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Faculty of Economics and Business

Cohesiveness in group decision making problems: its measurement and its achievement

María Teresa González Arteaga

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Cohesiveness in group decision making problems: its measurement and its achievement

Thesis submitted by:

María Teresa González Arteaga

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Supervised by:
Rocío de Andrés Calle
José Carlos Rodríguez Alcantud

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ORIGINAL PUBLICATIONS

This thesis has been made in the format of compendium of publications according to Regulations for Doctoral Degrees of the University of Salamanca (which were approved on 15 February 2013). It is made of the following original publications:

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Title: *A cardinal dissensus measure based on the Mahalanobis distance.*

Authors: T. González-Arteaga[1], J.C.R. Alcantud[2], R. de Andrés Calle[2].

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[1] Department of Statistics and Operational Research, University of Valladolid, Valladolid, Spain.
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Authors: T. González-Arteaga[1], R. de Andrés Calle[2], and F. Chiclana[3].

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PUBLICATION III:

Title: *A new consensus ranking approach for correlated ordinal information based on Mahalanobis distance*

Authors: T. González-Arteaga[1], J.C.R. Alcantud[2], R. de Andrés Calle[2].

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La Dra. **Rocío de Andrés Calle** y el Dr. **Jose Carlos Rodríguez Alcantud**, ambos profesores del área de Fundamentos de Análisis Económico, Departamento de Economía e Historia Económica de la Universidad de Salamanca

Certifican:

Que **María Teresa González Arteaga**, licenciada en Matemáticas, ha realizado bajo nuestra dirección la tesis doctoral titulada "**Cohesiveness in group decision making problems: its measurement and its achievement**" y que se recoge en esta memoria para optar al grado de Doctor en Economía. Dicha tesis doctoral reúne las condiciones necesarias para ser defendida y optar a la Mención de Doctorado Internacional por la Universidad de Salamanca.

Y para que así conste y tenga los efectos oportunos, expedimos y firmamos este certificado en Salamanca, a 24 de noviembre de 2016.

Dra. Rocío de Andrés Calle　　　Dr. Jose Carlos Rodríguez Alcantud

La Dra. **Rocío de Andrés Calle** y el Dr. **Jose Carlos Rodríguez Alcantud**, ambos profesores del área de Fundamentos de Análisis Económico, Departamento de Economía e Historia Económica de la Universidad de Salamanca

Autorizan:

Que la tesis doctoral titulada "**Cohesiveness in group decision making problems: its measurement and its achievement**" sea presentada en la modalidad de compendio de artículos/publicaciones (Comisión de Doctorado y Posgrado de la Universidad de Salamanca, 15 de febrero de 2013).

Y para que así conste, a los efectos legales, expiden y firman el presente certificado en Salamanca, a 24 de noviembre de 2016.

Dra. Rocío de Andrés Calle | Dr. Jose Carlos Rodríguez Alcantud

A la memoria de mi madre, Teresa

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Finally, to my mind, it is imperative to acknowledge all those courageous women that struggled for gender equality, especially women's right to High Education.

Abstract

The general objective of this doctoral thesis is to develop novel approaches for measuring cohesiveness / consensus and for accomplishing social consensus solutions in group decision making problems. In this line, this thesis expects to broaden the scope of the traditional and related methodologies.

These general issues are then addressed in the three following contributions.

In the first contribution, the problem of measuring the degree of consensus/dissensus in a context where experts or agents express their opinions on alternatives or issues by means of cardinal evaluations is studied. The assumption of considering cardinal evaluations to measure the cohesiveness had not been previously examined in literature. To this end, a new class of distance-based consensus methods, the *family of the Mahalanobis dissensus measures* for profiles of cardinal values is proposed. The main advantage of this proposal is that it takes into account the effects of differences in scale and possible interrelated issues. Moreover, some meaningful properties of the Mahalanobis dissensus measures are set forth. Finally, an application over a real empirical example is presented and discussed.

In the second contribution, a new approach to the measurement of consensus based on the Pearson correlation coefficient is studied under the assumption of

experts' opinions modelled via reciprocal preference relations. The new *correlation consensus degree* measures the concordance between the intensities of preference for pairs of alternatives. Although a detailed study of the formal properties of the new correlation consensus degree shows that it verifies relevant and desirable properties common either to distance or to similarity functions, it is also proved that it is different to traditional consensus measures. In order to emphasise the novelty of our work, an application to Clinical Decision-Making realm is presented.

In the third contribution, three basic essentials are addressed: the management of experts' opinions when they are expressed by ordinal information; the measurement of the degree of dissensus among such opinions; and the achievement of a group solution that conveys the minimum dissensus to the experts' group. Accordingly, a new procedure to codify ordinal information is characterised. Likewise, a new measurement of the degree of dissensus among individual preferences based on the Mahalanobis distance is designed in such a way that it is especially indicated for the case of possibly correlated alternatives. Finally, a procedure to obtain a social consensus solution, that also includes the possibility of alternatives that are correlated, is investigated. In addition, we examine the main traits of the dissensus measurement as well as the social solution proposed. The operational character and intuitive interpretation of these approaches are illustrated by an explanatory example.

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Introduction

Every day, people make decisions and most of them are made subconsciously, but also they often play an active role in decision making process. Many times decisions are not an individual issue, however, they are question of a group of people. Since the processes involved in decision-making are quite complex, getting consistent group decisions is a hard challenge. Accordingly, several research topics have emerged in several sciences to deal with problems that arise in this scenario like *Psychology, Political Science, Economics*, *Computer Science*, and so on.

Among many possible aspects that could be investigation target in this context, this dissertation focuses on cohesiveness in group decision making problems. The concept of group cohesion has a wide appeal in *Psychology* and *Sociology*. Much of this interest is due to the belief that keeping members' group together is important to get its successful performance. We would like to note that, on one hand, the possibilities of achieving high levels of cohesion are, to a considerable extent, a function of particular social or pre-group conditions (Braaten (1991)). On the other hand, there are several interpersonal aspects that affect group cohesiveness like that members' group closeness, group size, entry difficulties, etc. (Eisenberg (2007)). All these elements about social or interpersonal cohesion are not the standpoint in this doctoral thesis.

From another point of view, consensus is a multi-facet term as it has been described in Martínez-Panero's survey (Martínez-Panero (2011)). In this survey, several perspectives and aspects of consensus and how it is used within different formal developments like consensus measures in Social Choice Theory, Decision Making Theory and applications in Biomathematics are presented. In this dissertation the term cohesiveness refers to the degree of agreement among individual opinions of a group of experts or agents (a society) over a set of alternatives or issues. In this sense, the term consensus is considered in a similar form to Martínez-Panero.

Then, this thesis lies within the cross area between Social Choice Theory and Decision Making Theory. In this regard, it should be pointed out that this cross subject area has impacts within several fields of *Economics, Computer Science, Health Sciences*, and *Political Science*.

Group cohesiveness from the Social Choice Theory was first dealt in the literature by Bosch (2005) with the notion of consensus measure within a group where several issues are involved. Alcalde-Unzu and Vorsatz (2013), García-Lapresta and Pérez-Román (2011) and Alcantud et al. (2013b), among others, extended and provided axiomatic supports of consensus measures in the sense introduced by Bosch.

From the viewpoint of the Decision Making Theory and its applications, consensus measurement and its reaching in a group of experts are a prominent and active research areas. It is worth mentioning for instance the works provided by Kacprzyk and Fedrizzi (1988), Fodor and Roubens (1994) and Herrera-Viedma et al. (2014), among others.

Despite of all fruitful methodologies proposed to measure and achieve consensus, there are some challenges that the existing approaches hardly address.

In this sense we can mention as such challenges the measuring of cohesiveness when experts' opinions are expressed by means of cardinal evaluations and the explicit consideration of possible cross-related alternatives in the problem. From another perspective, it should be emphasised that most of the aforementioned methodologies are based on distance or similarity functions. This fact could be a limitation in some particular cases.

Taking into account the previous challenges and restrictions, the research performed in this doctoral thesis has focused on overcoming them. Concretely, the specific objectives of this thesis are listed below.

Objectives

The general objective of this doctoral thesis is to develop novel approaches for measuring cohesiveness/consensus and for accomplishing social consensus solutions in group decision making problems. In this line, this thesis expects to broaden the coverage of traditional approaches. This general objective can be detached into the following specific ones:

- Developing cohesiveness measures having explicitly regard to possible cross-related alternatives with the intent to cover lacunae and to complement the existing literature in the measurement and the achievement of cohesiveness.
- Opening a new via to measure cohesiveness that is neither a distance function nor a similarity function unlike the traditional consensus measures.
- Building consensus/dissensus measures from a theoretical point of view assuming different frameworks where experts or agents express their evaluations on the alternatives by means of diverse formats: ordinal information, cardinal evaluations and reciprocal preference relations.

- The detailed study of formal and desirable properties of the proposed measures.
- Defining a novel procedure to obtain a social consensus solution that includes the possible cross-relation among the alternatives and satisfies desirable properties.
- Showing practical applications of the innovative methodologies in order to prove the applicability and interest of our approaches in real life situations.

Structure of the doctoral thesis

This doctoral thesis is divided in several chapters which are structured as follows.

Chapter 1: *Background and literature review.* This chapter contains a short literature review of how the cohesiveness/consensus in a group has been addressed. Due to the subject of this dissertation, we focus on related literature from Social Choice Theory and Group Decision Making Theory.

The next three chapters contain the complete publications arising from the research carried out and therefore, the main contributions of this doctoral thesis.

Chapter 2: *A cardinal dissensus measure based on the Mahalanobis distance.* In this work, after introducing basic notation and definitions, the *class of the Mahalanobis dissensus measures* for profiles of cardinal evaluations and their main properties are set forth. Then, a comparison of several Mahalanobis dissensus measures is provided. To end, a practical application with discussion is given.

Chapter 3: *A new measure of consensus with reciprocal preference relations: The correlation consensus degree.* This contribution starts with a brief overview of different approaches in literature to measure group cohesiveness. Then, a new approach for measuring the consensus for reciprocal preference relations, *the correlation consensus degree* is formally built and its main properties are examined.

Finally, a practical application to Shared Clinical Decision Making of the proposed methodology is presented.

Chapter 4: *A new consensus ranking approach for correlated ordinal information based on Mahalanobis distance.* First, the problem of transforming ordinal information about individual preferences into numerical vectors is addressed. Secondly, we introduce the basic definition of dissensus measure and the Mahalanobis class of dissensus measures together with their main traits. Thirdly, we set forth the definition of our proposal of *Mahalanobis social consensus solution* that allows possible correlated alternatives and we prove some of its properties. Eventually, a visually appealing example is solved.

Chapter 5: *Concluding remarks and future research.* A summary of this thesis is presented, highlighting its major contributions and sketching its future lines of enquiry.

In addition to these five chapters, there are two appendices to fulfil the specific academic regulations for this doctoral thesis format. The first appendix, *Publication quality indicators*, encloses some quality indicators of scientific journals realising the scientific publications included in this doctoral thesis. The second appendix provides a *Spanish summary* of the main aspects of this doctoral thesis.

The dissertation finishes with a list of references which lists all the literature cited in the entire thesis.

Publications derived from this doctoral thesis included in the Journal Citation Report (JCR)

- T. González-Arteaga, J.C.R. Alcantud, R. de Andrés Calle (2016).A cardinal dissensus measure based on the Mahalanobis distance. *European Journal of Operational Research*, 251, 575-585. DOI: 10.1016/j.ejor.2015.11.019.

- T. González-Arteaga, R. de Andrés Calle and F. Chiclana (2016). A new measure of consensus with reciprocal preference relations: The correlation consensus degree. *Knowledge-Based Systems*, 107, 104–116. DOI: 10.1016/j.knosys.2016.06.002.

- T. González-Arteaga, J.C.R. Alcantud, R. de Andrés Calle (2016). A new consensus ranking approach for correlated ordinal information based on Mahalanobis distance. *Information Sciences*, 372, 546-564. DOI: 10.1016/j.ins.2016.08.071.

Dissemination of results

In addition to the aforementioned publications, several results of this dissertation and related works have been presented, in various stages of the research, in several scientific conferences and workshops that are specified below:

International meetings:

- ASSET 2013, *Annual Meeting of the Association of Southern European Economic Theorists.* Bilbao (Spain), 2013.
- EUROFUSE 2013, *Workshop on Uncertainty and Imprecision Modelling in Decision Making.* Oviedo (Spain), 2013.
- *Summer School of Interdisciplinary Analysis of Voting Rules.* Caen (France), 2014.
- FLINS 2016, *Conference on Uncertainty Modelling in Knowledge Engineering and Decision Making.* Roubaix (France), 2016

Spanish meetings:

- *Encuentro de la Red Española de Elección Social*. Málaga (Spain), 2013 and Madrid (Spain), 2014.
- CAEPIA 2015, *Conferencia de la Asociación Española de Inteligencia Artificial*, LODISCO 2015, *Simposio sobre Lógica Difusa y Soft Computing*. Albacete (Spain), 2015.
- ESTYLF 2016, *Congreso Español sobre Tecnologías y Lógica Fuzzy*. San Sebastian (Spain), 2016.
- CAEPIA 2016, *Conferencia de la Asociación Española de Inteligencia Artificial*, LODISCO 2016, *Simposio sobre Lógica Difusa y Soft Computing*. Salamanca (Spain), 2016.

Chapter 1

Background and literature review

This chapter contains a brief overview on recent literature about group cohesiveness from the perspective of Social Choice Theory and Decision Making Theory. Nevertheless, for the sake of completeness, it should be mentioned there are some other research areas that propose different definitions and applications of consensus measures like those which can be seen for instance in Jaime et al. (2014) and López-Molina et al. (2016), among others.

To the best of our knowledge, one of the first analysis about cohesiveness is found in Hays (1960). In this earlier contribution, Hays proposed an "analysis of agreement" among a group of rankings and a method for obtaining a rank order of "best fit" to such a group rankings. Another former example is the approach provided by Day and McMorris (1985) where a formalization of a consensus index was introduced as a measure of agreement among "profile objects". Both contributions may not be clearly included in one of the aforementioned theories due to the fact that it is sometimes cumbersome to determine a boundary between them.

From now on, we will review separately the related literature to both theories with the aim of providing a clear and overall background on the subject of this doctoral thesis.

The cohesiveness from Social Choice Theory

Bosch's Ph.D. thesis (Bosch (2005)) is considered the first serious analysis of consensus measurement from an Arrovian perspective in Social Choice Theory. In his work, both absolute and intrinsic measures of consensus were proposed, analysed and axiomatically characterised. Bosch introduced the notion of consensus measure within a group of experts where several issues are involved. Concretely, he defined a consensus measure as a function that assigns a number in the unit interval to each profile of individual strict linear orderings verifying some properties.

From the point of view of considering consensus among a family of voters, McMorris and Powers (2009) characterised consensus rules defined on hierarchies. Alcalde-Unzu and Vorsatz (2010, 2013, 2016) went one step further by characterizing axiomatically some absolute cohesiveness measures under the assumption of agents order alternatives linearly. Later on, García-Lapresta and Pérez-Román (2011) focused on how to measure consensus using complete preorders on alternatives (indifferences are permitted) and introduced a class of consensus measures based on seven well-known distance functions.

Apart from the above contributions, Alcantud et al. (2013a,c) characterised a class of consensus measure, the *referenced consensus measure*, that permits to produce a numerical social evaluation from ordinal individual information. They also contributed to the formal and computational analysis of such measures by focusing on two relevant and specific cases: the Borda and the Copeland rules under a Kemeny-type measure.

Furthermore, Alcantud, de Andrés Calle and Cascón in Alcantud et al. (2013b, 2015) took a different position. They studied the case where agents have dichotomous opinions on alternatives providing a model that does not necessarily require pairwise comparisons. These proposals are conceptually rich to the aim of assuring axiomatic support to these consensus indexes.

In addition, it is worth mentioning the work Erdamar et al. (2014) due to the fact they presented a distance-based approach to measure the degree of consensus considering two different types of information: approval information about alternatives as well as rankings of them.

The cohesiveness from Decision Making Theory

In Decision Making Theory and its applications, consensus measurement and its reaching (for simplicity we will call “consensus problem”) have attracted research attention for a long time and consequently, a considerable amount of contributions have been made under several frameworks and methodologies.

In group decision making problems, the way in which experts express their opinions about alternatives plays an important role. Then, it is essential to establish a suitable framework for each problem. Generally speaking, experts’ opinions can be express by means of different formats. Therefore, it is possible to distinguish ordinal and cardinal information. The former being more extensively used, while cardinal information is not so common in the subject being addressed in this dissertation.

Ordinal information implies that experts order alternatives by linear orders, complete preorders or partial orderings. We may cite, among the great variety of contributions in the literature, as examples of works that address the "consensus problem" under ordinal information: Cook and Kress (1991), González-Pachón and Romero (2001), Emond and Mason (2002), Cook (2006), Leyva López and Alvarez Carrillo (2015) and Amodio et al. (2016).

Cardinal information implies that experts assign numerical values to each alternative. This kind of information has mainly been dealt under the Utility Theory, e.g., Keeney and Kirkwood (1975), Farquhar (1984) and Fishburn (1994). Nonetheless, there are some contributions related to "consensus problem" managing cardinal information. In this respect, it should be noted the approaches proposed in Herrera-Viedma et al. (2002), González-Pachón and Romero (2009), González-Pachón et al. (2014) and so on.

Apart from ordinal and cardinal information, experts can express their opinions by means of different preference structures. One of the most related classical reference is the contribution Kacprzyk and Fedrizzi (1988). This work introduces the concept of "degree of consensus" in the sense of expressing the degree to which "most of" individuals in a group agree to "almost all of" options. The point of departure is a set of individual fuzzy preference relations. Moreover, taking into account the called "consensus problem", it may be appropriate to refer to the works Nurmi (1981), Kacprzyk and Fedrizzi (1988), Herrera-Viedma et al. (2007) for reciprocal fuzzy preference relations, Saaty (1980) for multiplicative preference relations, Wu and Chiclana (2014a) for interval-valued fuzzy reciprocal preference relations, Herrera et al. (1996), Ben-Arieh and Chen (2006), Cabrerizo et al. (2009)) and Sun and Ma (2015) for linguistic preference relations, Xu and Liao (2015) for intuitionistic fuzzy preference relations, Wu and Chiclana (2014b) for triangular fuzzy complementary preference relations, García-Lapresta and Pérez-Román (2016a,b)) for linguistic terms.

