

A diabatic motion of two-component BPS kinks

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Abstract

The low energy dynamics of degenerated BPS domain walls arising in a generalized Wess-Zumino model is described as geodesic motion in the space of these topological walls.

1 Introduction

According to a fruitful idea by Manton, geodesics in the moduli space determine the slow motion of topological defects, [1]. The adiabatic principle [2] has been successfully applied to black holes [3], magnetic monopoles [4], and self-dual vortices, both in Higgs, [5], and Chern-Simons-Higgs, [6]-[7]-[8]-[9], models. Recently, the moduli space of BPS domain walls has been discussed by Tong [10] and, following Manton's method, the low energy dynamics of solitons has been studied by Townsend and Portugues [11] in variations of the Wess-Zumino model.

In this paper we shall study the low energy dynamics of BPS kinks living in a topological sector of a super-symmetric $(1+1)$ -dimensional system proposed by Bazeia and co-workers in [13]. In [14] Shifm an and Voloshin have shown that the $(1+1)D$ system comes from the dimensional reduction of a generalized Wess-Zumino model with two chiral superfields; in this latter case, the kink solutions appear as BPS domain walls.

From a one-dimensional perspective, the system encompasses several topological sectors and Shifm an and Voloshin in [14] discovered that there exists a degenerate family of BPS domain walls in a distinguished sector. In [15], three of us found that for some critical values of the coupling parameter there exist more degenerate kink families in other topological sectors, although the new walls are generically non-BPS.

Each BPS domain wall, however, seems to be made from two basic walls, which belong to other topological sectors. This interesting structure has been explored by Sakai and Sugisaka, who found in [16] an intriguing bound-state of wall/anti-wall pairs. The aim of this work is to describe how the basic kinks move in the space of BPS topological kinks. To achieve this goal we shall apply Manton's method and we shall thus extend the applicability of the adiabatic principle to one-dimensional topological defects.

The organization of the paper is as follows: in Section 2 we briefly summarize the general framework of (1+1)D super-symmetric field theory. Section 3 is devoted to describing the moduli space of BPS super-kink solutions. In Section 4 we unveil the metric inherited from the adiabatic kink motion in the space of kink solutions, determine the geodesic orbits, and describe the low speed motion of BPS kinks. Finally, in Section 5 we interpret these results from the point of view of moving walls.

2 $N = 1$ super-symmetric (1+1)-dimensional field-theoretical systems

Using non-dimensional space-time coordinates and fields as defined in Reference [15], we consider Bose, $\hat{\psi}^a(x) = \sum_{a=1}^2 \hat{\psi}^a(x) e_a$, and Fermi,

$$\hat{\psi}(x) = \sum_{a=1}^2 \begin{pmatrix} \hat{\psi}_1^a(x) \\ \hat{\psi}_2^a(x) \end{pmatrix} e_a \quad ;$$

fields. Here, $x = (x^0; x^1)$ are local coordinates in $R^{1,1}$ space-time; $e_a e_b = \delta_{ab}$ is an orthonormal basis in R^2 internal (iso-spin) space, and the Fermi fields belong to the Majorana representation of the $Spin(1;1;R)$ group: iff $\gamma = 2g$, $g = \text{diag}(1; -1)$, is the Clifford algebra of $R^{1,1}$, we choose $\gamma^0 = \gamma^2$; $\gamma^1 = i\gamma^3$. Also, $\gamma^5 = \gamma^0\gamma^1 = \gamma^3$, with $\gamma^1; \gamma^2; \gamma^3$ the 2×2 Pauli matrices.

The canonical quantization procedure dictates the equal-time commutation/anticommutation relations among the fields and their momenta. Defining $\hat{\psi}(x) = \frac{\partial \hat{\psi}}{\partial x^0}(x)$, we have that:

$$[\hat{\psi}^a(x); \hat{\psi}^b(y)] = i\delta^{ab} \delta(x-y) \quad ; \quad \{\hat{\psi}^a(x); \hat{\psi}^b(y)\} = \delta^{ab} \delta(x-y) \quad ; \quad (1)$$

where $i; = 1; 2$ are Majorana spinor indices and a natural system of units $\hbar = c = 1$ has been chosen.

Interacting $N = 1$ super-symmetric field theory is built from the normal-ordered super-charge operator:

$$\hat{Q} = \int dx : \hat{\psi}^0(x) \partial \hat{\psi}(x) + i \hat{\psi}^0(x) \tilde{r} \hat{W}(x) : \quad ; \quad (2)$$

Interactions come from the gradient of the super-potential $\tilde{r} \hat{W}(x) = \sum_{a=1}^2 \frac{\partial \hat{W}}{\partial x^a}(x) e_a$ and the super-symmetry algebra:

$$[\hat{Q}; \hat{Q}] = 2(\hat{P}^0 - 2i\hat{T}^1) \quad ; \quad ; \quad ; = 1; 2 \quad (3)$$

encompasses the energy,

$$\hat{P}^0 = \frac{1}{2} \int dx : 4 \hat{\psi}^2 + \frac{\partial \hat{\psi}}{\partial x} \frac{\partial \hat{\psi}}{\partial x} + \tilde{r} \hat{W} \tilde{r} \hat{W} - i \hat{\psi}^1 \frac{\partial \hat{\psi}}{\partial x} + \hat{\psi}^2 \tilde{r} \hat{W} : \quad ; \quad (4)$$

