TWO-DIMENSIONAL SUPERSYMMETRY: FROM SUSY QUANTUM MECHANICS TO INTEGRABLE CLASSICAL MODELS

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Two known 2-dim SUSY quantum mechanical constructions - the direct generalization of SUSY with first-order supercharges and Higher order SUSY with second order supercharges - are combined for a class of 2-dim quantum models, which are not amenable to separation of variables. The appropriate classical limit of quantum systems allows us to construct SUSY-extensions of original classical scalar Hamiltonians. Special emphasis is placed on the symmetry properties of the models thus obtained - the explicit expressions of quantum symmetry operators and of classical integrals of motion are given for all (scalar and matrix) components of SUSY-extensions. Using Grassmanian variables, the symmetry operators and classical integrals of motion are written in a unique form for the whole Superhamiltonian. The links of the approach to the classical Hamilton-Jacobi method for related "flipped" potentials are established.

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1. Introduction

Supersymmetric Quantum Mechanics (SUSY QM) [1] is a new framework for analyzing non-relativistic quantum problems. In particular, it helps to investigate the spectral prop-

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erties of different quantum models as well as to generate new systems with given spectral characteristics (quantum design).

Much less attention has been paid to SUSY QM as a tool to study individual hidden symmetries of the superpartner Hamiltonians. This problem is reasonable for considering either one-dimensional quantum systems with internal degrees of freedom (with matrix potentials) or systems of higher spatial dimensionality. Only for these classes of systems may the symmetry operators, which are in involution with the Hamiltonian (and are independent of it), exist. Thus, the quantum models in one-dimensional SUSY QM with matrix potentials and the higher-dimensional (in particular, two-dimensional) models with scalar and/or matrix potentials are extremely attractive.

Regarding SUSY QM systems with matrix potentials, we refer to papers [2]. The SUSY QM systems with an arbitrary $(d > 1)$ dimensionality of space[d] were constructed and investigated in [4], [5]. Such models include both the scalar and matrix components of the Superhamiltonian. The latter are interesting either for a description of interacting non-relativistic particles with spin [6], [7], or for developing supersymmetric quantum field theory on a spatial lattice [8]. The appearance both of scalar and matrix Hamiltonians in a unique Superhamiltonian provides an opportunity to consider (starting from a given scalar Schrodinger operator) SUSY extensions that correspond to the systems with internal degrees of freedom. Some SUSY extensions of this type were considered in [9] (Calogero-like models of N particles on a line) and in [10] (Coulomb potential in d dimensions).

Among multidimensional SUSY QM models, those most developed are the two-dimensional ones. Namely, precisely for these systems second-order supercharges were used to build the higher-order deformation of SUSY algebra [11], [12], [13]. In the framework of this HSUSY QM generalization of conventional Witten's superalgebra[e] one can avoid the appearance of any matrix components of the Superhamiltonian, so that two scalar two-dimensional Schrodinger operators are intertwined by second-order supercharges. As a by-product of this construction, each of the intertwined Hamiltonians obeys the hidden symmetry: the differential operators of fourth order in derivatives exist, which are not reducible to the Hamiltonian and commute with the Hamiltonian, [11], [12], [16], [13]. In the two-dimensional context this

[d] Instead of dimensionality of the space d can be also interpreted as a number of particles, for example in Calogero-like models of interacting N particles on a line [3].

[e] One-dimensional HSUSY QM was investigated in detail in [14], [15].

means the complete integrability of the system.

Another direction in which to investigate SUSY QM models involves their connections with classical counterparts. Initially, the arbitrary-space-dimensionality generalization [5] of SUSY QM, mentioned above, was obtained by canonical quantization of a suitably chosen multi-dimensional system of Classical Mechanics. Then the quasi-classical limit of some supersymmetrical quantum models investigated afforded new insight into the properties of the classical models obtained. In the one-dimensional case, this limit led to a new SWKB quantization rule [17], which turned out to be more useful than the standard WKB rule. In the two-dimensional case, the quasi-classical limit provided an alternative effective method [18] for the construction of integrable systems in Classical Mechanics, which essentially enlarges the list of such models. The SUSY QM approach also provides new interesting links with the well-known Hamilton-Jacobi equation in Classical Mechanics [19].

In the present paper we shall combine both known two-dimensional SUSY QM constructions – the direct generalization of SUSY with first-order supercharges and HSUSY with second-order supercharges – in order to investigate the symmetry properties of the models obtained, both at quantum and classical level. The paper is organized as follows. In Section 2, known results about two-dimensional SUSY QM models and their connections with classical models, necessary for the original part of the paper, are briefly summarized. In Section 3, the particular case of two-dimensional models for second-order supercharges with intermediate twists are studied: the particular models of generalized Morse and Poschl-Teller potentials are presented within this common framework. In Section 4, the integrability of these models is extended onto their matrix – both quantum and classical – superpartners. In Section 5, the links with classical Hamilton-Jacobi equation are considered.

2. Two-dimensional SUSY Quantum Mechanics

2.1. The representation of SUSY algebra with first-order supercharges

In the two-dimensional case $\vec{x} = (x_1; x_2)$, the direct (i.e. of first order in derivatives) generalization of SUSY QM satisfies [4], [5] the conventional Witten's [20] SUSY algebra

$$\{\hat{Q}^+,\hat{Q}^-\}=\hbar\hat{H}; \qquad \{\hat{Q}^+,\hat{Q}^+\}=\{\hat{Q}^-,\hat{Q}^-\}=0; \qquad [\hat{Q}^\pm,\hat{H}]=0: \qquad (1)$$

by the 4×4 matrix operators:

