Explicit computations of low lying eigenfunctions for the quantum trigonometric Calogero-Sutherland model related to the exceptional algebra E_7

J. Fernández Núñez, W. García Fuertes, A. M. Perelomov*
Departamento de Física, Facultad de Ciencias, Universidad de Oviedo, E-33007 Oviedo, Spain

Abstract

In a previous paper [1] we have studied the characters and Clebsch-Gordan series for the exceptional Lie algebra E_7 by relating them to the quantum trigonometric Calogero-Sutherland Hamiltonian with coupling constant $\kappa = 1$. Now we extend that approach to the case of general κ.

1 Introduction

The Calogero-Sutherland models [2, 3] related to the root systems of the simple Lie algebras [4, 5, 6] have been deeply investigated during the last two decades. Originally introduced on purely theoretical grounds, this class of models have however found a number of relevant applications in such diverse fields as condensed matter physics, supersymmetric Yang-Mills theory or black-hole physics. On the mathematical side, an interesting feature of the quantum version of this kind of models is that their energy eigenfunctions provide a natural generalization of several types of hypergeometric functions to the multivariable case. For the potential $v(q) = \kappa(\kappa - 1)\sin^{-2}(q)$ and special values of the coupling constant, these eigenfunctions are related to some othogonal functional systems of particular interest in the theory of Lie algebras and symmetric spaces: for $\kappa = 1$ we obtain the characters of the irreducible representations of the algebra, while for $\kappa = 0$ the corresponding monomial symmetric functions arise; other values of κ lead to zonal spherical functions in symmetric spaces associated to the Lie algebra: in particular, for E_7, $\kappa = \frac{1}{2}$ gives these functions for the symmetric space EV [7, 6]. The Calogero-Sutherland Hamiltonian appears in this way as a natural unified tool for the computation of all these objects.

The Calogero-Sutherland Hamiltonian associated to the root system of a simple Lie algebra can be written as a second-order differential operator whose variables are the characters of the fundamental representations of the algebra. As it was shown in the papers [8, 9, 10], and later in [11, 12, 13, 14, 15, 16, 17], this approach gives the possibility of developing some systematic procedures to solve the Schrödinger equation and determine important properties of the eigenfunctions, such as recurrence relations or generating functions for some subsets of them. The approach has been used for classical algebras of A_n and D_n type, for the exceptional algebra E_6, and recently also for E_7 for the special value of the coupling constant for which the eigenfunctions are proportional to the characters of the irreducible representations of the algebra. The aim of this paper is to show how to generalize the treatment given in [1] to arbitrary values of the coupling constant and to extend some of the particular results found there to the general case.

* On leave of absence from the Institute for Theoretical and Experimental Physics, 117259, Moscow, Russia. Current E-mail address: perelomo@dftuz.unizar.es

2 The Calogero–Sutherland Hamiltonian for E_7 in Weyl-invariant variables

The trigonometric Calogero–Sutherland model related to the root system R of a simply-laced Lie algebra of rank r is the quantum system in an Euclidean space $\mathbb{R}^r$ defined by the standard Hamiltonian operator

$$H = \frac{1}{2}\sum_{j=1}^{r} p_j^2 + \sum_{\alpha\in R^+} \kappa(\kappa-1)\sin^{-2}(\alpha,q); \tag{1}$$

where $q=(q_j)$ is a cartesian coordinate system and $p_j = -i\partial_{q_j}$; R^+ is the set of the positive roots of the algebra, and κ is the coupling constant. The (non-normalized) ground state wave function is

$$\Psi_0(q) = \prod_{\alpha\in R^+} \sin^{\kappa}(\alpha,q); \tag{2}$$

while the excited states are indexed by the highest weights $\lambda = \sum m_i\lambda_i \in P^+$ (P^+ is the cone of dominant weights) of the irreducible representations of the algebra, that is, by the r-tuple of non-negative integers $m=(m_1;\dots;m_r)$. Looking for solutions Ψ_m of the Schrödinger equation in the form

$$\Psi_m(q) = \Psi_0(q)\,\Phi_m(q); \tag{3}$$

we are led to the eigenvalue problem

$$\Delta\Phi_m = \varepsilon_m(\kappa)\,\Phi_m\;; \tag{4}$$

where Δ is the linear differential operator

$$\Delta = -\frac{1}{2}\sum_{j=1}^{r}\partial_{q_j}^2 - \kappa\sum_{\alpha\in R^+}\cot(\alpha,q)(\alpha,\nabla_q): \tag{5}$$

Due to the Weyl symmetry of the Hamiltonian, to solve the eigenvalue problem (4) it is convenient to express the operator Δ in a set of independent W-invariant variables such as $z_k = \chi_{\lambda_k}(q)$, the characters of the irreducible representations of the algebra. The operator Δ in the z-variables has the structure:

$$\Delta = -\sum_{j,k} a_{jk}(z)\partial_{z_j}\partial_{z_k} + \sum_j \left(b_j^0(z) + \kappa\, b_j^1(z)\right)\partial_{z_j}; \tag{6}$$

but to fix the coefficients by direct change of variables is very cumbersome. As explained in [1], a different procedure, based on the computation of the quadratic Clebsch-Gordan series and the second order characters of E_7, is possible. In [1], we applied this procedure to compute $a_{jk}(z)$ and $b_j^0(z)+b_j^1(z)$, thus finding the operator Δ for the special case $\kappa = 1$. The remaining task is to compute $b_j^0(z)$ and $b_j^1(z)$ separately.

To accomplish that task a new piece of information is required: we need to know all the first order symmetric monomials for E_7 given as a function of the z-variables. To obtain them, we will rely on the expansions of the fundamental characters of E_7 in terms of monomial functions computed by Loris and Sasaki in [18]. In the notation of [1], these expansions are

$$\begin{aligned}
z_1 &= M_{\lambda_1} + 7;\\
z_2 &= M_{\lambda_2} + 6M_{\lambda_7};\\
z_3 &= M_{\lambda_3} + 5M_{\lambda_6} + 22M_{\lambda_1} + 77;\\
z_4 &= M_{\lambda_4} + 4M_{\lambda_1+\lambda_6} + 15M_{\lambda_2+\lambda_7} + 15M_{2\lambda_1} + 45M_{2\lambda_7} + 50M_{\lambda_3} + 145M_{\lambda_6} + 390M_{\lambda_1} + 980;\\
z_5 &= M_{\lambda_5} + 5M_{\lambda_1+\lambda_7} + 21M_{\lambda_2} + 71M_{\lambda_7};\\
z_6 &= M_{\lambda_6} + 6M_{\lambda_1} + 27;\\
z_7 &= M_{\lambda_7}:
\end{aligned}$$

To invert these formulas to compute the fundamental monomial functions, we have to proceed in increasing order of the height of the dominant weights associated to the characters. Once a first-order monomial function is known, we compute the corresponding $b_k^0(z)$ and $b_k^1(z)$ following the procedure described in [1]. This makes it possible to use the part of the operator already known in each step to compute the second order monomial functions in advance, i.e. before they are needed to obtain the next fundamental monomial function. With this strategy, it is easy to nd

$$
\begin{aligned}
M_{1} &= z_1 \quad 7;\\
M_{2} &= z_2 \quad 6z_7;\\
M_{3} &= z_3 \quad 5z_6 + 8z_1 + 2;\\
M_{4} &= z_4 \quad 4z_1z_6 + 9z_2z_7 + 9z_1^2 + 9z_7^2 \quad 14z_3 \quad 39z_6 \quad 22z_1 \quad 18;\\
M_{5} &= z_5 \quad 5z_1z_7 + 14z_2 + 15z_7;\\
M_{6} &= z_6 \quad 6z_1 + 15;\\
M_{7} &= z_7;
\end{aligned}
$$

and, therefore,

$$
\begin{aligned}
b_1^0(z) + \; b_1^1(z) &= \quad 28 + 4z_1 + \quad (28 + 68z_1)\\
b_2^0(z) + \; b_2^1(z) &= 7z_2 \quad 24z_7 + \quad (98z_2 + 24z_7)\\
b_3^0(z) + \; b_3^1(z) &= 8 \quad 56z_1 + 12z_3 \quad 20z_6 + \quad (\; 8 + 56z_1 + 132z_3 + 20z_6)\\
b_4^0(z) + \; b_4^1(z) &= \quad 72 + 72z_1 \quad 24z_1^2 + 24z_3 + 24z_4 \quad 16z_6 \quad 16z_1z_6 \quad 24z_2z_7 + 36z_7^2 +\\
&\quad (72 \quad 72z_1 + 24z_1^2 \quad 24z_3 + 192z_4 + 16z_6 + 16z_1z_6 + 24z_2z_7 \quad 36z_7^2)\\
b_5^0(z) + \; b_5^1(z) &= \quad 28z_2 + 15z_5 \quad 4z_7 \quad 20z_1z_7 + \quad (28z_2 + 150z_5 + 4z_7 + 20z_1z_7)\\
b_6^0(z) + \; b_6^1(z) &= \quad 48 \quad 24z_1 + 8z_6 + \quad (48 + 24z_1 + 104z_6)\\
b_7^0(z) + \; b_7^1(z) &= 3z_7 + 54 \; z_7:
\end{aligned}
$$

This completes the computation of . In the rest of the paper we present some results obtained through the use of this operator.

