories, kinks in the field theory, should also satisfy the asymptotic conditions:

$$\lim_{\tau\to\pm\infty}\frac{dq_a}{d\tau} = 0; \qquad \lim_{\tau\to\pm\infty} q_a(\tau) = \pm\delta_{a1} \tag{11}$$

We shall use the Hamiltonian formalism to integrate the mechanical system. The canonical momenta $p_a(\tau) = \frac{\partial L}{\partial \dot{q}_a} = \frac{dq_a}{d\tau}(\tau)$, together with the positions $q_a(\tau)$, form a system of local coordinates in phase space. We should bear in mind that $p_a(\tau) = \frac{d\phi_a}{dx}$ when going back to the field theory. The mechanical Hamiltonian

$$I_1 = \frac{1}{2}\vec{p}\cdot\vec{p} - \frac{1}{2}(\vec{q}\cdot\vec{q} - 1)^2 - \sum_{a=1}^{N}\frac{1}{2}\sigma_a^2 q_a^2 \tag{12}$$

leads to the system of canonical equations

$$\frac{dq_a}{d\tau} = \{I_1, q_a\}; \qquad \frac{dp_a}{d\tau} = \{I_1, p_a\}$$

equivalent to (10). Given any two functions $F(\vec{q},\vec{p})$, $G(\vec{q},\vec{p})$ in phase space, the Poisson bracket is defined in the usual way:

$$\{F, G\} = \sum_{a=1}^{N}\left(\frac{\partial F}{\partial q_a}\frac{\partial G}{\partial p_a} - \frac{\partial F}{\partial p_a}\frac{\partial G}{\partial p_a}\right)$$

Obviously $\frac{dI_1}{d\tau} = 0$, but our mechanical system is full of other invariants. In fact, as early as 1919 Garnier [20] solved the motion equations and described periodic trajectories in terms of Theta functions: the kink trajectories of finite "action" correspond to a limiting case and are the separatrices between the periodic trajectories and unbounded motion. More recently Grosse, and other authors [21] have shown that the functions:

$$K_a = \sum_{b=1; b\neq a}^{N} \frac{1}{\sigma_b^2 - \sigma_a^2} l_{ab}^2 + p_a^2 + (2 - \sigma_a^2) q_a^2 - q_a^2 \sum_{b=1}^{N} q_b^2 \tag{13}$$

$$l_{ab} = p_a q_b - p_b q_a$$

are first integrals in involution:

$$\{I_1, K_a\} = 0 \qquad \{K_a, K_b\} = 0$$

There is a set of $N+1$ invariants in involution: $I_1, K_1, K_2, \dots, K_N$. The dynamical system is not superintegrable, however, because there are only N-independent invariants: $K_1 + K_2 + \dots + K_N = 2I_1 + 1$. According to the Liouville theorem, the N-dimensional mechanical system is completely integrable and all trajectories can be found, at least in principle.

At this point we pause to explain the singular nature of the deformation of the linear $O(N)$-sigma model chosen from among many possibilities. The $\frac{\sigma_a^2}{2}\phi_a^2$ terms explicitly break the $O(N)$-symmetry of the linear sigma model; the case $\sigma_a^2 = 0$, $\forall a = 1, 2, \dots, N$. In the mechanical system the $O(N)$ internal transformations become ordinary rotations. The angular momentum components, l_{ab}, conserved in the limit $\sigma_a^2 = 0$, $\forall a$, are no longer 'time'-independent if $\sigma_a^2 \neq 0$. There are, however, N invariants K_a, which in the limit $\sigma_a = 0$, $\forall a$, are given in terms of the $O(N)$-invariants: the r Casimir invariants and the r generators of the Cartan sub-algebra, where $r = \frac{N}{2}$ or $\frac{N-1}{2}$ if N-even or -odd is the rank of the group. A warning: in the N = odd case, the energy must be added to the other $N-1$ invariants built from the Cartan sub-algebra and the Casimir invariants. For any N, the maximally asymmetric chosen deformation is special because it retains enough symmetry to solve the mechanical system. There is no Lie algebra associated with K_a however; since the invariants are quadratic in q_a, p_a, the action of K_a in the phase space, given by $\{K_a, q_b\}$ and $\{K_a, p_b\}$, is non-linear.

In (1+1)-dimensional field theory, the energy-momentum tensor:

$$T^{\mu\nu} = \frac{\partial L}{\partial(\partial_\mu \phi_a)} \partial^\nu \phi_a - g^{\mu\nu} L$$

is divergenceless due to invariance under space-time translations. $P^\mu = \int dx\, T^{0\mu}$ are thus conserved quantities whatever the values of σ_a^2. The $O(N)$ "isospin" currents however,

$$J_a^\mu = \sum_{b,c=1}^{N} c_{abc}\, \phi_b \partial^\mu \phi_c$$

are only divergenceless if $\sigma_a = 0$, $\forall a$. The c_{abc} are the Lie $O(N)$ structure constants and the charges $Q_a = \int dx J_a^0$ are not conserved if there is no symmetry with respect to the transformation generated by them. For static configurations, we have

$$\int dx\, T^{00} = E = J; \quad T^{10} = T^{01} = 0; \qquad T^{11} = I_1$$

$$J_a^0 = 0; \qquad J_a^1 = \sum_{b,c=1}^{N} C_{abc} l_{bc}$$

In terms of the 'isospin' currents, the invariants K_a can be written as:

$$K_a = \sum_{b=1;b\neq a}^{N} \frac{1}{\sigma_b^2 - \sigma_a^2} \left(\sum_{c=1}^{N} C_{abc} J_c^1 \right) \left(\sum_{d=1}^{N} C_{abd} J_d^1 \right) + \left(\frac{\partial \phi_a}{\partial x} \right)^2 + (2 - \sigma_a^2)\phi_a^2 - \sigma_a^2 \sum_{b=1}^{N} \phi_b^2$$

We expect that the time-evolution occurs in such a way that there is some equation of non-linear character

$$F\left(\frac{\partial L_a}{\partial t}, \frac{\partial K_a}{\partial x} \right) = 0$$

between K_a and

$$L_a = \sum_{b=1;b\neq a}^{N} \frac{1}{\sigma_b^2 - \sigma_a^2} \left(\sum_{c=1}^{N} C_{abc} J_c^0 \right) \left(\sum_{d=1}^{N} C_{abd} J_d^0 \right) + \left(\frac{\partial \phi_a}{\partial t} \right)^2 + (2 - \sigma_a^2)\phi_a^2 - \sigma_a^2 \sum_{b=1}^{N} \phi_b^2$$

which reduces to $\frac{\partial J_a^0}{\partial t} = \frac{\partial J_a^1}{\partial x}$ when $\phi_a = 0, \forall a$. The situation is analogous to that occurring between conformal field theories and models with infinite-dimensional algebraic symmetry as in (1+1)-dimensional Toda field theories and Toda affine models [22]. There are two differences: (1) the conformal group is infinite dimensional in (1+1)-dimensions. We have only one finite-dimensional group $O(N)$ and thus we can solve only the static limit of the field theory model. (2) Due to the non-linear character of the deformation of the $O(N)$ Lie generators, we do not even have a finite-dimensional Lie algebra.

2.3 The Hamilton-Jacobi equation and kink trajectories

The K_a invariants defined in (13) are quadratic in the momenta, but not orthogonal (they contain terms in $p_a p_b, a \neq b$). Therefore, the Stackel theorem can not be applied to assure Hamilton-Jacobi separability. This problem is surpassed in our dynamical system with the choice of some suitable system of coordinates. The appropriate system is provided by elliptic Jacobi coordinates, with a choice of separation constants determined by the deformation parameters giving mass to the Goldstone bosons; $\sigma_a^2 = 1 - \gamma_a^2; \ \forall a = 1, 2, \ldots, N$. Thus we define:

$$q_a^2 = \frac{\prod_{b=1}^{N} (\sigma_a^2 - \lambda_b)}{\prod_{b=1;b\neq a}^{N} (\sigma_a^2 - \sigma_b^2)} = \frac{\Pi(\sigma_a^2)}{A'(\sigma_a^2)} \tag{14}$$

ruling the change of coordinates from Cartesian, $\vec{q} \equiv (q_1, \ldots, q_N)$ to elliptic $\vec{\lambda} \equiv (\lambda_1, \ldots, \lambda_N)$. In the Appendix, it is explained how the elliptic variables are split:

$$-\infty < \lambda_1 < \sigma_N^2 < \lambda_2 < \sigma_{N-1}^2 < \ldots < \sigma_2^2 < \lambda_N < 1 \tag{15}$$

Notice that formula (14) coincides with formula (72) in the Appendix if we change q_a by q_{N-a+1} and choose $r_{N-a+1} = \sigma_a^2$.

Together with formula (14), this splitting means that the change of coordinates produces a map from a sub-space of $\mathbb{R}^N$, characterized as the set of points which are not invariants under the Z_2^N group generated by $q_a \to (-1)^{\delta_{ab}} q_a; b = 1, \dots, N$, to the interior of the infinite parallelepiped $P_N(\infty)$ obtained by replacing the inequalities in (15) by equalities: $-\infty < \lambda_1 \leq -\sigma_N^2 \leq \cdots \leq -\sigma_2^2 \leq \lambda_N \leq -1$. Notice that in this map 2^N regular points in $\mathbb{R}^N$ go to a single point in the interior of $P_N(\infty)$; Singular points lie in the $\mathbb{R}^m$, $m = 0, 1, \dots, N-1$, sub-spaces that are invariant under the action of some non-trivial element of $G = Z_2^N$. These singular sub-spaces are mapped into the boundary of $P_N(\infty)$.

The standard length of an interval in Euclidean space is expressed in elliptic coordinates in the form

$$ds^2 = \sum_{a=1}^{N} dq_a dq_a = \sum_{a=1}^{N} g_{aa}(\vec{\lambda}) d\lambda_a d\lambda_a$$

because the metric $g_{aa}(\vec{\lambda})$, as derived in the Appendix, is:

$$g_{aa}(\vec{\lambda}) = \frac{1}{4}\frac{f_a(\vec{\lambda})}{A(\lambda_a)}; \quad g_{ab}(\vec{\lambda}) = 0; \forall a \neq b$$

where $A(\lambda_a) = \prod_{b=1}^{N}(\lambda_a + \sigma_b^2)$, and [1]

$$f_a(\vec{\lambda}) = f_a(\lambda_1, \lambda_2, \dots, \lambda_N) = \prod_{\substack{b=1 \\ b \neq a}}^{N} (\lambda_a - \lambda_b)$$

Therefore, the Lagrangian reads:

$$\begin{aligned} L &= \frac{1}{2}\sum_{a=1}^{N}\left(\frac{dq_a}{d\tau}\right)^2 - U(q_1, \dots, q_N) \\ &= \frac{1}{2}\sum_{a=1}^{N} g_{aa}(\vec{\lambda})\left(\frac{d\lambda_a}{d\tau}\right)^2 - U(\lambda_1, \dots, \lambda_N) \end{aligned} \quad (16)$$

where the potential in elliptic coordinates is:

$$U(\lambda_1, \dots, \lambda_N) = \sum_{a=1}^{N} \frac{1}{2}\frac{\lambda_a^{N+1} - (\sigma - 1)\lambda_a^N + (1 - \sigma + \tau)\lambda_a^{N-1}}{f_a(\vec{\lambda})} \quad (17)$$

$$\sigma = \sum_{a=1}^{N} \sigma_a^2; \qquad \tau = \sum_{a=1; a<b}^{N} \sum_{b=2}^{N} \sigma_a^2 \sigma_b^2$$

The computation of $U(\vec{\lambda})$ is highly non-trivial and requires the use of formulas that follow the Jacobi Lemma, such as (76), (77), (78), etc.

