Effect of scalings and translations on the supersymmetric quantum mechanical structure of soliton systems

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Abstract

We investigate a peculiar supersymmetry of the pairs of reflectionless quantum mechanical systems described by n-soliton potentials of a general form that depends on n scaling and n translation parameters. We show that if all the discrete energy levels of the subsystems are different, the superalgebra, being insensitive to translation parameters, is generated by two supercharges of differential order $2n$, two supercharges of order $2n+1$, and two bosonic integrals of order $2n+1$ composed from Lax integrals of the partners. The exotic supersymmetry undergoes a reduction when r discrete energy levels of one subsystem coincide with any r discrete levels of the partner, the total order of the two independent intertwining generators reduces then to $4n-2r+1$, and the nonlinear superalgebraic structure acquires a dependence on r relative translations. For a complete pairwise coincidence of the scaling parameters which control the energies of the bound states and the transmission scattering amplitudes, the emerging isospectrality is detected by a transmutation of one of the Lax integrals into a bosonic central charge. Within the isospectral class, we reveal a special case giving a new family of finite-gap first order Bogoliubov-de Gennes systems related to the AKNS integrable hierarchy.

1 Introduction

Solitons and related topologically nontrivial objects such as kinks, instantons, vortices, monopoles and domain walls play an important role in diverse areas of physics, engineering and biology [1, 2, 3]. Darboux and Bäcklund transformations, with their origin in the theory of the linear Sturm-Liouville problem and classical differential geometry, proved to be very effective in their study [4, 5]. Darboux transformations [4], on the other hand, underlie the construction of supersymmetric quantum mechanics [6, 7]. Via the Bogomolny bound and the associated first order Bogomolny-Prasad-Sommerfield equations [8, 9], supersymmetry, in turn, turns out to be closely related with the topological solitons [10, 11, 12].

Solitons and their periodic analogs appear as solutions of classical nonlinear integrable field equations, and by means of Lax representation [13] are related with reflectionless and periodic finite-gap quantum systems [14, 15]. As both families of quantum systems are characterized by nontrivial, higher derivative integrals of motion, one could expect that supersymmetric extensions of them should possess some peculiar properties. This is indeed the case [16, 17, 18, 19, 20, 21], and exotic supersymmetric structures of reflectionless and finite-gap systems found recently some interesting physical applications [22, 23, 24, 25, 26].

The most known example of reflectionless systems is given by a hierarchy of Pöschl-Teller potentials. The Schrödinger Hamiltonian with one-, two-, and, in general, n bound states Pöschl-Teller reflectionless potentials controls, particularly, the stability of kinks in sine-Gordon, φ^4 and other exotic (1+1)-dimensional field theory models [1, 3, 27, 28, 29, 30, 31, 32]. These systems also appear in Gross-Neveu model [33, 34]. The indicated hierarchy represents, however, only a very restricted case of a general family of n-soliton potentials. The latter corresponds to $2n$-parametric solutions of the Korteweg-de Vries (KdV) equation [2, 4, 35].

More explicitly, the Schrödinger operator is at the heart of the inverse scattering transform method of solving the classical KdV equation, for which the reflectionless potentials V_n provide the particle-like, n-soliton solutions. On the other hand, the Schrödinger Hamiltonians $H = -\frac{d^2}{dx^2} + V_n$ with reflectionless potentials V_n control the stability of the above mentioned kink solutions in (1+1)-dimensional field theories, and their certain supersymmetric quantum mechanical structure proved particularly to be very useful in the computing of the kink mass quantum shifts, see ref. [36].

In the present paper we study the exotic supersymmetry that appears in the pairs of reflectionless systems described by n-soliton potentials of the most general form. Namely, we investigate a peculiar supersymmetric quantum mechanical structure of the class of one-dimensional systems described by a matrix 2×2 Hamiltonian

$$\mathcal{H} = \begin{pmatrix} -\frac{d^2}{dx^2} + V_+(x) & 0 \\ 0 & -\frac{d^2}{dx^2} + V_-(x) \end{pmatrix}, \qquad (1.1)$$

with

$$V_+(x) = V_n(x, \vec{\kappa}, \vec{\tau}) \quad \text{and} \quad V_-(x) = V_n(x, \vec{\kappa}\,', \vec{\tau}\,') \qquad (1.2)$$

to be n-soliton solutions of the KdV equation, each depending on the sets of n scaling parameters, denoted here as $\vec{\kappa}$ and $\vec{\kappa}\,'$, and n translation parameters, $\vec{\tau}$ and $\vec{\tau}\,'$. One of the possible (but not unique, see below) physical interpretations of the system (1.1), (1.2) is that it can be considered as a Hamiltonian of non-relativistic spin-1/2 particle with spin-dependent forces of a special form (not inducing spin flips).

A non-soliton system of a general form (1.1), with arbitrary chosen potentials $V_+(x)$ and $V_-(x)$, has just a trivial integral given by the diagonal Pauli matrix σ_3. For a special choice of potentials $V_\pm = W^2(x) \pm \frac{dW}{dx}$, this trivial symmetry is extended for supersymmetric structure related to non-trivial additional integrals of motion $Q_1 = -i\frac{d}{dx}\sigma_1 + \sigma_2 W(x)$, $Q_2 = i\sigma_3 Q_1$. They generate a linear in $\mathcal{H}$, Lie superalgebraic structure $\{Q_a, Q_b\} = 2\delta_{ab}\mathcal{H}$, $[\mathcal{H}, Q_a] = 0$, $a, b = 1, 2$, with the integral σ_3 playing a role of the $\mathbb{Z}_2$-grading operator, $[\sigma_3, \mathcal{H}] = 0$, $\{\sigma_3, Q_a\} = 0$. It is such a linear superalgebraic structure that appears, particularly, in the Landau problem for non-relativistic electron, where superpotential is a linear function $W(x) = \omega x$, and (1.1) takes a form of the superoscillator Hamiltonian, see [7]. The existence of the linear supersymmetric structure is equivalent to the condition that the upper and lower components of the matrix Hamiltonian, $H_\pm = -\frac{d^2}{dx^2} + V_\pm$, are related by the Darboux intertwining generators, $H_+A_+ = A_+H_-$, $H_-A_- = A_-H_+$, being the first order differential operators $A_+ = \frac{d}{dx} + W(x)$ and $A_- = A_+^\dagger = -\frac{d}{dx} + W(x)$. With this observation, the construction can be generalized to nonlinear supersymmetry if the potentials V_+ and V_- are such that the corresponding partner Hamiltonians are connected by the intertwining relations of the same form, but with A_+ and $A_- = A_+^\dagger$ to be differentials operators of order $\ell > 1$. If this happens, the system $\mathcal{H}$ possesses nilpotent supercharges $Q_+ = A_+\sigma_+ = \frac{1}{2}(Q_2 + iQ_1)$ and $Q_- = A_-\sigma_- = Q_+^\dagger$, $[Q_\pm, H] = 0$, $Q_\pm^2 = 0$, where $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$. They generate a nonlinear supersymmetry of the form $\{Q_a, Q_b\} = 2\delta_{ab}P_\ell(\mathcal{H})$, where $P_\ell(\mathcal{H})$ is an order ℓ polynomial. The simplest example of a system with nonlinear supersymmetry is provided by a generalized superoscillator system $\mathcal{H} = b^+b^- + \ell\frac{1}{2}(1 + \sigma_3)$, for which $A_+ = (b^-)^\ell$, $b^\pm$ are the usual creation-annihilation bosonic oscillator operators, and the order ℓ polynomial is $P_\ell(\mathcal{H}) = \prod_{j=0}^{\ell-1}(H - j\omega)$, see ref. [37].

The peculiarity of the system (1.1), (1.2) we study here is that the n-soliton potentials (1.2) are reflectionless. By a known construction based on Crum-Darboux transformations, such potentials can be obtained from a free particle system, which possesses a momentum integral $p = -i\frac{d}{dx}$. It will be shown that, as a consequence, the n-soliton extended system is described by an exotic supersymmetric structure that includes not only one but two pairs of Z_2-odd (anti-diagonal) matrix supercharges, and two Z_2-even (diagonal) additional nontrivial bosonic integrals being differential operators of order $2n + 1$. The supercharges in general case are higher order matrix differential operators, two of which are of the even order $2r$, and other two supercharges are of the odd order $2l + 1$ such that $2(r + l) \geq 2n$. Corresponding superalgebra generated by four supercharges is nonlinear, and includes in its structure those additional nontrivial bosonic integrals of motion which are nothing else as a Crum-Darboux dressed form of the free particle momentum operator. The supercharges also have a nature of the dressed integrals of motion of the free spin-1/2 particle described by the Hamiltonian (1.1) with $V_+ = V_- = 0$. We shall show that such a peculiar supersymmetric structure of the extended n-soliton systems experiences radical changes in dependence on relation between the two sets of the scaling and translation parameters of the partner potentials: the differential order of supercharges can change, and in the completely isospectral case when $\vec{\kappa} = \vec{\kappa}\,'$, one of the additional bosonic integrals transforms into the central charge of the corresponding nonlinear superalgebra. Analyzing different faces of supersymmetry restructuring, we detect, particularly, a special family of supersymmetric n-soliton partner potentials when one pair of supercharges reduces to the matrix first order differential operators. These first order supercharges and $\mathcal{H}$ form between themselves a linear superalgebra corresponding to the broken supersymmetry. In such a case, one of the first order supercharges can be reinterpreted as a first order Hamiltonian of a Dirac particle. The reinterpretation provides us then with new kink-anti-kink type solutions for the Gross-Neveu model by means of the first order Bogoliubov-de Gennes system, in which a superpotential takes a meaning of a condensate, an order parameter, or a gap function depending on the physical context.

The paper is organized as follows. In the next Section, we review the general construction of soliton potentials with the help of Crum-Darboux transformations, summarize the basic properties of the corresponding reflectionless quantum systems, and formulate precisely the problems related to supersymmetry of soliton systems (1.1), (1.2) to be studied here. Section 3 is devoted to the analysis of supersymmetry of non-isospectral pairs of reflectionless $n = 1$ systems with different bound state energy levels given in terms of non-equal scaling parameters $\kappa_1 \neq \kappa_1'$. In Section 4 we investigate the changes this supersymmetric structure undergoes in the isospectral case $\kappa_1 = \kappa_1'$. Section 5 generalizes the results of Section 3 for the case of $n > 1$-soliton pairs with completely broken isospectrality. To clarify the supersymmetry picture in extended $n > 1$ systems with partially broken and exact isospectralities, we study in detail the case of $n = 2$ in Section 6. In Section 6.1 we review the properties of the generic $n = 2$ reflectionless systems to identify the ingredients to be important for further analysis. Then, in Section 6.2, we discuss a generalization of Crum-Darboux transformations that is related to alternative factorizations of the basic Crum-Darboux generators of order $n > 1$. The results of Sections 6.1 and 6.2 are employed in Sections 6.3 and 6.4 for analysis of supersymmetry in extended $n = 2$ systems with partial isospectrality breaking. Finally, in Sections 6.5, 6.6 and 6.7 we investigate the most tricky case of supersymmetry in two-soliton extended systems with exact isospectrality. We do this first in Section 6.5 for a particular case of exact isospectrality with a common virtual $n = 1$ subsystem. In Section 6.6 we investigate a generic case of exact isospectrality, within which we detect yet another, very special, particular case. The latter is studied in Section 6.7, and provides us with a new, first order finite-gap system belonging to the AKNS hierarchy [38, 15]. In Section 7 we discuss how the results on partially broken and exact isospectralities are generalized for the systems (1.1), (1.2) with $n > 2$. In Section 8 we consider an interpretation of the system (1.1), (1.2) as a non-relativistic spin-1/2 particle with spin-dependent forces. We conclude the paper with discussion of the obtained results and their possible developments and applications in Section 9.

2 Family of reflectionless n-soliton systems

A Crum-Darboux transformation of order n, $n = 1, 2, \ldots$, applied to a quantum free particle generates a system characterized by the Hamiltonian [4]

$$H_n = H_0 + V_n(x), \qquad V_n = -2\frac{d^2}{dx^2}\ln W_n\,. \tag{2.1}$$

Here $H_0 = -\frac{d^2}{dx^2}$ is a free particle Hamiltonian, and $W_n = W(\psi_1, \ldots, \psi_n)$ is a Wronskian of its eigenfunctions $\psi_1(x)$, ..., $\psi_n(x)$, $H_0\psi_j = E_j^{(0)}\psi_j$,

$$W(f_1, \ldots, f_n) = \det \mathcal{A}, \qquad \mathcal{A}_{ij} = \frac{d^{i-1}}{dx^{i-1}}f_j, \qquad i, j = 1, \ldots, n. \tag{2.2}$$

A simple choice of $\psi_j(x)$ in the form of the unidirectional plane waves e^{ik_jx}, which are eingensolutions of H_0, produces the Wronskian of the form $W_n(x) = const \cdot e^{i(k_1+\ldots+k_n)x}$, and, therefore, $V_n = 0$. If we take a linear independent set of linear combinations of left- and right- moving plane waves $\psi_j(x) = e^{ik_jx} + c_je^{-ik_jx}$ with $c_j \neq 0$ for all $j = 1, \ldots, n$, we obtain a nontrivial potential $V_n \not\equiv 0$, which satisfies a higher order stationary g-KdV, $g = 2n + 1$, (Novikov) equation being a nonlinear ordinary differential equation with a linear highest derivative d^gV_n/dx^g term [39, 40]. (2.1) belongs then to a class of finite-gap, or algebro-geometric systems [1]. For real k_j, the emergent 'finite-gap' potential $V_n(x)$ has, however, singularities on $\mathbb{R}$ and does not disappear at $x = \pm\infty$. An appropriate choice of the free particle non-physical eigenfunctions (corresponding to certain linear combinations of the left- and right- moving plane waves evaluated at imaginary momenta),

$$\psi_j = \begin{cases} \cosh \kappa_j(x + \tau_j), & j = \text{odd} \\ \sinh \kappa_j(x + \tau_j), & j = \text{even} \end{cases}, \qquad 0 < \kappa_1 < \kappa_2 < \ldots < \kappa_{j-1} < \kappa_n\,, \tag{2.3}$$

of energies $E_j^{(0)} = -\kappa_j^2$, $j = 1, \ldots, n$, gives rise to a nodeless Wronskian $W_n(x)$. A non-singular $2n$-parametric potential

$$V_n = V_n(x; \kappa_1, \tau_1, \ldots, \kappa_n, \tau_n) \tag{2.4}$$

corresponds then to a *reflectionless* (Bargmann) system H_n with $n + 1$ non-degenerate states, separated by n gaps, n of which, of energies $E_j^{(n)} = -\kappa_j^2$, $j = 1, \ldots, n$, are the bound states, while the non-degenerate state of zero energy, $E = 0$, lies at the bottom of the doubly degenerate continuous spectrum with $E > 0$. From another perspective, reflectionless potential $V_n(x; \kappa_1, \tau_1, \ldots, \kappa_n, \tau_n)$ describes n-soliton solutions of the KdV equation.

[1]Finite-gap *periodic* systems are given by the Its-Matveev representation of the form (2.1) but with $W(x)$ substituted by a Riemann's theta function [41]. If such a periodic potential is real and regular on $\mathbb{R}$, the spectrum of Schrödinger (Hill) operator is organized in valence and a conductance bands separated by gaps. (2.1) with reflectionless, n-soliton potential (2.4) can be considered then as the infinite period limit of a periodic or almost periodic finite-gap system. In the indicated limit, the valence bands shrink, some of which can merge in this process, and transform into the non-degenerate discrete energy levels of the bound states of a resulting soliton potential; the semi-infinite conductance band turns into the continuous part of the spectrum of a reflectionless system. Quantum systems with periodic n-gap and non-periodic n-soliton potentials (whose discrete energy levels and continuous spectrum are also separated by n gaps) are characterized by the existence of the differential operator of order $2n+1$, related with a higher order Novikov equation, that commutes with a Hamiltonian, see below. A free particle can be treated in this picture as a zero-gap system (of an arbitrary period), for which the corresponding first order differential operator is just the momentum integral $p = -i\frac{d}{dx}$. For the theory of finite-gap and soliton systems including historical aspects, see [14, 42].

Eigenstates of H_n, $H_n\psi[n;\lambda]=\lambda\psi[n;\lambda]$, different from the physical bound states, are generated from eigenfunctions $\psi[0;\lambda]$ of the free particle, $H_0\psi[0;\lambda]=\lambda\psi[0;\lambda]$, $\lambda\neq-\kappa_j^2$,

$$\psi[n;\lambda]=\frac{W(\psi_1,\ldots,\psi_n,\psi[0;\lambda])}{W(\psi_1,\ldots,\psi_n)}\,, \tag{2.5}$$

where ψ_j are given by Eq. (2.3). Physical non-degenerate bound states of H_n with $\lambda=-\kappa_j^2$ are obtained by the same prescription (2.5) under the choice $\psi[0;\lambda]=\sinh\kappa_j(x+\tau_j)$ for odd j, and $\psi[0;\lambda]=\cosh\kappa_j(x+\tau_j)$ for even j. The lowest non-degenerate state of the continuous part of the spectrum of H_n corresponds to the eigenstate $\psi[0;0]=1$ of H_0.

Transmission scattering amplitudes $a[n;k]$ for the continuous part of the spectrum $E=k^2$, $k>0$, of reflectionless system H_n are defined by the scaling parameters κ_j [4],

$$a[n;k]=\prod_{j=1}^{n}\frac{k-i\kappa_j}{k+i\kappa_j}\,. \tag{2.6}$$

The states (2.5) have an alternative but equivalent representation, $\psi[n;\lambda]=A_n\ldots A_1\psi[0;\lambda]$, generated by an n-sequence of the first order Darboux transformations,

$$\psi[j;\lambda]\equiv\psi[(\kappa,\tau)_{(j)};\lambda]=A_j\psi[j-1;\lambda]\,, \tag{2.7}$$

where $(\kappa,\tau)_{(j)}$ denotes the set of $2j$ parameters $\kappa_1,\tau_1,\ldots,\kappa_j,\tau_j$, and $A_j=A_j[(\kappa,\tau)_{(j)}]$ are the first order differential operators defined recursively in terms of the states (2.3) by

$$A_1=\psi_1\frac{d}{dx}\frac{1}{\psi_1}=\frac{d}{dx}-\kappa_1\tanh\kappa_1(x+\tau_1)\,, \tag{2.8}$$

$$A_j=(A_{j-1}\ldots A_1\psi_j)\frac{d}{dx}\frac{1}{(A_{j-1}\ldots A_1\psi_j)}=\frac{d}{dx}-\left(\frac{d}{dx}\ln(A_{j-1}\ldots A_1\psi_j)\right)\,. \tag{2.9}$$

The first order operator A_j annihilates the state $A_{j-1}\ldots A_1\psi_j$, that is a nonphysical eigenstate of H_{j-1} of eigenvalue $-\kappa_j^2$. As inverse to (2.7), there is, up to an overall multiplicative constant, a relation

$$\psi[j-1;\lambda]=A_j^\dagger\psi[j;\lambda]\,. \tag{2.10}$$

The zero mode of the first order operator $A_j^\dagger$ is $1/(A_{j-1}\ldots A_1\psi_j)$. It is the ground state of H_j of the energy $-\kappa_j^2$.