$$\hat{H}=\begin{pmatrix} H^{(0)}(\vec{x}) & 0 & 0 \\ 0 & H^{(1)}_{ik}(\vec{x}) & 0 \\ 0 & 0 & H^{(2)}(\vec{x}) \end{pmatrix}; \quad i,k=1,2; \quad \hat{Q}^+=(\hat{Q}^-)^\dagger=\hbar^{-1/2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ q_1^- & 0 & 0 & 0 \\ q_2^- & 0 & 0 & 0 \\ 0 & s_1^+ & s_2^+ & 0 \end{pmatrix}; \qquad (2)$$

where two scalar Hamiltonians $H^{(0)}, H^{(2)}$ and one 2×2 matrix Hamiltonian $H^{(1)}_{ik}$ of the Schrödinger type take on a quasi-factorized form:

$$\begin{aligned} H^{(0)} &= q_l^+ q_l^- = -\hbar^2\partial_l^2 + V^{(0)}(\vec{x}) = -\hbar^2\partial_l^2 + \left(\partial_l\chi(\vec{x})\right)^2 - \hbar\partial_l^2\chi(\vec{x}); \quad \partial_l^2 \equiv \partial_1^2+\partial_2^2; \\ H^{(2)} &= s_l^+ s_l^- = -\hbar^2\partial_l^2 + V^{(2)}(\vec{x}) = -\hbar^2\partial_l^2 + \left(\partial_l\chi(\vec{x})\right)^2 + \hbar\partial_l^2\chi(\vec{x}); \qquad (3) \\ H^{(1)}_{ik} &= q_i^- q_k^+ + s_i^- s_k^+ = -\hbar^2\delta_{ik}\partial_l^2 + \delta_{ik}\left(\left(\partial_l\chi(\vec{x})\right)^2 - \hbar\partial_l^2\chi(\vec{x})\right) + 2\hbar\partial_i\partial_k\chi(\vec{x}); \end{aligned}$$

the components of the supercharges being of first order in derivatives:

$$q_l^\pm \equiv \mp\hbar\partial_l + (\partial_l\chi(\vec{x})); \quad s_l^\pm \equiv \epsilon_{lk} q_k^\mp; \qquad (4)$$

where $\partial_i \equiv \partial/\partial x_i$ and summation over repeated indices is assumed. The Planck constant was restored in formulas (2) – (4), and the (normalizable or unnormalizable) zero-energy wave functions of the scalar Hamiltonians $H^{(0),(2)}$ are now written as $\exp(\mp\chi/\hbar)$.

The quasi-factorization in (3) ensures that the last equation in (1) holds, and leads to the following intertwining relations for the components of the Superhamiltonian (2):

$$H^{(0)}q_i^+ = q_k^+ H^{(1)}_{ki}; \quad H^{(1)}_{ik}q_k^- = q_i^- H^{(0)}; \quad H^{(1)}_{ik}s_k^- = s_i^- H^{(2)}; \quad H^{(2)}s_i^+ = s_k^+ H^{(1)}_{ki}: \qquad (5)$$

These relations play the main role in the SUSY QM approach and, in particular, they lead to the connections between the spectrum of the matrix Hamiltonian and the spectra of two scalar ones. We remark that $H^{(0)}$ and $H^{(2)}$ are not intertwined with each other and are not (in general) isospectral since $q_k^+ s_k^- \equiv 0$.

2.2. Second-order supercharges

An alternative opportunity to include two-dimensional scalar Hamiltonians in the SUSY QM framework is based on the supercharges of second order in derivatives [11], [12], [13]:

$$Q^+ = (Q^-)^\dagger = g_{ik}(\vec{x})\hbar^2\partial_i\partial_k + C_i(\vec{x})\hbar\partial_i + B(\vec{x}) \tag{6}$$

where g_{ik}, C_i, B are arbitrary real functions. In this case, two scalar Hamiltonians $H^{(0)}, \tilde{H}^{(0)}$ are intertwined directly without any (matrix) intermediary:

$$\tilde{H}^{(0)}(\vec{x})Q^+ = Q^+H^{(0)}(\vec{x}); \qquad H^{(0)}(\vec{x})Q^- = Q^-\tilde{H}^{(0)}(\vec{x}). \tag{7}$$

Although no method to find the general solution of the intertwining relations (7) has been proposed, a certain number of models obeying these relations have been found for the cases of hyperbolic (Lorentz) and degenerate metric g_{ik}. One very important specific property of all these models, which follows from the intertwining relations (7), is their integrability. Indeed, both Hamiltonians possess the symmetry operators $R^{(0)}, \tilde{R}^{(0)}$ of fourth order in derivatives [11], [12]:

$$[R^{(0)}, H^{(0)}] = 0; \quad [\tilde{R}^{(0)}, \tilde{H}^{(0)}] = 0; \qquad R^{(0)} = Q^-Q^+; \qquad \tilde{R}^{(0)} = Q^+Q^-; \tag{8}$$

which are not, in general[f], polynomials of $H^{(0)}, \tilde{H}^{(0)}$.

For the case of Lorentz (hyperbolic) metric $g_{ik} = \mathrm{diag}(1, -1)$ [11] – [18], the intertwining relations (7) can be rewritten in a reduced form:

$$\partial_-(C_-F) = -\partial_+(C_+F); \tag{9}$$

where $x_\pm = (x_1 \pm x_2)/\sqrt{2}$; functions $C_{1,2}$ were found to satisfy $C_\pm \equiv C_1 \mp C_2 \equiv C_\pm(\sqrt{2}x_\pm)$; and $F(\vec{x}) = F_1(2x_1) + F_2(2x_2)$. Thus, the potentials $V^{(0),(1)}$ and the supercharges $Q^\pm$ are expressed in terms of the functions $C_\pm(\sqrt{2}x_\pm)$ and $F_1(2x_1), F_2(2x_2)$:

$$\begin{aligned} V^{(0)}, \tilde{V}^{(0)} &= \pm\frac{1}{2}\hbar\left(C_+'(\sqrt{2}x_+) + C_-'(\sqrt{2}x_-)\right) + \frac{1}{8}\left(C_+^2(\sqrt{2}x_+) + C_-^2(\sqrt{2}x_-)\right) + \\ &\quad + \frac{1}{4}\left(F_2(2x_2) - F_1(2x_1)\right); \end{aligned} \tag{10}$$

$$Q^+ = \hbar^2(\partial_1^2 - \partial_2^2) + C_1\hbar\partial_1 + C_2\hbar\partial_2 + B; \tag{11}$$

$$B = \frac{1}{4}\left(C_+(\sqrt{2}x_+)C_-(\sqrt{2}x_-) + F_1(2x_1) + F_2(2x_2)\right); \tag{12}$$