the momentum ,

$$\hat{P}_1 = \int dx : 4 \frac{\partial \hat{\phi}}{\partial x} + \frac{i}{2} \frac{\partial \hat{\psi}}{\partial x} : ; \quad (5)$$

and the anomalous topological/central charge,

$$\hat{T} = \int dx : 4 \frac{\partial \hat{\phi}}{\partial x} \hat{\phi} + \frac{1}{4} \hat{\psi}^2 : ; \quad (6)$$

operators. In formulas (4)-(5)-(6) we have defined $\hat{\phi}(x) = \hat{\psi}(x)^0$, and

$$\hat{W} = \hat{\phi} \hat{\phi} = \sum_{a=1}^2 \sum_{b=1}^2 e_a e_b \frac{\partial^2 \hat{W}}{\partial x^a \partial x^b} ; \quad \hat{W} = \hat{\phi} \hat{\phi} = \sum_{a=1}^2 \frac{\partial^2 \hat{W}}{\partial x^a \partial x^a}$$

are respectively the Hessian and Laplacian operators applied to \hat{W} . Note that there is no anomaly in the central charge if the super-potential is harmonic, the condition for the existence of $N = 2$ super-symmetry.

The usual definition of BPS states in this system, [12], is derived from (3):

$$\hat{P}_0 = \frac{1}{2} \hat{Q}_1 \hat{Q}_2 + \hat{j} \hat{j} ; \quad \hat{Q}_1 \hat{Q}_2 \text{ BPS } i = 0 \quad (7)$$

as the requirement of minimal energy in each super-selection sector. To find these states a variational method, using the coherent states

$$\hat{\phi}(x) \tilde{\psi}(x); \tilde{\psi}(x) i = 0 ; \hat{\phi}(x) \tilde{\psi}(x); \tilde{\psi}(x) i = \tilde{\psi}(x) \hat{\phi}(x); \tilde{\psi}(x) i ; \hat{\phi}(x) \tilde{\psi}(x); \tilde{\psi}(x) i = \tilde{\psi}(x) \hat{\phi}(x); \tilde{\psi}(x) i$$

as trial states, is conventionally applied. $\tilde{\psi}(x)$ and $\tilde{\psi}(x)$ are respectively scalar and Majorana spinor static classical field configurations and, on these states, the (7) BPS condition becomes:

$$\tilde{\psi}(x) \frac{d\tilde{\psi}}{dx} - \tilde{\psi} W - \tilde{\psi}(x); \tilde{\psi}(x) i = 0 ; \quad \tilde{\psi}(x) = \tilde{\psi}_1(x) \tilde{\psi}_2(x) \quad (8)$$

because the expectation values in this kind of state of normal-ordered operator functionals are equal to their classical counterparts, see e.g. [17].

Thus, the BPS states are the coherent states for which the scalar field configurations correspond to the flow lines of $\text{grad} W$, whereas the corresponding 1 eigen-spinor $\tilde{\psi}$ configurations are zero. Moreover, super-symmetry forces the surviving (non-null) spinor configurations to satisfy the equation:

$$\frac{d\tilde{\psi}}{dx}(x) = \tilde{W}(\tilde{\psi}) \tilde{\psi}(x) \quad (9)$$

3 The BPS kink moduli space

Our choice of super-potential is:

$$W(\phi) = \frac{1}{4} \left(\frac{\phi}{2} \right)^2 - \frac{1}{3} \left(\frac{\phi}{2} \right)^3 + \frac{1}{4} \left(\frac{\phi}{2} \right)^4 - \frac{1}{2} \left(\frac{\phi}{2} \right)^2 \left(\frac{\phi}{2} \right)^2 \quad :$$

Thus, we deal with the super-symmetric extension of the BNRT model analyzed in [15] for the special value of the parameter $\mu = \frac{1}{2}$.

In Reference [15] the BPS kinks were shown to be in one-to-one correspondence with the kink orbits:

$$\frac{1}{2} + \frac{1}{2} \frac{1}{2} = \frac{1}{4} + c \frac{1}{2} \quad :$$

The explicit analytical expressions for these families of BPS kink/anti-kink solutions are:

$$\phi^K(x; a; b) = \frac{1}{2} \frac{\sinh \left(\frac{\phi}{2} (x+a) \right)}{\cosh \left(\frac{\phi}{2} (x+a) \right) + b^2} e_1 + \frac{b}{b^2 + \cosh \left(\frac{\phi}{2} (x+a) \right)} e_2 A \quad (11)$$

for arbitrary integration constants $a; b \in \mathbb{R}$. The constant b is defined in terms of c ($\mu = \frac{1}{2}$) as: $b = \frac{1}{1-4c}$, see [15]. Thus, the plane is the space S^K of BPS kinks: there is a one-to-one correspondence between the points $(a; b) \in S^K \subset \mathbb{R}^2$ and the space of topological kink solutions of the field equations. The ϕ_1 - and ϕ_2 -components of these solitary waves are shown in Figure 1 for $a = 0$ and several particular values of b . The dependence on a of the kink profile is trivial; the value of a only determines the "center" of the kink.