A reflectionless j-soliton Hamiltonian H_j admits two factorization representations

$$H_j=A_{j+1}^\dagger A_{j+1}-\kappa_{j+1}^2=A_jA_j^\dagger-\kappa_j^2\,. \tag{2.11}$$

In particular, the free particle 0-gap Hamiltonian $H_0=-\frac{d^2}{dx^2}$ has an alternative representation $H_0=A_1^\dagger A_1-\kappa_1^2$. From (2.11) there follow intertwining relations

$$A_jH_{j-1}=H_jA_j\,,\qquad A_j^\dagger H_j=H_{j-1}A_j^\dagger\,,\qquad j=1,\ldots,n\,. \tag{2.12}$$

Let us take now a pair of n-soliton reflectionless systems,

$$H_n=H_n(\kappa_1,\tau_1,\ldots,\kappa_n,\tau_n)\quad\text{and}\quad H_n'=H_n(\kappa_1',\tau_1',\ldots,\kappa_n',\tau_n')\,, \tag{2.13}$$

and consider the extended matrix 2×2 Hamiltonian of the form (1.1) with $H_+=H_n$ and $H_-=H_n'$. Two sets of parameters are supposed to be completely different, or may partially coincide. If the two sets of the scaling parameters κ_j, $j = 1, \ldots, n$, and $\kappa'_{j'}$, $j' = 1, \ldots, n$, do not coincide, the two subsystems have not only different spectra of bound states, but in accordance with (2.6), their transmission amplitudes are also different. If, moreover, $\kappa_j \neq \kappa'_{j'}$ for all $j, j' = 1, \ldots, n$, all the energy levels of bound states for two n-soliton reflectionless systems are different, and their transmission amplitudes are given by rational functions of k with different zeroes and poles. Having in mind that the factorization relations (2.11) and the associated intertwining relations (2.12) are reformulated in terms of supersymmetric quantum mechanics construction, one can put a question:

- What a supersymmetric structure is associated with reflectionless pair (2.13) in a *completely non-isospectral* case[2] characterized by inequalities $\kappa_j \neq \kappa'_{j'}$ for all $j, j' = 1, \ldots, n$?

Such a kind of supersymmetry of the pairs of reflectionless systems was not investigated yet in the literature, but, instead, supersymmetry of the pairs ($H_+ = H_j$, $H_- = H_{j+l}$), $l \geq 1$, belonging to the same Darboux chain (2.12) is usually considered. In particular, the pairs of reflectionless Pöschl-Teller systems, see below, appear in the context of shape-invariance [43, 44, 7], they also emerge in the infinite-period limit of finite-gap periodic crystal structures [22, 24]. Supersymmetry of reflectionless Pöschl-Teller pairs (H_j, H_{j+l}) was studied recently from the perspective of AdS/CFT holography and Aharonov-Bohm effect [45].

A special choice of the parameteres

$$\kappa_j = \kappa'_j = j\kappa, \qquad \tau_j = \tau, \qquad \tau'_j = \tau', \quad j = 1, \ldots, n\,, \tag{2.14}$$

results in two copies of the n-soliton potentials $V_n = -n(n+1)\kappa^2 \mathrm{sech}^2 \kappa(x+\tau)$ and $V'_n = -n(n+1)\kappa^2 \mathrm{sech}^2 \kappa(x+\tau')$, which describe two mutually shifted reflectionless Pöschl-Teller systems with n bound states. Since the partner potentials under the choice (2.14) have exactly the same form, this corresponds to a particular case of a shape-invariance, whose analog in the case of periodic supersymmetric systems was called by Dunne and Feinberg '*self-isospectrality*' [17]. The exotic nonlinear supersymmetry of the simplest isospectral pair ($H_+ = H_1$, $H_- = H'_1$) with $\kappa_1 = \kappa'_1$, $\tau_1 \neq \tau'_1$ was investigated and applied for the description of the kink and kink-anti-kink solutions of the Gross-Neveu model [46, 24]. One can expect that the self-isospectral pair of reflectionless Pöschl-Teller systems with $n > 1$ bound states should also be described by some not studied yet exotic nonlinear supersymmetric structure.

In a more general case of the choice $\kappa_j = \kappa'_j$, $j = 1, \ldots, n$, different from (2.14), the partners with $\vec{\tau} \neq \vec{\tau}\,'$, $\vec{\tau} = (\tau_1, \ldots, \tau_n)$, are completely isospectral, their bound states energies and transmission amplitudes coincide, but the potentials have different form. We then arrive at the natural questions related to that formulated above:

- How the supersymmetric structure of a general, non-isospectral case detects the coincidence of some of the scaling parameters of two systems in (2.13)?
- Particularly, for a partial coincidence of the bound states energy levels, does the supersymmetry distinguish the coincidence of the scaling parameters of the same level, $\kappa_j = \kappa'_j$, from that corresponding to the case when distinct levels, $\kappa_j = \kappa'_{j'}$ with $j \neq j'$, coincide?
- Is the case of a complete isospectrality of the two systems, $\kappa_j = \kappa'_j$, $j = 1, \ldots, n$, detected somehow by supersymmetric structure?
- Does the case of self-isospectrality possess some special characteristics from the viewpoint of supersymmetry in comparison with a general case of isospectral systems with different form of potentials?

[2] Using this term we neglect the fact that the continuous (scattering) parts of the spectra of the partner systems are the same, $E \geq 0$.

In what follows, we study a peculiar supersymmetric structure of the pair (2.13), and, particularly, respond the highlighted questions.

3 Supersymmetry of $n = 1$ reflectionless pair with distinct scalings

We first investigate the supersymmetric structure of the extended system

$$\mathcal{H}_1 = \begin{pmatrix} H_1 & 0 \\ 0 & H_1' \end{pmatrix} \tag{3.1}$$

described by the pair of $n = 1$ reflectionless Pöschl-Teller Hamiltonians $H_1 = H_1(\kappa, \tau)$ and $H_1' = H_1(\kappa', \tau')$ with $\kappa \neq \kappa'$ and arbitrary displacement parameters τ and τ'. This will allow us to trace how the restructuring of supersymmetry happens in the self-isospectral case $\kappa = \kappa'$, and to form a base for further analysis for $n > 1$, where we will restore index 1, omitted here to simplify notations, in the scaling and translation parameters.

The choice of a non-physical eigenstate $\psi_1(\kappa, \tau) = \cosh \kappa(x + \tau)$, $\kappa > 0$, $\tau \in \mathbb{R}$, of H_0 produces a Hamiltonian of $n = 1$ reflectionless Pöschl-Teller system

$$H_1 = -\frac{d^2}{dx^2} - \frac{2\kappa^2}{\cosh^2 \kappa(x + \tau)}\,, \tag{3.2}$$

and first order operators A_1 and $A_1^\dagger$ defined by Eq. (2.8). Operators A_1 and $A_1^\dagger$ factorize the shifted for an additive constant Hamiltonians H_0 and H_1,

$$H_1 = A_1 A_1^\dagger - \kappa^2, \qquad H_0 = A_1^\dagger A_1 - \kappa^2\,, \tag{3.3}$$

and intertwine them,

$$A_1^\dagger H_1 = H_0 A_1^\dagger\,, \qquad A_1 H_0 = H_1 A_1\,. \tag{3.4}$$

A degenerate pair of eigenstates in the continuous part, $E = k^2$, $k > 0$, of the spectrum of H_1 is constructed from the free particle plane wave states,

$$\psi_1^{\pm k} = A_1(\kappa, \tau) e^{\pm ikx} = (\pm ik - \kappa \tanh \kappa(x + \tau)) e^{\pm ikx}\,. \tag{3.5}$$

The lowest non-degenerate state with $E = 0$ corresponds to a boundary case $k = 0$ of (3.5),

$$\psi_1^0 = \tanh \kappa(x + \tau)\,. \tag{3.6}$$

Another, bound non-degenerate state

$$\psi_1^{-\kappa^2} = \kappa\, \mathrm{sech}\, \kappa(x + \tau) \tag{3.7}$$

of energy $E = -\kappa^2$ is obtained from the partner, $\tilde{\psi}_1(\kappa, \tau) = \sinh \kappa(x+\tau)$, of non-physical eigenstate $\psi_1(\kappa, \tau) = \cosh \kappa(x + \tau)$ of H_0, $\psi_1^{-\kappa^2}(\kappa, \tau) = A_1(\kappa, \tau)\tilde{\psi}_1(\kappa, \tau)$.

Based on intertwining relations (3.4) and their analog for the system $H_1' = H_1(\kappa', \tau')$, we construct the second order operator

$$Y_2 = Y_2(\kappa, \tau, \kappa', \tau') = A_1(\kappa, \tau) A_1^\dagger(\kappa', \tau') = A_1 {A_1'}^\dagger\,, \qquad Y_2^\dagger = Y_2(\kappa', \tau', \kappa, \tau) = Y_2'\,, \tag{3.8}$$

that intertwines the partner Hamiltonians of the extended system (3.1), $Y_2 H_1' = H_1 Y_2$. Taking into account that H_0 has an integral $p = -i\frac{d}{dx}$, one can obtain yet another, third order intertwining operator,

$$X_3 = X_3(\kappa, \tau, \kappa', \tau') = A_1 \frac{d}{dx} {A_1'}^\dagger\,, \qquad X_3^\dagger(\kappa, \tau, \kappa', \tau') = -X_3(\kappa', \tau', \kappa, \tau) = -X_3'\,, \tag{3.9}$$

$X_3 H_1' = H_1 X_3$, which is independent from the second order intertwiner Y_2.

Intertwining relations in the reverse direction are obtained by a change $\kappa, \tau \leftrightarrow \kappa', \tau'$, that corresponds to a Hermitian conjugation of the corresponding relations, $Y_2^\dagger H_1 = H_1' Y_2^\dagger$, $X_3^\dagger H_1 = H_1' X_3^\dagger$, see Fig. 1a.

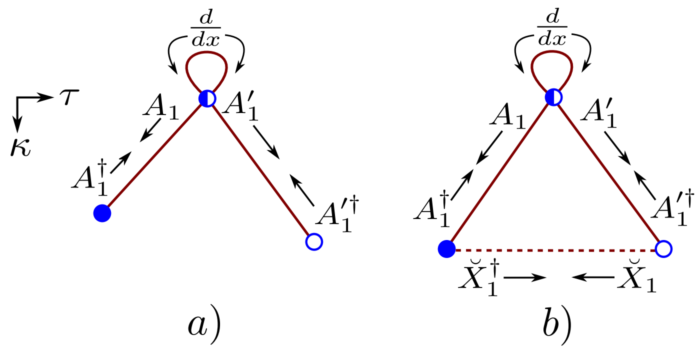


Figure 1: a) Non-isospectral one-soliton Hamiltonians H_1 (blue dot) and H_1' (white dot) are intertwined by the second, Y_2 and $Y_2^\dagger$, and the third, X_3 and $X_3^\dagger$, order Crum-Darboux operators via a virtual translation-invariant free particle system H_0 (half blue/half white dot). b) In the isospectral case $\kappa = \kappa'$, a direct 'tunneling' channel for intertwining by the first order operators $\breve{X}_1$ and $\breve{X}_1^\dagger$ is opened. In both cases, Lax integrals Z_3 and Z_3', being the dressed forms of the free particle integral $\frac{d}{dx}$, are the 'self-intertwining' generators for H_1 and H_1'.

The free particle integral $p = -i\frac{d}{dx}$ and intertwining relations (3.4) also generate a nontrivial integral for the $n = 1$ reflectionless Pöschl-Teller subsystem $H_1(\kappa, \tau)$,

$$Z_3 = Z_3(\kappa, \tau) = A_1 \frac{d}{dx} A_1^\dagger \,, \qquad Z_3^\dagger = -Z_3 \,, \tag{3.10}$$

and the analogous integral, $Z_3' = A_1' \frac{d}{dx} A_1'^\dagger$, for $H_1(\kappa', \tau')$. Integral (3.10) is a nontrivial operator of a Lax pair for stationary KdV equation in the non-periodic case.

Here and in what follows, the odd and even order intertwining operators are denoted by X and Y, respectively, while the odd order integrals of the corresponding reflectionless systems are denoted by Z; the lower index indicates the differential order of these operators.

Integral (3.10) detects both physical non-degenerate states of $H_1(\kappa, \tau)$ by annihilating them $Z_3 \psi_1^0(\kappa, \tau) = Z_3 \psi_1^{-\kappa^2}(\kappa, \tau) = 0$. The third state of its kernel is a non-physical eigenstate $\tilde{\psi}_1^{-\kappa^2}(x) = \psi_1^{-\kappa^2}(x) \int dx / (\psi_1^{-\kappa^2}(x))^2$ of H_1 of energy $-\kappa^2$, which is a linear combination of the physical bound state $\psi_1^{-\kappa^2}(x)$ of the same energy and of a non-physical eigenstate $\psi_1(\kappa, \tau) = \cosh \kappa(x + \tau)$ of H_0.

The extended system (3.1) has an obvious integral of motion σ_3. The intertwining relations together with integral (3.10) allow us to identify the nontrivial Hermitian integrals for the system $\mathcal{H}_1$,

$$\mathcal{Q}_{1;1} = \begin{pmatrix} 0 & Y_2 \\ Y_2^\dagger & 0 \end{pmatrix}, \quad \mathcal{Q}_{1;2} = i\sigma_3 \mathcal{Q}_{1;1}, \qquad \mathcal{S}_{1;1} = \begin{pmatrix} 0 & X_3 \\ X_3^\dagger & 0 \end{pmatrix}, \quad \mathcal{S}_{1;2} = i\sigma_3 \mathcal{S}_{1;1}, \tag{3.11}$$

$$\mathcal{P}_{1;1} = -i \begin{pmatrix} Z_3 & 0 \\ 0 & Z_3' \end{pmatrix}, \qquad \mathcal{P}_{1;2} = \sigma_3 \mathcal{P}_{1;1} \,. \tag{3.12}$$

As $\sigma_3^2 = \mathbb{1}$, we can take the integral $\Gamma = \sigma_3$ as a $\mathbb{Z}_2$-grading operator. It classifies then $\mathcal{P}_{1;a}$, $a = 1, 2$, as bosonic integrals, $[\sigma_3, \mathcal{P}_{1;a}] = 0$, while the integrals (3.11) are identified as fermionic supercharges, $\{\sigma_3, \mathcal{Q}_{1;a}\} = \{\sigma_3, \mathcal{S}_{1;a}\} = 0$, of the supersymmetric structure of the extended system $\mathcal{H}_1$. There are other possibilities to choose Γ, which are based on reflection operators and classify the nontrivial integrals of the extended system in a way different from that prescribed by the choice $\Gamma = \sigma_3$. The alternative choices for Γ find some interesting physical applications, see [22, 24, 46, 47], and we return to the discussion of this point in the last Section.

Operators (3.11) and (3.12) are the Darboux-dressed integrals of the extended system described by the Hamiltonian $\mathcal{H}_0 = \text{diag}(H_0, H_0)$ composed from two copies of the free particle Hamiltonian H_0. The system $\mathcal{H}_0$ possesses the set of 2×2 matrix Hermitian integrals

$$\mathcal{I}_0 = \sigma_a,\ \ \epsilon_{ab}\sigma_b p,\ \ \mathbb{1}p,\ \ \sigma_3 p, \qquad a=1,2\,. \tag{3.13}$$

The Darboux dressing,

$$\mathcal{I}_1 = \mathcal{D}_1\mathcal{I}_0\mathcal{D}_1^\dagger\,, \qquad \mathcal{D}_1 = \text{diag}\left(A_1(\kappa,\tau), A_1(\kappa',\tau')\right), \tag{3.14}$$

transforms them into the integrals (3.11) and (3.12) of $\mathcal{H}_1$.

We find the superalgebraic structure of the system $\mathcal{H}_1$ by employing the intertwining and factorization relations (3.4) and (3.3). It is given by the following nontrivial (anti)-commutation relations:

$$\{\mathcal{Q}_a, \mathcal{Q}_b\} = 2\delta_{ab}\mathbb{P}_1(\mathcal{H}_1,\kappa)\mathbb{P}_1(\mathcal{H}_1,\kappa')\,, \quad \{\mathcal{S}_a, \mathcal{S}_b\} = 2\delta_{ab}\mathcal{H}_1\mathbb{P}_1(\mathcal{H}_1,\kappa)\mathbb{P}_1(\mathcal{H}_1,\kappa')\,, \tag{3.15}$$

$$\{\mathcal{S}_a, \mathcal{Q}_b\} = 2\epsilon_{ab}\mathbb{P}_1(\mathcal{H}_1,\mathcal{K})\mathcal{P}_1\,, \tag{3.16}$$

$$[\mathcal{P}_1, \mathcal{S}_a] = i\mathcal{H}_1\mathbb{P}_0^-(\mathcal{H}_1,\kappa,\kappa')\mathcal{Q}_a\,, \qquad [\mathcal{P}_1, \mathcal{Q}_a] = -i\mathbb{P}_0^-(\mathcal{H}_1,\kappa,\kappa')\mathcal{S}_a\,, \tag{3.17}$$

$$[\mathcal{P}_2, \mathcal{S}_a] = i\mathcal{H}_1\mathbb{P}_1^+(\mathcal{H}_1,\kappa,\kappa')\mathcal{Q}_a\,, \qquad [\mathcal{P}_2, \mathcal{Q}_a] = -i\mathbb{P}_1^+(\mathcal{H}_1,\kappa,\kappa')\mathcal{S}_a\,, \tag{3.18}$$

where $\mathbb{P}_1(\mathcal{H}_1,\kappa) = \mathcal{H}_1 + \kappa^2\cdot\mathbb{1}$, $\mathbb{P}_1(\mathcal{H}_1,\mathcal{K}) = \mathcal{H}_1 + \mathcal{K}^2$, $\mathcal{K} = \text{diag}\,(\kappa',\kappa)$,

$$\mathbb{P}_0^-(\mathcal{H}_1,\kappa,\kappa') = \mathbb{P}_1(\mathcal{H}_1,\kappa) - \mathbb{P}_1(\mathcal{H}_1,\kappa') = (\kappa^2 - \kappa'^2)\cdot\mathbb{1}\,, \tag{3.19}$$

$\mathbb{P}_1^+(\mathcal{H}_1,\kappa,\kappa') = \mathbb{P}_1(\mathcal{H}_1,\kappa) + \mathbb{P}_1(\mathcal{H}_1,\kappa') = 2\mathcal{H}_1 + (\kappa^2+\kappa'^2)\cdot\mathbb{1}$, and to simplify the formulae, we omitted the index $n=1$ in the supercharges and bosonic integrals. Though in the final expression for $\mathbb{P}_0^-$ in (3.19) the dependence on $\mathcal{H}_1$ disappears, it is indicated here in the arguments having in mind a further generalization for the $n>1$ case, where this structure is substituted for the polynomial of order $n-1$ in Hamiltonian.

The $n=1$ extended reflectionless system (3.1) is described therefore by a nonlinear superalgebra generated by four fermionic supercharges, $\mathcal{Q}_{1;a}$ and $\mathcal{S}_{1;a}$, and by two bosonic integrals [3], $\mathcal{P}_{1;a}$. The fermionic integrals are constructed from the intertwining operators of the second and third orders, whose composition produces nontrivial third order integrals of Lax pairs of the $n=1$ non-isospectral subsystems. In this supersymmetric structure, Hamiltonian plays a role of the multiplicative central charge. The nonlinear superalgebra depends here on the scaling parameters κ and κ' via the polynomials $\mathbb{P}_1$, $\mathbb{P}_1^+$ and $\mathbb{P}_0^-$, but does not depend on the displacement parameters τ and τ'.

