N = 2 Supersymmetric Kinks and real algebraic curves

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Abstract

The kinks of the (1+1)-dimensional Wess-Zumino model with polynomic superpotential are investigated and shown to be related to real algebraic curves.

PACS: 11.27.+d, 11.30.Pb, 12.60.Jv.
Keywords: (1+1)-dimensional Wess-Zumino model, N = 2 Supersymmetry, BPS Kinks.

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The dimensional reduction of the (3+1)-dimensional Wess-Zumino model, produces an interesting (1+1)-dimensional Bose-Fermi system; this eld theory enjoys N = 2 extended supersymmetry provided that the interactions are introduced via a real harmonic superpotential, see [1]. In a recent paper [2] Gibbons and Townsend have shown the existence of domain-wall intersections in the (3+1)D WZ model, the authors relying on the supersymmetry algebra of the (2+1)D dimensional reduction of the system. Although the domain-wall junctions are two-dimensional structures, their properties are reminiscent of the one-dimensional kinks from which they are made. In this letter we shall thus describe the kinks of the underlying (1+1)-dimensional system.

The basic elds of the theory are:

- Two real bosonic elds, ${}^a(x)$, $a = 1;2$ that can be assembled in the complex eld: $(x) = {}^1(x) + i\,{}^2(x) \; 2 \; \mathrm{Maps}(R^{1;1};C)$. $x = (x^0;x^1)$ are local coordinates in the $R^{1;1}$ Minkowski space, where we choose the metric g ; $g^{00} = g^{11} = 1; g^{12} = g^{21} = 0$.

- Two Majorana spinor elds ${}^a(x)$, $a = 1;2$. We work in a Majorana representation of the Cliord algebra f ; g = 2g ,

$$^0 = \;^2; \qquad ^1 = i\,^1; \qquad ^5 = \;^0\;^1 = \;^3$$

where ^{1}, ^{2}, ^{3} are the Pauli matrices, such that $^a = \;^a$. We also dene the adjoint spinor as $(x) = {}^t(x)\;^0$ and consider Majorana-Weyl spinors: ${}^a(x) = \frac{1}{2}\;{}^5\;{}^a(x)$ with only one non-zero component.

Interactions are introduced through the holomorphic superpotential: $W(\;) = \frac{1}{2}\;W^1(\,^1;\,^2) + iW^2(\,^1;\,^2)$. One could in principle start from the supercharges:

$$\hat{Q}^{BC} = \int dx^1 \left[\sum_{a;b} f^B\;{}^{ab}\,(\partial_0\;{}^a \quad \partial_1\;{}^a)\;{}^b \quad \sum_c f^C\;{}^{bc}\,\frac{\partial W^C}{\partial\;{}^c}\;{}^a \right]$$

where W^B, $B = 1;2$, are respectively the real part if $B = 1$ and the imaginary part if $B = 2$ of $W(\;)$ and f^B is either the identity or the complex structure endomorphism in R^2 [3]:

$$f^{B=1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad f^{B=2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} :$$

Nevertheless, the Cauchy-Riemann equations:

$$\frac{\partial W^1}{\partial \phi^1} = \frac{\partial W^2}{\partial \phi^2} \qquad\qquad \frac{\partial W^1}{\partial \phi^2} = -\frac{\partial W^2}{\partial \phi^1}; \tag{1}$$

tell us that the theory is fully described by choosing either W^1 or W^2. We thus set $W^C = W^1$ and find the basic SUSY charges to be $\hat{Q}^{B1}_{\pm} = Q^B_{\pm}$:

$$Q^B_{\pm} = \int dx^1 \sum_{a,b} \left\{ f^{B\,ab} (\partial_0 \phi^a \pm \partial_1 \phi^a)\psi^b_{\pm} \mp \frac{\partial W^1}{\partial \phi^b}\psi^a_{\mp} \right\} \tag{2}$$