This sequence of Figures shows us the main characteristics of the different BPS kinks. If $b = 0$ only

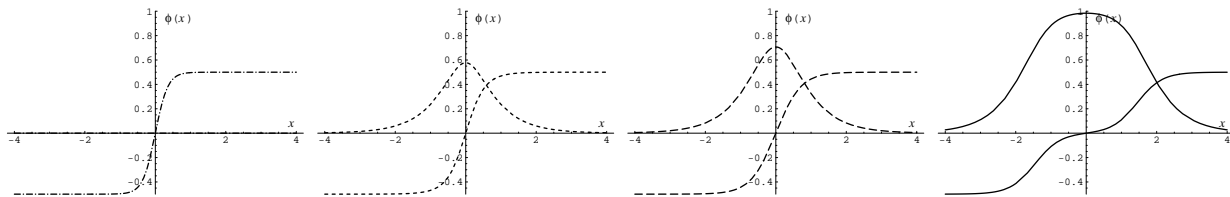


Figure 1: Solitary waves (11) corresponding to: (a) $b = 0$, (b) $b = \frac{1}{2}$, (c) $b = 1$ and (d) $b = \frac{1}{30}$.

one of the two field components ϕ_1^K is different from zero and the kink configuration interpolates between the minimum, $A_+ = \frac{1}{2} e_1$, and the maximum, $A_- = \frac{1}{2} e_1$, of W , the vacua of the theory. Therefore, the $(a; 0)$ kinks are topological solutions with one non-null component and, for obvious reasons, in the literature they are called TK1 kinks. If $b \neq 0$, ϕ_2^K is also different from zero but the field configurations still interpolate between the $A_+ = \frac{1}{2} e_1$ and $A_- = \frac{1}{2} e_1$ vacua. All these topological kinks have the two field components different from zero and they are thus called TK2 kinks. Note that changing b to $-b$ merely amounts to changing ϕ_2^K to $-\phi_2^K$.

The associated BPS fermionic form factors, the solutions of (9) for ψ^K , are:

$$\psi^K(x) = 0 \quad ; \quad \psi^K[x;a;b;d;f] = d \frac{\partial \psi^K}{\partial a}[x;a;b] + f \frac{\partial \psi^K}{\partial b}[x;a;b] \quad ; \quad (12)$$

where $d;f$ are real integration constants and $\psi^K = \psi_1^K \quad \psi_2^K = 2; \psi^K = \frac{1}{1}$ are constant eigen-spinors of σ^1 . (See Figures 2 and 3, where the fermionic partners of the bosonic solitary waves for the same values of the b parameter as above, are plotted).

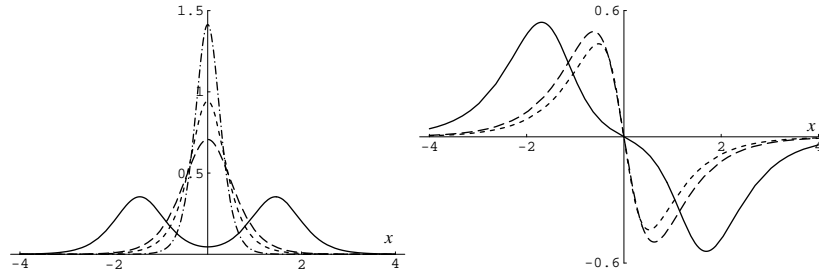


Figure 2: First (a) and second (b) components of $\frac{\partial \psi^K}{\partial a}$.

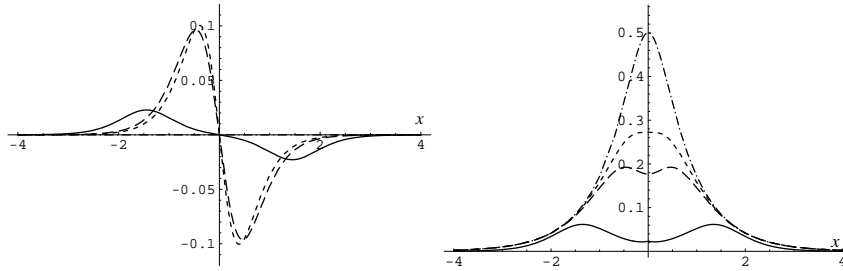


Figure 3: First (a) and second (b) components of $\frac{\partial \psi^K}{\partial b}$.

The Majorana-Weyl spinors shown in (12) span the tangent space to the space of BPS kink solutions $T S^K \simeq T R^2$ because they are linear combinations of the bosonic zero modes. Moreover, these fermionic configurations also solve the static Dirac equation of the coupled system and therefore, do not contribute to $\langle \psi(x); \hat{P}_0; \bar{\psi}(x); \psi(x) \rangle$; the BPS super-kinks, formed by the combination of the bosonic and fermionic solutions, also saturate the topological bound.

The properties of a solitary wave determined by a point in S^K are encoded in the bosonic energy density:

$$E^K[x;a;b] = \frac{\partial \psi^K}{\partial x} \cdot \frac{\partial \bar{\psi}^K}{\partial x} = \frac{4 + 7b^2 \cosh[2 \frac{p}{2}(x+a)] + 4b^4 \cosh[4 \frac{p}{2}(x+a)] + b^2 \cosh[6 \frac{p}{2}(x+a)]}{2(b^2 + \cosh[2 \frac{p}{2}(x+a)])^4} \quad (13)$$

Figure 4 shows a plot of $E^K[x;a;b]$ for $a = 0$ and the same values of b as in the above Figures. The identity $E^K[x;a;b] = E^K[x;a;-b]$ is a consequence of the invariance of the theory under the action of the group $G = Z_2 \times Z_2$ generated by the reflections $x \rightarrow -x$; $t \rightarrow -t$; $\phi \rightarrow \phi + \pi$. It merely replaces kinks by antikinks but taking a quotient by $\phi \rightarrow \phi + \pi$ both in (10) and (11) we find the moduli space $M^K \times S^K = Z_2$ of BPS kinks. Thus, a and b^2 are good coordinates in M^K , which is isomorphic to the upper half-plane: $H = (x; t) \in [0; \infty)$.