In addition, there are contributions where different experts can express their opinions on the alternatives using different preference structures like Valls and Torra (2000), Herrera-Viedma et al. (2002), Fedrizzi et al. (2010) and Dong and Zhang (2014).

Regarding methodologies, most of the contributions on "consensus problem" are based on distance and similarity functions. We can point out some of them like Kemeny and Snell (1962), Cook (2006), García-Lapresta and Pérez-Román (2011) and Chiclana et al. (2013) that make use of Kemeny, Mannhattan, Dice and Cosine distance functions. Association measures are less widely used than distance functions but it is also possible them in Cook and Seiford (1982), Emond and Mason (2002), Goodman and Kruskal (1979), Kendall and Gibbons (1990), where Kendall's coefficient, Goodman-Kruskal's index and Spearman's coefficient are handled.

Once the measurement of consensus has been provided, it is relevant finding the "solution" which more agreement conveys to the group. Traditionally, the achievement of a global solution (or social solution) has been considered as an aggregation problem of experts' opinions. Different methods have been proposed and analysed to this end. Borda first examined this problem in a voting context (Borda (1781)) and, afterwards, Kendall revised Borda's method in a statistical framework (Kendall (1962)). Kemeny and Snell approached the problem from a different direction, and employed the distance metric form for measuring ranking agreement (Kemeny and Snell (1962)).

Moreover, other authors also proposed alternative distance-based aggregation rules for obtaining social solutions among them it should be cited the works e.g., Saari and Merlin (2000), Ratliff (2001, 2002)),Klamler (2004, 2008), Meskanen and Nurmi (2006)) and (Eckert and Klamler (2011)).

Cook and Seiford (Cook and Seiford (1982)) developed a method based on "minimum variance" for determining a consensus ranking. This approach includes the Borda-Kendall method if ties are not allowed. Their proposal is connected to the Euclidean distance. Later, González-Pachón and Romero (González-Pachón and Romero (1999)) developed a general framework for distance-based consensus models under the assumption of a generic l_p metric. These authors also deals with group decision-making problems where experts can give partial orders on alternatives within distance-based methodology with l_p metric (González-Pachón and Romero (2001)). They also designed socially optimal decisions in a consensus scenario (González-Pachón and Romero (2011)).

Recently, the contribution of Pérez-Fernández, Rademaker and De Baets (Pérez-Fernández et al. (2017)) presents the search for the closest profile of rankings in a consensus state like an optimization problem based on monometric functions instead of distance functions.

Besides of building social solutions in the vein previously mentioned, there have been developed a lot of approaches in which there is a moderator to drive the consensus reaching process. In this spirit, it may be pointed out Herrera et al. (1996), Ben-Arieh and Chen (2006), Herrera-Viedma et al. (2007), Parreiras et al. (2012), Chiclana et al. (2013), Palomares and Martínez (2014), Wu et al. (2015), Gong et al. (2015) and Liu et al. (2015), among numerous contributions.

Chapter 2

Publication I: A cardinal dissensus measure based on the Mahalanobis distance

T. González-Arteaga, J.C.R. Alcantud, R. de Andrés Calle. A cardinal dissensus measure based on the Mahalanobis distance. European Journal of Operational Research, 251, Issue 2, 575-585. 2016. DOI: 10.1016/j.ejor.2015.11.019.

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Decision Support

A cardinal dissensus measure based on the Mahalanobis distance

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ABSTRACT

In this paper we address the problem of measuring the degree of consensus/dissensus in a context where experts or agents express their opinions on alternatives or issues by means of cardinal evaluations. To this end we propose a new class of distance-based consensus model, the family of the Mahalanobis dissensus measures for profiles of cardinal values. We set forth some meaningful properties of the Mahalanobis dissensus measures. Finally, an application over a real empirical example is presented and discussed.

1. Introduction

In Decision Making Theory and its applications, consensus measurement and its reaching in a society (i.e., a group of agents or experts) are relevant research issues. Many studies investigating the aforementioned subjects have been carried out under several frameworks (see Cabrerizo, Moreno, Pérez, & Herrera-Viedma, 2010; Dong, Xu, & Li, 2008; Dong, Xu, Li, & Feng, 2010; Dong & Zhang, 2014; Fedrizzi, Fedrizzi, & Marques Pereira, 2007; Fu & Yang, 2012; Herrera-Viedma, Herrera, & Chiclana, 2002; Liu, Liao, & Yang, 2015; Palomares, Estrella-Liébana, Martínez, & Herrera, 2014; Wu & Chiclana, 2014a, 2014b; Wu, Chiclana, & Herrera-Viedma, 2015 among others) and based on different methodologies (Chiclana, Tapia García, del Moral, & Herrera-Viedma, 2013; Cook, 2006; Eklund, Rusinowska, & de Swart, 2008; Eklund, Rusinowska, & Swart, 2007; Fedrizzi et al., 2007; Fu & Yang, 2010, 2011; Gong, Zhang, Forrest, Li, & Xu, 2015; González-Pachón & Romero, 1999; Liu et al., 2015; Palomares & Martínez, 2014 among others).

Since the seminal contribution by Bosch (2005) several authors have addressed the consensus measurement topic from an axiomatic perspective. Earlier analyses can be mentioned, e.g., Hays (1960) or Day and McMorris (1985). This issue is also seen as the problem of combining a set of ordinal rankings to obtain an indicator of their 'consensus', a term with multiple possible meanings (Martínez-Panero, 2011).

Generally speaking, the usual axiomatic approaches assume that each individual expresses his or her opinions through ordinal preferences over the alternatives. A group of agents is characterized by the set of their preferences – their preference profile. Then a consensus measure is a mapping which assigns to each preference profile a number between 0 and 1. The assumption is made that the higher the values, the more consensus in the profile.

Technical restrictions on the preferences provide various approaches in the literature. In most cases the agents are presumed to linearly order the alternatives (see Bosch, 2005 or Alcalde-Unzu & Vorsatz, 2013). Since this assumption seems rather demanding (especially as the number of alternatives grows), an obvious extension is to allow for ties. This is the case where the agents have complete preorders on the alternatives (e.g., García-Lapresta & Pérez-Román, 2011). Alcantud, de Andrés Calle, and Cascón (2013a, 2015) take a different position. They study the case where agents have dichotomous opinions on the alternatives, a model that does not necessarily require pairwise comparisons.

Notwithstanding the use of different ordinal preference frameworks, the problem of how to measure consensus is an open-ended question in several research areas. This fact is due to that methodology used in each case is a relevant element in the problem addressed. To date various methods have been developed to measure consensus under ordinal preference structures based on distances and association measures like Kemeny's distance, Kendall's coefficient, Goodman-Kruskal's index and Spearman's coefficient among others (see e.g., Cook & Seiford, 1982; Goodman & Kruskal, 1979; Kemeny, 1959; Kendall & Gibbons, 1990; Spearman, 1904).

In this paper we first tackle the analysis of coherence that derives from profiles of cardinal rather than ordinal evaluations. Modern

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0377-2217/

convention applies the term cardinal to measurements that assign significance to differences (cf., Basu, 1982; Chiclana, Herrera-Viedma, Alonso, & Herrera, 2009; High & Bloch, 1989). In contrast ordinal preferences only permit to order the alternatives from best to worst without any additional information. To see how this affects the analysis of our problem, let us consider a naive example of a society with two agents. They evaluate two public goods with monetary amounts. One agent gives a value of 1€ for the first good and 2€ for the second good. The other agent values these goods at 10€ and 90€ respectively. If we only use the ordinal information in this case, we should conclude that there is unanimity in the society: all members agree that 'good 2 is more valuable than good 1'. However the agents disagree largely. Therefore, the subtleties of cardinality clearly have an impact when we aim at measuring the cohesiveness of cardinal evaluations.

Unlike previous references, we adopt the notion of dissensus measure as the fundamental concept. This seems only natural because it resembles more the notion of a "measure of statistical dispersion", in the sense that 0 captures the natural notion of unanimity as total lack of variability among agents, and then increasingly higher numbers mean more disparity among evaluations in the profile.[1]

In order to build a particular dissensus measure we adopt a distance-based approach. Firstly, one computes the distances between each pair of individuals. Then all these distances are aggregated. In our present proposal the distances (or similarities) are computed through the Mahalanobis distance (Mahalanobis, 1936). We thus define the class of Mahalanobis dissensus measures.

The Mahalanobis distance plays an important role in Statistics and Data Analysis. It arises as a natural generalization of the Euclidean distance. A Mahalanobis distance accounts for the effects of differences in scales and associations among magnitudes. Consequently, building on the well-known performance of the Mahalanobis distance, our novel proposal seems especially fit for the cases when the measurement units of the issues are different, e.g., performance appraisal processes when employees are evaluated attending to their productivity and their leadership capacity; or where the issues are correlated. For example, evaluation of related public projects. An antecedent for the weaker case of profiles of preferences has been provided elsewhere, cf. Alcantud, de Andrés Calle, and González-Arteaga (2013b), and an application to comparisons of real rankings on universities worldwide is developed. Here we apply our new indicator to a real situation, namely, economic forecasts made by several agencies. Since the forecasts concern economic quantities, they have an intrinsic value which is naturally cardinal and also there are relations among them.

The paper is structured as follows. In Section 2, we introduce basic notation and definitions. In Section 3, we set forth the class of the Mahalanobis dissensus measures and their main properties. Section 4 provides a comparison of several Mahalanobis dissensus measures. Next, a practical application with discussion is given in Section 5. Finally, we present some concluding remarks. Appendices contain proofs of some properties and a short review in matrix algebra.

2. Notation and definitions

This section is devoted to introduce some notation and a new concept in order to compare group cohesiveness: namely, dissensus measures. Then, a comparison with the standard approach is made. We partially borrow notation and definitions from Alcantud et al. (2013b). In addition, we use some elements of matrix analysis that we recall in Appendix B to make the paper self-contained.

Let $X = \{x_1, \ldots, x_k\}$ be the finite set of k issues, options, alternatives, or candidates. It is assumed that X contains at least two options, i.e., the cardinality of X is at least 2. Abusing notation, on occasions we refer to issue x_s as issue s for convenience. A population of agents or experts is a finite subset $\mathbf{N} = \{1, 2, \ldots, N\}$ of natural numbers. To avoid trivialities we assume $N > 1$.

We consider that each expert evaluates each alternative by means of a quantitative value. The quantitative information gathered from the set of N experts on the set of k alternatives is summarized by an $N \times k$ numerical matrix M:

$$M = \left(M_{ij}\right)_{N\times k}$$

We write M_i to denote the evaluation vector of agent i over the issues (i.e., row i of M) and M^j to denote the vector with all the evaluations for issue j (i.e., column j of M). For convenience, $(1)_{N\times k}$ denotes the $N \times k$ matrix whose cells are all equal to 1 and $\mathbf{1}_N$ denotes the column vector whose N elements are equal to 1. We write $\mathbb{M}_{N\times k}$ for the set of all $N \times k$ real-valued matrices. Any $M \in \mathbb{M}_{N\times k}$ is called a *profile*.

Any permutation σ of the experts $\{1, 2, \ldots, N\}$ determines a profile M^σ by permutation of the rows of M: row i of the profile M^σ is row $\sigma(i)$ of the profile M. Similarly, any permutation π of the alternatives $\{1, 2, \ldots, k\}$ determines a profile ${}^\pi M$ by permutation of the columns of M: column i of the profile ${}^\pi M$ is column $\pi(i)$ of the profile M.

For each profile $M \in \mathbb{M}_{N\times k}$, its restriction to *subprofile* on the issues in $I \subseteq X$, denoted M^I, arises from exactly selecting the columns of M that are associated with the respective issues in I (in the same order). And for simplicity, if $I = \{j\}$ then $M^I = M^{\{j\}} = M^j$ is column j of M. Any partition $\{I_1, \ldots, I_s\}$ of $\{1, 2, \ldots, k\}$, that we identify with a partition of X, generates a *decomposition*of M into subprofiles $M^{I_1}, \ldots, M^{I_s}$.[2]

A profile $M \in \mathbb{M}_{N\times k}$ is *unanimous* if the evaluations for all the alternatives are the same across experts. In matrix terms, the columns of $M \in \mathbb{M}_{N\times k}$ are constant, or equivalently, all rows of the profile are coincident.

An *expansion* of a profile $M \in \mathbb{M}_{N\times k}$ of $\mathbf{N}$ on $X = \{x_1, \ldots, x_k\}$ is a profile $\bar{M} \in \mathbb{M}_{\bar{N}\times k}$ of $\bar{\mathbf{N}} = \{1, \ldots, N, N+1, \ldots, \bar{N}\}$ on $X = \{x_1, \ldots, x_k\}$, such that the restriction of $\bar{M}$ to the first N experts of $\mathbf{N}$ coincides with M.

Finally, a *replication* of a profile $M \in \mathbb{M}_{N\times k}$ of the society $\mathbf{N}$ on $X = \{x_1, \ldots, x_k\}$ is the profile $M \uplus M \in \mathbb{M}_{2N\times k}$ obtained by duplicating each row of M, in the sense that rows t and $N+t$ of $M \uplus M$ are coincident and equal to row t of M, for each $t = 1, \ldots, N$.

We now define a dissensus measure as follows:

Definition 1. A *dissensus measure* on $\mathbb{M}_{N\times k}$ is a mapping defined by $\delta : \mathbb{M}_{N\times k} \to [0, \infty)$ with the property:

(i) *Unanimity*: for each $M \in \mathbb{M}_{N\times k}$, $\delta(M) = 0$ if and only if the profile $M \in \mathbb{M}_{N\times k}$ is unanimous.
We also define a *normal* dissensus measure as a dissensus measure that additionally verifies:
(ii) *Anonymity*: $\delta(M^\sigma) = \delta(M)$ for each permutation σ of the agents and $M \in \mathbb{M}_{N\times k}$.
(iii) *Neutrality*: $\delta({}^\pi M) = \delta(M)$ for each permutation π of the alternatives and $M \in \mathbb{M}_{N\times k}$.

This definition does not attempt to state dissensus by opposition to consensus. The literature usually deals with a formulation of consensus where the higher the index, the more coherence in the society's opinions. The terms consensus and dissensus should not be taken as formal antonyms, especially because a universally accepted definition of consensus is not available and we do not intend to give an absolute concept of dissensus. However, consensus measures in the sense of Bosch (see Bosch, 2005, Definition 3.1) verify anonymity and neutrality (see also Alcantud et al., 2013b, Definition 1), and from

[1] As a remote antecedent of this position, we note that statistically variance-based methods are commonly employed to measure consensus of verbal opinions (cf., Hoffman, 1994, and Mejias, Shepherd, Vogel, & Lazaneo, 1996.)

[2] A partition of a set S is a collection of pairwise disjoints non-empty subsets of S whose union is S.

a *purely technical* viewpoint, they relate to dissensus measures as follows.

Lemma 1. *If μ is a consensus measure then $1-\mu$ is a normal dissensus measure. Conversely, if δ is a normal dissensus measure then $\frac{1}{\delta+1}$ is a consensus measure.*

Proof 1. We just need to recall that the mapping $i : [0, \infty) \longrightarrow (0, 1]$ given by $i(x) = \frac{1}{x+1}$ is strictly decreasing. □

3. The class of Mahalanobis dissensus measures and its properties

In this section we introduce a broad class of dissensus measures that depends on a reference matrix, namely the Mahalanobis dissensus measures. We also give its more prominent properties.

Our interest is to cover the specific characteristics in cardinal profiles, like possible differences in scales, and correlations among the issues. Before providing our main definition, we recover the definition of the Mahalanobis distance on which our measure is based.

Definition 2. Let $\Sigma \in \mathbb{M}_{k\times k}$ be a positive definite matrix and let us assume that x and y vectors from $\mathbb{R}^k$ are row vectors. The Mahalanobis (squared) distance on $\mathbb{R}^k$ associated with Σ is defined by [3]

$$d_\Sigma(x, y) = (x-y)\Sigma^{-1}(x-y)^t$$

The off-diagonal elements of Σ permit to account for cross relations among the issues or alternatives. Through the diagonal elements different measurement scales can be incorporated. The Σ matrix contains variances and covariances among random variables when the Mahalanobis distance is used in Statistical Data Analysis.

Definition 3. Let $\Sigma \in \mathbb{M}_{k\times k}$ be a positive definite matrix. The *Mahalanobis dissensus measure* on $\mathbb{M}_{N\times k}$ associated with Σ is the mapping $\delta_\Sigma : \mathbb{M}_{N\times k} \to \mathbb{R}$ given by

$$\delta_\Sigma(M) = \frac{1}{C_N^2} \cdot \sum_{i<j} d_\Sigma(M_i, M_j) = \frac{1}{C_N^2} \cdot \sum_{i<j} (M_i - M_j)\Sigma^{-1}(M_i - M_j)^t \tag{1}$$

for each profile $M \in \mathbb{M}_{N\times k}$ on k alternatives, where $C_N^2 = \frac{N(N-1)}{2}$ is the number of non-ordered pairs of the N agents.

Note that the above expression is the average of all distances between the evaluation vectors provided by all pairs of agents according to the Mahalanobis distance associated with Σ (Definition 2).

It is immediate to check that δ_Σ verifies conditions i) and ii) for each positive definite Σ matrix. But δ_Σ fails to satisfy neutrality like the following example proves.

Example 1. Let $\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $k = 2$ and $N = 2$. Then $\Sigma^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$. For $M = \begin{pmatrix} 1 & -1 \\ 3 & 0 \end{pmatrix}$ one has $M_1 = (1, -1)$ and $M_2 = (3, 0)$. Then

$$\delta_\Sigma(M) = \frac{1}{C_2^2} \cdot \left((1-3,\ -1-0)\ \Sigma^{-1}\ (1-3,\ -1-0)^t\right) = \frac{9}{2}.$$

If the columns of M are permuted in order to obtain ${}^\pi M = \begin{pmatrix} -1 & 1 \\ 0 & 3 \end{pmatrix}$, then

$$\delta_\Sigma({}^\pi M) = \frac{1}{C_2^2} \cdot \left((-1-0,\ 1-3)\ \Sigma^{-1}\ (-1-0,\ 1-3)^t\right) = 3.$$

Therefore

$$\delta_\Sigma({}^\pi M) = 3 \neq \frac{9}{2} = \delta_\Sigma(M),$$

which proves that δ_Σ does not verify neutrality.