[1] The standard notation in the literature on elliptic Jacobi coordinates is $\Psi'(\lambda_a) = \prod_{\substack{b=1 \\ b \neq a}}^{N} (\lambda_a - \lambda_b)$, see Appendix. We shall use $f_a(\vec{\lambda})$ in the main text instead of $\Psi'(\lambda_a)$, to stress the fact that this quantity depends on all the components of $\vec{\lambda}$

The canonical momenta associated to the λ_a variables are:

$$\pi_a = \sum_{b=1}^{N} g_{ab}(\tilde{\sigma})\frac{d\lambda_b}{d\tau} = g_{aa}(\tilde{\sigma})\frac{d\lambda_a}{d\tau}$$

and, through the standard Legendre transformation, we write the Hamiltonian:

$$H = \frac{1}{2}\sum_{a=1}^{N}\frac{4A(\lambda_a)}{f_a(\tilde{\sigma})}\pi_a^2 + U(\tilde{\sigma}) \tag{18}$$

The key point is that H can be written in Stackel's form:

$$H = \sum_{a=1}^{N}\frac{H_a}{f_a(\tilde{\sigma})} = \tag{19}$$
$$= \sum_{a=1}^{N}\frac{\left[2A(\lambda_a)\pi_a^2 - \frac{1}{2}\lambda_a^{N+1} - (\sigma^2-1)\lambda_a^N + (1-\sigma^2+\omega^2)\lambda_a^{N-1}\right]}{f_a(\tilde{\sigma})}$$

such that the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial \tau} + H\left(\frac{\partial S}{\partial \lambda_1},\ldots,\frac{\partial S}{\partial \lambda_N};\lambda_1,\ldots,\lambda_N\right) = 0 \tag{20}$$

is completely separable. We now prove this last statement.

Fixing $H = I_1$, the first integral of energy, and having in mind the expression (19) of H, we write the solution of (20) as:

$$S = -I_1\tau + \sum_{a=1}^{N} S_a(\lambda_a). \tag{21}$$

Therefore, (20) reduces to:

$$I_1 = H\left(\frac{dS_1}{d\lambda_1},\ldots,\frac{dS_N}{d\lambda_N};\lambda_1,\ldots,\lambda_N\right) = \sum_{a=1}^{N}\frac{H_a}{f_a(\tilde{\sigma})} \tag{22}$$

The Hamilton-Jacobi PDE equation (20) becomes equivalent to the system of non-coupled ordinary differential equations

$$H_a\left(\frac{dS_a}{d\lambda_a};\lambda_a\right) = \gamma_1\lambda_a^{N-1} + \gamma_2\lambda_a^{N-2} + \cdots + \gamma_{N-1}\lambda_a + \gamma_N \tag{23}$$

where

$$H_a\left(\frac{dS_a}{d\lambda_a};\lambda_a\right) = 2A(\lambda_a)\left(\frac{dS_a}{d\lambda_a}\right)^2 - \frac{1}{2}\lambda_a^{N+1} - (\sigma^2-1)\lambda_a^N + (1-\sigma^2+\omega^2)\lambda_a^{N-1}; \tag{24}$$

due to the identity

$$I_1 = I_1\sum_{a=1}^{N}\frac{\lambda_a^{N-1}}{f_a(\tilde{\sigma})} + \gamma_2\sum_{a=1}^{N}\frac{\lambda_a^{N-2}}{f_a(\tilde{\sigma})} + \cdots + \gamma_N\sum_{a=1}^{N}\frac{1}{f_a(\tilde{\sigma})} \tag{25}$$

–observe that $\beta_1 = I_1$ – which follows from the Jacobi Lemma and the subsequent relations (Appendix)

$$\sum_{a=1}^{N}\frac{\lambda_a^{N-1}}{f_a(\vec{\lambda})} = 1\ ; \qquad \sum_{a=1}^{N}\frac{\lambda_a^{N-i}}{f_a(\vec{\lambda})} = 0; \forall i = 2;\dots;N$$

Alternatively, one could take a more direct approach to show formula (23). We start from (22), written explicitly as

$$\frac{H_1(\lambda_1)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\cdots(\lambda_1-\lambda_N)} + \frac{H_2(\lambda_2)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)\cdots(\lambda_2-\lambda_N)} + \cdots + \frac{H_N(\lambda_N)}{(\lambda_N-\lambda_1)(\lambda_N-\lambda_2)\cdots(\lambda_N-\lambda_{N-1})} = I_1 \qquad (26)$$

Here, each $H_a(\lambda_a)$ is of the form (24) due to the ansatz (21). Multiplying (26) by $(\lambda_1-\lambda_2)$ and setting $\lambda_1 = \lambda_2$ one sees that $H_1(\lambda) \equiv H_2(\lambda)$, and hence by symmetry, all $H_a(\lambda)$, $a = 1;2;\dots;N$ are identical. It suffices therefore to find $H_1(\lambda)$. Multiplying (26) by $f_1(\vec{\lambda})$ one obtains

$$H_1(\lambda_1) + P_2(\vec{\lambda})H_2(\lambda_2) + \cdots + P_N(\vec{\lambda})H_N(\lambda_N) = I_1(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\cdots(\lambda_1-\lambda_N); \qquad (27)$$

where each $P_a(\vec{\lambda})$, $a = 2;3;\dots;N$, is a polynomial of degree $N-1$ in λ_1. Differentiating (27) N times with respect to λ_1 yields

$$\frac{d^N H_1(\lambda_1)}{d\lambda_1^N} = N!\, I_1 \qquad (28)$$

whence it follows that $H_1(\lambda)$ is a degree N polynomial in λ with leading coefficient I_1, in agreement with equation (23).

The set of separation constants $\beta_i; i = 1;2;\dots;N$ is another system of first integrals in involution. They can be expressed in the elliptic phase space $T^*P_N(1)$ as functions of λ_a and $\pi_a = \frac{dS_a}{d\lambda_a}$ by solving the linear system of equations (23) in the unknown β_i, which is a Vandermonde system. The $N-1$ roots of the polynomial

$$\lambda_a^{N-1} + \frac{\beta_2}{I_1}\lambda_a^{N-2} + \dots + \frac{\beta_N}{I_1} = (\lambda_a - F_2)(\lambda_a - F_3)\dots(\lambda_a - F_N)$$

together with the energy $F_1 = I_1$ form another system of invariants in involution. Both systems are related through the identities $\beta_1 = I_1$ and:

$$\beta_a = (-1)^a I_1 \sum_{i_1<i_2<\dots<i_a} F_{i_1}F_{i_2}\dots F_{i_a}\ ;\ i = 2;3;\dots;N$$

Therefore, all the separation constants β_a are proportional to the "particle" energy I_1.

Defining the polynomial $B(\lambda_a)$ in the form:

$$B(\lambda_a) = \lambda_a^{N+1} - (\sigma - 1)\lambda_a^{N} + (1 - \sigma + \dots + 2\beta_1)\lambda_a^{N-1} + 2\beta_2\lambda_a^{N-2} + \dots + 2\beta_N$$

the solution of the differential equation (23) is a quadrature:

$$S_a(\lambda_a) = \frac{1}{2}\,\mathrm{sign}\left(\frac{dS_a}{d\lambda_a}\right)\int\sqrt{\frac{B(\lambda_a)}{A(\lambda_a)}}\,d\lambda_a \qquad (29)$$

and the general solution of the Hamilton-Jacobi equation reads:

$$S = \gamma_1 + \sum_{a=1}^{N} \frac{1}{2}\,\mathrm{sign}\left(\frac{dS_a}{d\lambda_a}\right) \int \sqrt{\frac{B(\lambda_a)}{A(\lambda_a)}}\, d\lambda_a \qquad (30)$$

The explicit integration of the quadratures in (29) requires the theory of Theta functions of genus depending on N. The action of the associated trajectories is infinite because they are either periodic or unbounded. The asymptotic conditions (4) that guarantee finite action to continuous trajectories satisfying them also require that the energy used by the particle in these trajectories should be zero. This is so because $I_1|_{\tau=-\infty} = 0$ and, being an invariant of the evolution, $I_1 = 0; \forall\tau$.

We recall that the trajectories of finite action in an evolution lasting an infinite time are the kinks of the parent field theory system: one just trades the finite action of the trajectory for finite energy of the non-linear wave. Therefore, the kinks are the trajectories obtained when all the separation constants in (23) are zero: $\gamma_a = 0; \forall a$. These are the separatrices between bounded and unbounded motion and the integrals in (29) are easier to compute.

The explicit trajectories are also provided by the Hamilton-Jacobi principle, through the set of equations:

$$\beta_a = \frac{\partial S}{\partial \gamma_a} \; ; \quad a = 1, 2, \ldots, N$$

where the β_a are integration constants. In the hypersurface of the phase space determined by $\gamma_1 = \cdots = \gamma_N = 0$, the first equation

$$\beta_1 = \tau + \sum_{a=1}^{N} \frac{1}{2}\,\mathrm{sign}(\lambda_a) \int \frac{\lambda_a^{N-1} d\lambda_a}{|A(\lambda_a)|} \sqrt{\frac{A(\lambda_a)}{\lambda_a^{N+1} - (\sigma - 1)\lambda_a^{N} + (1 - \sigma + \sigma^2)\lambda_a^{N-1}}} \qquad (31)$$

rules the time-dependence of the particle in its journey through the orbit. From the field theoretical point of view, it provides the kink form factor. The other $N-1$ equations, $i = 2, 3, \ldots, N$,

$$\beta_i = \sum_{a=1}^{N} \frac{1}{2}\,\mathrm{sign}(\lambda_a) \int \frac{\lambda_a^{N-i} d\lambda_a}{|A(\lambda_a)|} \sqrt{\frac{A(\lambda_a)}{\lambda_a^{N+1} - (\sigma - 1)\lambda_a^{N} + (1 - \sigma + \sigma^2)\lambda_a^{N-1}}} \qquad (32)$$

determine the orbit in $P_N(1-\sigma)$, the intersection of $N-1$ hypersurfaces in the configuration space. Therefore, there is a $N-1$-dimensional family of kinks parametrized by the finite values of β_i.

Although (31) and (32) identify all the separatrix trajectories of the mechanical system and henceforth all the kink solutions of the deformed linear $O(N)$-sigma model, an explicit description of such solitary waves is difficult for two reasons: (1). (31) and (32) form a system of transcendent equations of impossible analytical resolution. (2). Even if it were possible, expressing back the solution in Cartesian coordinates through (72) for $N \geq 3$ is another impossible task by analytical means.

3 N = 3

To gain insight into the nature of the different kinks of the model, in this Section we shall address in full detail the $N = 3$ case. We shall deal with a (1+1)-dimensional field theory including three scalar fields which transform according to a vector representation of the O(3) group. The structure of the solitary wave solutions of the $N = 3$ system is extremely rich from different points of view and shows the behavioural pattern of the general case with N-component fields.