[f] It has been proved [12] that only for Laplacian (elliptic) metric $g_{ik} = \delta_{ik}$ can the symmetry operators be reduced to second-order operators, and the corresponding Hamiltonians $H^{(0)}, \tilde{H}^{(0)}$ allow the separation of variables.

where the prime denotes the derivative of function with respect to its argument. A set of particular solutions of (9) was obtained in [12], [18], [21].

3. Two-dimensional models with twisted reducibility of supercharges

In the previous Section it was shown that two different constructions with very different properties exist in two-dimensional SUSY QM. The first one (Subsection 2.1.) includes two scalar Hamiltonians $H^{(0)}, H^{(2)}$ (only one of them has normalizable[g] zero-energy ground state wave function $\Psi_0(\vec{x}) = \exp(-\frac{\chi(\vec{x})}{\hbar})$) and their 2×2 matrix partner $H^{(1)}_{ik}$. The second one (Subsection 2.2.) contains only the scalar Hamiltonians $H^{(0)}, \tilde{H}^{(0)}$ with no information about their ground-state energy in advance, and both $H^{(0)}$ and $\tilde{H}^{(0)}$ a priori (by construction) obey the important property of integrability with the symmetry operators $R^{(0)}, \tilde{R}^{(0)}$ of fourth order in derivatives (see (8)). The natural idea is to unite all the above tempting properties by combining these two constructions, i.e. by identifying the original Hamiltonian $H^{(0)}$ as the same in both approaches. More precisely: let $H^{(0)}$ of the form (3) have the superpartners $H^{(1)}_{ik}$ and $H^{(2)}$ in the first-order scheme, and at the same time the superpartner $\tilde{H}^{(0)}$ in the second-order scheme. It is known [12] that the simplest, reducible or quasi-factorizable, form of the second order supercharges $Q^+ = (Q^-)^\dagger = q_i^+ \tilde{q}_i^-$, which is suitable for the construction described, leads to the R separation of variables, and therefore it is not considered here. All other models (excluding the case of elliptic metric $g_{ik} = \delta_{ik}$ in $Q^\pm$) have been proved [12] not to be amenable to separation of variables; they have nontrivial fourth-order symmetry operators (8). The main idea to achieve the identification of $H^{(0)}$ in the two approaches is to consider a class of models with second-order supercharges, which are quasi-factorizable, but with an intermediate twist transformation (see also [21]):

$$Q^- = (Q^+)^\dagger = q_i^+ U_{ik} \tilde{q}_k^- ; \qquad (13)$$

where U_{ik} is a constant unitary matrix, $q^\pm$ were defined in (4), and

$$\tilde{q}_k^\pm \equiv \mp\hbar\partial_k + (\partial_k \tilde{\chi}(\vec{x}))$$

[g] For the case of supersymmetry not broken spontaneously.

with some new superpotential ~: Such a generalization of the notion of reducibility (we shall call it twisted reducibility) is somehow reminiscent of the "gluing with shift" recipe in one-dimensional scalar [14] and matrix [2] HSUSY QM. The intertwining relations (7) with supercharges (13) and the general expression for matrix U_{ik}:

$$U = \ _0 \ _0 + i \vec{!} \ \vec{!} \ ; \quad \ _0^2 + \ \vec{!} \ ^2 = 1; \quad \ _0; \ _i 2 R ;$$

($_i$ are the Pauli matrices and $_0$ is the unit matrix) give the system of four linear and one nonlinear equations for two functions $= \frac{1}{2}(\quad \sim)$:

$$_3 \qquad + 2 \ _1 @_1 @_2 \quad = 0; \qquad _1 \quad _+ \quad 2 \ _3 @_1 @_2 \ _+ \quad = 0; \tag{14}$$

$$_2 \quad _+ \quad 2 \ _0 @_1 @_2 \quad = 0; \qquad _0 \qquad + 2 \ _2 @_1 @_2 \ _+ \quad = 0; \tag{15}$$

$$(@_k \quad)(@_k \ _+) = 0; \tag{16}$$

where $\quad @_1^2 \quad @_2^2$. Precisely the last equation (16) is obviously most di cult to solve. Both the solutions of linear partial di erential equations (14)–(15) and the form of (16) depend crucially on the values of $_i$ chosen. For the most sets of 's, including the general case with all $_i \neq 0$ as well as almost all degenerate cases with some $_i$ vanishing, the corresponding potentials allow the separation of variables and are ignored here.

The only exception to the above rule, and therefore the most interesting quantum models, corresponds to the case[h] when $_0 = \ _1 = \ _2 = 0; \ _3 \neq 0$, i.e. $U = \ _3$. Then, the metric of supercharges Q is hyperbolic, i.e. Q belong to the class discussed in Subsection 2.2. For these models (due to (14) – (15) only), the supercharges are represented in terms of four arbitrary real functions $_{1,2}$, :

$$= \ _+(x_+) + \quad (x \ ); \qquad _+ = \ _1(x_1) + \ _2(x_2); \tag{17}$$

Hence, the last equation (16) rewritten via ^{0} takes the form of the functional equation:

$$_1(x_1)[\ _+(x_+) + \quad (x \ )] + \ _2(x_2)[\ _+(x_+) \quad (x \ )] = 0: \tag{18}$$

[h] The system with $_0 = \ _2 = \ _3 = 0; \ _1 \neq 0$, i.e. $U = \ _1$ leads to analogous results with the substitution $\leftrightarrow$.