3 Some explicit results on the low lying eigenfunctions of the systems

In this Section, we present some results on the rst and second order polynomials and generalized quadratic Clebsch-Gordan series. Because some formulas are too long, we give the complete results, in a form suitable for use in Mathematica or Maple, in the adjoint les results31.txt { results35.txt, which are accessible through the \source" format of this document.

3.1 Second order monomial symmetric functions

Once we know , we can compute its eigenfunctions by means of the iterative algorithms given in [1]. In particular, we can obtain the monomial symmetric functions for E_7 by simply taken = 0 in these algorithms. We present here the list of the second order monomial functions obtained in that way.

$$
\begin{aligned}
M_{2000000} &= z_1^2 \quad 2z_3 \quad 2z_1 \quad 7\\
M_{1100000} &= z_1z_2 \quad 5z_5 + 3z_1z_7 \quad 23z_7\\
M_{0200000} &= z_2^2 \quad 2z_4 \quad 2z_2z_7 \quad 2z_1^2 \quad 6z_7^2 + 4z_3 + 14z_6 + 4z_1 + 12\\
M_{1010000} &= z_1z_3 \quad 3z_4 \quad z_1z_6 + 6z_2z_7 \quad 3z_1^2 \quad 9z_7^2 \quad 9z_3 + 4z_6 + 20z_1 + 32\\
M_{0110000} &= z_2z_3 \quad 4z_1z_5 + 5z_2z_6 + 4z_1^2z_7 \quad 4z_3z_7 \quad 17z_1z_2 \quad 16z_6z_7 + 41z_5 + 13z_1z_7 + 12z_2 + 7z_7\\
M_{0020000} &= z_3^2 \quad 2z_1z_4 + 2z_2z_5 \quad 2z_3z_6 \quad 7z_6^2 + 12z_5z_7 \quad 2z_1^3 + 4z_1z_3 + 2z_1z_7^2 \quad 10z_4 \quad 2z_1z_6 + 10z_1^2\\
&+ \quad 10z_7^2 \quad 16z_3 \quad 22z_6 \quad 16z_1 \quad 8\\
M_{1001000} &= z_1z_4 \quad 4z_2z_5 \quad 4z_1^2z_6 + 10z_3z_6 + 9z_1z_2z_7 + 14z_6^2 \quad 34z_5z_7 + 9z_1^3 \quad 21z_2^2 \quad 39z_1z_3 \quad 21z_1z_7^2\\
&+ \quad 66z_4 + 54z_1z_6 + 23z_2z_7 + 36z_1^2 \quad 22z_7^2 \quad 54z_3 \quad 24z_6 \quad 56z_1 \quad 24\\
M_{0101000} &= z_2z_4 \quad 3z_3z_5 + 2z_1z_2z_6 \quad 2z_2^2z_7 + 5z_1z_3z_7 + 5z_5z_6 \quad 14z_4z_7 \quad 19z_1z_6z_7 \quad 12z_1^2z_2 + 15z_2z_3
\end{aligned}
$$

$$+ \quad 17z_2z_7^2 + 25z_1z_5 + 19z_1^2z_7 + 42z_7^3 + 5z_2z_6 \quad 29z_3z_7 \quad 27z_1z_2 \quad 133z_6z_7 + 131z_5 + 10z_1z_7$$
$$+ \quad 40z_2 \quad 43z_7$$
$$M_{0011000} = z_3z_4 \quad 3z_1z_2z_5 + 5z_2^2z_6 + 2z_1z_3z_6 + 5z_5^2 \quad 7z_4z_6 + 5z_1^2z_2z_7 \quad 10z_1z_6^2 \quad 14z_2z_3z_7 + 10z_1z_5z_7$$
$$12z_1^2z_3 \quad 3z_1^2z_7^2 \quad 5z_1z_2^2 + 24z_3^2 + 5z_1z_4 \quad 2z_2z_6z_7 + 6z_1^2z_6 + z_2z_5 + 11z_3z_7^2 + 10z_3z_6$$
$$+ \quad 15z_6z_7^2 \quad 28z_1z_2z_7 \quad 24z_6^2 + 40z_2^2 + 4z_5z_7 + 21z_1^3 \quad 15z_1z_3 + 19z_1z_7^2 \quad 16z_4 \quad 54z_1z_6$$
$$48z_1^2 + 17z_2z_7 \quad 5z_7^2 + 7z_3 + 31z_6 \quad 16z_1 \quad 22$$
$$M_{0002000} = z_4^2 \quad 2z_2z_3z_5 + 2z_3^2z_6 + 2z_1z_5^2 + 2z_1z_2^2z_6 \quad 2z_2^3z_7 \quad 4z_1z_4z_6 \quad 2z_1z_2z_3z_7 \quad 2z_2z_5z_6 + 6z_2z_4z_7$$
$$8z_1^2z_6^2 + 12z_3z_6^2 + 12z_1^2z_5z_7 \quad 20z_3z_5z_7 \quad 6z_1^2z_4 + 2z_1z_2z_6z_7 + 2z_6^3 + 2z_2^2z_3 + 2z_1^3z_7^2$$
$$4z_5z_6z_7 + 9z_2^2z_7^2 \quad 4z_1z_3z_7^2 + 12z_3z_4 \quad 4z_1z_2z_5 \quad 16z_2^2z_6 \quad 14z_4z_7^2 \quad 4z_1^3z_6 + 2z_5^2 + 8z_1z_3z_6$$
$$+ \quad 2z_1^2z_2z_7 + 30z_4z_6 + 9z_1^4 \quad 18z_2z_7^3 + 12z_1z_6^2 \quad 6z_2z_3z_7 \quad 16z_1z_5z_7 \quad 36z_1^2z_3 + 4z_1z_2^2 \quad 8z_1^2z_7^2$$
$$+ \quad 26z_3^2 + 32z_2z_6z_7 + 16z_3z_7^2 + 32z_1z_4 \quad 28z_2z_5 + 14z_1^2z_6 + 9z_7^4 \quad 22z_6z_7^2 \quad 16z_1^3 \quad 5z_6^2$$
$$8z_1z_2z_7 + 32z_1z_3 \quad 16z_2^2 + 52z_5z_7 \quad 48z_4 \quad 10z_1z_7^2 + 36z_2z_7 + 42z_7^2 + 20z_1^2 \quad 8z_3 \quad 88z_6$$
$$8z_1 \quad 60$$
$$M_{1000100} = z_1z_5 \quad 5z_2z_6 \quad 5z_1^2z_7 + 15z_3z_7 + 9z_1z_2 + 19z_6z_7 \quad 54z_5 \quad 29z_1z_7 + 30z_2 + 56z_7$$
$$M_{0100100} = z_2z_5 \quad 4z_3z_6 + 4z_6^2 + 5z_1z_2z_7 \quad 7z_2^2 + 4z_1z_3 \quad 4z_5z_7 \quad 10z_4 \quad 16z_1z_7^2 + 4z_1z_6 + 29z_2z_7$$
$$+ \quad 54z_7^2 + 6z_1^2 \quad 12z_3 \quad 90z_6 \quad 76z_1 + 4$$
$$M_{0010100} = z_3z_5 \quad 4z_1z_2z_6 + 9z_2^2z_7 + 5z_1z_3z_7 + 5z_5z_6 \quad 12z_4z_7 \quad 11z_1z_6z_7 + 4z_1^2z_2 \quad 25z_2z_3 + 16z_1z_5$$
$$4z_1^2z_7 \quad 5z_2z_6 + 19z_3z_7 \quad 2z_1z_2 + 31z_6z_7 \quad 46z_5 \quad 26z_1z_7 \quad 40z_2 + 74z_7$$
$$M_{0001100} = z_4z_5 \quad 3z_2z_3z_6 + 2z_1z_5z_6 + 5z_3^2z_7 + 5z_2z_6^2 + 5z_1z_2^2z_7 \quad 7z_1z_4z_7 \quad 7z_2^3 \quad 12z_1z_2z_3 \quad 14z_2z_5z_7$$
$$+ \quad 27z_2z_4 \quad 10z_1^2z_6z_7 + 10z_3z_6z_7 \quad 2z_1z_2z_7^2 \quad 3z_6^2z_7 + 21z_1^2z_5 \quad 17z_3z_5 + 10z_1z_2z_6 + 11z_5z_7^2$$
$$+ \quad 5z_1^3z_7 \quad 8z_5z_6 + 15z_1z_7^3 + 6z_2^2z_7 \quad 5z_1z_3z_7 \quad 5z_1^2z_2 \quad 17z_4z_7 + 16z_2z_3 \quad 26z_1z_6z_7 \quad 16z_1z_5$$
$$45z_2z_7^2 \quad z_1^2z_7 + 45z_2z_6 + 5z_7^3 + 31z_3z_7 \quad 26z_1z_2 + 14z_6z_7 + 42z_5 \quad 26z_1z_7 + 110z_2 \quad 26z_7$$
$$M_{0000200} = z_5^2 \quad 2z_4z_6 + 2z_2z_3z_7 \quad 2z_3^2 \quad 2z_1z_5z_7 \quad 2z_1z_2^2 \quad 7z_1^2z_7^2 + 4z_1z_4 + 12z_3z_7^2 + 2z_2z_5 + 14z_1^2z_6$$
$$24z_3z_6 + 2z_1z_2z_7 + 2z_6z_7^2 + 4z_1^3 \quad 4z_6^2 + 2z_5z_7 + 14z_2^2 + 10z_1z_7^2 \quad 8z_1z_3 \quad 28z_4 \quad 24z_1z_6$$
$$26z_2z_7 + 17z_7^2 \quad 20z_1^2 + 32z_3 \quad 8z_6 + 32z_1 \quad 32$$
$$M_{1000010} = z_1z_6 \quad 6z_2z_7 + 9z_7^2 \quad 6z_1^2 + 21z_3 + 6z_6 \quad 27z_1 + 27$$
$$M_{0100010} = z_2z_6 \quad 5z_3z_7 + 3z_6z_7 + 9z_1z_2 \quad 7z_5 \quad 19z_1z_7 + 15z_2 + 7z_7$$
$$M_{0010010} = z_3z_6 \quad 5z_6^2 \quad 5z_1z_2z_7 + 14z_2^2 + 9z_1z_3 + 15z_5z_7 \quad 27z_4 + 19z_1z_7^2 \quad 49z_1z_6 \quad 14z_2z_7 \quad 27z_1^2$$
$$4z_7^2 + 42z_3 + 79z_6 + 12z_1 \quad 42$$
$$M_{0001010} = z_4z_6 \quad 4z_2z_3z_7 \quad 4z_1z_6^2 + 9z_3^2 + 10z_1z_5z_7 + 9z_2z_6z_7 + 9z_1z_2^2 + 14z_1^2z_7^2 \quad 18z_1z_4 \quad 34z_3z_7^2$$
$$33z_2z_5 \quad 39z_1^2z_6 + 72z_3z_6 \quad 5z_1z_2z_7 \quad 21z_6z_7^2 \quad 18z_1^3 + 34z_6^2 + 22z_5z_7 \quad 14z_2^2 + 18z_1z_7^2$$
$$+ \quad 48z_1z_3 \quad 7z_4 \quad 16z_2z_7 \quad 61z_7^2 \quad 27z_1^2 + 18z_3 + 151z_6 + 150z_1 + 26$$
$$M_{0000110} = z_5z_6 \quad 3z_4z_7 \quad z_1z_6z_7 + 5z_2z_3 + 6z_2z_7^2 \quad 2z_1z_5 \quad 11z_2z_6 \quad 9z_7^3 \quad 9z_1^2z_7 + 9z_3z_7 + 7z_1z_2$$
$$+ \quad 25z_6z_7 \quad 16z_5 + 26z_1z_7 \quad 60z_2 + 16z_7$$
$$M_{0000020} = z_6^2 \quad 2z_5z_7 + 2z_4 \quad 6z_1^2 \quad 2z_7^2 + 12z_3 + 4z_6 + 12z_1 + 17$$
$$M_{1000001} = z_1z_7 \quad 7z_2 + 8z_7$$
$$M_{0100001} = z_2z_7 \quad 6z_3 \quad 6z_7^2 + 14z_6 + 16z_1 \quad 28$$
$$M_{0010001} = z_3z_7 \quad 5z_6z_7 \quad 6z_1z_2 + 20z_5 + 24z_1z_7 \quad 49z_2 \quad 12z_7$$
$$M_{0001001} = z_4z_7 \quad 4z_1z_6z_7 \quad 5z_2z_3 + 9z_2z_7^2 + 14z_1z_5 \quad 5z_2z_6 + 9z_7^3 + 19z_1^2z_7 \quad 44z_3z_7 \quad 14z_1z_2$$
$$53z_6z_7 + 54z_5 + 42z_1z_7 \quad 40z_2 \quad 144z_7$$
$$M_{0000101} = z_5z_7 \quad 4z_4 \quad 5z_1z_7^2 + 8z_1z_6 + 11z_2z_7 + 12z_1^2 \quad 3z_7^2 \quad 42z_3 \quad 18z_6 \quad 4$$
$$M_{0000011} = z_6z_7 \quad 3z_5 \quad z_1z_7 + 7z_2 \quad 11z_7$$
$$M_{0000002} = z_7^2 \quad 2z_6 \quad 2$$