From the canonical quantization rules

$$[\phi^a(x_1), \dot{\phi}^b(y_1)] = i\delta^{ab}\delta(x_1 - y_1) = \{\psi^a(x_1), i\psi^b(y_1)\} \tag{3}$$

one checks that the $N = 2$ extended supersymmetric algebra

$$\{Q^B_{\pm}, Q^C_{\pm}\} = 2\delta^{BC} P_{\pm} \qquad\qquad \{Q^B_{+}, Q^C_{-}\} = -(-1)^B(\delta^{BC} 2T + \epsilon^{BC} 2\tilde{T}) \tag{4}$$

is closed by the four generators $Q^B_{\pm}$, defined in (2). Here

$$\begin{aligned} P_{\pm} &= \frac{1}{2}\int dx^1 \sum_a \left[(\partial_0 \phi^a \pm \partial_1 \phi^a)(\partial_0 \phi^a \pm \partial_1 \phi^a) \mp 2i\psi^a_{\pm}\partial_1 \psi^a_{\pm} \right] + \\ &+ \frac{1}{2}\int dx^1 \sum_a \left[\frac{\partial W^1}{\partial \phi^a}\frac{\partial W^1}{\partial \phi^a} \mp 2i \sum_b \frac{\partial^2 W^1}{\partial \phi^a \partial \phi^b}\psi^a_{+}\psi^b_{-} \right] \end{aligned}$$

are the light-cone momenta and

$$T = \int dx^1 \left(\frac{\partial W^1}{\partial \phi^1}\frac{\partial \phi^1}{\partial x^1} + \frac{\partial W^1}{\partial \phi^2}\frac{\partial \phi^2}{\partial x^1} \right) = \int dW^1 = W^1(\infty) - W^1(-\infty)$$

$$\tilde{T} = \int dx^1 \left(\frac{\partial W^1}{\partial \phi^1}\frac{\partial \phi^2}{\partial x^1} - \frac{\partial W^1}{\partial \phi^2}\frac{\partial \phi^1}{\partial x^1} \right) = \int d\tilde{W}^1 = W^2(\infty) - W^2(-\infty)$$

the central extensions.

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From the SUSY algebra one deduces,

$$2P_0 = 2|T| + (Q^B_{+} \mp (-1)^B Q^B_{-})^2 = 2|\tilde{T}| + (Q^B_{\pm} \mp (-1)^B \epsilon^{BC} Q^C_{\mp})^2;$$

see [4]. We thus define the charge operators on zero momentum states:

$$\begin{aligned} \tilde{Q}^1 &= Q^1_{+} - Q^1_{-} = \int dx^1 \sum_a \left(\partial_1 \phi^a - \frac{\partial W^1}{\partial \phi^a} \right)(\psi^a_{+} - \psi^a_{-}) \\ \tilde{Q}^2 &= Q^1_{-} - Q^2_{-} = \int dx^1 \sum_a \left(\partial_1 \phi^a - \sum_b f^{2\,ab}\frac{\partial W^1}{\partial \phi^b} \right)\left(\psi^a_{-} - \sum_c f^{2\,ac}\psi^c_{-} \right) \end{aligned}$$

Spatially extended coherent states built from the solutions of any of the two systems of first order equations, [5]:

$$\frac{d\phi^1}{dx^1} = \frac{\partial W^1}{\partial \phi^1} \qquad\qquad \frac{d\phi^2}{dx^1} = \frac{\partial W^1}{\partial \phi^2} \tag{5}$$

$$\frac{d\phi^1}{dx^1} = \frac{\partial W^1}{\partial \phi^2} \qquad\qquad \frac{d\phi^2}{dx^1} = -\frac{\partial W^1}{\partial \phi^1} \tag{6}$$

have minimum energy because they are respectively annihilated by $\tilde{Q}^1$ (system (5)) and $\tilde{Q}^2$ (system (6)).