It is reasonable to formulate here the important specific property of solution (17). The superpotential

$$\chi(\vec{x}) = \chi_+ + \chi_- = \mu_1(x_1) + \mu_2(x_2) + \mu_+(x_+) + \mu_-(x_-); \qquad (19)$$

leads to an expression for the quantum potential $V^{(0)}(\vec{x})$ (see the first of Eq.(3)), which also has the form of the sum:

$$V^{(0)}(\vec{x}) = \left(\partial_1 \chi(\vec{x})\right)^2 - \hbar\partial_1^2\chi(\vec{x}) = v_1(x_1) + v_2(x_2) + v_+(x_+) + v_-(x_-);; \qquad (20)$$

with $v_{1,2} = \mu'^2_{1,2} \mp \hbar\mu''_{1,2}$, $v_\pm = \mu'^2_\pm \mp \hbar\mu''_\pm$. It may be seen that both terms in quantum potential (20) separately have the form of the sums as in (19). Therefore, at the quasi-classical limit $V^{(0)}_{cl}$ (see Sections 4 and 5 below), where only the first term $\left(\partial_1\chi(\vec{x})\right)^2$ survives, the potential is also represented in a form similar to (20) but with truncated $v_{1,2,+,-}$. Both in the quantum and classical contexts, form (20) seems to be typical for a wide class of integrable two-dimensional models, considered within very different approaches in the literature (see [22], [18], as examples). This is why the following statement might be useful (at least, in the classical framework). Thus, if the general solution for the superpotential $\chi(\vec{x})$ in relation

$$V^{(0)}_{cl}(\vec{x}) = \left(\partial_1\chi(\vec{x})\right)^2 = v_1(x_1) + v_2(x_2) + v_+(x_+) + v_-(x_-) \qquad (21)$$

is of the form of (19), precisely the functional equation (18) must be fulfilled. This equation ensures the mutual cancellation of crossed terms in (21) and is therefore very important for this class of model. The general solution of (18) was found by D.Nishnianidze[i] (see [21]):

$$\mu'^2_{1,2} = a\mu^4_{1,2} + b\mu^2_{1,2} + c; \qquad x = \int \frac{d\mu_1}{\sqrt{a\mu_1^4 + b\mu_1^2 + c}}; \quad a;b;c = \mathrm{const}; \qquad (22)$$

and explicit expressions for $\mu_\pm$ can be obtained from (18). Among the functions that satisfy conditions (22) there exists a set of solutions of Eq.(18) possessing the periodicity property. For example,

$$\mu_+(x) = A\frac{\mathrm{sn}(ax|k)\,\mathrm{cn}(ax|k)}{\mathrm{dn}(ax|k)}; \qquad \mu_-(x) = A\frac{\mathrm{dn}(ax|k)}{k^2\,\mathrm{sn}(ax|k)\,\mathrm{cn}(ax|k)}; \qquad (23)$$

$$\mu_1(x) = \mu_2(x) = B\,\mathrm{dn}(\sqrt{2}ax|k);$$

where $A;B;a$ are real constants and $\mathrm{sn}(\cdot|k)$, $\mathrm{cn}(\cdot|k)$ and $\mathrm{dn}(\cdot|k)$ are Jacobi elliptic functions [23] with modulus k. They are doubly periodic on the complex plane of argument x, but

[i] Private communication.

in the case $k = 1$ the real period becomes infinite and the elliptic functions turn into the hyperbolic functions sinh and cosh. Restricting ourselves in (22) to non-periodic functions on a whole plane $(x_1; x_2)$ satisfying (22), which do not happens in systems with separation of variables, two families of models exist (see [21]). One of them is represented by the two-dimensional Morse potential, with $\chi_1 = \chi_2 = Be^{-\alpha x}$; $\chi_+ = 2A$; $\chi_- = 2A\coth(\alpha x/\sqrt{2})$:

$$V^{(0)} = \alpha^2(B^2 e^{-2\alpha x_1} + \hbar B\alpha e^{-\alpha x_1}) + \alpha^2(B^2 e^{-2\alpha x_2} + \hbar B\alpha e^{-\alpha x_2}) + 2A\left(2A + \frac{\hbar\alpha}{\sqrt{2}}\right)\left[\sinh\left(\frac{\alpha(x_1 - x_2)}{2}\right)\right]^{-2} + 8A^2 . \quad (24)$$

and the second by the two-dimensional Pöschl-Teller potential with $\chi_1 = -\chi_2 = A\left(\sinh(\sqrt{2}\alpha x)\right)^{-1}$; $\chi_+ = \chi_- = B\tanh(\alpha x)$:

$$V^{(0)} = \alpha^2 B^2 - \frac{\alpha B(\alpha B + \hbar\alpha)}{\cosh^2(\frac{\alpha}{\sqrt{2}}(x_1 + x_2))} + \alpha^2 B^2 - \frac{\alpha B(\alpha B + \hbar\alpha)}{\cosh^2(\frac{\alpha}{\sqrt{2}}(x_1 - x_2))} + \alpha A\left[\frac{A - \hbar\alpha\sqrt{2}\cosh(\sqrt{2}\alpha x_1)}{\sinh^2(\sqrt{2}\alpha x_1)} + \frac{A + \hbar\alpha\sqrt{2}\cosh(\sqrt{2}\alpha x_2)}{\sinh^2(\sqrt{2}\alpha x_2)}\right] . \quad (25)$$

Other members of these families can be obtained by using two discrete symmetries of solutions of Eq.(18):

$$S_1 : \quad \left(\chi_1(x_1); \chi_2(x_2); \chi_+(x_+); \chi_-(x_-)\right) \to \left(\chi_+(x_1); \chi_-(x_2); \chi_1(x_+); \chi_2(x_-)\right) ;$$

$$S_2 : \quad \left(\chi_1(x_1); \chi_2(x_2); \chi_+(x_+); \chi_-(x_-)\right) \to \left(\chi_1(x_1); -\chi_2(x_2); \chi_+^{-1}(x_+); \chi_-^{-1}(x_-)\right) \quad (26)$$

and different combinations thereof.

