

The generating function for a particular class of characters of $SU(n)$

W. García Fuertes, A. M. Perelomov

Departamento de Física, Facultad de Ciencias, Universidad de Oviedo, E-33007 Oviedo, Spain

Abstract

We compute the generating function for the characters of the irreducible representations of $SU(n)$ whose associated Young diagrams have only two rows with the same number of boxes. The result is given by formulae (11), (14), (25)–(27) and is a rational determinantal expression in which both the numerator and the denominator have a simple structure when expressed in terms of Schur polynomials.

1 Introduction

Among the integrable quantum mechanical systems known to date, those related to the root systems of finite dimensional simple Lie algebras form a prominent class [8]. They constitute, in particular, a natural framework to extend many classical systems of orthogonal polynomials to the case of several independent variables. A much studied example is the trigonometric Calogero–Sutherland model related to A_{n-1} [1], [13], whose eigenfunctions provide a natural generalization to n variables of the Gegenbauer polynomials. A number of properties of these generalized Gegenbauer polynomials are known [9], [10], [11], [12]. The polynomials depend on a continuous parameter λ , which is related to the coupling constant in the Hamiltonian, and are determined by $n-1$ quantum numbers. Several special values for these quantities are interesting, among which we mention two: first, when the λ parameter goes to unity, the generalized Gegenbauer polynomials converge to the characters of the irreducible representations of $SU(n)$; second, when only the first quantum number is non-vanishing, the polynomials become those of Jack [6].

In the task of deepening our understanding of the properties of the generalized Gegenbauer polynomials, the computation of the generating function would be one important milestone. This is known only for the simplest A_1 and A_2 cases [12]. Nevertheless, in some circumstances one can extract useful information from the generating function of some particular subsets of the whole system of orthogonal polynomials. The generating function of Jack polynomials, for instance, has been recently used as the starting point of an inductive proof of the structure of the derivatives of the generalized Gegenbauer polynomials [3]. This

On leave of absence from the Institute for Theoretical and Experimental Physics, 117259, Moscow, Russia. Current E-mail address: perelomov@dfuz.unizar.es

function is a simple deformation of the generating function of the characters of the irreducible representations of $SU(n)$ obtained by taking $t = 1$ in the Jack polynomials. This shows how the knowledge of the generating function of some specific classes of irreducible characters of $SU(n)$ can be a valuable clue for studying the corresponding generating function for the generalized G egebauer polynomials. In this spirit, the purpose of this article is to compute the generating function of a subset of irreducible characters of $SU(n)$ which represents the immediate next step in complexity starting from the characters related to Jack polynomials.

2 Computation of the generating function

As stated in the introduction, our goal is to compute

$$F(t; z_i) = \sum_{k=0}^{\infty} P_k(z_j) t^k \quad (1)$$

where $P_k(z_j)$ is the character of the irreducible representation of $SU(n)$ whose associated Young diagram has only two rows of length k , i.e.

$$P_k(z_j) = \chi_{(k, 0, \dots, 0)}(z_j) \quad (2)$$

We explain the notation. We use $\chi_{(k_1, k_2, \dots, k_n)}$ to designate the character of the irreducible representation of $SU(n)$ with Young diagram containing k_i boxes in the i -th row; z_j is the j -th elementary symmetric polynomial in the coordinates x_1, x_2, \dots, x_n of the maximal torus of $SU(n)$:

$$z_j = \sum_{i_1 < i_2 < \dots < i_j} x_{i_1} x_{i_2} \dots x_{i_j} \quad (3)$$

There exists a simple relation between the z_j -symbols and the generalized G egebauer polynomials for $t = 1$, namely

$$\chi_{(k_1, k_2, \dots, k_n)} = z_n^{k_n} P_{(k_1, k_2, \dots, k_{n-1}, k_n)}^1 \quad (4)$$

Finally, we give two convenient formulae for computing the characters. First, directly in terms of the x_j , we have the Weyl character formula [14]

$$\chi_{(k_1, k_2, \dots, k_n)}(x_j) = \frac{1}{j} x^{n+k_1-1} x^{n+k_2-2} \dots x^{k_n} j \quad (5)$$

in which the shorthand notation

$$j = \begin{matrix} x_1^1 & x_1^2 & & x_1^n \\ x_2^1 & x_2^2 & & x_2^n \\ \vdots & \vdots & \ddots & \vdots \\ x_n^1 & x_n^2 & & x_n^n \end{matrix} \quad (6)$$

is used, and the denominator is the Vandermonde determinant

$$= j^{n-1} x^{n-2} \dots j = \prod_{i < j} (x_i - x_j) \quad (7)$$

And second, as functions of z_j , they can be expressed through the second Giambelli identity [7]: if $(l_1; l_2; \dots; l_m)$ is the conjugate partition to $(k_1; k_2; \dots; k_n)$,

$$_{k_1, k_2, \dots, k_n} (z_j) = \begin{array}{cccc} z_{l_1} & z_{l_1+1} & z_{l_1+2} & \dots & z_{l_1+m-1} \\ z_{l_2-1} & z_{l_2} & z_{l_2+1} & \dots & z_{l_2+m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{l_m-m+1} & & & & z \end{array} \quad (8)$$

where it is understood that $z_0 = 1$ and $z_j = 0$ if $j > n$ or $j < 0$.