3.2 Expansion of second order characters in monomial functions

As the name suggests, the orthogonal system of monomial symmetric functions is the simplest one among the different classes of symmetric polynomials associated to the Lie algebra E_7: each monomial symmetric function is nothing but the sum of all the monomials associated to one orbit of the Weyl group on the weight lattice. Now, we can easily expand other polynomials associated to the root system of E_7 in the basis of the monomial symmetric functions. In fact, the method is the same which we have described in [1] for the computation of Clebsch-Gordan series. In particular, the coefficients in the decomposition of characters in monomial symmetric functions are interesting in that they give the multiplicities of the weights in the corresponding irreducible representations. As an example, we present such decomposition for all the second order characters.

$$
\begin{aligned}
{}_{2000000} &= M_{2000000} + M_{0010000} + 4M_{0000010} + 17M_{1000000} + 63M_{0000000} \\
{}_{1100000} &= M_{1100000} + 4M_{0000100} + 16M_{1000001} + 56M_{0100000} + 171M_{0000001} \\
{}_{0200000} &= M_{0200000} + M_{0001000} + 3M_{1000010} + 11M_{0100001} + 10M_{2000000} + 36M_{0000002} + 34M_{0010000} \\
&+ 96M_{0000010} + 248M_{1000000} + 603M_{0000000} \\
{}_{1010000} &= M_{1010000} + 2M_{0001000} + 8M_{1000010} + 24M_{0100001} + 32M_{2000000} + 64M_{0000002} + 78M_{0010000} \\
&+ 208M_{0000010} + 544M_{1000000} + 1344M_{0000000} \\
{}_{0110000} &= M_{0110000} + 3M_{1000100} + 10M_{0100010} + 10M_{2000001} + 30M_{0010001} + 90M_{1100000} + 80M_{0000011} \\
&+ 231M_{0000100} + 570M_{1000001} + 1344M_{0100000} + 3024M_{0000001} \\
{}_{0020000} &= M_{0020000} + M_{1001000} + 2M_{0100100} + 3M_{2000010} + 7M_{0010010} + 19M_{1100001} + 20M_{0000020} \\
&+ 46M_{0000101} + 10M_{3000000} + 49M_{0200000} + 56M_{1010000} + 104M_{1000002} + 125M_{0001000} \\
&+ 291M_{1000010} + 682M_{2000000} + 638M_{0100001} + 1338M_{0000002} + 1402M_{0010000} + 2908M_{0000010} \\
&+ 5938M_{1000000} + 11844M_{0000000} \\
{}_{1001000} &= M_{1001000} + 3M_{0100100} + 4M_{2000010} + 10M_{0010010} + 30M_{1100001} + 25M_{0000020} + 75M_{0000101} \\
&+ 15M_{3000000} + 84M_{0200000} + 90M_{1010000} + 180M_{1000002} + 213M_{0001000} + 507M_{1000010} \\
&+ 1149M_{0100001} + 1185M_{2000000} + 2484M_{0000002} + 2565M_{0010000} + 5439M_{0000010} \\
&+ 11265M_{1000000} + 22680M_{0000000} \\
{}_{0101000} &= M_{0101000} + 2M_{0010100} + 6M_{1100010} + 20M_{0200001} + 15M_{1010001} + 15M_{0000110} + 42M_{0001001} \\
&+ 96M_{1000011} + 40M_{2100000} + 114M_{0110000} + 220M_{0100002} + 256M_{1000100} + 565M_{2000001} \\
&+ 480M_{0000003} + 575M_{0100010} + 1240M_{0010001} + 2624M_{1100000} + 2580M_{0000011} \\
&+ 5340M_{0000100} + 10589M_{1000001} + 20524M_{0100000} + 38864M_{0000001} \\
{}_{0011000} &= M_{0011000} + 2M_{1100100} + 5M_{0200010} + 6M_{1010010} + 5M_{0000200} + 14M_{0001010} + 15M_{2100001} \\
&+ 33M_{1000020} + 37M_{0110001} + 83M_{1000101} + 40M_{2010000} + 180M_{2000002} + 94M_{1200000} \\
&+ 100M_{0020000} + 215M_{1001000} + 180M_{0100011} + 467M_{2000010} + 456M_{0100100} + 375M_{0010002} \\
&+ 958M_{0010010} + 750M_{0000012} + 1964M_{1100001} + 1920M_{0000020} + 3963M_{0200000} + 3850M_{0000101} \\
&+ 1010M_{3000000} + 4005M_{1010000} + 7374M_{1000002} + 7700M_{0001000} + 14642M_{1000010} \\
&+ 27546M_{2000000} + 27263M_{0100001} + 49698M_{0000002} + 50206M_{0010000} + 90408M_{0000010} \\
&+ 160642M_{1000000} + 281268M_{0000000} \\
{}_{0002000} &= M_{0002000} + M_{0110100} + 2M_{0020010} + 2M_{1000200} + 2M_{1200010} + 5M_{0300001} + 5M_{1001010} \\
&+ 11M_{1110001} + 11M_{0100110} + 27M_{0101001} + 12M_{2000020} + 25M_{2200000} + 23M_{0010020} \\
&+ 23M_{2000101} + 54M_{0010101} + 25M_{1020000} + 52M_{2001000} + 109M_{1100011} + 45M_{0000030} \\
&+ 64M_{0210000} + 45M_{3000002} + 210M_{0000111} + 225M_{0200002} + 210M_{1010002} + 129M_{0011000} \\
&+ 258M_{1100100} + 520M_{0200010} + 408M_{0001002} + 750M_{1000012} + 105M_{3000010} + 499M_{0000200} \\
&+ 501M_{1010010} + 960M_{2100001} + 968M_{0001010} + 215M_{4000000} + 1365M_{0100003} + 1787M_{1000020} \\
&+ 1854M_{0110001} + 3376M_{1000101} + 1830M_{2010000} + 3524M_{1200000} + 6055M_{2000002} \\
&+ 3525M_{0020000} + 6085M_{0100011} + 10760M_{0010002} + 6350M_{1001000} + 11358M_{0100100}
\end{aligned}
$$