Nevertheless, if the Σ matrix is adapted according to a specific permutation of the alternatives then the Mahalanobis disensus measure verifies a kind of "soft" neutrality like the following result proves.

Proposition 1. *Let $\Sigma \in \mathbb{M}_{k\times k}$ be a positive definite matrix. For each profile $M \in \mathbb{M}_{N\times k}$ and each permutation π of the alternatives, i.e., a permutation of $\{1, \ldots, k\}$,*

$$\delta_\Sigma(M) = \delta_{\Sigma^\pi}({}^\pi M)$$

where $\Sigma^\pi = P_\pi^t\ \Sigma\ P_\pi$ and P_π is the permutation matrix corresponding to π.

Proof 2. Using the definition of Mahalanobis dissensus measure (Definition 3), it is sufficient to prove that $d_{\Sigma^\pi}({}^\pi M_i, {}^\pi M_j) = d_\Sigma(M_i, M_j)$

$$\begin{aligned} d_{\Sigma^\pi}({}^\pi M_i, {}^\pi M_j) &= ({}^\pi M_i - {}^\pi M_j)\ (\Sigma^\pi)^{-1}\ ({}^\pi M_i - {}^\pi M_j)^t \\ &= (M_i P_\pi - M_j P_\pi)\ (P_\pi^t \Sigma P_\pi)^{-1}\ (M_i P_\pi - M_j P_\pi)^t \\ &= (M_i - M_j)\ P_\pi\ P_\pi^t\ \Sigma^{-1} P_\pi\ P_\pi^t\ (M_i - M_j)^t \\ &= (M_i - M_j)\Sigma^{-1}(M_i - M_j)^t \\ &= d_\Sigma(M_i, M_j). \end{aligned}$$

We have only used the fact that the permutation matrix P_π is orthogonal. □

3.1. Some particular specifications

Some special instances of Mahalanobis dissensus measures have specific interpretations.

- If we have a single issue or alternative, then $M \in \mathbb{M}_{N\times 1}$ is a vector and Σ can be identified as a number $c > 0$. Then

  $$\delta_c(M) = \frac{1}{C_N^2} \cdot \sum_{i<j} \frac{1}{c}(M_i - M_j)^2 = \frac{1}{c} \cdot \frac{2N}{N-1} \cdot S_M^2$$

  where S_M^2 is the sample variance of M.[4] Therefore the dissensus for a single issue is the result of correcting its sample variance by a factor of $\frac{1}{c} \cdot \frac{2N}{N-1}$.
- If Σ is the identity, then $\delta_I(M) = \frac{1}{C_N^2} \cdot \sum_{i<j} \sum_{r=1}^k (M_{ir} - M_{jr})^2$. This expression uses the square of the Euclidean distance between real-valued vectors, thus it recovers a version of the consensus measure for ordinal preferences based on this distance (Cook & Seiford, 1982). Henceforth δ_I is called the *Euclidean dissensus measure*.
- If $\Sigma = diag(c_{11}, \ldots, c_{kk})$ is a diagonal matrix then $d_\Sigma(M_i, M_j)$ gives the weighted average of the square of the differences in assessments for each alternative between agents i and j, where the weight attached to alternative r is $\frac{1}{c_{rr}}$:

  $$\begin{aligned} \delta_\Sigma(M) &= \frac{1}{C_N^2} \cdot \sum_{i<j} d_\Sigma(M_i, M_j) \\ &= \frac{1}{C_N^2} \cdot \sum_{i<j} \left(\sum_{r=1}^k \frac{1}{c_{rr}} \cdot (M_{ir} - M_{jr})^2 \right) \\ &= \sum_{r=1}^k \frac{1}{c_{rr}} \cdot \delta_I(M^r). \end{aligned}$$

[3] Our choice of $d_\Sigma(x, y)$ coincides with the original Mahalanobis' definition (see Mahalanobis, 1936). In order to exploit the inclusion of the Euclidean distance, some authors work with $\sqrt{d_\Sigma(x, y)}$ instead. In both cases we have distances on $\mathbb{R}^k$.

[4] In order to check this, we use a well-known property of the variance: given a vector $x = (x_1, x_2, \ldots, x_n)$, whose mean is $\bar{x}$, $S_x^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2 = \frac{1}{2n^2}\sum_{i=1}^n \sum_{j=1}^n (x_i - x_j)^2$.

This particular specification of the dissensus measure allows us to incorporate different weights to the alternatives. This fact increases the richness of the analysis in comparison with the (square of the) Euclidean distance. Furthermore, if $\Sigma = \lambda I$ for some $\lambda > 0$, then Proposition 2 below gives additional relationships.

Proposition 2 gives the relation between the Euclidean dissensus measure and the Mahalanobis dissensus measure associated with a matrix which is a multiple of the identity matrix, $\Sigma = \lambda I$.

Proposition 2. *For each profile $M \in \mathbb{M}_{N\times k}$ and $\lambda > 0$,*

$$\delta_{\lambda I}(M) = \delta_I\left(\frac{1}{\sqrt{\lambda}} \cdot M\right) = \frac{1}{\lambda} \cdot \delta_I(M).$$

Proof 3. Using Definition 3, the assertion is direct if we check $d_{\lambda I}(M_i, M_j) = d_I(\frac{1}{\sqrt{\lambda}} \cdot M_i, \frac{1}{\sqrt{\lambda}} \cdot M_j)$ and $d_{\lambda I}(M_i, M_j) = \frac{1}{\lambda} \cdot d_I(M_i, M_j)$.

$$\begin{aligned} d_{\lambda I}(M_i, M_j) &= (M_i - M_j)(\lambda I)^{-1}(M_i - M_j)^t \\ &= (M_i - M_j)\begin{pmatrix} \frac{1}{\lambda} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{\lambda} \end{pmatrix}(M_i - M_j)^t \\ &= \sum_{r=1}^{k} \frac{1}{\lambda} \cdot (M_{ir} - M_{jr})^2 = \sum_{r=1}^{k}\left(\frac{1}{\sqrt{\lambda}} \cdot M_{ir} - \frac{1}{\sqrt{\lambda}} \cdot M_{jr}\right)^2 \\ &= \left(\frac{1}{\sqrt{\lambda}} \cdot M_i - \frac{1}{\sqrt{\lambda}} \cdot M_j\right)\begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \\ &\quad \times \left(\frac{1}{\sqrt{\lambda}} \cdot M_i - \frac{1}{\sqrt{\lambda}} \cdot M_j\right)^t \\ &= d_I\left(\frac{1}{\sqrt{\lambda}} \cdot M_i, \frac{1}{\sqrt{\lambda}} \cdot M_j\right). \end{aligned}$$

$$\begin{aligned} d_{\lambda I}(M_i, M_j) &= \frac{1}{\lambda} \cdot \sum_{r=1}^{k}(M_{ir} - M_{jr})^2 \\ &= \frac{1}{\lambda} \cdot (M_i - M_j)\begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}(M_i - M_j)^t \\ &= \frac{1}{\lambda} \cdot d_I(M_i, M_j). \end{aligned}$$

□

3.2. Some properties of the class of Mahalanobis dissensus measures

Measuring cohesiveness by means of the Mahalanobis dissensus measure ensures some interesting operational features. We proceed to examine them. The proofs of these properties are given in Appendix A.

Let $M \in \mathbb{M}_{N\times k}$ denote a profile and let $\Sigma, \Sigma_1, \Sigma_2 \in \mathbb{M}_{k\times k}$ be positive definite matrices. The following properties hold true:

1. *Neutrality*. A dissensus measure δ_Σ verifies neutrality if and only if the associated Σ matrix is a diagonal matrix whose diagonal elements are the same. Formally:
   $\delta_\Sigma(M) = \delta_\Sigma(^\pi M)$ any profile $M \in \mathbb{M}_{N\times k}$ and any permutation π of $\{1, \ldots, k\}$, if and only if $\Sigma = diag\{\lambda, \ldots, \lambda\}$ for some $\lambda > 0$.
2. *Oneness*. If for a particular size N of a society the Mahalanobis dissensus measures associated with two matrices coincide for all possible profiles, then the corresponding dissensus measures are equal. Formally:
   If for a fixed N it is the case that $\delta_{\Sigma_1}(M) = \delta_{\Sigma_2}(M)$ for each profile $M \in \mathbb{M}_{N\times k}$, then $\Sigma_1 = \Sigma_2$, i.e., for each N' and $M' \in \mathbb{M}_{N'\times k}$, it is also the case that
   $\delta_{\Sigma_1}(M') = \delta_{\Sigma_2}(M')$.
3. *Cardinal transformations*. In contrast to ordinal assessments, cardinal evaluations are dependent on scales. So an important question arises about if the scale choice disturbs the cohesiveness measures. In this regard, once we update the reference matrix accordingly, the Mahalanobis dissensus measures associated to Σ do not vary. This fact happens even if we modify the scales of all issues in different way. In addition, a simple translation of each issue by adding a number does not change the cohesiveness measure. Formally:
   Let $a = (a_1, \ldots, a_k)^t$ be a column vector and $B = diag(b_1, \ldots, b_k)$ be a diagonal matrix. The affine transformation of the profile $M \in \mathbb{M}_{N\times k}$ is $M^* = \mathbf{1}_N a^t + MB$, $M^* \in \mathbb{M}_{N\times k}$. Its columns are defined by $M^{*j} = a_j \cdot \mathbf{1}_N + b_j \cdot M^j$ and its rows are defined by $M_i^* = (a_1 + b_1 M_{i1}, \ldots, a_k + b_k M_{ik}) = a + M_i B$.
   If $M^* = \mathbf{1}_N a^t + MB$ is a positive affine transformation of the profile $M \in \mathbb{M}_{N\times k}$ and $\Sigma^* = B\Sigma B^t$ is the corresponding adjusted Σ, then
   $\delta_{\Sigma^*}(M^*) = \delta_\Sigma(M)$.
4. *Replication monotonicity*. When a non-unanimous society is replicated, its dissensus measure increases. That is, if $M \in \mathbb{M}_{N\times k}$ is a non-unanimous profile then
   $$\delta_\Sigma(M \uplus M) = \left(\frac{2N-2}{2N-1}\right) \cdot \delta_\Sigma(M)$$
   therefore
   $\delta_\Sigma(M \uplus M) > \delta_\Sigma(M)$.
   We can note that the difference between such measures is negligible for large societies. In addition, if we have an unanimous profile $M \in \mathbb{M}_{N\times k}$ then by Definition 1 i), δ_Σ verifies
   $\delta_\Sigma(M \uplus M) = \delta_\Sigma(M) = 0$.
5. *Splitting the set of alternatives*. Suppose that the set of alternatives is divided in two (or more) subgroups, in such way that we do not consider any possible cross-effect among subgroups (perhaps because we know that there is not interdependence). Then the computation can be simplified by referring to measures of the dissensus in sub-profiles as follows.
   Given $\Sigma = \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{pmatrix}$, where $\Sigma_{11} \in \mathbb{M}_{r\times r}$, $\Sigma_{22} \in \mathbb{M}_{(k-r)\times(k-r)}$, for each profile $M = (M^{I_1}, M^{I_2})$ where $M^{I_1} \in \mathbb{M}_{N\times r}$, $M^{I_2} \in \mathbb{M}_{N\times(k-r)}$
   $\delta_\Sigma(M) = \delta_{\Sigma_{11}}(M^{I_1}) + \delta_{\Sigma_{22}}(M^{I_2})$.

Remark 1. Note that if the Σ matrix was originally a block diagonal matrix in the form $\Sigma = diag(\Sigma_{11}, \ldots, \Sigma_{ss})$, then it is possible to take the corresponding partition of the set of alternatives, $X = I_1 \cup I_2 \cup \ldots \cup I_s$. Consequently, the original profile $M \in \mathbb{M}_{N\times K}$ can be rewritten like $M = (M^{I_1}, M^{I_2}, \ldots, M^{I_s})$. Then

$$\delta_\Sigma(M) = \sum_{i=1}^{s} \delta_{\Sigma_{ii}}(M^{I_i}).$$

6. *Adding alternatives*. An*extension* of a profile $M \in \mathbb{M}_{N\times k}$ is a new profile, $M^* \in \mathbb{M}_{N\times(k+r)}$, such that M^* includes r new alternatives. Under this assumption, M^* can be seen as a profile with two subgroups, the initial and the new alternatives, $M^* = (M, M^{new}) \in \mathbb{M}_{N\times(k+r)}$. If the aforementioned subgroups of alternatives are not related then Property 5 applies. Consequently,
   $\delta_{\Sigma^*}(M^*) = \delta_\Sigma(M) + \delta_{\Sigma^{new}}(M^{new})$

where $\Sigma^* = \begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma^{new} \end{pmatrix}$ and $\Sigma^{new} \in \mathbb{M}_{r\times r}$ is the associated matrix to the dissensus measure for the r new alternatives.
In the particular case where all the new alternatives added to the profile M are evaluated equally by all agents,

$$\delta_{\Sigma^*}(M^*) = \delta_\Sigma(M),$$

irrespective of Σ^{new} because unanimous profiles produce dissensus measures equal to zero. This particular case is defined like a property called "independence of irrelevant alternatives" in Alcantud, de Andrés Calle, and Cascón (2013a).

7. *Adding agents to the society.* Suppose that a new agent is added to the society, and then the Mahalonabis dissensus measure of the enlarged society does not decrease. In addition, the increment is minimal when the"average agent" is added up. Formally:
Let $M \in \mathbb{M}_{N\times k}$ be a profile and $\bar{M} \in \mathbb{M}_{(N+1)\times k}$ be its expansion after incorporating the evaluations of a new agent. The Mahalanobis dissensus measure for $\bar{M}$ is

$$\delta_\Sigma(\bar{M}) = \frac{N-1}{N+1}\cdot\delta_\Sigma(M) + \frac{1}{C^2_{N+1}}\cdot\sum_{i=1}^{N} d_\Sigma(M_i, \bar{M}_{N+1})$$

where $\bar{M}_{N+1}$ is the row of $\bar{M}$ which incorporates the new agent's assessments for the alternatives.
If the assessments of the new agent coincide with the average of the original agents' evaluations for each alternative, then the minimal increment of the dissensus measure is obtained.

Remark 2. A particular case is when the Mahalanobis dissensus measure is zero, or equivalently, there exits unanimity. If we include a new agent whose evaluations coincide with the assessments of the original agents, the Mahalanobis dissensus measure continues being zero.

4. Comparison of Mahalanobis dissensus measures

In practical situations we could potentially use various Mahalanobis dissensus measures for profiles of cardinal information.[5] Hence it is worth studying the relations among evaluations achieved when we vary the reference matrices. This section addresses this point.

Theorems 1 and 2 below identify conditions on matrices that ensure consistent comparisons between Mahalanobis dissensus measures, whatever the number of agents. Based on these theorems, a final result gives bounds for the Mahalanobis dissensus measure.

Along this section $\Sigma_1, \Sigma_2 \in \mathbb{M}_{k\times k}$ denote two positive definite matrices and $d_{\Sigma_1}, d_{\Sigma_2}$ denote the corresponding Mahalanobis (squared) distances on $\mathbb{R}^k$ associated to Σ_1 and Σ_2. Let $\lambda_1^{(i)} \geq \lambda_2^{(i)} \geq \cdots \geq \lambda_k^{(i)} > 0$ be the eigenvalues of Σ_i, $i = 1, 2$.

Theorem 1. *If there exists N for which each profile $M \in \mathbb{M}_{N\times k}$ verifies $\delta_{\Sigma_1}(M) \geq \delta_{\Sigma_2}(M)$ then*

$$\lambda_i^{(1)} \leq \lambda_i^{(2)} \quad \text{for } i = 1, \ldots, k \tag{2}$$

Proof 4. We take a profile $M \in \mathbb{M}_{k\times k}$ with $M_i = 0$ for $i = 2, 3, \ldots, N$ and $M_1 = x \in \mathbb{R}^k$. By assumption

$$\delta_{\Sigma_1}(M) = \frac{1}{C^2_N}\cdot d_{\Sigma_1}(x,0) \;\geq\; \delta_{\Sigma_2}(M) = \frac{1}{C^2_N}\cdot d_{\Sigma_2}(x,0).$$

Consequently, the hypothesis is reduced to $d_{\Sigma_1}(x,0) \geq d_{\Sigma_2}(x,0)$ for $x \in \mathbb{R}^k$. It means

$$x\Sigma_1^{-1}x^t \geq x\Sigma_2^{-1}x^t \;\Rightarrow\; x\left(\Sigma_1^{-1} - \Sigma_2^{-1}\right)x^t \geq 0 \quad \text{for } x \in \mathbb{R}^k.$$

Then $(\Sigma_1^{-1} - \Sigma_2^{-1})$ is a non-negative definite matrix. Now we use the result included in Appendix B (see Point 11) to finish the proof:

$$\Sigma_1^{-1} \geq \Sigma_2^{-1} \Longrightarrow \frac{1}{\lambda_i^{(1)}} \geq \frac{1}{\lambda_i^{(2)}} \Longrightarrow \lambda_i^{(1)} \leq \lambda_i^{(2)} \text{ for } i = 1, 2, \ldots, k.$$

□

The converse of Theorem 1 is not always true like Example 2 below shows. Nevertheless, Theorem 2 below proves that a partial converse of Theorem 1 holds true under a technical restriction on the definite matrices.

