# Invariants in Supersymmetric Classical Mechanics

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Abstract

The bosonic second invariant of Super Liouville models in supersymmetric classical mechanics is described.

## 1 Introduction.

The search for kinks in N-component (1 + 1)-dimensional field theory is a very difficult endeavour, see Reference [1] pp. 23–24. Mathematically, this problem is equivalent to a N-dimensional mechanical system via dimensional reduction to (0+1)-dimensions of space-time. Therefore, only if the mechanical system is completely integrable is there a hope of finding all the kinks of the field theory that are in one-to-one correspondence with the separatrix trajectories (developing finite action in infinite time) of the mechanical system. The authors have investigated this line of research on one-dimensional topological defects in several earlier papers, see [2].

After the seminal paper of Olive and Witten on extended supersymmetry algebras and solitons, [3], the paradigmatic N = 1 kinks have been understood as BPS states of N = 1 supersymmetric quantum field theory. In (1+1)-dimensions, the N = 1 theory is based in an extended supersymmetric algebra because there exist Majorana-Weyl fermions that could be used to generalize the purely bosonic theory supporting kinks to a Bose-Fermi system enjoying supersymmetry. Very recently, domain walls have been studied in the Wess-Zumino model, [4], and N = 1 SUSY QCD, [5]. These theories arise in the low energy limit of string theory and the domain walls can be seen as membranes or other extended structures. On the other hand, both the WZ model and SQCD contain more than one real scalar field. Thus, their dimensional reduction is a (1 + 1)-dimensional either N = 1 or N = 2 supersymmetric field theory. Because the kinks are the actual domain walls if they are looked at from a (3 + 1)D point of view, in order to identify the

BPS kinks (hence the domain walls, hence the d-branes) it is convenient to study $N = 2$ or $N = 4$ supersymmetric classical mechanics, starting from the finite action trajectories of these mechanical systems.

Accordingly, in this work we shall study the $N = 2$ supersymmetric extension of the $N=2$ Liouville systems. These supersymmetric extensions of completely integrable mechanical systems arise in the dimensional reduction of $(1+1)$-dimensional bosonic theories that have a very rich manifold of kinks. Our aim is to elucidate wheter these supersymmetric models in classical mechanics have a second invariant; if the answer is affirmative, then in priciple all the finite action trajectories that give the kinks and their supersymmetric partners can be found.

## 2 Supersymmetric Classical Mechanics

Let us start with a Grassman algebra $B_L$ with generators $\theta^i_a \in B_L$, $a = 1,2$, $i = 1,\dots,N$, that satisfy the anti-commutation rules:

$$\theta^i_a\theta^j_b + \theta^j_b\theta^i_a = 0$$

Any element of $B_L$ is a combination of the $L = 2N$ generators $\theta^i_a$ that can be written in the form:

$$b = b_0 1 + \sum b^{a_1\dots a_m}_{i_1\dots i_m}\,\theta^{i_1}_{a_1}\dots\theta^{i_m}_{a_m}\ ;$$

where the coefficients $b^{a_1\dots a_m}_{i_1\dots i_m}$ are real numbers. We shall distinguish between odd and even elements of $B_L$ according to the parity of the number of Grassman generators.

### 2.1 Lagrangian Formalism

The configuration space of the system is $C = R^N \times (B_{2N})$. If we choose $(x^i;\theta^i_1;\theta^i_2)$ as local coordinates in $C$, the Lagrangian of our system has the "supernatural" form:

$$L = \frac{1}{2}\dot{x}^j\dot{x}^j - U(\vec{x}) + \frac{i}{2}\theta^j_a\dot{\theta}^j_a + iR_{jk}(\vec{x})\,\theta^j_1\theta^k_2$$

Observe that the Lagrangian is defined on even elements of $C$ of two types: there is a bosonic contribution collecting the kinetic and potential energies of the $x^i$ coordinates. There are also two terms that contain fermionic variables: besides the Grassman kinetic energy, there is a Yukawa coupling $R_{jk}$ that we assume to be symmetric in the $j,k$ indices, see [6]. The Euler-Lagrange equations

$$\ddot{x}^k + \frac{\partial U}{\partial x^k} - i\frac{\partial R_{jl}}{\partial x^k}\theta^j_1\theta^l_2 = 0 \qquad \dot{\theta}^i_1 = -R_{ij}\theta^j_2 \qquad \dot{\theta}^i_2 = R_{ji}\theta^j_1$$

determine the dynamics of our mechanical system. Via Noether's theorem, one finds the energy as the invariant associated to invariance under time translations:

$$I = H = \frac{1}{2}\dot{x}^j\dot{x}^j + U(\vec{x}) - iR_{jk}(\vec{x})\psi_1^j\psi_2^k \qquad (1)$$

## 2.2 Hamiltonian Formalism

We shall now briefly discuss the Hamiltonian formalism [7]. The usual definition of generalized momentum is extended to the Grassman variables:

$$\pi_a^j = L\frac{\partial}{\partial\dot{\psi}_a^j} = \frac{i}{2}\psi_a^j \qquad (2)$$

We note the dependence of the fermionic generalized momenta on their Grassman variables. In the 6N-dimensional phase space $T^*C$ with coordinates $(x^i; \psi_1^i; \psi_2^i; p_i; \pi_1^i; \pi_2^i)$, the definition of Grassman generalized momenta (2) provides 2N second class constraints. Instead of working in the reduced 4N-dimensional phase space (the Grassman sub-space of the phase space coincides with the Grassman sub-space of the configuration space because the Lagrangian is first-order in the velocities) we implement the constraints in the Hamiltonian by means of Grassman-Lagrange multipliers [8]:

$$H_T = \frac{1}{2}\dot{x}^j\dot{x}^j + U(\vec{x}) - iR_{jk}(\vec{x})\psi_1^j\psi_2^k - \left(\pi_a^j - \frac{i}{2}\psi_a^j\right)\lambda_a^j$$