$+ \; 11320M_{2000010} + 2440M_{0000004} + 18700M_{0000012} + 20031M_{3000000} + 19977M_{0010010}$

$+ \; 34569M_{0000020} + 34769M_{1100001} + 60006M_{1010000} + 60004M_{0200000} + 59439M_{0000101}$

$+ \; 101592M_{0001000} + 100299M_{1000002} + 170142M_{1000010} + 281804M_{0100001} + 461353M_{0000002}$

$+ \; 282527M_{2000000} + 462702M_{0010000} + 750988M_{0000010} + 1208053M_{1000000} + 1925763M_{0000000}$

$_{1000100} \; = \; M_{1000100} + 4M_{0100010} + 5M_{2000001} + 15M_{0010001} + 50M_{1100000} + 44M_{0000011} + 139M_{0000100}$

$+ \; 365M_{1000001} + 910M_{0100000} + 2145M_{0000001}$

$_{0100100} \; = \; M_{0100100} + 3M_{0010010} + 10M_{0000020} + 10M_{1100001} + 35M_{0200000} + 29M_{1010000} + 30M_{0000101}$

$+ \; 88M_{0001000} + 80M_{1000002} + 223M_{1000010} + 545M_{0100001} + 1260M_{0000002} + 538M_{2000000}$

$+ \; 1262M_{0010000} + 2800M_{0000010} + 5976M_{1000000} + 12341M_{0000000}$

$_{0010100} \; = \; M_{0010100} + 3M_{1100010} + 9M_{0200001} + 10M_{1010001} + 10M_{0000110} + 28M_{0001001} + 72M_{1000011}$

$+ \; 29M_{2100000} + 169M_{0100002} + 79M_{0110000} + 196M_{1000100} + 374M_{0000003} + 464M_{2000001}$

$+ \; 458M_{0100010} + 1029M_{0010001} + 2258M_{1100000} + 2198M_{0000011} + 4708M_{0000100} + 9574M_{1000001}$

$+ \; 18998M_{0100000} + 36774M_{0000001}$

$_{0001100} \; = \; M_{0001100} + 2M_{0110010} + 6M_{1000110} + 5M_{0020001} + 15M_{0100020} + 5M_{1200001} + 14M_{1001001}$

$+ \; 14M_{0300000} + 34M_{1110000} + 37M_{0100101} + 91M_{0101000} + 33M_{2000011} + 83M_{0010011} + 180M_{1100002}$

$+ \; 180M_{0000021} + 78M_{2000100} + 203M_{0010100} + 437M_{1100010} + 375M_{0000102} + 170M_{3000001}$

$+ \; 914M_{0000110} + 750M_{1000003} + 929M_{0200001} + 905M_{1010001} + 1834M_{2100000} + 1858M_{0001001}$

$+ \; 3723M_{0110000} + 3635M_{1000011} + 7156M_{1000100} + 6949M_{0100002} + 13480M_{2000001} + 13524M_{0100010}$

$+ \; 12954M_{0000003} + 25015M_{0010001} + 45599M_{1100000} + 45368M_{0000011} + 81502M_{0000100}$

$+ \; 143470M_{1000001} + 249025M_{0100000} + 426280M_{0000001}$

$_{0000200} \; = \; M_{0000200} + M_{0001010} + 2M_{0110001} + 3M_{1000020} + 5M_{0020000} + 7M_{1000101} + 19M_{0100011}$

$+ \; 5M_{1200000} + 20M_{2000002} + 16M_{1001000} + 46M_{0010002} + 46M_{0100100} + 41M_{2000010} + 110M_{0010010}$

$+ \; 250M_{1100001} + 104M_{0000012} + 94M_{3000000} + 254M_{0000020} + 549M_{0000101} + 560M_{0200000}$

$+ \; 1150M_{1000002} + 539M_{1010000} + 1159M_{0001000} + 2362M_{1000010} + 4700M_{0100001} + 9126M_{0000002}$

$+ \; 4678M_{2000000} + 9103M_{0010000} + 17256M_{0000010} + 32022M_{1000000} + 58324M_{0000000}$

$_{1000010} \; = \; M_{1000010} + 5M_{0100001} + 20M_{0000002} + 6M_{2000000} + 21M_{0010000} + 70M_{0000010} + 212M_{1000000}$

$+ \; 588M_{0000000}$

$_{0100010} \; = \; M_{0100010} + 4M_{0010001} + 16M_{0000011} + 15M_{1100000} + 51M_{0000100} + 149M_{1000001} + 399M_{0100000}$

$+ \; 999M_{0000001}$

$_{0010010} \; = \; M_{0010010} + 5M_{0000020} + 4M_{1100001} + 14M_{0200000} + 15M_{1010000} + 15M_{0000101} + 47M_{0001000}$

$+ \; 44M_{1000002} + 133M_{1000010} + 343M_{0100001} + 350M_{2000000} + 828M_{0000002} + 845M_{0010000}$

$+ \; 1957M_{0000010} + 4366M_{1000000} + 9387M_{0000000}$

$_{0001010} \; = \; M_{0001010} + 3M_{0110001} + 4M_{1000020} + 9M_{0020000} + 10M_{1000101} + 30M_{0100011} + 9M_{1200000}$

$+ \; 25M_{2000002} + 27M_{1001000} + 75M_{0010002} + 78M_{0100100} + 69M_{2000010} + 193M_{0010010} + 449M_{1100001}$

$+ \; 180M_{0000012} + 165M_{3000000} + 460M_{0000020} + 1014M_{0000101} + 1008M_{0200000} + 2169M_{1000002}$

$+ \; 999M_{1010000} + 2184M_{0001000} + 4549M_{1000010} + 9198M_{0100001} + 18063M_{0000002} + 9189M_{2000000}$

$+ \; 18114M_{0010000} + 34807M_{0000010} + 65475M_{1000000} + 120771M_{0000000}$

$_{0000110} \; = \; M_{0000110} + 2M_{0001001} + 8M_{1000011} + 5M_{0110000} + 24M_{0100002} + 19M_{1000100} + 59M_{0100010}$

$+ \; 64M_{0000003} + 54M_{2000001} + 154M_{0010001} + 374M_{1100000} + 384M_{0000011} + 879M_{0000100}$

$+ \; 1958M_{1000001} + 4193M_{0100000} + 8694M_{0000001}$

$_{0000020} \; = \; M_{0000020} + M_{0000101} + 2M_{0001000} + 4M_{1000002} + 9M_{1000010} + 29M_{0100001} + 30M_{2000000}$