The flow in $R^2 \simeq C$ of the solutions of (5) is given by:

$$\frac{d\phi^2}{d\phi^1} = \frac{\partial W^1}{\partial \phi^2}\left(\frac{\partial W^1}{\partial \phi^1}\right)^{-1} \Rightarrow \frac{\partial W^1}{\partial \phi^2}d\phi^1 - \frac{\partial W^1}{\partial \phi^1}d\phi^2 = dW^2 = 0$$

If $W(\phi)$ is polynomic in ϕ, the solutions of (5) live on the real algebraic curves determined by the equation:

$$W^2(\phi^1;\phi^2) = \gamma_\star \qquad (7)$$

where $\gamma_\star$ is a real constant. Similimodo, the solution flow of (6) in C,

$$\frac{d\phi^2}{d\phi^1} = -\frac{\partial W^1}{\partial \phi^1}\left(\frac{\partial W^1}{\partial \phi^2}\right)^{-1} \Rightarrow \frac{\partial W^1}{\partial \phi^1}d\phi^1 + \frac{\partial W^1}{\partial \phi^2}d\phi^2 = dW^1 = 0$$

runs on the real algebraic curves:

$$W^1(\phi^1;\phi^2) = \gamma \qquad (8)$$

where γ is another real constant. There are two observations: (I) Solutions of system (5) live on curves for which W^2 = constant and solutions of (6) have support on curves for which W^1 = constant. (II) The curves that support the solutions of (5) are orthogonal to the curves related to the solutions of (6).

Assume that $W(\phi)$ has a discrete set of extrema, forming the vacuum orbit of the system: $\frac{\partial W}{\partial \phi}\Big|_{v^{(i)}} = 0$, $i = 1;2;\dots;n$. Kinks are solutions of (5) and/or (6) such that they tend to $v^{(i_\pm)}$ when x_1 reaches $\pm\infty$. $v^{(i_+)}$ and $v^{(i_-)}$ thus belong either to curves (7) or (8), and this fixes the values of γ or $\gamma_\star$ for which the real algebraic curves support kinks. In Reference [6] a general proof based in singularity theory of the existence of these soliton solutions, that counts its number, is achieved. The energies of the states grown from kinks are $P_0 = |T| = |W^1(v^{(i_+)}) - W^1(v^{(i_-)})|$ for solutions of (5) and $P_0 = |\tilde{T}| = |W^2(v^{(i_+)}) - W^2(v^{(i_-)})|$ for solutions of (6). The kink form factor is obtained from a quadrature: one replaces either (7) or (8) in the first equation of (5) or (6) and integrates.

Therefore, the fermionic charges $\tilde{Q}^1$ and $\tilde{Q}^2$ are annihilated on coherent states $|K^1\rangle$ and $|K^2\rangle$ that correspond to the tensor product of the quantum antikink/kink, living respectively on curves W^2 = constant and W^1 = constant, with its supersymmetric partners (the translational mode times a constant spinor). We find

$$\tilde{Q}^1|K^1\rangle = \int dx_1 \sum_a \left[\partial_1\phi^a_{K^1} - \frac{\partial W^1}{\partial \phi^a}\Big|_{K^1}\right]\partial_1\phi^a_{K^1}\begin{pmatrix}1\\ -1\end{pmatrix}|K^1\rangle = 0$$

$$\tilde{Q}^2_+|K^2_+\rangle = \int dx_1 \sum_a \left[\partial_1\phi^a_{K^2_+} + \sum_b \varepsilon^{ab}\frac{\partial W^1}{\partial \phi^b}\Big|_{K^2_+}\right]\begin{pmatrix}\partial_1\phi^a_{K^2_+}\\ \sum_c \varepsilon^{ac}\partial_1\phi^c_{K^2_+}\end{pmatrix}|K^2_+\rangle = 0$$

$$\tilde{Q}^2|K^2\rangle = \int dx_1 \sum_a \left[\partial_1\phi^a_{K^2} - \sum_b \varepsilon^{ab}\frac{\partial W^1}{\partial \phi^b}\Big|_{K^2}\right]\begin{pmatrix}\sum_c \varepsilon^{ac}\partial_1\phi^c_{K^2}\\ \partial_1\phi^a_{K^2}\end{pmatrix}|K^2_+\rangle = 0$$

on solutions of (7) and/or (8); the SUSY kinks are thus $\frac{1}{4}$-BPS states. The energy of these states does not receive quantum corrections [1], because $N = 2$ supersymmetry forbids any anomaly in the central charges.