Example 2. Let us consider a particular case of two matrices

$$\Sigma_1 = \begin{pmatrix} 0.18 & -0.16 \\ -0.16 & 0.42 \end{pmatrix} \qquad \Sigma_2 = \begin{pmatrix} 0.60 & 0.20 \\ 0.20 & 0.30 \end{pmatrix}$$

whose eigenvalues verify $\lambda_i^{(1)} \leq \lambda_i^{(2)}$ for $i = 1, 2$ because $\lambda_1^{(1)} = 0.5$, $\lambda_2^{(1)} = 0.1$ and $\lambda_1^{(2)} = 0.7$, $\lambda_2^{(2)} = 0.2$.

Let $M \in \mathbb{M}_{2\times 2}$ be the profile $M = \begin{pmatrix} 4 & 60 \\ 0 & 0 \end{pmatrix}$. The Mahalanobis dissensus measures for M associated with Σ_1 and Σ_2 produce

$$\delta_{\Sigma_1}(M) = 14630.4 \leq 14777.14 = \delta_{\Sigma_2}(M).$$

Therefore it is not true that $\delta_{\Sigma_1}(M) \geq \delta_{\Sigma_2}(M)$ holds throughout.

Theorem 2. *If $\Sigma_1, \Sigma_2 \in \mathbb{M}_{k\times k}$ are commutable matrices and their eigenvalues verify $\lambda_1^{(1)} \leq \lambda_k^{(2)}$ then*

$$\delta_{\Sigma_1}(M) \geq \delta_{\Sigma_2}(M)$$

for each size N and each profile $M \in \mathbb{M}_{N\times k}$.

Proof 5. Assuming $\Sigma_1, \Sigma_2 \in \mathbb{M}_{k\times k}$ are commutable, we can apply Point 12 in Appendix B to Σ_1^{-1} and Σ_2^{-1}. Consequently, there exists an orthonormal matrix $Q \in \mathbb{M}_{k\times k}$ such that

$$Q^t\Sigma_1^{-1}Q = D_1 \quad \text{and} \quad Q^t\Sigma_2^{-1}Q = D_2$$

being $D_1, D_2 \in \mathbb{M}_{k\times k}$ diagonal matrices. It is possible to select Q in such a way that the diagonal elements of D_1 verify $\frac{1}{\lambda_1^{(1)}} \leq \cdots \leq \frac{1}{\lambda_k^{(1)}}$. Thus

$$D_1 = diag\left(\frac{1}{\lambda_1^{(1)}}, \ldots, \frac{1}{\lambda_k^{(1)}}\right) \quad \text{and} \quad D_2 = diag\left(\frac{1}{\lambda_{\pi(1)}^{(2)}}, \ldots, \frac{1}{\lambda_{\pi(k)}^{(2)}}\right),$$

where π is a permutation of $\{1, 2, \ldots, k\}$.[6]

Let $x, y \in \mathbb{R}^k$ be two row vectors. Since Q is an orthonormal matrix, there exists a vector $z \in \mathbb{R}^k$ such that $(x-y)^t = Qz$

$$d_{\Sigma_1}(x,y) = (x-y)\Sigma_1^{-1}(x-y)^t = z^tQ^t\Sigma_1^{-1}Qz = z^tD_1z = \sum_{j=1}^{k}\frac{1}{\lambda_j^{(1)}}z_j^2$$

$$d_{\Sigma_2}(x,y) = (x-y)\Sigma_2^{-1}(x-y)^t = z^tQ^t\Sigma_2^{-1}Qz = z^tD_2z = \sum_{j=1}^{k}\frac{1}{\lambda_{\pi(j)}^{(2)}}z_j^2$$

From premise that $\lambda_1^{(1)} \leq \lambda_k^{(2)}$ we have

$$\frac{1}{\lambda_k^{(1)}} \geq \cdots \geq \frac{1}{\lambda_1^{(1)}} \geq \frac{1}{\lambda_k^{(2)}} \geq \cdots \geq \frac{1}{\lambda_1^{(2)}}.$$

Thus $\frac{1}{\lambda_j^{(1)}} \geq \frac{1}{\lambda_{\pi(j)}^{(2)}}$ for $j = 1, 2, \ldots, k$ and as a result it is obtained

$$\sum_{j=1}^{k}\frac{1}{\lambda_j^{(1)}}z_j^2 \geq \sum_{j=1}^{k}\frac{1}{\lambda_{\pi(j)}^{(2)}}z_j^2.$$

In consequence, $d_{\Sigma_1}(x,y) \geq d_{\Sigma_2}(x,y)$.

[5] This is the case of our real example in Section 5 below.

[6] When Q does not lead to a diagonal matrix with properly ordered eigenvalues, we change Q for $Q' = QP^t$, P being a permutation matrix. Q' is also an orthogonal matrix (see Appendix B, Point 10) which simultaneously diagonalizes Σ_1^{-1} and Σ_2^{-1}. In addition, we get a diagonal matrix D_1^* with the same eigenvalues that D_1 but in the proper order.

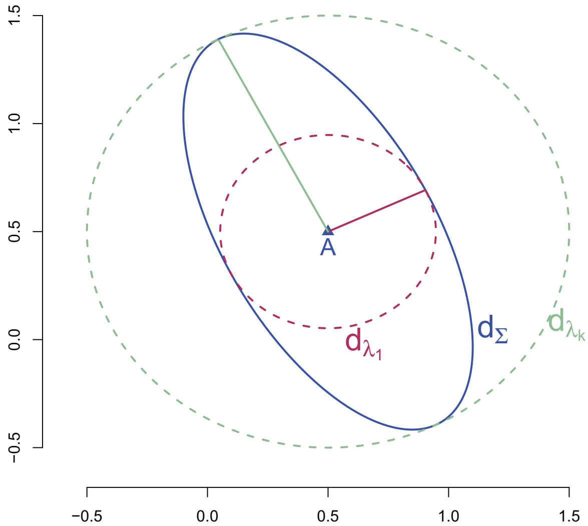

Fig. 1. Curves of equidistance to point A with d_{Σ} (ellipse), $d_{\lambda_1 I}$ and $d_{\lambda_k I}$ (circumferences).

Now, using Definition 3 the theorem is proven. □

Example 3 below shows the relevance of hypothesis on the eigenvalues (Theorem 2).

Example 3. Considering Σ_1 and Σ_2 from Example 2, we observe that they are commutable matrices:

$$\Sigma_1\,\Sigma_2 = \begin{pmatrix} 0.18 & -0.16 \\ -0.16 & 0.42 \end{pmatrix}\begin{pmatrix} 0.6 & 0.2 \\ 0.2 & 0.3 \end{pmatrix} = \begin{pmatrix} 0.076 & -0.012 \\ -0.012 & 0.094 \end{pmatrix}$$

$$\Sigma_2\,\Sigma_1 = \begin{pmatrix} 0.6 & 0.2 \\ 0.2 & 0.3 \end{pmatrix}\begin{pmatrix} 0.18 & -0.16 \\ -0.16 & 0.42 \end{pmatrix} = \begin{pmatrix} 0.076 & -0.012 \\ -0.012 & 0.094 \end{pmatrix}$$

We can see that the assumption $\lambda_1^{(1)} \le \lambda_k^{(2)}$ even if $k = 2$ does not imply $\lambda_i^{(1)} \le \lambda_i^{(2)}$ for $i = 1, 2$ (see Eq. (2), Theorem 1):

$$\begin{aligned} \lambda_1^{(1)} = 0.5 \quad &< \quad 0.7 = \lambda_1^{(2)}, \\ \lambda_2^{(1)} = 0.1 \quad &< \quad 0.2 = \lambda_2^{(2)}, \\ \lambda_1^{(1)} = 0.5 \quad &> \quad 0.2 = \lambda_2^{(2)}. \end{aligned}$$

Example 4 bellow reveals that the commutativity of Σ_1 and Σ_2 is not superfluous in the statement of Theorem 2.

Example 4. Let us consider $\Sigma_1 = \begin{pmatrix} 0.05 & 0 \\ 0 & 0.1 \end{pmatrix}$ and $\Sigma_2 = \begin{pmatrix} 0.6 & 0.2 \\ 0.2 & 0.3 \end{pmatrix}$, with $\lambda_2^{(1)} = 0.05$, $\lambda_1^{(1)} = 0.1$ and $\lambda_2^{(2)} = 0.2$, $\lambda_1^{(2)} = 0.7$. These eigenvalues satisfy $\lambda_1^{(1)} \le \lambda_2^{(2)}$ and Σ_1 and Σ_2 matrices are not commutable:

$$\Sigma_1\,\Sigma_2 = \begin{pmatrix} 0.03 & 0.01 \\ 0.02 & 0.03 \end{pmatrix} \ne \begin{pmatrix} 0.03 & 0.02 \\ 0.01 & 0.03 \end{pmatrix} = \Sigma_2\,\Sigma_1$$

Let $M \in \mathbb{M}_{2\times 2}$ be a specific profile, $M = \begin{pmatrix} 4 & 60 \\ 0 & 0 \end{pmatrix}$. The Mahalanobis dissensus measures for M associated with Σ_1 and Σ_2 produce $\delta_{\Sigma_1}(M) = 360.8 \le 14,777.14 = \delta_{\Sigma_2}(M)$. Therefore it is not true that $\delta_{\Sigma_1}(M) \ge \delta_{\Sigma_2}(M)$ holds throughout.

Theorems 1 and 2 can be extended to r positive definite matrices $\Sigma_1, \ldots, \Sigma_r$ as a matter of course.

Apart from Theorems 1 and 2, the following corollary reveals that the Mahalanobis dissensus measure associated to Σ is confined within bounds depending only on the extreme eigenvalues of Σ.

Corollary 1. *Let $\Sigma \in \mathbb{M}_{k\times k}$ be a positive definite matrix with eigenvalues $\lambda_1 \ge \cdots \ge \lambda_k$, it is verified*

$$\delta_{\lambda_1 I}(M) \le \delta_{\Sigma}(M) \le \delta_{\lambda_k I}(M)$$

or equivalently

$$\frac{1}{\lambda_1} \cdot \delta_I(M) \le \delta_{\Sigma}(M) \le \frac{1}{\lambda_k} \cdot \delta_I(M)$$

for each N and for each $M \in \mathbb{M}_{N\times k}$.

Proof 6. This result is straightforward from Theorem 2. Observe that such Theorem can be applied because $\lambda_k I$ (resp., $\lambda_1 I$) and M are commutable matrices and the eigenvalues of the diagonal matrix $\lambda_k I$ (resp., $\lambda_1 I$) are all equal to λ_k (resp. λ_1). Proposition 2 is used. □

Fig. 1 illustrates the previous corollary regarding the distances used for $\delta_{\lambda_1 I}$, δ_{Σ} and $\delta_{\lambda_k I}$. We can observe that all points on the ellipse have the same Mahalanobis distance to point A, namely d_{Σ}. Moreover, distance d_{Σ} is always between the values of the corresponding distances $d_{\lambda_1 I}$ and $d_{\lambda_k I}$.

5. Discussion on practical application using a real example

In this section we fully develop a real example. It aims at giving an explicit application of our proposal and discussing some of its features.

We are interested in assessing the cohesiveness of the forecasts of various magnitudes for the Spanish Economy in 2014: GDP (Gross Domestic Product), Unemployment Rate, Public Deficit, Public Debt and Inflation. These forecasts have been published by different institutions and organizations, and each one was made at around the same time. Specifically, three waves of forecasts were published in

Table 1
Forecasts for several magnitudes for the Spanish Economy for the year 2014 published in Spring of 2013.

	GDP	U. Rate	P. Deficit	P. Debt	Inflation
IMF	0.70	26.40	−6.90	97.60	1.50
OECD	0.40	28.00	−6.40	97.00	0.40
European Commission	0.90	26.40	−7.00	91.30	0.80
BBVA research	0.90	26.40	−5.70	96.30	1.20
FUNCAS	0.50	26.00	−4.60	99.20	1.60

Abbreviations: Unemployment Rate (U. Rate), Public Deficit/Debt (P. Deficit/Debt), Banco Bilbao Vizcaya Argentaria Reseach (BBVA Research), Fundación de las Cajas de Ahorros (FUNCAS).

Table 2
Forecasts for several magnitudes for the Spanish Economy for the year 2014 published in Autumn of 2013.

	GDP	U. Rate	P. Deficit	P. Debt	Inflation
IMF	0.20	26.70	−5.80	99.10	1.50
OECD	0.50	26.30	−6.10	98.00	0.50
European Commission	0.50	26.40	−5.90	99.90	0.90
BBVA Research	0.90	25.60	−5.80	98.50	1.10
FUNCAS	1.00	25.90	−5.90	100.50	1.30

Abbreviations: Unemployment Rate (U. Rate), Public Deficit/Debt (P. Deficit/Debt), Banco Bilbao Vizcaya Argentaria Reseach (BBVA Research), Fundación de las Cajas de Ahorros (FUNCAS).

Table 3
Forecasts for several magnitudes for the Spanish Economy for the year 2014 published in Spring of 2014.

	GDP	U. Rate	P. Deficit	P. Debt	Inflation
IMF	0.90	25.50	−5.89	98.80	0.50
OECD	1.00	25.40	−5.50	98.30	0.10
European Commission	1.10	25.50	−5.60	103.80	0.10
BBVA Research	1.10	25.10	−5.80	98.40	1.10
FUNCAS	1.20	25.10	−6.00	100.00	0.10

Abbreviations: Unemployment Rate (U. Rate), Public Deficit/Debt (P. Deficit/Debt), Banco Bilbao Vizcaya Argentaria Reseach (BBVA Research), Fundación de las Cajas de Ahorros (FUNCAS).

the Spring of 2013 (Table 1), Autumn of 2013 (Table 2) and Spring of 2014 (Table 3).

We intend to measure the cohesiveness of the aforementioned predictions. Since they are expressed by cardinal valuations, we need to go beyond the traditional analyses referred to in this paper. To this purpose, we first gather the data corresponding to Tables 1–3 in the profiles $M^{(S)}, M^{(A)}, M^{(IS)} \in \mathbb{M}_{5\times 5}$, respectively. Next, we select a suitable reference matrix and finally we make the computations of the Mahalanobis dissensus measures.

5.1. Reference matrix

Once the profiles have been fixed, the following step to compute their Mahalanobis dissensus measures is to avail oneself of a suitable reference matrix Σ. The choice of such a matrix can easily raise controversy. Nevertheless, we can learn from the role of the Σ matrix in the Mahalanobis distance from a statistical point of view. This matrix contains the variances and covariances among the statistical variables, therefore, those characteristics are brought into play in this distance. We recall that covariances (or corresponding correlations) among variables reveal their interdependence. In statistics, this Σ matrix is usually unknown and it is estimated from a sample. One exception is the unlikely case when the data are generated by a known multivariate probability distribution. This is not the case of our example.

Therefore we employ a reference matrix Σ that captures the variances and covariances among the macroeconomic magnitudes of the Spanish Economy. It seems natural to produce such a matrix from historical macroeconomic data corresponding to the issues under inspection. Table 4 contains such recorded data, and Table 5 gives the corresponding correlation coefficients. [7] These values are depicted in Fig. 2. Each ellipse represents the correlation between a pair of variables. The ellipses slant upward (resp., downward) show a positive (resp., negative) correlation. Moreover, the narrower the ellipse

Table 4
Past data for the Spanish Economy (2001–2012). Source: Spanish National Statistics Institute (INE) and Bank of Spain.

Year	GDP	U. Rate	P. Deficit	P. Debt	Inflation
2001	3.70	10.55	0.50	55.60	2.70
2002	2.70	11.47	0.20	52.60	3.50
2003	3.10	11.48	0.30	48.80	3.00
2004	3.30	10.97	0.10	46.30	3.00
2005	3.60	9.16	−1.30	43.20	3.40
2006	4.10	8.51	−2.40	39.70	3.50
2007	3.50	8.26	−1.90	36.30	2.80
2008	0.90	11.33	4.50	40.20	4.10
2009	−3.70	18.01	11.20	53.90	−0.30
2010	−0.30	20.06	9.70	61.50	1.80
2011	0.40	21.64	9.40	69.30	3.20
2012	−1.40	25.03	10.60	84.20	2.40

Table 5
Correlations between macroeconomic magnitudes for historial data.

	GDP	U. Rate	P. Deficit	P. Debt	Inflation
GDP	1.00	−0.81	−0.94	−0.59	0.73
U. Rate	−0.81	1.00	0.93	0.92	−0.46
P. Deficit	−0.94	0.93	1.00	0.75	−0.60
P. Debt	−0.59	0.92	0.75	1.00	−0.30
Inflation	0.73	−0.46	−0.60	−0.30	1.00

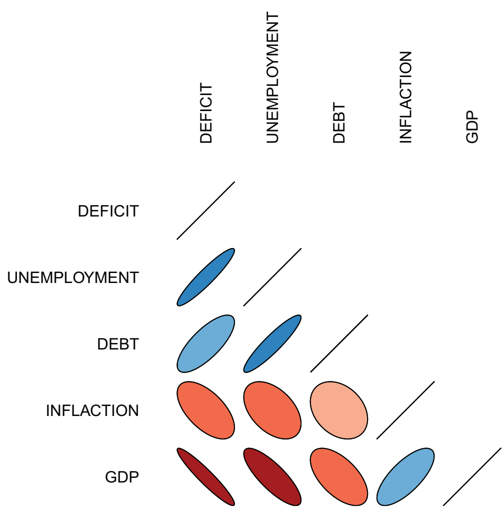

Fig. 2. A depiction of the correlation matrix of the Spanish macroeconomic data from 2001 to 2012.

[7] Given two vectors $X = (x_1, \ldots, x_n)'$ and $Y = (y_1, \ldots, y_n)'$ with $\bar{x}$ and $\bar{y}$ their respective means, the correlation coefficient between X and Y is computed by $cor(X, Y) = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}$.

Table 6
Dissensus between pairs of agents for the profiles of forecasts published in Spring of 2013 (in descending order), Autumn of 2013 and Spring of 2014.

		Spring 2013	Autumn 2013	Spring 2014
OECD	FUNCAS	23.18	2.85	0.27
European Comm.	FUNCAS	19.65	1.21	0.61
IMF	FUNCAS	9.31	3.63	1.15
OECD	BBVA Research	8.05	2.50	2.04
European Comm.	BBVA Research	5.62	1.25	2.86
IMF	OECD	5.24	2.76	0.93
IMF	European Comm.	4.87	1.47	2.62
BBVA Research	FUNCAS	4.52	0.12	2.10
IMF	BBVA Research	3.31	4.11	0.86
OECD	European Comm.	0.79	0.60	1.61

the stronger correlation represented. For example, the pair formed by GDP and Public Deficit holds the strongest negative correlation.