3.1 The general solution of the Hamilton-Jacobi equation

The Hamiltonian of the underlying dynamical system reads:

$$H = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) - \frac{1}{2}\left(q_1^2 + q_2^2 + q_3^2 - 1\right)^2 - \frac{\sigma_2^2}{2}q_2^2 - \frac{\sigma_3^2}{2}q_3^2 \tag{33}$$

in Cartesian coordinates. To write the Hamiltonian in elliptic coordinates, note that for $N = 3$ we have:

$$\gamma = 1 + \sigma_2^2 + \sigma_3^2 \, ; \qquad \beta = \sigma_2^2\sigma_3^2 + \sigma_2^2 + \sigma_3^2$$

$$A(\lambda_a) = (\lambda_a - 1)(\lambda_a - \sigma_2^2)(\lambda_a - \sigma_3^2)$$

$$f'(\lambda_1) = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3); \quad f'(\lambda_2) = (\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3); \quad f'(\lambda_3) = (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)$$

Hence, $H = \sum_{a=1}^{3} \frac{1}{f'(\lambda_a)} H_a$, where

$$H_a = -2A(\lambda_a)\pi_a^2 - \frac{1}{2}\left[\lambda_a^2(\lambda_a - \sigma_2^2)(\lambda_a - \sigma_3^2)\right] \tag{34}$$

The separatrix trajectories, those in one-to-one correspondence with solitary waves of kink type in the encompassing field theory, are fully determined by the equations (32) restricted to the $N = 3$ case:

$$C_2 = \left(\frac{\sqrt{1-\lambda_1}+\sigma_2}{\sqrt{1-\lambda_1}-\sigma_2}\right)^{\sigma_3 \mathrm{sign}(\pi_1)} \left(\frac{\sqrt{1-\lambda_1}-\sigma_3}{\sqrt{1-\lambda_1}+\sigma_3}\right)^{\sigma_2 \mathrm{sign}(\pi_1)} \left(\frac{\sqrt{1-\lambda_2}+\sigma_2}{\sqrt{1-\lambda_2}-\sigma_2}\right)^{\sigma_3 \mathrm{sign}(\pi_2)} \left(\frac{\sqrt{1-\lambda_2}-\sigma_3}{\sqrt{1-\lambda_2}+\sigma_3}\right)^{\sigma_2 \mathrm{sign}(\pi_2)} \left(\frac{\sqrt{1-\lambda_3}+\sigma_2}{\sqrt{1-\lambda_3}-\sigma_2}\right)^{\sigma_3 \mathrm{sign}(\pi_3)} \left(\frac{\sqrt{1-\lambda_3}-\sigma_3}{\sqrt{1-\lambda_3}+\sigma_3}\right)^{\sigma_2 \mathrm{sign}(\pi_3)} \tag{35}$$

where $C_2 = \exp\{2\gamma_2\sigma_2\sigma_3(\sigma_2^2 - \sigma_3^2)\}$ is constant, and:

$$C_3 = \left(\frac{\sqrt{1-\lambda_1}-1}{\sqrt{1-\lambda_1}+1}\right)^{\sigma_2\sigma_3(\sigma_2^2-\sigma_3^2)\mathrm{sign}(\pi_1)} \left(\frac{\sqrt{1-\lambda_1}+\sigma_2}{\sqrt{1-\lambda_1}-\sigma_2}\right)^{\sigma_3\sigma_3^2\mathrm{sign}(\pi_1)} \left(\frac{\sqrt{1-\lambda_1}-\sigma_3}{\sqrt{1-\lambda_1}+\sigma_3}\right)^{\sigma_2\sigma_2^2\mathrm{sign}(\pi_1)} \left(\frac{\sqrt{1-\lambda_2}-1}{\sqrt{1-\lambda_2}+1}\right)^{\sigma_2\sigma_3(\sigma_2^2-\sigma_3^2)\mathrm{sign}(\pi_2)}$$

$$\left(\frac{\sqrt{1-{}_2}+{}_2}{\sqrt{1-{}_2}\quad{}_2}\right)^{{}_3\,{}_3^2\,\mathrm{sign}({}_2)}\left(\frac{\sqrt{1-{}_2}\quad{}_3}{\sqrt{1-{}_2}+{}_3}\right)^{{}_2\,{}_2^2\,\mathrm{sign}({}_2)}$$

$$\left(\frac{\sqrt{1-{}_3}\quad 1}{\sqrt{1-{}_3}+1}\right)^{{}_2\,{}_3({}_2^2\quad{}_3^2)\mathrm{sign}({}_3)}\left(\frac{\sqrt{1-{}_3}+{}_2}{\sqrt{1-{}_3}\quad{}_2}\right)^{{}_3\,{}_3^2\,\mathrm{sign}({}_3)} \qquad (36)$$

$$\left(\frac{\sqrt{1-{}_3}\quad{}_3}{\sqrt{1-{}_3}+{}_3}\right)^{{}_2\,{}_2^2\,\mathrm{sign}({}_3)}$$

with $C_3 = \exp\{2\;{}_3\;{}_2\;{}_3\;{}_2^2\;{}_3^2({}_2^2\quad{}_3^2)\}$.

Integration of (31) in the $N = 3$ case shows the time-table of the particle in each trajectory, or, the kink form factor:

$$C_1(\ ) = \left(\frac{\sqrt{1-{}_1}\quad{}_2}{\sqrt{1-{}_1}+{}_2}\right)^{{}_3\,{}_2^2\,\mathrm{sign}({}_1)}\left(\frac{\sqrt{1-{}_1}+{}_3}{\sqrt{1-{}_1}\quad{}_3}\right)^{{}_2\,{}_3^2\,\mathrm{sign}({}_1)}$$

$$\left(\frac{\sqrt{1-{}_2}+{}_2}{\sqrt{1-{}_2}\quad{}_2}\right)^{{}_3\,{}_2^2\,\mathrm{sign}({}_2)}\left(\frac{\sqrt{1-{}_2}\quad{}_3}{\sqrt{1-{}_2}+{}_3}\right)^{{}_2\,{}_3^2\,\mathrm{sign}({}_2)} \qquad (37)$$

$$\left(\frac{\sqrt{1-{}_3}\quad{}_2}{\sqrt{1-{}_3}+{}_2}\right)^{{}_3\,{}_2^2\,\mathrm{sign}({}_3)}\left(\frac{\sqrt{1-{}_3}+{}_3}{\sqrt{1-{}_3}\quad{}_3}\right)^{{}_2\,{}_3^2\,\mathrm{sign}({}_3)}$$

if $C_1(\ ) = \exp\{2({}_1+\ )({}_3^2\quad{}_2^2)\;{}_2\;{}_3\}$. Therefore, there is a family of kinks parametrized by the integration constants ${}_2$, ${}_3$: it corresponds to the family of curves in $P_3(1\ )$ determined by the intersection of the surfaces defined by (35) and (36). The third constant, ${}_1$, fixes the center of the kink, the point where the energy density reaches its maximum value.

Better intuition of the kink shapes requires an interpretation of the solutions described by equations (35) and (36) in Cartesian coordinates. We shall describe how is this achieved in the next sub-sections, but before this it is convenient to note some details of the change of coordinates from Cartesian to elliptic in R^3:

$$q_1^2 = \frac{1}{{}_2^2\,{}_3^2}(1\quad{}_1)(1\quad{}_2)(1\quad{}_3)$$

$$q_2^2 = \frac{1}{{}_2^2({}_3^2\quad{}_2^2)}({}_2^2\quad{}_1)({}_2^2\quad{}_2)({}_2^2\quad{}_3) \qquad (38)$$

$$q_3^2 = \frac{1}{{}_3^2({}_2^2\quad{}_3^2)}({}_3^2\quad{}_1)({}_3^2\quad{}_2)({}_3^2\quad{}_3)$$

The change of coordinates is singular at the three R^2 coordinate planes; $q_1 = 0$, $q_2 = 0$ and $q_3 = 0$. The image of the $q_1 = 0$ plane is a unique face, ${}_3 = 1$, of the $P_3(1\ )$ parallelepiped:

$$1 < {}_1 \quad {}_3^2 \quad {}_2 \quad {}_2^2 \quad {}_3 \quad 1 \qquad (39)$$

The $q_2 = 0$ plane, however, is mapped into faces ${}_2 = {}_2^2$ and ${}_3 = {}_2^2$, while the $q_3 = 0$ plane goes to faces ${}_2 = {}_3^2$ and ${}_1 = {}_3^2$ of $P_3(1\ )$. Observe that $g_{11}({}_3^2;{}_2;{}_3) = g_{22}({}_1;{}_3^2;{}_3) = g_{22}({}_1;{}_2^2;{}_3) = g_{33}({}_1;{}_2;{}_2^2) = g_{33}({}_1;{}_2;1) = 1$. The whole R^3 space is mapped in $P_3(\infty)$. Due to the symmetry under the group $G = Z_2 \otimes Z_2 \otimes Z_2$ generated by $q_a \to -q_a$, the mapping (38) is eight to one in regular points of R^3: to any point in the interior of $P_3(\infty)$ correspond eight points in R^3 away from the coordinate planes. These planes are fixed loci of some subgroup of G.

The asymptotic conditions (4) in q_a restrict the motion to the compact sub-space D^3 of R^3 bounded by the tri-axial ellipsoid:

$$q_1^2 + \frac{q_2^2}{\sigma_2^2} + \frac{q_3^2}{\sigma_3^2} = 1 \qquad (40)$$

and are satisfied by finite action and zero energy trajectories. Elliptic coordinates are best suited for demonstrating such a restriction. In this coordinate system D^3 is mapped to the finite parallelepiped $P_3(0)$:

$$0 \le \lambda_1 \le \sigma_3^2 \le \lambda_2 \le \sigma_2^2 \le \lambda_3 \le 1 \qquad (41)$$

The unique non-singular face of $P_3(0)$ with respect to the change of coordinates is $\lambda_1 = 0$ and the inverse image of this face is the ellipsoid (40). The asymptotic conditions (4) force $\pi_b = 0; \forall b$, and thus, by (23), $H_a = 0; \forall a$, for the finite action solutions. λ_2 and λ_3 are bounded, see (41). Thus, we focus on,

$$H_1 = 0 \Rightarrow \frac{1}{2}\pi_1^2 + \frac{1}{8}\frac{\lambda_1^2}{\lambda_1 - 1} = 0 \qquad (42)$$

Equation (42) describes the motion of a particle with zero energy moving under the influence of a potential

$$V(\lambda_1) = \frac{1}{4}\frac{\lambda_1^2}{\lambda_1 - 1}; \qquad -\infty < \lambda_1 \le \sigma_3^2$$
$$= -\infty; \qquad \sigma_3^2 < \lambda_1 < 1$$

$V(\lambda_1)$ has a maximum at $\lambda_1 = 0$ and goes to $-\infty$ when λ_1 tends to $-\infty$; therefore, bounded motion occurs only in the $\lambda_1 \in [0;\sigma_3^2]$ interval and the trajectories giving rise to kinks lie in $P_3(0)$, seen in elliptic coordinates, or D^3 in Cartesian space.

In Figure 1 the whole picture is depicted and we notice the following important elements of the dynamics:

–Points: (1) the origin. This is a fixed point of $G = Z_2^3$ and thus only one point O in D^3 is mapped to the vertex O in $P_3(\infty)$. (2) Points B, C, D: these are the intersection points of the three distinguished ellipses, $q_1^2 + \frac{q_2^2}{\sigma_2^2} = 1$, $q_1^2 + \frac{q_3^2}{\sigma_3^2} = 1$ and $\frac{q_2^2}{\sigma_2^2} + \frac{q_3^2}{\sigma_3^2} = 1$, in the ellipsoid (40). They are fixed points under the action of a sub-group Z_2^2 of G and thus, two points in the boundary of D^3 are mapped to a single point in the boundary of $P_3(0)$. D is the point where the two vacuum points $v^\pm$ are mapped and hence it is a very important point of the dynamics: every finite action trajectory starts and ends at D. (3) Points F_1, F_2, F_3, the foci of the above ellipses. Again two points in D^3 are mapped in a unique point in $P_3(0)$. (4) The umbilicus A of the ellipsoid $\lambda_1 = 0$ is another characteristic point; in Cartesian coordinates A corresponds to four points in the boundary of D^3 because they are invariant only under a Z_2 sub-group of G.

–Curves:

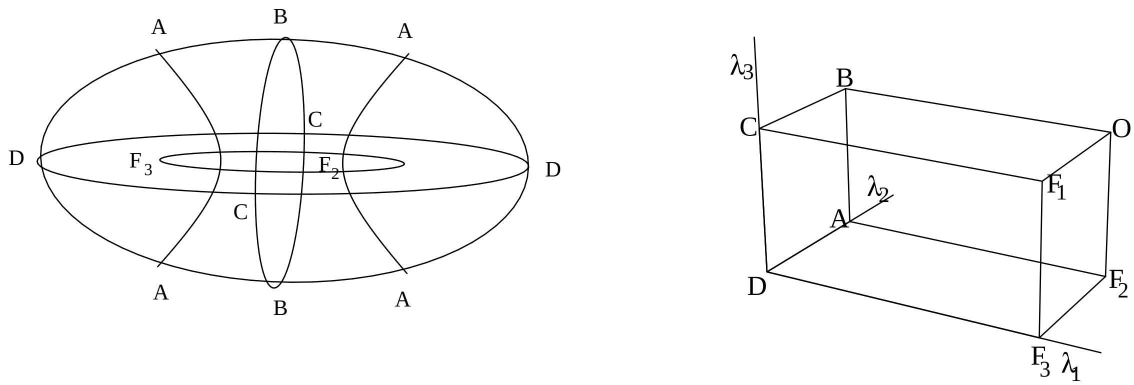

Figure 1: (a) The domain D^3 in R^3: Cartesian coordinates (b) The domain $P_3(0)$ in R^3: Jacobi elliptic coordinates

The ellipse with foci F_2

$$\frac{q_1^2}{\sigma_3^2} + \frac{q_2^2}{\sigma_3^2 - \sigma_2^2} = 1 \qquad (43)$$

in the $q_3 = 0$ plane passing through F_3 and F_1. This is the edge $\lambda_2 = \sigma_3^2 = \lambda_1$ in $P_3(0)$. Observe that four points on the ellipse (43) are mapped to one point in the edge of $P_3(0)$, because it is invariant under a Z_2 sub-group of G. The map leading to F_3 and F_1 is, however, two to one: the invariance group of these points is bigger, $Z_2^2 \subset G$.