3

We focus on the case in which the potential is:

$$U(\phi) = \frac{1}{2}\sum_a \frac{\partial W^1}{\partial \phi^a}\frac{\partial W^1}{\partial \phi^a} = \frac{1}{2}\left[1 - 2(\phi_1^2 + \phi_2^2)^{\frac{n-1}{2}}\cos\left((n-1)\arctan\frac{\phi_2}{\phi_1}\right) + (\phi_1^2 + \phi_2^2)^{n-1}\right]$$

see [2] and [7]. In polar variables in the R^2 internal space,

$$\rho(x) = +\sqrt{[\phi^1(x)]^2 + [\phi^2(x)]^2}\,; \qquad \theta(x) = \arctan\frac{\phi^2(x)}{\phi^1(x)}$$

the potential reads:

$$U(\rho;\theta) = \frac{1}{2}\left[1 - 2\rho^{n-1}\cos(n-1)\theta + \rho^{2(n-1)}\right] \tag{9}$$

There is symmetry under the $D_{2(n-1)} \simeq Z_2 \times Z_{n-1}$ dihedral group: $\theta' = -\theta$, $\theta' = \theta + \frac{2\pi j}{n-1}$, $j = 0,1,2,\dots,n-2$. In Cartesian coordinates, these transformations form the $D_{2(n-1)}$ sub-group of $O(2)$ given by:

(1) $\phi_2^{\prime} = -\phi_2$; $\phi_1^{\prime} = \phi_1$

(2) $\phi_1{}' = \cos\frac{2\pi j}{n-1}\phi_1 - \sin\frac{2\pi j}{n-1}\phi_2$; $\phi_2{}' = \sin\frac{2\pi j}{n-1}\phi_1 + \cos\frac{2\pi j}{n-1}\phi_2$

The vacuum orbit is the set of $(n-1)$-roots of unity:

$$\mathcal{M} = \left\{ v^{(k)} = e^{i\frac{2\pi k}{n-1}} \right\} = \frac{D_{2(n-1)}}{Z_2} = Z_{n-1}\,. \tag{10}$$

When the $v^{(k)}$ vacuum is chosen to quantize the theory, the symmetry under the $D_{2(n-1)}$ group is spontaneously broken to the Z_2 sub-group generated by $\theta' = -\theta + \frac{4\pi k}{n-1}$; this transformation leaves a fixed point, $v^{(k)}$, if n is even and two fixed points, $v^{(k)}$ and $v^{(k+\frac{n-1}{2})}$, if n is odd.

The Z_{n-1}-symmetry allows for the existence of $(n-1)$ harmonic superpotentials that are equivalent: $W^{(j)}(\psi) = \frac{1}{2}\left(\psi^{(j)} - \frac{(\psi^{(j)})^n}{n}\right)$, $\psi^{(j)} = e^{i\frac{2\pi j}{n-1}}\psi$, all of them leading to the same potential U. Thus:

$$W^{(j)1} = \rho\cos\theta(j) - \frac{1}{n}\rho^n\cos n\theta(j) \qquad W^{(j)2} = \rho\sin\theta(j) - \frac{1}{n}\rho^n\sin n\theta(j)$$

where $\theta(j) = \theta + \frac{2\pi j}{n-1}$. There is room for closing the $N=2$ supersymmetry algebra (4) in $n-1$ equivalent forms: define the $n-1$ equivalent sets of SUSY charges:

$$Q^{(j)B} = \int dx^1 \sum_{a,b} f^{B\,ab}\left\{(\partial_0\phi^{(j)a} - \partial_1\phi^{(j)a})\chi^{(j)b} - \frac{\partial W^{(j)1}}{\partial\phi^{(j)a}}\chi^{(j)b}\right\};$$

also in terms of the "rotated" fermionic fields $\chi^{(j)a}$, and the corresponding central charges $T^{(j)}$ and $\tilde{T}^{(j)}$. Observe that the $N=2$ supersymmetry is unbroken, while the choice of vacuum that spontaneously breaks the Z_{n-1} symmetry does not affect the physics, which is the same for different values of j.

