Phases of dual superconductivity and confinement in softly broken $\mathcal{N}=2$ supersymmetric Yang–Mills theories

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We study the electric flux tubes that undertake color confinement in $\mathcal{N}=2$ supersymmetric Yang–Mills theories softly broken down to $\mathcal{N}=1$ by perturbing with the first two Casimir operators. The relevant Abelian Higgs model is not the standard one due to the presence of an off-diagonal coupling among different magnetic $U(1)$ factors. We perform a preliminary study of this model at a qualitative level. BPS vortices are explicitely obtained for particular values of the soft breaking parameters. Generically however, even in the ultrastrong scaling limit, vortices are not critical but live in a "hybrid" type II phase. Also, ratios among string tensions are seen to follow no simple pattern. We examine the situation at the half Higgsed vacua and find evidence for solutions with the behaviour of superconducting strings. In some cases they are solutions to BPS equations.

PACS numbers: 11.27.+d 11.30.Pb 12.38.Aw

Preprint numbers: US-FT/26-99 FFUOV-99/13

arXiv:hep-th/0001184v2 19 Jun 2000

I. INTRODUCTION

Certainly, one of the most beautiful ideas in the context of quantum chromodynamics (QCD) is the confinement mechanism envisaged by 't Hooft [1] and Mandelstam [2] through the condensation of light monopoles. In essence it states that the QCD vacuum should behave as a *dual* superconductor where magnetic order takes place, and electric flux tubes form thus producing color confinement. In the context of QCD it stands for a kind of descriptive scheme, as long as it is not known how magnetically charged quanta can arise and condense in the effective low energy theory. In this respect, the idea of Abelian projection proposed by 't Hooft has received increasing support from numerical simulations on the lattice in the last few years [3]. Even in the continuum, recent work using a novel parametrization of QCD [4], points in the direction of the above scenario for color confinement [5]. From the analytical side, the understanding of non-perturbative phenomena in four dimensional quantum field theory has been put several steps forward since Seiberg and Witten constructed an exact solution for the low energy dynamics of $SU(2)$ $\mathcal{N}=2$ supersymmetric Yang-Mills theory [6]. In particular, it was possible for them to show that the mechanism of color confinement devised by 't Hooft and Mandelstam takes place when supersymmetry is broken down to $\mathcal{N}=1$. These results were soon extended to the case of $SU(N)$ [7–9]. Furthermore, when $\mathcal{N}=2$ supersymmetry is softly broken down to $\mathcal{N}=0$, the same mechanism has been shown to persist [10].

In spite of the fact that these results are well known, not much attention has been paid to the actual solutions in the strong coupling limit corresponding to electric flux lines that would undertake quark confinement. In Ref. [8], it was shown that this sort of vortices should have a spectrum of string tensions that distinguishes among different factors in the magnetic $U(1)^{N-1}$ theory arising in the infrared. The same result was found in the framework of the M-theory fivebrane version of QCD, also named MQCD [11]. The string tension of the $N-1$ electric flux tubes T_k , $k=1,...,N-1$, is given –up to a dimensionful factor which is different for each theory but independent of k–, by a dimensionless function $f_N(k)=\sin(\pi k/N)$. This function is somehow universal as long as the soft breaking perturbation is carried by a single Casimir operator [8,11,12]. Even in that case, the problem of finding such solutions in the particular model that emerges in this context has not yet been addressed in detail, [1] probably due to the

[1] Except for $SU(2)$, both at the maximal singularity of the Coulomb branch for pure gauge [13], and on the Higgs branch in the theory with massive fundamental matter [14].

naive expectation that the effective theory consists of $N-1$ copies of the standard $U(1)$ Abelian Higgs model. It was already shown by Douglas and Shenker that the magnetic $U(1)$ factors of the infrared quantum theory describing the neighborhood of the monopole singularities are coupled [8]. The existence of these off-diagonal couplings, $\tau_{ij}^{\rm off}$, was confirmed in two different frameworks. First, they appear in the expression of the Donaldson–Witten functional for gauge group $SU(N)$ [15]. Moreover, these couplings were shown to satisfy a stringent constraint coming from the Whitham hierarchy formulation of the Seiberg–Witten solution in Ref. [16] where, in addition, a general ansatz for $\tau_{ij}^{\rm off}$ is given.

In this paper, we extend the work [13] to the case of $SU(N)$, $\mathcal{N}=2$ supersymmetric Yang-Mills theory softly broken to $\mathcal{N}=1$. The analysis is performed in a "peculiar" scaling limit (named "ultrastrong" in [11]). We show that, even in that limit, generically there are no BPS electric flux tubes. A perturbative analysis leads to the conclusion that the phase of dual superconductivity is of type II *i.e.* there is a short range repulsive force between different vortices. This fact supports the expectation that indeed electric flux lines are safely confined into stable flux tubes, a feature of the confinement mechanism which is not *a priori* granted. [2] It is worth mentioning in this respect that numerical simulations in lattice QCD seem to point out that the type of Abelian Higgs model behind the picture of dual superconductivity is a critical one between type I and type II [18].

The plan of the paper is as follows. The setup of the problem is given in Section II where some aspects of the low-energy dynamics of $\mathcal{N}=2$ supersymmetric gauge theories softly broken down to $\mathcal{N}=1$ are reviewed. We emphasize the existence of non-vanishing couplings between the different $U(1)$ factors –even at the maximal singularity–, which play an essential rôle in our results. In Section III, we show that the string tension of vortex-like configurations obeys a Bogomol'nyi bound in the ultrastrong scaling limit. However, there are no BPS electric vortices in the system unless the complex phases of the soft breaking parameters corresponding to different Casimir operators are aligned. Even in this case, we show that the string tensions of the resulting BPS vortices are governed by a dimensionless function $f_N(k)$ which is different from the one obtained in [8,11], the latter being recovered as a particular limit of our system corresponding to a single quadratic $\mathcal{N}=1$ perturbation. In Section IV, we focus for convenience on the group $SU(3)$ and analyze the critical vortex solutions in certain simplified cases. We speculate about the full spectrum of such configurations. A perturbative analysis of the dynamics expected for nearly critical vortices is performed in sections V and VI by means of energetic arguments. This analysis reveals the existence of repulsive forces among vortices corresponding to different magnetic $U(1)$ factors. Thereafter we refer to this phase as an "hybrid" type II phase. In Section VII, the half Higgsed vacua are considered. The similarities and differences with the model proposed by Witten to describe cosmic superconducting strings [19] are discussed. We find solutions to the Bogomol'nyi equations with the behavior of superconducting strings. Finally, Section VIII is devoted to our conclusions and further remarks.

II. INFRARED DYNAMICS AT MAXIMAL SINGULARITIES

The quantum moduli space of vacua $\mathcal{M}_\Lambda$ of $SU(N)$, $\mathcal{N}=2$ supersymmetric gauge theory has a singular locus given by hypersurfaces of complex codimension one which may intersect with each other [7]. Along each of these hypersurfaces, an extra massless degree of freedom –whose quantum numbers can be read off from the monodromy matrix corresponding to a closed path encircling the singularity–, must be included into the effective action. At the intersections, many states become simultaneously massless. Of special interest are those singularities where $N-1$, *i.e.* the maximum allowed number of mutually local states, become massless. They are accordingly called maximal singularities. [3]

The addition of a microscopic superpotential breaks supersymmetry and leads to an $\mathcal{N}=1$ theory

$$W_{\mathcal{N}=1} = \sum_{k=2}^{N} \frac{1}{k}\, \lambda_k\ {\rm Tr}\Phi^k\ . \tag{1}$$

[2] It could happen, for example, that the electric vortices result to be unstable, and their core grows and smears in such a way that they do not lead to confinement of electric charges [17].

[3] In $SU(3)$, for example, the singular locus is given by the complex curves

$$4u^3 - 27(v \pm 2\Lambda^3)^2 = 0\ ,$$

where $u = 1/2\langle {\rm Tr}\phi^2\rangle$ and $v = 1/3\langle {\rm Tr}\phi^3\rangle$ are the gauge invariant order parameters constructed out of the scalar field belonging to the $\mathcal{N}=2$ vector supermultiplet, and Λ is the quantum dynamical scale. Higher intersections of these curves lead to the so-called Z_2 and Z_3 singularities, given respectively by the points $\{u^3 = 27\Lambda^6, v=0\}$ and $\{u=0, v^2 = 4\Lambda^6\}$.