$+ \; 84M_{0000002} + 80M_{0010000} + 209M_{0000010} + 510M_{1000000} + 1197M_{0000000}$

$_{1000001} \; = \; M_{1000001} + 6M_{0100000} + 27M_{0000001}$

$$
\begin{aligned}
{}_{0100001} &= M_{0100001} + 5M_{0010000} + 6M_{0000002} + 22M_{0000010} + 75M_{1000000} + 225M_{0000000} \\
{}_{0010001} &= M_{0010001} + 5M_{0000011} + 5M_{1100000} + 20M_{0000100} + 66M_{1000001} + 196M_{0100000} + 531M_{0000001} \\
{}_{0001001} &= M_{0001001} + 4M_{1000011} + 4M_{0110000} + 15M_{0100002} + 14M_{1000100} + 45M_{0100010} \\
&+ 45M_{0000003} + 40M_{2000001} + 125M_{0010001} + 319M_{1100000} + 325M_{0000011} + 784M_{0000100} \\
&+ 1809M_{1000001} + 4004M_{0100000} + 8529M_{0000001} \\
{}_{0000101} &= M_{0000101} + 3M_{0001000} + 5M_{1000002} + 13M_{1000010} + 45M_{0100001} + 39M_{2000000} + 135M_{0000002} \\
&+ 129M_{0010000} + 351M_{0000010} + 879M_{1000000} + 2079M_{0000000} \\
{}_{0000011} &= M_{0000011} + 2M_{0000100} + 10M_{1000001} + 35M_{0100000} + 111M_{0000001} \\
{}_{0000002} &= M_{0000002} + M_{0000010} + 5M_{1000000} + 21M_{0000000}
\end{aligned}
$$

3.3 First order polynomials

The iterative methods given in [1] allow us to solve the Schrodinger equation (4) for general . The eigenfuntions are polynomials. In this and the next subsection, we present a partial list of such polynomials of rst and second order.

$$
\begin{aligned}
P_{1000000}(z) &= z_1 + \frac{7(\ 1+\ )}{1+17} \\
P_{0100000}(z) &= z_2 + \frac{6(\ 1+\ )z_7}{1+11} \\
P_{0010000}(z) &= z_3 + \frac{5(\ 1+\ )z_6}{1+7} + \frac{8(\ 1+\ )(\ 1+8\ )z_1}{(1+7\ )(1+8\ )} + \frac{2(\ 1+\ )(\ 1\quad 159\ +136\ ^2)}{(1+7\ )(1+8\ )(1+11\ )} \\
P_{0001000}(z) &= z_4 + \frac{4(\ 1+\ )z_1z_6}{1+5} + \frac{3(\ 1+\ )(\ 3+5\ )z_2z_7}{(1+5\ )^2} + \frac{9(\ 1+\ )(1+22\ +5\ ^2)z_7^2}{(1+5\ )^2(1+7\ )} \\
&+ \frac{9(\ 1+\ )(\ 1+3\ )z_1^2}{(1+5\ )(1+7\ )} + \frac{2(\ 1+\ )(\ 7+11\ +385\ ^2+175\ ^3)z_3}{(1+5\ )^3(1+7\ )} \\
&+ \frac{(\ 1+\ )(78+1255\ +3653\ ^2\quad 6295\ ^3+3325\ ^4)z_6}{(1+5\ )^3(1+7\ )(2+11\ )} \\
&+ \frac{2(\ 1+\ )(\ 22\quad 755\quad 3477\ ^2+11255\ ^3+175\ ^4)z_1}{(1+5\ )^3(1+7\ )(2+11\ )} \\
&+ \frac{2(\ 1+\ )(18+365\ +8123\ ^2\quad 2045\ ^3+2275\ ^4)}{(1+5\ )^3(1+7\ )(2+11\ )} \\
P_{0000100}(z) &= z_5 + \frac{5(\ 1+\ )z_1z_7}{1+7} + \frac{7(\ 1+\ )(\ 4+7\ )z_2}{(1+7\ )(2+13\ )} + \frac{5(\ 1+\ )(\ 6\quad 137\ +56\ ^2)z_7}{(1+7\ )(1+8\ )(2+13\ )} \\
P_{0000010}(z) &= z_6 + \frac{6(\ 1+\ )z_1}{1+9} + \frac{15(\ 1+\ )(\ 1+5\ )}{(1+9\ )(1+13\ )} \\
P_{0000001}(z) &= z_7
\end{aligned}
$$

3.4 Second order polynomials

$$
\begin{aligned}
P_{2000000}(z) &= z_1^2 + \frac{2z_3}{1+} + \frac{10\ z_6}{(1+\ )(1+4\ )} + \frac{2(\ 3\quad 6\quad 119\ ^2+28\ ^3)z_1}{(1+\ )(1+4\ )(3+17\ )} \\
&+ \frac{42\quad 459\quad 290\ ^2\quad 3205\ ^3+196\ ^4}{(1+\ )(1+4\ )(2+17\ )(3+17\ )} \\
P_{1100000}(z) &= z_1z_2 + \frac{5z_5}{1+4} + \frac{(6\quad 95\ +24\ ^2)z_1z_7}{(1+4\ )(2+11\ )} + \frac{28(\ 1+\ )\ (\ 26+11\ )z_2}{(1+4\ )(2+11\ )(3+17\ )} \\
&+ \frac{(\ 1+\ )(138+365\quad 9979\ ^2+1176\ ^3)z_7}{(1+4\ )(1+7\ )(2+11\ )(3+17\ )}
\end{aligned}
$$

$$P_{0200000}(z) = z_2^2 + \frac{2z_4}{1+\ } + \frac{8\ z_1 z_6}{(1+\ )(1+3\ )} + \frac{6(\ 1+\ )(1\ \ +6\ ^2)z_2 z_7}{(1+\ )(1+3\ )(3+11\ )} + \frac{2(1+23\ ^2)z_1^2}{(1+\ )(1+3\ )(1+5\ )}$$

$$+ \frac{18(\ 1+\ )(2+13\ \ 7\ ^2+6\ ^3)z_7^2}{(1+\ )(1+3\ )(2+11\ )(3+11\ )} + \frac{4(3+23\ +141\ ^2+493\ ^3+180\ ^4)z_3}{(1+\ )(1+3\ )(1+4\ )(1+5\ )(3+11\ )}$$

$$+ \frac{2(\ 42\ \ 537\ \ 2397\ ^2\ \ 6715\ ^3\ \ 14529\ ^4+2380\ ^5)z_6}{(1+\ )(1+3\ )(1+4\ )(1+5\ )(2+11\ )(3+11\ )}$$

$$+ \frac{4(\ 6\ \ 99\ \ 205\ ^2\ \ 1021\ ^3\ \ 12161\ ^4+2572\ ^5)z_1}{(1+\ )(1+3\ )(1+4\ )(1+5\ )(2+11\ )(3+11\ )}$$

$$+ \frac{4(\ 1+\ )(18+325\ +2143\ ^2+2067\ ^3\ \ 14045\ ^4+22012\ ^5)}{(1+\ )(1+3\ )(1+4\ )(1+5\ )(1+7\ )(2+11\ )(3+11\ )}$$

$$P_{1010000}(z) = z_1 z_3 + \frac{3z_4}{1+2\ } + \frac{(\ 2\ \ 35\ +10\ ^2)z_1 z_6}{(1+2\ )(2+7\ )} + \frac{6(\ 1+\ )(2+15\ )z_2 z_7}{(1+2\ )(1+4\ )(2+7\ )}$$

$$+ \frac{(\ 18\ \ 47\ \ 704\ ^2+256\ ^3)z_1^2}{(1+2\ )(2+7\ )(3+16\ )} + \frac{9(\ 2+17\ )z_7^2}{(1+2\ )(1+4\ )(2+7\ )}$$

$$+ \frac{(\ 216\ \ 2054\ +1429\ ^2+33210\ ^3+15224\ ^4+6272\ ^5)z_3}{(1+2\ )(1+4\ )(2+7\ )(3+16\ )(4+17\ )}$$

$$+ \frac{2(\ 1+\ )(\ 48+1190\ +6697\ ^2\ \ 9260\ ^3+2240\ ^4)z_6}{(1+2\ )(1+4\ )(2+7\ )(3+16\ )(4+17\ )}$$

$$+ \frac{12(\ 1+\ )(\ 80\ \ 100\ +7708\ ^2+27189\ ^3\ \ 30716\ ^4+12736\ ^5)z_1}{(1+2\ )(1+4\ )(2+7\ )(2+11\ )(3+16\ )(4+17\ )}$$

$$+ \frac{4(\ 1+\ )(\ 384\ \ 6676\ \ 12672\ ^2+67253\ ^3\ \ 75612\ ^4+7616\ ^5)}{(1+2\ )(1+4\ )(2+7\ )(2+11\ )(3+16\ )(4+17\ )}$$