On the basis of Table 4, we compute the corresponding variance-covariance matrix $\mathbf{\Sigma}$[8].

$$\mathbf{\Sigma} = \begin{pmatrix} 6.11 & -11.49 & -12.43 & -20.19 & 2.03 \\ -11.49 & 32.74 & 28.43 & 72.42 & -2.97 \\ -12.43 & 28.43 & 28.41 & 55.41 & -3.60 \\ -20.19 & 72.42 & 55.41 & 190.52 & -4.73 \\ 2.03 & -2.97 & -3.60 & -4.73 & 1.28 \end{pmatrix}$$

5.2. Computation of the dissensus

Now we calculate the Mahalanobis dissensus measures associated with $\mathbf{\Sigma}$ for the profiles of the forecasts for the Spanish Economy, namely, $M^{(S)}$, $M^{(A)}$ and $M^{(IS)}$.

We obtain the following Mahalanobis dissensus measures associated with the aforementioned $\mathbf{\Sigma}$:

$$\delta_{\mathbf{\Sigma}}(M^{(S)}) = 8.45, \quad \delta_{\mathbf{\Sigma}}(M^{(A)}) = 2.05, \quad \delta_{\mathbf{\Sigma}}(M^{(IS)}) = 1.51.$$

Note that the measure of the dissensus decreases along the time. This is what we intuitively expect, since the latter forecasts rest on more accurate and factual information.

Apart from the measure of the cohesiveness of the profiles, our proposal also produces a measure of divergence among the evaluations of different agents on a set of issues. We can answer questions like "*Are the predictions of the European Commission for the Spanish Economy similar to the predictions of the BBVA Reseach?*" or "*Is the previous comparison more or less similar than the comparison between the predictions of the BBVA Research vs. the predictions of the IMF?*". Table 6 provides these items for comparison.

5.3. Other simpler approaches: drawbacks or limitations

The choice of the reference matrix is a key point in the application of the Mahalanobis dissensus measure. As an explanatory exercise in this subsection we discuss on the more simplistic approaches where naive reference matrices are employed. If we use the identity matrix as the reference matrix (for example, because we lack data to make a better inference), then we get a Mahalanobis dissensus measure which gives the same importance to the differences in all the issues (see Section 3.1). However the choice of the identity matrix as the reference matrix discards much relevant information. We note the variance of the Public Debt is 190.52, while Inflation has a variance of 1.28 (see $\mathbf{\Sigma}$). So, a difference of one unit in the forecasts from two agents does not signify the same if such a difference corresponds to Inflation or to Public Debt.

[8] Let X be a $n \times k$ matrix whose columns have means $\overline{X}_i$, $i = 1, \ldots, k$. The cells of the variance-covariance matrix are $\Sigma_{ij} = \frac{1}{n-1}\sum_{r=1}^{n}(x_{ri} - \overline{X}_i)(x_{rj} - \overline{X}_j)$.

Table 7
Dissensus for several profiles of economic forecasts for the Spanish Economy for the year 2014. Data published in Spring of 2013, Autumn of 2013 and in Spring of 2014.

		Profiles		
Reference matrix		$M^{(S)}$ Spring 2013	$M^{(A)}$ Autumn 2013	$M^{(IS)}$ Spring 2014
Σ	$\delta_{\mathbf{\Sigma}}$	8.45	2.05	1.51
Diagonal	$\delta_{\mathbf{\Sigma}_\sigma}$	0.61	0.29	0.37
Identity	δ_I	21.59	2.97	11.20

We could alternatively employ as the reference matrix, the diagonal matrix with the variances of the issues, that is,

$$\mathbf{\Sigma}_\sigma = diag(6.11, 32.74, 28.41, 190.52, 1.28).$$

In this case, we remove the effects of the interdependence among the economic magnitudes on the dissensus measure.

In order to check that an inconvenient choice of the reference matrix easily produces misleading conclusions. Table 7 shows the dissensus measures derived from the three matrices mentioned above, $\mathbf{\Sigma}$, I and $\mathbf{\Sigma}_\sigma$. The dissensus $\delta_{\mathbf{\Sigma}}$ is decreasing along time as previously reported. This intuitively appealing feature is not captured when we utilize simpler matrices. Consequently, introducing corrections due to variances or to cross-effects is crucial for a reliable final analysis.

6. Concluding remarks

We explore the problem of measuring the degree of cohesiveness in a setting where experts express their opinions on alternatives or issues by means of cardinal evaluations. We use the general concept of dissensus measure and introduce one particular formulation based on the Mahalanobis distance for numerical vectors, namely the Mahalanobis dissensus measure.

We provide some properties which make our proposal appealing. We emphasize that the Mahalanobis dissensus measure on the profiles with k issues or alternatives is scale-independent for each issue and it accounts for cross-relations of issues. In addition, the comparison between different Mahalanobis dissensus measures can be made through the eigenvalues of their associated matrices.

We illustrate our proposal with a real numerical application about forecasts for several magnitudes for the Spanish Economy. We discuss the relevance of the choice of the reference matrix in this context.

Acknowledgments

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Appendix A. Proofs of properties in Section 3.2

Proof of property 1. Neutrality. Let us first prove sufficiency. If $\Sigma = diag\{\lambda, \ldots, \lambda\}$ for a value $\lambda > 0$, the thesis is straightforward from the Definition 3.

Let us now prove necessity. Due to the fact that δ_Σ verifies neutrality for any profile $M \in \mathbb{M}_{N\times k}$ and for any permutation π of $\{1, \ldots, k\}$

$$\delta_\Sigma(M) = \delta_\Sigma(^{\pi}M),$$

it must be deduced $\Sigma = diag\{\lambda, \ldots, \lambda\}$ for a value $\lambda > 0$.

Let $M \in \mathbb{M}_{2\times k}$ be a particular profile such that $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$.

The dissensus measure for $M \in \mathbb{M}_{2\times k}$ is given by $\delta_\Sigma(M) = (M_1 - M_2)\Sigma^{-1}(M_1 - M_2)^t$ according to Definition 3. If M is permuted by means of π, we obtain the matrix ${}^\pi M \in \mathbb{M}_{2\times k}$ and consequently its dissensus measure is $\delta_\Sigma({}^\pi M) = ({}^\pi M_1 -{}^\pi M_2)\Sigma^{-1}({}^\pi M_1 -{}^\pi M_2)^t$.

According to Point 10 in Appendix B, we can write ${}^\pi M = MP_\pi$, being $P_\pi \in \mathbb{M}_{k\times k}$ the corresponding permutation matrix. Consequently,

$$\begin{aligned}\delta_\Sigma({}^\pi M) &= (M_1P_\pi - M_2P_\pi)\Sigma^{-1}(M_1P_\pi - M_2P_\pi)^t \\ &= (M_1 - M_2)P_\pi\Sigma^{-1}P_\pi^t(M_1 - M_2)^t.\end{aligned}$$

Since δ_Σ verifies neutrality, $\delta_\Sigma(M) = \delta_\Sigma({}^\pi M)$ for any $M \in \mathbb{M}_{2\times k}$,

$$\Sigma^{-1} = P_\pi\Sigma^{-1}P_\pi^t.$$

Using the spectral decomposition (see Appendix B, Points 15 and 16) Σ^{-1} can be written as $\Sigma^{-1} = \Gamma D_\lambda^{-1}\Gamma^t$ for a unique orthogonal matrix Γ. Therefore

$$\Sigma^{-1} = P_\pi\Sigma^{-1}P_\pi^t = P_\pi\Gamma D_\lambda^{-1}\Gamma^tP_\pi^t.$$

Observe that the matrix $P_\pi\Gamma$ is orthogonal because it is the product of two orthogonal matrices. Since the spectral decomposition assures that Γ is unique, it must be

$$\Gamma = P_\pi\Gamma$$

for every $P_\pi \in \mathbb{M}_{k\times k}$ permutation matrix. Note that this equation implies that performing any permutation of the rows of Γ produces Γ.

Therefore Γ must be the identity matrix, i.e., $\Gamma = I$.

We can now deduce

$$\Sigma^{-1} = \Gamma D_\lambda^{-1}\Gamma^t = D_\lambda^{-1},$$

$$\Sigma^{-1} = P_\pi\Gamma D_\lambda^{-1}\Gamma^tP_\pi^t = P_\pi D_\lambda^{-1}P_\pi^t.$$

Thus we conclude that Σ is a diagonal matrix.

Let us now prove that the diagonal elements of $\Sigma = D_\lambda$ are all equal.

From the above equalities of Σ^{-1}, it is verified $D_\lambda^{-1} = P_\pi D_\lambda^{-1}P_\pi^t$, for any permutation π of $\{1, \ldots, k\}$.

For the particular permutation matrix

$$P_\pi = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

we obtain $P_\pi D_\lambda^{-1}P_\pi^t = diag\{\lambda_2^{-1}, \lambda_1^{-1}, \ldots, \lambda_k^{-1}\}$ and given that $D_\lambda^{-1} = P_\pi D_\lambda^{-1}P_\pi^t$, it must be $\lambda_1 = \lambda_2$. A routine modification of the argument proves $\lambda_1 = \lambda_j$, $j = 3, \ldots, k$. □

Proof of property 2. Oneness. Let N be a fixed value. We take a profile $M \in \mathbb{M}_{N\times k}$ with $M_i = 0$ for $i = 2, \ldots, N$ and $M_1 = x \in \mathbb{R}^k$ any row vector. For this particular profile the hypothesis $\delta_{\Sigma_1}(M) = \delta_{\Sigma_2}(M)$ reduces to $d_{\Sigma_1}(x, 0) = d_{\Sigma_2}(x, 0)$. It means that, $x\left(\Sigma_1^{-1} - \Sigma_2^{-1}\right)x^t = 0$ and $\left(\Sigma_1^{-1} - \Sigma_2^{-1}\right)$ is a non-negative definite matrix.

Let c_{ij} be the elements of the matrix $\left(\Sigma_1^{-1} - \Sigma_2^{-1}\right)$. Considering the ith row vector of the canonical base $e_i = (0, \ldots, 1, \ldots, 0)$ then, $e_i\left(\Sigma_1^{-1} - \Sigma_2^{-1}\right)e_i^t = c_{ii} = 0$. Therefore $c_{11} = \cdots = c_{kk} = 0$ and $trace\left(\Sigma_1^{-1} - \Sigma_2^{-1}\right) = 0$. As a consequence, using Appendix B (Point 13), $\Sigma_1 = \Sigma_2$. □

Proof of property 3. Cardinal transformations. Let $a = (a_1, \ldots, a_k)^t$ be a column vector and $B = diag(b_1, \ldots, b_k)$ be a diagonal matrix. The affine transformation of the profile $M \in \mathbb{M}_{N\times k}$ is $M^* = \mathbf{1}_N\, a^t + M\,B$, $M^* \in \mathbb{M}_{N\times k}$. Its columns are defined by $M^{*j} = a_j \cdot \mathbf{1}_N + b_j \cdot M^j$ and its rows are defined by $M_i^* = (a_1 + b_1M_{i1}, \ldots, a_k + b_kM_{ik}) = a + M_i\,B$.

Let $\Sigma^* = B\Sigma B^t$ be the Σ matrix updated according to the affine transformation. Then, all elements σ_{ij}^* of Σ^* and all elements σ_{ij} of Σ are related by $\sigma_{ij}^* = b_ib_j\sigma_{ij}$. Due to the fact that B is a diagonal matrix, $B = B^t$ and $(\Sigma^*)^{-1} = B^{-1}\Sigma^{-1}B^{-1}$. We now proceed to compute the Mahalanobis distance under the previous remarks:

$$\begin{aligned}d_{\Sigma^*}(M_i^*, M_j^*) &= (M_i^* - M_j^*)(\Sigma^*)^{-1}(M_i^* - M_j^*)^t \\ &= (a + M_iB - a - M_jB)(B\Sigma B^t)^{-1}\,(a + M_iB - a - M_jB)^t \\ &= (M_i - M_j)\,BB^{-1}\Sigma^{-1}B^{-1}B\,(M_i - M_j)^t \\ &= (M_i - M_j)\Sigma^{-1}(M_i - M_j)^t \\ &= d_\Sigma(M_i, M_j).\end{aligned}$$

Based on the previous distance, we obtain:

$$\delta_{\Sigma^*}(M^*) = \frac{1}{C_N^2} \cdot \sum_{i<j} d_{\Sigma^*}(M_i^*, M_j^*) = \frac{1}{C_N^2} \cdot \sum_{i<j} d_\Sigma(M_i, M_j) = \delta_\Sigma(M)$$

□

Proof of property 4. Replication monotonicity. Let us compute the Mahalanobis dissensus measure for $M \uplus M$.

$$\begin{aligned}\delta_\Sigma(M \uplus M) &= \frac{1}{C_{2N}^2} \cdot \sum_{i=1}^{2N}\sum_{\substack{j=1\\ i<j}}^{2N} d_\Sigma((M \uplus M)_i, (M \uplus M)_j) \\ &= \frac{1}{C_{2N}^2} \cdot \left(\sum_{i=1}^{N}\sum_{\substack{j=1\\ i<j}}^{N} d_\Sigma(M_i, M_j) + \sum_{i=1}^{N}\sum_{j=N+1}^{2N} d_\Sigma(M_i, M_j)\right) \\ &+ \frac{1}{C_{2N}^2} \cdot \left(\sum_{i=N}^{2N}\sum_{\substack{j=1\\ i<j}}^{2N} d_\Sigma(M_i, M_j)\right) = \frac{1}{C_{2N}^2} \cdot C_N^2 \cdot \delta_\Sigma(M) \\ &+ \frac{1}{C_{2N}^2} \cdot \sum_{i=1}^{N}\sum_{r=1}^{N} d_\Sigma(M_i, M_{N+r}) \\ &+ \frac{1}{C_{2N}^2} \cdot \sum_{i=1}^{N}\sum_{\substack{j=1\\ i<j}}^{N} d_\Sigma(M_i, M_j) \\ &= \frac{1}{C_{2N}^2} \cdot \left(4C_N^2 \cdot \delta_\Sigma(M)\right) = \left(\frac{2N-2}{2N-1}\right) \cdot \delta_\Sigma(M)\end{aligned}$$

Therefore

$$\delta_\Sigma(M \uplus M) = \left(\frac{2N-2}{2N-1}\right) \cdot \delta_\Sigma(M)$$

and in particular

$$\delta_\Sigma(M \uplus M\,) > \delta_\Sigma(M).$$

□

Proof of property 5. Splitting the set of alternatives. We set $X = I_1 \cup I_2 = \{x_1, \ldots, x_r\} \cup \{x_{r+1}, \ldots, x_k\}$ as a partition of the alternatives. Given $\Sigma = \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{pmatrix}$, where $\Sigma_{11} \in \mathbb{M}_{r\times r}$, $\Sigma_{22} \in \mathbb{M}_{(k-r)\times(k-r)}$, for each profile $M = (M^{I_1}, M^{I_2})$ where $M^{I_1} \in \mathbb{M}_{N\times r}$, $M^{I_2} \in \mathbb{M}_{N\times(k-r)}$. Recalling Point 5 in Appendix B

$$\Sigma^{-1} = \begin{pmatrix} \Sigma_{11}^{-1} & 0 \\ 0 & \Sigma_{22}^{-1} \end{pmatrix}.$$

We are now in a position to calculate $d_\Sigma(M_i, M_j)$, the Mahalanobis distance between a pair of agents i and j:

$$\begin{aligned}d_\Sigma(M_i, M_j) &= (M_i^{I_1} - M_j^{I_1})\Sigma_{11}^{-1}(M_i^{I_1} - M_j^{I_1})^t \\ &+ (M_i^{I_2} - M_j^{I_2})\Sigma_{22}^{-1}(M_i^{I_2} - M_j^{I_2})^t \\ &= d_{\Sigma_{11}}(M_i^{I_1}, M_j^{I_1}) + d_{\Sigma_{22}}(M_i^{I_2}, M_j^{I_2}).\end{aligned}$$

Using Definition 3, the Mahalanobis dissensus measure on M associated with Σ is given by

$$\begin{aligned}\delta_\Sigma(M) &= \frac{1}{C_N^2}\cdot\sum_{i<j} d_\Sigma(M_i, M_j)\\ &= \frac{1}{C_N^2}\cdot\sum_{i<j}\left(d_{\Sigma_{11}}(M_i^{I_1}, M_j^{I_1}) + d_{\Sigma_{22}}(M_i^{I_2}, M_j^{I_2})\right)\\ &= \delta_{\Sigma_{11}}(M^{I_1}) + \delta_{\Sigma_{22}}(M^{I_2}).\end{aligned} \tag{A.1}$$

It is easy to check that this property holds true for any size of the partition. We set $X = I_1 \cup I_2 \cup \ldots \cup I_s$ as a partition of the alternatives. Considering not cross-effects among the subsets of the alternatives, the $\Sigma \in \mathbb{M}_{k\times k}$ matrix has a block diagonal form, $\Sigma = diag(\Sigma_{11}, \ldots, \Sigma_{ss})$. Analogously, a profile $M \in \mathbb{M}_{N\times k}$ can be written as $M = (M^{I_1}, M^{I_2}, \ldots, M^{I_s})$. Then

$$\delta_\Sigma(M) = \sum_{i=1}^{s} \delta_{\Sigma_{ii}}(M^{I_i}).$$

□

Proof of property 6. Adding alternatives. The proof is straightforward from Eq. (A.1). □