There are two regimes in the b^2 -parameter classified by the dependence on b^2 of the critical points of the energy density, i.e. the zeroes of:

$$\frac{\partial E^K}{\partial x}[x;a;b] = 2 \frac{\partial}{\partial x} \left[\frac{K}{1} \frac{\partial^2 K}{\partial x^2} \right] + \frac{\partial}{\partial x} \left[\frac{K}{2} \frac{\partial^2 K}{\partial x^2} \right] = \frac{4 \sqrt{2} \sinh \frac{\sqrt{2}}{2}(x+a)}{(b^2 + \cosh \frac{\sqrt{2}}{2}(x+a))^5} P_3(\cosh \frac{\sqrt{2}}{2}(x+a)) \quad (14)$$

$$P_3(\cosh \frac{\sqrt{2}}{2}(x+a)) = b^2 \cosh^3 \frac{\sqrt{2}}{2}(x+a) - b^4 \cosh^2 \frac{\sqrt{2}}{2}(x+a) + (3b^2 + 4b^6) \cosh \frac{\sqrt{2}}{2}(x+a) + 5b^4 - 4 :$$

Note that (14) relates the shape of the energy density to the shape of the kink profile.

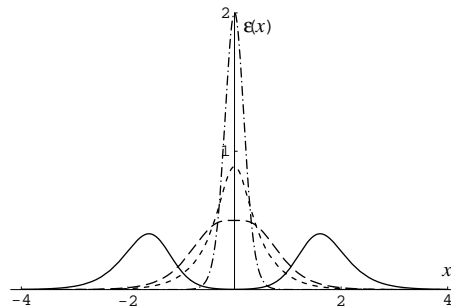


Figure 4: Energy density $E^K[x;0;b]$.

A part from the obvious solution, $x = -a$, i.e. $\sinh \frac{\sqrt{2}}{2}(x+a) = 0$, $8b^2$, we can classify the solutions of $\frac{\partial E^K}{\partial x}[x;a;b] = 0$ in terms of the roots of the cubic polynomial $P_3(\cosh \frac{\sqrt{2}}{2}(x+a))$. Writing P_3 as $P_3(\cosh \frac{\sqrt{2}}{2}(x+a)) = b^2 P(u)$, where $P(u)$ is the bi-cubic polynomial $P(u) = (u^2)^3 + (b^2 + 3)(u^2)^2 - (4b^4 - 2b^2 - 6)(u^2) - 4b^4 + b^2 - 1 - \frac{1}{b^2}$ in the variable $u^2(x) = 1 + \cosh \frac{\sqrt{2}}{2}(x+a)$, a classical analysis of the roots of $P(u)$, based on the Cardano and the Vieta formulae and use of Rolle's theorem, shows that:

$P(u)$ has no real roots in $u(x)$ if $b^2 \in [0; 1]$. Thus, $\frac{\partial E^K}{\partial x}[x;a;b] = 0$ only for $x = -a$, which is the only critical point of E^K as a function of x . Moreover, $E^K[-a;a;b] = \frac{2}{(b^2 + 1)^2}$ is the maximum value of E^K on the real line; b^2 therefore measures the height of the solitary wave energy density, see Figure 4.

Things are more interesting if $b^2 \in (1; \infty)$: P has two real roots. As a cubic polynomial in u^2 , P has a single positive root $r(b^2)$ that depends on the value of b^2 ; hence, $u = \pm \sqrt{r}$ are

the real roots of $P(u)$. Besides $x = a$, which is a minimum, two other critical points of E^K arise at $x = a \pm m(b^2)$ if $b^2 > 1$, where $m(b^2) = \frac{1}{2\sqrt{2}} \operatorname{arccosh}(1 + r(b^2))$.

These two points are maxima of E^K and the solitary wave is made from two lumps if $b^2 > 1$. The distance between the peaks grows with b^2 , and b^2 must be understood as the relative coordinate of a system of two "particles" if $b^2 > 1$, whereas a is still the center of mass coordinate.

Alternatively, one can trust Mathematica and just look at Figure 4.

Exactly at the $b = 1$ limits, the solutions

$$\tilde{B}[x; x_0; ;] = (1) \frac{1}{4} (1 - \tanh(\frac{\sqrt{2}}{2}(x + x_0))) e_1 + \frac{r}{2} \frac{1}{(1 + \tanh(\frac{\sqrt{2}}{2}(x + x_0)))} e_2 ; \quad (15)$$

(; ; = 0; 1) living in other topological sectors, appear, see Figure 5. The integration constant $x_0 \in \mathbb{R}$ determines the center of mass of the lump: $x_{CM} = x_0 + \frac{1}{2\sqrt{2}} \ln \frac{1 + \sqrt{17}}{8}$ - the shift with respect to what one expects for the center of mass is seen in Figure 5(b). Note that the inflection points in $\frac{B}{1}$ and $\frac{B}{2}$ do not coincide. $\tilde{B}[x; x_0; ;]$ are also flow lines of $\operatorname{grad} W$ but start or end at a saddle point e_2 of W - the other two vacua of the theory - . The shape of the energy density