$$P_{0110000}(z) = z_2 z_3 + \frac{4z_1 z_5}{1+3\ } + \frac{5(\ 1+\ )(\ 2+3\ )z_2 z_6}{(1+3\ )(2+7\ )} + \frac{4(\ 1+7\ )z_1^2 z_7}{(1+3\ )(1+5\ )}$$

$$+ \frac{6(\ 4+51\ +311\ ^2+41\ ^3+105\ ^4)z_3 z_7}{(1+3\ )(1+5\ )(2+7\ )(3+11\ )}$$

$$+ \frac{2(\ 1+\ )(51+397\ \ 6\ ^2\ \ 2992\ ^3+2640\ ^4)z_1 z_2}{(1+3\ )(1+4\ )(1+5\ )(2+7\ )(3+11\ )}$$

$$+ \frac{6(\ 16+17\ +673\ ^2\ \ 245\ ^3+75\ ^4)z_6 z_7}{(1+3\ )(1+5\ )(2+7\ )(3+11\ )}$$

$$+ \frac{2(\ 1+\ )(\ 123\ \ 821\ +1018\ ^2+10196\ ^3+1280\ ^4)z_5}{(1+3\ )(1+4\ )^2(1+5\ )(2+7\ )(3+11\ )}$$

$$+ \frac{6(13\ \ 156\ \ 2853\ ^2\ \ 2376\ ^3+27380\ ^4\ \ 11328\ ^5+1920\ ^6)z_1 z_7}{(1+3\ )(1+4\ )^2(1+5\ )(2+7\ )(3+11\ )}$$

$$+ \frac{6(\ 1+\ )(\ 12\ \ 595\ \ 2610\ ^2+7325\ ^3\ \ 2248\ ^4+1360\ ^5)z_2}{(1+3\ )(1+4\ )^2(1+5\ )(2+7\ )(3+11\ )}$$

$$+ \frac{2(\ 1+\ )(\ 42\ \ 3713\ \ 46855\ ^2\ \ 49890\ ^3+620062\ ^4\ \ 178192\ ^5+24480\ ^6)z_7}{(1+3\ )(1+4\ )^2(1+5\ )(2+7\ )(2+11\ )(3+11\ )}$$

$$P_{1000100}(z) = z_1 z_5 + \frac{5z_2 z_6}{1+4\ } + \frac{5(\ 1+\ )z_1^2 z_7}{1+7\ } + \frac{15(\ 1+\ )(1+6\ )z_3 z_7}{(1+4\ )^2(1+7\ )}$$

$$+ \frac{(\ 1+\ )(\ 27\ \ 320\ +112\ ^2)z_1 z_2}{(1+4\ )^2(3+13\ )} + \frac{(\ 1+\ )(19+309\ +1182\ ^2)z_6 z_7}{(1+4\ )^2(1+5\ )(1+7\ )}$$

$$+ \frac{(\ 1+\ )(648+8119\ +26227\ ^2+12296\ ^3+7280\ ^4)z_5}{(1+4\ )^2(1+5\ )(3+13\ )(4+17\ )}$$

$$+ \frac{2(\ 174\ \ 4381\ \ 20915\ ^2+61085\ ^3+363609\ ^4\ \ 239624\ ^5+42000\ ^6)z_1 z_7}{(1+4\ )^2(1+5\ )(1+7\ )(3+13\ )(4+17\ )}$$

$$+\ \frac{10(36+449\quad 4256^{2}\quad 63640^{3}\quad 162486^{4}+81791^{5}\quad 107534^{6}+13720^{7})z_2}{(1+4)^2(1+5)^2(1+7)(3+13)(4+17)}$$

$$+\ \frac{(1+)(672\quad 17905\quad 81245^{2}+497285^{3}+2671037^{4}\quad 1392420^{5}+98000^{6})z_7}{(1+4)^2(1+5)^2(1+7)(3+13)(4+17)}$$

$$P_{1000010}(z) = z_1z_6+\frac{6z_2z_7}{1+5}+\frac{9(1+7)z_7^2}{(1+5)(1+8)}+\frac{6(1+)z_1^2}{1+9}+\frac{3(1+)(7+55)z_3}{(1+5)^2(1+9)}$$

$$+\ \frac{(18+1213+15375^{2}+51579^{3}\quad 15985^{4}+12600^{5})z_6}{(1+5)^2(1+8)(1+9)(3+17)}$$

$$+\ \frac{3(54\quad 775+4734^{2}+107248^{3}+344272^{4}\quad 252825^{5}+121400^{6})z_1}{(1+5)^2(1+8)(1+9)(2+13)(3+17)}$$

$$+\ \frac{3(54\quad 245\quad 9444^{2}+22702^{3}+391238^{4}\quad 115305^{5}+35000^{6})}{(1+5)^2(1+8)(1+9)(2+13)(3+17)}$$

$$P_{0100010}(z) = z_2z_6+\frac{5z_3z_7}{1+4}+\frac{(6\quad 95+24^{2})z_6z_7}{(1+4)(2+11)}+\frac{6(1+)(3+4)z_1z_2}{(1+4)(2+9)}$$

$$+\ \frac{2(42+643+4519^{2}+600^{3})z_5}{(1+4)(2+9)(2+11)(3+13)}$$

$$+\ \frac{2(1+)(114+113\quad 11559^{2}\quad 46218^{3}+4680^{4})z_1z_7}{(1+4)(1+5)(2+9)(2+11)(3+13)}$$

$$+\ \frac{(180\quad 3864\quad 20509^{2}+80848^{3}\quad 4515^{4}+16500^{5})z_2}{(1+4)(1+5)(2+9)(2+11)(3+13)}$$

$$+\ \frac{2(1+)(42+2347+38355^{2}\quad 27714^{3}+4500^{4})z_7}{(1+4)(1+5)(2+9)(2+11)(3+13)}$$

$$P_{0000020}(z) = z_6^2+\frac{2z_5z_7}{1+}+\frac{2(1+)z_4}{(1+)(1+2)}+\frac{10\ z_1z_7^2}{(1+)(1+4)}+\frac{4\ (13+43+10^{2}+24^{3})z_1z_6}{3(1+)(1+2)(1+3)(1+4)}$$

$$+\ \frac{2(1+)\ (7+36)z_2z_7}{(1+)(1+2)(1+3)(1+4)}+\frac{2(1+)(6+47+59^{2}\quad 88^{3}+48^{4})z_1^2}{(1+)(1+2)(1+3)(1+4)(2+9)}$$

$$+\ \frac{2(1+)(1\quad 8+30^{2})z_7^2}{(1+)(1+2)(1+3)(1+4)}+\frac{4(6+29+36^{2}+199^{3}+60^{4})z_3}{(1+)(1+2)(1+3)(1+4)(2+9)}$$

$$+\ \frac{4(18+213\quad 140^{2}\quad 1100^{3}+11362^{4}+2787^{5}+2700^{6})z_6}{3(1+)(1+2)(1+3)(1+4)(2+9)(3+13)}$$

$$+\ \frac{4(18+285+949^{2}\quad 2675^{3}\quad 5493^{4}+33950^{5}+1646^{6}+3000^{7})z_1}{(1+)(1+2)(1+3)(1+4)(1+5)(2+9)(3+13)}$$

$$+\ \frac{(1+)(204\quad 4208\quad 37487^{2}\quad 140165^{3}\quad 24391^{4}+655613^{5}\quad 66238^{6}+75000^{7})}{(1+)(1+2)(1+3)(1+4)(1+5)(2+9)(2+13)(3+13)}$$

$$P_{1000001}(z) = z_1z_7+\frac{7z_2}{1+6}+\frac{(16\quad 191+42^{2})z_7}{(1+6)(2+17)}$$

$$P_{0100001}(z) = z_2z_7+\frac{6z_3}{1+5}+\frac{6(1+)z_7^2}{1+11}+\frac{2(7\quad 31+290^{2})z_6}{(1+5)(1+6)(1+11)}$$

$$+\ \frac{16(1\quad 26\quad 119^{2}+398^{3})z_1}{(1+5)(1+6)(1+7)(1+11)}+\frac{4(1+)(7\quad 51+826^{2})}{(1+5)(1+6)(1+7)(1+11)}$$

$$P_{0010001}(z) = z_3z_7+\frac{5(1+)z_6z_7}{1+7}+\frac{6z_1z_2}{1+5}+\frac{10(1+)(4+25)z_5}{(1+5)(1+7)(2+9)}$$

$$+\ \frac{2(1+)(72\quad 1183\quad 4188^{2}+2880^{3})z_1z_7}{(1+5)(1+7)(2+9)(3+16)}$$