Notice that those contributions in (1) with $k > 3$ are non-renormalizable. However, this does not necessarily mean that they do not affect the low-energy dynamics. They could be dangerously irrelevant operators [20,21]. We will not discuss these subtleties here, and shall restrict ourselves to the case of up to cubic perturbations,

$$W_{\mathcal{N}=1} = \frac{1}{2}\ \mu\ \mathrm{Tr}\Phi^2 + \frac{1}{3}\ \nu\ \mathrm{Tr}\Phi^3\ . \tag{2}$$

This breaking is soft, in fact renormalizable. The continuum vacuum degeneracy is lifted except for a given set of points which depend on the actual values of the parameters μ and ν. [4]

Let us focus on the low-energy effective field theory near a maximal point that we choose, for simplicity, to be that with real quadratic Casimir, $u = N\Lambda^2$. This is a dual $\mathcal{N} = 2$ supersymmetric gauge system with gauge group $U(1)^{N-1}$ which includes both chiral multiplets $\Psi_i^D = (\chi_i^D, V_i^D)$, as well as hypermultiplets $H_i = (M_i, \tilde{M}_i)$ that correspond to the monopoles that become light in that patch of the moduli space. One can choose a homology basis for the cycles on the auxiliary curve such that each monopole has a unit of charge with respect to each dual gauge field. The quantities χ_i^D, M_i and $\tilde{M}_i$ are chiral $\mathcal{N} = 1$ superfields, while V_i^D are $\mathcal{N} = 1$ vector superfields (and W_α^{Di} their corresponding superfield strengths). For completeness, we give also the $\mathcal{N} = 0$ content of these superfields

$$\begin{aligned} \chi_i^D &= (\phi_i^D, \psi_i, F_i) & \qquad V_i^D &= ((A_\mu^D)_i, \lambda_i, D_i) \\ M_i &= (\phi_{m_i}, \psi_{m_i}, F_{m_i}) & \qquad \tilde{M}_i &= (\tilde{\phi}_{m_i}, \tilde{\psi}_{m_i}, \tilde{F}_{m_i}) \end{aligned}$$

where the notation for fermionic, bosonic and auxiliary components is the standard one. Setting $a_i^D \equiv \langle \phi_i^D \rangle$, the coordinates at the point of maximal singularity that we are focusing on are $a_i^D = 0$. The dominant piece of the $\mathcal{N} = 2$ low energy effective Lagrangian is given in terms of a holomorphic function $\mathcal{F}$, called the effective prepotential

$$\begin{aligned} \mathcal{L}_{eff}^{N=2} = &\frac{1}{4\pi}\mathrm{Im}\left[\int d^4\theta\ \frac{\partial \mathcal{F}(\chi^D)}{\partial \chi_i^D}\chi_i^{D\dagger} + \frac{1}{2}\int d^2\theta\ \frac{\partial^2 \mathcal{F}(\chi^D)}{\partial \chi_i^D \partial \chi_j^D} W_\alpha^{Di} W^{D\alpha j}\right] \\ &+ \int d^4\theta \{M_i^\dagger e^{2V_i^D} M_i + \tilde{M}_i^\dagger e^{-2V_i^D}\tilde{M}_i\} + \mathrm{Re}\ \int d^2\theta\ W(\chi^D, M, \tilde{M})\ . \end{aligned} \tag{3}$$

The monopole fields have been "integrated in" in order to soak up the singularity of the effective action when $a_i^D = \langle \phi_i^D \rangle \to 0$ where M_i becomes massless. The effective superpotential at low energies is

$$W(\chi^D, M, \tilde{M}) = \sqrt{2} M_i \chi_i^D \tilde{M}_i + \mu\, \mathcal{U}(\chi^D) + \nu\ \mathcal{V}(\chi^D)\ , \tag{4}$$

the last two terms being the effective contribution of the supersymmetry breaking superpotential (2). In fact, $\mathcal{U}$ and $\mathcal{V}$ are the Abelian superfields arising respectively from the quadratic and cubic Casimir operators in the low-energy theory. The vacuum expectation value of their lowest components, U and V, are the holomorphic coordinates in $\mathcal{M}_\Lambda$, $\langle U \rangle = u$ and $\langle V \rangle = v$.

Written in component fields, the bosonic sector of the system is described by the Lagrangian

$$\begin{aligned} \mathcal{L}_{eff,\ B}^{\mathcal{N}=1} = &-\frac{1}{4} b_{ij} (F_{\mu\nu})_i (F^{\mu\nu})_j + (D_\mu \phi_{m_i})^* D^\mu \phi_{m_i} + D_\mu \tilde{\phi}_{m_i} (D^\mu \tilde{\phi}_{m_i})^* + b_{ij} \partial_\mu \phi_i^{D^*} \partial^\mu \phi_j^D \\ &- \left[\frac{1}{2} b_{ij} D_i D_j + b_{ij} F_i^* F_j + F_{m_i}^* F_{m_i} + \tilde{F}_{m_i}^* \tilde{F}_{m_i}\right]\ , \end{aligned} \tag{5}$$

where the auxiliary fields are solved as

$$D_i = -b_{ij}^{-1}(|\phi_{m_j}|^2 - |\tilde{\phi}_{m_j}|^2) \qquad F_i = -b_{ij}^{-1}\left(\sqrt{2}\phi_{m_j}\tilde{\phi}_{m_j} + C_j\right) \tag{6}$$

$$F_{m_i} = -\sqrt{2}\phi_i^{D^*}\tilde{\phi}_{m_i}^* \qquad \tilde{F}_{m_i} = -\sqrt{2}\phi_i^{D^*}\phi_{m_i}^*\ , \tag{7}$$

whereas field strengths and covariant derivatives are given by

[4] For instance, in the case of $SU(3)$, the theory has generically five $\mathcal{N} = 1$ vacua, three of which are the maximal Z_2 points. In the limit $\mu \to 0$, the remaining vacua approach the Z_3 points [9].

$$(F_{\mu\nu})_i = \partial_\mu (A^D_\nu)_i - \partial_\nu (A^D_\mu)_i \ , \tag{8}$$

$$D_\mu \phi_{m_i} = \partial_\mu \phi_{m_i} + i(A^D_\mu)_i \phi_{m_i} \qquad D_\mu \tilde{\phi}_{m_i} = \partial_\mu \tilde{\phi}_{m_i} - i(A^D_\mu)_i \tilde{\phi}_{m_i} \ . \tag{9}$$

Concerning C_j in (6), it stands for

$$C_j(\phi^D) = \mu U_j(\phi^D) + \nu V_j(\phi^D) \equiv |C_j| e^{i\beta_j} \ , \tag{10}$$

where U_j and V_j are the derivatives of U and V with respect to ϕ^D_j [8],

$$U_j(\phi^D) = u^{(0)}_j \Lambda + \sum_{p\geq 1} u^{(p)}_j(\phi^D)\Lambda^{1-p} \ , \qquad u^{(0)}_j = -2j \sin\hat{\theta}_j \tag{11}$$

$$V_j(\phi^D) = v^{(0)}_j \Lambda^2 + \sum_{p\geq 1} v^{(p)}_j(\phi^D)\Lambda^{2-p} \ , \qquad v^{(0)}_j = -2j \sin 2\hat{\theta}_j \ , \tag{12}$$

while $u^{(p)}_j(\phi^D)$ and $v^{(p)}_j(\phi^D)$ are homogeneous polynomials in ϕ^D_i of degree p, so that C_j are regular functions in the vicinity of the maximal singularity. Finally, b_{ij} is ($\frac{1}{4\pi}$ times) the imaginary part of the period matrix τ^D_{ij},

$$\tau^D_{ij}(\phi^D) = \frac{\partial^2 \mathcal{F}}{\partial \phi^D_i \partial \phi^D_j} = \frac{1}{2\pi i} \log\left(\frac{\phi^D_i}{\Lambda_i}\right) \delta_{ij} + \tau^{\rm off}_{ij} + \mathcal{O}\left(\frac{\phi^D}{\Lambda}\right) \ , \tag{13}$$

where $\Lambda_j = \Lambda \sin\hat{\theta}_j$ and $\hat{\theta}_j = j\pi/N$. When expanding around the vacuum expectation value $a^D_i = \langle \phi^D_i \rangle$, $\tau^D_{ij}(\phi^D)$ yields the effective coupling constant matrix. The logarithmic singularity when $a^D_i = 0$ corresponds to the perturbative running of the dual coupling constant up to the maximal point, displaying the asymptotic freedom of the dual description. The coupling flows to zero due to the fact that the quantum fluctuations of massless monopoles have been integrated out. This is fine as long as one is interested only in searching for vacuum solutions. Then M and $M^\dagger$ in (3)–(4) stand for the zero modes of the monopole field (see the discussion in [10]). Here, however, in order not to run into double counting of degrees of freedom we should introduce, on physical grounds, an infrared cut off for the monopole loop integrals. In each $U(1)$ factor the natural energy scale is set by the soft breaking parameters $a^D_i \sim |C^{(0)}_i|^{1/2}$ with

$$C^{(0)}_i = \mu u^{(0)}_i \Lambda + \nu v^{(0)}_i \Lambda^2 = -2i\Lambda(\mu \sin\hat{\theta}_i + \nu\Lambda \sin 2\hat{\theta}_i) \ , \tag{14}$$

and the perturbative couplings of each monopole to its corresponding dual vector field,

$$\frac{4\pi}{g^2_{Di}} \simeq -\frac{1}{4\pi} \log\left(\frac{|C^{(0)}_i|}{\Lambda^2_i}\right) \ , \tag{15}$$

show logarithmic variations among different $U(1)$ factors. [5]