The Hamilton equations are:

$$\dot{x}^j = \frac{\partial H_T}{\partial p_j}; \qquad \dot{p}_j = -\frac{\partial H_T}{\partial x^j}; \qquad \dot{\psi}_a^j = -\frac{\partial H_T}{\partial \pi_a^j}; \qquad \dot{\pi}_a^j = -\frac{\partial H_T}{\partial \psi_a^j}$$

Note the difference in sign between the bosonic and fermionic canonical equations. Solving for the Lagrange multipliers, we find

$$H_T = \frac{1}{2}\dot{x}^j\dot{x}^j + U(\vec{x}) + R_{jk}(\vec{x})(\pi_2^k\psi_1^j - \pi_1^j\psi_2^k);$$

which provides the right Hamiltonian flow in the phase space.

Defining the Poisson brackets for the generic functions F and G in phase space in the usual way:

$$\{F,G\}_P = \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q^j} - \frac{\partial F}{\partial q^j}\frac{\partial G}{\partial p_j} + iF\frac{\overleftarrow{\partial}}{\partial\psi_a^j}\frac{\overrightarrow{\partial}}{\partial\psi_a^j}G - \frac{1}{2}F\frac{\overleftarrow{\partial}}{\partial\pi_a^j}\frac{\overrightarrow{\partial}}{\partial\psi_a^j}G - \frac{1}{2}F\frac{\overleftarrow{\partial}}{\partial\psi_a^j}\frac{\overrightarrow{\partial}}{\partial\pi_a^j}G - \frac{i}{4}F\frac{\overleftarrow{\partial}}{\partial\pi_a^j}\frac{\overrightarrow{\partial}}{\partial\pi_a^j}G;$$

the canonical equations read:

$$\frac{dx^j}{dt} = \{H_T, x^j\}_P \qquad \frac{d\theta^j_a}{dt} = \{H_T, \theta^j_a\}_P$$

$$\frac{dp_j}{dt} = \{H_T, p_j\}_P \qquad \frac{d\pi^j_a}{dt} = \{H_T, \pi^j_a\}_P$$

In general, the time-dependence of any observable $F$ is ruled by:

$$\frac{dF}{dt} = \{H_T, F\}_P$$

Therefore, physical observables are constants of motion or invariants if and only if:

$$\{H_T, I\}_P = 0$$

In practical terms, it is better to work on the reduced phase space and define the reduced Poisson brackets as:

$$\{F, G\}_P = \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q^j} - \frac{\partial F}{\partial q^j}\frac{\partial G}{\partial p_j} + iF\,\frac{\overleftarrow{\partial}}{\partial\theta^j_a}\frac{\overrightarrow{\partial}}{\partial\theta^j_a}G$$

Thus, we receive the following Poisson structure:

$$\{p_j, x^k\}_P = \delta^k_j \qquad \{x^j, x^k\}_P = \{p_j, p_k\}_P = 0 \qquad \{\theta^j_a, \theta^k_b\}_P = i\,\delta^{jk}\delta_{ab}$$

and the canonical equations and the invariant observables are referred to the reduced Hamiltonian $H$ given in (1).

## 2.3 Supersymmetry

The question arises: are there transformations in the configuration or phase space such that they mix the bosonic and fermionic variables and leave invariant the Lagrangian or the Hamiltonian?. If the answer is yes, then the mechanical system can be said to enjoy supersymmetry [9]. Instead of using the elegant superfield/superspace formalism, we take a direct approach. Consider the following infinitesimal variations in the configuration space defined in terms of a Grassman parameter $\epsilon$:

$$\text{Variation 1:}\quad \begin{cases} \delta_1 x^j = \epsilon\,\theta^j_1 \\ \delta_1 \theta^j_1 = i\epsilon\,\dot{x}^j \\ \delta_1 \theta^j_2 = -i\epsilon\, f^j(\vec{x}) \end{cases} \qquad \text{Variation 2:}\quad \begin{cases} \delta_2 x^j = \epsilon\,\theta^j_2 \\ \delta_2 \theta^j_1 = i\epsilon\, g^j(\vec{x}) \\ \delta_2 \theta^j_2 = i\epsilon\,\dot{x}^j \end{cases}$$

The variations induced on the Lagrangian are:

$$\delta_1 L = \frac{d}{dt}\left(\frac{1}{2}\dot{x}^j\epsilon\psi_1^j + \frac{1}{2}f^j\epsilon\bar{\psi}_2^j\right) - \epsilon\left(R_{kj}\dot{x}^k + \frac{1}{2}\dot{f}^j\right)\psi_2^j$$
$$- \epsilon\left(\frac{\partial V}{\partial x^j} + R_{jk}f^k\right)\psi_1^j + i\frac{\partial R_{jk}}{\partial x^l}\epsilon\psi_1^l\psi_1^j\psi_2^k$$
$$\delta_2 L = \frac{d}{dt}\left(\frac{1}{2}\dot{x}^j\epsilon\psi_2^j - \frac{1}{2}g^j\epsilon\bar{\psi}_1^j\right) + \epsilon\left(R_{kj}\dot{x}^k + \frac{1}{2}\dot{g}^j\right)\psi_1^j$$
$$- \epsilon\left(\frac{\partial V}{\partial x^j} + R_{jk}g^k\right)\psi_2^j + i\frac{\partial R_{jk}}{\partial x^l}\epsilon\psi_2^l\psi_1^j\psi_2^k$$