After these preliminaries, we turn back to (1). According to (5), we can write

$$F = \frac{1}{2S_n} \sum_{k=0}^X \text{sgn}(\dots) x_{(1)}^{n-1+k} x_{(2)}^{n-2+k} x_{(3)}^{n-3} \dots (x_{(1)} x_{(n)})^k \quad (9)$$

If we interchange the summatories, we get an alternating sum of simple geometric progressions, thus

$$F = \frac{1}{2S_n} \sum_{k=0}^X \frac{x_{(1)}^{n-1} x_{(2)}^{n-2} \dots (x_{(1)} x_{(n)})^k}{1 - t x_{(1)} x_{(2)}}; \quad (10)$$

which we will write as

$$F = \frac{g}{f} \quad (11)$$

with

$$f = \prod_{i < j} (1 - t x_i x_j) \quad (12)$$

$$g = \frac{1}{2S_n} \sum_{k=0}^X \text{sgn}(\dots) x_{(1)}^{n-1} x_{(2)}^{n-2} \dots (x_{(1)} x_{(n)})^k \frac{f}{1 - t x_{(1)} x_{(2)}} \quad (13)$$

f and g are polynomials in t of respective degrees $N = \frac{1}{2}n(n-1)$ and $N-1$. It is obvious that the coefficients of f are homogeneous symmetric polynomials in x_i over the integers. As F is a series of polynomials of the same kind, the statement turns out to be also true for g . Our next task is to compute f and g in closed form; we would like, in particular, to express their coefficients in the most natural basis for homogeneous symmetric polynomials in the present context: the irreducible characters of $SU(n)$, or Schur polynomials.

In fact, the result for f is known: it was obtained by Weyl in the course of his computation of the characters of the symplectic groups. The Weyl result is [14]

$$f = \frac{1}{k!} x^{n-1}; x^{n-2} + x^n; x^{n-3} + x^{n+1}; \dots; 1 + x^{2n-2} \prod_{j > k} p_{\bar{t}x} \quad (14)$$

Given this expression, we can obtain the coefficients in t in

$$f = 1 + Z_1 t + Z_2 t^2 + Z_3 t^3 + \dots + (Z_N) t^N \quad (15)$$

by expanding the determinant in the numerator of (14) in such a way that all terms have monomial columns and collecting terms of the same order in t . This gives Z_d as a sum of determinantal quotients of the form

$$Z = \frac{1}{k!} x^{n+r}; x^{n+r-1}; \dots; x^{n+1}; x^n; \dots; x^{n-d+2}; \dots; x^{n-d+r-2}; \dots; 1 \prod_{j > k} p_{\bar{t}x} \quad (16)$$

where the hat over a term means that the term is absent and the sum is extended to all possible combinations such that

$$\sum_{r_1 + \dots + r_n = d} \hat{r}_1 > \hat{r}_2 > \dots > \hat{r}_n \geq 0 \quad (17)$$

From (16), the Weyl character formula gives $z = \sum_{k_1, k_2, \dots, k_n} z_{k_1, k_2, \dots, k_n}$ with

$$k_j = \frac{\prod_{r=1}^j (j+r-j+1) \prod_{r=1}^{j-1} (r+1-j)}{\prod_{r=1}^j (r+k) \prod_{r=1}^{j-1} (r+k+3-k-j)} \prod_{k=1}^r (k+1) \quad (18)$$

A closer inspection shows that this structure corresponds to a Young diagram of rank r with $j+r-j+1$ boxes in the j -th row and in which the number of rows with number of boxes greater or equal to j is $1+j+r-j+1, j=1;2;\dots;r$. Therefore, the Young diagram associated to k_1, k_2, \dots, k_n is $(r; r-1; \dots; 1; j, r+1; r-1+1; \dots; 1+1)$ in Frobenius notation, see [7], and, by identifying each diagram with its associated $SU(n)$ character, we can write

$$Z_d = \sum_{P_d} (1; 1; 2; 1; \dots; r-1; j, 1; 2; \dots; r) \quad (19)$$

where P_d is the set of partitions of d such that $1 > 2 > \dots > r \geq 1$.