Even in the close vicinity of the singularity, different magnetic $U(1)$ factors are coupled through $\tau^{\rm off}_{ij}$ [7,8]. Exactly at the singularity, *i.e.* at $a^D_i = 0$, the generic expression proposed in [16] for these off-diagonal couplings is

$$\tau^{\rm off}_{ij} = \frac{2i}{N^2\pi} \sum_{k=1}^{N-1} \sin k\hat{\theta}_i \sin k\hat{\theta}_j \sum_{p,q=1}^{N} \tau^{(0)}_{pq} \cos k\theta_p \cos k\theta_q \ , \tag{16}$$

where $\tau^{(0)}_{pq}$ is given by

$$\tau^{(0)}_{pq} = \delta_{pq} \sum_{k\neq p} \log(2\cos\theta_p - 2\cos\theta_k)^2 - (1-\delta_{pq}) \log(2\cos\theta_p - 2\cos\theta_q)^2 \ , \tag{17}$$

[5] In other words, we are dealing here with a macroscopic (classical) theory of the Ginzburg–Landau type, and we should consider the coupling constant of the M_i and $\tilde{M}^\dagger_i$ *classical* fields to V^D_i: wave-particle duality connects g_D with the running coupling constant of the quantum theory through the formula $\hbar g_{Di} = g_{Di}(a^D_i \sim |C^{(0)}_i|^{1/2})$, the strong coupling limit becoming the classical limit for the magnetically charged quanta [22].

with $\theta_p = (p-1/2)\pi/N$ and $p,q = 1,...,N$. In the case of $SU(3)$, for example, $\tau_{12}^{\rm off} = i/\pi \log 2$ [7,8,16]. These interactions are also present in the effective potential obtained from the terms in square brackets of (5),

$$V_{eff} = \frac{1}{2} b_{ij}^{-1}(\phi^D)(|\phi_{m_i}|^2 - |\tilde{\phi}_{m_i}|^2)(|\phi_{m_j}|^2 - |\tilde{\phi}_{m_j}|^2) + 2|\phi_i^D|^2(|\phi_{m_i}|^2 + |\tilde{\phi}_{m_i}|^2)$$
$$+ b_{ij}^{-1}(\phi^D)(\sqrt{2}\phi_{m_i}\tilde{\phi}_{m_i} + C_i(\phi^D))(\sqrt{2}\phi_{m_j}\tilde{\phi}_{m_j} + C_j(\phi^D))^* \ . \qquad (18)$$

Notice that, b_{ij}^{-1} being positive definite, the potential is either positive or zero. Given the expectation values of the complex scalars,

$$\langle \phi_i^D \rangle = a_i^D \qquad \langle \phi_{m_i} \rangle = m_i \qquad \langle \tilde{\phi}_{m_i} \rangle = \tilde{m}_i \ , \qquad (19)$$

$\mathcal{N}=1$ supersymmetric vacua are in one to one correspondence with zeroes of V_{eff}:

$$\sqrt{2} m_i \tilde{m}_i = -C_i(a^D) \ , \qquad (20)$$

$$m_i a_i^D = \tilde{m}_i a_i^D = 0 \ , \qquad (21)$$

$$|m_i| = |\tilde{m}_i| \ . \qquad (22)$$

$i = 1, 2, ... N-1$. From (21) we learn that monopole condensation can only occur at hypersurfaces where $a_i^D = 0$ for some i. At the maximal singularity, every a_i^D vanishes, and it is clear from (20)–(22) that $N-1$ monopoles have a chance to condense. While soft breaking is parametrized by μ and ν, monopole condensation is controlled by C_i. If for some j we have $a_j^D = 0$ and adjust $C_j^{(0)} = 0$, the corresponding $U(1)$ remains unbroken ($m_j = \tilde{m}_j = 0$), and the vacuum is said to be "partially Higgsed". Summarizing, the Higgs vacuum $\mathcal{H}$ at the maximal point is given by

$$\mathcal{H} = \{ m_i, \tilde{m}_i \ / \ |m_i|^2 = |\tilde{m}_i|^2 = |C_i^{(0)}|/\sqrt{2} \ , \ \ \tilde{m}_j = -e^{i\beta_j^{(0)}} m_j^* \} \ , \qquad (23)$$

with $C_i^{(0)} = |C_i^{(0)}| e^{i\beta_i^{(0)}}$ given in equation (14). Since the absolute phases of m_i are not fixed, it has the topology of a torus of genus $g = N-1$. Equation (23) shows that the scalar components of the monopole superfields condense in the vacua placed at the maximal points. Although the presence of condensation suggests that confinement indeed takes place, some further analysis is required before this can be definitively established. An important question to be answered is whether the collimation of the electric (or dual magnetic) flux lines is energetically favored or not. This is a dynamical issue that goes beyond the simple vacuum analysis.

III. BOGOMOL'NYI BOUND IN THE ULTRASTRONG SCALING LIMIT

The resulting effective theory we have arrived at, in the bosonic sector, is an Abelian $(N-1)$–Higgs model with coupled $U(1)$ factors and a quite non-standard Higgs potential. The search for stable vortex solutions in the complete system is a hard problem. On general grounds, one should not expect to have BPS string solutions in spite of the fact that $\mathcal{N}=1$ supersymmetry is enough, generically, to have BPS vortices in four dimensions [23,24]. At least, this is the case of $\mathcal{N}=1$ QCD, where the strings are conserved modulo N so they cannot carry an additive conserved quantity such as a central charge [25]. There is a limit, however, in which the system simplifies and admits BPS vortices [11]. It happens whenever the condensation parameters (10) are independent of ϕ^D, something that corresponds to linear perturbations in the superpotential (4), *i.e.* Fayet–Iliopoulos terms. This kind of terms together with properly normalized quartic potentials are known to lead to Abelian Higgs models that admit BPS vortices [24,26,27]. Taking into account (11)–(12), one should consider $\Lambda \to \infty$ and small values of the soft breaking parameters $\mu \to 0$ and $\nu \to 0$, such that $\mu\Lambda$ and $\nu\Lambda^2$ remain finite. In this "ultrastrong" limit, $C_i(\phi_D) \to C_i^{(0)}$ are constants, and one can easily check that setting $a_i^D = \langle \phi_i^D \rangle = 0$ is a consistent constraint. One may then study the existence of extended solutions in the remaining fields.