$$+\ \frac{28(1+)(21\quad 260\quad 333^{2}+3056^{3})z_2}{(1+5)(1+7)(2+9)(2+11)(3+16)}$$

$$+ \frac{2(\ 72 \quad 4430 \quad 9270\ ^2 + 121301\ ^3 \quad 54561\ ^4 + 12240\ ^5)z_7}{(1+5\ )(1+7\ )(2+9\ )(2+11\ )(3+16\ )}$$

$$P_{0000101}(z) = z_5 z_7 + \frac{4z_4}{1+3} + \frac{5(\ 1+\ )z_1 z_7^2}{1+7} + \frac{8(\ 1 \quad + 22\ ^2)z_1 z_6}{(1+3\ )(1+4\ )(1+7\ )}$$

$$+ \frac{(\ 1+\ )(\ 33 \quad 239\ + 84\ ^2)z_2 z_7}{(1+3\ )(1+4\ )(3+13\ )} + \frac{12(\ 1+\ )(1+12\ )z_1^2}{(1+3\ )(1+4\ )(1+7\ )}$$

$$+ \frac{(\ 9+327\ + 3089\ ^2 \quad 1267\ ^3 + 420\ ^4)z_7^2}{(1+3\ )(1+4\ )(1+7\ )(3+13\ )} + \frac{12(\ 1+\ )(\ 21 \quad 118\ + 8\ ^2)z_3}{(1+3\ )(1+4\ )(2+9\ )(3+13\ )}$$

$$+ \frac{4(27+1050\ + 8824\ ^2 + 18278\ ^3 \quad 11811\ ^4 + 25872\ ^5)z_6}{(1+3\ )(1+4\ )(1+5\ )(1+7\ )(2+9\ )(3+13\ )}$$

$$+ \frac{48(\ 1+\ )(\ 89 \quad 1205 \quad 3637\ ^2 + 2159\ ^3)z_1}{(1+3\ )(1+4\ )(1+5\ )(1+7\ )(2+9\ )(3+13\ )}$$

$$+ \frac{8(3 \quad 203\ + 3439\ ^2 + 24833\ ^3 \quad 17242\ ^4 + 10290\ ^5)}{(1+3\ )(1+4\ )(1+5\ )(1+7\ )(2+9\ )(3+13\ )}$$

$$P_{0000011}(z) = z_6 z_7 + \frac{3z_5}{1+2} + \frac{(\ 2 \quad 43\ + 12\ ^2)z_1 z_7}{(1+2\ )(2+9\ )} + \frac{7(\ 1+\ )(2+19\ )z_2}{(1+2\ )(1+5\ )(2+9\ )}$$

$$+ \frac{2(\ 1+\ )(22 \quad 2 \quad 1052\ ^2 + 375\ ^3)z_7}{(1+2\ )(1+5\ )(2+9\ )(2+13\ )}$$

$$P_{0000002}(z) = z_7^2 + \frac{2z_6}{1+} + \frac{12\ z_1}{(1+\ )(1+5\ )} + \frac{2(1+59\ ^2)}{(1+\ )(1+5\ )(1+9\ )}$$

3.5 Generalized quadratic Clebsch-Gordan series

For each , the product of polynomials can be decomposed as a linear combination of polynomials of the same . The terms entering in this decomposition are exactly the same entering in the product of characters, i.e. in the corresponding Clebsch-Gordan series, while the coefficients are rational functions of . The method for computing these coefficients was explained in [1]. Here we give some of the quadratic Clebsch-Gordan series for general .

$$P_{1000000} \quad P_{1000000} = P_{2000000} + \frac{2}{1+} P_{0010000} + \frac{10(1+3\ )}{(1+4\ )(1+7\ )} P_{0000010}$$

$$+ \frac{24(4+103\ + 547\ ^2 + 696\ ^3)}{(1+8\ )(1+9\ )(1+17\ )(3+17\ )} P_{1000000}$$

$$+ \frac{252(1+3\ )(1+5\ )(1+8\ )(1+18\ )}{(1+11\ )(1+13\ )(1+17\ )^2(2+17\ )}$$

$$P_{1000000} \quad P_{0100000} = P_{1100000} + \frac{5}{1+4} P_{0000100} + \frac{32(1+2\ )(1+12\ )}{(1+7\ )(1+11\ )(2+11\ )} P_{1000001}$$

$$+ \frac{42(1+3\ )(1+14\ )(5+101\ + 74\ ^2)}{(1+6\ )(1+11\ )(2+13\ )(1+17\ )(3+17\ )} P_{0100000}$$

$$+ \frac{144(1+2\ )(1+3\ )(1+5\ )(1+18\ )}{(1+7\ )(1+8\ )(1+11\ )^2(2+17\ )} P_{0000001}$$

$$P_{1000000} \quad P_{0010000} = P_{1010000} + \frac{3}{1+2} P_{0001000} + \frac{16(1+2\ )(1+8\ )}{(1+5\ )(1+7\ )(2+7\ )} P_{1000010}$$

$$+ \frac{15(1+\ )(1+3\ )(1+11\ )}{(1+4\ )(1+5\ )^2(1+7\ )} P_{0100001} + \frac{48(1+2\ )(1+4\ )(2+17\ )}{(1+7\ )(1+8\ )(1+9\ )(3+16\ )} P_{2000000}$$

$$+ \frac{24(1+11\ )(\ 5 \quad 187 \quad 2180\ ^2 \quad 9982\ ^3 \quad 18875\ ^4 \quad 11395\ ^5 + 1800\ ^6)}{(1+\ )(1+5\ )^3(1+7\ )(1+8\ )(1+17\ )(4+17\ )} P_{0010000}$$

$$+ \frac{240(1+\ )(1+2\ )(1+3\ )(1+12\ )(1+13\ )}{(1+6\ )(1+7\ )^2(1+8\ )(2+11\ )(3+17\ )} P_{0000010}$$

$$+ \frac{384(1+2\ )(1+3\ )(1+4\ )^2(1+5\ )(1+12\ )(1+14\ )(1+17\ )}{(1+7\ )^2(1+8\ )^2(1+9\ )(1+11\ )(2+11\ )(2+13\ )(3+17\ )} P_{1000000}$$

$$P_{1000000} \quad P_{0000100} = P_{1000100} + \frac{5}{1+4} P_{0100010} + \frac{10(1+\ \ )(1+9\ )}{(1+4\ )^2(1+7\ )} P_{0010001}$$

$$+ \frac{240(1+\ \ )(1+2\ )(1+10\ )}{(1+5\ )(2+9\ )(2+13\ )(3+13\ )} P_{1100000} + \frac{32(1+2\ )(1+3\ )(1+12\ )}{(1+5\ )(1+7\ )^2(2+11\ )} P_{0000011}$$

$$+ \frac{90(1+10\ )(\ 16\quad 562\quad\ 5649\ ^2\quad 18716\ ^3\quad 18653\ ^4 + 3276\ ^5)}{(1+4\ )(1+7\ )(2+9\ )(2+13\ )(3+13\ )(1+17\ )(4+17\ )} P_{0000100}$$

$$+ \frac{240(1+\ \ )(1+2\ )(1+3\ )(2+7\ )(1+10\ )(1+12\ )(2+17\ )}{(1+5\ )(1+7\ )^2(1+8\ )(2+9\ )(2+11\ )(2+13\ )(3+16\ )} P_{1000001}$$

$$+ \frac{420(1+\ \ )(1+2\ )(1+3\ )(1+4\ )(2+7\ )(1+11\ )(1+12\ )(1+14\ )}{(1+5\ )^2(1+6\ )(1+7\ )(1+8\ )(2+11\ )(2+13\ )^2(3+17\ )} P_{0100000}$$

$$P_{1000000} \quad P_{0000010} = P_{1000010} + \frac{6}{1+5} P_{0100001} + \frac{27(1+3\ )}{(1+8\ )(1+11\ )} P_{0000002} + \frac{15(1+\ \ )(1+11\ )}{(1+5\ )^2(1+9\ )} P_{0010000}$$

$$+ \frac{48(1+3\ )(1+13\ )(\ 2\quad 45\quad\ 109\ ^2 + 6\ ^3)}{(1+\ \ )(1+6\ )(1+7\ )(1+9\ )(1+17\ )(3+17\ )} P_{0000010}$$

$$+ \frac{120(1+\ \ )(1+3\ )(1+4\ )(1+6\ )(1+14\ )(1+17\ )}{(1+5\ )(1+7\ )(1+8\ )(1+9\ )^2(1+13\ )(2+13\ )} P_{1000000}$$

$$P_{1000000} \quad P_{0000001} = P_{1000001} + \frac{7}{1+6} P_{0100000} + \frac{54(1+3\ )(1+18\ )}{(1+11\ )(1+17\ )(2+17\ )} P_{0000001}$$