[3]There are four bosonic integrals if one counts the integrals $\mathcal{H}_1$ and σ_3.

4 Supersymmetry of the $n = 1$ self-isospectral pair

For the isospectral extended system $\mathcal{H}_1$ with $\kappa = \kappa'$, the partner potentials have the same form and are mutually displaced. This $n = 1$ self-isospectral case is special from the viewpoint of supersymmetric structure. As follows from (3.17) and (3.19), for $\kappa = \kappa'$ the integral $\mathcal{P}_{1;1}$, composed from the third order integrals of Lax pairs of superpartner subsystems, commutes with all the integrals, and so, transmutes into a bosonic central charge of the nonlinear superalgebra. We show now that the supersymmetric structure in this case undergoes even more radical changes.

For $\kappa = \kappa'$ the following reduction takes place [4] :

$$X_3(\kappa,\tau,\kappa,\tau') = \left(H_1(\kappa,\tau) + \kappa^2\right)\breve{X}_1(\kappa,\tau,\tau') - \mathcal{C}(\kappa,\tau-\tau')Y_2(\kappa,\tau,\kappa,\tau')\,, \tag{4.1}$$

where

$$\begin{aligned}\breve{X}_1(\kappa,\tau,\tau') &= \frac{d}{dx} - \kappa\tanh\kappa(x+\tau) + \kappa\tanh\kappa(x+\tau') + \mathcal{C}(\kappa,\tau-\tau') &(4.2)\\ &= A_1(\kappa,\tau) - A_1^\dagger(\kappa,\tau') + A_\mathcal{C}^\dagger(\kappa,\tau-\tau')\,, &(4.3)\end{aligned}$$

$$A_\mathcal{C}(\kappa,\tau-\tau') = \frac{d}{dx} + \mathcal{C}(\kappa,\tau-\tau')\,,\qquad \mathcal{C}(\kappa,\tau-\tau') = \kappa\coth\kappa(\tau-\tau')\,. \tag{4.4}$$

Relation (4.1) means that for $\tau \neq \tau'$, the first order operator $\breve{X}_1 = \breve{X}_1(\kappa,\tau,\tau')$ should be taken as a basic odd order intertwining operator instead of $X_3(\kappa,\tau,\kappa,\tau')$,

$$\breve{X}_1 H_1(\kappa,\tau') = H_1(\kappa,\tau)\breve{X}_1\,,\qquad \breve{X}_1^\dagger(\kappa,\tau,\tau') = -\breve{X}_1(\kappa,\tau',\tau) = -\breve{X}_1'\,. \tag{4.5}$$

Note that in the limit $\tau' \to \pm\infty$, we have $H_1' \to H_0$ and $\breve{X}_1 \to A_1$, while for $\tau \to \pm\infty$, $H_1 \to H_0$ and $\breve{X}_1 \to -A_1'^\dagger$. This is coherent with the intertwining relations (3.4).

Because of (4.1), the third order integrals $\mathcal{S}_{1;a}$ are reducible, $\mathcal{S}_{1;a} = (\mathcal{H}_1 + \kappa^2)\breve{\mathcal{S}}_{1;a} - \mathcal{C}\mathcal{Q}_{1;a}$, and have to be changed for the first order irreducible integrals

$$\breve{\mathcal{S}}_{1;1} = \begin{pmatrix} 0 & \breve{X}_1 \\ \breve{X}_1^\dagger & 0 \end{pmatrix},\qquad \breve{\mathcal{S}}_{1;2} = i\sigma_3\breve{\mathcal{S}}_{1;1}\,. \tag{4.6}$$

Integrals $\breve{\mathcal{S}}_{1;a}$ correspond, in accordance with (3.14), to the dressed form of the integrals $\breve{s}_a = \epsilon_{ab}\sigma_b p + \mathcal{C}\sigma_a$, of the extended free particle system $\mathcal{H}_0 = \mathrm{diag}(H_0, H_0)$, $\mathcal{D}\breve{s}_a\mathcal{D}^\dagger = \breve{\mathcal{S}}_{1;a}(\mathcal{H}_1 + \kappa^2)$. Alternatively, the first order matrix operator $\breve{s}_1 = \sigma_2 p + \mathcal{C}\sigma_1$, or $\breve{s}_2 = i\sigma_3\breve{s}_1$, can be considered as a first order Hamiltonian of the free Dirac particle of mass $|\mathcal{C}|$ in (1+1) dimensions, while its dressed form, $\breve{\mathcal{S}}_{1;1}$, can be identified as a Bogoliubov-de Gennes Hamiltonian describing the kink-antikink solution in the Gross-Neveu model [33]. Function $\Delta(\xi,\lambda) = \kappa\left(\tanh(\xi-\lambda) - \tanh(\xi+\lambda) + \coth 2\lambda\right)$, that appears in the structure of $\breve{X}_1$ with $\xi = \kappa(x + \frac{\tau+\tau'}{2})$ and $\lambda = -\kappa\frac{\tau-\tau'}{2}$, has then a sense of a gap function [23].

The following relations are valid:

$$\breve{X}_1\breve{X}_1^\dagger = H_1(\kappa,\tau) + \mathcal{C}^2\,, \tag{4.7}$$

$$\breve{X}_1 A_1(\kappa,\tau') = A_1(\kappa,\tau)A_\mathcal{C}(\kappa,\tau-\tau')\,,\qquad A_1^\dagger(\kappa,\tau)\breve{X}_1 = A_\mathcal{C}(\kappa,\tau-\tau')A_1^\dagger(\kappa,\tau')\,. \tag{4.8}$$

[4] A reduction of the third order intertwining generators was discussed in a general form in [48], however, with giving no special attention to a peculiar supersymmetric structure we study here; see also [49].

The employment of (4.7), (4.8) together with (4.5) gives nontrivial nonlinear superalgebraic relations

$$\{\breve{\mathcal{S}}_{1;a}, \breve{\mathcal{S}}_{1;b}\} = 2\delta_{ab}h_{\mathcal{C}}\,, \qquad \{\mathcal{Q}_{1;a}, \mathcal{Q}_{1;b}\} = 2\delta_{ab}h_\kappa^2\,, \tag{4.9}$$

$$\{\breve{\mathcal{S}}_{1;a}, \mathcal{Q}_{1;b}\} = 2\delta_{ab}\mathcal{C}h_\kappa + 2\epsilon_{ab}\mathcal{P}_{1;1}\,, \tag{4.10}$$

$$[\mathcal{P}_{1;2}, \breve{\mathcal{S}}_{1;a}] = 2i(h_{\mathcal{C}}\mathcal{Q}_{1;a} - \mathcal{C}h_\kappa\breve{\mathcal{S}}_{1;a})\,, \qquad [\mathcal{P}_{1;2}, \mathcal{Q}_{1;a}] = 2ih_\kappa(\mathcal{C}\mathcal{Q}_{1;a} - h_\kappa\breve{\mathcal{S}}_{1;a})\,, \tag{4.11}$$

which substitute nontrivial superalgebraic relations (3.15), (3.16), (3.17) and (3.18) of the general, non-isospectral case $n = 1$. Here we denoted $h_\kappa = \mathcal{H}_1 + \kappa^2$, $h_{\mathcal{C}} = \mathcal{H}_1 + \mathcal{C}^2$. As $\mathcal{C}^2 > \kappa^2$, the spectrum of $h_{\mathcal{C}}$ is strictly positive, and the Lie sub-superalgebra generated by the first order supercharges $\breve{\mathcal{S}}_{1;a}$ corresponds to a broken $N = 2$ supersymmetry. The $\mathcal{P}_{1;1}$ commutes now with all the supercharges in accordance with the observation made at the beginning of the Section.

While the third order intertwining operator (3.9) is well defined at $\kappa = \kappa'$, $\tau = \tau'$ and reduces to the integral $Z_3(\kappa, \tau)$ of $H_1(\kappa, \tau)$, the first order intertwining operator $\breve{X}_1(\kappa, \tau, \tau')$ in the limit $\tau' \to \tau$ reduces to the operator $\frac{d}{dx}$ shifted for an infinite additive constant term $\pm\infty$ in dependence on which side the difference $(\tau - \tau')$ tends to zero. In this case extended Hamiltonian (3.1) reduces just to the two identical copies of the Pöschl-Teller Hamiltonians, $\mathcal{H}_1(\kappa, \tau) = \mathrm{diag}(H_1(\kappa, \tau), H_1(\kappa, \tau))$. The integrals $\breve{\mathcal{S}}_{1;a}$, $a = 1, 2$, can be renormalized multiplying them by $1/\mathcal{C}(\kappa, \tau - \tau')$, and taking a limit $\tau' \to \tau$. In such a way they are reduced to the trivial integrals σ_a, $a = 1, 2$, of $\mathcal{H}_1(\kappa, \tau)$. The second order intertwining operator (3.8) reduces in the limit $\tau' \to \tau$ to $H_1(\kappa, \tau) + \kappa^2$, and the second order supercharges $\mathcal{Q}_{1;a}$ are reduced to the same trivial integrals σ_a multiplied by a shifted for a constant Hamiltonian, $Q_{1;a} \to (\mathcal{H}_1(\kappa, \tau) + \kappa^2 \cdot \mathbb{1})\sigma_a$. The only nontrivial integrals we have in the limit $\tau' \to \tau$ are the bosonic third order integrals $\mathcal{P}_{1;a}(\kappa, \tau)$.

The special case of self-isospectrality in the $n = 1$ extended system $\mathcal{H}_1$ is detected, therefore, by a radical change of nonlinear supersymmetric structure. One of the bosonic integrals, $\mathcal{P}_{1;1}$, turns into a central charge, and two third order supercharges are substituted for the supercharges of the first order. The reduction of the order of the half of the supercharges at $\kappa = \kappa'$ originates from relation (4.1) and is accompanied by appearance of dependence of the superalgebraic structure on the distance between mutually shifted one-soliton partner potentials by means of a constant $\mathcal{C} = \kappa \coth \kappa(\tau - \tau')$. In other words, one can say that in a generic case $\kappa \neq \kappa'$, the H_1 and H_1' are intertwined by the third order operators X_3 and $X_3^\dagger$, side by side with the second order operators Y_2 and $Y_2^\dagger$, via the free particle (zero gap) system, and the supersymmetric structure does not feel a relative distance $\tau - \tau'$ between the corresponding one-soliton subsystems because of the translation invariance of H_0. For $\kappa = \kappa'$, a kind of a 'tunneling' channel is opened: the one-soliton subsystems are intertwined then directly by the first order operators $\breve{X}_1$ and $\breve{X}_1^\dagger$, and the modified supersymmetric structure detects a 'tunneling distance' $\tau - \tau'$, see Fig. 1b.

5 Supersymmetry of an $n > 1$ extended system: complete isospectrality breaking

The discussion of supersymmetric structure for extended system composed from two subsystems having $n \geq 2$ bound states requires to distinguish three cases:

- Complete isospectrality breaking, when $\kappa_i \neq \kappa_j'$ for all $i, j = 1, \ldots, n$, with no restriction on displacement parameters τ_i and τ_j'.
- Partial isospectrality breaking, in which some, but not all, scaling parameters κ_i and κ_j' of the two subsystems coincide.

- Exact isospectrality, that is characterized by the complete coincidence of the sets of the scaling parameters, $\vec{\kappa} = \vec{\kappa}'$, accompanied by a restriction $\vec{\tau} \neq \vec{\tau}'$.

The case of a complete isospectrality breaking for $n > 1$ is a direct generalization of that for $n = 1$ case with $\kappa_1 \neq \kappa_1'$, which was studied in Section 3. It is discussed in the present Section. Other two cases are more involved. Though they generalize somehow the picture of the one-soliton case ($n = 1$) with $\kappa_1 = \kappa_1' = \kappa$, investigated in the previous Section, the corresponding analysis for $n > 1$ requires a generalization of the described Crum-Darboux transformations scheme. New peculiarities appear there, and those two cases deserve a separate consideration. To understand the picture, we study the case of $n = 2$ in the next Section, and then in Section 7 the results will be extended for a generic case of $n \geq 2$.

With these comments in mind, let us consider an extended system

$$\mathcal{H}_n = \begin{pmatrix} H_n & 0 \\ 0 & H_n' \end{pmatrix}, \tag{5.1}$$

composed from a completely non-isospectral pair $H_n = H_n(\vec{\kappa}, \vec{\tau})$ and $H_n' = H_n(\vec{\kappa}\,', \vec{\tau}\,')$ of the form (2.13), where $\vec{\kappa} = (\kappa_1, \ldots, \kappa_n)$, $\vec{\tau} = (\tau_1, \ldots, \tau_n)$, and we assume that there is no coincidence in the sets of the scaling parameters of the two subsystems, $\kappa_j \neq \kappa_{j'}'$ for all $j, j' = 1, \ldots, n$, see Fig. 2a.

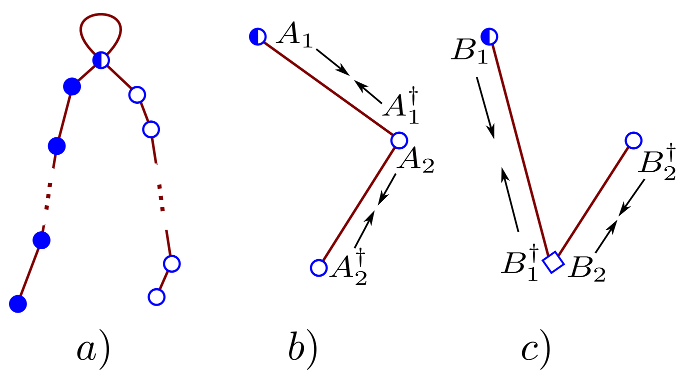


Figure 2: a) An $n > 1$ pair with complete isospectrality breaking. Each subsystem, H_n and H_n', is specified by indicating the set of intermediate, virtual, systems in the plane $\kappa - \tau$ via which the edge points are connected to the free particle by means of the first order Darboux generators A_j and $A_j^\dagger$, not shown here. Figs. b) and c) illustrate two alternative representations for the same $n = 2$ system, that is related to the two different factorizations of the second order Crum-Darboux generator $\mathbb{A}_2$. In the case b) the virtual system is regular, while in the case c) it is singular. So, a system is specified not only by indication of the set of points in the $\kappa - \tau$ plane, but also by the path via these points to a free system H_0.

Following the general picture described in Section 2, Hamiltonian $H_n = H_n(\vec{\kappa}, \vec{\tau})$ can be intertwined with a free particle Hamiltonian H_0 by order n differential operators $\mathbb{A}_n = \mathbb{A}_n(\vec{\kappa}, \vec{\tau})$ and $\mathbb{A}_n^\dagger = \mathbb{A}_n^\dagger(\vec{\kappa}, \vec{\tau})$,

$$\mathbb{A}_n(\vec{\kappa}, \vec{\tau}) = A_n((\kappa, \tau)_n) A_{n-1}((\kappa, \tau)_{n-1}) \ldots A_1(\kappa_1, \tau_1) \,, \tag{5.2}$$

defined in terms of Darboux generators (2.8), (2.9),

$$\mathbb{A}_n H_0 = H_n \mathbb{A}_n, \qquad \mathbb{A}_n^\dagger H_n = H_0 \mathbb{A}_n^\dagger \,. \tag{5.3}$$

Making use of these relations, we construct an order $2n$ operator

$$Y_{2n} = Y_{2n}(\vec{\kappa},\vec{\tau};\vec{\kappa}\,',\vec{\tau}\,') = \mathbb{A}_n\mathbb{A}_n'^{\,\dagger}\,, \qquad Y_{2n}^\dagger = Y_{2n}(\vec{\kappa}\,',\vec{\tau}\,';\vec{\kappa},\vec{\tau}) = Y_{2n}'\,, \tag{5.4}$$

where $\mathbb{A}_n' = \mathbb{A}_n(\vec{\kappa}\,',\vec{\tau}\,')$, and two operators of the order $2n+1$, X_{2n+1} and Z_{2n+1},

$$X_{2n+1}(\vec{\kappa},\vec{\tau};\vec{\kappa}\,',\vec{\tau}\,') = \mathbb{A}_n\frac{d}{dx}\mathbb{A}_n'^{\,\dagger}\,, \qquad X_{2n+1}^\dagger = -X_{2n+1}(\vec{\kappa}\,',\vec{\tau}\,';\vec{\kappa},\vec{\tau}) = -X_{2n+1}'\,, \tag{5.5}$$

$$Z_{2n+1} = Z_{2n+1}(\vec{\kappa},\vec{\tau}) = \mathbb{A}_n\frac{d}{dx}\mathbb{A}_n^\dagger\,, \qquad Z_{2n+1}^\dagger = -Z_{2n+1}\,. \tag{5.6}$$

Operators Y_{2n} and X_{2n+1} intertwine the components of the matrix Hamiltonian $\mathcal{H}_n$ [5],

$$Y_{2n}H_n' = H_nY_{2n}\,, \qquad X_{2n+1}H_n' = H_nX_{2n+1}\,, \tag{5.7}$$

while $Z_{2n+1}(\vec{\kappa},\vec{\tau})$ is an integral for $H_n(\vec{\kappa},\vec{\tau})$,

$$[Z_{2n+1},H_n] = 0\,. \tag{5.8}$$

Taking into account that the coefficients of the $(2n+1)$ order differential operator Z_{2n+1} may be expressed in terms of the potential V_n and its derivatives of the order less than $2n+1$ [40], relation (5.8) means that the potential V_n satisfies a higher stationary g-KdV equation with $g = 2n+1$, mentioned in Section 2.

In correspondence with an identity $Z_{2n+1}^2 = -H_n\prod_{j=1}^{j=n}(H_n+\kappa_j^2)^2$, the integral Z_{2n+1} detects all the physical non-degenerate states of H_n of energies $E=0$ and $E_j = -\kappa_j^2$ by annihilating them. These are constructed from the free particle non-degenerate eigenstate $\psi_0^0 = 1$, $\psi_n^0 = \mathbb{A}_n1$, and non-physical partners of the states (2.3), $\tilde{\psi}_1 = \sinh\kappa_1(x+\tau_1)$, $\tilde{\psi}_2 = \cosh\kappa_2(x+\tau_2), \ldots$, $\psi_n^{-\kappa_j^2} = \mathbb{A}_n\tilde{\psi}_j$, $j = 1,\ldots,n$. Other n states of the kernel of Z_{2n+1} are non-physical partners of the bound states $\psi_n^{-\kappa_j^2}$, $\tilde{\psi}_n^{-\kappa_j^2}(x) = \psi_n^{-\kappa_j^2}(x)\int dx/(\psi_n^{-\kappa_j^2}(x))^2$.

With the described operators, we construct six matrix integrals $\mathcal{Q}_{n;a}$, $\mathcal{S}_{n;a}$ and $\mathcal{P}_{n;a}$ for the extended system $\mathcal{H}_n$ in the form similar to that in (3.11) and (3.12) by changing Y_2, X_3 and Z_3 for, respectively, Y_{2n}, X_{2n+1} and Z_{2n+1}. As in the $n=1$ case, these integrals correspond to a dressed form of the integrals of the extended free particle system $\mathcal{H}_0$ obtained by means of Eq. (3.14) with the change of $\mathcal{D}_1$ for $\mathcal{D}_n = \text{diag}\,(\mathbb{A}(\vec{\kappa},\vec{\tau}),\mathbb{A}_n(\vec{\kappa}\,',\vec{\tau}\,'))$.