4. Supersymmetric extensions of scalar Hamiltonians and their integrability

In the previous Section we presented the explicit forms (24), (25) of the terms in Eq.(20) for a certain class of quantum integrable Hamiltonians. In this Section we shall build their classical and quantum SUSY extensions and we shall also demonstrate their integrability properties.

4.1. The classical limit for $H^{(0)}$

First, we shall consider for $H^{(0)}$ its classical limit $H^{(0)}_{cl}$, for which the integral of motion R_{cl} exists: $\{H^{(0)}_{cl}; R^{(0)}_{cl}\}_P = 0$ ($\{\ ;\ \}_P$ denotes standard Poisson brackets). This can be done by the simple limit procedure $\hbar \to 0$ in Eq.(3). The practical recipe is as follows. One has to replace all operators $i\hbar\partial_i$ by momenta p_i and skip all derivatives of functions, which include $\hbar$ as a multiplier. One thus obtains[j]:

$$H^{(0)}_{cl} = p_jp_j + {}^2_1 + {}^2_2 + {}^2_+ + {}^2 ;$$

$$Q_{cl} = p_1^2 \quad p_2^2 \quad i\sqrt{2}({}_+ + \quad)p_1 \quad i\sqrt{2}({}_+ \quad)p_2 + {}^2_1 \quad {}^2_2 \quad 2 {}_+ \quad :$$

The integral of motion has the form $R^{(0)}_{cl} = Q^+_{cl}Q_{cl}$: Its involution with the Hamiltonian can be checked either by direct calculation or by a simpler two-step procedure, proposed in general form in [24]. It is instructive to perform it in the context of the models considered here. First, one has to prove the intermediate relations:

$$\{H^{(0)}_{cl};Q_{cl}\}_P = \quad 2i({}^0_+ + \quad {}^0)Q_{cl}: \qquad (27)$$

From the definition of Poisson brackets

$$\{H^{(0)}_{cl};Q_{cl}\}_P = \quad 2i({}^0_+ + \quad {}^0)(p_1^2 \quad p_2^2 \quad i\sqrt{2}({}_+ + \quad)p_1 \quad i\sqrt{2}({}_+ \quad)p_2 \quad 2 {}_+ \quad)$$
$$2i\sqrt{2}({}_+ ({}_1 {}^0_1 \quad {}_2 {}^0_2) + \quad ({}_1 {}^0_1 + {}_2 {}^0_2)): \qquad (28)$$

The last term can be transformed with the use of the functional equation (18):

$$ {}_+ ({}_1 {}^0_1 \quad {}_2 {}^0_2) + \quad ({}_1 {}^0_1 + {}_2 {}^0_2) = \frac{1}{\sqrt{2}} {}_+ \partial \quad {}^2_1 + {}^2_2 + \frac{1}{\sqrt{2}} \quad \partial_+ \quad {}^2_1 + {}^2_2 =$$
$$= \frac{1}{\sqrt{2}} \partial \;({}_+ ({}_1 + {}_2)^2) + \partial_+ (\quad ({}_1 \quad {}_2)^2) \quad 2 {}_+ \partial \;({}_1 {}_2) + 2 \quad \partial_+ ({}_1 {}_2) =$$
$$= \frac{1}{\sqrt{2}} \quad \partial \;(\quad ({}^2_1 \quad {}^2_2)) \quad \partial_+ ({}_+ ({}^2_1 \quad {}^2_2)) \quad 2 {}_+ \partial \;({}_1 {}_2) + 2 \quad \partial_+ ({}_1 {}_2) =$$
$$= \quad \frac{1}{\sqrt{2}} ({}^0_+ + \quad {}^0)({}^2_1 \quad {}^2_2):$$

Substituting this into (28), we prove (27). Then, (27) leads to the involution of $R^{(0)}_{cl}$ with $H^{(0)}_{cl}$:

$$\{H^{(0)}_{cl};Q^+_{cl}Q_{cl}\}_P = \{H^{(0)}_{cl};Q^+_{cl}\}_P Q_{cl} + Q^+_{cl}\{H^{(0)}_{cl};Q_{cl}\}_P = 0:$$

It is easy to check that the same classical limit procedure for the second scalar Hamiltonian $H^{(2)}$ and for the components of the matrix $H^{(1)}_{ik}$ in (3) leads to the simple results: $H^{(2)}_{cl} =$

[j] Here and below we will omit the arguments of ${}_{1;2}$ and $\;$, which implies that ${}_{1;2} \quad {}_{1;2}(x_{1;2})$ and $\;(x \;)$:

$H^{(0)}_{cl}$ and $H^{(1)}_{ik;cl} = \delta_{ik} H^{(0)}_{cl}$. Naturally, the corresponding integrals of motion coincide too: $R^{(2)}_{cl} = R^{(0)}_{cl}$ and $R^{(1)}_{ik;cl} = \delta_{ik} R^{(0)}_{cl}$. In the next Subsection we shall construct another classical limit of $\hat{H}$ that will also include Grassmanian dynamical variables in addition to p_j and x_j, and this can be interpreted as a SUSY-extension of $H^{(0)}_{cl}$.