On the other hand, we can show that g is given by a determinant expression very similar to (14):

$$g = \frac{1}{x_1^{n-1} x_2^{n-2} x_3^{n-3} x_4^{n-4} + x_1^n x_2^{n-5} + x_1^{n+1} + \dots; 1 + x_1^{2n-4} x_2^{n-3} x_3^{n-4} x_4^{n-5} + \dots} \quad (20)$$

To see this, it is convenient to rescale temporarily t to the unity, to write $g = \frac{Q_1}{D_1}$ and to consider first the case $x_1 = x_2^{-1}$. From (13) we get

$$\begin{aligned} Q_1|_{x_1=x_2^{-1}} &= [(x_1^n x_2^{n-2} x_3^{n-3} \dots x_n + \text{perm}) (x_2^n x_1^{n-2} x_3^{n-3} \dots x_n + \text{perm})] \\ &\quad f[(1/x_3) \dots (1/x_n)] [(1/x_1^{-1} x_3) \dots (1/x_n)] \prod_{j,k=3}^n (1-x_j x_k) g = \\ &= (x_1^{-1} x_2) [x_3^{n-3} x_4^{n-4} \dots x_n + \text{perm}] \\ &\quad f[(1/x_3) \dots (1/x_n)] [(1/x_1^{-1} x_3) \dots (1/x_n)] \prod_{j,k=3}^n (1-x_j x_k) g \quad (21) \end{aligned}$$

where perm refers to the permutations of the powers of x_3, x_4, \dots, x_{n-1} including signature. By taking x_k^{n-2} as a common factor in the k row of the numerator of (20) for $k=1$ to n , we write:

$$\begin{aligned} Q_2 &= x_1^{n-1} x_2^{n-2} x_3^{n-3} x_4^{n-4} + x_1^n x_2^{n-5} + x_1^{n+1} + \dots; 1 + x_1^{2n-4} x_2^{n-3} x_3^{n-4} x_4^{n-5} + \dots \\ &= (x_1 x_2 \dots x_n)^{n-2} x_1^{-1} x_2^{-2} x_3^{-3} x_4^{-4} + \dots; y_1^{-2} y_2^{-3} \dots y_{n-2}^{-2} \quad (22) \end{aligned}$$

with $y_j = x_j + x_j^{-1}$. If we take $x_1 = x_2^{-1}$ in this expression and subtract in the determinant the second from the first row, we obtain through the Vandermonde formula (7)

$$Q_2 \Big|_{x_1=x_2^{-1}} = (1)^{\frac{(n-1)(n-2)}{2}} (x_1 \ x_2) (x_3 x_4 \ \dots \ x_n) x_2^{-2} (y_2 \ y_3 \ \dots \ y_n) \prod_{\substack{i,j=3 \\ i < j}}^n (y_i \ y_j); \quad (23)$$

The use of the identities

$$\begin{aligned} (1 - x_2^{-1} x_j)(1 - x_2 x_j) &= x_j (y_2 - y_j) \\ y_k - y_j &= (x_k x_j)^{-1} (x_k - x_j)(1 - x_k x_j) \end{aligned} \quad (24)$$

in (23) transforms the right side of this equation in exactly the last member of (21); therefore $Q_1 \Big|_{x_1=x_2^{-1}} = Q_2 \Big|_{x_1=x_2^{-1}}$. As Q_1 and Q_2 are symmetric polynomials, this implies that $Q_1 - Q_2 = fP$ where P is a symmetric polynomial, but from (12) and the definitions of Q_1 and Q_2 , one can easily check that the total degree of the left side of this equation is necessarily lower than the total degree of f , so that $P = 0$. This concludes the proof of (20).

If we now write

$$g = \sum_{k=0}^{\infty} G_k t^k \quad (25)$$

a computation completely analogous to that leading from (14) to (19) allows us to write G_k as follows

$$G_k = \sum_{Q_{k+j;j}} (1)^{k+j} (1 \ 3; 2 \ 3; \dots; j \ 3j_1; 2; \dots; j); \quad (26)$$

where $Q_{k+j;j}$ is the set of partitions of $k+j$ with j terms and satisfying $1 > 2 > \dots > j \geq 3$. From (14) and (20), we give the final formula for the desired generating function:

$$F = \frac{\sum_{j \geq 1} \sum_{Q_{k+j;j}} (1)^{k+j} (1 \ 3; 2 \ 3; \dots; j \ 3j_1; 2; \dots; j) x^{k+j}}{\sum_{j \geq 1} \sum_{Q_{k+j;j}} (1)^{k+j} (1 \ 3; 2 \ 3; \dots; j \ 3j_1; 2; \dots; j) x^{k+j}}; \quad (27)$$

3 Differential equations for f and g

We will deduce in this section two differential equations satisfied by f and g . These equations can be taken as the basis for an alternative approach for the computation of these quantities.