There is symmetry with respect to the variations $\delta_1$ and $\delta_2$ if and only if $\delta_1 L$ and $\delta_2 L$ are pure divergences. This happens if

$$U(\vec{x}) = \frac{1}{2}\frac{\partial W}{\partial x^j}\frac{\partial W}{\partial x^j} \qquad R_{jk} = -\frac{\partial^2 W}{\partial x^j \partial x^k} \qquad f^j = g^j = \frac{\partial W}{\partial x^j};$$

where $W(\vec{x})$ is a function defined in the bosonic piece of the configuration space called the superpotential. A supersymmetric Lagrangian has the form,

$$L = \frac{1}{2}\dot{x}^j\dot{x}^j + \frac{i}{2}\psi_a^j\dot{\psi}_a^j - \frac{1}{2}\frac{\partial W}{\partial x^j}\frac{\partial W}{\partial x^j} - i\frac{\partial^2 W}{\partial x^j\partial x^k}\psi_1^j\psi_2^k$$

The Noether charges associated with the symmetries with respect to $\delta_1$ and $\delta_2$ are respectively

$$Q_1 = \dot{x}^j\psi_1^j - \frac{\partial W}{\partial x^j}\psi_2^j \qquad Q_2 = \dot{x}^j\psi_2^j + \frac{\partial W}{\partial x^j}\psi_1^j$$

Going to Hamiltonian formalism, one easily checks that these fermionic supercharges close the $N=2$ superalgebra

$$\{Q_1, Q_1\}_P = 2H \qquad \{Q_1, Q_2\}_P = 0 \qquad \{Q_2, Q_2\}_P = 2H$$

We immediately find a bosonic invariant, the Hamiltonian $H$ itself, and two fermionic constants of motion–the supercharges $Q_1$ and $Q_2$; their Poisson brackets with $H$ are zero. If the number of bosonic degrees of freedom is $N$, it is easy to check that there is other bosonic invariant, $I_3 = \prod_{i=1}^{N}\psi_1^i\psi_2^i$ [10].

If the bosonic piece of the configuration space is a general Riemannian manifold $M^N$ equipped with a metric tensor $g_{ij}$, the Grassman variables $\psi_a^i$ are the components of a contravariant vector. The definition of the supercharges is generalized in the form:

$$Q_1 = g_{jk}\dot{x}^j\psi_1^k - \frac{\partial W}{\partial x^j}\psi_2^j \qquad Q_2 = g_{jk}\dot{x}^j\psi_2^k + \frac{\partial W}{\partial x^j}\psi_1^j$$

The supersymmetric algebra dictates the form of the Hamiltonian

$$H = \frac{1}{2}g_{jk}\dot{x}^j\dot{x}^k + \frac{1}{2}g^{jk}\frac{\partial W}{\partial x^j}\frac{\partial W}{\partial x^k} + iW_{j;k}\vartheta_1^j\vartheta_2^k;$$

if

$$W_{j;k} = \frac{\partial^2 W}{\partial x^j \partial x^k} - \Gamma^l_{jk}\frac{\partial W}{\partial x^l};$$

and the inverse Legendre transformation leads to the Lagrangian

$$L = \frac{1}{2}g_{jk}\dot{x}^j\dot{x}^k + \frac{i}{2}g_{jk}\vartheta_a^j D_t\vartheta_a^k + \frac{1}{4}R_{jkln}\vartheta_1^j\vartheta_2^l\vartheta_1^k\vartheta_2^n - \frac{1}{2}g^{jk}\frac{\partial W}{\partial x^j}\frac{\partial W}{\partial x^k} - iW_{j;k}\vartheta_1^j\vartheta_2^k; \quad (3)$$

where the covariant derivative is defined as $D_t\vartheta_a^j = \dot{\vartheta}_a^j + \Gamma^j_{lk}\dot{x}^l\vartheta_a^k$.

## 3 From Liouville to SuperLiouville Models

We shall focus on mechanical systems of two bosonic degrees of freedom; $N = 2$. In particular, we shall analyze the supersymmetric extensions of the classical Liouville models. These models are completely integrable and their classical invariants are well known [11]. Our goal is to study what the invariants are in their supersymmetric extensions.

In fact, the classical Liouville models are not only completely integrable but Hamilton-Jacobi separable. The Hamilton-Jacobi principle thus provides all the solutions of the dynamics in these models. This is achieved by using appropriate coordinate systems. There are four possibilities:

**Liouville Models of Type I:** Let us consider the map $\phi: D \to \mathbb{R}^2$, where $D$ is an open sub-set of $\mathbb{R}^2$ with coordinates $(u;v)$. These variables are the elliptic coordinates of the bosonic system if the map is defined as: $\phi(x^1) = \frac{1}{\sigma}uv$, $\phi(x^2) = \frac{1}{\sigma}\sqrt{(u^2-\sigma^2)(\sigma^2-v^2)}$ and $u \in [\sigma;\infty)$, $v \in [-\sigma;\sigma]$. In the new variables, the Lagrangian of a Liouville model of Type I reads:

$$L = \frac{1}{2}\frac{u^2-v^2}{u^2-\sigma^2}\dot{u}\dot{u} + \frac{1}{2}\frac{u^2-v^2}{\sigma^2-v^2}\dot{v}\dot{v} - \frac{u^2-\sigma^2}{u^2-v^2}f(u) - \frac{\sigma^2-v^2}{u^2-v^2}g(v) \quad (4)$$

Observe that apart from a common factor the contribution to the Lagrangian of the $u$ and $v$ variables splits completely. The common factor can be interpreted as a metric (of zero curvature): $g_{ij} = (u^2-v^2)\delta_{ij}$.