4.2. The SUSY extension of classical scalar Hamiltonians

It is well-known [5], [8] that the 2D representation of SUSY algebra, reviewed in Section 2, can be obtained by the canonical quantization from the classical system with the Hamiltonian:

$$H_{cl} = p_l p_l + (\partial_l \chi(\vec{x}))(\partial_l \chi(\vec{x})) + 2i\partial_i\partial_l \chi(\vec{x})\, \psi^1_i \psi^2_l, \qquad (29)$$

where ψ^1_i and ψ^2_j are Grassmanian anticommuting variables: $\{\psi^\alpha_i, \psi^\beta_j\} = 0$ $(i,j = 1,\dots,d;\ \alpha,\beta = 1,2)$. One can define the Poisson bracket on the phase space of the system with classical bosonic and fermionic variables [25] as follows:

$$\{F, G\}_P = \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial x^j} - \frac{\partial F}{\partial x^j}\frac{\partial G}{\partial p_j} + iF \frac{\overleftarrow{\partial}}{\partial \psi^\alpha_i}\frac{\overrightarrow{\partial}}{\partial \psi^\alpha_i} G,$$

such that the canonical brackets are $\{p_i, x_j\}_P = \delta_{ij}$; $\{\psi^\alpha_i, \psi^\beta_j\}_P = i\delta^{\alpha\beta}\delta_{ij}$. Thus, the Hamiltonian is involved in the SUSY algebra:

$$\{Q^\alpha, Q^\beta\}_P = i\delta^{\alpha\beta} H_{cl}, \qquad \{H_{cl}, Q^\alpha\}_P = 0$$

with classical supercharges

$$Q^1 = p_j \psi^1_j - (\partial_j \chi)\psi^2_j, \qquad Q^2 = p_j \psi^2_j + (\partial_j \chi)\psi^1_j.$$

To quantize this model, one has to introduce the bosonic operators $\hat{p}_i$ and $\hat{x}_j$ with canonical commutation relations, and fermionic ones $\hat{\psi}^\alpha_j$ obeying $\{\hat{\psi}^\alpha_i, \hat{\psi}^\beta_j\} = \hbar\delta_{ij}\delta^{\alpha\beta}$. At this stage, it is convenient to introduce the fermionic operators $\hat{\psi}_j = (\sqrt{2})^{-1}(\hat{\psi}^1_j - i\hat{\psi}^2_j)$ with anticommutation relations $\{\hat{\psi}^+_i, \hat{\psi}_j\} = \hbar\delta_{ij}$ and $\{\hat{\psi}_i, \hat{\psi}_j\} = \{\hat{\psi}^+_i, \hat{\psi}^+_j\} = 0$. These can be treated as creation/annihilation operators in the system of d spin $1/2$ fermions. Thus, the quantum counterpart of the Hamiltonian (29) – the Superhamiltonian – in terms of these operators takes the form:

$$\hat{H} = -\hbar^2\partial_j\partial_j + [(\partial_j \chi)(\partial_j \chi) - \hbar(\partial_j\partial_j \chi)] + 2(\partial_i\partial_j \chi)\hat{\psi}^+_i \hat{\psi}_j. \qquad (30)$$

Together with the quantum supercharges $\hat{Q}^{\pm} = \mp i(\sqrt{2})^{-1}(\hat{Q}^1 \pm i\hat{Q}^2) = (\mp i\hat{p}_j + (\partial_j \chi))\hat{\psi}^{\mp}_j$ this generates the algebra (1). To reproduce the matrix form (2) – (3) one should choose the matrix representation for the creation and annihilation operators $\hat{\psi}^{\pm}_j$. For $d = 2$, these operators are 4×4 matrices, and a possible choice is as follows [5]: $\hat{\psi}^+_1 = \hbar^{1/2}(E_{2,1} - E_{4,3})$, $\hat{\psi}^+_2 = \hbar^{1/2}(E_{3,1} + E_{4,2})$ and $\hat{\psi}_j = (\hat{\psi}^+_j)^{\dagger}$ (here matrices $E_{m,n}$ are defined as $(E_{m,n})_{ik} \equiv \delta_{mi}\delta_{nk}$). Thus, we have obtained a matrix realization of two-dimensional SUSY QM (1) – (3) by means of the canonical quantization of a certain classical model (29).

One can see that in this representation $\hat{H}$ has a block-diagonal structure. The origin of this feature of the model is the conservation of the fermion number $[\hat{H}, \hat{N}] = 0$, with fermion number operator $\hat{N} = \hat{\psi}^+_j \hat{\psi}_j$. Therefore, each component of the Superhamiltonian acts in a space of states with a fixed fermion number.

In our case $d = 2$, this structure is rather simple. Let us define a basis in the state space: the vacuum $|00\rangle$, which is annihilated by $\hat{\psi}_i$, and the excited states $|10\rangle = \hbar^{-1/2}\hat{\psi}^+_1|00\rangle$; $|01\rangle = \hbar^{-1/2}\hat{\psi}^+_2|00\rangle$; $|11\rangle = \hbar^{-1}\hat{\psi}^+_2\hat{\psi}^+_1|00\rangle$ (the $\hbar$-dependent multipliers provide the proper normalization of the state vectors: $\langle mn|mn\rangle = 1$, $\forall m, n = 0, 1$). Thus $H^{(0)}$ acts in one-dimensional space with a fermion number of 0 and a single basis element $|00\rangle$; for $H^{(2)}$, the fermion number is 2 and the basis element is $|11\rangle$. The Hamiltonian $H^{(1)}_{ik}$ acts in two-dimensional state space $\{|01\rangle, |10\rangle\}$, where the fermion number is 1.

The conclusion to be drawn from this derivation is as follows. Having the classical system with $H^{(0)}_{cl} = p_j p_j + (\partial_j \chi)(\partial_j \chi)$, one can construct its SUSY extension of the form (29). This classical SUSY extension can be quantized canonically to obtain the quantum Superhamiltonian (30), the original $H^{(0)}_{cl}$ being the classical limit of $H^{(0)}$ – the first scalar component of the quantum Superhamiltonian. In our case, $H^{(0)}_{cl}$ was integrable (see previous Subsection), and we shall explicitly find the integral of motion R_{cl} for its quantum SUSY-extension (Superhamiltonian). An analogous problem was investigated by alternative methods in [26], but for a much more narrow class of models (amenable to separation of variables).