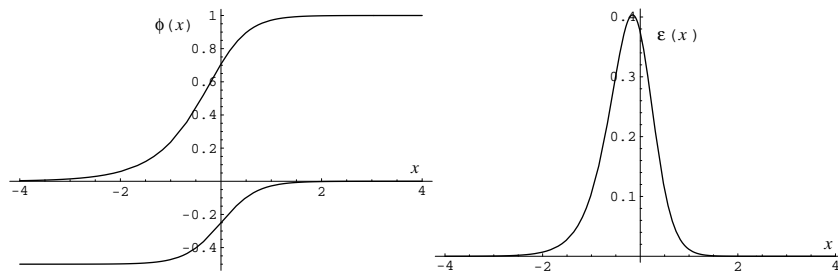


Figure 5: One of the solitary waves (15) (a) and its energy density (b).

for large values of $|b|$ suggests that the \tilde{B} solitary waves are the basic kinks in the topological sector of the \tilde{K} family. Near $|b| = 1$ the identity

$$\tilde{K}[x; a; b] \Big|_{b=1} \approx \tilde{B}[x; x_0^+; 1; 0] \theta(x - a) + \tilde{B}[x; x_0^-; 0; 1] \theta(x - a) ; \quad (16)$$

$$x_0 = a \pm m(b^2) + \frac{1}{2\sqrt{2}} \ln \frac{1 + \sqrt{17}}{8}$$

where $\theta(z)$ is the Heaviside step function, approximately holds. In fact, one can prove that the approximation becomes exact at the $|b| = 1$ limit, where the two \tilde{B} kinks become infinitely separated. Thus, they are not accessible in the kink space S^K but belong to the boundary; the circle of infinite radius in $\mathbb{R}^2: @S^K = S^1_1$.

Besides $b = 0$ and $b = 1$, there are two other special points in S^K : $b = \frac{1}{2}$. The corresponding kink orbits are the upper and lower half-ellipses $\frac{x^2}{1} + \frac{y^2}{2} = \frac{1}{4}$ and we shall use the term TK 2E to refer to these two-component kinks, which form the frontier in M^K between solitary waves carrying one or two lumps of energy density.

4 Geodesic motion in the kink space S^K

In our framework, the adiabatic principle is equivalent to restricting time-evolution to the subspace of BPS kink states. The expectation value of the kinetic energy in these states -there is no contribution of the fermionic variables to the kinetic energy because the Dirac Lagrangian is first-order in time derivatives- is:

$$\begin{aligned}
 K &= \frac{1}{2} \int_{-1}^1 dx \left[\dot{\phi}^K [x; a(t); b(t)] \hat{\phi}^K(x) \hat{\phi}^K(x) : \dot{\psi}^K [x; a(t); b(t)] \right] = \\
 &= \frac{1}{2} g_{aa}(a; b) \dot{a}^2 + g_{ab}(a; b) \dot{a} \dot{b} + \frac{1}{2} g_{bb}(a; b) \dot{b}^2
 \end{aligned}
 \tag{17}$$

where:

$$\begin{aligned}
 g_{aa}(a; b) &= \int_{-1}^1 dx \frac{\partial \phi^K}{\partial a} \frac{\partial \phi^K}{\partial a}; & g_{ab}(a; b) &= \int_{-1}^1 dx \frac{\partial \phi^K}{\partial a} \frac{\partial \phi^K}{\partial b}; & g_{bb}(a; b) &= \int_{-1}^1 dx \frac{\partial \phi^K}{\partial b} \frac{\partial \phi^K}{\partial b}
 \end{aligned}
 \tag{18}$$

We think of K as the Lagrangian for geodesic motion in the kink space S^K with a metric inherited from the dynamics of the bosonic zero modes. Alternatively, the fermionic partners of the BPS kinks acquire a direct geometrical meaning; they are the zweig-beins ("the square root") of the metric in the sense that,

$$\begin{aligned}
 g_{11}(a; b) &= \int_{-1}^1 dx \phi^K [x; a; b; 1; 0] \phi^K [x; a; b; 1; 0]; & g_{22}(a; b) &= \int_{-1}^1 dx \phi^K [x; a; b; 0; 1] \phi^K [x; a; b; 0; 1] \\
 g_{12}(a; b) &= \int_{-1}^1 dx \phi^K [x; a; b; 1; 0] \phi^K [x; a; b; 0; 1] = g_{21}(a; b) :
 \end{aligned}$$

The metric tensor is Z_2 -symmetric -invariant under $b \rightarrow -b$:

$$\begin{aligned}
 g_{aa}(a; b) &= \frac{2^{\frac{p-2}{2}}}{3}; & g_{ab}(a; b) &= 0; & g_{bb}(a; b) &= \frac{2^{\frac{p-2}{2}}}{3} h(b) \\
 h(b) &= \frac{1}{4(b^4 - 1)^2} (4 + 2b^6 - 5b^2 + 3 \frac{\arctan \frac{1}{b^2}}{1 - b^4})
 \end{aligned}
 \tag{19}$$

Computation of the integrals (18) has been performed by changing variables to $y = \exp[2^{\frac{p-2}{2}}(x+a)]$ in such a way that the quadratures reduce to rational definite integrals in y . The transition from one to two lumps is seen in formula (19) in the trading of \arctan by iarctanh that happens at

$b = 1$. $h(b)$, however, is also real for $|b| > 1$ because the denominator becomes purely imaginary in this regime, see Figure 6.