The (bosonic part of the) effective Lagrangian adopts the following form:

$$\mathcal{L}_{eff,\ B}^{\mathcal{N}=1} = -\frac{1}{4} b_{ij}^{(0)} F_{\mu\nu}^i F^{j\,\mu\nu} + (D_\mu \phi_{m_i})^* D^\mu \phi_{m_i} + D_\mu \tilde{\phi}_{m_i} (D^\mu \tilde{\phi}_{m_i})^*$$
$$- \left[\frac{1}{2} b_{ij}^{(0)} D_i^{(0)} D_j^{(0)} + b_{ij}^{(0)} F_i^{(0)*} F_j^{(0)} \right] \ , \qquad (24)$$

where $b_{ij}^{(0)}$ stands for the actual value of b_{ij} at the maximal singularity and $D_i^{(0)}$, $F_i^{(0)}$ are obtained from (6) by replacing b_{ij} with $b_{ij}^{(0)}$. It is now feasible to give an expression *à la Bogomol'nyi* [28] for the energy per unit length corresponding to static and magnetically neutral ($(A_0^D)_i = 0$) vortex-like configurations (*i.e.* configurations with translational symmetry along one axis) by means of the remainder $\mathcal{N} = 1$ supersymmetry [23,24] (see also [26] where a multi Higgs system has been treated). Indeed, the energy density can be rearranged as follows:

$$\mathcal{E}_{eff} = \frac{1}{2} b_{ij}^{(0)} \left(F_{12}^i \pm D_i^{(0)} \right) \left(F_{12}^j \pm D_j^{(0)} \right) + |(D_1 \pm i D_2)\phi_{m_i}|^2 + \left| (D_1 \pm i D_2)\tilde{\phi}_{m_i} \right|^2 + b_{ij}^{(0)} F_i^{(0)} F_j^{(0)^*} \mp \epsilon_{ab} \partial_a \mathcal{J}_b \ , \tag{25}$$

where the last term, corresponding to the current $\mathcal{J}_b = -i(\phi_{m_i}^* D_b \phi_{m_i} + \tilde{\phi}_{m_i}^* D_b \tilde{\phi}_{m_i})$, does not contribute to the string tension for finite energy configurations. It is easier to analyze this system in a different set of variables, obtained from the above ones by means of an $SU(2)_R$ transformation yielding

$$D_i^{(0)} \longrightarrow \hat{D}_i^{(0)} = -\sqrt{2}\ \mathrm{Re}(e^{i\alpha} F_i^{(0)}) \ , \tag{26}$$

$$\sqrt{2} F_i^{(0)} \longrightarrow \sqrt{2} \hat{F}_i^{(0)} = -e^{-i\alpha} \left(D_i^{(0)} + i\sqrt{2}\ \mathrm{Im}(e^{i\alpha} F_i^{(0)}) \right) \ , \tag{27}$$

$$\phi_{m_i} \longrightarrow \hat{\phi}_{m_i} = -\frac{i}{\sqrt{2}} (\phi_{m_i} - e^{-i\alpha} \tilde{\phi}_{m_i}^*) \ , \tag{28}$$

$$\tilde{\phi}_{m_i}^* \longrightarrow \hat{\tilde{\phi}}_{m_i}^* = -\frac{i}{\sqrt{2}} e^{i\alpha} (\phi_{m_i} + e^{-i\alpha} \tilde{\phi}_{m_i}^*) \ . \tag{29}$$

The tension $\sigma_{eff} = \int d^2x\ \mathcal{E}_{eff}$ now reads

$$\sigma_{eff} = \int d^2x \left[\frac{1}{2} b_{ij}^{(0)} \left(F_{12}^i \pm \hat{D}_i^{(0)} \right) \left(F_{12}^j \pm \hat{D}_j^{(0)} \right) + \left| (D_1 \pm i D_2)\hat{\phi}_{m_i} \right|^2 + \left| (D_1 \pm i D_2)\hat{\tilde{\phi}}_{m_i} \right|^2 + b_{ij}^{(0)} \hat{F}_i^{(0)} \hat{F}_j^{(0)\ *} \mp \sqrt{2} F_{12}^i\ \mathrm{Re}(e^{i\alpha} C_i^{(0)}) \right] \ . \tag{30}$$

The last term breaks explicitely $SU(2)_R$ symmetry. Finiteness of the string tension demands regularity of the fields on $\mathbb{R}^2$, and vanishing of the potential energy, field strenghts and covariant derivatives at infinity. Altogether, these requirements make the space of solutions to split into $\mathbb{Z}^{N-1}$ disconnected pieces that differ by the winding numbers of each ϕ_{m_i} over the border of the plane. The electric-fluxes label these sectors. In particular, in the $(n_1, n_2, ..., n_{N-1})$–sector they are

$$\Phi_j = -\int d^2x\ F_{12}^j = 2\pi n_j \ , \qquad j = 1, 2, ..., N-1 \ . \tag{31}$$

The string tension of possible vortex configurations with topologically quantized (n_1, n_2) electric flux, exhibits a Bogomol'nyi bound

$$\sigma_{eff} \geq 4\sqrt{2}\pi\Lambda \sum_i |(\mu\ \sin\hat{\theta}_i + \nu\Lambda\ \sin 2\hat{\theta}_i) \cos(\alpha + \beta_i^{(0)})\ n_i| \ , \tag{32}$$

which is saturated for configurations solving the following set of first order equations:

$$F_{12}^i = \pm\sqrt{2}\ \mathrm{Re}(e^{i\alpha} F_i^{(0)}) \ , \qquad D_i^{(0)} + i\sqrt{2}\ \mathrm{Im}(e^{i\alpha} F_i^{(0)}) = 0 \ , \tag{33}$$

$$(D_1 \pm i D_2)\hat{\phi}_{m_i} = 0 \ , \qquad (D_1 \pm i D_2)\hat{\tilde{\phi}}_{m_i} = 0 \ . \tag{34}$$

The second equation in (33) implies

$$|\phi_{m_j}| = |\tilde{\phi}_{m_j}| \qquad \mathrm{Im}\left(e^{i\alpha} \left[\sqrt{2} \phi_{m_j} \tilde{\phi}_{m_j} + C_j^{(0)} \right] \right) = 0 \ . \tag{35}$$

These constraints should hold at any point, in particular, at zeroes of the Higgs field. Thus

$$\phi_{m_i} = -e^{i\beta_i^{(0)}} \tilde{\phi}_{m_i}^* \ . \tag{36}$$

with $\alpha+\beta_i^{(0)}=0$ or π. Consequently, for $i\neq j$, $\beta_{ij}^{(0)}\equiv\beta_i^{(0)}-\beta_j^{(0)}=0$ or π. Summarizing, *there are no BPS electric vortices in the system unless the complex numbers $C_i^{(0)}$ are aligned or anti–aligned.* This alignment, in turn, requires supersymmetry breaking parameters to have no relative complex phases. Notice that this corresponds to having a CP invariant bare Lagrangian. For definiteness, in the case of $SU(3)$ one easily sees that

$$C_1^{(0)}=\sqrt{3}\Lambda(\mu+\nu\Lambda)\qquad;\qquad C_2^{(0)}=\sqrt{3}\Lambda(\mu-\nu\Lambda)\ , \tag{37}$$

so that $\beta_{21}^{(0)}=0$ or π if and only if $\arg(\nu\Lambda)=\arg\mu+n\pi$ and $|\nu\Lambda|<|\mu|$ or $|\nu\Lambda|>|\mu|$ respectively.

A comment is in order at this point regarding the string tensions of unit vortices, whose existence will be discussed below. It is immediate to read, from (32), the string tension of electric vortices carrying a single flux quantum $n_k=1,\ n_{i\neq k}=0$. Up to a common factor, it is given by

$$T_k\propto\Lambda f_N(k)\qquad\qquad f_N(k)=|\mu\,\sin\hat{\theta}_k+\nu\Lambda\,\sin 2\hat{\theta}_k|\ . \tag{38}$$

This result makes clear the dependence of $f_N(k)$ on the supersymmetry breaking deformation entering the superpotential. It generalizes previous results in [8,11,12] and, in particular, it shows that for perturbations other than the quadratic one, the string tensions are modified with respect to those in the above mentioned results. In particular, notice that when μ and ν do not vanish it is possible to have different string tensions even in the case of $SU(3)$ and, in general, $T_k\neq T_{N-k}$.