The hyperbola

$$\frac{q_1^2}{\sigma_2^2} - \frac{q_3^2}{\sigma_3^2 - \sigma_2^2} = 1 \qquad (44)$$

in the $q_2 = 0$ plane passing through F_2 and A and having foci F_3 (the edge $\lambda_2 = \sigma_2^2 = \lambda_3$ in $P_3(0)$).

The above points and curves play a special rôle in the definition of the elliptic coordinates and are also "critical loci" of the dynamics.

3.2 Generic Kinks

The generic kinks of the $N = 3$ system are the trajectories given by the solutions of (35)-(36) for non-zero finite values of C_2 and C_3. The solutions of the implicit equations (35)-(36) cannot be graphically represented by means of the built-in functions of Mathematica. We use a numerical algorithm implemented in Mathematica to obtain the graphic portrait of the trajectories. The algorithm allows us to calculate an arbitrary number of points on the orbit. These points joined by straight segments provide a visualization of the trajectory. There is a special step and an iteration of routine steps in the procedure, which is based on the Newton-Raphson method.

First step. Identification of two points on the trajectory.

For given values of $C_2, C_3, \sigma_2, \sigma_3$ and a choice of signs, we set the first variable to the "point" $\lambda_1 = \bar{\lambda}_1$. (35)-(36) becomes a system of two equations in two unknowns that can be solved by the Newton-Raphson method with starting values ($^0_2$; $^0_3$). The outcome is a point P_1 ($_1$; $_2$; $_3$) on the trajectory. We repeat this operation starting from $_1$ = $_1$ + = $^0_1$ to nd a second point P_2 ($^0_1$; $^0_2$; $^0_3$) on the orbit.

$_1$, $^0_1$, $^0_2$ and $^0_3$ are chosen at random; good convergence is attained if these points belong to the middle zones of the variation ranges of $_1$, $_2$ and $_3$ or, at least, they are far away from the singularities on the faces of $P_3(0)$

Successive steps.

P_1 and P_2 provide an approximation of the curve by the secant line joining them. For some small 2 R^+, we choose $P_3^0 = P_1 +$ (P_2 P_1) as the starting value of the Newton-Raphson procedure applied to the solution of equations (35) and (36); we thus obtain the point P_3 on the curve. P_2 and P_3 lead to guesstimate by the same token another value P_4^0 that produce the next point P_4 on the orbit and now the iteration is obvious. Replacing by we travel along the opposite sense on the trajectory. The algorithm stops when one of the three variables $_1$, $_2$, $_3$ reaches its extreme value; it is applied independently on each stage, determined by the signs of $_a$ and the global trajectory is obtained by the demand for continuity.

We now describe the portrait of these orbits. Having xed $_2$ and $_3$, the corresponding kink trajectory is a non-plane curve in the interior of $P_3(0)$ that starts from the vacuum point D, reaches the top face BCF_1O and hits the edge AF_2. It then goes to the edge F_1F_3, back again to the top face, hits the edge AF_2 a second time, the top face a third time and ends at D: see Figure 2 and Figure 3. Varying $_2$ and $_3$ in the range of nite real numbers, other similar trajectories are obtained that hit the edges AF_2 and F_1F_3 at di erent points. Given a sense of time there therefore exists a two-parameter family of kink trajectories in one-to-one correspondence with the points in the interior of AF_2 and F_1F_3. It should be mentioned that a whole congruence of trajectories parametrized by the interior of F_1F_3 converges at one single point in the interior of AF_2 and viceversa.

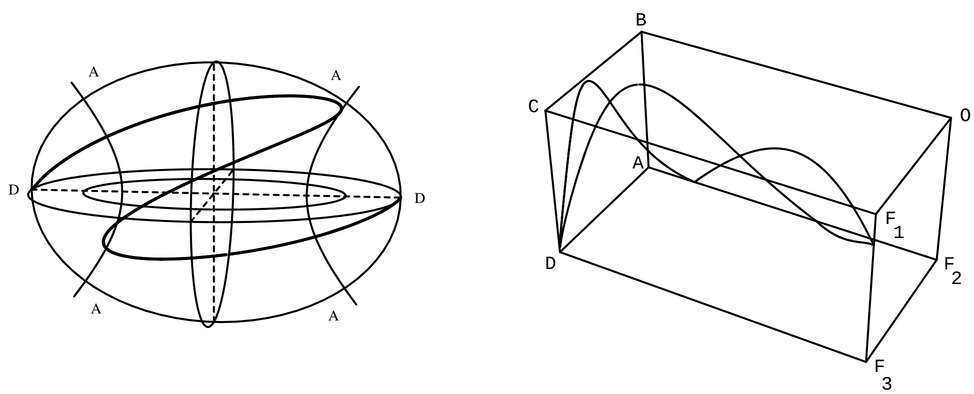

Figure 2: A generic kink drawn both in D^3 and in $P_3(0)$. Observe in D^3 that the generic kink is a heteroclinic trajectory

The translation of a generic kink trajectory to Cartesian coordinates is a delicate matter; due to the non-uniqueness of the mapping implied by the change of coordinates special care is necessary in the analysis of the trajectory near the special conics (43)-(44) where several

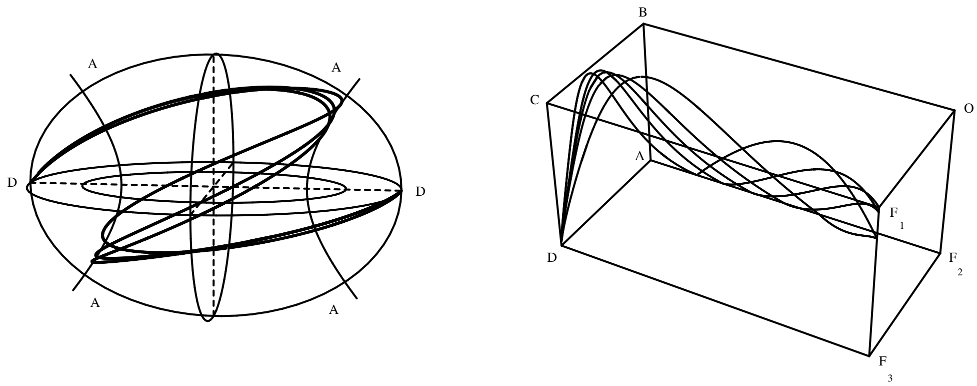

Figure 3: Several generic kinks, all of them intersecting once with the edge F_1F_3 and twice with the edge AF_2 of $P_3(0)$.

options are a priori possible. Since it requires continuity and derivability to the trajectories in the interior of the ellipsoid (40), the interior of D^3, the behaviour of the curve is mostly fixed. Two choices, the specification of $D = v^-$ as the starting point and the location of the intersection of the kink trajectory with the $q_1 = 0$ plane in the quadrant characterized by $q_2 > 0, q_3 > 0$, completely fix the itinerary.

There is a crossroad where the "particle" touches the $q_1 > 0$ branch of the hyperbola (44) and turns back towards the ellipse (43). There, the movement enters the $q_3 < 0$ half-space and the kink trajectory reaches the other branch, $q_1 < 0$, of the critical hyperbola after crossing the $q_2 = 0$ plane. At this stage, the particle makes its way for a third crossing of the $q_2 = 0$ plane and, finally, the journey ends at $D = v^+$. This kind of kink trajectory is therefore heteroclinic: it starts and ends at different unstable points, so that:

$$Q_1^T = \frac{1}{2}\int_{-1}^{1} d\ \frac{dq_1}{d} = 1; \quad Q_2^T = Q_3^T = 0 \tag{45}$$

We call these topological kinks TK3 because they have three non-null components:

$$q_1(\ ) = {}_1(x) \neq 0; \quad q_2(\ ) = {}_2(x) \neq 0; \quad q_3(\ ) = {}_3(x) \neq 0$$

It should be noted that a unique, apparently non-derivable, kink trajectory in elliptic coordinates corresponds to eight derivable trajectories in Cartesian coordinates: the choices of v^- or v^+ and $q_3 < 0$ or $q_3 > 0$, $q_2 < 0$ or $q_2 > 0$ as the starting point and initial quadrant give the eight possibilities.

The energy of a three-component topological kink is the action of the trajectory times $\frac{m^3}{2\sqrt{2}}$ and hence computable from formula (29) for the $N = 3$ case:

$$\frac{2\sqrt{2}}{m^3}E_{TK3} = \int_0^{\ _3^2} \frac{\ _1 d\ _1}{\sqrt{1-\ _1}} + \int_{\ _3^2}^{\ _2^2} \frac{\ _2^2\ _2 d\ _2}{\sqrt{1-\ _2}} + \int_{\ _2^2}^{1} \frac{\ _3^3\ _3 d\ _3}{\sqrt{1-\ _3}} \tag{46}$$

$$= \frac{4}{3} + \frac{2}{3}\Big[\ _3(3-\ _3^2) + \ _2(3-\ _2^2)\Big]$$

It is independent of $\ _2$ and $\ _3$ and hence the same for every kink in the TK3 family.

3.3 Enveloping Kinks

There is another family of $N = 3$ kinks living on the surface $M_3 \quad f(\ _1;\ _2;\ _3) = \ _1 = 0g$, the unique face of $P_3(0)$ where the elliptic coordinates are not singular. In M_3, the Hamiltonian becomes:

$$H = \sum_{a=2}^{3} \frac{H_a}{{}^0(\ _a)} = \sum_{a=2}^{3} \left(\frac{2A(\ _a)}{{}^0(\ _a)}\ {}^2_a \quad \frac{1}{2} \frac{{}^2_a(\ _a \quad {}^2_2)(\ _a \quad {}^2_3)}{{}^0(\ _a)} \right) \qquad (47)$$

and therefore there is a two-dimensional system hidden inside the $N = 3$ model which is Hamilton-Jacobi separable. The orbit equations,

$$C = \left(\frac{\sqrt{1 \quad _2 \quad _2}}{\sqrt{1 \quad _2 + \quad _2}} \right)^{_3\,\mathrm{sign}(\ _2)} \left(\frac{\sqrt{1 \quad _2 + \quad _3}}{\sqrt{1 \quad _2 \quad _3}} \right)^{_2\,\mathrm{sign}(\ _2)} \left(\frac{\sqrt{1 \quad _3 \quad _2}}{\sqrt{1 \quad _3 + \quad _2}} \right)^{_3\,\mathrm{sign}(\ _3)} \left(\frac{\sqrt{1 \quad _3 + \quad _3}}{\sqrt{1 \quad _3 \quad _3}} \right)^{_2\,\mathrm{sign}(\ _3)} \qquad (48)$$

are parametrized by only one real constant $\ _2$ ($C = e^{\ 2\ 3(\ {}^2_3 \quad {}^2_2)\ _2}$).

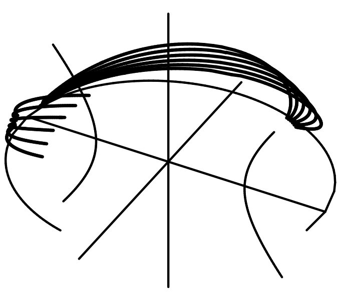

Figure 4: The NTK3 family

The Mathematica plot of these solutions is shown in Figure 4. Having xed $\ _2$, the corresponding kink trajectory is a plane curve in M_3 that starts from the vacuum point D, reaches the top edge BC, goes to the umbilicus A and then back to the edge BC, to end nally in the vacuum point D. The value of $\ _2$ determines the points in BC where the trajectory bounces back and thus the one-parameter family of this kind of kink trajectories is in one-to-one correspondence with the points in the interior of BC.