Applying factorization and intertwining relations, and products of corresponding generators collected in Appendix, we find that the superalgebra (3.15), (3.16), (3.17), (3.18) of the $n=1$ case is generalized for

$$\{\mathcal{Q}_{n;a},\mathcal{Q}_{n;b}\} = 2\delta_{ab}\mathbb{P}_n(\mathcal{H}_n,\vec{\kappa})\mathbb{P}_n(\mathcal{H}_n,\vec{\kappa}')\,,\ \{\mathcal{S}_{n;a},\mathcal{S}_{n;b}\} = 2\delta_{ab}\mathcal{H}_n\mathbb{P}_n(\mathcal{H}_n,\vec{\kappa})\mathbb{P}_n(\mathcal{H}_n,\vec{\kappa}')\,, \tag{5.9}$$

$$\{\mathcal{S}_{n;a},\mathcal{Q}_{n;b}\} = 2\epsilon_{ab}\mathbb{P}_n(\mathcal{H}_n,\vec{\mathcal{K}})\mathcal{P}_{n;1}\,, \tag{5.10}$$

$$[\mathcal{P}_{n;1},\mathcal{S}_{n;a}] = i\mathcal{H}_n\mathbb{P}_{n-1}^-(\mathcal{H}_n,\vec{\kappa},\vec{\kappa}')\mathcal{Q}_{n;a}\,, \qquad [\mathcal{P}_{n;1},\mathcal{Q}_{n;a}] = -i\mathbb{P}_{n-1}^-(\mathcal{H}_n,\vec{\kappa},\vec{\kappa}')\mathcal{S}_{n;a}\,, \tag{5.11}$$

$$[\mathcal{P}_{n;2},\mathcal{S}_{n;a}] = i\mathcal{H}_n\mathbb{P}_n^+(\mathcal{H}_n,\vec{\kappa},\vec{\kappa}')\mathcal{Q}_{n;a}\,, \qquad [\mathcal{P}_{n;2},\mathcal{Q}_{n;a}] = -i\mathbb{P}_n^+(\mathcal{H}_n,\vec{\kappa},\vec{\kappa}')\mathcal{S}_{n;a}\,, \tag{5.12}$$

where $\mathbb{P}_n(\mathcal{H}_n,\vec{\kappa}) = \prod_{j=1}^n(\mathcal{H}_n+\kappa_j^2\cdot\mathbb{1})$, $\mathbb{P}_n^+(\mathcal{H}_n,\vec{\kappa},\vec{\kappa}') = \mathbb{P}_n(\mathcal{H}_n,\vec{\kappa}) + \mathbb{P}_n(\mathcal{H}_n,\vec{\kappa}')$, $\mathbb{P}_{n-1}^-(\mathcal{H}_n,\vec{\kappa},\vec{\kappa}') = \mathbb{P}_n(\mathcal{H}_n,\vec{\kappa}) - \mathbb{P}_n(\mathcal{H}_n,\vec{\kappa}')$, $\mathbb{P}_n(\mathcal{H}_n,\vec{\mathcal{K}}) = \prod_{j=1}^n(\mathcal{H}_n+\mathcal{K}_j^2)$, $\mathcal{K}_j = \text{diag}\,(\kappa_j',\kappa_j)$.

[5] Intertwining relations through multi-step ladders of linear Darboux generators and their superalgebraic reducibility have been recently reviewed in [50], but in a very general and abstract form.

Operator $\mathbb{P}^{-}_{n-1}(\mathcal{H}_n,\vec{\kappa},\vec{\kappa}')$ is a polynomial of order $n-1$ in the extended Hamiltonian $\mathcal{H}_n$ that vanishes for $\vec{\kappa}=\vec{\kappa}'$. Then Eq. (5.11) signals that the supersymmetric structure of the $n>1$ reflectionless system $\mathcal{H}_n$ with exact isospectrality simplifies as in the case $n=1$: the integral $\mathcal{P}_{n,1}$ turns into bosonic central charge of the nonlinear superalgebra. Moreover, from the form of polynomial in $\mathcal{H}_n$ coefficients in superalgebra, one can expect that the supersymmetric structure should undergo some radical changes even in the case when not all the pairs of the scaling parameters coincide but only part of them. For instance, if $\kappa'_{j'}=\kappa_j$ for some indexes j' and j, which may coincide, $j'=j$, or may be different, $j'\neq j$, the same factor $(\mathcal{H}_n+\kappa_j^2\cdot\mathbb{1})$, or its square, appears in all the structure coefficients of the superalgebra. By analogy with the $n=1$ case this indicates that some fermionic supercharges may be substituted for supercharges of a lower differential order. To understand what changes the supersummetric structure undergoes in the cases of a partially broken or exact isospectrality, we investigate in detail the extended system (5.1) for the case of $n=2$ in the next Section.

6 Supersymmetry of the $n=2$ extended system

Explicit form of the supersymmetric structure for extended $n=2$ system with completely broken isospectrality follows as a particular case from a generic consideration of the previous Section. Before analyzing the partially broken and exact isospectrality cases, we first discuss some properties of the $n=2$ reflectionless system of the most general form. It is a particular case of such a system, described by the two-soliton Pöschl-Teller Hamiltonian, that appears in φ^4 field theoretical model with a double well potential, where it controls the stability of the kink and anti-kink solutions.

6.1 Generic reflectionless system with two bound states

Explicit form of the Hamiltonian of an $n=2$ reflectionless system of a general form is

$$H_2(\vec{\kappa},\vec{\tau}) = -\frac{d^2}{dx^2}+V_2(x;\vec{\kappa},\vec{\tau}), \tag{6.1}$$

$$V_2(x;\vec{\kappa},\vec{\tau}) = -2(\kappa_2^2-\kappa_1^2)^{-1}\left(\kappa_2^2\mathrm{csch}^2\kappa_2(x+\tau_2)+\kappa_1^2\mathrm{sech}^2\kappa_1(x+\tau_1)\right)w^2(x;\vec{\kappa},\vec{\tau}), \tag{6.2}$$

where

$$w(x;\vec{\kappa},\vec{\tau})=(\kappa_1^2-\kappa_2^2)\left(\kappa_2\coth\kappa_2(x+\tau_2)-\kappa_1\tanh\kappa_1(x+\tau_1)\right)^{-1}. \tag{6.3}$$

In the limit $\tau_2\to\pm\infty$, the two-soliton system (6.1) transforms into that of the one-soliton case,

$$V_2\to -2\kappa_1^2\mathrm{sech}^2\kappa_1(x+\tau_1\mp\xi_1), \tag{6.4}$$

where a shift parameter is defined by a relation $\sinh\kappa_1\xi_1=\kappa_1/\sqrt{\kappa_2^2-\kappa_1^2}$. In another limit, $\tau_1\to\pm\infty$, the two-soliton potential transforms into the one-soliton potential given by an expression of the form (6.4) but with the index 1 in the parameters changed for 2; the shift parameter ξ_2 is given then by a relation $\sinh\kappa_2\xi_2=\kappa_2/\sqrt{\kappa_2^2-\kappa_1^2}$. The indicated limits correspond to a picture of a two-soliton scattering described by the KdV equation, where the $n=1$ solitons of amplitudes $2\kappa_1^2$ and $2\kappa_2^2$ in such a process suffer asymptotically only temporal shifts [51].

Non-degenerate bound states of the system (6.1), $\psi_2^{-\kappa_j^2}=\mathbb{A}_2\tilde{\psi}_j$, $j=1,2$, of energies $E=-\kappa_1^2$ and $E=-\kappa_2^2$ are obtained from the partners, $\tilde{\psi}_1=\sinh\kappa_1(x+\tau_1)$ and $\tilde{\psi}_2=\cosh\kappa_2(x+\tau_2)$, of non-physical eigenstates $\psi_1=\cosh\kappa_1(x+\tau_1)$ and $\psi_2=\sinh\kappa_2(x+\tau_2)$ of H_0 by applying to them the second order composite operator

$$\mathbb{A}_2(\vec{\kappa},\vec{\tau})=A_2(\vec{\kappa},\vec{\tau})A_1(\kappa_1,\tau), \tag{6.5}$$

$$\psi_2^{-\kappa_1^2} = \kappa_1 \mathrm{sech}\kappa_1(x+\tau_1)\, w(x;\vec{\kappa},\vec{\tau})\,, \qquad \psi_2^{-\kappa_2^2} = -\kappa_2 \mathrm{csch}\,\kappa_2(x+\tau_2)\, w(x;\vec{\kappa},\vec{\tau})\,. \tag{6.6}$$

Here

$$A_2(\vec{\kappa},\vec{\tau}) = (A_1\psi_2)\frac{d}{dx}\frac{1}{(A_1\psi_2)} = -A_1^\dagger(\kappa_1,\tau_1) + w(x;\vec{\kappa},\vec{\tau})\,, \tag{6.7}$$

and A_1 is defined by Eq. (2.8). Function (6.3) satisfies the identities

$$dw/dx = \tfrac{1}{2}V_2\,, \tag{6.8}$$

$$w^2 + 2\kappa_1 w \tanh\kappa_1(x+\tau_1) = \tfrac{1}{2}V_2 + \kappa_2^2 - \kappa_1^2\,, \tag{6.9}$$

$$w^2 + 2\kappa_2 w \coth\kappa_2(x+\tau_2) = \tfrac{1}{2}V_2 + \kappa_1^2 - \kappa_2^2\,, \tag{6.10}$$

which will play a fundamental role in what follows.

The degenerate pairs of the states of the continuous part of the spectrum with $E = k^2 > 0$ are obtained from the plane wave states of the free particle, $\psi_2^{\pm k} = \mathbb{A}_2 e^{\pm ikx}$,

$$\psi_2^{\pm k} = \left[-(k^2+\kappa_1^2) + (\pm ik - \kappa_1\tanh\kappa_1(x+\tau_1))w(x;\vec{\kappa},\vec{\tau})\right] e^{\pm ikx}\,. \tag{6.11}$$

The boundary case $k=0$ gives a non-degenerate, zero energy edge state ψ_2^0 at the bottom of the continuous spectrum.

The particular case of reflectionless $n=2$ Pöschl-Teller system,

$$H_2(\kappa,\tau) = -\frac{d^2}{dx^2} - 6\kappa^2\mathrm{sech}^2\kappa(x+\tau),$$

is obtained by putting $\kappa_2 = 2\kappa_1 = 2\kappa$ and $\tau_2 = \tau_1 = \tau$. In this case, the function (6.3) and the operator (6.7) are reduced to $w = -3\kappa\tanh\chi$ and $A_2 = \frac{d}{dx} - 2\kappa\tanh\chi$, the indicated bound states are transformed, modulo overall multiplicative constants, into $\psi_2^{-\kappa^2} = \sinh\chi\,\mathrm{sech}^2\chi$ ($E=-\kappa^2$) and $\psi_2^{-4\kappa^2} = \mathrm{sech}^2\chi$ ($E=-4\kappa^2$), while the zero energy non-degenerate state is $\psi_2^0 = 1 - 3\tanh^2\chi$, where we use the notation $\chi = \kappa(x+\tau)$.

6.2 Generalized Crum-Darboux transformations scheme

We have constructed a generic $n=2$ reflectionless Hamiltonian (6.1) by employing a sequence of two Darboux transformations described in Section 2, namely, by using first the non-physical free particle state $\psi_1 = \cosh\kappa_1(x+\tau_1)$, and then the state $\psi_2 = \sinh\kappa_2(x+\tau_2)$. The same final result also can be achieved with the interchanged order of the indicated states. This corresponds to the alternative factorization of the second order operator (6.5),

$$\mathbb{A}_2 = B_2 B_1\,, \tag{6.12}$$

which intertwines H_2 with the free particle Hamiltonian, $\mathbb{A}_2 H_0 = H_2\mathbb{A}_2$, $\mathbb{A}_2^\dagger H_2 = H_0\mathbb{A}_2^\dagger$, see Fig. 2b, c. The first order operators B_1 and B_2 are obtained from A_1 and A_2 via the substitution

$$\kappa_1 \leftrightarrow \kappa_2\,, \quad \tau_1 \to \tau_2 + i\frac{\pi}{2\kappa_2} = \tilde{\tau}_2\,, \quad \tau_2 \to \tau_1 + i\frac{\pi}{2\kappa_1} = \tilde{\tau}_1\,. \tag{6.13}$$

This substitution leaves invariant the Hamiltonian (6.1), the second order intertwining operator $\mathbb{A}_2$, and the function (6.3). It also leaves invariant the states (6.11) of the continuous spectrum, including the non-degenerate edge state of zero energy, but interchanges the bound states (6.6), $\psi_2^{-\kappa_1^2} \to i\psi_2^{-\kappa_2^2}$, $\psi_2^{-\kappa_2^2} \to i\psi_2^{-\kappa_1^2}$. Transformation (6.13) changes, however, the first order intertwining operators A_1 and A_2, which are regular on $\mathbb{R}^1$, for the singular first order operators

$$B_1 = B_1(\kappa_2,\tau_2) = \frac{d}{dx} - \kappa_2\coth\kappa_2(x+\tau_2)\,, \quad B_2 = B_2(\vec{\kappa},\vec{\tau}) = -B_1^\dagger(\kappa_2,\tau_2) + w(x;\vec{\kappa},\vec{\tau})\,. \tag{6.14}$$

In terms of the first order operators (6.14) we have $H_0 = B_1^\dagger B_1 - \kappa_2^2$, $\tilde{H}_1 = H_1(\kappa_2, \tilde{\tau}_2) = B_1 B_1^\dagger - \kappa_2^2 = B_2^\dagger B_2 - \kappa_1^2$, and $H_2 = B_2 B_2^\dagger - \kappa_1^2$. This means that with the alternative factorization (6.12), the operator $\mathbb{A}_2$ intertwines Hamiltonian (6.1) with H_0 via the $n = 1$ system described by a *singular* Hamiltonian

$$\tilde{H}_1 = H_1(\kappa_2, \tilde{\tau}_2) = -\frac{d^2}{dx^2} + \frac{2\kappa_2^2}{\sinh^2 \kappa_2(x + \tau_2)}\,. \tag{6.15}$$

In what follows, singular Hamiltonian (6.15) will appear only as a virtual, or intermediate system, and the described generalization of the Crum-Darboux scheme will allow us to identify nontrivial intertwining operators for $n = 2$ extended system with partially broken and exact isospectrality. The picture with the alternative factorizations generalizes for the case $n > 2$. In this context it is worth to note that the change of the order of the free particle non-physical states (2.3) in the construction of a reflectionless system H_n, in comparison with that described in Section 2, corresponds to a certain permutation of the columns of the Wronskian (2.2). This produces no effect for potential in equation (2.1).

To conclude the discussion of the generalized Crum-Darboux transformations scheme, we present here the relations which are helpful for computation of the corresponding superalgebraic structures,

$$\breve{X}_1\,(\kappa, \tilde{\tau}, \tilde{\tau}')\,B_1' = B_1 A_{\mathcal{C}}(\kappa, \tau - \tau')\,, \qquad B_1^\dagger \breve{X}_1\,(\kappa, \tilde{\tau}, \tilde{\tau}') = A_{\mathcal{C}}(\kappa, \tau - \tau') B_1'^\dagger\,, \tag{6.16}$$

$$\breve{X}_1\,(\kappa, \tilde{\tau}, \tau')\,A_1' = B_1 A_{\mathcal{C}}(\kappa, \tilde{\tau} - \tau')\,, \qquad B_1^\dagger \breve{X}_1\,(\kappa, \tilde{\tau}, \tau') = A_{\mathcal{C}}(\kappa, \tilde{\tau} - \tau') A_1'^\dagger\,, \tag{6.17}$$

$$A_{\mathcal{C}}(\kappa, \tau - \tilde{\tau}') B_1'^\dagger = A_1^\dagger \breve{X}_1(\kappa, \tau, \tilde{\tau}')\,, \tag{6.18}$$

where $A_1 = A_1(\kappa, \tau)$, $A_1' = A_1(\kappa, \tau')$, $B_1 = B_1(\kappa, \tau)$, $B_1' = B_1(\kappa, \tau')$, $\tilde{\tau} = \tau + i\frac{\pi}{2\kappa}$, and $\tilde{\tau}' = \tau' + i\frac{\pi}{2\kappa}$. These identities can be obtained from (4.8) via the substitution (6.13).

6.3 Generic case of partial isospectrality breaking

Now we are in position to discuss the supersymmetric structure of the extended $n = 2$ systems with partially broken and exact isospectralities. We first consider three cases of partial isospectrality breaking, in which one discrete energy level $-\kappa_j^2$ of the subsystem H_2 coincides with any of the two discrete energy levels $-\kappa_{j'}^2$ of the partner Hamiltonian H_2', but the corresponding translation parameters are different, $\tau_j \neq \tau_{j'}'$. All these cases are described by a similar supersymmetric structure. Then, in the next subsection, we analyze the superalgebraic structure of the same three cases but with coinciding associated translation parameters, $\tau_j = \tau_{j'}'$.

We start with the case of partial isospectrality breaking characterized by the conditions

- $$\kappa_1 = \kappa_1', \quad \tau_1 \neq \tau_1', \qquad \kappa_2 \neq \kappa_2', \quad \text{no restrictions on } \tau_2,\, \tau_2'\,, \tag{6.19}$$

see Fig. 3a.

The subsystems $H_2 = H_2(\kappa_1, \tau_1, \kappa_2, \tau_2)$ and $H_2' = H_2(\kappa_1, \tau_1', \kappa_2', \tau_2')$ of the extended matrix Hamiltonian $\mathcal{H}_2$ are related by irreducible intertwining operators of orders 4 and 3, $Y_4 H_2' = H_2 Y_4$, $Y_4^\dagger H_2 = H_2' Y_4^\dagger$, $\breve{X}_3^A H_2' = H_2 \breve{X}_3^A$, $\breve{X}_3^{A\dagger} H_2 = H_2' \breve{X}_3^{A\dagger}$. Y_4 is given, in correspondence with the generic form (5.4), by $Y_4 = \mathbb{A}_2 \mathbb{A}_2'^\dagger$, while

$$\breve{X}_3^A = A_2(\kappa_1, \kappa_2; \tau_1, \tau_2) \breve{X}_1(\kappa_1; \tau_1, \tau_1') A_2^\dagger(\kappa_1, \kappa_2'; \tau_1', \tau_2') = A_2 \breve{X}_1 A_2'^{\,\dagger} \tag{6.20}$$

appears instead of the fifth order intertwining operator $X_5 = \mathbb{A}_2 \frac{d}{dx} \mathbb{A}_2'^\dagger$ because of the reduction

$$X_5 = (H_2 + \kappa_1^2) \breve{X}_3^A - \mathcal{C}(\kappa_1, \tau_1 - \tau_1') Y_4\,. \tag{6.21}$$

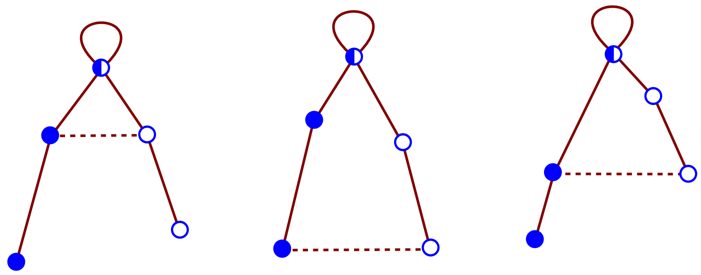


Figure 3: The $n = 2$ pairs with partially broken isospectrality.