IV. ALIGNED VACUA: CRITICAL VORTICES

We will focus hereafter on the case of $SU(3)$. When the constants μ and ν are fine tuned in such a way that the phases of the two complex energy scales $C_1^{(0)}$ and $C_2^{(0)}$ are either aligned or antialigned, *i.e.* $\beta_{21}^{(0)}=0$ or π respectively, we are at the self dual point. The Bogomol'nyi equations (33)–(34), after (36), read

$$F_{12}^i=\pm\frac{1}{2}b_{ij}^{(0)\,-1}\epsilon_j(|\varphi_j|^2-v_j^2)\ , \tag{39}$$
$$(D_1\pm i\epsilon_j D_2)\varphi_j=0\ , \tag{40}$$

where $\epsilon_j=e^{i(\alpha+\beta_j^{(0)})}=\pm1$ and $b_{ij}^{(0)}$ is

$$b_{ij}^{(0)}=\begin{pmatrix} g_{D,1}^{-2} & \frac{1}{4\pi^2}\log 2\\ \frac{1}{4\pi^2}\log 2 & g_{D,2}^{-2}\end{pmatrix}\ , \tag{41}$$

Also, we have performed, for convenience, some redefinitions of the fields, $\varphi_j=2\phi_{m_j}$, and parameters, $v_j^2=2\sqrt{2}|C_j^{(0)}|$. Let us further remark that Eq.(39) gives an unusual contribution to the electric field of each dual $U(1)$ factor from zeroes of both Higgs fields. This is a straight consequence of the presence of off-diagonal couplings and leads to interesting results. It is clear that solutions to (39)–(40) also satisfy the Euler-Lagrange equations. Without loss of generality, we can adjust α so that $\epsilon_1=+1,\epsilon_2\equiv\epsilon=e^{i\beta_{21}^{(0)}}=\pm1$. Let us focus on the BPS solutions with upper sign. The first order system can be written as

$$F_{12}^1=\quad\lambda_1(|\varphi_1|^2-v_1^2)-\epsilon\gamma\,(|\varphi_2|^2-v_2^2)\ , \tag{42}$$
$$F_{12}^2=-\gamma\,(|\varphi_1|^2-v_1^2)+\epsilon\lambda_2(|\varphi_2|^2-v_2^2)\ , \tag{43}$$
$$(D_1+\,iD_2)\,\varphi_1=0\ , \tag{44}$$
$$(D_1+i\epsilon D_2)\varphi_2=0\ , \tag{45}$$

with

$$\lambda_i=b_{ii}^{(0)\,-1}=\left(1-\frac{g_{D,1}^2 g_{D,2}^2}{16\pi^2}\log^2 2\right)^{-1}\ \frac{g_{D,i}^2}{2} \tag{46}$$

$$\gamma=b_{12}^{(0)\,-1}=\frac{\log 2}{8\pi^2}(g_{D,1}^2\lambda_2+g_{D,2}^2\lambda_1)\ . \tag{47}$$

Note that, as we are in the weak g_D-coupling regime, $\gamma < \lambda_i$. Naively one would suspect that in the scaling limit we are interested, the system diagonalizes. Notice however the important fact that the relative factor between λ_i and γ vanishes only logarithmically. Hence, for example, setting $|C_i|/\Lambda_i^2 \sim 10^{-10}$ in (15) yields $\gamma \sim (\log 2/5)\lambda_i \sim 0.13\,\lambda_i$.

The topology of the configuration space determines global properties of the solutions in two ways: the quantization of the fluxes is due either to the asymptotics of the A_j fields or to the existence of a prescribed number of zeroes of the φ_j. These global inputs should be made compatible with the differential equations, as it happens in the Abelian Higgs model. In the present situation things are less clear; from Eqs.(44)–(45), where no mixing between both $U(1)$s shows up, one reads the electric fluxes using Stokes theorem and the asymptotics of A_j. On the other hand, Eqs.(42)–(43) mix the factors and both φ_1 and φ_2 contribute together to each F^i_{12}. In this respect, our system is quite awkward as compared with other non-diagonal models as, for example, non-relativistic non-abelian Chern-Simons theories [29], in which the same mixing appears in the field strength and covariant derivative equations. Here, there is mixing in the former but not in the latter, and given such an asymmetry, it is much more difficult to show whether the local equations and the global conditions reconcile or not.

On general grounds, it is reasonable to expect that the equations (42)–(45) will exhibit solutions in the topological sector (n_1, n_2) with n_1, n_2 representing the integrated flux of an "ensemble" of noninteracting vortices located at different (maybe coincident) positions. Indeed, the smallness of the ratio γ/λ_i suggests to consider this system as a pertubation of the diagonal situation, so that the above solutions would come out from continuous deformations of the standard critical Abrikosov vortices. Only in some simple cases, the question about the existence of solutions can be answered by taking advantage of known results from the standard Abelian Higgs model. This will be done in the following two situations

- **Solutions of type** $(n, 0)$ **and** $(0, n)$.

  Clearly it will be enough to prove existence of one type, say $(n, 0)$. Assume therefore that $\varphi_2 = |\varphi_2|e^{i\xi_2}$ is nowhere vanishing on the finite transverse plane. As usual, (45) couples ξ_2 and A_2. So, if $|\varphi_2|$ has nowhere a zero, regularity of the phase enforces A_2 to have vanishing circulation around any loop. By Stokes theorem $F^2_{12} = 0$ everywhere, an inserting this back into (43) yields a constraint that correlates the profiles of $|\varphi_1|$ and $|\varphi_2|$,

  $$|\varphi_2|^2 = \epsilon\frac{\gamma}{\lambda_2}(|\varphi_1|^2 - v_1^2) + v_2^2\ . \tag{48}$$

  Existence of the required vortex profile for $|\varphi_1|$ can be proved by inserting (48) into (42), which leads to the standard Bogomol'nyi equations for the critical Abelian Higgs model (after a suitable re-normalization of the Higgs field)

  $$F^1_{12} = \lambda_1\left(1 - \frac{\gamma^2}{\lambda_2}\right)(|\varphi_1|^2 - v_1^2)\ , \tag{49}$$

  $$(D_1 + iD_2)\varphi_1 \ = \ 0\ . \tag{50}$$

  We learn from (48) that if $|\varphi_1|^2$ ranges from 0 (at the origin) up to v_1^2 (at infinity), $|\varphi_2|^2$ will correspondingly interpolate between $-\epsilon\frac{\gamma}{\lambda_2}v_1^2 + v_2^2$ and v_2^2. To remain consistent with our initial asumption that $|\varphi_2|$ vanished nowhere we must set either $\beta^{(0)}_{21} = 0$ with $v_2^2 > \frac{\gamma}{\lambda_2}v_1^2$, or else $\epsilon = -1$, *i.e.* , $\beta^{(0)}_{21} = \pi$. We observe that the latter possibility is less contrived.

- **Solutions of type** (n, n) **for a single perturbation**.

  Let us briefly consider the case of $SU(3)$ $\mathcal{N} = 2$ supersymmetric Yang–Mills theory softly broken to $\mathcal{N} = 1$ only by means of a single Casimir operator, *i.e.* $\mu = 0$ or $\nu = 0$. In both cases, $\beta^{(0)}_{21} = 0$ or π, and the theory is critical. Moreover, $\lambda_1 = \lambda_2 \equiv \lambda$, $C^{(0)}_1 = C^{(0)}_2$ and hence $v_1 = v_2 \equiv v$, so that the Bogomol'nyi equations have an almost trivial solution of vorticity (n, n) (or $(n, -n)$), by imposing the ansatz $\varphi_j \equiv \varphi$, $A_j \equiv A$ (or $\varphi_2^* = \varphi_1 \equiv \varphi$, $-A_2 = A_1 \equiv A$) in the case $\beta^{(0)}_{21} = 0$ (or π). The system is again reduced, after a suitable normalization of the Higgs field, to the critical Abelian Higgs model

  $$F_{12} = \pm(\lambda - \epsilon\gamma)(|\varphi|^2 - v^2)\ , \tag{51}$$

  $$(D_1 \pm iD_2)\varphi \ = \ 0\ , \qquad \epsilon = e^{i\beta^{(0)}_{21}}\ . \tag{52}$$

  It is crucial, for the system to admit regular solutions, that $\gamma < \lambda$ as it indeed happens. As it is well known, the general solution to this sytem represents an assembly of n separated vortices centered at the zeroes of φ. In our case, every such zero is doubled and we have assemblies of n couples of superimposed vortices of both $U(1)$ fields.