In Cartesian coordinates the enveloping kinks are trajectories that unfold on the ellipsoid (40):

$$q_1^2 + \frac{q_2^2}{{}^2_2} + \frac{q_3^2}{{}^2_3} = 1$$

The starting point is either $D = \ v^+$ or $D = \ v$ and the trajectories also end in either $D = \ v^+$ or $D = \ v$. Associated with \homoclinic" trajectories, the corresponding kinks are \non-topological": $Q_1^T = Q_2^T = Q_3^T = 0$. The three Cartesian components q_a di er from zero and the appropriate name for this kind of solitary wave is a non-topological kink of three components, NTK3 for short. Every NTK3 trajectory on its way from $D = v^-$ to $D = v^-$ crosses the umbilicus point of the ellipsoid. Note that, again, eight trajectories in the Cartesian space R^3 correspond to one trajectory in $P_3(0)$: the particle has the freedom to choose the points v^+ or v^- as base points of the curve. Having fixed one of them, the trajectory may develop in the half-ellipsoids determined in (45) by $q_3 \geq 0$ or $q_3 \leq 0$ and, finally, there are two travelling senses in each orbit.

Also, the energy of a three-component non-topological kink is essentially the action of the NTK3 trajectory:

$$\frac{2\sqrt{2}}{m^3} E_{NTK3} = \int_{\sigma_3^2}^{\sigma_2^2} \frac{2 d\lambda_2}{\sqrt{1-\lambda_2}} + \int_{\sigma_2^2}^{1} \frac{2\lambda_3 d\lambda_3}{\sqrt{1-\lambda_3}} = 2\left(\sigma_2 + \sigma_3 - \frac{\sigma_2^2 + \sigma_3^2}{3}\right) \qquad (49)$$

according to the Hamilton-Jacobi theory.

3.4 Embedded Kinks

Three-component topological and non-topological kinks arise as genuine solitary waves in the $N = 3$ model. Restriction to the $q_3 = 0$ and/or $q_2 = 0$ planes shows that the $N = 2$ system is included twice, once in each plane, in the $N = 3$ model. Therefore, all the solitary waves of the $N = 2$ model are embedded twice as kinks of the larger $N = 3$ system. The embedded kinks live on the $q_2 = 0$ and $q_3 = 0$ planes, i.e. the faces of $P_3(0)$ where the elliptic coordinate system is singular.

I. Embedded kinks in the $q_2 = 0$ plane

Both $\lambda_2 = \sigma_2^2$ and $\lambda_3 = \sigma_2^2$ give $q_3 = 0$, see (38), and hence this coordinate plane in R^3 is the union of the two faces, $\lambda_2 = \sigma_2^2$ and $\lambda_3 = \sigma_2^2$, of $P_3(0)$. Therefore, in

$$M_{\sigma_2\sigma_3} = \left\{ (\lambda_1; \lambda_2; \lambda_3) / \lambda_3 = \sigma_2^2 \right\} \sqcup \left\{ (\lambda_1; \lambda_2; \lambda_3) / \lambda_2 = \sigma_2^2 \right\} = M^1_{\sigma_2\sigma_3} \sqcup M^2_{\sigma_2\sigma_3}$$

we expect to find all the kinks of the $N = 2$ case.

In $M^1_{\sigma_2\sigma_3}$, $\lambda_3 = \sigma_2^2$, we are in the face of $P_3(0)$ such that $0 < \lambda_1 < \sigma_3^2 < \lambda_2 < \sigma_2^2$, and

$$q_1^2 = \frac{1}{\sigma_3^2}(1-\lambda_1)(1-\lambda_2)\,; \qquad q_3^2 = \frac{1}{\sigma_3^2}(\sigma_3^2-\lambda_1)(\sigma_3^2-\sigma_2^2)$$

The Hamiltonian also reduces to the $N = 2$ Hamiltonian

$$H = \frac{2A(\lambda_1)}{f'(\lambda_1)}\pi_1^2 - \frac{1}{2f'(\lambda_1)}\lambda_1^2(\lambda_1-\sigma_2^2)(\lambda_1-\sigma_3^2)$$
$$\frac{2A(\lambda_2)}{f'(\lambda_2)}\pi_2^2 - \frac{1}{2f'(\lambda_2)}\lambda_2^2(\lambda_2-\sigma_2^2)(\lambda_2-\sigma_3^2)$$

and the Hamilton-Jacobi method prescribes the equation

$$e^{2\ _3\ ^2_3\ _2} = \left(\frac{\sqrt{1\ \ _1}\ \ _3}{\sqrt{1\ \ _1}+\ _3}\ \frac{\sqrt{1\ \ _1}+1}{\sqrt{1\ \ _1}\ 1}\right)^{\ _3 ! \,\mathrm{sign}(\ _1)} \left(\frac{\sqrt{1\ \ _2}\ \ _3}{\sqrt{1\ \ _2}+\ _3}\ \frac{\sqrt{1\ \ _2}+1}{\sqrt{1\ \ _2}\ 1}\right)^{\ _3 ! \,\mathrm{sign}(\ _2)} \tag{50}$$

as ruling the portion of the trajectories at this face, bounded by the edges AD, AF_2, F_2F_3 and F_3D (see Figure 5).

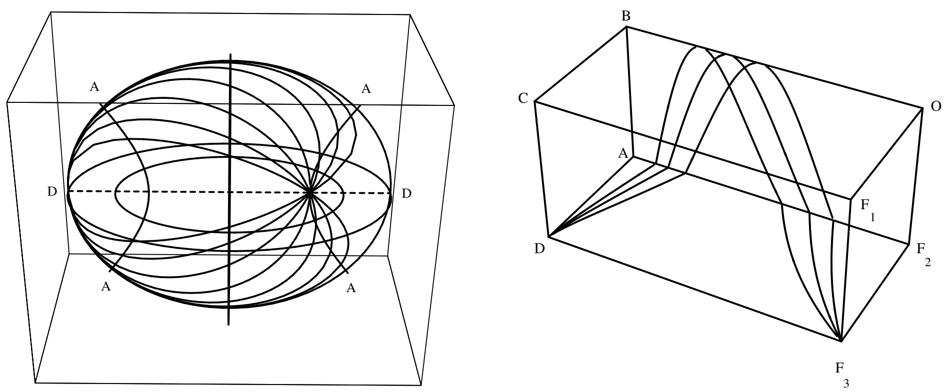

Figure 5: Several NTK2 $_3$ kink trajectories of the N = 2 system embedded in the $q_2 = 0$ plane

In $M^2_{2\ _3}$, $_2 = \ ^2_2$ and the face in the boundary of $P_3(0)$ is $0 < \ _1 < \ ^2_3 < \ ^2_2 < \ _3 < 1$. The trajectory equations at this face are:

$$e^{2\ _3\ ^2_3\ _2} = \left(\frac{\sqrt{1\ \ _1}\ \ _3}{\sqrt{1\ \ _1}+\ _3}\ \frac{\sqrt{1\ \ _1}+1}{\sqrt{1\ \ _1}\ 1}\right)^{\ _3 ! \,\mathrm{sign}(\ _1)} \left(\frac{\sqrt{1\ \ _3}\ \ _3}{\sqrt{1\ \ _3}+\ _3}\ \frac{\sqrt{1\ \ _3}+1}{\sqrt{1\ \ _3}\ 1}\right)^{\ _3 ! \,\mathrm{sign}(\ _3)} \tag{51}$$

and the boundary is formed by the edges AF_2, F_2O, OB and BA.

For nite values of $_2$, some kink trajectories given by (50)-(51) are depicted in Figure 5. It may be observed that the trajectory starts at D and then runs through the face $M^1_{2\ _3}$ until the edge AF_2. From this point, the particle enters the $M^2_{2\ _3}$ face (here the path is not derivable), reaches the BO edge and comes back to the AF_2 edge. This is the second point of non-di erentiability re-entering the trajectory the $M^1_{2\ _3}$ face. All the trajectories then meet at the vertex F_3 and come back in a symmetric way to end in the D point. In Cartesian coordinates, these kink trajectories start and end in either $D = v^+$ or $D = v$, do not leave the $q_2 = 0$ plane, and cross either the focus ($q_1 = \ _3$, $q_3 = 0$) or ($q_1 = \ _3$, $q_3 = 0$). We therefore call them NTK2 $_3$ because they are two-component non-topological kinks, merely the family of NTK2 kinks of the N = 2 model, embedded this way within the manifold of kinks of the $N=3$ system. There are four trajectories of this kind inside the ellipsoid (45) in $\mathbb{R}^3$ per trajectory in the boundary of $P_3(0)$: there is freedom to choose v^+ or v^- and the sense of travel in each orbit. The $NTK2_{\sigma_3}$ kinks are fixed points of the Z_2 sub-group of $G=Z_2^{\,3}$ generated by $q_2 \to -q_2$, that, however, does not leave invariant the $TK3$ and the $NTK3$ trajectories. The energy of these solutions is:

$$\frac{2\sqrt{2}}{m^3}E_{NTK2_{\sigma_3}} = \int_0^{\sigma_3^2}\frac{\lambda_1 d\lambda_1}{\sqrt{1-\lambda_1}} + \int_{\sigma_3^2}^{\sigma_2^2}\frac{2\lambda_2 d\lambda_2}{\sqrt{1-\lambda_2}} + \int_{\sigma_2^2}^{1}\frac{2\lambda_3 d\lambda_3}{\sqrt{1-\lambda_3}}$$
$$= \frac{4}{3} + 2\sigma_3\left(1-\frac{\sigma_3^2}{3}\right) \qquad (52)$$

There is a limiting case to this family of kinks: a trajectory along the DA and AB edges and back to D through the same way. Elliptic coordinates are even more singular on the edges, but the dynamical system reduces to a one-dimensional Hamiltonian system which can be integrated analytically. We have a two-step trajectory:

At the DA edge, $\lambda_1=0$ and $\lambda_3=\sigma_2^2$, the canonical equations (after use of the first integral) reduce to:

$$\frac{d\lambda_2}{d\tau} = \pm 2(\lambda_2-\sigma_3^2)\sqrt{1-\lambda_2} \qquad (53)$$

with the solution

$$\lambda_1^{TK2_{\sigma_3}}(\tau)=0;\quad \lambda_2^{TK2_{\sigma_3}}(\tau)=1-\sigma_3^2\tanh^2(\sigma_3\tau);\quad \lambda_3^{TK2_{\sigma_3}}(\tau)=\sigma_2^2 \qquad (54)$$

for $\tau\in(-\infty,-\frac{1}{\sigma_3}\mathrm{arctanh}\frac{\sigma_2}{\sigma_3}]\sqcup[\frac{1}{\sigma_3}\mathrm{arctanh}\frac{\sigma_2}{\sigma_3},\infty)$. The second step occurs on the AB edge, where, again, the canonical equations reduce to a single differential equation: if $\lambda_1=0$ and $\lambda_2=\sigma_2^2$,

$$\frac{d\lambda_3}{d\tau} = \pm 2(\lambda_3-\sigma_3^2)\sqrt{1-\lambda_3} \qquad (55)$$

has the solution

$$\lambda_1^{TK2_{\sigma_3}}(\tau)=0;\quad \lambda_2^{TK2_{\sigma_3}}(\tau)=\sigma_2^2;\quad \lambda_3^{TK2_{\sigma_3}}(\tau)=1-\sigma_3^2\tanh^2(\sigma_3\tau) \qquad (56)$$

for $\tau\in\left[-\frac{1}{\sigma_3}\mathrm{arctanh}\frac{\sigma_2}{\sigma_3},\frac{1}{\sigma_3}\mathrm{arctanh}\frac{\sigma_2}{\sigma_3}\right]$. The corresponding kinks in Cartesian coordinates are $TK2_{\sigma_3}$ and $TK2_{\sigma_3}$, the four two-component topological kinks of the $N=2$ model:

$$\begin{pmatrix} q_1^{TK2_{\sigma_3}}(\tau)\\ q_2^{TK2_{\sigma_3}}(\tau)\\ q_3^{TK2_{\sigma_3}}(\tau)\end{pmatrix} = \begin{pmatrix}\tanh(\sigma_3\tau)\\ 0\\ \sigma_3\,\mathrm{sech}(\sigma_3\tau)\end{pmatrix} \qquad \begin{pmatrix} \phi_1^{TK2_{\sigma_3}}(x,t)\\ \phi_2^{TK2_{\sigma_3}}(x,t)\\ \phi_3^{TK2_{\sigma_3}}(x,t)\end{pmatrix} = \begin{pmatrix}\tanh(\sigma_3 x)\\ 0\\ \sigma_3\,\mathrm{sech}(\sigma_3 x)\end{pmatrix}$$
$$(57)$$