**Liouville Models of Type II:** In polar coordinates $\phi(x^1) = \rho\cos\varphi$, $\phi(x^2) = \rho\sin\varphi$, $\rho \in [0;\infty)$, $\varphi \in [0;2\pi)$ the Lagrangian of the Liouville models of Type II reads:

$$L = \frac{1}{2}\dot{\rho}\dot{\rho} + \frac{1}{2}\rho^2\dot{\varphi}\dot{\varphi} - f(\rho) - \frac{1}{\rho^2}g(\varphi) \quad (5)$$

Again, besides the metric factor $g_{11} = 1$, $g_{22} = \;^2$, the contributions of and appear completely separated in the Lagrangian.

Liouville Models of Type III: Parabolic coordinates $u \in (\;\infty;\infty)$, $v \in [0;\infty)$ are defined through the map $:D \to R^2$ such that $(x^1) = \frac{1}{2}(u^2 \; v^2)$, $(x^2) = uv$. A Liouville model of Type III obeys a Lagrangian of the form:

$$L = \frac{1}{2}(u^2 + v^2)(\underline{u}\,\underline{u} + \underline{v}\,\underline{v}) \quad \frac{1}{u^2 + v^2}(f(u) + g(v)) \tag{6}$$

There is a metric factor $g_{ij} = (u^2 + v^2)\;_{ij}$ and separate contributions of u and v to L.

Liouville Models of Type IV: In these models the Lagrangian is directly separated in Cartesian coordinates

$$L = \frac{1}{2}\underline{x}^1\,\underline{x}^1 + \frac{1}{2}\underline{x}^2\,\underline{x}^2 \quad f(x^1) \quad g(x^2) \tag{7}$$

and a Euclidean metric $g_{ij} = \;_{ij}$ can be understood.

The definition of SuperLiouville models is a two step process:

(i) Define a supersymmetric N=2 Lagrangian system on a Riemannian manifold $M^2$, which is $R^2$ equipped with the metric induced by the maps , and for Types I, II, and III, and the Euclidean metric for Type IV. Consider also Grassman variables that transform as $\#^i_a = \frac{@x^{0i}}{@x^j}\;^j_a$ under these changes of coordinates.

(ii) A model defined in this way is a SuperLiouville model if the superpotential splits in such a manner that the bosonic part of the Lagrangian coincides with the Lagrangian of a Liouville model.

SuperLiouville Models of Type I: The Lagrangian of the bosonic sector of this Type of model contains two contributions:

$$L_B = \frac{1}{2}\frac{u^2 \; v^2}{u^2 \;\; ^2}\underline{u}\,\underline{u} + \frac{1}{2}\frac{u^2 \; v^2}{^2 \; v^2}\underline{v}\,\underline{v} \quad \frac{1}{2}\frac{u^2 \;\; ^2}{u^2 \; v^2}\left(\frac{@W}{@u}\right)^2 \quad \frac{1}{2}\frac{^2 \; v^2}{u^2 \; v^2}\left(\frac{@W}{@v}\right)^2;$$

where the superpotential W provides the potential $U_B$ through the identity,

$$U_B = \frac{1}{2}\frac{u^2 \;\; ^2}{u^2 \; v^2}\left(\frac{@W}{@u}\right)^2 + \frac{1}{2}\frac{^2 \; v^2}{u^2 \; v^2}\left(\frac{@W}{@v}\right)^2$$

In the fermionic sector the Lagrangian takes the form $L_F = T_F + L^I_{BF}$, with

$$T_F = \frac{i}{2}\frac{u^2 \; v^2}{u^2 \;\; ^2}\#^u_a D_t \#^u_a + \frac{i}{2}\frac{u^2 \; v^2}{^2 \; v^2}\#^v_a D_t \#^v_a$$

The Bose-Fermi interaction adds to the Lagrangian the Yukawa terms

$$\begin{aligned} L^{I}_{FB} \;=\;& i\left(\frac{\partial^2 W}{\partial u \partial u} + \frac{{}^2 \; v^2}{(u^2 \; v^2)(u^2 \quad {}^2)}\left(u\frac{\partial W}{\partial u} \; v\frac{\partial W}{\partial v}\right)\right)\psi^u_1\psi^u_2 \\ & i\left(\frac{\partial^2 W}{\partial u \partial v} + \frac{1}{u^2 \; v^2}\left(v\frac{\partial W}{\partial u} \; u\frac{\partial W}{\partial v}\right)\right)(\psi^u_1\psi^v_2 + \psi^v_1\psi^u_2) \\ & i\left(\frac{\partial^2 W}{\partial v \partial v} + \frac{(u^2 \quad {}^2)}{(u^2 \; v^2)({}^2 \; v^2)}\left(u\frac{\partial W}{\partial u} \; v\frac{\partial W}{\partial v}\right)\right)\psi^v_1\psi^v_2 \end{aligned}$$

Definition: A system in supersymmetric classical mechanics is a SuperLiouville model of Type I if the map given by the change from Cartesian to elliptic coordinates acting on the Cartesian superpotential is such that:

$$W_C = W_1(u) \; W_2(v)$$

SuperLiouville Models of Type II: The bosonic Lagrangian is:

$$L_B = \frac{1}{2}\underline{\quad} + \frac{1}{2}{}^2\underline{\quad} \quad \frac{1}{2}\left(\frac{\partial W}{\partial}\right)^2 \quad \frac{1}{2\,{}^2}\left(\frac{\partial W}{\partial}\right)^2$$

and the superpotential $W$ is related to the potential through

$$U_B = \frac{1}{2}\left(\frac{\partial W}{\partial}\right)^2 + \frac{1}{2\,{}^2}\left(\frac{\partial W}{\partial}\right)^2$$