As follows from (6.20), the reduction (6.21) is related to the opening of a 'tunneling channel' via the virtual isospectral pair of $n = 1$ systems $H_1(\kappa_1, \tau_1)$ and $H_1(\kappa_1, \tau_1')$.

Taking the products of the described intertwining operators and Lax integrals of order 5, $Z_5 = \mathbb{A}_2 \frac{d}{dx} \mathbb{A}_2^\dagger$, $Z_5' = \mathbb{A}_2' \frac{d}{dx} \mathbb{A}_2'^\dagger$, $[Z_5, H_2] = 0$, $[Z_5', H_2'] = 0$, presented in Appendix, we find the superalgebraic structure of the system $\mathcal{H}_2$ with partially broken isospectrality (6.19). It is displayed below in the form that unifies (6.19) with two other similar cases.

A partial isospectrality breaking with coinciding ground state energy levels,

- $$\kappa_2 = \kappa_2', \quad \tau_2 \neq \tau_2', \qquad \kappa_1 \neq \kappa_1', \quad \text{no restrictions on } \tau_1,\, \tau_1'\,, \tag{6.22}$$

is similar to the previous case, see Fig. 3b. Intertwining operator Y_4 and integrals Z_5 and Z_5' are given by generic formulae with restriction (6.22). The third order irreducible intertwining operator can be obtained from (6.20) via the substitution (6.13),

$$\breve{X}_3^B = B_2 \breve{X}_1(\kappa_2; \tilde{\tau}_2, \tilde{\tau}_2') B_2'^\dagger\,, \tag{6.23}$$

where B_2 and B_2' are given by Eq. (6.14) with $\kappa_2' = \kappa_2$, while

$$\begin{aligned} \breve{X}_1(\kappa_2; \tilde{\tau}_2, \tilde{\tau}_2') &= \frac{d}{dx} - \kappa_2 \coth \kappa_2(x + \tau_2) + \kappa_2 \coth \kappa_2(x + \tau_2') + \mathcal{C}(\kappa_2, \tau_2 - \tau_2') \\ &= B_1(\kappa_2, \tau_2) - B_1^\dagger(\kappa_2, \tau_2') + A_{\mathcal{C}}^\dagger(\kappa_2, \tau_2 - \tau_2')\,. \end{aligned} \tag{6.24}$$

Though all the three first order operators that appear in factorization of $\breve{X}_3$ in (6.23) are singular, the third order intertwining operator itself is regular on $\mathbb{R}^1$. This follows just from the reduction relation for the fifth order intertwining operator for the case (6.22) under consideration,

$$X_5 = (H_2 + \kappa_2^2)\breve{X}_3^B - \mathcal{C}(\kappa_2, \tau_2 - \tau_2')Y_4\,. \tag{6.25}$$

The third order intertwining operator (6.23) realizes the intertwining between H_2 and H_2' by means of a 'tunneling channel' via a pair of singular $n = 1$ Hamiltonians $H_1(\kappa_2, \tilde{\tau}_2)$ and $H_1(\kappa_2, \tilde{\tau}_2')$ of the form (6.15) [6].

The supersymmetric structure for partial isospectrality breaking

- $$\kappa_1 = \kappa_2', \qquad \kappa_2 \neq \kappa_1', \quad \text{no restrictions on } \tau_{1,2},\, \tau_{1,2}'\,, \tag{6.26}$$

[6] By shifting the argument $x \to x + i\delta$, where δ is a real constant, one can translate all the consideration for the case of $\mathcal{PT}$-symmetric quantum systems [52] with $\tilde{H}_1$ and $\tilde{H}_1'$ to be regular isospectral Hamiltonians, see [53].

see Fig. 3c, is generated in a similar way. Here, the third order irreducible intertwining operator is

$$\breve{X}_3^{AB} = A_2\breve{X}_1(\kappa_1;\tau_1,\tilde{\tau}_2')B_2'^{\dagger}\,, \tag{6.27}$$

where $B_2' = B_2(\kappa_1',\kappa_2,\tau_1',\tau_2')$ is given by Eq. (6.14), and $\tilde{\tau}_2' = \tau_2' + i\frac{\pi}{2\kappa_1}$. In this case, we have a reduction relation

$$X_5 = (H_2+\kappa_1^2)\breve{X}_3^{AB} - \mathcal{C}(\kappa_1,\tau_1-\tilde{\tau}_2')Y_4\,. \tag{6.28}$$

Unlike the two previous cases, $\mathcal{C}(\kappa_1,\tau_1-\tilde{\tau}_2') = \kappa_1\tanh\kappa_1(\tau_1-\tau_2')$ is regular for any values of τ_1 and τ_2' associated with coinciding scaling parameters [7].

The superalgebra for the described three cases of partial isospectrality breaking can be presented in a unified form

$$\{\breve{S}_a,\breve{S}_b\} = 2\delta_{ab}h_dh_{d'}h_{\mathcal{C}_l}\,,\qquad \{Q_a,Q_b\} = 2\delta_{ab}h_i^2h_dh_{d'}\,, \tag{6.29}$$

$$\{\breve{S}_a,Q_b\} = 2\delta_{ab}\mathcal{C}_lh_ih_dh_{d'} + 2\epsilon_{ab}h_{d',d}\mathcal{P}_1\,, \tag{6.30}$$

$$[\mathcal{P}_1,\breve{S}_a] = i(\kappa_d^2-\kappa_d'^2)(h_{\mathcal{C}_l}Q_a-\mathcal{C}_lh_i\breve{S}_a)\,,\quad [\mathcal{P}_1,Q_a] = i(\kappa_d^2-\kappa_d'^2)h_i(\mathcal{C}_lQ_a-h_i\breve{S}_a), \tag{6.31}$$

$$[\mathcal{P}_2,\breve{S}_a] = i(h_d+h_{d'})(h_{\mathcal{C}_l}Q_a-\mathcal{C}_lh_i\breve{S}_a)\,,\quad [\mathcal{P}_2,Q_a] = i(h_d+h_{d'})h_i(\mathcal{C}_lQ_a-h_i\breve{S}_a)\,. \tag{6.32}$$

Here $h_i = \mathcal{H}_2+\kappa_i^2$, $h_d = \mathcal{H}_2+\kappa_d^2$, $h_{d'} = \mathcal{H}_2+\kappa_d'^2$, $h_{d',d} = \mathcal{H}_2+\text{diag}(\kappa_d'^2,\kappa_d^2)$, $h_{\mathcal{C}_l} = \mathcal{H}_2+\mathcal{C}_l^2$, $l=1,2,3$, κ_i denotes the coinciding scaling parameter of the pair, κ_d and κ_d' correspond to other, not coinciding scaling parameters of the subsystems H_2 and H_2', respectively, while $\mathcal{C}_1 = \mathcal{C}(\kappa_1,\tau_1-\tau_1')$ for (6.19), $\mathcal{C}_2 = \mathcal{C}(\kappa_2,\tau_2-\tau_2')$ for (6.22), and $\mathcal{C}_3 = \mathcal{C}(\kappa_1,\tau_1-\tilde{\tau}_2')$ for the case (6.26). Notation $\breve{S}_a$ reflects the reduction $S_{2;a} = (\mathcal{H}_2+\kappa_i^2)\breve{S}_a - \mathcal{C}_lQ_a$ of the supercharges constructed in terms of X_5 and $X_5^\dagger$, and to simplify notations, we do not supply the supercharges with index l, and omitted the index $n=2$ in all the integrals.

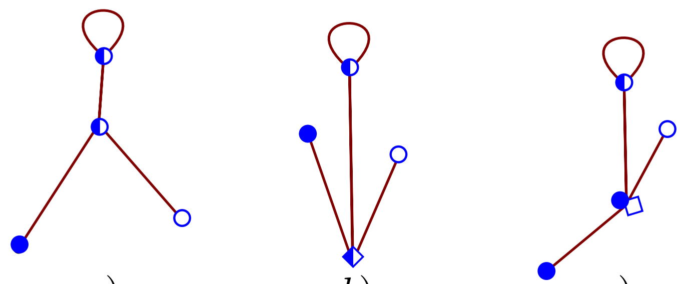
The fact of a partial isospectrality breaking is reflected here in the superalgebraic structure. On the one hand, relations (6.29), (6.30) and (6.32) are similar to superalgebraic structure (4.9), (4.10) and (4.11) of the $n=1$ isospectral case. At the same time, the commutators in (6.31), being of the nature of those in (3.17) for the $n=1$ non-isospectral family of the systems, show that a 'non-centrality' character of the Lax matrix integral $\mathcal{P}_{2;1}$ is measured by the scale of isospectrality breaking, $\kappa_d^2-\kappa_d'^2$.

6.4 Partial isospectrality breaking with coinciding associated translation parameters

Let us discuss now the supersymmetry of the systems with partial isospectrality breaking, in which one discrete energy level, $\kappa_j = \kappa_{j'}'$, and the associated translation parameters, $\tau_j = \tau_{j'}'$, coincide, see Fig. 4a, b, c. The two cases corresponding to either $\kappa_1=\kappa_1'$ or $\kappa_2=\kappa_2'$ are similar. For them, supersymmetry undergoes restructuring, and is generated by intertwining operators of the second, $\breve{Y}_2$, and fifth, X_5, orders, and by the fifth order integrals Z_5 and Z_5'. The fifth order operators, X_5 and Z_5, in this case include in their structure the third order integral of the corresponding common virtual $n=1$ system.

For the sake of definiteness, consider the case $\kappa_1=\kappa_1'$, $\tau_1=\tau_1'$, $\kappa_2\neq\kappa_2'$. We have $X_5 = A_2Z_3A_2'^{\dagger}$, $Z_5 = A_2Z_3A_2^{\dagger}$, and $Z_5' = A_2'Z_3A_2'^{\dagger}$, where $Z_3 = Z_3(\kappa_1,\tau_1) = A_1(\kappa_1,\tau_1)\frac{d}{dx}A_1^\dagger(\kappa_1,\tau_1)$ is the third order Lax integral for the common Pöschl-Teller virtual system $H_1(\kappa_1,\tau_1)$. The second order

[7] The operator (6.27) intertwines H_2' and H_2 via the virtual $n=1$ systems $\tilde{H}_1'$ and H_1 of different, singular and regular, nature. After the imaginary shift mentioned in the previous footnote, the latter pair will transform into regular $n=1$ reflectionless Pöschl-Teller $\mathcal{PT}$-symmetric Hamiltonians.



a) *b*) *c*)

Figure 4: The pairs with partially broken isospectrality, in which the translation parameters associated with the equal scaling parameters do coincide. In the case a), a common virtual system corresponds to a regular $n = 1$ reflectionless Pöschl-Teller system. In the case b) such a common virtual system is singular. In the case c), the partners can be intertwined via a pair of $n = 1$ virtual systems, one of which is singular.

intertwining operator has a form $\breve{Y}_2^A = A_2 A_2'^{\dagger}$, with $A_2 = A_2(\kappa_1, \kappa_2, \tau_1, \tau_2)$ and $A_2' = (\kappa_1, \kappa_2', \tau_1, \tau_2')$, and the fourth order intertwining operator $Y_4 = \mathbb{A}_2 \mathbb{A}_2'^{\dagger}$ of a generic case reduces as

$$Y_4 = (H_2 + \kappa_1^2)\breve{Y}_2^A \,. \tag{6.33}$$

The second order operator $\breve{Y}_2^A$ can be obtained also from the third order operator (6.20) of the case (6.19) considered above. Indeed, multiplying (6.20) by $-\mathcal{C}^{-1}(\kappa_1, \tau_1 - \tau_1')$, and taking a limit $\tau_1' \to \tau_1$, we get $\breve{Y}_2^A$. So, the change of supersymmetric structure is related here to a singular nature of $\mathcal{C}(\kappa_1, \tau_1 - \tau_1')$ in the limit $\tau_1' \to \tau_1$. Another case, with $\kappa_2 = \kappa_2'$, $\tau_2 = \tau_2'$, $\kappa_1 \neq \kappa_1'$ is treated in a similar way, and superalgebraic structure for these two cases can be presented in a unified form:

$$\{S_a, S_b\} = 2\delta_{ab}\mathcal{H}_2 h_i^2 h_d h_{d'} , \qquad \{\breve{Q}_a, \breve{Q}_b\} = 2\delta_{ab} h_d h_{d'} , \tag{6.34}$$

$$\{S_a, \breve{Q}_b\} = 2\epsilon_{ab} h_{d',d}\mathcal{P}_1 \,, \tag{6.35}$$

$$[\mathcal{P}_1, S_a] = i(\kappa_d^2 - \kappa_d'^2)\mathcal{H}_2 h_i^2 \breve{Q}_a \,, \quad [\mathcal{P}_1, \breve{Q}_a] = -i(\kappa_d^2 - \kappa_d'^2) S_a \,, \tag{6.36}$$

$$[\mathcal{P}_2, S_a] = i\mathcal{H}_2 h_i^2 (h_d + h_{d'})\breve{Q}_a \,, \quad [\mathcal{P}_2, \breve{Q}_a] = -i(h_d + h_{d'}) S_a \,. \tag{6.37}$$

Notation $\breve{Q}_a$ reflects here the reduction $Q_{2;a} = (\mathcal{H}_2 + \kappa_i^2)\breve{Q}_a$, and, again, we omitted the index $n = 2$ in the specification of nontrivial integrals.

The case $\kappa_1 = \kappa_2'$, $\tau_1 = \tau_2'$, $\kappa_1 \neq \kappa_2'$ is different from the two previous ones because the corresponding parameter-dependent function $\mathcal{C}(\kappa_1, \tau_1 - \tilde{\tau}_2') = \kappa_1 \tanh \kappa_1(\tau_1 - \tau_2')$ is non-singular for any values of $\tau_1 - \tau_2'$, and, moreover, turns into zero at $\tau_1 = \tau_2'$. Here, the intertwining operators are Y_4, and $\breve{X}_3^{AB}$ given by Eq. (6.27) with $\tau_1 = \tau_2'$. A non-singular nature of the latter is seen from (6.28). The superalgebra for this case is obtained directly from (6.29)–(6.32) just by putting there $\mathcal{C}_3 = 0$. Though here the irreducible intertwining generators are different in comparison with the previous two cases, the resulting superalgebra (6.29)–(6.32) with $\mathcal{C}_3 = 0$ has a similar form to (6.34)–(6.37). Notice also a remarkable similarity of (6.34)–(6.37) with the superalgebra (3.15)–(3.17) of the $n = 1$ non-isospectral case.

We see that in all the three cases of partial breaking of isospectrality with corresponding coinciding translation parameters (associated with coinciding discrete energy levels), the superalgebraic structure does not depend on the two remaining translation parameters associated with the second, different discrete energy levels.

In all the cases of partial isospectrality breaking described in this and previous subsections, the total order of the two basic intertwining operators is the same, $3+4=2+5=7$, being less in 2 in comparison with the complete isospectrality breaking case.

6.5 Exact isospectrality with a common virtual $n=1$ system

The supersymmetric structure of the systems with exact isospectrality, $\kappa_1=\kappa_1'$ and $\kappa_2=\kappa_2'$, depends on whether the corresponding translation parameters are different, $\tau_j\neq\tau_j'$, $j=1,2$, or they coincide in one of the pairs [8]. The analysis of the second case, see Fig. 5a, b, is more simple, and we first consider it supposing, for the sake of definiteness, that $\tau_1=\tau_1'$, $\tau_2\neq\tau_2'$. The

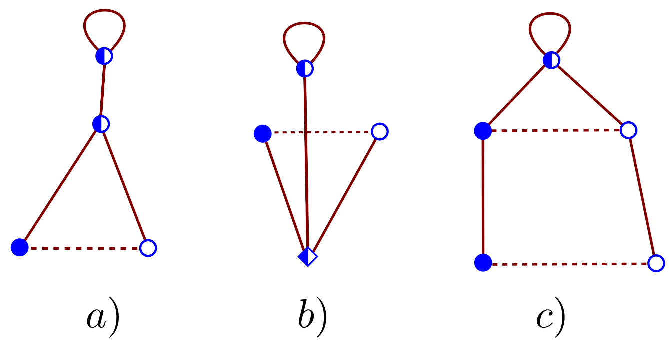


Figure 5: The $n=2$ isospectral pairs with a common regular, (a), or singular, (b), virtual system. A general case of the $n=2$ isospectral pair with $\tau_j\neq\tau_j'$, $j=1,2$, is illustrated by c).

intertwining operators for such an isospectral system with a common regular virtual $n=1$ system $H_1(\kappa_1,\tau_1)$ are

$$\breve{Y}_2^A=A_2A_2'^{\dagger}\,,\qquad \breve{X}_3^B=B_2\breve{X}_1(\kappa_2,\tilde{\tau}_2,\tilde{\tau}_2')B_2'^{\dagger}\,,\tag{6.38}$$

where in A_2' and B_2' we assume that $\kappa_j'=\kappa_j$, $j=1,2$, and $\tau_1'=\tau_1$, $\tau_2'\neq\tau_2$. They can be obtained here via the reduction relations of generic intertwining operators,

$$X_5=(H_2+\kappa_2^2)\breve{X}_3^B-\mathcal{C}(\kappa_2,\tau_2-\tau_2')Y_4\,,\qquad Y_4=(H_2+\kappa_1^2)\breve{Y}_2^A\,.\tag{6.39}$$

The intertwining generators $\breve{Y}_2^B$ and $\breve{X}_3^A$, and the corresponding reduction relations for the exact isospectrality case $\kappa_j=\kappa_j'$, $j=1,2$, $\tau_2'=\tau_2$, $\tau_1'\neq\tau_1$ are obtained from (6.38) and (6.39) by changing $\kappa_1\leftrightarrow\kappa_2$, $\tau_1\leftrightarrow\tau_2$, $\tau_1'\leftrightarrow\tau_2'$, and $A_2\leftrightarrow B_2$.

The nontrivial relations of superalgebraic structure for the isospectral case with $\tau_1=\tau_1'$, $\tau_2\neq\tau_2'$ are

$$\{\breve{\mathcal{S}}_a,\breve{\mathcal{S}}_b\}=2\delta_{ab}h_{\mathcal{C}_2}h_1^2,\quad \{\breve{\mathcal{Q}}_a,\breve{\mathcal{Q}}_b\}=2\delta_{ab}h_2^2,\tag{6.40}$$

$$\{\breve{\mathcal{S}}_a,\breve{\mathcal{Q}}_b\}=2\delta_{ab}\mathcal{C}_2h_1h_2+2\epsilon_{ab}\mathcal{P}_1,\tag{6.41}$$

$$[\mathcal{P}_2,\breve{\mathcal{S}}_a]=2ih_1(h_{\mathcal{C}_2}h_1\breve{\mathcal{Q}}_a-\mathcal{C}_2h_2\breve{\mathcal{S}}_a),\quad [\mathcal{P}_2,\breve{\mathcal{Q}}_a]=2ih_2(\mathcal{C}_2h_1\breve{\mathcal{Q}}_a-h_2\breve{\mathcal{S}}_a),\tag{6.42}$$

where $\mathcal{C}_2=\kappa_2\coth\kappa_2(\tau_2-\tau_2')$, $h_i=\mathcal{H}_2+\kappa_i^2$, $i=1,2$, and $h_{\mathcal{C}_2}=\mathcal{H}_2+\mathcal{C}_2^2$. The superalgebra for the isospectral case with $\tau_2=\tau_2'$, $\tau_1\neq\tau_1'$ is obtained from the displayed one by changing $\mathcal{C}_2\to\mathcal{C}_1$, $h_1\leftrightarrow h_2$ in the right hand side expressions. The supersymmetry (6.40), (6.41), (6.42) has the structure similar to that for the $n=1$ isospectral case.