In establishing the equations, we will take advantage of two differential operators of a class introduced in [4], [5], [2]:

$$\begin{aligned} D_1 &= \sum_{p=1}^n z_p^{-1} \frac{\partial}{\partial z_p} \\ D_2 &= \frac{1}{2} \left[\sum_{p=2}^n z_p^{-2} \frac{\partial}{\partial z_p} + D_1^2 \right] \end{aligned}$$

The action of these operators on Schur polynomials can be most simply described in graphical terms: D_1 applied to the Schur polynomial S_λ gives the sum of all Schur polynomials whose associated Young diagrams are that of λ with one box removed. D_2 does similarly but, in

this case, the sum is over the Young diagrams obtained by removing two boxes not in the same row in all possible ways.

From (12), we get

$$\ln f = \sum_{i < j}^X (x_i x_j + \frac{1}{2} x_i^2 x_j^2 + \frac{1}{3} x_i^3 x_j^3 + \dots) \quad (28)$$

and, therefore

$$f = \exp \left[\sum_{k=1}^X \frac{m_k}{k} t^k g \right]; \quad m_k = \sum_{i < j}^{X^n} x_i^k x_j^k \quad (29)$$

or, alternatively

$$f = \exp \left[\sum_{k=1}^X \frac{p_k^2 - p_{2k}}{2k} t^k g \right]; \quad p_k = \sum_{i=1}^{X^n} x_i^k \quad (30)$$

This gives $\frac{\partial f}{\partial p_1} = p_1 t f$. But $p_1 = z_1$ and $\frac{\partial}{\partial p_1} = D_1$ [2], [7], hence

$$D_1 f = z_1 t f \quad (31)$$

or, using (15)

$$D_1 Z_j = z_1 Z_{j-1} \quad (32)$$

To find the differential equation for g , we use (30) to write

$$\left(\frac{1}{2} \frac{\partial^2}{\partial p_1^2} - \frac{\partial}{\partial p_2} \right) f = \left[\frac{1}{2} t^2 (p_1^2 + p_2) - t \right] f \quad (33)$$

and, as $z_2 = \frac{1}{2} (p_1^2 - p_2)$ and $D_2 = \frac{1}{2} \frac{\partial^2}{\partial p_1^2} - \frac{\partial}{\partial p_2}$ [2], [7], we conclude that

$$D_2 f = [t^2 (z_1^2 - z_2) - t] f \quad (34)$$

Now,

$$D_2 g = D_2 (f F) = (D_2 f) F + f (D_2 F) + (D_1 f) (D_1 F) \quad (35)$$

The first term is proportional to g . The second too: as the diagrams in F consist only of two identical lines, we get

$$D_2 F = t F \quad (36)$$

The third term gives a contribution proportional to $D_1 g$, because

$$D_1 g = D_1 (f F) = t z_1 g + f D_1 F \quad (37)$$

The differential equation is therefore

$$D_2 g + t z_1 D_1 g + t^2 z_2 g = 0 \quad (38)$$

or, alternatively

$$D_2 G_j + z_1 D_1 G_{j-1} + z_2 G_{j-2} = 0 \quad (39)$$

Acknowledgments

A.M.P. would like to express his gratitude to the Department of Physics of the University of Oviedo for the hospitality during his stay as a Visiting Professor. The work of W.G.F. has been partially supported by grant BFM 2000 0357 (DGICYT, Spain).

References

- [1] Calogero F., *J. Math. Phys.* 12, 419{436, 1971.
- [2] Foulkes, H., *J. Lond. Math. Soc.* 24, 136{143, 1949.
- [3] García Fuertes W. and Perełomov A.M., *Theor. Math. Phys.*, to appear.
- [4] Hammond, J., *Proc. Lond. Math. Soc.* 13, 79{84, 1882.
- [5] Hammond, J., *Proc. Lond. Math. Soc.* 14, 119{129, 1883.
- [6] Jack H., *Proc. Roy. Soc. Edinb. A* 69, 1{18, 1970.
- [7] Macdonald I.G., *Symmetric Functions and Hall Polynomials*, Clarendon Press, Oxford, 1995.
- [8] Olshanetsky M.A. and Perełomov A.M., *Phys. Rep.* 94, 313{404, 1983.
- [9] Perełomov A.M., *J. Phys. A* 31, L31{L37, 1998.
- [10] Perełomov A.M., Ragoucy E. and Zaugg Ph., *J. Phys. A* 31, L559{L565, 1998.
- [11] Perełomov A.M., *J. Phys. A* 32, 8563{8576, 1999.
- [12] Perełomov A.M., in *Proc. Clusthal Conference 1999, Lie Theory and Its Applications in Physics III*, World Scientific, Singapore, 2000.
- [13] Sutherland B., *Phys. Rev. A* 4, 2019{2021, 1972.
- [14] Weyl, H., *The Classical Groups, their Invariants and Representations*, Princeton Univ. Press, Princeton, 1946.