Proof of property 7. Adding agents to the society. Let $M \in \mathbb{M}_{N\times k}$ be a profile on X of the society **N**, $\bar{M} \in \mathbb{M}_{(N+1)\times k}$ an expansion of M by adding the evaluations of a new agent, $\bar{M}_{N+1}$. Then

$$\begin{aligned}\delta_\Sigma(\bar{M}) &= \frac{1}{C_{N+1}^2}\cdot\sum_{i<j} d_\Sigma(\bar{M}_i, \bar{M}_j) = \frac{1}{C_{N+1}^2}\cdot\sum_{i=1}^{N+1}\sum_{\substack{j=1\\ i<j}}^{N+1} d_\Sigma(\bar{M}_i, \bar{M}_j)\\ &= \frac{1}{C_{N+1}^2}\cdot\left(\sum_{i=1}^{N}\sum_{\substack{j=1\\ i<j}}^{N} d_\Sigma(M_i, M_j) + \sum_{i=1}^{N} d_\Sigma(M_i, \bar{M}_{N+1})\right)\\ &= \frac{1}{C_{N+1}^2}\cdot\left(C_N^2\cdot\delta_\Sigma(M) + \sum_{i=1}^{N} d_\Sigma(M_i, \bar{M}_{N+1})\right)\\ &= \frac{N-1}{N+1}\cdot\delta_\Sigma(M) + \frac{1}{C_{N+1}^2}\cdot\sum_{i=1}^{N} d_\Sigma(M_i, \bar{M}_{N+1}).\end{aligned}$$

Now we have to minimize $\delta_\Sigma(\bar{M})$. Obviously, the vector which minimizes $\delta_\Sigma(\bar{M})$ is the vector that gathers the opinion of the agent $N+1$ in the profile $\bar{M}$. For simplicity we recall $\bar{M}_{N+1}$ like $x \in \mathbb{R}^k$. From $\delta_\Sigma(\bar{M})$ expression, it is enough to resolve

$$\min_x \sum_{i=1}^{N} d_\Sigma(M_i, x) = \min_x \sum_{i=1}^{N}\left(M_i\Sigma^{-1}M_i^t - 2M_i\Sigma^{-1}x^t + x\Sigma^{-1}x^t\right),$$

or equivalently,

$$\min_x \sum_{i=1}^{N}\left(-2M_i\Sigma^{-1}x^t + x\Sigma^{-1}x^t\right).$$

We solve it by the standard method using Point 14 in Appendix B.

$$\begin{aligned}&\frac{\partial}{\partial x}\sum_{i=1}^{N}\left(-2M_i\Sigma^{-1}x^t + x\Sigma^{-1}x^t\right)\\ &= -2\sum_{i=1}^{N}\left(M_i\Sigma^{-1}\right)^t + \sum_{i=1}^{N} 2\Sigma^{-1}x^t\\ &= -2\left(\sum_{i=1}^{N}\Sigma^{-1}M_i^t\right) + 2N\Sigma^{-1}x^t\\ &= -2\Sigma^{-1}\left(\sum_{i=1}^{N} M_i^t - Nx^t\right) = 0.\end{aligned}$$

$$\sum_{i=1}^{N} M_i^t - Nx^t = 0 \Longrightarrow x = \frac{1}{N}\sum_{i=1}^{N} M_i.$$

Due to the fact that the second derivative is $2N\Sigma^{-1}$, a positive definite matrix, we have a minimum in $x = \frac{1}{N}\sum_{i=1}^{N} M_i$. □

Appendix B. Review in matrix algebra

This appendix contains some technical results and background material of matrix analysis which are particularly useful in this paper. Let A be a real matrix of order $n \times n$.

1. A diagonal matrix A with diagonal elements $a_{11}, a_{22}, \ldots, a_{nn}$ is represented as $A = diag(a_{11}, a_{22}, \ldots, a_{nn})$.
2. The trace of a matrix A of dimension $n \times n$ is the sum of its diagonal elements, i.e., $trace(A) = \sum_{i=1}^{n} = a_{ii}$.
3. Two matrices A and B of dimensions $n \times n$ are commutable if $AB = BA$. It is also said that they commute. We say that a family of $n \times n$ matrices $A_1, A_2, \ldots, A_k$ is a commutable family if for any $i, j \in \{1, \ldots, k\}$, A_i and A_j commute.
4. A matrix A is orthogonal if $A^TA = AA^T = I$, i.e. $A^{-1} = A^T$.
5. The inverse matrix of a partitioned matrix $A = \begin{pmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}$, where A_{11} and A_{22} are non-singular, is
$$\begin{pmatrix} (A_{11} - A_{21}A_{22}^{-1}A_{12})^{-1} & -A_{11}A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}\\ -(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}\end{pmatrix}.$$
6. Let v be a vector $n \times 1$. A symmetric matrix A is a positive semi-definite matrix (or non-negative definite matrix) if $v^tAv \geq 0$ and A is a positive definite matrix if $v^tAv > 0$ for all non-zero vector v.
7. If there exist a scalar λ and a non-zero vector γ such that $A\gamma = \lambda\gamma$, we call them an eigenvalue of A and an associated eigenvector, respectively.
8. There are up to n eigenvalues $\lambda_1, \ldots, \lambda_n$ of A. If A is a positive semi-definite matrix, its eigenvalues are all non-negative.
9. If A is a positive definite matrix, its eigenvalues $\lambda_1, \ldots, \lambda_n$ are positive values and A^{-1} has eigenvalues $\lambda_1^{-1}, \ldots, \lambda_n^{-1}$.
10. A permutation matrix of order $n \times n$ is a square matrix obtained from the same size identity matrix by a permutation of rows. Let π be a permutation of {1, 2, ..., k} and let e_i be the ith vector of the canonical base of $\mathbb{R}^n$, that is, $e_{ij} = 1$ if $i = j$, $e_{ij} = 0$ otherwise. We define the permutation matrix P_π whose rows are $e_{\pi(i)}$. We rearrange the corresponding rows (resp. columns) of A using the permutation π by left (resp., right) multiplication, $P_\pi A$ (resp., AP_π). Every row and every column of a permutation matrix contain exactly one nonzero entry, which is 1. A product of permutation matrices is again a permutation matrix. The inverse of a permutation matrix is again a permutation matrix. In fact, $P^{-1} = P^t$.
11. Let A and B be $p \times p$ symmetric matrices. If $A - B$ is a non-negative definite matrix, then it is expressed as $A \geq B$. In this case $ch_i(A) \geq ch_i(B)$ for $i = 1, \ldots, p$, where $ch_i(A)$ denotes the ith characteristic root of a symmetric matrix A, arranged in increasing order (Fujikoshi, Ulyanov, & Shimizu, 2010, pp. 497 (A.1.9)).
12. *A theorem on a simultaneous diagonalizable family of matrices.* A set consisting of symmetric $n \times n$ matrices, $A_1, \ldots, A_r$, is simultaneously diagonalizable by an orthogonal matrix if and only if they commute in pairs, that is to say, for each $i \neq j$, $A_iA_j = A_jA_i$. Simultaneously diagonalizable means that there exists an orthogonal matrix U such that $U^tA_iU = D_i$ where D_i is a diagonal matrix for every A_i in the set (HorZXn & Johnson, 2010, pp. 52, theorem 1.3.19) and (Harville, 1997, pp. 561).
13. If a non-negative definite matrix has trace equal to zero, then this matrix is zero (Harville, 1997, pp. 238).

14. If A is a symmetric matrix $n \times n$, x and b are vectors of length n, then
$$\frac{\partial Ax}{\partial x} = \frac{\partial x^t A}{\partial x} = A\,; \quad \frac{\partial b^t x}{\partial x} = \frac{\partial x b^t}{\partial x} = b\,; \qquad \frac{\partial x^t Ax}{\partial x} = 2 \cdot Ax.$$
See Harville (1997).
15. Spectral decomposition. Let Σ be a $k \times k$ real symmetric matrix. There exists an orthogonal matrix $\Gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$, whose column vectors γ_i are the normalized eigenvectors of Σ, $\gamma_i^t \gamma_i = 1$. Its eigenvalues are $\lambda_1, \ldots, \lambda_k$. It is verified that $\Gamma^t \Sigma \Gamma = D_\lambda$ where $D_\lambda = diag(\lambda_1, \ldots, \lambda_k)$ is a diagonal matrix with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$. In this way Γ is unique. [9] We note that $\Sigma = \Gamma D_\lambda \Gamma^t$, that is, $\Sigma = \sum_{i=1}^{k} \lambda_i \gamma_i \gamma_i^t$.
16. When Σ is a positive semi-definite matrix, all its characteristic roots or eigenvalues are real and greater than or equal to zero. Accordingly, the inverse of Σ is $\Sigma^{-1} = \Gamma D_\lambda^{-1} \Gamma^t$.

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[9] With this ordering D_λ is unique. If all the eigenvalues are different, Γ is unique. In other case Γ is unique except for a postfactor (a matrix which allows a different base for eigenvalues).

Chapter 3

Publication II: A new measure of consensus with reciprocal preference relations: The correlation consensus degree

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A new measure of consensus with reciprocal preference relations: The correlation consensus degree

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ABSTRACT

The achievement of a 'consensual' solution in a group decision making problem depends on experts' ideas, principles, knowledge, experience, etc. The measurement of consensus has been widely studied from the point of view of different research areas, and consequently different consensus measures have been formulated, although a common characteristic of most of them is that they are driven by the implementation of either distance or similarity functions. In the present work though, and within the framework of experts' opinions modelled via reciprocal preference relations, a different approach to the measurement of consensus based on the Pearson correlation coefficient is studied. The new correlation consensus degree measures the concordance between the intensities of preference for pairs of alternatives as expressed by the experts. Although a detailed study of the formal properties of the new correlation consensus degree shows that it verifies important properties that are common either to distance or to similarity functions between intensities of preferences, it is also proved that it is different to traditional consensus measures. In order to emphasise novelty, two applications of the proposed methodology are also included. The first one is used to illustrate the computation process and discussion of the results, while the second one covers a real life application that makes use of data from Clinical Decision-Making.

1. Introduction

Consensus reaching is an important component in decision making processes, and indeed it plays a key role in the resolution process of group decision making problems. One of the most significant current discussion in consensus research concerns the measurement and achievement of consensus from both a theoretical and applied points of view. On the one hand, establishing and characterising different methodologies to measure consensus have been addressed from a Social Choice perspective [1,3,13]. On the other hand, within the Decision Making Theory framework, modelling group decision making problems in order to reach a higher level of cohesiveness has been managed successfully [15,32,34,38,39,65]. Outside of these main areas, it is possible to find other methodologies that use the idea of consensus in different ways to the aforementioned ones, with [41,46] being representative examples of these methodologies.

Despite the productive research on this area, consensus measurement is still an open-ended research question because the methodology to use in each case is an essential component of the problem. Up to now most studies on consensus measurement have focused on the use of distance/similarity function based measures and association measures, respectively. Among the distance functions used, and worth highlighting, are the Kemeny, Mahalanobis, Mannhattan, Jacard, Dice and Cosine distance functions [1,4,6,17,19,29,31]. Association measures are less widely used than distance functions but it is also possible to find the use of some of them such as the Kendall's coefficient, the Goodman-Kruskal's index and the Spearman's coefficient [18,24,35,44,58]. In this paper we focus on establishing a new consensus measure following the tradition of association measures. Our proposal is based on the original statistical correlation concept, the *Pearson correlation coefficient*. Therefore, this new measure is an alternative to the use of the aforementioned approaches. The Pearson correlation coefficient plays an important role in Statistics and Data Analysis and it is extensively used as a measure of the degree of linear dependence between two variables. It is easy to interpret as well as invariant to certain changes in the variables [52,55,57]. Specifically, in this paper the notion of dependence among elements from correlation

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coefficient as a measure of the cohesiveness between opinions is adopted. This seems natural because the measurement of consensus resembles the notion of a "measure of statistical correlation", in the sense that the maximum value 1 captures the notion of unanimity as a perfect relationship among agents' preferences (experts' preferences follow the same direction), while the minimum value −1 captures the notion of total disagreement (experts' preferences present a negative relationship). Furthermore, the higher the cohesiveness between experts' preferences, the more positive correlated the preferences are. Similarly, the lower the cohesiveness between experts' preferences, the more negative correlated the preferences are.

This new consensus measure will be developed within assumptions of experts' opinions or preferences being expressed by means of reciprocal preference relations, a framework that is currently of interest to the research community in decision theory under uncertainty [7,27,28,45]. Under reciprocal preference relations, on the one hand and as it was mentioned above, the new proposed approach inherits advantages of previous approaches based on traditional distance/similarity and association measures. On the other hand, maximum consensus traditionally represents the case when experts provide the same preference intensities for each possible pair of alternatives. This, though, is not the only possible scenario of maximum consensus. Indeed, the proposal here put forward addresses this issue satisfactorily because maximum possible cohesiveness or consensus between experts' opinions does not necessary imply that all reciprocal preference relations have to coincide, and therefore all experts do not necessary need to have the same preference intensities in all possible pairs of alternatives. It is sufficient, though, that experts rank alternatives in the same way. To support all these claims, a set of properties verified by the new proposed measure of consensus, the *correlation consensus degree*, are proved. These properties ensure the suitability of the correlation consensus degree. Furthermore, in order to emphasise novelty, two applications of the proposed methodology are also included. The first one is used to illustrate the computation process and discussion of the results, while the second one covers a real life application that makes use of data from Clinical Decision-Making.

The rest of the paper is organised as follows. Section 2 contains a brief overview of the different approaches in literature to measure group cohesiveness. The basic notation and preliminaries are presented in Section 3. Section 4 provides the new approach to consensus measurement based on the Pearson correlation coefficient. In Section 5, properties of the new correlation consensus degree are studied. Section 6 presents two practical applications of the proposed methodology. Finally, some concluding remarks and future research are presented in Section 7.

2. Consensus measurement in the literature

A considerable amount of literature has been published on measuring and reaching consensus in group decision making problems. Consensus measurement is a prominent and active research subject in several areas such as Social Choice Theory and Decision Making Theory. A brief overview of how this issue has been addressed in recent literature from the aforementioned research areas is provided.

From the Social Choice Theory, the first serious discussions and analysis of consensus measurement from an Arrovian perspective emerged with Bosch's PhD Thesis [13], where both absolute and intrinsic measures of consensus were proposed, analysed and axiomatically characterised. From the point of view of considering consensus among a family of voters, McMorris and Powers [48] characterised consensus rules defined on hierarchies, while García-Lapresta and Pérez-Román [29] focused on how to measure consensus using complete preorders on alternatives and introduced a class of consensus measures based on seven well-known distances. Subsequently, Alcalde-Unzu and Vorstatz in [1] characterised a family of linear and additive consensus measures, whereas in [2] new ways to measure the similarity of preferences in a group of individuals were suggested. Alcantud, de Andrés Calle and Cascón [3] studied and characterised a class of consensus measure, called *referenced consensus measure*, that permits to produce a numerical social evaluation from purely ordinal individual information. This measure has to be specified by means of a voting mechanism and a measure of agreement between profiles of orderings and individual orderings. Moreover, Alcantud, de Andrés Calle and Cascón in [5] contributed to the formal and computational analysis of the aforementioned referenced consensus measure by focusing on two relevant and specific cases: the Borda and the Copeland rules under a Kemeny-type measure. There are, however, situations where each member of a population classifies a list of options as either acceptable or non-acceptable; either agree or disagree, etc., and therefore generating a dichotomous preference structure. Under this assumption, Alcantud, de Andrés Calle and Cascón [4] proposed the concept of *approval consensus measure* and gave axiomatic characterisations of two generic classes of such approval consensus measures. Alcantud, de Andrés Calle and González-Arteaga [6] introduced the use of the Mahalanobis distance for the analysis of the cohesiveness of a group of complete preorders and proved that arbitrary codifications of the preferences are incompatible with their formulation although affine transformations permit to compare profiles on the basis of such proposal. Finally, it is worth mentioning a distance-based approach to measure the degree of consensus considering approval information about alternatives as well as the rankings of them suggested by Erdamar et al. in [25].

From the Decision Making Theory, a considerable amount of contributions have been made since the 1980's. As such, it is worth mentioning the first preliminary work on reaching consensus and its measurements carried out by Kacprzyk and Fedrizzi [42], in which the concept of "degree of consensus" in the sense of expressing the degree to which "most of" the individuals in a group agree to "almost all of" the options. The point of departure of this paper being that the experts' opinions are expressed by fuzzy preference relations. Within this framework of preference representation, different consensus measurement based on similarity measures have been put forward by Herrera-Viedma, et al. [37] and Wu and Chiclana [63] for both complete and incomplete information environments. The case when experts' opinions are expressed by means of linguistic assessments has been extensively studied and it is worth mentioning the works of Ben-Arieh and Chen [12], Cabrerizo, Alonso and Herrera-Viedma [14], García-Lapresta, Pérez-Román [30], Herrera, Herrera-Viedma and Verdegay [36], Herrera-Viedma, et al. [40], Pérez-Asurmendi and Chiclana [53] and Wu, Chiclana and Herrera-Viedma [65]. Finally, models to reach consensus where experts assess their preferences using different preference representation structures (preference orderings, utility functions, multiplicative preference relations and fuzzy preference relations) have also been studied and proposed by Dong and Zhang [23], Fedrizzi et al. [26] and Herrera-Viedma, Herrera and Chiclana [39]. The problem of measuring and reaching consensus with intuitionistic fuzzy preference relations and triangular fuzzy complementary preference relations have also been covered in detail by Wu and Chiclana in [62,64].

To conclude, Table 1 summarises and classifies the approaches that have been reviewed in this Section.

3. Preliminaries

This Section briefly presents the main concepts needed to make the paper self-contained, and as such a short review of

Table 1
Summary table of studies related to consensus measures.