[8] The case when both pairs of translation parameters coincide corresponds to $\mathcal{H}_2$ composed from the two copies of the same Hamiltonian H_2 . Such a system $\mathcal{H}_2$ is described by a trivial supersymmetric structure to be similar to that discussed for $n=1$ case in Section 4, with integral Z_3 changed for Z_5.

As it is expected, the integral $\mathcal{P}_{2;1}$ transmutes here into the bosonic central charge of nonlinear superalgebra. The total order of the basic irreducible intertwining generators reduces in two in comparison with the partially broken isospectrality case and equals the order 5 of Lax integrals Z_5 and Z'_5. In correspondence with this, the anticommutator of the second order, $\breve{\mathcal{Q}}_{2;a}$, and the third order, $\breve{\mathcal{S}}_{2;a}$, supercharges taken with different values of indexes a and b is equal to the central charge $\mathcal{P}_{2;1}$ up to a numerical, Hamiltonian-independent, coefficient, see Eq. (6.41). The superalgebraic structure also detects the difference of the non-coinciding translation parameters.

6.6 Generic case of $n = 2$ exact isospectrality

Consider a generic case of exact isospectrality characterized by the relations $\kappa_1 = \kappa'_1$, $\kappa_2 = \kappa'_2$, $\tau_1 \neq \tau'_1$, $\tau_2 \neq \tau'_2$, see Fig. 5c. The second order intertwining operator X_5 possesses then two distinct reductions, (6.21) and (6.25), in which it is necessary to put in addition, respectively, $\kappa_2 = \kappa'_2$ and $\kappa_1 = \kappa'_1$. The existence of the two third order intertwining operators means that a generic isospectral case is described by the basic intertwining operators of the orders 2 and 3, to which the intertwining operators X_5 and Y_4 are reduced. To see this, we note that $\breve{X}^A_3$ and $\breve{X}^B_3$ are the third order operators with the same coefficient -1 before the leading derivative term. Then the difference of these two operators has to be an intertwining differential operator of the second order. This implies that the coefficient before the leading second order derivative term in the latter should be a constant. Taking into account that $A_2 = -A^\dagger_1 + w$ and $B_2 = -B^\dagger_1 + w$, and employing relations (4.8) and (6.16), we find

$$\breve{X}^A_3 - \breve{X}^B_3 = (\mathcal{C}_1 - \mathcal{C}_2)G_2 + (\kappa_2^2 - \kappa_1^2)\hat{X}_1\,, \tag{6.43}$$

where $\mathcal{C}_1 = \kappa_1 \coth \kappa_1(\tau_1 - \tau'_1)$, and $\mathcal{C}_2 = \kappa_2 \coth \kappa_2(\tau_2 - \tau'_2)$,

$$G_2 = -\frac{d^2}{dx^2} + (w' - w)\frac{d}{dx} + \frac{dw'}{dx} + ww' + w\kappa_2 \coth \kappa_2(x + \tau_2) + w'\kappa_2 \coth \kappa_2(x + \tau'_2) + \kappa_2^2\,, \tag{6.44}$$

$$\hat{X}_1 = \frac{d}{dx} + (w - w') + \mathcal{C}_1\,. \tag{6.45}$$

In (6.44) and (6.45) w corresponds to the function (6.3), and w' is the same function but with τ_j changed for τ'_j, $j = 1, 2$. From (6.43) it follows immediately that the case $\mathcal{C}_1 = \mathcal{C}_2$ is special, and we shall consider it in the next subsection. So, till the end of this subsection we suppose that

$$\mathcal{C}_1 \neq \mathcal{C}_2\,. \tag{6.46}$$

We obtain then the second order intertwining operator

$$\hat{Y}_2 = \frac{\breve{X}^A_3 - \breve{X}^B_3}{\mathcal{C}_1 - \mathcal{C}_2} = G_2 + \frac{\kappa_2^2 - \kappa_1^2}{\mathcal{C}_1 - \mathcal{C}_2}\hat{X}_1\,. \tag{6.47}$$

The operator (6.47) intertwines H'_2 and H_2, $\hat{Y}_2 H'_2 = H_2 \hat{Y}_2$, and satisfies the relation $\hat{Y}^\dagger_2 = \hat{Y}'_2$, where $\hat{Y}'_2$ corresponds to $\hat{Y}_2$ with the interchanged translation parameters τ_j and τ'_j, $j = 1, 2$. $\hat{Y}^\dagger_2$ generates the intertwining relation in the reverse direction. Operator $\hat{Y}_2$, and any of two third order operators, $\breve{X}^A_3$ or $\breve{X}^B_3$, play now a role of independent intertwining generators. It is more convenient, however, to take a linear combination

$$\hat{X}_3 = \frac{\mathcal{C}_2 \breve{X}^A_3 - \mathcal{C}_1 \breve{X}^B_3}{\mathcal{C}_2 - \mathcal{C}_1}\,, \tag{6.48}$$

different from that in (6.43), as a third order intertwining generator to be independent from $\hat{Y}_2$. Using Eqs. (6.21) and (6.25), we find that the generic intertwining operators X_5 and Y_4 are reduced here as follows,

$$(\mathcal{C}_1-\mathcal{C}_2)X_5 = \left((\mathcal{C}_1-\mathcal{C}_2)H_2+\mathcal{C}_1\kappa_2^2-\mathcal{C}_2\kappa_1^2\right)\hat{X}_3+(\kappa_2^2-\kappa_1^2)\mathcal{C}_1\mathcal{C}_2\hat{Y}_2\,, \qquad (6.49)$$

$$(\mathcal{C}_1-\mathcal{C}_2)Y_4 = (\kappa_2^2-\kappa_1^2)\hat{X}_3+\left((\mathcal{C}_1-\mathcal{C}_2)H_2+\mathcal{C}_1\kappa_1^2-\mathcal{C}_2\kappa_2^2\right)\hat{Y}_2\,. \qquad (6.50)$$

Proceeding from the relations (6.49), (6.50) and the relations, presented in the Appendix, which correspond to the products of operators X_5, Y_4 and Z_5 with the imposed isospectrality relations $\kappa_j=\kappa_j'$, $j=1,2$, one can find all the products of the irreducible intertwining operators $\hat{Y}_2$, $\hat{X}_3$, $\hat{Y}_2^\dagger$, $\hat{X}_3^\dagger$, and Lax operators Z_5 and Z_5'. With these, one can compute the superalgebra generated by the second order, $\breve{\mathcal{Q}}_{2;a}$, and third order, $\breve{\mathcal{S}}_{2;a}$, supercharges constructed in terms of $\hat{Y}_2$ and $\hat{X}_3$ following the same rules as we used before, and by the fifth order bosonic integrals $\mathcal{P}_{2;a}$. There is another, more simple way to compute the superalgebra. Having in mind that fermionic supercharges are matrix differentials operators of orders 2 and 3, the alternative form of superalgebra is generated by taking a linear combination of them, $F_a^A=\mathcal{C}_1\breve{\mathcal{Q}}_{2;a}+\breve{\mathcal{S}}_{2;a}$ and $F_a^B=\mathcal{C}_2\breve{\mathcal{Q}}_{2;a}+\breve{\mathcal{S}}_{2;a}$, constructed from $\breve{X}_3^A$ and $\breve{X}_3^B$ in correspondence with relations (6.43) and (6.48),

$$F_1^{A,B}=\begin{pmatrix} 0 & \breve{X}_3^{A,B} \\ \breve{X}_3^{A,B\dagger} & 0\end{pmatrix}, \qquad F_2^{A,B}=i\sigma_3F_1^{A,B}\,. \qquad (6.51)$$

Modifying further the notations, $F_a^{(1)}=F_a^A$, $F_a^{(2)}=F_a^B$, and using the product relations of the operators $\breve{X}_3^{A,B}$, their conjugate, $\breve{X}_3^{A,B\dagger}$, and Lax operators Z_5 and Z_5', see Appendix, we present nonzero superalgebraic relations in a compact form,

$$\{F_a^{(i)},F_b^{(j)}\}=2\delta_{ab}h_{ij}h_ih_j+2\epsilon_{ab}\epsilon^{ij}\Delta\mathcal{C}\,\mathcal{P}_1, \qquad (6.52)$$

$$[\mathcal{P}_2,F_a^{(j)}]=\frac{2i}{\Delta\mathcal{C}}\left((-1)^jh_1h_2h_{12}F_a^{(j)}+\epsilon^{jk}h_{jj}h_k^2F_a^{(k)}\right). \qquad (6.53)$$

Here $h_i=\mathcal{H}_2+\kappa_i^2$, $h_{ij}=\mathcal{H}_2+\mathcal{C}_i\mathcal{C}_j$, $i,j=1,2$, $\Delta\mathcal{C}=\mathcal{C}_2-\mathcal{C}_1$, and no summation in the indexes i and j is implied in the right hand sides.

Again, the integral $\mathcal{P}_1=\mathcal{P}_{2;1}$ transmutes here into the bosonic central charge, and the structure coefficients depend on both relative translation parameters via $\mathcal{C}_1$ and $\mathcal{C}_2$.

The nonzero superalgebraic relations for the third, $\breve{\mathcal{S}}_{2;a}$, and second, $\breve{\mathcal{Q}}_{2;a}$, order supercharges and bosonic integrals $\mathcal{P}_{2;2}$ can now easily be obtained from (6.52) and (6.53) by employing the relations $\breve{\mathcal{Q}}_{2;a}=(F_a^{(2)}-F_a^{(1)})/\Delta\mathcal{C}$, $\breve{\mathcal{S}}_{2;a}=(\mathcal{C}_2F_a^{(1)}-\mathcal{C}_1F_a^{(2)})/\Delta\mathcal{C}$. The superalgebra has the same structure (4.9), (4.10), (4.11) as for the $n=1$ isospectral case, but with Hamiltonian-dependent coefficients of a more complicated form.

6.7 Special case of isospectrality with $\mathcal{C}_1=\mathcal{C}_2$

Let us consider the special case of isospectrality characterized by the relation

$$\mathcal{C}_1=\mathcal{C}_2\,. \qquad (6.54)$$

Equation (6.54) means that there is a special correlation between relative displacements $\tau_1-\tau_1'$ and $\tau_2-\tau_2'$ and scaling parameters, $\kappa_1\coth\kappa_1(\tau-\tau_1')=\kappa_2\coth\kappa_2(\tau_2-\tau_2')$. In correspondence with this relation, we may take an $n=2$ system H_2 defined by arbitrary parameters $\kappa_2>\kappa_1$, and arbitrary, but finite, τ_1 and τ_2. Particularly, we can choose the $n=2$ Pöschl-Teller system defined by the relations $\kappa_2 = 2\kappa_1$ and $\tau_1 = \tau_2$. The partner Hamiltonian H_2' is given then by the same scaling parameters, the finite parameter τ_2' may be chosen in an arbitrary way with the only restriction $\tau_2' \neq \tau_2$, while τ_1' is fixed uniquely, $\tau_1' = \tau_1 - \frac{1}{\kappa_1}\text{arccoth}\,(\frac{\kappa_2}{\kappa_1}\coth\kappa_2(\tau_2-\tau_2'))$.

As a consequence of relation (6.43), here a difference $\breve{X}_3^A - \breve{X}_3^B$ reduces to the first order intertwining operator (6.45), which satisfies a relation $\hat{X}_1^\dagger(\vec{\kappa},\vec{\tau},\vec{\tau}\,') = -\hat{X}_1(\vec{\kappa},\vec{\tau}\,',\vec{\tau}) = -\hat{X}_1'$. Moreover, we will show below that each of the third order intertwining operators $\breve{X}_3^A$ and $\breve{X}_3^B$ is reducible, and so, here the irreducible intertwining operators are $\hat{X}_1$ and Y_4.

As the intertwining generator $\hat{X}_1$ is the first order differential operator, let us define a superpotential W by means of

$$\hat{X}_1 = \frac{d}{dx} + W\,, \qquad W = w - w' + \mathcal{C}_1\,. \tag{6.55}$$

In accordance with relations (6.8), (6.9), (6.10), we have $W^2 + W' = V_2 + \mathcal{C}_1^2$, $W^2 - W' = V_2' + \mathcal{C}_1^2$, and then

$$\hat{X}_1\hat{X}_1^\dagger = H_2 + \mathcal{C}_1^2, \qquad \hat{X}^\dagger\hat{X}_1 = H_2' + \mathcal{C}_1^2\,, \tag{6.56}$$

and $\hat{X}_1 H_2' = H_2\hat{X}_1$, $\hat{X}_1^\dagger H_2 = H_2'\hat{X}_1^\dagger$. The first order intertwining operator $\hat{X}_1$ has a form similar to that of the operator $\check{X}_1$ in the $n=1$ isospectral case. The superpotential $W(x)$ plays here a role of the gap function Δ mentioned there in the context of its relation to the Bogoliubov-de Gennes system.

Operator $\hat{X}_1$ together with the first order operators $\breve{X}_1^A = \breve{X}_1(\kappa_1,\tau_1,\tau_1')$, $\breve{X}_1^B = \breve{X}_1(\kappa_2,\tau_2,\tau_2')$ satisfies in addition the identities

$$\breve{X}_1^A A_2'^\dagger = A_2^\dagger\hat{X}_1\,, \quad A_2\breve{X}_1^A = \hat{X}_1 A_2'\,, \qquad \breve{X}_1^B B_2'^\dagger = B_2^\dagger\hat{X}_1\,, \quad B_2\breve{X}_1^A = \hat{X}_1 B_2'\,. \tag{6.57}$$

Let us stress that like (6.56), these relations are valid only in the special isospectral case (6.54). Employing them, we find that the third order intertwining generators $\breve{X}_3^A$ and $\breve{X}_3^B$ are reducible,

$$\breve{X}_3^A = (H_2 + \kappa_2^2)\hat{X}_1\,, \qquad \breve{X}_3^B = (H_2 + \kappa_1^2)\hat{X}_1\,. \tag{6.58}$$

As a consequence, the fifth order generic intertwining operator also is reducible, $X_5 = (H_2 + \kappa_1^2)(H_2+\kappa_2^2)\hat{X}_1 - \mathcal{C}_1 Y_4$.

Applying the product relations (A.32)-(A.35) collected in Appendix, we can compute the superalgebra generated by the fermionic supercharges $\hat{\mathcal{S}}_{2;a}$ constructed from $\hat{X}_1$ and $\hat{X}_1^\dagger$, by the supercharges $\mathcal{Q}_{2;a}$ composed from Y_4 and $Y_4^\dagger$, and by the bosonic integrals $\mathcal{P}_{2;a}$ constructed from Lax operators Z_5 and Z_5'. The nontrivial (anti) commutations relations are

$$\{\hat{\mathcal{S}}_a,\hat{\mathcal{S}}_b\} = 2\delta_{ab}h_{\mathcal{C}_1}\,, \qquad \{\mathcal{Q}_a,\mathcal{Q}_b\} = 2\delta_{ab}h_1^2h_2^2\,, \tag{6.59}$$

$$\{\hat{\mathcal{S}}_a,\mathcal{Q}_b\} = 2\delta_{ab}\mathcal{C}_1h_1h_2 + 2\epsilon_{ab}\mathcal{P}_1\,, \tag{6.60}$$

$$[\mathcal{P}_2,\hat{\mathcal{S}}_a] = 2i(h_{\mathcal{C}_1}\mathcal{Q}_a - \mathcal{C}_1h_1h_2\hat{\mathcal{S}}_a)\,, \quad [\mathcal{P}_2,\mathcal{Q}_a] = 2ih_1h_2(\mathcal{C}_1\mathcal{Q}_a - h_1h_2\hat{\mathcal{S}}_a)\,, \tag{6.61}$$

where $h_i = \mathcal{H}_2 + \kappa_i^2$, $i=1,2$, $h_{\mathcal{C}_1} = \mathcal{H}_2 + \mathcal{C}_1^2$, and we omitted the index $n=2$ in the integrals.

Supercharges $\hat{\mathcal{S}}_{2;a}$, $a=1,2$, generate a Lie sub-superalgebra of $N=2$ supersymmetry. Since $\mathcal{C}_1^2 = \mathcal{C}_2^2 > \kappa_2^2$, it corresponds to the spontaneously broken phase. However, a peculiarity of the extended system $\mathcal{H}_2$ is that it has a structure of centrally extended $N=4$ nonlinear superasymmetry with the two additional fourth order supercharges $\mathcal{Q}_{2;a}$, and two bosonic integrals $\mathcal{P}_{2;a}$. Again, the integral $\mathcal{P}_{2;1}$ plays here the role of the central charge. As in a generic isospectral case from the previous subsection, the sum of differential orders of the basic irreducible intertwining operators equals 5 and coincides with the order of Lax operators. Again, the superalgebra (6.59), (6.60), (6.61) has a remarkable similarity with that for the $n=1$ isospectral case.

We conclude that with a chosen subsystem H_2, Eq. (6.54) defines a one-parametric family, in which τ_2', , $\tau_2' \neq \tau_2$, is a free parameter of the exactly isospectral system H_2'. Such a family of the Schrödinger pairs is described by the supersymmetry with the two first order supercharges, two supercharges of order four, and two bosonic integrals of differential order five, one of which is a central charge. This generalizes the $n = 1$ self-isospectral case discussed in Section 4 for the case of $n = 2$ isospectral, but not self-isospectral, pairs.

7 Partially broken and exact isospectralities in $n > 2$ systems

The analysis of partially broken and exact isospectralities can be generalized for n-soliton extended systems with $n > 2$. The case $n = 2$ considered in the previous Section shows that the concrete structure of supersymmetry, namely its irreducible generators and coefficients in the superalgebra, depends not only on how many scaling parameters coincide, but also on whether they correspond to the same or different ordinal numbers of discrete energy levels of subsystems. It also depends on relative translation parameters associated with the corresponding coinciding discrete energy levels, and may change in the cases when such relative translation parameters turn into zero, or are correlated via equalities of the form (6.54). Correspondingly, a concrete form of supersymmetric structure is rather variable, but the general picture can be summarized as follows. The $n > 2$ pair is characterized by two irreducible basic intertwining operators, one of which is a differential operator of odd order, while another is of even order. Each n-soliton subsystem also is characterized by a nontrivial integral to be a differential Lax operator of order $2n + 1$. The orders of irreducible intertwining operators satisfy the following rules. As we saw, the case of complete isospectrality breaking, when all the scaling parameters of one subsystem are different from those of the second subsystem, the supersymmetric pair is characterized by intertwining operators, X_{2n+1} and Y_{2n}, of differential orders $|X_{2n+1}| = 2n + 1$ and $|Y_{2n}| = 2n$. The sum of their differential orders, $4n + 1$, coincides with the order of the composite differential operator of the form $(H_n)^n Z_n$. When any pair of the scaling parameters of the subsystems coincides, the total order of the two basic irreducible intertwining operators decreases in such a way that $|XY^\dagger| = |(H_n)^{n-1}P| = 4n - 1$. Any new coincidence of some new pair of scaling parameters decreases the total order of $XY^\dagger$ in two. Finally, in the case of exact isospectrality, when all the n pairs of the scaling parameters coincide, we have $|XY^\dagger| = |Z_n| = (4n + 1) - 2n = 2n + 1$.