Thus, the enveloping kinks of the $N=2$ model are also embedded in the $N=3$ system. The energy for these solutions and their anti-kinks is:

$$\frac{2\sqrt{2}}{m^3}E_{TK2_{\sigma_3}} = \int_{\sigma_3^2}^{\sigma_2^2}\frac{\lambda_2 d\lambda_2}{\sqrt{1-\lambda_2}} + \int_{\sigma_2^2}^{1}\frac{\lambda_3 d\lambda_3}{\sqrt{1-\lambda_3}} = 2\sigma_3\left(1-\frac{\sigma_3^2}{3}\right) \qquad (58)$$

In the $q_2=0$ plane there is still one trajectory that is even more singular: it is a three step path running on the edges DF_3, F_3F_2, F_2O and back to D through the same way. The canonical equations and its solutions in the three steps are:

1. $\lambda_2 = \sigma_3^2$ and $\lambda_3 = \sigma_2^2$.

$$\frac{d\lambda_1}{d\tau} = -2\lambda_1\sqrt{1-\lambda_1}; \qquad \tau \in (-\infty; -\mathrm{arctanh}\,\sigma_3] \sqcup [\mathrm{arctanh}\,\sigma_3; \infty)$$

$$\lambda_1^{TK1}(\tau) = 1 - \tanh^2\tau \; ; \quad \lambda_2^{TK1}(\tau) = \sigma_3^2; \quad \lambda_3^{TK1}(\tau) = \sigma_2^2$$

2. $\lambda_1 = \sigma_3^2$ and $\lambda_3 = \sigma_2^2$.

$$\frac{d\lambda_2}{d\tau} = -2\lambda_2\sqrt{1-\lambda_2}; \qquad \tau \in [-\mathrm{arctanh}\,\sigma_3; \mathrm{arctanh}\,\sigma_2] \sqcup [\mathrm{arctanh}\,\sigma_2; \mathrm{arctanh}\,\sigma_3]$$

$$\lambda_1^{TK1}(\tau) = \sigma_3^2; \quad \lambda_2^{TK1}(\tau) = 1 - \tanh^2\tau \; ; \quad \lambda_3^{TK1}(\tau) = \sigma_2^2$$

3. $\lambda_1 = \sigma_3^2$ and $\lambda_2 = \sigma_2^2$.

$$\frac{d\lambda_3}{d\tau} = -2\lambda_3\sqrt{1-\lambda_3}; \qquad \tau \in [-\mathrm{arctanh}\,\sigma_2; \mathrm{arctanh}\,\sigma_2]$$

$$\lambda_1^{TK1}(\tau) = \sigma_3^2; \quad \lambda_2^{TK1}(\tau) = \sigma_2^2; \quad \lambda_3^{TK1}(\tau) = 1 - \tanh^2\tau$$

Only one Cartesian component is different from zero:

$$\begin{pmatrix} q_1^{TK1}(\tau) \\ q_2^{TK1}(\tau) \\ q_3^{TK1}(\tau) \end{pmatrix} = \begin{pmatrix} \pm\tanh\tau \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \phi_1^{TK1}(x;t) \\ \phi_2^{TK1}(x;t) \\ \phi_3^{TK1}(x;t) \end{pmatrix} = \begin{pmatrix} \pm\tanh x \\ 0 \\ 0 \end{pmatrix} \tag{59}$$

and hence the one-component topological kink of the $N = 1$ model is embedded first in the manifold of kinks of the $N = 2$ model, and then in the $N = 3$ system. There are two kinks of this kind in Cartesian coordinates which are mapped in a unique trajectory in the boundary of $P_3(0)$. The TK1 trajectories are fixed points of the Z_2^2 sub-group of G generated by $q_2 \to -q_2$ and $q_3 \to -q_3$. The energy is

$$\frac{2\sqrt{2}}{m^3}E_{TK1} = \int_0^{\sigma_3^2} \frac{\lambda_1 d\lambda_1}{\sqrt{1-\lambda_1}} + \int_{\sigma_3^2}^{\sigma_2^2} \frac{\lambda_2 d\lambda_2}{\sqrt{1-\lambda_2}} + \int_{\sigma_2^2}^{1} \frac{\lambda_3 d\lambda_3}{\sqrt{1-\lambda_3}} = \frac{4}{3} \tag{60}$$

Embedded Kinks in the $q_3 = 0$ plane

The DF_3, F_3F_2 and F_2O edges form the intersection of the $q_2 = 0$ and $q_3 = 0$ planes. Therefore, the TK1 kinks also live in the $q_3 = 0$ plane. There is another maximally singular trajectory living on the "edge" in the $q_3 = 0$ plane:

At the DC edge, $\lambda_2 = \sigma_3^2$ and $\lambda_1 = 0$, the canonical equations for the finite action trajectories are:

$$\frac{d\lambda_3}{d\tau} = -2(\lambda_3 - \sigma_2^2)\sqrt{1-\lambda_3} \tag{61}$$

The path

$$\lambda_1^{TK2_2}(\tau) = 0; \quad \lambda_2^{TK2_2}(\tau) = \sigma_3^2; \quad \lambda_3^{TK2_2}(\tau) = 1 - \bar{\sigma}_2^2\tanh^2(\bar{\sigma}_2\tau) \tag{62}$$

solves (61) and runs when goes from $-\infty$ to $+\infty$ from D to D passing through the vertex C at $= 0$. In Cartesian coordinates we recover the four two-component topological kinks of the $N = 2$ model, now embedded in the $q_3 = 0$ plane:

$$\begin{pmatrix} q_1^{TK2_2}(\ ) \\ q_2^{TK2_2}(\ ) \\ q_3^{TK2_2}(\ ) \end{pmatrix} = \begin{pmatrix} \tanh(\ _2\ ) \\ _2^2\,\mathrm{sech}(\ _2\ ) \\ 0 \end{pmatrix} \qquad \begin{pmatrix} {}_1^{TK2_2}(x;t) \\ {}_2^{TK2_2}(x;t) \\ {}_3^{TK2_2}(x;t) \end{pmatrix} = \begin{pmatrix} \tanh(\ _2 x) \\ _2^2\,\mathrm{sech}(\ _2 x) \\ 0 \end{pmatrix} \tag{63}$$

These are heteroclinic trajectories that produce the $TK2_2$ and $TK2_2$ topological kinks. The energy is:

$$\frac{_2\sqrt{2}}{m^3}E_{TK2_2} = \int_{_2^2}^{1} \frac{_3 d\ _3}{\sqrt{1-\ _3}} = 2\ _2\left(1 - \frac{_2^2}{3}\right) \tag{64}$$

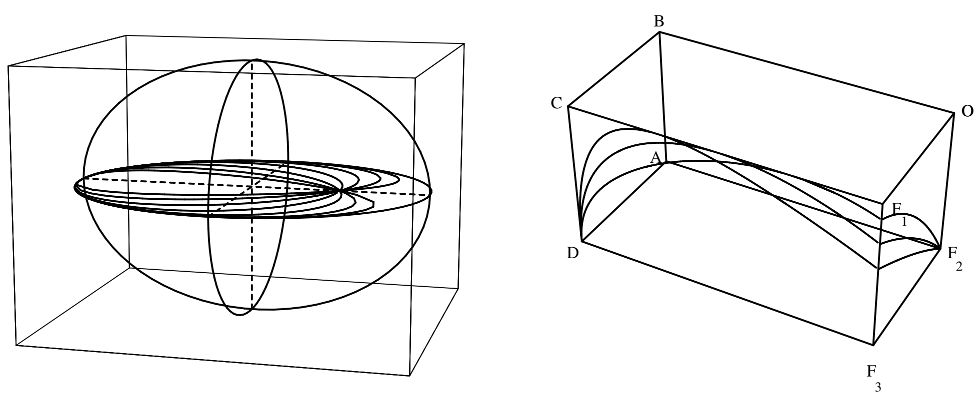

Figure 6: $NTK2_2$ kink trajectories of the $N = 2$ system embedded in the $q_3 = 0$ plane

Of course, the full manifold of kinks of the $N = 2$ model is embedded in the $q_3 = 0$ plane : the set of kinks of the $N = 3$ system is completed by the two-component non-topological kinks living in the $q_3 = 0$ plane, see Figure 6. The $q_3 = 0$ plane is mapped to the union of two faces in the boundary of $P_3(0)$:

$$M_{2_2} = \left\{(\ _1;\ _2;\ _3) = \ _1 = \ _3^2\right\} \sqcup \left\{(\ _1;\ _2;\ _3) = \ _2 = \ _3^2\right\} = M^1_{2_2} \sqcup M^2_{2_2}$$

1. In $M^1_{2_2}$, $_1 = \ _3^2$ implies $q_3 = 0$. Therefore:

$$\begin{aligned} q_1^2 &= \frac{1}{_2^2}(1 - \ _2)(1 - \ _3) \\ q_2^2 &= -\frac{1}{_2^2}(\ _2^2 - \ _2)(\ _2^2 - \ _3) \end{aligned}$$

is a well defined change of coordinates in the range $_3^2 < \ _2 < \ _2^2 < \ _3 < 1$. In this region, the interior of the ellipse (43), the trajectories providing kinks are given by the equations:

$$e^{2\ \ 2\ \frac{2}{2}\ 2} = \left(\frac{\sqrt{1\ \ \ _2}\ \ \ _2}{\sqrt{1\ \ \ _2}+\ \ _2}\ \frac{\sqrt{1\ \ \ _2}+1}{\sqrt{1\ \ _2}\ \ 1}\right)^{2\ \mathrm{sign}(\ _2)} \left(\frac{\sqrt{1\ \ \ _3}\ \ \ _2}{\sqrt{1\ \ \ _3}+\ \ _2}\ \frac{\sqrt{1\ \ \ _3}+1}{\sqrt{1\ \ _2}\ \ 1}\right)^{2\ \mathrm{sign}(\ _3)} \quad (65)$$

2. In $M^2_{2\ _2}$, $\ _2 = \ ^2_3$ also implies $q_3 = 0$. In the range $0 < \ _1 < \ ^2_3 < \ ^2_2 < \ _3 < 1$ the change of coordinates is defined as

$$q_1^2 = \frac{1}{\ ^2_2}(1\ \ \ _1)(1\ \ \ _3)$$

$$q_1^2 = \frac{1}{\ ^2_2}(\ ^2_2\ \ \ _1)(\ ^2_2\ \ \ _3)$$

The kink trajectories satisfy the equations:

$$e^{2\ \ 2\ \frac{2}{2}\ 2} = \left(\frac{\sqrt{1\ \ \ _1}\ \ \ _2}{\sqrt{1\ \ \ _1}+\ \ _2}\ \frac{\sqrt{1\ \ \ _1}+1}{\sqrt{1\ \ _1}\ \ 1}\right)^{2\ \mathrm{sign}(\ _1)} \left(\frac{\sqrt{1\ \ \ _3}\ \ \ _2}{\sqrt{1\ \ \ _3}+\ \ _2}\ \frac{\sqrt{1\ \ \ _3}+1}{\sqrt{1\ \ _2}\ \ 1}\right)^{2\ \mathrm{sign}(\ _3)} \quad (66)$$

The features of this kind of kinks are identical to the characteristics of the two-component non-topological kinks that exist in the $q_2 = 0$ plane. The only difference is that they have support in the faces $M^1_{2\ _2}$ and $M^2_{2\ _2}$ instead of $M^1_{2\ _3}$ and $M^2_{2\ _3}$ and we therefore call them $NTK2_{\ _2}$. They meet at the vertex F_2, and therefore at the foci $(q_1 = \ \ _2; 0; 0)$ in R^3; see Figure 6. The energy is:

$$\frac{2\sqrt{2}}{m^3}E_{NTK2_{\ _2}} = 2\left[\int_0^{\ ^2_3}\frac{\ _1 d\ _1}{\sqrt{1\ \ \ _1}} + \int_{\ ^2_3}^{\ ^2_2}\frac{\ _2 d\ _2}{\sqrt{1\ \ \ _2}} + \int_{\ ^2_2}^{1}\frac{\ _3 d\ _3}{\sqrt{1\ \ \ _3}}\right] =$$

$$= \frac{4}{3} + 2\ _2\left(1\ \ \frac{\ ^2_2}{3}\right) \quad (67)$$

In sum: the manifold of kinks of the $N = 2$ model is embedded twice in the $N = 3$ system, once in the $q_2 = 0$ plane and other in the $q_3 = 0$ plane. They are sewn togehter by the common TK1, embedded from the $N = 1$ model. The embedded kinks fill the gaps left by the TK3 families of kinks in the interior of the ellipsoid (45) and also develop through the curves left by the NTK3 families on the boundary of D^3. D^3 is thus a "totally" geodesic manifold with respect to the separatrices between bounded and unbounded motion in the $N = 3$ dynamical system. The NTK3 family form the envelop of the separatrices and the NTK3 kinks are themselves separatrices.