$$P_{0100000} \quad P_{0100000} = P_{0200000} + \frac{2}{1+\ \ } P_{0001000} + \frac{8(1+2\ )}{(1+3\ )(1+5\ )} P_{1000010}$$

$$+ \frac{12(5+84\ \ + 255\ ^2 + 160\ ^3)}{(1+5\ )^2(1+11\ )(3+11\ )} P_{0100001} + \frac{32(1+2\ )(1+4\ )}{(1+5\ )(1+7\ )(1+9\ )} P_{2000000}$$

$$+ \frac{144(1+2\ )(1+3\ )(1+5\ )(1+12\ )}{(1+7\ )(1+8\ )(1+11\ )^2(2+11\ )} P_{0000002}$$

$$+ \frac{40(1+2\ )^2(\ 1\quad 22\quad\ 59\ ^2 + 10\ ^3)}{(1+\ \ )(1+4\ )(1+5\ )^3(1+11\ )} P_{0010000}$$

$$+ \frac{160(1+2\ )^2(1+3\ )(1+12\ )(\ 3\quad 64\ \ + 11\ ^2)}{(1+\ \ )(1+6\ )(1+7\ )(1+11\ )^2(2+11\ )(3+17\ )} P_{0000010}$$

$$+ \frac{192(1+2\ )(1+3\ )(1+4\ )(1+14\ )(5+101\ \ + 74\ ^2)}{(1+7\ )(1+8\ )(1+9\ )(1+11\ )^2(2+13\ )(3+17\ )} P_{1000000}$$

$$+ \frac{1152(1+2\ )(1+3\ )(1+4\ )(1+5\ )(1+6\ )(1+18\ )}{(1+7\ )(1+9\ )(1+11\ )^2(1+13\ )(1+17\ )(2+17\ )}$$

$$P_{0100000} \quad P_{0000010} = P_{0100010} + \frac{5}{1+4} P_{0010001} + \frac{32(1+2\ )(1+12\ )}{(1+7\ )(1+11\ )(2+11\ )} P_{0000011}$$

$$+ \frac{30(1+\ \ )(1+10\ )}{(1+5\ )(1+9\ )(2+9\ )} P_{1100000}$$

$$+ \frac{60(1+10\ )(\ 3\quad 56\quad\ 119\ ^2 + 18\ ^3)}{(1+4\ )(1+9\ )(2+9\ )(1+11\ )(3+13\ )} P_{0000100}$$

$$+ \frac{48(1+2\ )(1+3\ )(14+531\ \ + 6042\ ^2 + 23399\ ^3 + 24354\ ^4)}{(1+5\ )(1+7\ )(1+9\ )(2+9\ )(1+11\ )(2+11\ )(3+16\ )} P_{1000001}$$

$$+ \frac{420(1+2\ )(1+3\ )(1+4\ )(1+12\ )(1+14\ )(\ 3\quad 64\ \ + 11\ ^2)}{(1+5\ )(1+6\ )(1+9\ )(1+11\ )(2+11\ )(1+13\ )(2+13\ )(3+17\ )} P_{0100000}$$

$$+ \frac{864(1+\ \ )(1+2\ )(1+3\ )(1+4\ )(1+14\ )(1+18\ )}{(1+7\ )(1+8\ )(1+9\ )(1+11\ )^2(2+13\ )(2+17\ )} P_{0000001}$$

$$
\begin{aligned}
P_{0100000}\quad P_{0000001} &= P_{0100001} + \frac{6}{1+5}P_{0010000} + \frac{16(1+2\ )(1+13\ )}{(1+6\ )(1+7\ )(1+11\ )}P_{0000010}\\
&+ \frac{32(1+2\ )(1+4\ )(1+17\ )}{(1+7\ )(1+8\ )(1+9\ )(1+11\ )}P_{1000000}
\end{aligned}
$$

$$
\begin{aligned}
P_{0010000}\quad P_{0000001} &= P_{0010001} + \frac{6}{1+5}P_{1100000} + \frac{20(1+\ \ )(1+10\ )}{(1+4\ )(1+7\ )(2+9\ )}P_{0000100}\\
&+ \frac{48(1+2\ )(1+3\ )(1+12\ )(2+17\ )}{(1+7\ )^2(1+8\ )(2+11\ )(3+16\ )}P_{1000001}\\
&+ \frac{252(1+\ \ )(1+3\ )(1+4\ )(1+12\ )(1+14\ )}{(1+6\ )(1+7\ )(1+8\ )(2+11\ )(2+13\ )(3+17\ )}P_{0100000}
\end{aligned}
$$

$$
\begin{aligned}
P_{0000100}\quad P_{0000001} &= P_{0000101} + \frac{4}{1+3}P_{0001000} + \frac{8(1+2\ )(1+9\ )}{(1+4\ )(1+5\ )(1+7\ )}P_{1000010}\\
&+ \frac{90(1+\ \ )(1+3\ )(1+11\ )}{(1+5\ )^2(2+13\ )(3+13\ )}P_{0100001}\\
&+ \frac{40(1+\ \ )(1+2\ )(2+7\ )(1+10\ )(1+11\ )}{(1+5\ )^3(1+7\ )(2+9\ )(2+13\ )}P_{0010000}\\
&+ \frac{96(1+2\ )(1+3\ )(1+4\ )(2+7\ )(1+9\ )(1+12\ )(1+13\ )}{(1+5\ )(1+6\ )(1+7\ )^2(1+8\ )(2+11\ )(2+13\ )(3+17\ )}P_{0000010}
\end{aligned}
$$

$$
\begin{aligned}
P_{0001000}\quad P_{0000001} &= P_{0001001} + \frac{5}{1+4}P_{0110000} + \frac{12(1+\ \ )(1+8\ )}{(1+3\ )(1+5\ )(2+7\ )}P_{1000100}\\
&+ \frac{60(1+\ \ )(1+2\ )(1+6\ )(1+8\ )}{(1+4\ )(1+5\ )^2(2+7\ )(3+11\ )}P_{0100010}\\
&+ \frac{20(1+\ \ )(1+2\ )(2+5\ )(1+8\ )(1+9\ )(3+16\ )}{(1+4\ )^2(1+5\ )^3(3+11\ )(4+15\ )}P_{0010001}\\
&+ \frac{30(1+\ \ )(1+2\ )(1+3\ )^2(1+8\ )(1+10\ )(3+17\ )}{(1+4\ )^2(1+5\ )^4(2+9\ )(3+13\ )}P_{1100000}\\
&+ \frac{40(1+\ \ )(1+2\ )(1+3\ )(2+5\ )(1+8\ )(1+9\ )(1+10\ )(3+10\ )(2+13\ )}{(1+4\ )^3(1+5\ )^3(2+9\ )(2+11\ )(3+13\ )(4+17\ )}P_{0000100}
\end{aligned}
$$

$$
\begin{aligned}
P_{0000100}\quad P_{0000001} &= P_{0000101} + \frac{4}{1+3}P_{0001000} + \frac{8(1+2\ )(1+9\ )}{(1+4\ )(1+5\ )(1+7\ )}P_{1000010}\\
&+ \frac{90(1+\ \ )(1+3\ )(1+11\ )}{(1+5\ )^2(2+13\ )(3+13\ )}P_{0100001}\\
&+ \frac{40(1+\ \ )(1+2\ )(2+7\ )(1+10\ )(1+11\ )}{(1+5\ )^3(1+7\ )(2+9\ )(2+13\ )}P_{0010000}\\
&+ \frac{96(1+2\ )(1+3\ )(1+4\ )(2+7\ )(1+9\ )(1+12\ )(1+13\ )}{(1+5\ )(1+6\ )(1+7\ )^2(1+8\ )(2+11\ )(2+13\ )(3+17\ )}P_{0000010}
\end{aligned}
$$

$$
\begin{aligned}
P_{0000010}\quad P_{0000001} &= P_{0000011} + \frac{3}{1+2}P_{0000100} + \frac{20(1+3\ )(1+10\ )}{(1+7\ )(1+9\ )(2+9\ )}P_{1000001}\\
&+ \frac{42(1+\ \ )(1+4\ )(1+14\ )}{(1+5\ )(1+6\ )(1+9\ )(2+13\ )}P_{0100000}\\
&+ \frac{108(1+3\ )(1+4\ )(1+6\ )(1+14\ )(1+18\ )}{(1+8\ )(1+9\ )(1+11\ )(1+13\ )(2+13\ )(2+17\ )}P_{0000001}
\end{aligned}
$$

$$
\begin{aligned}
P_{0000001}\quad P_{0000001} &= P_{0000002} + \frac{2}{1+\ \ }P_{0000010} + \frac{12(1+4\ )}{(1+5\ )(1+9\ )}P_{1000000}\\
&+ \frac{56(1+4\ )(1+8\ )}{(1+9\ )(1+13\ )(1+17\ )}
\end{aligned}
$$

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