The j pairs of first-order systems of equations:

$$\frac{d\rho}{dx_1} = \sin\theta(j) - \rho^{n-1}\sin n\theta(j) \qquad \rho^2\frac{d\theta(j)}{dx_1} = \rho\cos\theta(j) - \rho^n\cos n\theta(j) \tag{11}$$

$$\frac{d\rho}{dx_1} = \cos\theta(j) - \rho^{n-1}\cos n\theta(j) \qquad \rho^2\frac{d\theta(j)}{dx_1} = -\rho\sin\theta(j) + \rho^n\sin n\theta(j) \tag{12}$$

correspond to (5) and (6) for this particular case. The solutions lie respectively on the algebraic curves

$$\rho\sin\theta(j) - \frac{1}{n}\rho^n\sin n\theta(j) = \gamma \tag{13}$$

$$\rho\cos\theta(j) - \frac{1}{n}\rho^n\cos n\theta(j) = \gamma \tag{14}$$

which form two families of orthogonal lines in R^2. In the family of curves (13) there are kinks joining the vacua $v^{(k)}$ and $v^{(k')}$ if and only if:

$$\sin\frac{2\pi(k+j)}{n-1} - \frac{1}{n}\sin\frac{2\pi(k+j)n}{n-1} = \sin\frac{2\pi(k'+j)}{n-1} - \frac{1}{n}\sin\frac{2\pi(k'+j)n}{n-1} = \gamma^K \tag{15}$$

This xes the value of $_? = {}^K_?$ for which the algebraic curve supports a topological kink. Similimodo,

$$\cos\frac{2\ (k+j)}{n\quad 1}\quad\frac{1}{n}\cos\frac{2\ (k+j)n}{n\quad 1} = \cos\frac{2\ (k^0+j)}{n\quad 1}\quad\frac{1}{n}\cos\frac{2\ (k^0+j)n}{n\quad 1} = {}^K \qquad (16)$$

is the value of the constant if the kink belong to the orthogonal family (14). Solutions of (15) and/or (16) exist, respectively, if and only if

$$2(k+k^0+2j) = n\quad 1 \bmod 2(n\quad 1) \qquad (17)$$

and/or

$$k+k^0+2j = 0 \bmod n\quad 1 \qquad (18)$$

Given the kink curves, the kink form factors are obtained in the following way:
One solves for in (13) or (14),

$$+\frac{2\ j}{n\quad 1} = h({}^K;\ ) \qquad ; \qquad +\frac{2\ j}{n\quad 1} = h_?({}^K_?;\ ) \qquad (19)$$

and plugs these expressions into the rst equation of (12) or (11),

$$\frac{d}{dx_1} = \sin h({}^K;\ )\quad {}^{n\ 1}\sin[nh({}^K;\ )]\quad ;\quad \frac{d}{dx_1} = \cos h_?({}^K_?;\ )\quad {}^{n\ 1}\cos[nh_?({}^K_?;\ )]$$

which are immediately integrated by quadratures: if a is an integration constant

$$\int\frac{d}{\sin h({}^K;\ )\quad {}^{n\ 1}\sin[nh({}^K;\ )]} = (x_1+a) \qquad (20)$$

$$\int\frac{d}{\cos h_?({}^K_?;\ )\quad {}^{n\ 1}\cos[nh_?({}^K_?;\ )]} = (x_1+a) \qquad (21)$$

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We rst consider the lower odd cases, only for $W^{(j=0)}$. The other kinks are obtained by application of a $Z_{n\ 1}$ rotation.

n = 3:

- Superpotential: $W(\ ) = \frac{1}{2}\qquad\frac{^3}{3}$

$$W^1 = {}_1\quad\frac{{}^3_1}{3} + {}_1\ {}^2_2 \qquad W^2 = {}_2\quad {}^2_1\ {}_2 + \frac{{}^3_2}{3}$$

- Potential: $U({}_1;{}_2) = \frac{1}{2}[(\qquad 1)^2 + 4\ {}^2_2]$
- Vacuum orbit: $M = \frac{D_2}{Z_2} = \{v^0 = 1; v^1 = \quad 1\}$
- Real algebraic curves:

$${}_1\quad\frac{{}^3_1}{3} + {}_1\ {}^2_2 = \quad ; \qquad {}_2\quad {}^2_1\ {}_2 + \frac{{}^3_2}{3} = {}_?$$