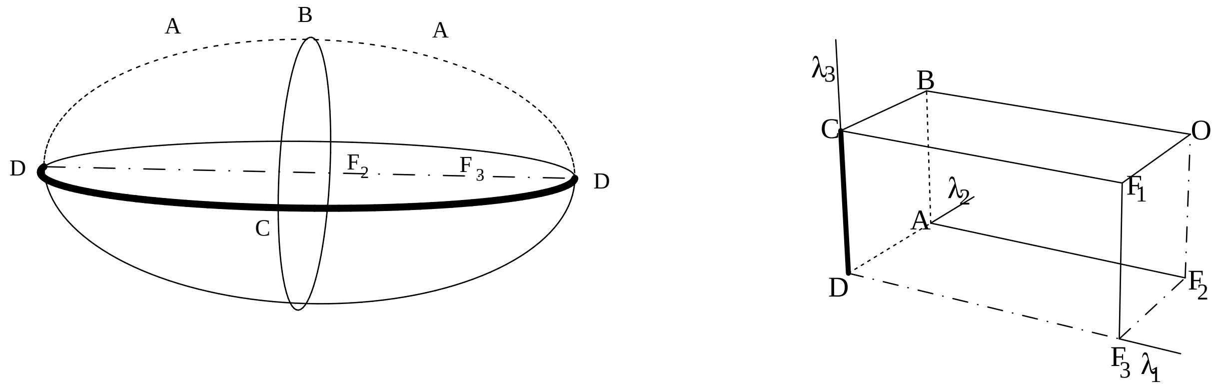

Figure 7: Plot of the "singular" topological kinks $TK2_2$ (solid line), $TK2_3$ (broken line) and $TK1$ (dash-dotted line) in both Cartesian and Elliptic coordinates.

4 Further Comments

We now infer the general structure of the kink manifold of the linear $O(N)$-sigma model from the pattern shown by the $O(2)$- and $O(3)$-sigma models. We can safely state that all the kink trajectories live in the sub-manifold $D^N \subset R^N$ determined by the inequality:

$$q_1^2 + \frac{q_2^2}{\sigma_2^2} + \cdots + \frac{q_N^2}{\sigma_N^2} \leq 1$$

There are three categories:

1. Generic Kinks

   A. There exists a family of generic kinks parametrized by $N-1$ real constants that live in the interior of D^N. The intersection loci of the generic kinks are the singular quadrics:

$$\frac{q_1^2}{\sigma_N^2} + \frac{q_2^2}{\sigma_N^2-\sigma_2^2} + \cdots + \frac{q_{N-1}^2}{\sigma_N^2-\sigma_{N-1}^2} = 1;\quad q_N = 0;\quad \lambda_1 = \lambda_2 = \sigma_N^2$$

$$\frac{q_1^2}{\sigma_{N-1}^2} + \frac{q_2^2}{\sigma_{N-1}^2-\sigma_2^2} + \cdots + \frac{q_{N-2}^2}{\sigma_{N-1}^2-\sigma_{N-2}^2} - \frac{q_N^2}{\sigma_N^2-\sigma_{N-1}^2} = 1;\quad q_{N-1} = 0;\quad \lambda_2 = \lambda_3 = \sigma_{N-1}^2$$

$$\vdots$$

$$\frac{q_1^2}{\sigma_2^2} - \frac{q_3^2}{\sigma_3^2-\sigma_2^2} - \cdots - \frac{q_N^2}{\sigma_N^2-\sigma_2^2} = 1;\quad q_2 = 0;\quad \lambda_{N-1} = \lambda_N = \sigma_2^2$$

   B. Generic kinks are non-topological–hence $NTKN$–if N even, and topological–hence TKN–if N is odd.

2. Enveloping kinks.

   A. The restriction of the dynamical system to the boundary ∂D^N of D^N, the hyper-ellipsoid:

$$q_1^2 + \frac{q_2^2}{\sigma_2^2} + \cdots + \frac{q_N^2}{\sigma_N^2} = 1$$

provides a family of enveloping kinks parametrized by $N-2$ real constants. Recalling that in elliptic coordinates ∂D^N is characterized by the equation $\lambda_1 = 0$, the intersection loci of this congruence are the umbilical sub-manifolds:

$$\lambda_1 = 0 \;;\quad \lambda_a = -\sigma^2_N \, \lambda_{a+1} = -\lambda_{a+1} \;;\; \forall a = 2,3,\ldots,N-1$$

of dimension $N-3$ of the hyper-ellipsoid ∂D^N.

B. Enveloping kinks are topological, hence TKN, if N is even, and non-topological, hence NTKN, if N is odd.

3. Embedded Kinks.

On the $N-1$ $\mathbb{R}^{N-1}$ sub-manifolds determined by the conditions $q_a = 0$, if $a = 2$ or 3 or $\ldots$ or N, the dynamical system reduces to the mechanical system that arises in the linear $O(N-1)$-sigma model. Thus, the kink manifold of the $N-1$ case is included $N-1$ times in the $O(N)$-model, filling the holes left in the interior of D^N by the generic kinks, and also covering in ∂D^N the sub-spaces which are not covereded by the enveloping kinks. Each $N-1$ kink sub-manifold is not, however, included $(N-1)\cdot(N-2)$ times in the $O(N)$-model because the $\mathbb{R}^{N-2}$ sub-spaces are intersections of the $N-1$ $\mathbb{R}^{N-1}$, defined above. The $N-1$ kink manifolds are not separated but sewn togehter through the $N-2$ kink sub-manifolds. This is a iterative process in such a way that the kink manifold of the $O(N-r)$-sigma model is included $\binom{N-1}{r}$ times in the kink manifold of the $O(N)$ system.

One can ask what happens if a continuous sub-group $O(r)$ of $O(N)$ survives as symmetry group of the system. This happens if the deformation is chosen in such a way that $0 < \sigma_2^2 = \sigma_3^2 = \cdots = \sigma_r^2 < \sigma_{r+1}^2 < \cdots < \sigma_N^2 < 1$. In this case we obtain a sub-manifold of kinks from $O(r)$ rotations around the q_1 axis of the kink manifold of the $N = 2$ system that lives in the $q_1:q_2$ plane. The remaining kinks correspond to the solitary waves of the $N = r-1$ system defined in the orthogonal $\mathbb{R}^{N-r+1}$ sub-space. Also the deformations where $1 < \sigma^2_{r+1} < \cdots < \sigma^2_N$ are easy to understand. Finite action trajectories spread out in the domain in $\mathbb{R}^N$ bounded by the hyper-hyperboloid:

$$q_1^2 + \frac{q_2^2}{\sigma_2^2} + \cdots + \frac{q_r^2}{\sigma_r^2} - \frac{q_{r+1}^2}{|\sigma_{r+1}^2|} - \cdots - \frac{q_N^2}{|\sigma_N^2|} = 1.$$

The kink manifold of this system is the kink manifold of the $N = r$ model defined in the sub-space $\mathbb{R}^r \subset \mathbb{R}^N$ such that $q_{r+1} = \cdots = q_N = 0$.

Finally we consider a mild deformation of our model by introducing asymmetries in the non-harmonic terms of the potential energy and also adapting the quadratic terms in a suitable manner:

$$V = \frac{1}{2}\left(\phi_1^2 + (1+\varepsilon_2)\,\phi_2^2 + \cdots + (1+\varepsilon_N)\,\phi_N^2 - 1\right)^2 + \\ + \frac{\sigma_2^2}{2}\phi_2^2 + \cdots + \frac{\sigma_N^2}{2}\phi_N^2 + \frac{\gamma_2}{4}\phi_2^4 + \cdots + \frac{\gamma_N}{4}\phi_N^4$$

The new non-dimensional constants ϵ_a and σ_a are defined in terms of the old γ_a's through:

$$1+\epsilon_a = \frac{\gamma_a(\gamma_a+1)}{2} \; ; \quad \sigma_a = \frac{\gamma_a^3(2+\gamma_a)}{2} \; ; \quad a = 2,3,\dots,N$$

Among the kinks of the deformed linear O(N)-sigma model only the following survive as solitary wave solutions of this perturbed system:

(a) The TK1.

$$\phi_1 = \tanh x \; ; \quad \phi_2 = \dots = \phi_N = 0$$

(b) All the TK2 kinks. On the ellipse,

$$\phi_1^2 + \frac{1+\epsilon_a}{1-\sigma_a^2}\,\phi_a^2 = 1$$

the TK2ϕ_a and TK2ϕ_a configurations,

$$\phi_1 = \tanh \sigma_a x \; ; \quad \phi_a = \sqrt{\frac{1-\sigma_a^2}{1+\epsilon_a}}\,\mathrm{sech}\, \sigma_a x \; ; \quad \phi_b = 0 \; ; \; \forall b \neq a, b \neq 1$$

are solutions of the field equations. The amazing fact is that in this deformation of the O(N)-linear sigma model, discussed by Bazeia et al. if $N = 2$ [12], the energy of all these kinks is the same:

$$E_{TK1} = E_{TK2\phi_2} = \dots = E_{TK2\phi_N} = \frac{4}{3\sqrt{2}}\frac{m^3}{\lambda^2}$$

On one hand, we have a deformation of the linear O(N)-sigma model that exhibits a complex variety of kinks; on the other hand, another deformation of the O(N)-model rejects almost every kink but the simplest ones as solutions, and all of the surviving kinks are degenerated in energy.

5 Outlook

The developments disclosed in this paper suggest a general strategy in the search for kinks in two space-time dimensional field theories. When the fields have N components assembled in a vector representation of the O(N) group, we focus on systems with symmetry breaking to a discrete sub-group of O(N) which has more than one element. If the dynamical system that determines the localized static solutions is completely integrable, all the solitary waves can be found, at least in principle. Particularly interesting is the situation where the $N-1$ invariants in involution with the mechanical energy act non-trivially on the manifold of localized solutions and the orbit is a continuous space. One can then perturb such a system, loosing in the perturbation many of the solitary wave solutions: only few of the localized static solutions survive as kinks of the perturbed (more realistic) model.

We finally list several interesting questions that will be postponed for future research:

- study of the structure of the kink manifold of the deformed linear O(N)-sigma model as a moduli space seems to be worthwhile.

A detailed analysis of the sum rules between the energies of the different kinds of kinks is necessary to fix the structure mentioned above.

A treatment a la Bogomolny is also possible. This allows for a supersymmetric extension of the model in such a way that the kinks become BPS states.

The difficult problem remains of determining the stability of the different kinds of kinks. Application of the Morse index theorem helps in finding the stability properties, which in turn provide information about the quantization of these topological defects.

6 Acknowledgements

The authors are grateful to Askold Perelomov for teaching them the magic of the elliptic Jacobi coordinates and their relationship to dynamical problems on ellipsoids.