4.3. Integrals of motion for the quantum and classical SUSY extensions

We start from construction of the quantum integral of motion $\hat{R}$ for the Superhamiltonian (30): $[\hat{H}, \hat{R}] = 0$, i.e. of conserved operators $R^{(i)}$ for each component of the Superhamiltonian:

$$[H^{(0)}, R^{(0)}] = 0; \quad [H^{(2)}, R^{(2)}] = 0; \tag{31}$$

$$[H^{(1)}, R^{(1)}] = 0; \tag{32}$$

(note that the last commutator is the matrix one). The explicit expression for $R^{(0)}$ can be obtained from (8) and (13):

$$R^{(0)} = q_i^+ U_{ik} q_k^- q_m^+ U_{ml} q_l^- : \tag{33}$$

One can obtain $R^{(2)}$ from $R^{(0)}$ by the substitutions $q_j^+ \to q_j^-$ and $q_j^- \to q_j^+$ (since $H^{(0)}$ turns to $H^{(2)}$ after the substitution $\chi(\vec{x}) \to -\chi(\vec{x})$):

$$R^{(2)} = q_i^- U_{ik} q_k^+ q_m^- U_{ml} q_l^+ : \tag{34}$$

With respect to $R^{(1)}$, the form of intertwining relations (5) tells us how to build this symmetry operator explicitly. One can check that the following matrix operator of sixth order in derivatives

$$R^{(1)}_{ik} = q_i^- R^{(0)} q_k^+ ; \tag{35}$$

satisfies Eq.(32).

It is clear from the material of the previous Subsection that knowledge of $R^{(i)}$; $(i = 0,1,2)$ provides a symmetry operator $\hat{R}$ for the whole block-diagonal Superhamiltonian $\hat{H}$. In order to construct an all-sector expression for $\hat{R}$:

$$\hat{R} = R^{(0)} P^{(0)} + R^{(2)} P^{(2)} + R^{(1)}_{ik} P^{(1)}_{ik} ; \tag{36}$$

we introduce the corresponding projection operators $P^{(i)}$ $(i = 0,1,2)$ onto subspaces with definite fermion numbers. The scalar projectors $P^{(0)}$ and $P^{(2)}$ give unity when acting on $|00\rangle$ and $|11\rangle$; respectively, and give zero otherwise. The components $P^{(1)}_{ik}$ $(i,k = 1,2)$ of the 2×2 matrix projector operators $P^{(1)}$ transform the k-th component of the state vector into its i-th component[k] and are zero on other components of the state vector. One can check directly that an explicit form of these operators P leads to:

$$\hat{R} = R^{(0)}\hbar^{-2}\hat{\psi}_1\hat{\psi}_2\hat{\psi}_1^+\hat{\psi}_2^+ + R^{(2)}\hbar^{-2}\hat{\psi}_1^+\hat{\psi}_2^+\hat{\psi}_1\hat{\psi}_2 + R^{(1)}_{11}\hbar^{-2}\hat{\psi}_1^+\hat{\psi}_2\hat{\psi}_1\hat{\psi}_2^+ + R^{(1)}_{22}\hbar^{-2}\hat{\psi}_1\hat{\psi}_2^+\hat{\psi}_1^+\hat{\psi}_2 + R^{(1)}_{12}\hbar^{-1}\hat{\psi}_1^+\hat{\psi}_2 + R^{(1)}_{21}\hbar^{-1}\hat{\psi}_2^+\hat{\psi}_1 ; \tag{37}$$

[k] By definition, "the first component" is $|10\rangle$; and "the second component" is $|01\rangle$.

where $R^{(i)}$ are given by (33) – (35). Naturally, (37) can be simplified by employing anti-commutation relations for $\hat{}_i$. One should not be confused by the presence of the negative powers of the Planck constant in Eq.(37) since they disappear in all matrix elements for $\hat{R}$.

Let us prove straightforwardly that the $\hat{R}$ constructed commutes with $\hat{H}$. The Super-hamiltonian can be presented similarly to (36):

$$\hat{H} = H^{(0)}P^{(0)} + H^{(2)}P^{(2)} + H^{(1)}_{ik}P^{(1)}_{ik};$$

where $H^{(i)}$ are given by Eqs.(3). By definition $[P^{(0)};P^{(2)}] = [P^{(1)}_{ik};P^{(2)}] = [P^{(1)}_{ik};P^{(0)}] = 0$, and therefore, due to (31), we have:

$$[\hat{H};\hat{R}] = [H^{(1)}_{ij}P^{(1)}_{ij};R^{(1)}_{kl}P^{(1)}_{kl}]:$$

Employing the explicit form (37) of $P^{(i)}$ and Eq.(32)), one can see that the commutator in the rhs vanishes, completing our proof.

From expression (37), its classical limit R_{cl} can be constructed by means of the substitution $\hbar^{1=2}\hat{}_j \; ! \; {}_j$ for fermionic operators. Finally, the total classical integral of motion reads:

$$\begin{aligned} R_{cl} &= R^{(0)}_{cl} + \frac{i}{2}(R^{(1)}_{11\,cl} \quad R^{(1)}_{22\,cl})({}^1_1 {}^2_1 \quad {}^1_2 {}^2_2) + \frac{1}{2}(R^{(1)}_{21\,cl} \quad R^{(1)}_{12\,cl})({}^1_1 {}^1_2 + {}^2_1 {}^2_2) + \\ &\quad + \frac{i}{2}(R^{(1)}_{12\,cl} + R^{(1)}_{21\,cl})({}^2_1 {}^1_2 + {}^1_1 {}^2_2) + (R^{(1)}_{11\,cl} + R^{(1)}_{22\,cl} \quad 2R^{(0)}_{cl}) {}^1_1 {}^2_1 {}^1_2 {}^2_2; \end{aligned} \qquad (38)$$

where its components are:

$$\begin{aligned} R^{(0)}_{cl} &= R^{(2)}_{cl} = p_1^2 \quad p_2^2 + {}^2_1 \quad {}^2_2 \quad 2\,{}_+ \quad {}^2 + 2\,((\,{}_+ + \quad)p_1 \quad (\,{}_+ \quad)p_2)^2; \\ R^{(1)}_{11\,cl} &= p_1^2 + {}^2_1 + \frac{1}{2}(\,{}^2_+ + {}^2) + 2\sqrt{2}\,{}_+ \quad R^{(0)}_{cl}; \\ R^{(1)}_{22\,cl} &= p_2^2 + {}^2_2 + \frac{1}{2}(\,{}^2_+ \quad {}^2) \quad 2\sqrt{2}\,{}_+ \quad R^{(0)}_{cl}; \\ R^{(1)}_{12\,cl} &= ip_1 + {}_1 + \frac{1}{\sqrt{2}}(\,{}_+ + \quad) \quad ip_2 + {}_2 + \frac{1}{\sqrt{2}}(\,{}_+ \quad) \quad R^{(0)}_{cl}; \\ R^{(1)}_{21\,cl} &= ip_2 + {}_2 + \frac{1}{\sqrt{2}}(\,{}_+ \quad) \quad ip_1 + {}_1 + \frac{1}{\sqrt{2}}(\,{}_+ + \quad) \quad R^{(0)}_{cl}: \end{aligned} \qquad (39)$$

5. The flipped potentials and the classical Hamilton-Jacobi equation

In this Section we shall establish links between the Hamilton-Jacobi equations of Classical Mechanics and the equation for the superpotential. Starting from Eq.(21), one can see that the condition for the classical Hamiltonian

$$H_{cl} = p_1^2 + V(\vec{x})$$

to be supersymmetric with superpotential $\Phi(\vec{x})$ takes the form:

$$V(\vec{x}) = \frac{\partial \Phi(\vec{x})}{\partial x_1}\frac{\partial \Phi(\vec{x})}{\partial x_1}: \tag{40}$$

On the other hand, for the Hamiltonian $h_{cl} = p_1^2 + U(\vec{x})$ with potential $U(\vec{x})$ and the classical action functional S, the well known Hamilton-Jacobi equation [27] reads:

$$\frac{\partial S}{\partial t} + h_{cl}\left(\frac{\partial S}{\partial x_1};\frac{\partial S}{\partial x_2};\cdots\frac{\partial S}{\partial x_d};x_1;x_2;\cdots{}_d\right) = 0: \tag{41}$$

There being no explicit dependence on time in h_{cl}, one looks for its solutions of the form $S(t;x_1;x_2;\cdots{}_d) = W(x_1;x_2;\cdots{}_d) - Et$, and the time-independent Hamilton-Jacobi equation becomes:

$$E = (\partial_1 W)^2 + U(\vec{x}); \tag{42}$$

where $W(x_1;x_2;\cdots{}_d)$ is usually referred to as the Hamilton characteristic function. Solutions of (42) in the case $E = 0$ are obviously connected with those of (40):

$$\Phi(\vec{x}) = \pm iW(\vec{x}): \tag{43}$$

Eq.(42), with zero energy, can alternatively be thought of as a condition for the "flipped" classical potential $V = -U$ to be supersymmetric, i.e. V should satisfy (40), with Φ and W related by (43). Thus, we find

$$W(x_1;x_2) = \mp i[\phi_1(x_1) + \phi_2(x_2) + \phi_+(x_+) + \phi_-(x_-)]$$

as the Hamilton characteristic function of the system. The system of equations of motion for $E = 0$

$$\dot{x}_1 = \frac{\partial W}{\partial x_1} = \mp i\left(\phi_1' + \frac{1}{\sqrt{2}}\phi_+' + \frac{1}{\sqrt{2}}\phi_-'\right)$$
$$\dot{x}_2 = \frac{\partial W}{\partial x_2} = \mp i\left(\phi_2' + \frac{1}{\sqrt{2}}\phi_+' - \frac{1}{\sqrt{2}}\phi_-'\right) \tag{44}$$

is not amenable to separation of variables and in general has (non-physical) complex solutions. One may see that system (44) becomes real, and bona fide solutions exist for the specific complexification of the Poschl-Teller model (25): namely, with purely imaginary .
In contrast to the complexification of two-dimensional Morse potential in [28], this one is PT-invariant.

Analogous classical systems with "flipped" potentials were investigated in [26] for the case of $d = 2$ integrable models of the Liouville type. For these systems the Hamilton-Jacobi equations were separable in elliptic, polar, parabolic and Cartesian coordinates. The structure of related supersymmetric models (also with separation of variables) in the quantum domain has been investigated in [19] via canonical quantization. In particular, it was shown that there are two essentially different supersymmetric extensions (two different superpotentials) for a given separable classical solution of the Hamilton-Jacobi equation. In the present paper our strategy is just the opposite. Namely, starting from scalar quantum $d = 2$ systems which do not allow for separation of variables, but do have non-trivial symmetry operators, we construct their quantum SUSY-extension. Then, we describe corresponding classical SUSY-extended systems and their integrals of motion. Finally, the link between this kind of classical system and the Hamilton-Jacobi approach for related systems with "flipped" potential provides the integrability of these "flipped" systems too. The necessary explicit expressions for integrals of motion can be obtained from (38), (39) by the substitution $\to i$, which is equivalent to (43). It should be remarked that (analogously to [19]) besides an arbitrary common sign in (43) there is the additional non-uniqueness of the superpotential for this class of model. Indeed, equation (40) has two independent solutions for fixed original classical potential: $(\vec{x})$ and $\tilde{}(\vec{x})$: To prove this statement, one has to check that $(\partial_1 \ )^2 = (\partial_1 \tilde{\ })^2$ for $= {}_1 + {}_2 + {}_+ +$ and $\tilde{\ } = {}_1 + {}_2 \quad {}_+$ (see (17)) due to the nonlinear equation (16).

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