As an example, consider a generic case of exact isospectrality for the pair of the reflectionless soliton systems, each having three bound states. In this case, the composite operator $\mathbb{A}_3$ has 6 different factorizations in dependence on the order of the free particle non-physical states ψ_j, $j = 1, 2, 3$, which are used to generate a 3-soliton system. For instance, factorization $\mathbb{A}_3 = A_3^{(3)} A_2^{(2)} A_1^{(1)}$ corresponds to that described in Section 5, while $\mathbb{A}_3 = A_3^{(3)} A_2^{(1)} A_1^{(2)}$ corresponds to alternative factorization like that described in Section 6.2, with $A_1^{(2)}$ constructed in terms of the state ψ_2, $A_2^{(1)}$ constructed recursively in terms of $A_1^{(2)}$ and ψ_1, and finally, $A_3^{(3)}$ is constructed recursively by employing $A_1^{(2)}$, $A_2^{(1)}$ and ψ_3. In other words, the upper index indicates here the index of a state ψ_j we use to construct the first order Darboux operator of the generation marked by the lower index. The factorizations different from the standard one $\mathbb{A}_3 = A_3^{(3)} A_2^{(2)} A_1^{(1)}$ correspond to permutations of columns in the Wronskian (2.2), and in accordance with Eq. (2.1), do not produce any effect on the final form of the three-soliton potential V_3. Employing the information on intertwining operators of the $n = 2$ case, we construct three intertwining operators of order 5, $\breve{X}_5^{(1)} = A_3^{(3)} \breve{X}_3^{(12)} A_3'^{(3)\dagger}$, $\breve{X}_5^{(2)} = A_3^{(1)} \breve{X}_3^{(23)} A_3'^{(1)\dagger}$, and $\breve{X}_5^{(3)} = A_3^{(2)} \breve{X}_3^{(31)} A_3'^{(2)\dagger}$, where $\breve{X}_3^{(12)} = A_2 \breve{X}_1^{(1)} A_2'^{(2)\dagger}$ and $\breve{X}_1^{(1)} = A_1^{(1)} - A_1'^{(1)\dagger} - A_{\mathcal{C}_1}$ is the first order operator constructed in accordance with Eq. (4.2), and $\mathcal{C}_r = \kappa_r \coth \kappa_r(\tau_r - \tau_r')$, $r = 1, 2, 3$. The generic intertwining operator of order

7 reduces as

$$X_7 = (H_3 + \kappa_r^2)\breve{X}_5^{(r)} - \mathcal{C}_r Y_6\,, \quad r = 1,2,3\,. \tag{7.1}$$

Taking $(\breve{X}_5^{(1)} - \breve{X}_5^{(2)})$ and $(\breve{X}_5^{(2)} - \breve{X}_5^{(3)})$, we get two intertwining operators of order 4, $\breve{Y}^{(12)}$ and $\breve{Y}^{(23)}$, in which the coefficients before leading derivative term d^4/dx^4 will be constants. Presenting $\breve{Y}^{(12)}$ and $\breve{Y}^{(23)}$ in a normal form, with leading coefficients to be equal to 1, and taking a difference of the resulting fourth order differential operators, we get irreducible intertwining operator of order 3. Taking any one of the obtained two fourth order operators, we identify finally a pair of the basic irreducible intertwining operators $\hat{X}_3$ and $\hat{Y}_4$ of orders 3 and 4. Three identities in (7.1) allow us then, on the one hand, to express the generic intertwining operators X_7 and Y_6, which are reducible here, in terms of $\hat{X}_3$ and $\hat{Y}_4$ multiplied by certain polynomials in H_3. On the other hand, the same identities (7.1) indicate that the cases with $\mathcal{C}_1 = \mathcal{C}_2$ and/or $\mathcal{C}_2 = \mathcal{C}_3$ are peculiar. Coherently with the analysis of the previous Section, one can expect that in the special case $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}_3$ the basic irreducible intertwining operators are of orders 1 and 6. The analysis of this special case requires a separate consideration and we do not present it here, but only note that a corresponding isospectral pair is constructed similarly to the case of the $n = 2$. Namely, the scaling, $\kappa_3 > \kappa_2 > \kappa_1$, and translation, τ_1, τ_2 and τ_3, parameters of the subsystem H_3 are taken arbitrarily, the scaling parameters of the partner system H_3' are the same, and parameter τ_3' can take any finite value restricted by the condition $\tau_3' \neq \tau_3$. The relation $\mathcal{C}_2 = \mathcal{C}_3$ defines τ_2' uniquely in terms of the already chosen parameters, and then the equality $\mathcal{C}_1 = \mathcal{C}_2$ fixes uniquely the remaining displacement parameter τ_1'.

8 Spin-1/2 particle interpretation

In this section, following ref. [22], we discuss shortly a spin-1/2 particle interpretation of the studied class of the soliton systems (1.1), (1.2). This, particularly, will shed a new light on a peculiarity of the special family of isospectral n-soliton systems characterized by the first order supercharges.

Consider a non-relativist particle (electron) of mass $m = \frac{1}{2}$, charge $e = -1$ and gyromagnetic ratio $g = 2$ confined to a plane in the presence of electric field described by a scalar potential $\phi(x, y)$ and perpendicular magnetic field $B_z(x, y)$. The system is described by the Pauli Hamiltonian

$$H = (-i\tfrac{d}{dx} + A_x)^2 + (-i\tfrac{d}{dy} + A_y)^2 + \sigma_3 B_z - \phi\,. \tag{8.1}$$

Let us assume that electric and magnetic fields are homogeneous in the direction y, $\phi = \phi(x)$, $B_z = B_z(x)$, and choose $A_x = 0$, $A_y = a(x)$. Then $B_z = \frac{da}{dx}$, and the spinor wave function can be taken in the form $\Psi(x, y) = e^{iky}\psi(x)$. The action of the Hamiltonian (8.1) on a spinor $\psi(x)$ reduces to the matrix Hamiltonian of the form (1.1) with $V_\pm(x) = (k + a(x))^2 - \phi \pm \frac{da}{dx}$. Our system (1.1), (1.2) corresponds to the scalar electric potential and magnetic field of a special form

$$\phi(x) = (a(x) + k)^2 - \frac{1}{2}(V_n + V_n'), \quad B_z(x) = \frac{da}{dx} = \frac{1}{2}(V_n - V_n')\,, \tag{8.2}$$

given by the n-soliton, reflectionless potentials V_n and V_n'. Taking into account Eq. (2.1), the potentials $\phi(x)$ and $a(x)$ can be written in terms of the corresponding Wronskians as

$$\phi(x) = (a(x) + k)^2 + \tfrac{d^2}{dx^2}\ln\left(W_n W_n'\right), \quad a(x) = \tfrac{d}{dx}\ln\left(\tfrac{W_n'}{W_n}\right) + c_0\,, \tag{8.3}$$

where c_0 is an integration constant. Therefore, a spin-1/2 particle in the plane subjected to homogeneous in y direction electric and magnetic fields of the special form (8.2) is described by an exotic supersymmetry that was investigated and described in the previous sections.

Let us show now that the systems (1.1), (1.2) constructed from the special isospectral pairs of the n-soliton potentials, which are characterized by the first first order supercharges (alongside with the supercharges of order $2n$ and bosonic integrals $\mathcal{P}_{n,a}$, $a = 1, 2$, being differential operators of order $2n+1$), correspond to a case of a zero electric field, i. e. a constant scalar potential ϕ. First, consider a one-soliton case for which $V_1 = -2\,\mathrm{sech}^2\kappa(x+\tau)$ and $V_1' = -2\,\mathrm{sech}^2\kappa(x+\tau')$. For it, $W_1 = \cosh\kappa(x+\tau)$ and $W_1' = \cosh\kappa(x+\tau')$. Putting the integration constant $c_0 = \kappa\coth\kappa(\tau-\tau') - k$, we obtain

$$a(x) = -\Delta(x) - k, \qquad \Delta(x) = \kappa(\tanh\kappa(x+\tau) - \tanh\kappa(x+\tau') - \coth\kappa(\tau-\tau')), \tag{8.4}$$

that, up to the constant term $-k$, coincides exactly with the superpotential that appears in the first order intertwining operator (4.2). The trigonometric identity

$$1 - \tanh\alpha\tanh\beta - \coth(\alpha-\beta)(\tanh\alpha - \tanh\beta) = 0 \tag{8.5}$$

gives then $\phi = \kappa^2\coth^2\kappa(\tau-\tau')$, that is a square of the constant $\mathcal{C}$ defined in Eq. (4.4).

In the same way, for the special $n = 2$ case discussed in Section 6.7, we find $a(x) = W(x) - k$, where $W(x)$ is the superpotential appearing in the first order intertwining operator (6.55), and the scalar electric potential reduces to the square of the constant $\mathcal{C}_1 = \kappa_1\coth\kappa_1(\tau-\tau')$, $\phi = \mathcal{C}_1^2$. This picture with disappearing electric field is also valid for special isospectral n-soliton systems with $n > 2$, which were briefly discussed in the previous section.

It is interesting to note that electric field can also be eliminated in the self-isospectral case of reflectionless Pöschl-Teller systems having $n > 1$ bound states, that corresponds to the pair of mutually shifted soliton potentials $V_n = -n(n+1)\kappa^2\,\mathrm{sech}^2\kappa(x+\tau)$ and $V_n' = -n(n+1)\kappa^2\,\mathrm{sech}^2\kappa(x+\tau')$ with $n > 1$. This, however, can be done by the price of changing the gyromagnetic ratio $g = 2$ corresponding to the Pauli Hamiltonian (8.1), to the value $g_n = \sqrt{2n(n+1)}$. Indeed, changing the magnetic term in (8.1) for $\frac{1}{2}g_n\sigma_3B_z$, analogous analysis with employing the identity (8.5) results in $a(x) = -\frac{1}{2}g_n\Delta(x) - k$, where $\Delta(x)$ is the same as in Eq. (8.4), and $\phi = \frac{1}{2}n(n+1)\kappa^2\coth^2\kappa(\tau-\tau')$. According to the discussion in Sections 6.6 and 7, a matrix system (1.1), (1.2) with mutually shifted reflectionless Pöschl-Teller potentials is characterized by the pairs of the supercharges to be differential operators of orders n and $n+1$. This picture can be contrasted with a nonlinear supersymmetric structure appearing in the Landau problem for a charged spin-1/2 particle with special values of the gyromagnetic ratio $g = 2n$, see ref. [54], where supersymmetry is generated by a pair of the supercharges to be differential operators of order n.

9 Discussion and outlook

A generic supersymmetric quantum mechanical system with a 2×2 matrix Hamiltonian, whose components are intertwined either by first order Darboux or higher order Crum-Darboux differential operators, is described by two fermionic supercharges constructed from the intertwining generators. The supercharges together with the matrix Hamiltonian generate, respectively, either linear or nonlinear $N = 2$ superalgebra. For the linear supersymmetry (in the sense of superalgebra), the system has one non-degenerate zero energy level corresponding to the ground state in the case of the non-broken supersymmetry, or only degenerate energy levels if the supersymmetry is broken. For nonlinear supersymmetry case the picture is more complicated, and the system can possess $0 \leq \ell \leq n$ non-degenerate states if nonlinear supersymmetry is of order n, see [37], [50] and references therein.

We studied a special class of reflectionless systems with super-partners having the same number n of discrete energy levels in their spectra. Each of super-partner potentials describes an n-soliton solution of a nonlinear KdV equation that depends on n scaling and n translation parameters, and satisfies corresponding higher stationary equation of the KdV hierarchy. Because of the peculiar, soliton nature of the composite matrix Hamiltonians, their supersymmetric structure, on the one hand, turns out to be more rich in comparison with a generic case, and, on the other hand, it experiences essential changes depending on relation between the two sets of $2n$ parameters that characterize the partner n-soliton potentials.

It is worth to stress here that according to the terminology we used, the complete isospectrality breaking for a pair of n-soliton potentials $V_n = V_n(\kappa_1, \ldots, \kappa_n, \tau_1, \ldots, \tau_n)$ and $V'_n = V_n(\kappa'_1, \ldots, \kappa'_n, \tau'_1, \ldots, \tau'_n)$ means that $\kappa_j \neq \kappa'_{j'}$ for all $j, j' = 1, \ldots, n$, and so, the energies of their bound states, $E_j = -\kappa_j^2$ and $E'_j = -\kappa'^2_j$, have no coincidence, i.e. the extended system (1.1), (1.2) in this case has $2n$ discrete non-degenerate levels. At the same time, the lowest, zero energy level at the bottom of the continuous part of the spectrum of the extended system is doubly degenerate, while all the energy levels with $E > 0$ inside the continuous spectrum are four-fold degenerate.

There are four supercharges in the system (1.1), (1.2), two of which are composed from intertwining generators X_{2k+1} and $X^\dagger_{2k+1}$ to be differential operators of the odd order $2k+1 \leq 2n+1$, while two other fermionic integrals are constructed from intertwining generators Y_{2l} and $Y^\dagger_{2l}$ of the even order $2l \leq 2n$, such that in general case the total order, $|X_{2k+1}| + |Y_{2l}|$, of the basic irreducible intertwining operators satisfies a relation $2n+1 \leq (2k+1) + 2l \leq 4n+1$. The system also possesses two bosonic diagonal matrix integrals composed from nontrivial Lax operators of the n-soliton subsystems, Z_{2n+1} and Z'_{2n+1}, which are differential operators of order $2n+1$ being the Crum-Darboux dressed form of the free particle momentum $p = -i\frac{d}{dx}$. Operator Z_{2n+1} (Z'_{2n+1}) detects all the physical non-degenerate states of the subsystem H_n (H'_n) by annihilating them.

When the two sets of the scaling parameters are completely different, we have a complete isospectrality breaking, and the irreducible intertwining generators are of the orders $2n+1$ and $2n$. In this case X_{2n+1} and Y_{2n} intertwine the partner Hamiltonians H_n and H'_n via a virtual free particle system. Operator Y_{2n} detects all the bound states of the H'_n subsystem, by annihilating them, while X_{2n+1} makes the same job and, additionally, annihilates the non-degenerate state of the zero energy at the bottom of the continuous spectrum. The eigenstates of the H'_n not annihilated by these intertwining operators are transformed by them into the corresponding eigenstates of the H_n. The operators $X^\dagger_{2n+1}$ and $Y^\dagger_{2n}$ do the same with the eigenstates of H_n. The anticommutator between the supercharges of differential orders $2n+1$ and $2n$ generates the diagonal Lax integral $\mathcal{P}_{n;1} = -i\mathrm{diag}\,(Z_{2n+1}, Z'_{2n+1})$ multiplied by the order n polynomial of the matrix Hamiltonian. Both bosonic integrals, $\mathcal{P}_{n;1}$ and $\mathcal{P}_{n;2} = \sigma_3\mathcal{P}_{n;1}$, commute nontrivially with the supercharges. The Hamiltonian $\mathcal{H}_n$ of the system plays a role of the multiplicative central charge of the nonlinear superalgebra, whose structure is insensible to the translation parameters of the potentials.

In the simplest case of $n = 1$, when the scaling parameters κ_1 and κ'_1 of the partner potentials coincide, a kind of a channel for a direct 'tunneling' between the partners is opened, the third order operator X_3 is substituted for the operator X_1 of the first order, that intertwines H_1 and H'_1 directly, without communication via the virtual free particle system, and bosonic integral $\mathcal{P}_{1;1}$ transmutes into the central charge of the superalgebra, whose structure starts to depend on the 'tunneling distance' $\tau_1 - \tau'_1$. Operator X_1 transforms now all the physical eigenstates of the H'_1 subsystem into the corresponding eigenstates of the H_1. In the case $n > 1$, each time when any two discrete energy levels of the partner subsystems coincide, the basic intertwining operators X and Y undergo a reduction, decreasing their total differential order in two, and a dependence on a relative translation parameter associated with a pair of coinciding scaling parameters appears in the superalgebraic structure. The details of restructuring of supersymmetry generators depend on whether the discrete energy levels of the partners of the same or different ordinal numbers do coincide. A structure of supersymmetry also suffers abrupt changes in the orders of the basic irreducible intertwining operators, leaving invariant their total sum, when the coincidence of translation parameters, associated with the coinciding scaling parameters, happens. The supersymmetry also experiences a restructuring for another kind of correlation, $\kappa_j \coth \kappa_j(\tau_j - \tau_j') = \kappa_{j'} \coth \kappa_{j'}(\tau_{j'} - \tau_{j'}')$, $j \neq j'$, between the translation parameters associated with the coinciding pairs of discrete energy levels of the different ordinal numbers, $j \neq j'$.

Only in the case of the exact isospectrality of the partners, when all their discrete energy levels coincide pairwise, and as a consequence, their transmission scattering amplitudes also coincide, the bosonic integral $\mathcal{P}_{n;1}$ transmutes into the central charge of the superalgebra. In this case the total order $2n+1$ of the two basic irreducible intertwining operators X and Y coincides with the differential order of bosonic integrals. A particular case of such a situation corresponds to a self-isospectral pair of Pöschl-Teller systems.

From the viewpoint of supersymmetric structure we investigated, the self-isospectral Pöschl-Teller pairs possess, however, no special properties when $n > 1$, though the special subfamily of the extended systems with exact isospectrality that we detected corresponds to a generalization of the $n = 1$ self-isospectral case. For $n > 1$, those special isospectral pairs with the scaling and translation parameters correlated by means of $(n-1)$ equalities $\kappa_1 \coth \kappa_1(\tau_1 - \tau_1') = \kappa_j \coth \kappa_j(\tau_j - \tau_j')$, $j = 2, \ldots, n$, are described by the basic irreducible intertwining generators X_1 and Y_{2n}. For $n > 1$, the corresponding isospectral partner potentials have a form different from each other, and if one of them is chosen to be a reflectionless Pöschl-Teller potential with $n > 1$ bound states, an isospectral partner does not belong to the Pöschl-Teller hierarchy of potentials. More precisely, we identified and investigated in detail supersymmetric structure of such a special pair in the case $n = 2$, while we provided here only general indications that the same happens for $n > 2$. The special family of the completely isospectral pairs of n-soliton systems with $n > 2$ requires a separate consideration and will be presented elsewhere. The property $|X_1| = 1$ means that any of the two hermitian supercharges composed from the irreducible intertwining generators X_1 and $X_1^\dagger$ may be identified as a first order, Dirac type, Bogoliubov-de Gennes finite-gap Hamiltonian that belongs to the AKNS integrable hierarchy. From another perspective, we also observed the peculiarity of the special family of completely isospectral pairs with $|X_1| = 1$ from the viewpoint of interpretation of the matrix Hamiltonian (1.1), (1.2) in terms of the non-relativistic spin-1/2 particle system. In this context, we showed that all the family of self-isospectral reflectionless Pöschl-Teller systems also is special.

Analyzing the changes of supersymmetric structure associated with a coincidence of the scaling parameters, or, that is the same, of the bound states energies, we referred to the opening of tunneling channels conventionally. This might correspond nevertheless to real tunneling processes in some applications of the exotic supersymmetry, particularly, related to instantons.

We discussed the exotic supersymmetric structure from the standpoint of a usual Shrödinger equation that corresponds to a potential problem for a particle with a constant mass. It would be interesting to reinterpret the results from a perspective of a quantum problem for a particle with a position-dependent mass [55] having in mind possible applications for condensed matter physics.