Appendix: Elliptic coordinates

Given any set of N real positive numbers such that $0 < r_1 < r_2 < \cdots < r_N$, let us consider the equation:

$$\sum_{a=1}^{N} \frac{q_a^2}{r_a - \lambda} = 1 \tag{68}$$

The left-hand member $Q_\lambda(\vec{q}) = \sum_{a=1}^{N} \frac{q_a^2}{r_a - \lambda}$ of this equation can be understood either as a function of $\mathbb{R}^N$, for fixed $\lambda \in \mathbb{C}$, or as a function of the complex variable λ, for fixed $\vec{q} \in \mathbb{R}^N$. From (68) one immediately deduces:

$$1 - Q_\lambda(\vec{q}) = 1 + \sum_{a=1}^{N} \frac{q_a^2}{\lambda - r_a} = \frac{\prod_{a=1}^{N}(\lambda - \lambda_a)}{\prod_{a=1}^{N}(\lambda - r_a)} = 0 \tag{69}$$

Therefore, the N roots λ_a of the polynomial in the numerator of $1 - Q_\lambda(\vec{q})$, a rational function of λ, are the roots of equation (68). The roots λ_a are also real numbers and $1 - Q_\lambda(\vec{q})$ is a rational function such that $\lambda_1 < r_1 < \lambda_2 < \cdots < r_{N-1} < \lambda_N < r_N$, see Figure 8. To prove this point one needs to study $1 - Q_\lambda(\vec{q})$ along the λ-real axis, near the poles $\lambda = r_a$, using Bolzano's theorem.

Definition. The elliptic coordinates of the point $\vec{q} \equiv (q_1, \ldots, q_N) \in \mathbb{R}^N$ are the roots $\vec{\lambda}_E \equiv (\lambda_1, \ldots, \lambda_N) \in P^N(\infty)$ of $Q_\lambda(\vec{q}) = 1$.

$P^N(\infty) \subset \mathbb{R}^N$ is the open sub-space of $\mathbb{R}^N$ given by: $-\infty < \lambda_1 < r_1$, $r_1 < \lambda_2 < r_2$, $\ldots$, $r_{N-1} < \lambda_N < r_N$. The solution of (68) for $\lambda = \lambda_1 \equiv$ constant $\in (-\infty; r_1)$ is, geometrically, a quadric surface, a hyperellipsoid, when $\vec{q}$ varies in $\mathbb{R}^N$. The family of quadrics obtained by taking $\lambda = \lambda_a \equiv$ constant $\in (r_{a-1}; r_a)$, $a \geq 2$, corresponds to a family of hyper-hyperboloids of every possible signature in $\mathbb{R}^N$, if $Q_\lambda(\vec{q})$ is considered as a function of $\vec{q}$.

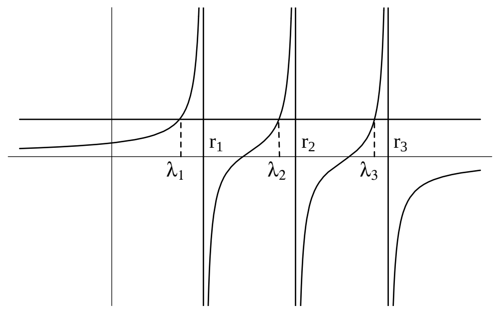

Figure 8: Plot of the function $y = \sum_{a=1}^{3} \frac{q_a^2}{r_a - \lambda}$ and the function $y = 1$, see (68), for fixed values of q_a and r_a.

It is convenient to denote the products in (69) as:

$$\Lambda(\lambda) = \prod_{a=1}^{N} (\lambda - \lambda_a)\,; \qquad A(\lambda) = \prod_{a=1}^{N} (\lambda - r_a)$$

so that

$$1 + \sum_{a=1}^{N} \frac{q_a^2}{\lambda - r_a} = \frac{\prod_{a=1}^{N} (\lambda - \lambda_a)}{\prod_{a=1}^{N} (\lambda - r_a)} \equiv \frac{\Lambda(\lambda)}{A(\lambda)} \tag{70}$$

An explicit formula for defining q_a^2 as a function of the λ's, $\forall a$, is obtained by applying the residue theorem to both members of the equation (70):

$$\begin{aligned} &\mathrm{Res}(1 - Q(\lambda))|_{\lambda = r_a} = q_a^2 \\ &\mathrm{Res}\left(\frac{\Lambda(\lambda)}{A(\lambda)}\right)_{\lambda = r_a} = \frac{\Lambda(r_a)}{A'(r_a)}\,; \quad A'(r_a) = \left.\frac{dA(\lambda)}{d\lambda}\right|_{\lambda = r_a} \end{aligned} \tag{71}$$

Therefore:

$$q_a^2 = \frac{\Lambda(r_a)}{A'(r_a)} = \frac{\prod_{b=1}^{N} (r_a - \lambda_b)}{\prod_{b=1; b \neq a}^{N} (r_a - r_b)} \tag{72}$$

and we see that the transformation $(q_1, \ldots, q_N) \to (\lambda_1, \ldots, \lambda_N)$ is 2^N to 1.

Inverting (72) to express λ_a as a function of the q's, $\forall a$, requires one to solve an algebraic equation in λ_a with powers up to λ_a^N. This is easy for $N = 2$, possible, but very difficult for $N = 3, 4$ using Cardano's formulas, and impossible if $N \geq 5$. For this reason, another derivation of (72) is useful, which in passing allows one to show identities between Cartesian and elliptic coordinates that make practical computations possible. To do this, notice that (70) implies:

$$\Phi(\lambda) = \prod_{a=1}^{N}(\lambda - r_a) + \sum_{a=1}^{N} q_a^2 \prod_{b=1;b\neq a}^{N}(\lambda - r_b) \tag{73}$$

Setting $\lambda = r_c$ in (73) one immediately derives (72). More important, expanding the two members of (73) in a power series in λ, we obtain:

$$\lambda^N - \lambda^{N-1}\left(\sum_{a=1}^{N} r_a + \sum_{a=1}^{N} q_a^2\right) + \lambda^{N-2}\left(\sum_{b<a}\sum_{a=1}^{N} r_a r_b + \sum_{a=1}^{N} q_a^2 \sum_{b=1;b\neq a}^{N} r_b\right) + \cdots$$

$$= \lambda^N - \left(\sum_{a=1}^{N}\mu_a\right)\lambda^{N-1} + \left(\sum_{b<a}\sum^{N}\mu_a\mu_b\right)\lambda^{N-2} + \cdots + (-1)^N \prod_{a=1}^{N}\mu_a$$

Equalizing the coefficients of the terms with the same power of λ in the last equation we have N non trivial identities. We shall use the equalities between the coefficients of λ^{N-1} and λ^{N-2}:

$$\sum_{a=1}^{N}\mu_a = \sum_{a=1}^{N} r_a + \sum_{a=1}^{N} q_a^2 \tag{74}$$

$$\sum_{b<a}\sum^{N}\mu_a\mu_b = -\sum_{a=1}^{N} r_a q_a^2 + \sum_{a=1}^{N} q_a^2 \sum_{b=1}^{N} r_b + \sum_{b<a}\sum^{N} r_b r_a \tag{75}$$

Another important tool using elliptic coordinates is the Jacobi lemma:

Lemma. The expression

$$\sum_{a=1}^{N}\frac{\mu_a^s}{(\mu_a-\mu_1)(\mu_a-\mu_2)\cdots\overset{(a)}{\cdots}\cdots(\mu_a-\mu_N)}$$

where $(\mu_1,\ldots,\mu_N)$ are N real numbers such that $\mu_1 < \mu_2 < \cdots < \mu_N$ is equal to 0 if $s \leq N-2$ and 1 for $s = N-1$.

Proof:

Consider the function

$$f_s(z) = \frac{z^s}{\prod_{a=1}^{N}(z-\mu_a)}$$

which has N poles in the complex plane at $z = \mu_a, \forall a$, and another pole at infinity. If Γ is a closed curve which is the boundary of a region D of the complex plane containing all the finite poles of $f_s(z)$, the residue theorem tells us that:

$$\frac{1}{2\pi i}\oint_\Gamma f_s(z)dz = \sum_{a=1}^{N}\mathrm{Res}(f_s)(\mu_a) = -\mathrm{Res}(f_s)(\infty) = \frac{1}{2\pi i}\oint f_s(z)dz$$

Also,

$$\sum_{a=1}^{N}\mathrm{Res}(f_s)(\mu_a) = \sum_{a=1}^{N}\frac{\mu_a^s}{(\mu_a-\mu_1)(\mu_a-\mu_2)\cdots\overset{(a)}{\cdots}\cdots(\mu_a-\mu_N)}$$

$$\mathrm{Res}(f_s)(\infty) = \begin{cases} 0; & \forall s < N-1 \\ -1; & s = N-1 \end{cases}$$

and the lemma is proved.

Applying this result to the choice $\lambda_a = \mu_a$, $\forall a = 1,\ldots,N$, we obtain new identities

$$\sum_{a=1}^{N} \frac{\mu_a^s}{\phi'(\mu_a)} = 0, \forall s < N-1 \tag{76}$$

$$\sum_{a=1}^{N} \frac{\mu_a^{N-1}}{\phi'(\mu_a)} = 1 \tag{77}$$

because $\phi'(\mu_a) = (\mu_a - \mu_1)(\mu_a - \mu_2)\cdots\overset{(a)}{\cdots}\cdots(\mu_a - \mu_N)$. Alternatively if we take $\lambda_a = r_a$, the lemma implies that:

$$\sum_{a=1}^{N} \frac{r_a^s}{A'(r_a)} = 0, \forall s < N-1; \qquad \sum_{a=1}^{N} \frac{r_a^{N-1}}{A'(r_a)} = 1 \tag{78}$$

An important identity obtained from the lemma is:

$$\sum_{a=1}^{N} \frac{q_a^2}{(r_a - \mu_b)(r_a - \mu_c)} = 0; \forall b, c \tag{79}$$

The normal vectors to the family of quadrics (68)

$$Q_\mu(\vec q) = \sum_{a=1}^{N} \frac{q_a^2}{r_a - \mu} = 1$$

at the point $\vec q \equiv (q_1,\ldots,q_N)$, are

$$\vec n(\mu) \equiv (n_1(\mu),\ldots,n_N(\mu)) = \left(\frac{q_1}{r_1 - \mu},\ldots,\frac{q_N}{r_N - \mu}\right)$$

Observe that (79) implies

$$\sum_{a=1}^{N} n_a(\mu_b) n_a(\mu_c) = 0; \forall a, b$$

Therefore, all the quadrics are orthogonal with each other and the elliptic coordinates form an orthogonal system. The standard Euclidean metric in Cartesian coordinates can be expressed in elliptic coordinates in the form:

$$ds^2 = \sum_{a=1}^{N} dq_a^2 = \sum_{a=1}^{N}\sum_{b=1}^{N} g_{ab}(\vec\mu_E)\, d\mu_a\, d\mu_b$$

Derivation of the two members of equation (72) leads to:

$$\frac{2dq_a}{q_a} = (-1)^N \sum_{b=1}^{N} \frac{d\mu_b}{r_a - \mu_b}$$

and, using the Jacobi Lemma,

$$4dq_a^2 = q_a^2 \sum_{b=1}^{N} \frac{d\lambda_b^2}{r_a - \lambda_b}$$

Finally, we have:

$$g_{aa} = \frac{1}{4}\frac{\Lambda'(\lambda_a)}{A(\lambda_a)} = \frac{1}{4}\frac{\prod_{b\neq a;b=1}^{N}(\lambda_a - \lambda_b)}{\prod_{b=1}^{N}(\lambda_a - r_b)}; \; g_{ab} = 0; \forall a \neq b \tag{80}$$

The kinetic energy of a natural dynamical system in elliptic coordinates is;

$$T = \frac{1}{2}\sum_{a=1}^{N} \dot{q}_a^2 = \frac{1}{2}\sum_{a=1}^{N} g_{aa}\dot{\lambda}_a^2 \tag{81}$$

In terms of the canonical momentum $\pi_a = \frac{\partial T}{\partial \dot{\lambda}_a}$, T reads:

$$T = \sum_{a=1}^{N} \pi_a \dot{\lambda}_a - T = \frac{1}{2}\sum_{a=1}^{N} \frac{1}{g_{aa}}\pi_a^2 = 2\sum_{a=1}^{N} \frac{A(\lambda_a)}{\Lambda'(\lambda_a)}\pi_a^2 \tag{82}$$

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