Also, self-dual configurations in which the center of the vortices of different types split apart, can be easily constructed along the lines in [30,31]. To see this, we perturb one of the solutions just described for $\beta_{21}^{(0)} = 0$

$$\varphi_j' = \varphi_j + \delta\varphi_j \ , \qquad\qquad A_j' = A_j + \delta A_j \ , \tag{53}$$

and linearize the self-duality equations to get

$$-4i\partial_z \delta A_1 - 2\lambda\varphi^* \delta\varphi_1 + 2\gamma\varphi^*\delta\varphi_2 = 0 \ , \tag{54}$$
$$-4i\partial_z \delta A_2 + 2\gamma\varphi^* \delta\varphi_1 - 2\lambda\varphi^*\delta\varphi_2 = 0 \ , \tag{55}$$
$$ig_D\varphi\delta A_j + (\partial_{\bar z} + ig_D A_j)\delta\varphi_j = 0 \ , \tag{56}$$

where we use the notation $\partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$, $A_j = \frac{1}{2}[(A_1)_j + i(A_2)_j]$, $j = 1, 2$, and fix the gauge conditions as

$$\partial_c(\delta A_c)_1 = -\lambda|\varphi|^2\delta\Omega_1 + \gamma|\varphi|^2\delta\Omega_2 \ , \tag{57}$$
$$\partial_c(\delta A_c)_2 = \gamma|\varphi|^2\delta\Omega_1 - \lambda|\varphi|^2\delta\Omega_2 \ . \tag{58}$$

By writing $\delta\varphi_j = \varphi\xi_j$ and using (56), the vector perturbations are found to be $\delta A_j = \frac{i}{g_D}\partial_{\bar z}\xi_j$ and the system of linearized equations reduces to

$$\nabla^2 W_\pm = 2(\lambda \mp \gamma)g_D|\varphi|^2 W_\pm \ , \tag{59}$$

with $W_\pm = \xi_1 \pm \xi_2$. Notice that in both equations $(\lambda \mp \gamma)g_D > 0$. Although they have not regular square-integrable solutions, we can admit singular ones provided the singularities of ξ_j fit with the zeroes of φ in such a way that $\delta\varphi_j$ is well-behaved. Take for instance the case of a radially symmetric solution of vorticity n centered at the origin of the complex plane. Then, for small z

$$\varphi(z, \bar z) \simeq z^n \ , \tag{60}$$

and a singularity of $W_\pm$ at the origin is harmless if its order is lower or equal than n. Equation (59) has indeed solutions with such a behaviour [32]. To be exact, two sets of linearly independent self-dual perturbations $W_\pm^m(z, \bar z), m = 1, 2, 3, ..., n$ with

$$W_\pm^m(z, \bar z) \simeq z^{-m}, \qquad z \simeq 0 \ . \tag{61}$$

In particular, if we consider $W_\pm = -aW_\pm^m$, we get, near the origin,

$$\xi_1 \simeq -az^{-m} \qquad\qquad \xi_2 \simeq 0 \ , \tag{62}$$

so that

$$\varphi_1' \simeq z^{n-m}(z^m - a) \qquad\qquad \varphi_2' \simeq z^n \ . \tag{63}$$

This perturbation realizes the splitting of a (n, n) vortex at the origin into a $(n - m, n)$ at that point and m $(1, 0)$ vortices located at the m roots of the coefficient a. The analysis for $\beta_{21}^{(0)} = \pi$ (*i.e.* $\epsilon = -1$) is totally equivalent and yields nothing but vortices of type 1 and anti-vortices of type 2 or viceversa, moving freely with respect to each other.

For the general analysis, following Jaffe and Taubes [33], the Higgs fields should be "couched" as

$$\varphi_j \equiv v_j e^{\frac{1}{2}(u_j + i\Omega_j)} \ , \tag{64}$$

to recast the Higgs system in the following form

$$\nabla^2 u_1 = \ \ 2\lambda_1 v_1^2(e^{u_1} - 1) - 2\epsilon\gamma v_2^2(e^{u_2} - 1) + \varepsilon_{bc}\partial_b\partial_c\Omega_1 \ , \tag{65}$$
$$\nabla^2 u_2 = -2\gamma v_1^2(e^{u_1} - 1) + 2\epsilon\lambda_2 v_2^2(e^{u_2} - 1) + \varepsilon_{bc}\partial_b\partial_c\Omega_2 \ . \tag{66}$$

The gauge fields are determined by

$$(A_c)_1 = -\frac{1}{2}(\partial_c\Omega_1 + \varepsilon_{ca}\partial_a u_1) \ , \tag{67}$$

$$(A_c)_2 = -\frac{\epsilon}{2}(\partial_c\Omega_2 + \varepsilon_{ca}\partial_a u_2) \ . \tag{68}$$

At each (n_1, n_2) sector, regularity implies that φ_j has exactly n_j zeroes on $\mathbb{C}$, say $z_1^j, z_2^j, \ldots, z_{n_j}^j$. Also, these are the only points at which the singularities of the phases can occur. We can then choose the particular gauge

$$\Omega_j(z,\bar z) = 2\sum_{l=1}^{n_j} \arg(z - z_l^j) \ , \tag{69}$$

in which the problem reduces to

$$\nabla^2 u_1 = \ \ 2\lambda_1 v_1^2(e^{u_1}-1) - 2\epsilon\gamma v_2^2(e^{u_2}-1) + 4\pi\sum_{l=1}^{n_1}\delta(z-z_l^1) \ , \tag{70}$$

$$\nabla^2 u_2 = -2\gamma v_1^2(e^{u_1}-1) + 2\epsilon\lambda_2 v_2^2(e^{u_2}-1) + 4\pi\sum_{l=1}^{n_2}\delta(z-z_l^2) \ , \tag{71}$$

where both u_j should vanish at space infinity. The general analysis is involved, and usually goes through by numerical relaxation techniques or hard Sovolev estimates.

V. HYBRID TYPE II VORTICES

By itself, the Abelian Higgs model we are dealing with is worth a detailed analysis. For the moment, and awaiting a sounder analytical or numerical study of its solutions, aside from the two simplified samples considered above little can be said about the generic (n_1, n_2) vortex solution. An interesting peculiarity comes from the fact that there are only two overall choices of signs available in equations (39) and (40): either upper or lower sign have to be taken simultaneously on all the equations or, else, the bound (32) will not be saturated. This should be contrasted with the situation in the standard diagonal Abelian Higgs model, where each $U(1)$ can be conjugated independently. To better grasp what is going on let us consider the Bogomol'nyi equations (42)-(45) with $\beta_{21}^{(0)} = 0$

$$F_{12}^1 = \pm(\lambda_1 W_1 - \gamma W_2) \ , \tag{72}$$

$$(D_1 \pm iD_2)\,\varphi_1 = 0 \ , \tag{73}$$

$$F_{12}^2 = \pm(\lambda_2 W_2 - \gamma\, W_1) \ , \tag{74}$$

$$(D_1 \pm iD_2)\varphi_2 = 0 \ , \tag{75}$$

with $W_i = (|\varphi_i|^2 - v_i^2)$. If $\gamma << \lambda_1, \lambda_2$, $(\pm n_1, \pm n_2)$ vortex with $n_1, n_2 > 0$ come from solutions to the previous equations with the upper (lower) sign which should correspond to deformations of analogous configurations in the case $\gamma = 0$. In the diagonal limit $\gamma = 0$ the vortex-antivortex solutions $(\pm n_1, \mp n_2)$ would also solve the previous equations but with a choice of sign for (72)–(73) and the opposite one for (74)–(75). If $\gamma \neq 0$, as is now the case, solutions with this second choice of sign do not saturate the bound (32) and, indeed, there is an energy remnant coming from the off-diagonal piece $\mathcal{E} = \pi\,|n_1 v_1^2 + n_2 v_2^2| + \delta\mathcal{E}$

$$\delta\mathcal{E} = \int d^2x\, \delta\sigma_{eff} = \int d^2x\, 2b_{12}^{(0)} F_{12}^1 F_{12}^2 = \frac{\log 2}{2\pi^2}\int d^2x\, F_{12}^1 F_{12}^2 \tag{76}$$

For anti-aligned magnetic fields, this extra term is negative and tends to increase the overlap by attracting the cores of vortices of different kind.

A similar reasoning can be carried out of $\beta_{21}^{(0)} = \pi$. In this case, the equations read

$$F_{12}^1 = \pm(\lambda_1 W_1 + \gamma W_2) \ , \tag{77}$$

$$(D_1 \pm iD_2)\,\varphi_1 = 0 \ , \tag{78}$$

$$F_{12}^2 = \mp(\lambda_2 W_2 + \gamma\, W_1) \ , \tag{79}$$

$$(D_1 \mp iD_2)\varphi_2 = 0 \ , \tag{80}$$

and critical configurations are naturally of the form $(\pm n_1, \mp n_2)$, $n_1, n_2 \geq 0$ saturating the bound $\mathcal{E} = \pi\, |n_1 v_1^2 - n_2 v_2^2\,|$. Here, in contrast, vortex-vortex solutions of the form $(\pm n_1, \pm n_2)$ would lead to the same energy surplus as in (76). But now $\delta\mathcal{E} \geq 0$ for aligned magnetic fields, and this term decreases by minimizing the overlap, hence by taking the cores far appart.

In summary, to a first approximation, we see that, if not neutral, vortex-vortex (vortex-antivortex) configurations behave repulsively (attractively) as in type II superconductors. Since this interaction involves vortices of different $U(1)$'s, we speak of an "hybrid type II" phase.