Author(s)/Year	Framework	Measurement methodology
Consensus measures in Social Choice Theory		
Bosch [13], 2005	Ordinal Inf.	Based on different distances
McMorris and Powers [48], 2009	Ordinal Inf.	
García-Lapresta and Pérez-Román [29], 2011	Ordinal Inf.	
Alcalde and Vorsatz [1], 2013	Ordinal Inf.	
Alcantud, de Andrés Calle and Cascón [3] [5], 2013	Ordinal Inf.	
Alcantud, de Andrés Calle and Cascón [4], 2013	Dichotomous Inf.	
Alcantud, de Andrés Calle and González-Arteaga [6], 2013	Ordinal Inf.	
Erdamar, et al. [25], 2014	Ordinal Inf.	
Alcalde and Vorsatz [2], 2015	Ordinal Inf.	
Consensus measures in decision making theory		
Kacprzyk and Fedrizzi [42], 1988	Fuzzy Inf.	Based on collective solution
Fedrizzi et al. [26], 2010	Fuzzy Inf.	
Herrera-Viedma et al. [37], 2007	Incomplete Fuz. Inf.	
Herrera, Herrera-Viedma and Verdegay [36], 1996	Linguistic Inf.	
Herrera-Viedma et al. [40], 2005	Linguistic Inf.	
Cabrerizo, Alonso and Herrera-Viedma [14], 2009	Linguistic Inf.	
Wu and Chiclana [62–64], 2014	Incomplete Fuz. and Ling. Inf.	
García-Lapresta, Pérez-Román and Falcó [30], 2015	Linguistic Inf.	
Wu, Chiclana and Herrera-Viedma [65], 2015	Incomplete Linguistic Inf.	
Herrera-Viedma, Herrera and Chiclana [39], 2002	Different Inf.	Based on individual solution
Ben-Arieh and Chen [12], 2006	Linguistic Inf.	
Dong and Zhang [23], 2014	Different Inf.	

the terminology and the concept of fuzzy binary relation are presented. The interested reader is advice to consult the following [7–9,27,28,45,50,60].

Definition 1. Let X be a non empty set. A *fuzzy binary relation* $\mathcal{P}$ on X is a fuzzy subset of the Cartesian product $X \times X$ characterised by its membership function $\mu_{\mathcal{P}} : X \times X \longrightarrow [0, 1]$, where $\mu_{\mathcal{P}}(x_1, x_2) = p_{ij}$ represents the strength of the relation between x_1 and x_2.

Henceforth, X is a finite set $X = \{x_1 \ldots, x_n\}$ $(n > 2)$, whose elements will be referred to as alternatives. Abusing notation, on occasions alternative x_i will be represented simply as i for convenience.

Definition 2. A *reciprocal preference relation* on X is a fuzzy binary relation $\mathcal{P}$ where $\mu_{\mathcal{P}}(x_i, x_j) = p_{ij} \in [0, 1]$ represents the partial preference intensity of element i over j and that verifies the following property: $p_{ij} + p_{ji} = 1$ $\forall x_i, x_j \in X$.

In order to realise the meaning of a reciprocal preference relation, we suppose the following common situation: an expert compares two alternatives x_i and x_j. In this specific context, the expert not only establishes that the alternative x_i is preferred to the alternative x_j, but also shows her/his intensity of preference between them by means of the value p_{ij}. So, the higher p_{ij}, the higher the preference intensity of alternative x_i over alternative x_j. Thus, $0 < p_{ij} < 0.5$ would indicate that x_j is preferred to x_i. If $p_{ij} = 0.5$ then alternatives x_i and x_j are equally preferred. When $0.5 < p_{ij} < 1$, x_i is preferred to x_j. Moreover, $p_{ij} = 0$ (resp. $p_{ij} = 1$) indicates that x_j (resp. x_i) is absolutely preferred to x_i (resp. x_j).

Let P be an $n \times n$ matrix that contains all the partial intensity degrees of a reciprocal preference relation on the set X:

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix},$$

verifying $0 \le p_{ij} \le 1$; $p_{ij} + p_{ji} = 1$ for $i, j \in \{1, \ldots, n\}$. The set of all these matrices $n \times n$ is denoted by $\mathbb{P}_{n\times n}$. Here it is also noticed that a reciprocal preference relation can also be mathematically represented by means of a vector, namely the *essential vector of preference intensities*.

Definition 3. The *essential vector of preference intensities*, V_P, of a reciprocal preference relation $P = (p_{ij})_{n\times n} \in \mathbb{P}_{n\times n}$ is the vector made up with the $\dfrac{n(n-1)}{2}$ elements above its main diagonal:

$$V_P = \left(p_{12}, p_{13}, \ldots, p_{1n}, p_{23}, \ldots, p_{2n}, \ldots, p_{(n-1)n}\right)$$
$$= \left(v_1, \ldots, v_r, \ldots, v_{n(n-1)/2}\right).$$

The reciprocity property of reciprocal preference relations allows the alternative definition of the essential vector of preference intensities of a reciprocal preference relation as the vector composed of the preference values below the main diagonal, $V_{Pt} = (p_{21}, p_{31}, \ldots, p_{n1}, p_{32}, \ldots, p_{n2}, \ldots, p_{n(n-1)})$.

4. A novel measurement of consensus based on the Pearson correlation coefficient

Based on the concept of correlation, specifically the Pearson correlation coefficient, this section introduces a new consensus measure for group decision making problems under reciprocal preference relations. First, we recall such a correlation coefficient and its properties as necessary to define the new correlation consensus degree and associated properties.

4.1. Pearson correlation coefficient

The measurement of the relationship strength among variables is an important issue in Statistical Analysis, and the *Pearson correlation coefficient* is a traditional tool used for that purpose [52,55].

Definition 4. Given a sample of n pairs of real values $\{(x_1, y_1), \ldots, (x_n, y_n)\}$, the *Pearson correlation coefficient* of the two n-dimensional vectors $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$, $cor(\mathbf{x}, \mathbf{y})$, is computed as

$$cor(\mathbf{x}, \mathbf{y}) = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ are the arithmetic means of $\mathbf{x}$ and $\mathbf{y}$, respectively.

The standard interpretation of the Pearson correlation coefficient states that positive coefficient values point out a positive tendency relationship between $\mathbf{x}$ and $\mathbf{y}$ i.e., $\mathbf{x}$ and $\mathbf{y}$ increase (decrease) in the same direction. Negative correlation coefficient values point out towards a reverse direction between $\mathbf{x}$ and $\mathbf{y}$. In addition, the nearer the absolute correlation coefficient value is to 1, the stronger and more linear the tendency is. The Pearson correlation coefficient verifies the following well-known properties [57]:

1. $cor(\mathbf{x},\mathbf{y}) \in [-1,1]\ \forall \mathbf{x},\mathbf{y} \in \mathbb{R}^n$.
2. $cor(\mathbf{x},\mathbf{y}) = cor(\mathbf{y},\mathbf{x})\ \forall \mathbf{x},\mathbf{y} \in \mathbb{R}^n$.
3. $cor(\mathbf{x},\mathbf{x}) = 1\ \forall \mathbf{x} \in \mathbb{R}^n$.
4. If $cor(\mathbf{x},\mathbf{y}) = 1$ then there exists a perfect positive linear correlation between $\mathbf{x}$ and $\mathbf{y}$, i.e. $\exists a \in \mathbb{R}, b \in \mathbb{R}^+ : \mathbf{y} = a\cdot \mathbf{1} + b\cdot \mathbf{x}$ where $\mathbf{1} = (1,\dots,1)$ is a vector of n ones. Respectively, if $cor(\mathbf{x},\mathbf{y}) = -1$ there exists a perfect negative linear correlation between $\mathbf{x}$ and $\mathbf{y}$.
5. Let $\mathbf{x}' = a\cdot\mathbf{1} + b\cdot\mathbf{x}$ and $\mathbf{y}' = c\cdot\mathbf{1} + d\cdot\mathbf{y}$ be two vectors with $a,b,c,d \in \mathbb{R}$, b and d non zero and of equal sign (both positive or both negative). Then, $cor(\mathbf{x}',\mathbf{y}') = cor(\mathbf{x},\mathbf{y})$.

4.2. A new consensus measure: Correlation consensus degree

From the Social Choice Theory perspective, the measurement of the degree of agreement in a group is associated the range [0, 1], with 0 representing total lack of agreement and 1 unanimous agreement [1,4,13]. Also, as aforementioned in Section 1, the measurement of the degree of cohesiveness in a group has been based on the notion of distance or similarity between opinions or preferences of the members of such group. In this paper, a new way to measure the degree of consensus in a group based on the Pearson correlation coefficient, the *correlation consensus degree* ($\mathcal{CCD}$), is proposed within the framework of opinions on a set of elements, alternatives or options being represented by reciprocal preference relations.

A set of agents or experts will be represented by a finite subset $\mathbf{E} = \{1,2,\dots,m\}$ of natural numbers, $m \geq 2$. Assume that the m experts provide their pairwise preferences on a finite set of n alternatives, $n \geq 3$, $X = \{x_1,\dots,x_n\}$ using fuzzy preference relations $\{P^{(1)},\dots,P^{(m)}\}$. As per Definition 3, the essential vector of preference intensities associated to $P^{(k)}$ will be denoted by $V_{P^{(k)}}$.

Definition 5. The *correlation consensus degree*, $\mathcal{CCD}$, for reciprocal preference relations is a mapping $\mathcal{CCD}: \mathbb{P}_{n\times n} \times \mathbb{P}_{n\times n} \to [0,1]$ that associates a pair of reciprocal preference relations ($P^{(1)}$, $P^{(2)}$) the following [0,1]-value:

$$\mathcal{CCD}(P^{(1)},P^{(2)}) = \frac{1}{2}\left(1 + cor(V_{P^{(1)}},V_{P^{(2)}})\right). \quad (1)$$

Given $P^{(1)},P^{(2)} \in \mathbb{P}_{n\times n}$, the elaborated expression of $\mathcal{CCD}(P^{(1)},P^{(2)})$ is

$$\mathcal{CCD}(P^{(1)},P^{(2)}) = \frac{1}{2}\left(1 + \frac{\sum_{r=1}^{n(n-1)/2}\left(v_r^{(1)} - \overline{V}_{P^{(1)}}\right)\left(v_r^{(2)} - \overline{V}_{P^{(2)}}\right)}{\sqrt{\sum_{r=1}^{n(n-1)/2}\left(v_r^{(1)} - \overline{V}_{P^{(1)}}\right)^2}\sqrt{\sum_{r=1}^{n(n-1)/2}\left(v_r^{(2)} - \overline{V}_{P^{(2)}}\right)^2}}\right)$$

where $\overline{V}_{P^{(1)}} = \frac{1}{n(n-1)/2}\sum_{r=1}^{n(n-1)/2} v_r^{(1)}$ and $\overline{V}_{P^{(2)}} = \frac{1}{n(n-1)/2}\sum_{r=1}^{n(n-1)/2} v_r^{(2)}$. [1]

Notice that the higher the value of $\mathcal{CCD}(P^{(1)},P^{(2)})$, the more positive correlated the reciprocal preferences of $P^{(1)}$ and $P^{(2)}$ are. The maximum possible value $\mathcal{CCD}(P^{(1)},P^{(2)}) = 1$ implies that $cor(V_{P^{(1)}},V_{P^{(2)}}) = 1$ which, contrary to previous consensus measures based on distance/similarity functions, does not necessarily implies that both reciprocal preference relations coincide. Consequently, $\mathcal{CCD}$ could be 1 even in cases when experts provide different preferences, although positive linearly correlated. On the other hand, the lower the value of $\mathcal{CCD}(P^{(1)},P^{(2)})$, the more negative correlated the reciprocal preference intensities are, with $\mathcal{CCD}(P^{(1)},P^{(2)}) = 0$ representing the case when preferences are negative linearly correlated. The following proposition reflects these limit cases:

Proposition 1. *Let $P^{(1)}$, $P^{(2)} \in \mathbb{P}_{n\times n}$ be two reciprocal preference relation matrices. Then $\mathcal{CCD}(P^{(1)},P^{(2)}) = 1$ (resp. $\mathcal{CCD}(P^{(1)},P^{(2)}) = 0$) if and only if $\exists a \in \mathbb{R}$, $b > 0$ (resp. $b < 0$) such that: $p_{ij}^{(2)} = a + b\cdot p_{ij}^{(1)}\ \forall i < j$; $p_{ij}^{(2)} = (1-a-b) + b\cdot p_{ij}^{(1)}\ \forall i > j$.*

Proof. Using Eq. (1), we have that $\mathcal{CCD}(P^{(1)},P^{(2)}) = 1$ if and only if $cor(V_{P^{(1)}},V_{P^{(2)}}) = 1$. Property 4 of the Pearson correlation coefficient (Section 4.1) implies that $\exists a \in \mathbb{R}, b \in \mathbb{R}^+$ such that $V_{P^{(2)}} = a\cdot\mathbf{1} + b\cdot V_{P^{(1)}}$, being $\mathbf{1} = (1,\dots,1)$ a vector of ones with suitable dimension, in this cases $n(n-1)/2$, i.e.:

$$p_{ij}^{(2)} = a + b\cdot p_{ij}^{(1)}\ \forall i < j.$$

When $j < i$, reciprocity of preferences means that

$$\begin{aligned} p_{ij}^{(2)} &= 1 - p_{ji}^{(2)} = 1 - (a + b\cdot p_{ji}^{(1)}) \\ &= 1 - (a + b\cdot(1 - p_{ij}^{(1)})) = (1-a-b) + b\cdot p_{ij}^{(1)}. \end{aligned}$$

The proof for the case when $\mathcal{CCD}(P^{(1)},P^{(2)}) = 0$ is obtained accordingly. □

Notice that if the set of alternatives is small, the experts can easily rank the alternatives and the possibility that the experts do it in a similar way (or opposite way) is high. Then, in this case the absolute value of the correlation coefficient tend to be close to 1. Meanwhile, when the set of alternatives is large, the experts may find it difficult to rank them (see [49]) and the possibility that the experts rank the alternatives in a similar way (or opposite way) is low. Then, in this case it is easy that the absolute value of the correlation coefficient becomes small.

The following proposition provides the sufficient condition for the correlation consensus degree to coincide for different pairs of reciprocal preference relations.

Proposition 2. *Let $P^{(1)}$, $P^{(2)} \in \mathbb{P}_{n\times n}$ be reciprocal preference relation matrices such that $\mathcal{CCD}(P^{(1)},P^{(2)}) = 1$, then*

$$\mathcal{CCD}(P,P^{(1)}) = \mathcal{CCD}(P,P^{(2)}) \quad \forall P \in \mathbb{P}_{n\times n}.$$

Proof. By Proposition 1, $\exists a \in \mathbb{R}$ and $b > 0$ such that $V_{P^{(2)}} = a\cdot\mathbf{1} + b\cdot V_{P^{(1)}}$. Applying Property 5 of the Pearson correlation coefficient (see Subsection 4.1) we have that $cor(V_P,V_{P^{(1)}}) = cor(V_P,V_{P^{(2)}})\ \forall P \in \mathbb{P}_{n\times n}$ and by Definition 5 it is equivalent to $\mathcal{CCD}(P,P^{(1)}) = \mathcal{CCD}(P,P^{(2)})\ \ \forall P \in \mathbb{P}_{n\times n}$. □

The measurement of the degree of agreement among the preferences expressed by two or more experts can be captured by using a summary measure like the mean of all possible correlation consensus degrees between all different pairs of experts' reciprocal preference relations. The use of aggregation functions to merge inputs into a single output has been extensively analysed in literature [11,28,33,43]. In the Decision Making context, the use of aggregation functions to derive the degree of agreement among a group of experts has been justified (see for example [11,29,31,43,47]). Recall that the main aim of considering aggregation functions is to produce an overall output that can be considered representative of the aggregated values by incorporating

[1] $\overline{V}_{P^{(i)}}$ summarizes the general level of uncertainty of the expert i on the set of alternatives.

desirable properties. The *arithmetic mean* has been widely investigated and it is considered the most common central tendency aggregation function. All these considerations are used to motivate the definition of the new correlation group consensus measure, $\mathcal{CD}$, within a reciprocal preference relation framework.

Definition 6. Let **E** be a group of m experts with associated fuzzy preference relation relations $P^{(1)}, \ldots, P^{(m)} \in \mathbb{P}_{n\times n}$ on a set of alternatives X. The *group consensus degree* among the set of experts is

$$\mathcal{CD}(\mathbf{E}) = \frac{2}{m(m-1)} \sum_{k=1}^{m-1} \sum_{l=k+1}^{m} \mathcal{CCD}(P^{(k)}, P^{(l)}).$$

4.3. Consistency under maximum correlation consensus degree

Given a reciprocal preference relation on a set of alternatives, the concept of *non-dominance* degree introduced by Orlovsky [51] has been extensively used to rank the alternatives [10,23,38,61,63,65,66]. In the following, and in order to improve the understanding of the proposed correlation consensus degree, the consistency of the correlation consensus degree with Orlovsky's non-dominance degree is proved. Specifically, it is proved that when two reciprocal preference relations have a CCD equal to 1 then their Orlovsky's non-dominance degree orderings of the set of alternatives coincide. First, the concept of non-dominance degree is provided.

Given a reciprocal preference relation on a finite set of alternatives X, $P = (p_{ij})_{n\times n} \in \mathbb{P}_{n\times n}$, when $p_{ji} - p_{ij} > 0$ then alternative x_i is dominated by alternative x_j. Formally, it can be stated that alternative x_i is dominated by alternative x_j at degree $d(x_i, x_j) = \max\{p_{ji} - p_{ij}, 0\}$. Thus, the value $1 - d(x_i, x_j) = 1 - \max\{p_{ji} - p_{ij}, 0\}$ represents the degree of non-dominance of alternative x_i by alternative x_j. The degree up to which x_i is not dominated by any of the elements of X is known as the non-dominance degree of alternative x_i. This is summarised in the following definition.

Definition 7. Let $P = (p_{ij})_{n\times n} \in \mathbb{P}_{n\times n}$ be a reciprocal preference relation on X. The *non-dominance degree* is a mapping $\mu_{ND} : X \longrightarrow [0, 1]$ such that

$$\mu_{ND}(x_i) = \min_{j:\, j\neq i} \left\{1 - d(x_i, x_j)\right\},$$

where $d(x_i, x_j) = \max\{p_{ji} - p_{ij}, 0\}$.

The aforementioned non-dominance degree can be used to provide a total ordering of alternatives by means of the following rule:

$$x_i \succeq x_j \Leftrightarrow \mu_{ND}(x_i) \geq \mu_{ND}(x_j).$$

Notice that $p_{ij} - p_{ji} = -(p_{ji} - p_{ij})$, and therefore to compute $d(x_j, x_i) = \max\{p_{ji} - p_{ij}, 0\}$ when $j > i$, we use $d(x_j, x_i) = \max\left\{-(p_{ji} - p_{ij}), 0\right\}$. Now we are in disposition of introduce the following result.