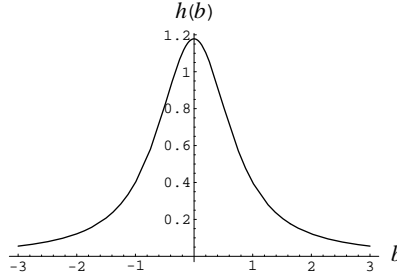


Figure 6: Graphic of the function $h(b)$.

There is an important difference with the low energy dynamics of other topological defects. In this case the moduli space M^K is an orbifold: the orbit of every point in the TK1 line ($a; b = 0$) $2 S^K$ by the action of the Z_2 -group is a single point whereas any other point in S^K is not invariant under the $b \rightarrow -b$ reflection. Therefore, geodesic motion in M^K would lead to non-smooth dynamics at the TK1 line. Fortunately, the generic orbit is a two-element set and one can safely pursue the analysis of geodesic motion in S^K .

The geodesics of a metric in the kink space S^K of the form :

$$ds^2 = \frac{2^{\frac{p-2}{2}}}{3} da^2 + h(b) db^2$$

are easily found (at least implicitly). Writing the metric in terms of a new variable, $b = \int \sqrt{\frac{2^{\frac{p-2}{2}}}{h(b)}} db$, we have :

$$\frac{3}{2^{\frac{p-2}{2}}} ds^2 = da^2 + db^2 \quad ; \quad db = \sqrt{\frac{2^{\frac{p-2}{2}}}{h(b)}} db$$

and the geodesic curves are straight lines in the $a-b$ plane. If k_1, k_2, k_1^0, k_2^0 , are integration constants, the geodesics are:

$$a(t) = k_1 t + k_2 \quad ; \quad b(t) = \int \sqrt{\frac{2^{\frac{p-2}{2}}}{h(b)}} db = k_1^0 t + k_2^0 \quad : \quad (20)$$

In terms of the new integration constants, $k_1 = \frac{k_1^0}{k_1}$; $k_2 = k_2^0 - k_1 k_2^0$, we can also represent the geodesic paths by writing b as a function of a :

$$b = \int \sqrt{\frac{2^{\frac{p-2}{2}}}{h(b)}} db = k_1 a + k_2 \quad : \quad (21)$$

We immediately identify a simple kind of geodesic motion: taking $k_1 = 0$ in (21), orbits with $b = \text{constant}$ are found. This motion corresponds to free displacement of the kink center of mass, for all types of solitary wave, without changing the shape of the kink profile, see Figure 7.

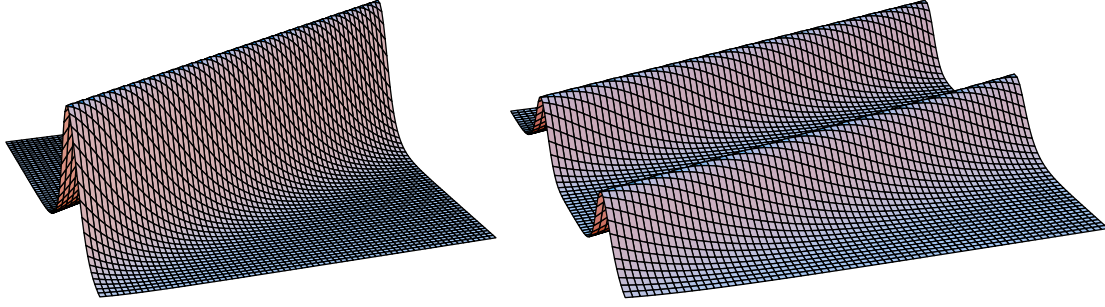


Figure 7: Energy density evolution along straight geodesic lines with $b = \text{constant}$: (a) $b = 0.9$, a single lump is moving (b) $b = 1.0$, synchronous motion of two lumps. Time runs from left to right

In a search for the more general geodesic motion, the problem is to find the explicit form of $h(b)db$. Because performing this integration explicitly is out of reach by analytical means, we propose two alternative ways for describing the geodesic motion of BPS kinks.

4.1 Numerical integration

In Figure 8(a) a Mathematica numerical plot of several geodesic orbits (21) is shown. β_1 has been set to 3, 2 and 1, whereas the freedom in β_2 has been fixed by setting the value of $b = 0.1$ -near the TK1 point in the space S^K - at the instant $t = \frac{k_2}{k_1}$. The features common to these generic geodesics are as follows: the starting point at $t = -1$ is the point in $\mathcal{E}S^K$ that corresponds to the $b \rightarrow 1$ limit of (16). Coming from very far apart, the two basic lumps begin to approach each other when $|\beta_j|$ decreases. This approach occurs simultaneously to a global displacement from β_1 of the center of mass: β_1 increases. The two kinks merge in a single TK2E $^-$ lump at the point $b = 1$ in the space S^K , and then move together, becoming higher and thinner until they become the TK1 kink when $b = 0$. From this point the composite lump moves towards the $\sim B_+$ kinks, becoming shorter and thicker until the critical value $b = 1$ is reached, where a metamorphosis of a TK2E $^+$ kink takes place, giving back rise to two separate lumps. The geodesic evolution is completed through the increasing separation of the two lumps, $|\beta_j|$ increasing, and the center of mass motion, β_1 increasing, asymptotically running towards the $b \rightarrow 1$ limit of the (16) configuration.