Let us discuss the peculiarities that arise whenever one tries to model confinement in the present scenario. First we fix some notation for convenience: the chromoelectric fluxes (n_1, n_2) of the basic vortices arising in the dual Meissner effect are $(1,0)$ ("vortex 1") and $(0,1)$ ("vortex 2"). In turn, quarks enter the system as external probes with chromoelectric charges (Q_1, Q_2) equal to $(1,0)$ ("red quark"), $(0,-1)$ ("blue quark") and $(-1,1)$ ("yellow quark"). (h_1, h_2) is the "monopole" basis of the Cartan algebra of the dual $\check{SU}(3)$ group and the fundamental BPS monopoles correspond to the simple co-roots of $SU(3)$. In other words, the chromomagnetic charges of the φ_i–field quanta is $h_i = 1, h_{j\neq i} = 0$. Consider now, for example, the case $\beta_{21}^{(0)} = 0$. According to our previous analysis, chromoelectric flux tubes of both $(1,0)$ and $(0,1)$ type form in response to parallel external electric fields $\vec{E}_1$ and $\vec{E}_2$. Vortices of type 1 end at pairs of red quark-antiquark and vortices of type 2 finish at pairs of blue antiquark-quark. There is therefore confinement of red and blue quarks in a critical phase between Type I and Type II superconductivity, whereas the yellow quark confinement occurs in a hybrid Type II phase. The weak repulsion between the vortex 1/antivortex 2 pair pull slightly apart the flux lines from each other. Thus, the quark/antiquark potential energy would increase slower than linearly with the distance, and one is allowed to expect deviations from the area law, but the force is still confining. If, instead, $\beta_{21}^{(0)} = \pi$, a pair of yellow quark-antiquark will now be joined by a stable and non-interacting vortex 1/antivortex 2 pair of flux tubes. In conclusion, the cases $\beta_{21}^{(0)} = 0$ or π can be physically distinguished by the behaviour of the yellow quark-antiquark force. At large separation W-pair production leads to instability of the string and the lowest string tension governs the large distance regime [8,11].

In the framework of condensed matter it is well known the fact that, in standard type II superconductivity on a finite piece of material, though mutually repelling, vortices tend to form a regular pattern by lying at the sites of a triangular lattice. This fact can be reproduced analytically by variational methods [34]. We expect a similar situation here, the difference being that now repulsion involves vortex cores of distinct Higgs fields. Upon substitution of (36) into (24), the exact second order equations with $\beta_{21}^{(0)} = \pi$, corresponding to vortices of type 1 and 2, in a finite piece of material

$$b_{11}^{(0)}\partial_a(F_{ab})_1 + b_{12}^{(0)}\partial_a(F_{ab})_2 = \frac{i}{2}(\varphi_1^* D_b\varphi_1 - \varphi_1 D_b\varphi_1^*)\ , \tag{81}$$

$$b_{22}^{(0)}\partial_a(F_{ab})_2 + b_{21}^{(0)}\partial_a(F_{ab})_1 = \frac{i}{2}(\varphi_2^* D_b\varphi_2 - \varphi_2 D_b\varphi_2^*)\ , \tag{82}$$

$$D_c D_c\varphi_1 = -\frac{1}{\sqrt{2}}\varphi_1^* b_{1j}^{(0)-1}(|\varphi_j|^2 - v_j^2)(-1)^{\beta_{1j}}\ , \tag{83}$$

$$D_c D_c\varphi_2 = -\frac{1}{\sqrt{2}}\varphi_2^* b_{2j}^{(0)-1}(|\varphi_j|^2 - v_j^2)(-1)^{\beta_{2j}}\ , \tag{84}$$

should now be supplemented with periodic boundary conditions. Thus, the system of differential equations is defined in a torus of modular parameter $\tau = L_2/L_1\, e^{i\theta}$. We have chosen the x_1-axis as the direction of the first L_1 periodicity; the length and direction of the second periodicity is determined by $L_2 e^{i\theta}$. Application of the Rayleigh-Ritz variational method as in [34] plus previous work on the rôle of Riemann Theta functions in magnetic systems [35], suggest the field configurations

$$\varphi_1 = \sum_{m_1\in\mathbf{Z}} C_{m_1}\exp[i n_1 m_1 \mathrm{Im} z - \frac{1}{2}(\mathrm{Re} z - n_1 m_1)^2]\ , \tag{85}$$

$$\varphi_2 = \sum_{m_2\in\mathbf{Z}} C_{m_2}\exp[i n_2 m_2 \mathrm{Im} z - \frac{1}{2}(\mathrm{Re} z - n_2 m_2)^2]\ , \tag{86}$$

where n_1, n_2 are integers and $z = \sqrt{g_D(\lambda-\gamma)}\ \left(\frac{x_1 + i x_2}{L_1}\right)$, as trial functions to model extremals of the energy. In fact, the choice of the coefficients C_{m_1} and C_{m_2} in such a way that

$$\varphi_1^{n_1}(z) = \exp\{-\pi n_1 \frac{(\mathrm{Im}z)^2}{\mathrm{Im}\tau}\} \prod_{l_1=1}^{n_1} \Theta \left[\begin{array}{c} 0 \\ \frac{l_1}{n_1} \end{array} \right] (z|\frac{\tau}{n_1}) \ , \tag{87}$$

$$\varphi_2^{n_2}(z) = \exp\{-\pi n_2 \frac{(\mathrm{Im}z)^2}{\mathrm{Im}\tau}\} \prod_{l_2=1}^{n_2} \Theta \left[\begin{array}{c} \frac{1}{2} \\ \frac{l_2}{n_2} + \frac{1}{2} \end{array} \right] (z|\frac{\tau}{n_2}) \ , \tag{88}$$

leads to (meta)-stable solutions to the field equations. Here $l_i = 1, \ldots, n_i$, and $\Theta[^a_b](z|\tau)$ are the Riemann Theta functions with characteristics, see [35] and references quoted therein.

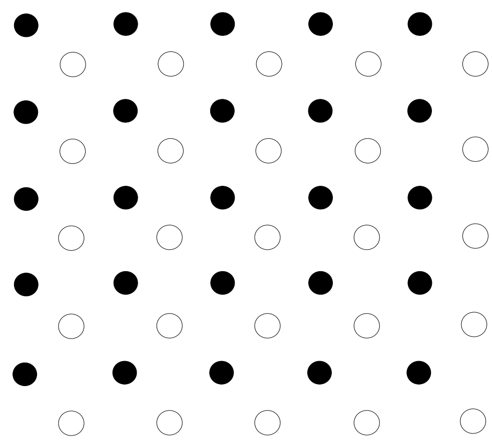

FIG. 1. The Type II hybrid lattice. Black and white circles represent the core of vortices corresponding to different $U(1)^{'s}$.

Notice that the solution describes n_1 chromoelectric vortices, located at the zeroes of $\varphi_1^{n_1}$, and n_2 vortices of the other kind centered around the zeroes of $\varphi_2^{n_2}$. It corresponds therefore to a hybrid static triangular lattice of vortices; see the Figure. One can check, from a dynamical point of view that a configuration like this where a vortex of type 1 is at the center of a square with vortices of type 2 at the vertices and viceversa is stable against small fluctuations.

VI. MISALIGNED VACUA

As discussed earlier, there are no BPS vortices in the generic case where soft breaking parameters are not aligned. We would however be interested in the response of the BPS configuration when an infinitesimal misalignment $\beta_{21}^{(0)} \equiv \varepsilon$ or $\beta_{21}^{(0)} \equiv \pi + \varepsilon$ is turned on. The dynamics of the system drives the configuration off the constraint (36) which, therefore, can no longer be imposed consistently. In fact, though the Higgs mechanism yields a critical mass spectra for any value of ε (an obvious consequence of supersymmetry), the eigenvectors do depend on this phase difference in such a way that when it is different from 0 or π, massive excitations do not respect the constraint surface (36).