In the fermionic sector, the Lagrangian takes the form:

$$T_F = \frac{i}{2}\psi_a D_t \psi_a + \frac{i}{2}{}^2\psi_a D_t \psi_a ;$$

and the Yukawa Bose-Fermi couplings are:

$$\begin{aligned} L^{I}_{FB} \;=\;& i\frac{\partial^2 W}{\partial \; \partial}\psi_1\psi_2 \quad i\left(\frac{\partial^2 W}{\partial \; \partial} + \frac{\partial W}{\partial}\right)\psi_1\psi_2 \\ & i\left(\frac{\partial^2 W}{\partial \; \partial} \quad \frac{1}{}\frac{\partial W}{\partial}\right)(\psi_1\psi_2 + \psi_1\psi_2) \end{aligned}$$

Definition: a system in supersymmetric classical mechanics is a SuperLiouville Model of Type II if the map given by the change from Cartesian to polar coordinates acting on the Cartesian superpotential produces:

$$\& W_C = W_1(\;) \; W_2(\;)$$

**SuperLiouville Models of Type III:** The bosonic Lagrangian is:

$$L_B = \frac{1}{2}(u^2+v^2)(\dot{u}\dot{u}+\dot{v}\dot{v}) - \frac{1}{2(u^2+v^2)}\left[\left(\frac{\partial W}{\partial u}\right)^2 + \left(\frac{\partial W}{\partial v}\right)^2\right]$$

The potential is determined from the superpotential $W$ as:

$$U_B = \frac{1}{2(u^2+v^2)}\left[\left(\frac{\partial W}{\partial u}\right)^2 + \left(\frac{\partial W}{\partial v}\right)^2\right]$$

The purely fermionic contribution to the Lagrangian is:

$$T_F = \frac{i}{2}(u^2+v^2)\psi^u_a D_t\psi^u_a + \frac{i}{2}(u^2+v^2)\psi^v_a D_t\psi^v_a$$

and the Yukawa couplings are:

$$\begin{aligned} L^I_{BF} = & -i\left[\frac{\partial^2 W}{\partial u\partial u} - \frac{1}{u^2+v^2}\left(u\frac{\partial W}{\partial u} - v\frac{\partial W}{\partial v}\right)\right]\psi^u_1\psi^u_2 \\ & -i\left[\frac{\partial^2 W}{\partial u\partial v} - \frac{1}{u^2+v^2}\left(u\frac{\partial W}{\partial v} + v\frac{\partial W}{\partial u}\right)\right](\psi^u_1\psi^v_2+\psi^v_1\psi^u_2) \\ & -i\left[\frac{\partial^2 W}{\partial v\partial v} + \frac{1}{u^2+v^2}\left(u\frac{\partial W}{\partial u} - v\frac{\partial W}{\partial v}\right)\right]\psi^v_1\psi^v_2 \end{aligned}$$

**Definition:** A system in supersymmetric classical mechanics is a SuperLiouville Model of Type III if the map given by the change from Cartesian to parabolic coordinates acting on the Cartesian superpotential is such that:

$$W_C = W_1(u) \pm W_2(v)$$

**SuperLiouville Models of Type IV:** Finally, the definition of SuperLiouville Model of Type IV is straightforward.

**Definition:** A system in supersymmetric classical mechanics belongs to Type IV SuperLiouville models if the superpotential $W(x^1;x^2)$ is of the form:

$$W(x^1;x^2) = W_1(x^1) \pm W_2(x^2)$$

The Lagrangian is:

$$L = \frac{1}{2}\dot{x}^j\dot{x}^j + \frac{i}{2}\psi^j_a\dot{\psi}^j_a - \frac{1}{2}\frac{\partial W}{\partial x^j}\frac{\partial W}{\partial x^j} - i\frac{\partial^2 W}{\partial x^1\partial x^1}\psi^1_1\psi^1_2 - i\frac{\partial^2 W}{\partial x^2\partial x^2}\psi^2_1\psi^2_2;$$

and the system can be be understood as a $N=2 \oplus N=2$ SUSY in $(0+1)$ dimensions.

As a common feature, observe that the potential is insensitive to the relative signs of the separated parts of the superpotential. Therefore, all the Liouville models are supersymmetrizable by means of two different superpotentials.

## 4 On the Bosonic Invariants

It is well known that Liouville models have a second invariant in involution with the energy –the first invariant– that guarantees complete integrability in the sense of the Liouville theorem. We shall now show that SuperLiouville models also have a second invariant of bosonic nature. Our strategy in the search of such an invariant, $[I;H]=0$, follows the general pattern shown in the literature: see [12]. The ansatz for invariants at the highest quadratic level in the momenta is:

$$
\begin{aligned}
I \;=\; & \frac{1}{2}H^{ij}p_ip_j + V(x_1;x_2) + F_{ij}\,{}^{i}_{1}\,{}^{j}_{2} + G_{ij}\,{}^{i}_{1}\,{}^{j}_{1} + J_{ij}\,{}^{i}_{2}\,{}^{j}_{2} + \\
& + L^{i}_{jk}p_i\,{}^{j}_{1}\,{}^{k}_{1} + M^{i}_{jk}p_i\,{}^{j}_{2}\,{}^{k}_{2} + N^{i}_{jk}p_i\,{}^{j}_{1}\,{}^{k}_{2} + S_{ijkl}\,{}^{i}_{1}\,{}^{k}_{2}\,{}^{j}_{1}\,{}^{l}_{2}
\end{aligned}
$$

Here, we assume that:

– i) $H^{ij}$ is a symmetric tensor depending on $x^i$. There are three independent functions to determine.