A closer look at the geodesic orbits near the TK1 point depicted in Figure 8(b) shows us that the bigger β_1 the shorter the time that the two lumps remain aggregated. At the $\beta_1 = 1$ limit, the geodesic curve becomes a vertical straight line in the space S^K and there is no motion of the center of mass at all. Different values of β_2 give different geodesic orbits by setting the kink point crossed at the instant $t = \frac{k_2}{k_1}$.

In sum, a generic orbit describes the following low energy dynamics: the motion starts from two infinitely separated $\sim B_+ [x; x_0; \dots]$ kinks (15) living in the topological sectors in which any configuration asymptotically connects the vacuum $\mathbf{v}^A = \frac{1}{2}\mathbf{e}_1$ with the vacuum $\mathbf{v}_+^B = \mathbf{e}_2$, and, \mathbf{v}_+^B

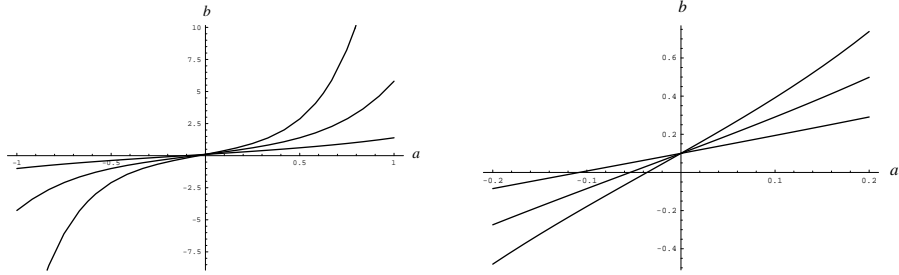


Figure 8: Geodesic orbits in the $a-b$ plane (a). Detail of the geodesic orbits (b).

with $\mathbf{v}_+^A = \frac{1}{2}\mathbf{e}_1$. The end point, however, corresponds to two \sim^{B+} $[x; x_0; \dots]$ kinks, also in nitely separated, (15) connecting $\mathbf{v}_+^A = \frac{1}{2}\mathbf{e}_1$ with $\mathbf{v}^B = \mathbf{e}_2$, and, \mathbf{v}^B with $\mathbf{v}_+^A = \frac{1}{2}\mathbf{e}_1$. The whole picture is synthesized in Figure 9. The energy density along the geodesic $k_1 = 2, k_2 = 1, k_3 = 0$ is plotted as a function of x and t , showing the adiabatic evolution of the two basic kinks. In the drawing, time runs from left to right and the spatial coordinate x grows from bottom to top.

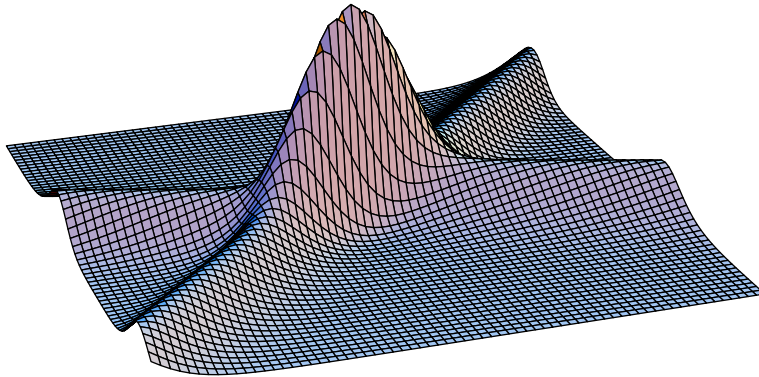


Figure 9: Evolution of energy density along a generic geodesic curve.

4.2 Asymptotic behaviour of geodesics

It is possible and indeed appropriate to find analytically the geodesics near the special points $b=0, b=1$ and $b=-1$ in S^K .

4.2.1 $b=0$

Close to the TK 1 point in S^K , the metric can be obtained approximately from the series expansion of $h(b)$ around $b=0$:

$$h(b) = \frac{3}{8} - 2b^2 + O(b^4) \quad :$$

The geodesic orbits are given in this region by :

$$a_1 + a_2 = \int^z \frac{p}{h(b)db} \quad \int^z \frac{r}{3 - 2b^2} db = \frac{4b^3 \sqrt{16b^2 + 3} \arcsin(\frac{4b}{3})}{16 \sqrt{2}} \quad ; \quad (22)$$

Therefore, equation (22) analytically describes how the TK1 kink is reached from kink configurations in its neighbourhood and vice-versa. Note that this statement is tantamount to saying that (22) determines how the two basic kinks become completely aggregated on the TK1 kink, $b = \frac{3}{4}$, and how they start to split, $b = \frac{3}{2}$.

4.2.2 $b = 1$

The expansion of $h(b)$ around the $b = 1$ points;

$$h(b) = \frac{1}{2} \frac{1}{b^2} + O\left(\frac{1}{b^6}\right) \quad ;$$

leads to the geodesic asymptotic behaviour

$$a_1 + a_2 = \int^z \frac{p}{h(b)db} \approx \frac{1}{2} \text{sign}(b) \ln |b| \quad ; \quad (23)$$

which shows how the two $\sim b^{-1} [x; 1; ;]$ kinks (15) are reached exponentially fast in a , or, how fast the two $\sim b^{-1} [x; 1; ;]$ lumps start to approach each other.