In the same vein as above, for small values of ε we will treat the system as a perturbation of the critical situation in which the net effect of the misalignment reflects itself in a force between the former noninteracting vortices. The shortcut to obtain the sign of this force is to split the energy (30) of the configuration as a BPS contribution plus an additional perturbation. Namely, after inserting the ansatz (36) into (24), solutions to (39)–(40) exhibit a string tension $\sigma_{eff} = \sigma_{eff}^{SD} + \delta\sigma_{eff}$ where σ_{eff}^{SD} is given in (32) and

$$\delta\sigma_{eff} = \epsilon \frac{\gamma\varepsilon^2}{8} \int d^2x \, (|\varphi_1|^2 - v_1^2)(|\varphi_2|^2 - v_2^2) \ . \qquad (\varepsilon << 1) \tag{89}$$

with $\epsilon = 1$ for $\beta_{21}^{(0)} = 0 + \varepsilon$ and $\epsilon = -1$ for $\beta_{21}^{(0)} = \pi + \varepsilon$. Consider a vortex configuration of type $(1, 1)$ where the zeroes of each Higgs field are well separated. Then, the above surplus of energy is positive for $\epsilon = 1$ and decreases as the cores are taken further appart and the overlap diminishes hence the interaction in this case is repulsive. When perturbing around the anti-aligned case, $\beta_{21}^{(0)} = \pi + \varepsilon$, the energy increment (89) reverses sign. Previously non-interacting, $(1, -1)$ antiparallel vortex configurations tend to increase the overlap in order to lower the perturbation, and hence the force is attractive.

In summary, when perturbing around the aligned or misaligned scenarios, the vortex configurations are not neutral anymore, and the interactions follow the pattern that was previously named "hybrid type II" where, if made of distinct U(1)'s, parallel vortices repel and antiparallel vortices attract.

VII. HALF HIGGSED VACUA

As pointed out in [9], for particular values of the soft breaking parameters μ and ν we have four instead of five vacua. This happens whenever one of the two half Higgsed vacua $\{a_1^D \neq 0, a_2^D = 0\}$ with $C_1(\mu,\nu) = 0$, or $(1 \leftrightarrow 2)$, meets and replaces one of the normal vacua at $\{a_1^D = 0, a_2^D = 0\}$. This possibility is actually achieved by turning off $C_i^{(0)}$ for $i = 1$ or 2. Since precisely at the Z_2 point we have (37), this amounts to μ and ν fulfilling $\mu = \mp\nu\Lambda$. Let us choose for definiteness, $C_2^{(0)} = 0$. Inserting this back into (18), the effective potential at the maximal point reads

$$V = \frac{1}{8}\lambda_1(|\varphi_1|^2 - v_1^2)^2 + \frac{1}{8}\lambda_2|\varphi_2|^4 - \frac{1}{4}\gamma\cos\beta_1|\varphi_2|^2(|\varphi_1|^2 - v_1^2)\ . \tag{90}$$

Observe that the phase of φ_2 is free. When $\cos\beta_1 < 0$ this is precisely the type of situation that was studied by Witten [19] and shown to lead to superconducting strings for specific ranges of parameters. Let us briefly recall the essence of the mechanism. As the vacuum equations (20) exhibit, only the first $U(1)$ is broken by the v.e.v. $\langle\varphi_1\rangle = v_1$, whereas the second $U(1)$ remains intact since $\langle\varphi_2\rangle = 0$. This is fine for vacuum solutions, but suppose now that φ_1 developes a vortex line. At the core of the vortex $\langle\varphi_1\rangle = 0$ and, in turn, it may become favorable that $\langle\varphi_2\rangle \neq 0$ there. Actually the model considered in [19] is slightly more general than ours involving the potential

$$V = \frac{1}{8}g(|\varphi_1|^2 - v^2)^2 + \frac{1}{4}\tilde{g}|\varphi_2|^4 + f|\varphi_1|^2|\varphi_2|^2 - m^2|\varphi_2|^2\ . \tag{91}$$

The detailed analysis of the dynamics showed that for parameters in the range $fv^2 \geq m^2$, instability actually takes over and the superconducting string indeed forms. We see easily that the present situation lies precisely at the boundary of the region of validity, since in our case $fv^2 - m^2 = 0$, and the induced mass term for φ_2 exactly vanishes. In [36], this situation was also studied and seen to yield a power law decay of the profile of φ_2 which leads to a long range scalar attractive interaction among vortices.

At this point we would not like to put forward too strong a claim, but simply point out the ocurrence of this coincidence among models. The possible existence and relevance of structures like superconducting strings in the microscopic context of confinement models should be handled with care. For example the question of quantum tunnelling will be certainly much more relevant here than for cosmic strings. Incidentally this question was also addressed in [36] where it was seen that these power law solutions are more stable than the usual ones.

As compared with Witten's model, the one here involves the additional feature of the non-diagonal kinetic term for the (dual) vector particles (*cf.* eq. (24)). But precisely the fact that the quadratic forms of kinetic term and potential are related paves the way to the possibility of rewritting the energy as a sum of squares (30). We may therefore expect vortex solutions of the superconducting type with dynamical properties of BPS configurations. We can check that this is indeed the case by looking at the smooth deformation of a generic (anti-)aligned scenario. [6] Let us follow a continuous line of anti-aligned ($\beta_{21}^{(0)} = \pi$) vacua $C_1^{(0)} \neq 0, C_2^{(0)} \to 0$. Precisely in this situation, (48) presents no obstruction to a smooth deformation of the $(n,0)$ solutions down to the situation where $v_2 = 0$. In this limit the profiles of $|\varphi_1|$ and $|\varphi_2|$ are correlated in such a way that both vanish at opposite ends. In fact, as $|\varphi_1|^2$ varies from zero up to v_1^2 far away, $|\varphi_2|$ interpolates between $\frac{\gamma}{\lambda_2}v_1^2 = (\log 2/8\pi^2)g_{D,1}^2 v_1^2$ at the origin (which need not be small !), and 0 at infinity. Moreover, since the phase of φ_2 is free, the same arguments of ref. [19] can be used to show that a persistent current occurs. We would call this a *BPS superconducting string solution*.

VIII. CONCLUDING REMARKS

The present paper is devoted to the low energy dynamics of $\mathcal{N} = 2$ supersymmetric gauge theories softly broken to $\mathcal{N} = 1$ by a superpotential containing up to cubic perturbations. The effective lagrangian in the neighborhood

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[6] As we approach the situation when $C_2 \to 0$, the parameters that enter (90) are such that $\gamma, \lambda_2 << \lambda_1$ (see eqns.(15) and (46)–(47)). Hence at very low energy the second $U(1)$ seemingly decouples. This is suggested by the $\mathcal{N} = 2$ exact effective solution, although it is reasonable to expect modifications of the renormalization group flow in the $\mathcal{N} = 1$ theory.

of maximal singularities of the quantum moduli space corresponds to an Abelian $U(1)^{N-1}$ multi Higgs system with couplings among different dual $U(1)$ factors. The case of $SU(3)$ has been analized in some detail. There are generically no BPS electric vortices in the system unless the soft breaking parameters have coincident complex phases (or they differ by π) and the ultrastrong scaling limit [11] is taken. We have seen that the effect over a BPS configuration of turning on an infinitesimal misalignment among these parameters is the appearance of a net repulsive force between parallel vortices corresponding to (zeroes of) different Higgs fields. In a finite piece of material, metastable solutions take place and vortices develope static triangular lattice. We call this phase "hybrid Type II" dual superconductivity.

When the theory is perturbed with a cubic superpotential, the ratio of string tensions differs from that computed in the quadratic case [8] both when the $\mathrm{Tr}\Phi^2$ perturbation is present or not. In the former case, we found that these ratios even depend on the supersymmetry breaking parameters. These results were obtained after imposing the ultrastrong scaling limit. It would be certainly interesting to know if similar results emerge in the context of MQCD. This is intriguing in the sense that string tensions in MQCD are given by the distance of D4-branes which, for a single Casimir perturbation, are stretched at the roots of unity over a circle of radius of order Λ [11], so one would not expect them to be modified (except, possibly, for a global factor due to an induced change in Λ) as compared to the purely quadratic case.

A natural extension of the present work involves the case of $\mathcal{N}=2$ supersymmetric theories softly broken down to $\mathcal{N}=0$, and possible soft breaking by higher than the two first Casimir operators. This program can be addressed within the Whitham approach to the Seiberg–Witten solution, where the slow-times of the hierarchy can be used as spurionic sources of soft supersymmetry breaking [21].

ACKNOWLEDGMENTS

We are pleased to thank José F. Barbón, A. González-Arroyo, Michael Douglas, Amihay Hanany and Marcos Mariño for interesting discussions. J.M. wants to thank J.J. Blanco Pillado for pointing out reference [36]. The work of J.D.E. has been supported by the National Research Council (CONICET) of Argentina and the Ministry of Education and Culture of Spain. The work of J.M. was partially supported by DGCIYT under contract PB96-0960 and European Union TMR grant ERBFM-RXCT960012.

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