– ii) $L^i_{jk}$ and $M^i_{jk}$ also depend only on $x^i$ and are antisymmetric in the indices $j$ and $k$: $L^i_{jk} = \; L^i_{kj}$, $M^i_{jk} = \; M^i_{kj}$. They include four independent functions.

– iii) $G_{ij}$ and $J_{ij}$ are antisymmetric functions of $x^i$ in the indices: $G_{ij} = \; G_{ji}$ and $J_{ij} = \; J_{ji}$. $F_{ij}(x^i)$, however, is neither symmetric nor antisymmetric; it contains four independent functions.

– iv) Finally, $S_{ijkl}(x^i)$ is antisymmetric in the exchange of the indices $i;j$ and $k;l$ and symmetric in the exchange of the pairs $ij;kl$. There is only one independent function to determine in this tensor.

The commutator with the Hamiltonian is:

$$
\begin{aligned}
[I;H] = \; & \frac{1}{2}\frac{\partial H^{jk}}{\partial x^l}p_lp_jp_k + \quad H^{lj}\frac{\partial^2 W}{\partial x^j\partial x^k}\frac{\partial W}{\partial x_k} + \frac{\partial V}{\partial x_l}\quad p_l + \\
+ \quad & iH^{nj}\frac{\partial^3 W}{\partial x^k\partial x^l\partial x^j} + \frac{\partial F_{kl}}{\partial x_i} + 2L^i_{km}\frac{\partial^2 W}{\partial x_m\partial x_l} + 2M^i_{ln}\frac{\partial^2 W}{\partial x_n\partial x^k}\quad p_n\,{}^{k}_{1}\,{}^{l}_{2} + \\
+ \quad & \frac{\partial^2 W}{\partial x_l\partial x^k}F_{lj} \quad M^n_{kj}\frac{\partial^2 W}{\partial x_n\partial x_l}\frac{\partial W}{\partial x^l}\quad {}^{k}_{2}\,{}^{j}_{2} + \quad \frac{\partial J_{lj}}{\partial x_k} + N^k_{mj}\frac{\partial^2 W}{\partial x_m\partial x^l}\quad p_k\,{}^{l}_{2}\,{}^{j}_{2} \\
& \frac{\partial^2 W}{\partial x^j\partial x_k}F_{nk} + L^l_{nj}\frac{\partial^2 W}{\partial x^l\partial x^k}\frac{\partial W}{\partial x_k}\quad {}^{n}_{1}\,{}^{j}_{1} + \quad \frac{\partial G_{nj}}{\partial x_l} \quad N^l_{nk}\frac{\partial^2 W}{\partial x_j\partial x_k}\quad p_l\,{}^{n}_{1}\,{}^{j}_{1} + \\
+ \; 2 \quad & G_{nj}\frac{\partial^2 W}{\partial x_j\partial x^k} + J_{kl}\frac{\partial^2 W}{\partial x^n\partial x_l} \quad \frac{1}{2}N^j_{nk}\frac{\partial^2 W}{\partial x^j\partial x^l}\frac{\partial W}{\partial x_l}\quad {}^{n}_{1}\,{}^{k}_{2} + \\
+ \; & \frac{\partial L^n_{jk}}{\partial x_l}p_lp_n\,{}^{j}_{1}\,{}^{k}_{1} + \frac{\partial M^n_{jk}}{\partial x_l}p_lp_n\,{}^{j}_{2}\,{}^{k}_{2} + \frac{\partial N^n_{jk}}{\partial x_l}p_lp_n\,{}^{j}_{1}\,{}^{k}_{2} + \\
& iN^n_{jk}\frac{\partial^3 W}{\partial x^n\partial x^l\partial x^m}\,{}^{j}_{1}\,{}^{k}_{2}\,{}^{l}_{1}\,{}^{m}_{2} \quad i\frac{\partial S_{njkl}}{\partial x_m}p_m\,{}^{n}_{1}\,{}^{k}_{2}\,{}^{j}_{1}\,{}^{l}_{2}
\end{aligned}
$$

Therefore, $[I;H] = 0$, and $I$ is a second invariant if and only if the following equations are satisfied:

| | |
|---|---|
| BOX 1 | a) $\frac{\partial H^{ij}}{\partial x^k} + \frac{\partial H^{kj}}{\partial x^i} = 0$ |
| BOX 2 | a) $H^{ij} \frac{\partial^2 W}{\partial x^j \partial x^k} \frac{\partial W}{\partial x_k} = \frac{\partial V}{\partial x_i}$ |
| BOX 3 | a) ${}^{jk} \frac{\partial L^i_{jk}}{\partial x_l} + {}^{jk} \frac{\partial L^l_{jk}}{\partial x_i} = 0$<br>b) ${}^{jk} \frac{\partial M^i_{jk}}{\partial x_l} + {}^{jk} \frac{\partial M^l_{jk}}{\partial x_i} = 0$ |
| BOX 4 | a) $H^{nj} \frac{\partial^3 W}{\partial x^k \partial x^l \partial x^j} + i \frac{\partial F_{kl}}{\partial x_n} + 2iL^n_{km} \frac{\partial^2 W}{\partial x_m \partial x_l} + 2iM^n_{lj} \frac{\partial^2 W}{\partial x_j \partial x^k} = 0$<br>b) ${}^{ij} \quad \frac{\partial^2 W}{\partial x_i \partial x^k} F_{kj} \quad M^k_{ij} \frac{\partial^2 W}{\partial x_k \partial x_l} \frac{\partial W}{\partial x^l} = 0$<br>c) ${}^{ij} \quad \frac{\partial^2 W}{\partial x^j \partial x_k} F_{ik} + L^l_{ij} \frac{\partial^2 W}{\partial x^k \partial x^l} \frac{\partial W}{\partial x_k} = 0$ |
| BOX 5 | a) ${}^{ij} \quad \frac{\partial G_{ij}}{\partial x_l} \quad N^l_{jk} \frac{\partial^2 W}{\partial x_i \partial x_k} = 0$<br>b) ${}^{ij} \quad \frac{\partial J_{ij}}{\partial x_k} + N^k_{m\,j} \frac{\partial^2 W}{\partial x_m \partial x^i} = 0$<br>c) $G_{ij} \frac{\partial^2 W}{\partial x_j \partial x^k} + J_{kl} \frac{\partial^2 W}{\partial x^i \partial x_l} \quad \frac{1}{2} N^j_{ik} \frac{\partial^2 W}{\partial x^j \partial x^l} \frac{\partial W}{\partial x_l} = 0$<br>d) $\frac{\partial N^i_{jk}}{\partial x_l} + \frac{\partial N^l_{jk}}{\partial x_i} = 0$<br>e) ${}^{ij} \, {}^{lk} N^m_{jk} \frac{\partial^3 W}{\partial x^i \partial x^l \partial x^m} = 0$ |
| BOX 6 | a) ${}^{ij} \, {}^{kl} \frac{\partial S_{ijkl}}{\partial x_m} = 0$ |

## 4.1 General properties of the solution

We deal with a overdetermined system of partial di erential equations: there are 31 PDE relating 15 unknown functions. Moreover, some sub-systems can be solved for some sub-set of functions. We proceed in a recurrent way:

– i) The equations in BOXES 1 and 2 are su cient to nd $H^{ij}$ and $V$. We recover the information about the second invariant of the purely bosonic sector: the Liouville model.

– ii) The equations in BOX 3 are solved if the independent components of $L^i_{jk}$ and $M^i_{jk}$ have the form ,

$$L^i_{12} = C^{ij}x_j + A_i \qquad\qquad M^i_{12} = D^{ij}x_j + B_i;$$

where $A_i$, $B_i$, $C$ y $D$ are constants.

– iii) The equations in BOX 4, together with the previous information, leads to the computation of $F_{ij}$. The identity $\frac{\partial^2 F_{kl}}{\partial x_1 \partial x_2} = \frac{\partial^2 F_{kl}}{\partial x_2 \partial x_1}$ and the equation 4a) requires that

$$^{mn}\frac{\partial}{\partial x_m}\left( L^n_{jk}\frac{\partial^2 W}{\partial x_j \partial x^l} + M^n_{jl}\frac{\partial^2 W}{\partial x^k \partial_j} + \frac{i}{2}H^{nj}\frac{\partial^3 W}{\partial x^j \partial x^k \partial x^l}\right) = 0$$

Moreover, if we restrict $F_{ij}$ to be symmetric under the exchange of indices and then identify $L^i_{jk} = M^i_{jk}$, equation 4b) becomes equal to 4c).

– iv) The equations of BOX 5 are satis ed if :

$$G_{ij} = J_{ij} = N_{ijk} = 0$$

– v) Equation 6a) by itself, BOX 6, sets the only independent component of $S_{ijkl}$ to be constant; $S_{1212}$ = cte. Then:

$$I_3 = {}^{1}_{1}\,{}^{2}_{1}\,{}^{1}_{2}\,{}^{2}_{2}$$

is a constant of motion, an invariant.

## 4.2 Invariants in SuperLiouville models

We now apply the previous results to the computation of the supersymmetric extensions of the second invariant of Liouville models. In general they have the form :

$$I_2 = I_2^{(B)} + I_2^{(F)};$$

where $I_2^{(B)}$ is the "body", already present in the Liouville model, and $I_2^{(F)}$ is the "soul"-containing Grassman variables- of the second invariant in the SuperLiouville models. We nd:

#### 4.2.1 SuperLiouville Models of Type I:

$$I_2^{(B)} = \frac{1}{2}\left[ x^2\underline{x}^1 \quad x^1\underline{x}^2 \right]^2 \quad {}^2\underline{x}^2\underline{x}^2 + \left( x^2\frac{\partial W}{\partial x^1} \quad x^1\frac{\partial W}{\partial x^2} \right)^2 \quad {}^2\frac{\partial W}{\partial x^2}\frac{\partial W}{\partial x^2}$$

$$I_2^{(F)} = i(x^2\underline{x}^1 \quad x^1\underline{x}^2)\,{}^1_a\,{}^2_a + i\left( 2x^1\frac{\partial^2 W}{\partial x^2\partial x^2} \quad \frac{\partial W}{\partial x^1} \quad x^2\frac{\partial^2 W}{\partial x^1\partial x^2} \right){}^2_1\,{}^2_2 +$$

$$+ \; i \left( x^1 x^2\frac{\partial^2 W}{\partial x^2\partial x^2} + x^2\frac{\partial W}{\partial x^1} + x^2x^2\frac{\partial^2 W}{\partial x^1\partial x^2} \right) ({}^1_1\,{}^2_2 + {}^2_1\,{}^1_2) +$$

$$+ \; i \left( x^2\frac{\partial W}{\partial x^2} \quad x^1x^2\frac{\partial^2 W}{\partial x^1\partial x^2} + x^2x^2\frac{\partial^2 W}{\partial x^1\partial x^1} \right){}^1_1\,{}^1_2$$