- Kink curve: $_? = 0$ ($W^2 = 0$), tantamount to $_2 = 0$.
- Kink form factor:
  a) Solutions of $\frac{d_1}{dx_1} = (1\quad {}^2_1)$ on $^2 = 0$: ${}^{K_1}_1(x_1) = \quad\tanh(x_1+a)$
- Kink energy: $P_0[{}^{K_1}] = jTj = W^1(v^0)\quad W^1(v^1) = \frac{4}{3}$
- Conserved SUSY charge: $Q^1\ K^1 = 0$

n = 5:

- Superpotential: $W(\ ) = \frac{1}{2}\qquad \frac{^{5}}{5}$

$$W^{1} = {}_{1}\ 1\quad \frac{^{4}_{1}}{5} + 2\,{}^{2}_{1}\,{}^{2}_{2}\quad {}^{4}_{2} \qquad\qquad W^{2} = {}_{2}\ 1\quad {}^{4}_{1} + 2\,{}^{2}_{1}\,{}^{2}_{2}\quad \frac{^{4}_{2}}{5}$$

- Potential: $U(\ _1;\ _2) = \frac{1}{2}[(\quad + 1)^2\quad 4\ {}^{2}_{1}][(\quad + 1)^2\quad 4\ {}^{2}_{2}]$
- Vacuum orbit: $M = \frac{D_4}{Z_2} = f v^0 = 1; v^1 = i; v^2 = \quad 1; v^3 = \quad i g$
- Real algebraic curves:

$${}_{1}\ 1\quad \frac{^{4}_{1}}{5} + 2\,{}^{2}_{1}\,{}^{2}_{2}\quad {}^{4}_{2}\ = \quad ;\qquad {}_{2}\ 1\quad {}^{4}_{1} + 2\,{}^{2}_{1}\,{}^{2}_{2}\quad \frac{^{4}_{2}}{5}\ = \ _{?}$$

- Kink curves: a) $_{?} = 0\quad _2 = 0$, b) $\ = 0\quad _1 = 0$.
- Kink form factor:
  - a) Solutions of $\frac{d\ _1}{dx_1} = 1\quad {}^{4}_{1}$ on $_2 = 0$: $\arctan\ {}^{K^1}_{1} + \operatorname{arctanh}\ {}^{K^1}_{1} = \quad 2(x_1 + a)$
  - b) Solutions of $\frac{d\ _2}{dx_1} = 1\quad {}^{4}_{2}$ on $_1 = 0$: $\arctan\ {}^{K^2}_{2} + \operatorname{arctanh}\ {}^{K^2}_{2} = \quad 2(x_1 + a)$
- Kink energies: (a) $P_0[\ ^{K^1}] = jT j = \ W^1(v^0)\quad W^1(v^2) = \frac{8}{5}$

  (b) $P_0[\ ^{K^2}] = j\tilde{T} j = \ W^2(v^1)\quad W^2(v^3) = \frac{8}{5}$
- Conserved SUSY charges: (a) $Q^1\ \ K^1 = 0$; (b) $\tilde{Q}^2\ \ K^2 = 0$

n = 7:

- Superpotential: $W[\ ] = \frac{1}{2}\qquad \frac{^{7}}{7}$

$$W^{1} = {}_{1}\quad \frac{^{7}_{1}}{7} + 3\,{}^{5}_{1}\,{}^{2}_{2}\quad 5\,{}^{3}_{1}\,{}^{4}_{2} + \ _1\,{}^{6}_{2} \qquad\qquad W^{2} = {}_{2}\quad {}^{6}_{1}\,{}_{2} + 5\,{}^{4}_{1}\,{}^{3}_{2}\quad 3\,{}^{2}_{1}\,{}^{5}_{2} + \frac{^{7}_{2}}{7}$$

- Potential: $U(\ _1;\ _2) = \frac{1}{2}\ (\quad)^6\quad 2(\ {}^{2}_{1}\quad {}^{2}_{2})\ (\quad)^2\quad 16\ {}^{2}_{1}\ {}^{2}_{2} + 1$
- Vacuum orbit:

  $M = \frac{D_6}{Z_2} = \quad v^0 = 1; v^1 = \frac{1}{2} + i\frac{\sqrt{3}}{2}; v^2 = \quad \frac{1}{2} + i\frac{\sqrt{3}}{2}; v^3 = \quad 1; v^4 = \quad \frac{1}{2}\quad i\frac{\sqrt{3}}{2}; v^5 = \frac{1}{2}\quad i\frac{\sqrt{3}}{2}$
- Real algebraic curves:

$${}_{1}\quad \frac{^{7}_{1}}{7} + 3\,{}^{5}_{1}\,{}^{2}_{2}\quad 5\,{}^{3}_{1}\,{}^{4}_{2} + \ _1\,{}^{6}_{2} = \quad ;\qquad {}_{2}\quad {}^{6}_{1}\,{}_{2} + 5\,{}^{4}_{1}\,{}^{3}_{2}\quad 3\,{}^{2}_{1}\,{}^{5}_{2} + \frac{^{7}_{2}}{7} = \quad _{?}$$

- Kink curves: there are two choices of $_{?}$ and three choices of for which one nds kink curves. The other kinks associated with the other superpotentials can be obtained by Z_6 rotations.
  - a) $_{?} = \frac{3\sqrt{3}}{7}$: kink curve joining v^1 with v^2

    $_{?} = \quad \frac{3\sqrt{3}}{7}$: kink curve joining v^4 with v^5
  - b) $\ = \frac{3}{7}$: kink curve joining v^1 with v^5

    $\ = \quad \frac{3}{7}$: kink curve joining v^2 with v^4
  - c) $\ = 0$: kink curve joining v^0 with v^3
- Kink energies: a) $P_0[\ ^{K^1}] = jT j = jW^1(v^k)\quad W^1(v^{k+1}) j = \frac{6}{7}$

  b) $P_0[\ ^{K^2}] = j\tilde{T} j = jW^2(v^k)\quad W^2(v^{k+2}) j = \frac{6\sqrt{3}}{7}$

  c) $P_0[\ ^{K^1}] = jT j = jW^2(v^k)\quad W^1(v^{k+3}) j = \frac{12}{7}$
- Conserved SUSY charges: (a) $Q^1\ \ K^1 = 0$. (b) and (c) $\tilde{Q}^2\ \ K^2 = 0$

We now study two even cases.

The first and most interesting model occurs for $n = 4$. Here, we find that the kink curves are straight lines in W-space (true for any n) and curved in ϕ-space, in agreement with Reference [8]:

– Superpotential: $W[\phi] = \frac{1}{2}\left(\phi - \frac{\bar{\phi}^4}{4}\right)$

$$W^1 = \phi_1 - \frac{\phi_1^4}{4} + \frac{3}{2}\phi_1^2\phi_2^2 - \frac{\phi_2^4}{4} \qquad W^2 = \phi_2\left(1 - \phi_1^3 + \phi_1\phi_2^2\right)$$

– Potential: $U(\phi_1;\phi_2) = \frac{1}{2}\left[(\phi\bar{\phi})^3 - 2\phi_1(\phi_1^2 - 3\phi_2^2) + 1\right]$

– Vacuum orbit: $M = \frac{D_3}{Z_2} = \{v^0 = 1; v^1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}; v^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}\}$

– Real algebraic curves:

$$\phi_1 - \frac{\phi_1^4}{4} + \frac{3}{2}\phi_1^2\phi_2^2 - \frac{\phi_2^4}{4} = \alpha \; ; \qquad \phi_2\left(1 - \phi_1^3 + \phi_1\phi_2^2\right) = \beta$$

– Kink curve: $\alpha = -\frac{3}{8}$

– Kink form factor: on the kink curve we find $\phi_1^{K_2} = f^{-1}[\pm(x + a)]$ where

$$f(\phi_1) = \int \frac{d\phi_1}{\sqrt{\frac{3}{2} + 4\phi_1 + 8\phi_1^4}\sqrt{3\phi_1^2 - \sqrt{\frac{3}{2} + 4\phi_1 + 8\phi_1^4}}}$$