As it was noted, by displacing the coordinate x for a pure imaginary constant, $x \to x + i\delta$, our analysis can be generalized for the case of $\mathcal{PT}$-symmetric quantum systems [52]. Such a generalization seems to deserve a special attention as it was proved to be useful for a particular case of supersymmetric extensions of reflectionless Pöschl-Teller and related systems, that helped recently to clarify some peculiarities in the $\mathcal{PT}$-symmetric quantum mechanics [53]. Particularly, $\mathcal{PT}$-symmetric generalization might be useful for applications in quantum optics.

As we mentioned, $n = 1$ and $n = 2$ reflectionless Pöschl-Teller systems control the stability of the kink solutions in the sine-Gordon, φ^4, and other exotic (1+1)-dimensional field theoretical models [9]. By considering the doublets of these fields with equal or different masses [56, 57], one could expect that the studied supersymmetric structure may reveal itself somehow at the level of the symmetries of the corresponding kink solutions.

We investigated exotic supersymmetry of soliton systems with the primary focus on its quantum mechanical aspects. The intriguing open question is whether it can be related somehow to a space-time symmetry of relativistic field systems having topological solitons. The developments in the Section IV of [36] seems to point towards a positive answer to this conjecture.

We discussed supersymmetric structure by choosing the diagonal Pauli matrix as a grading operator Γ. Alternative choices for Γ related to reflection operators are also possible. They provide the identification of the nontrivial integrals of motion as fermionic and bosonic generators in a way different from that described here. Particularly, the treatment of $\mathcal{P}_{n;a}$ as odd supercharges is possible, see [24, 37, 46, 47, 49, 58]. Supersymmetric structures for alternative choices of Γ can be computed by employing the product relations of the intertwining generators and Lax operators collected in Appendix. The alternative choices were useful for identification of the hidden supersymmetric structure in the systems described by the first order Bogoliubov-de Gennes Hamiltonian, particularly, in those associated with the Schrödinger $n = 1$ isospectral pair considered here [46, 24]. In this direction, it seems to be interesting to apply the results on a special case of the two-soliton pairs with exact isospectrality studied in Section 6.7 to the physics related to the Gross-Neveu model.

Finally, it would be interesting to generalize our analysis for finite-gap periodic systems, which also find many interesting applications in physics [23, 24, 26, 59, 60]. In that case it seems to be natural to restrict the considerations to the isospectral pairs.

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Appendix

Here we collect the products of the intertwining operators and Lax operators to be necessary for computing the concrete superalgebraic relations.

In the $n = 1$ non-isospectral case, $\kappa_1 \neq \kappa_1'$, the basic products of intertwining operators and Lax integrals are

$$X_3X_3^\dagger = H_1(H_1+\kappa_1^2)(H_1+\kappa_1'^2), \qquad X_3^\dagger X_3 = H_1'(H_1'+\kappa_1^2)(H_1'+\kappa_1'^2), \tag{A.1}$$

$$Y_2Y_2^\dagger = (H_1+\kappa_1^2)(H_1+\kappa_1'^2), \qquad Y_2^\dagger Y_2 = (H_1'+\kappa_1^2)(H_1'+\kappa_1'^2), \tag{A.2}$$

$$X_3Y_2^\dagger = -Y_2X_3^\dagger = (H_1+\kappa_1'^2)Z_3, \qquad Y_2^\dagger X_3 = -X_3^\dagger Y_2 = (H_1'+\kappa_1^2)Z_3', \tag{A.3}$$

$$Z_3X_3 = -H_1(H_1+\kappa_1^2)Y_2, \qquad X_3Z_3' = -H_1(H_1+\kappa_1'^2)Y_2, \tag{A.4}$$

$$Z_3Y_2 = (H_1+\kappa_1^2)X_3, \qquad Y_2Z_3' = (H_1+\kappa_1'^2)X_3\,, \tag{A.5}$$

$$Z_3Z_3^\dagger = -Z_3^2 = H_1(H_1+\kappa_1^2)^2, \qquad Z_3'Z_3'^\dagger = -Z_3'^2 = H_1'(H_1'+\kappa_1'^2)^2. \tag{A.6}$$

The products $X_3^\dagger Z_3$, $Z_3'X_3^\dagger$, $Y_2^\dagger Z_3$ and $Z_3'Y_2^\dagger$ are obtained by Hermitian conjugation of (A.4) and (A.5). They are given by expressions of the same form but multiplied by -1 because of the property

[9] Reflectionless n-soliton potentials of a general form like that analyzed in Section 6 for $n = 2$ also appear in stability equations for kink solutions in certain (1+1)-dimensional nonlinear field models, see [36].

$Z_3^\dagger = -Z_3$, and with substitutions $H_1 \to H_1'$, $X_3 \to X_3^\dagger$ and $Y_2 \to Y_2^\dagger$. Relations (A.6) are needed for computing of the superalgebraic structures in the case of alternative choices of the grading operator.

In the $n = 1$ isospectral case $\kappa_1 = \kappa_1'$, because of reduction (4.1), some relations are changed for

$$\breve{X}_1\breve{X}_1^\dagger = H_1 + \mathcal{C}^2, \quad \breve{X}_1^\dagger\breve{X}_1 = H_1' + \mathcal{C}^2, \tag{A.7}$$

$$\breve{X}_1 Y_2^\dagger = Z_3 + \mathcal{C}(H_1 + \kappa_1^2), \quad Y_2\breve{X}_1^\dagger = -Z_3 + \mathcal{C}(H_1 + \kappa_1^2), \tag{A.8}$$

$$Z_3\breve{X}_1 = \breve{X}_1 Z_3' = \mathcal{C}(H_1 + \kappa_1^2)\breve{X}_1 - (H_1 + \mathcal{C}^2)Y_2\,, \tag{A.9}$$

$$Z_3 Y_2 = Y_2 Z_3' = (H_1 + \kappa_1^2)((H_1 + \kappa_1^2)\breve{X}_1 - \mathcal{C}Y_2)\,. \tag{A.10}$$

The products $\breve{X}_1^\dagger Z_3 = Z_3'\breve{X}_1^\dagger$ and $Y_2^\dagger Z_3 = Z_3' Y_2^\dagger$ are obtained by Hermtian conjugation of (A.9) and (A.10) as in the non-isospectral case.

For a pair of n-soliton systems with complete isospectrality breaking the basic products are

$$Y_{2n}Y_{2n}^\dagger = \mathbb{P}_n\mathbb{P}_n'\,, \qquad X_{2n+1}X_{2n+1}^\dagger = H_n\mathbb{P}_n\mathbb{P}_n'\,, \tag{A.11}$$

$$X_{2n+1}Y_{2n}^\dagger = -Y_{2n}X_{2n+1}^\dagger = \mathbb{P}_n' Z_{2n+1}\,, \tag{A.12}$$

$$Z_{2n+1}Y_{2n} = \mathbb{P}_n X_{2n+1}\,, \qquad Y_{2n}Z_{2n+1}' = \mathbb{P}_n' X_{2n+1}\,, \tag{A.13}$$

$$Z_{2n+1}X_{2n+1} = -H_n\mathbb{P}_n Y_{2n}\,, \qquad X_{2n+1}Z_{2n+1}' = -H_n\mathbb{P}_n' Y_{2n}\,, \tag{A.14}$$

$$Z_{2n+1}^2 = -H_n\mathbb{P}_n\,, \tag{A.15}$$

where $\mathbb{P}_n = \prod_{l=1}^n (H_n + \kappa_l^2)$, $\mathbb{P}_n' = \prod_{l=1}^n (H_n + \kappa_l'^2)$. Other products of the type $X_{2n+1}^\dagger Y_{2n}$ etc. are obtained from these ones via the change $\kappa_j \leftrightarrow \kappa_j'$, $\tau_j \leftrightarrow \tau_j'$ with taking into account that $X_{2n+1}^\dagger = -X_{2n+1}'$, $Y_{2n}^\dagger = Y_{2n}'$ and $Z_{2n+1}^\dagger = -Z_{2n+1}$.

For three cases (6.19), (6.22), and (6.22) of $n = 2$ pairs with partial isospectrality breaking, the basic product relations are obtained from (A.11)–(A.15) by taking into account the reduction relations (6.21), (6.25) and (6.28). The latter are presented in the unified form $X_5 = h_{\kappa_i}\breve{X}_3^l - \mathcal{C}_l Y_4$, and then for each of three cases, distinguished by the index $l = 1, 2, 3$ for (6.19), (6.22), (6.26), respectively, we have

$$\breve{X}_3^l\breve{X}_3^{l\dagger} = h_{\mathcal{C}_l}h_{\kappa_d}h_{\kappa_d'}\,, \quad Y_4 Y_4^\dagger = h_{\kappa_i}^2 h_{\kappa_d}h_{\kappa_d'}, \tag{A.16}$$

$$\breve{X}_3^l Y_4^\dagger = h_{\kappa_d'}(Z_5 + \mathcal{C}_l h_{\kappa_i}h_{\kappa_d}), \qquad Y_4\breve{X}_3^{l\dagger} = h_{\kappa_d'}(-Z_5 + \mathcal{C}_l h_{\kappa_i}h_{\kappa_d}), \tag{A.17}$$

$$Z_5 Y_4 = h_{\kappa_i}h_{\kappa_d}(h_{\kappa_i}\breve{X}_3^l - \mathcal{C}_l Y_4), \quad Y_4 Z_5' = h_{\kappa_i}h_{\kappa_d'}(h_{\kappa_i}\breve{X}_3^l - \mathcal{C}_l Y_4), \tag{A.18}$$

$$Z_5\breve{X}_3^l = h_{\kappa_d}(\mathcal{C}_l h_{\kappa_i}\breve{X}_3^l - h_{\mathcal{C}_l}Y_4), \quad \breve{X}_3^l Z_5' = h_{\kappa_d'}(\mathcal{C}_l h_{\kappa_i}\breve{X}_3^l - h_{\mathcal{C}_l}Y_4), \tag{A.19}$$

where $h_\alpha = H_2 + \alpha^2$, $\alpha = \kappa_i,\, \kappa_d,\, \kappa_d',\, \mathcal{C}_l$.

The $n = 2$ partial isospectrality breaking case $\kappa_1 = \kappa_1'$, $\kappa_2 \neq \kappa_2'$, $\tau_1 = \tau_1'$, shown on Fig. 4a, is characterized by the following basic products of the intertwining and Lax operators,

$$\breve{Y}_2^A\breve{Y}_2^{A\dagger} = h_{\kappa_2}h_{\kappa_2'}\,, \qquad X_5 X_5^\dagger = H_2 h_{\kappa_1}^2 h_{\kappa_2}h_{\kappa_2'}\,, \tag{A.20}$$

$$X_5\breve{Y}_2^{A\dagger} = -\breve{Y}_2^A X_5^\dagger = h_{k_2'}Z_5\,, \qquad \breve{Y}_2^{A\dagger}X_5 = -X_5^\dagger\breve{Y}_2^A = h_{\kappa_2'}Z_5'\,, \tag{A.21}$$

$$Z_5 X_5 = -H_2 h_{\kappa_1}^2 h_{\kappa_2}\breve{Y}_2^A\,, \qquad X_5 Z_5' = -H_2 h_{\kappa_1}^2 h_{\kappa_2'}\breve{Y}_2^A\,, \tag{A.22}$$

$$Z_5\breve{Y}_2^A = h_{\kappa_2}X_5\,, \qquad \breve{Y}_2^A Z_5' = h_{\kappa_2'}X_5\,. \tag{A.23}$$

For the $n = 2$ isospectral case with a common $n = 1$ virtual system, when $\kappa_1 = \kappa_1'$, $\kappa_2 = \kappa_2'$, $\tau_1 = \tau_1'$, $\tau_2 \neq \tau_2'$, the basic products are

$$\breve{Y}_2^A\breve{Y}_2^{A\dagger} = h_{\kappa_2}^2, \quad \breve{Y}_2^{A\dagger}\breve{Y}_2^A = h_{\kappa_2}'^2, \quad \breve{X}_3^B\breve{X}_3^{B\dagger} = h_{\mathcal{C}_2}h_{\kappa_1}^2, \quad \breve{X}_3^{B\dagger}\breve{X}_3^B = h_{\mathcal{C}_2}'h_{\kappa_1}'^2, \tag{A.24}$$

$$\breve{X}_3^B \breve{Y}_2^{A\dagger} = Z_5 + \mathcal{C}_2 h_{\kappa_1} h_{\kappa_2}, \quad \breve{Y}_2^A \breve{X}_3^{B\dagger} = -Z_5 + \mathcal{C}_2 h_{\kappa_1} h_{\kappa_2}, \tag{A.25}$$

$$\breve{X}_3^{B\dagger} \breve{Y}_2^{A} = -Z_5' + \mathcal{C}_2 h'_{\kappa_1} h'_{\kappa_2}, \quad \breve{Y}_2^{A\dagger} \breve{X}_3^{B} = Z_5' + \mathcal{C}_2 h'_{\kappa_1} h'_{\kappa_2}, \tag{A.26}$$

$$Z_5 \breve{Y}_2^A = \breve{Y}_2^A Z_5' = h_{\kappa_2}^2 \breve{X}_3^B - \mathcal{C}_2 h_{\kappa_1} h_{\kappa_2} \breve{Y}_2^A, \quad \breve{Y}_2^{A\dagger} Z_5 = Z_5' \breve{Y}_2^{A\dagger} = \mathcal{C}_2 h'_{\kappa_1} h'_{\kappa_2} \breve{Y}_2^{A\dagger} - h'^2_{\kappa_2} \breve{X}_3^{B\dagger}, \tag{A.27}$$

$$Z_5 \breve{X}_3^B = \breve{X}_3^B Z_5' = \mathcal{C}_2 h_{\kappa_1} h_{\kappa_2} \breve{X}_3^B - h_{\mathcal{C}_2} h_{\kappa_1}^2 \breve{Y}_2^A, \ \breve{X}_3^{B\dagger} Z_5 = Z_5' \breve{X}_3^{B\dagger} = h'_{\mathcal{C}_2} h'^2_{\kappa_1} \breve{Y}_2^{A\dagger} - \mathcal{C}_2 h'_{\kappa_1} h'_{\kappa_2} \breve{X}_3^{B\dagger}. \tag{A.28}$$

Here $h_{\kappa_i} = H_2 + \kappa_i^2$, $h'_{\kappa_i} = H'_2 + \kappa_i^2$ $i = 1, 2$, $h_{\mathcal{C}_2} = H_2 + \mathcal{C}_2^2$, $h'_{\mathcal{C}_2} = H'_2 + \mathcal{C}_2^2$, and $\mathcal{C}_2 = \kappa_2 \coth \kappa_2 (\tau_2 - \tau'_2)$. Relations for the same isospectral case but with $\tau_2 = \tau'_2$, $\tau_1 \neq \tau'_1$ are obtained from these ones by interchanging $A \leftrightarrow B$, $\kappa_1 \leftrightarrow \kappa_2$, $\tau_1 \leftrightarrow \tau_2$, $\tau'_1 \leftrightarrow \tau'_2$ and by, correspondingly, changing $\mathcal{C}_2 \to \mathcal{C}_1$.

In generic $n = 2$ isospectral case, $\kappa_1 = \kappa'_1$, $\kappa_2 = \kappa'_2$, $\tau_1 \neq \tau'_1$, $\tau_2 \neq \tau'_2$, denoting $\breve{X}_3^{(1)} = \breve{X}_3^A$ and $\breve{X}_3^{(2)} = \breve{X}_3^B$, we have

$$\breve{X}_3^{(i)} \breve{X}_3^{(j)\dagger} = h_i h_j h_{ij} - (\mathcal{C}_i - \mathcal{C}_j) Z_5 \,, \qquad \breve{X}_3^{(i)\dagger} \breve{X}_3^{(j)} = h'_i h'_j h'_{ij} + (\mathcal{C}_i - \mathcal{C}_j) Z'_5 \,, \tag{A.29}$$

$$Z_5 \breve{X}_3^{(i)} = \breve{X}_3^{(i)} Z'_5 = -\frac{1}{\Delta \mathcal{C}} \left((-1)^i h_1 h_2 h_{12} \breve{X}_3^{(i)} + \epsilon^{ij} h_{ii} h_j^2 \breve{X}_3^{(j)} \right), \tag{A.30}$$

$$\breve{X}_3^{(i)\dagger} Z_5 = Z'_5 \breve{X}_3^{(i)\dagger} = \frac{1}{\Delta \mathcal{C}} \left((-1)^i h'_1 h'_2 h'_{12} \breve{X}_3^{(i)\dagger} + \epsilon^{ij} h'_{ii} h'^2_j \breve{X}_3^{(j)\dagger} \right), \tag{A.31}$$

where $h_i = H_2 + \kappa_i^2$, $h'_i = H'_2 + \kappa_i^2$, $h_{ij} = H_2 + \mathcal{C}_i \mathcal{C}_j$, $h'_{ij} = H'_2 + \mathcal{C}_i \mathcal{C}_j$, $\Delta \mathcal{C} = \mathcal{C}_2 - \mathcal{C}_1$, and no summation in $i, j = 1, 2$ is implied on the right hand sides.

For the $n = 2$ special isospectral case $\mathcal{C}_1 = \mathcal{C}_2$,

$$\hat{X}_1 \hat{X}_1^\dagger = h_{\mathcal{C}_1} \,, \qquad \hat{X}_1^\dagger \hat{X}_1 = h'_{\mathcal{C}_1} \,, \qquad Y_4 Y_4^\dagger = h_{\kappa_1}^2 h_{\kappa_2}^2 \,, \qquad Y_4^\dagger Y_4 = h'^2_{\kappa_1} h'^2_{\kappa_2} \,, \tag{A.32}$$

$$\hat{X}_1 Y_4^\dagger = Z_5 + \mathcal{C}_1 h_{\kappa_1} h_{\kappa_2} \,, \qquad Y_4 \hat{X}_1^\dagger = -Z_5 + \mathcal{C}_1 h_{\kappa_1} h_{\kappa_2} \,, \tag{A.33}$$

$$Z_5 \hat{X}_1 = \hat{X}_1 Z'_5 = \mathcal{C}_1 h_{\kappa_1} h_{\kappa_2} \hat{X}_1 - h_{\mathcal{C}_1} Y_4 \,, \quad \hat{X}_1^\dagger Z_5 = Z'_5 \hat{X}_1^\dagger = h'_{\mathcal{C}_1} Y_4^\dagger - \mathcal{C}_1 h'_{\kappa_1} h'_{\kappa_2} \hat{X}_1^\dagger \,, \tag{A.34}$$

$$Y_4 Z'_5 = Z_5 Y_4 = h_{\kappa_1} h_{\kappa_2} (h_{\kappa_1} h_{\kappa_2} \hat{X}_1 - \mathcal{C}_1 Y_4) \,, \ \ Z'_5 Y_4^\dagger = Y_4^\dagger Z_5 = h'_{\kappa_1} h'_{\kappa_2} (\mathcal{C}_1 Y_4^\dagger - h'_{\kappa_1} h'_{\kappa_2} \hat{X}_1^\dagger) \,, \tag{A.35}$$

where $h_{\kappa_i} = H_2 + \kappa_i^2$, $h'_{\kappa_i} = H'_2 + \kappa_i^2$ $i = 1, 2$, $h_{\mathcal{C}_1} = H_2 + \mathcal{C}_1^2$, $h'_{\mathcal{C}_1} = H'_2 + \mathcal{C}_1^2$.

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