#### 4.2.2 SuperLiouville Models of Type II:

$$I_2^{(B)} = \frac{1}{2}\left( x^2\underline{x}^1 \quad x^1\underline{x}^2 \right)^2 + \frac{1}{2}\left( x^2\frac{\partial W}{\partial x^1} \quad x^1\frac{\partial W}{\partial x^2} \right)^2$$

$$I_2^{(F)} = i\left( x^2\underline{x}^1 \quad x^1\underline{x}^2 \right){}^1_a\,{}^2_a + ix^2\left( x^2\frac{\partial W}{\partial x^1\partial x^1} \quad x^1\frac{\partial^2 W}{\partial x^1\partial x^2} \quad \frac{\partial W}{\partial x^2} \right){}^1_1\,{}^1_2 +$$

$$+ \; ix^2\left( x^2\frac{\partial^2 W}{\partial x^1\partial x^1} + \frac{\partial W}{\partial x^1} \quad x^1\frac{\partial^2 W}{\partial x^2\partial x^2} \right) ({}^1_1\,{}^2_2 + {}^2_1\,{}^1_2) +$$

$$+ \; ix^1\left( x^1\frac{\partial^2 W}{\partial x^2\partial x^2} \quad \frac{\partial W}{\partial x^1} \quad x^2\frac{\partial^2 W}{\partial x^1\partial x^2} \right){}^2_1\,{}^2_2$$

#### 4.2.3 SuperLiouville Models of Type III:

$$I_2^{(B)} = \left( x^1\underline{x}^2 \quad x^2\underline{x}^1 \right)\underline{x}^2 + \left( x^1\frac{\partial W}{\partial x^2} \quad x^2\frac{\partial W}{\partial x^1} \right)\frac{\partial W}{\partial x^2}$$

$$I_2^{(F)} = i\underline{x}^2\,{}^1_a\,{}^2_a \quad ix^2\frac{\partial^2 W}{\partial x^1\partial x^2}\,{}^1_1\,{}^1_2 \quad ix^2\frac{\partial^2 W}{\partial x^2\partial x^2}({}^1_1\,{}^2_2 + {}^2_1\,{}^1_2) +$$

$$+ \, i\left( 2x^1\frac{\partial^2 W}{\partial x^2\partial x^2} \quad \frac{\partial W}{\partial x^1} \quad x^2\frac{\partial^2 W}{\partial x^1\partial x^2} \right){}^2_1\,{}^2_2$$

#### 4.2.4 SuperLiouville Models of Type IV:

$$I_2^{(B)} = \frac{1}{2}\underline{x}^1\underline{x}^1 + \frac{1}{2}\frac{\partial W}{\partial x^1}\frac{\partial W}{\partial x^1} \qquad\qquad I_2^{(F)} = i\frac{\partial^2 W}{\partial x^1\partial x^1}\,{}^1_1\,{}^1_2$$

Finally, we briefly comment on the geometrical and physical meaning of the second invariant. Usually, it is related to transformation that is termed as a hidden symmetry. We see that by introducing the generalized momenta $\pi_j = \dot{x}_j + i\frac{\partial W}{\partial x^j}$, the second invariant of the Type I model is:

$$I_2^{(B)} = \frac{1}{2}\left[ (x^2\pi_1 - x^1\pi_2)^2 - \sigma^2 |\pi_2|^2 \right];$$

which is no more than the modulus of the generalized angular momentum to the square added to $\sigma^2$ times the square of $|\pi_2|$. Similar considerations are easily applied to the second invariant of the other Types. A generalized momentum such as $\pi_j$ can be obtained if one adds the complex topological piece:

$$L_T^{(B)} = i\dot{x}^j \frac{\partial W}{\partial x^j}$$

to the bosonic Lagrangian.

## References


[1] R. Rajaraman, Solitons and Instantons, North-Holland, 1982.

[2] A. Alonso Izquierdo, M.A. Gonzalez Leon and J. Mateos Guilarte, J. Phys. A 31 (1998), 209; Geometry and Physics (Zaragoza 1998), 15–35, An. Fís. Monogr., 5. CIEMAT, Madrid 1998.

[3] E. Witten, D. Olive, Phys. Lett. 78B n.1 (1978) 97.

[4] G. Gibbons and P. Townsend, Phys. Rev. Lett. 83 (1999) 1727.

[5] G. Dvali, M. Shifman, Nucl. Phys. B 504 (1997) 127–146.

[6] R. Casalbuoni, Nuovo Cim. 33A (1976) 389–431.

[7] G. Junker, Supersymmetric methods in quantum and Statistical Physics, Springer, 1996.

[8] P. Dirac, Lectures on quantum mechanics, Yeshiva Univ. Press, 1964.

[9] P. Freund, Introduction to Supersymmetry, Camb. Univ. Press, 1986.

[10] M.S. Plyushchay hep-th/9808130.

[11] J. Liouville, J. Math. Phys. Appl. 11 (1849), 345. A. Perelomov, Integrable Systems of Classical Mechanics and Lie Algebras, Birkhauser, (1992).

[12] J. Hietarinta, Phys. Lett. 93A (1982) 55; Phys. Lett. 96A (1983) 6. B. Grammaticos, B. Borizzi and B. Padjen, Phys. Lett. 89A (1982) 111; M.F. Rañada, Sistemas Sobreintegrables VI Workshop de Otoño: Geometría y Física. Salamanca 1997.