– Kink energy: $P_0[\phi^{K_2}] = |\tilde{T}| = |W^2(v^k) - W^2(v^{k+1})| = \frac{3\sqrt{3}}{4}$

– Conserved SUSY charge: $\tilde{Q}^2[\phi^{K_2}] = 0$

$n = 6$:

– Superpotential: $W[\phi] = \frac{1}{2}\left(\phi - \frac{\bar{\phi}^6}{6}\right)$

$$W^1 = \phi_1 - \frac{\phi_1^6}{6} + \frac{5}{2}\phi_1^4\phi_2^2 - \frac{5}{2}\phi_1^2\phi_2^4 + \frac{\phi_2^6}{6}$$

$$W^2 = \phi_2\left(1 - \phi_1^5 + \frac{10}{3}\phi_1^3\phi_2^2 - \phi_1\phi_2^4\right)$$

– Potential: $U(\phi_1;\phi_2) = \frac{1}{2}\left[(\phi\bar{\phi})^5 - 2\phi_1(\phi_1^4 + 5\phi_2^4 - 10\phi_1^2\phi_2^2) + 1\right]$

– Vacuum orbit: $M = \frac{D_5}{Z_2} = \{v^0 = 1; v^1 = e^{i\frac{2\pi}{5}}; v^2 = e^{i\frac{4\pi}{5}}; v^3 = e^{i\frac{6\pi}{5}}; v^4 = e^{i\frac{8\pi}{5}}\}$

– Real algebraic curves:

$$\phi_1 - \frac{\phi_1^6}{6} + \frac{5}{2}\phi_1^4\phi_2^2 - \frac{5}{2}\phi_1^2\phi_2^4 + \frac{\phi_2^6}{6} = \alpha \; ; \quad \phi_2\left(1 - \phi_1^5 + \frac{10}{3}\phi_1^3\phi_2^2 - \phi_1\phi_2^4\right) = \beta$$

– Kink curves: there are two values of α giving kink curves: a) $\alpha = \frac{5}{24}(1 + \sqrt{5})$: kink curve joining v^2 with v^3, b) $\alpha = \frac{5}{24}(-1 + \sqrt{5})$: kink curve joining v^1 with v^4. The other kink curves are obtained through Z_5 rotations.

– Kink energies: a) $P_0[\phi^{K_2}] = |\tilde{T}| = |W^2(v^k) - W^2(v^{k+1})| = \frac{5}{6}\sqrt{\frac{5 - \sqrt{5}}{2}}$

b) $P_0[\phi^{K_2}] = |\tilde{T}| = |W^2(v^k) - W^2(v^{k+2})| = \frac{5}{6}\sqrt{\frac{5 + \sqrt{5}}{2}}$

– Conserved SUSY charges: (a) and (b) $\tilde{Q}^2[\phi^{K_2}] = 0$

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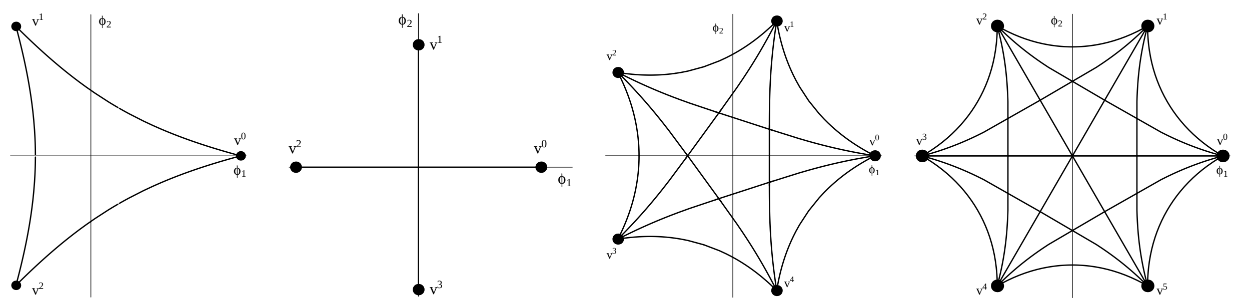

Figure 1: Kink curves in the n = 4, n = 5, n = 6 and n = 7 models