There is a subtlety: $\lim_{b \rightarrow 1} h(b) = 0$ and the metric ceases to be a rank-two tensor at these limits. Dynamically this breakdown of the geometrical meaning is due to the fact that by taking the $|b| \rightarrow 1$ limit we go out of S^K because the separation between the two basic kinks is infinite.

4.2.3 $b = 1$

There are still two other special points corresponding to the two-component topological TK2E kinks where the melting into a single lump $-b = 1-$ or the splitting into two lumps $-b = 1-$ take place. Despite appearances, the metric at $b = 1$ is regular and the series expansion of $h(b)$ in the vicinity of these points reads:

$$h(b) = \frac{2}{5} - \frac{4}{7}(b-1) + O((b-1)^2) \quad ;$$

Therefore, the geodesic equations

$$a_1 + a_2 = \int^z \frac{p}{h(b)db} \approx \frac{7}{6} - \frac{34}{35} - \frac{4}{7} b^{\frac{3}{2}}$$

analytically rule the low energy process of two-lump fusion into TK2E₋ / TK2E₊ fusion into two lumps in the vicinity of the TK2E₋ kink points of the S^K space.

5 From low-energy kink dynamics to the slow motion of domain walls

We finish this paper by offering some comments about the importance of the model that we have studied within the context of the low energy effective theories inspired in string/M theory. In Reference [14], the authors analyzed a generalized $N = 1$ super-symmetric Wess-Zumino model in $(3+1)$ -dimensional space-time with two chiral superfields, $\phi_1; \phi_2$, and interactions determined by the superpotential:

$$W(\phi_1; \phi_2) = \frac{4}{3} \phi_1^3 - \phi_1 + 2 \phi_1 \phi_2^2 \quad : \quad (24)$$

We implicitly assume non-dimensional field variables and that μ is a non-dimensional coupling constant between the two chiral superfields. Distinguishing among real and imaginary parts for the superfields, $\phi_1 = \phi_1 + i \psi_1$, $\phi_2 = \phi_2 + i \psi_2$, the real and imaginary parts of the superpotential $W = W^1 + iW^2$ read:

$$\begin{aligned} W^1(\phi_1; \phi_2) &= \frac{4}{3} \phi_1^3 - \phi_1 + 2 \left(\phi_1 \phi_2^2 \right) \quad (25) \\ W^2(\phi_1; \phi_2) &= \left[4 \phi_1^2 \frac{4}{3} \phi_1 + 2 \left(\phi_2^2 \phi_1 \right) \right] + 4 \phi_1 \phi_2^2 \quad : \end{aligned}$$

Therefore, the restriction to the real part of the model $\psi_1 = \psi_2 = 0; W^2 = 0$ leads to the BNRT system proposed in [13] and discussed from the point of view of kink defects in [15]. In fact, the BPS kinks described in [15] are in one-to-one correspondence with the BPS domain walls discovered in [14]. In particular, the moduli space of BPS kinks is identical to the moduli space of BPS walls of the generalized Wess-Zumino model and all that we have concluded for the slow motion of BPS kinks in the $\mu = \frac{1}{2}$ case can safely be translated to the adiabatic motion of BPS walls in the corresponding generalized Wess-Zumino model.

This combination of dimensional reduction and reality conditions poses a problem from a $(1+1)$ -dimensional perspective. We denote by $W = W^1$ the real part of the reduced superpotential. It has been shown in Reference [14] that there is a partner "quasi-superpotential", \tilde{W} , in the sense that the generalized Cauchy-Riemann equations

$$\frac{\partial W}{\partial \phi_a} = 2 \epsilon_{ab} j_2 \tilde{W}^{\frac{2}{j_2} + 1} \frac{\partial \tilde{W}}{\partial \phi_b} \quad ; \epsilon_{ab} = -\epsilon_{ba} \quad ; a = 1; 2 \quad ; b = 1; 2 \quad : \quad (26)$$

are satisfied. $(\tilde{W})^{\frac{2}{j_2} + 1}$ is the integrating factor used in Reference [15] to find the flow lines of $\text{grad} \tilde{W}$. In fact, the solutions of the first-order equations (9) in [15]—written here in (10) for $\mu = \frac{1}{2}$ —satisfy:

$$\frac{\partial W}{\partial \phi_2} d\phi_1 - \frac{\partial W}{\partial \phi_1} d\phi_2 = (\tilde{W})^{\frac{2}{j_2} + 1} d\tilde{W} = 0 \quad :$$

Thus, $d\tilde{W}$ is an exact one-form on the solutions (in \mathbb{R}^2) and \tilde{W} remains constant on the kink orbits. Only if $\mu = \frac{1}{2}$ will the dimensional reduction that we are considering coincide with the

outcome of the standard dimensional reduction in the genuine Wess-Zumino model. In this latter case, there is no need for any integrating factor because (26) becomes strictly the Cauchy-Riemann equations and W, \tilde{W} are conjugate harmonic functions that allow a complex super-potential to be built. If $\epsilon = 2$ this is not so and there is no possibility of obtaining $N = 2(1+1)$ -dimensional supersymmetry, which, in turn, means that one must expect one-loop corrections in the surface tension of the walls.

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