Proposition 3. *Let $P^{(1)}, P^{(2)} \in \mathbb{P}_{n\times n}$ be two reciprocal preference relation matrices such that $\mathcal{CCD}(P^{(1)}, P^{(2)}) = 1$ and $2a + b = 1$. The non-dominance based orderings of the set of alternatives derived from both reciprocal preference relation matrices are identical.*

Proof. Let $P^{(1)}, P^{(2)} \in \mathbb{P}_{n\times n}$ such that $\mathcal{CCD}(P^{(1)}, P^{(2)}) = 1$. By Proposition 1, $\exists a \in \mathbb{R}$ and $b > 0$ such that $p^{(2)}_{ij} = a + b \cdot p^{(1)}_{ij}$ $\forall i < j$ and $p^{(2)}_{ij} = 1 - p^{(2)}_{ji} = 1 - (a + b \cdot p^{(1)}_{ji}) = 1 - (a + b \cdot (1 - p^{(1)}_{ij})) = (1 - a - b) + b \cdot p^{(1)}_{ij}$ $\forall i < j$.

1. Notice that:
   (a) If $i < j$ then
   $$\begin{aligned} p^{(2)}_{ji} - p^{(2)}_{ij} &= [(1 - a - b) + b \cdot p^{(1)}_{ji}] - [a + b \cdot p^{(1)}_{ij}] \\ &= (1 - 2a - b) + b \cdot (p^{(1)}_{ji} - p^{(1)}_{ij}) \\ &= b \cdot (p^{(1)}_{ji} - p^{(1)}_{ij}). \end{aligned}$$
   (b) If $i > j$ then
   $$\begin{aligned} p^{(2)}_{ji} - p^{(2)}_{ij} &= [a + b \cdot p^{(1)}_{ji}] - [(1 - a - b) + b \cdot p^{(1)}_{ij}] \\ &= -(1 - 2a - b) + b \cdot (p^{(1)}_{ji} - p^{(1)}_{ij}) \\ &= b \cdot (p^{(1)}_{ij} - p^{(1)}_{ji}). \end{aligned}$$
   Thus:
   $\forall i, j:\ p^{(2)}_{ji} - p^{(2)}_{ij} = b \cdot (p^{(1)}_{ij} - p^{(1)}_{ji})$.
2. Let us denote by $\mu_{ND^{(1)}}(x_i)$ and $\mu_{ND^{(2)}}(x_i)$ the non-dominance choice degree associated to alternative x_i obtained from $P^{(1)}$ and $P^{(2)}$, respectively. It is:
   $$\mu_{ND^{(2)}}(x_i) = \min_{x_j \in X} \left\{1 - \max\{p^{(2)}_{ji} - p^{(2)}_{ij}, 0\}\right\}.$$
   Because $b > 0$ we have that $p^{(2)}_{ji} - p^{(2)}_{ij}$ and $p^{(1)}_{ij} - p^{(1)}_{ji}$ are both negative, both positive or both equal to zero. Therefore, it is:
   $$\begin{aligned} \max\{p^{(2)}_{ji} - p^{(2)}_{ij}, 0\} &= \max\{b \cdot (p^{(1)}_{ij} - p^{(1)}_{ji}), 0\} \\ &= b \cdot \max\{p^{(1)}_{ji} - p^{(1)}_{ij}, 0\}. \end{aligned} \tag{2}$$
   Let l be such that
   $$\begin{aligned} \mu_{ND^{(1)}}(x_i) &= \min_{j:\, j\neq i} \left\{1 - \max\{p^{(1)}_{ji} - p^{(1)}_{ij}, 0\}\right\} \\ &= 1 - \max\{p^{(1)}_{li} - p^{(1)}_{il}, 0\}. \end{aligned}$$
   The following inequalities yield:
   $$1 - \max\{p^{(1)}_{li} - p^{(1)}_{il}, 0\} \leq 1 - \max\{p^{(1)}_{ji} - p^{(1)}_{ij}, 0\} \quad \text{for } j = 1, \ldots, n.$$
   They can be re-written equivalently as
   $$\max\{p^{(1)}_{li} - p^{(1)}_{il}, 0\} \geq \max\{p^{(1)}_{ji} - p^{(1)}_{ij}, 0\} \quad \text{for } j = 1, \ldots, n.$$
   Consequently,
   $$1 - b \cdot \max\{p^{(1)}_{li} - p^{(1)}_{il}, 0\} \leq 1 - b \cdot \max\{p^{(1)}_{ji} - p^{(1)}_{ij}, 0\} \quad \text{for } j = 1, \ldots, n.$$
   Relation (2) implies that
   $$\begin{aligned} \mu_{ND^{(2)}}(x_i) &= \min_{j:\, j\neq i} \left\{1 - \max\{p^{(2)}_{ji} - p^{(2)}_{ij}, 0\}\right\} \\ &= 1 - \max\{p^{(2)}_{li} - p^{(2)}_{il}, 0\}. \end{aligned} \tag{3}$$
3. Finally, let us assume now that
   $\mu_{ND^{(1)}}(x_i) \leq \mu_{ND^{(1)}}(x_k)$.
   Then there exist l and s such that
   $$\begin{aligned} 1 - \max\{p^{(1)}_{li} - p^{(1)}_{il}, 0\} &= \mu_{ND^{(1)}}(x_i) \leq \mu_{ND^{(1)}}(x_k) \\ &= 1 - \max\{p^{(1)}_{sk} - p^{(1)}_{sk}, 0\} \end{aligned}$$
   The following inequality derives from it:
   $$1 - b \cdot \max\{p^{(1)}_{li} - p^{(1)}_{il}, 0\} \leq 1 - b \cdot \max\{p^{(1)}_{sk} - p^{(1)}_{sk}, 0\}.$$
   Applying again relation (2) and also expression (3), it can be concluded that
   $\mu_{ND^{(1)}}(x_i) \leq \mu_{ND^{(1)}}(x_k)$.

□

5. Formal properties of the new consensus measure

As shown in Subsection 4.2, given a set of m experts $\mathbf{E}$, the following correlation consensus degree matrix can be computed

$$\mathcal{CCD} = (\mathcal{CCD}_{ij})$$

with $\mathcal{CCD}_{ij} = \mathcal{CCD}(P^{(i)}, P^{(j)})$. The following important properties are verified:

Reflexivity: $\mathcal{CCD}_{ii} = 1\ \forall i$.
The proof is immediate from the properties of the Pearson correlation coefficient. This property rules out that the correlation consensus degree is a distance function, which will be pointed out at the end of this section.

Selfconsensus: $\mathcal{CCD}_{ij} \le \mathcal{CCD}_{ii}\ \forall i, j$.
In other words, the correlation consensus degree between one expert and herself/himself is not lower than the correlation consensus degree with another expert. This is obvious from Definition 5 and the reflexibity property above.

Reciprocity: It was mentioned in Definition 3 that the essential vector of preference intensities of a reciprocal preference relation may also be defined as the vector with elements the preference values below the main diagonal of the reciprocal preference relation. Denoting by $\mathcal{CCD}^t_{ij}$ the correlation consensus degree between the reciprocal preference relations $P^{(i)}$ and $P^{(j)}$ using essential vector of preference intensities below their main diagonal, respectively, we have that:

$$\mathcal{CCD}^t_{ij} = \mathcal{CCD}_{ij} \quad \text{for } i, j = 1, \ldots, m.$$

Indeed, because $P^{(i)}$ and $P^{(j)}$ are reciprocal then we have that $V_{p(i)^t} = \mathbf{1} - V_{P^{(i)}}$ and $V_{p(j)^t} = \mathbf{1} - V_{P^{(j)}}$, respectively. Applying Property 4 of the Pearson correlation coefficient (Subsection 4.1), it is true that $cor(V_{p(i)^t}, V_{p(j)^t}) = cor(V_{P^{(i)}}, V_{P^{(j)}})$, and consequently it is $\mathcal{CCD}^t_{ij} = \mathcal{CCD}_{ij}$.

Symmetry: $\mathcal{CCD}_{ij} = \mathcal{CCD}_{ji}$ for $i, j = 1, \ldots, n$. The proof is straightforward from the symmetry property of the Pearson correlation coefficient.

Transitivity under the maximum: If $\mathcal{CCD}_{ij} = 1$ and $\mathcal{CCD}_{jk} = 1$ then $\mathcal{CCD}_{ik} = 1$.
In other words, when an expert has maximum correlation consensus degree with two different experts, then these two experts have also maximum correlation consensus degree. Indeed, from Proposition 1 we have that $V_{P^{(j)}} = a \cdot \mathbf{1} + b \cdot V_{P^{(i)}}$ for some $a \in \mathbb{R}$ and $b > 0$ and $V_{P^{(k)}} = a' \cdot \mathbf{1} + b' \cdot V_{P^{(j)}}$ for some $a' \in \mathbb{R}$ and $b' > 0$. Consequently, it is: $V_{P^{(k)}} = a' \cdot \mathbf{1} + b' \cdot (a \cdot \mathbf{1} + b \cdot V_{P^{(i)}}) = a' \cdot \mathbf{1} + b'a \cdot \mathbf{1} + b'b \cdot V_{P^{(i)}}$, that is, $V_{P^{(k)}} = a'' \cdot \mathbf{1} + b'' \cdot V_{P^{(i)}}$ and because $b'' > 0$ it is $\mathcal{CCD}_{ik} = 1$.

Reversibility: The complementary reciprocal preference relation of a given reciprocal preference relation P, $\overline{P}$, is defined as follows: $\overline{P} = (1)_{n\times n} - P$. It is:

$$\mathcal{CCD}(P, \overline{P}) = 0.$$

It is obvious that $V_{\overline{P}} = \mathbf{1} - V_P$ and therefore applying Proposition 1 it is $\mathcal{CCD}(P, \overline{P}) = 0$.

The correlation consensus degree, $\mathcal{CCD}$, is neither a distance function, d, nor a similarity function, s. Firstly, $\mathcal{CCD}$ does not verify the property returning a zero value when an element is compared against itself, i.e. it does not verify $d(x, x) = 0$ [22]. Indeed, reflexivity property implies that $\mathcal{CCD}(P, P) = 1$ rather than $\mathcal{CCD}(P, P) = 0$. Secondly, a requirement for a similarity function [16,22] is that the similarity between two objects takes value 1 if and only if the two objects are equal, i.e. $s(x, y) = 1$ iff $x = y$. This is not the case for $\mathcal{CCD}$ as two reciprocal preference relations do not necessarily need to coincide to have maximum correlation consensus degree, as the Illustrative Example 6.1 shows next.

6. Practical applications and discussion

In this Section we show the flexibility and applicability of our proposal. After discussing the basis of the measure we exemplify its use by means of two examples. The first one is an illustrative example that shows the various steps in our procedure and the interpretation of the results. The second one is a real example based on patients' health preferences.

6.1. An illustrative example

In this illustrative example we establish the following problem. We consider a set X of four alternatives $X = \{x_1, x_2, x_3, x_4\}$ and a set of four agents or experts $\mathbf{E} = \{1, 2, 3, 4\}$, who provide the following reciprocal preference relations on X:

$$P^{(1)} = \begin{pmatrix} 0.50 & 0.10 & 0.20 & 0.30 \\ 0.90 & 0.50 & 0.35 & 0.40 \\ 0.80 & 0.65 & 0.50 & 0.45 \\ 0.70 & 0.60 & 0.65 & 0.50 \end{pmatrix}$$

$$P^{(2)} = \begin{pmatrix} 0.50 & 0.15 & 0.25 & 0.35 \\ 0.85 & 0.50 & 0.40 & 0.45 \\ 0.75 & 0.60 & 0.50 & 0.50 \\ 0.65 & 0.55 & 0.50 & 0.50 \end{pmatrix}$$

$$P^{(3)} = \begin{pmatrix} 0.50 & 0.75 & 0.55 & 0.35 \\ 0.25 & 0.50 & 0.25 & 0.15 \\ 0.45 & 0.75 & 0.50 & 0.05 \\ 0.65 & 0.80 & 0.95 & 0.50 \end{pmatrix}$$

$$P^{(4)} = \begin{pmatrix} 0.50 & 0.40 & 0.20 & 0.60 \\ 0.60 & 0.50 & 0.40 & 0.70 \\ 0.80 & 0.60 & 0.50 & 0.80 \\ 0.40 & 0.30 & 0.10 & 0.50 \end{pmatrix}$$

Once experts' preference matrices have been described we proceed to the computations.

Selection of essential vectors of intensities of preferences. For $P^{(1)}$ the elements above of the main diagonal are:

$$P^{(1)} = \begin{pmatrix} 0.50 & 0.10 & 0.20 & 0.30 \\ 0.90 & 0.50 & 0.35 & 0.40 \\ 0.80 & 0.65 & 0.50 & 0.45 \\ 0.70 & 0.60 & 0.65 & 0.50 \end{pmatrix}$$

Thus, it is

$V_{P^{(1)}} = (0.10,\ 0.20, 0.30, 0.35, 0.40, 0.45)$.

Similarly, the following essential vectors of intensities of preferences obtained:

$V_{P^{(2)}} = (0.15, 0.25, 0.35, 0.40, 0.45, 0.50)$,
$V_{P^{(3)}} = (0.75, 0.55, 0.35, 0.25, 0.15, 0.05)$,
$V_{P^{(4)}} = (0.40, 0.20, 0.60, 0.40, 0.70, 0.80)$.

Computation of the correlation consensus degree matrix. The correlation consensus degree of all different pairs of essential vectors are computed For example, for correlation coefficient between $V_{P^{(1)}}$ and $V_{P^{(2)}}$ is:

$$cor(V_{P^{(1)}}, V_{P^{(2)}}) = \frac{0.085}{\sqrt{0.085} \cdot \sqrt{0.085}} = 1.$$

Using Eq. (1), the correlation consensus degree between $P^{(1)}$ and $P^{(2)}$ would be $\mathcal{CCD}(P^{(1)}, P^{(2)}) = 1$.
The correlation consensus degree matrix in this case is:

$$\mathcal{CCD} = \begin{pmatrix} 1 & 1 & 0 & 0.879 \\ 1 & 1 & 0 & 0.897 \\ 0 & 0 & 1 & 0.121 \\ 0.879 & 0.897 & 0.121 & 1 \end{pmatrix}$$

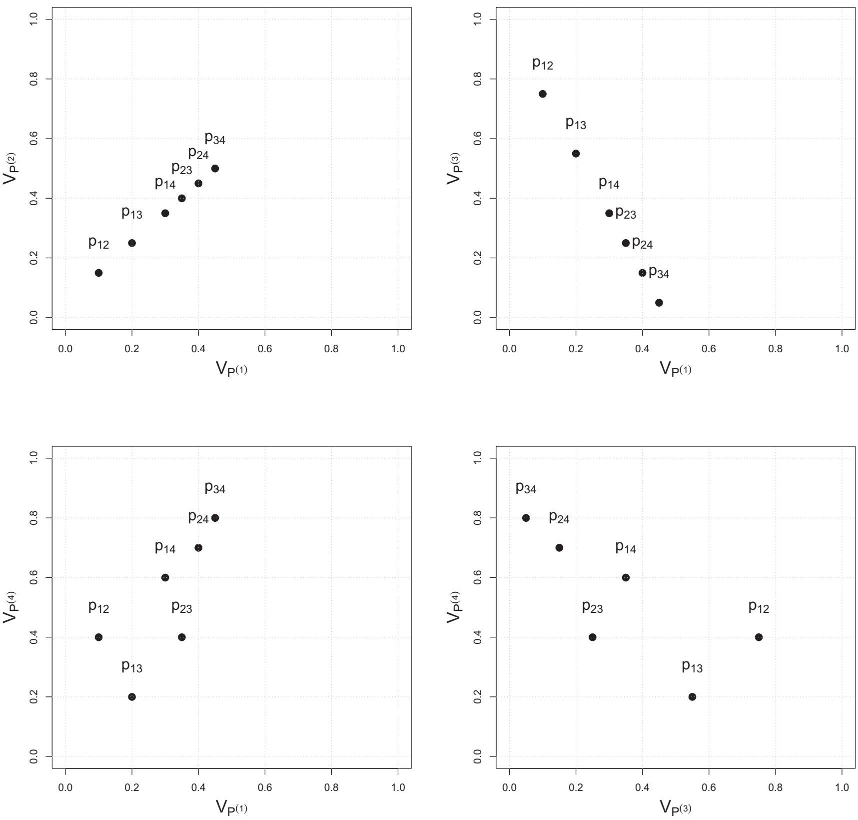

Fig. 1. Plots of essential vectors of intensities of preferences (Subsection 6.1). Top left: case where $\mathcal{CCD}(P^{(1)}, P^{(2)}) = 1$. Top right: case where $\mathcal{CCD}(P^{(1)}, P^{(3)}) = 0$. On the bottom plots, cases where $\mathcal{CCD}$ takes other non-extreme values.

Table 2
Control Preference Scale (CPS) [21].

Alternatives	Description
x_1	I prefer to make the final selection about which treatment I receive
x_2	I prefer to make the final selection of my treatment after seriously considering my doctor's opinion
x_3	I prefer that my doctor and I share responsibility for deciding which treatment is best for me
x_4	I prefer that my doctor makes the final decision about which treatment will be used, but seriously considers my opinion
x_5	I prefer to leave all decisions regarding my treatment to my doctor

Computation of the group consensus degree. Finally, the average of all correlation consensus degrees is computed to derive the group consensus degree:

$$\mathcal{CD}(\mathbf{E}) = \frac{2}{12} \cdot (1 + 0 + 0.879 + 0 + 0.879 + 0.121) = \frac{2}{12} \cdot 2.879 = 0.480$$

On discussion, it is worth pointing out the following interesting issues arising from the given example:

- There is one case when the correlation consensus degree between two experts is maximum, i.e is equal to 1, which happens for the pair of experts e_1 and e_2 ($\mathcal{CCD}(P^{(1)}, P^{(2)}) = 1$). As previously mentioned and this example illustrates, this does not necessarily imply that both experts have the same preferences on all the possible pairs of alternatives, but that their preferences are positive linearly correlated as the top left scatter plot of the essential vectors $V_{p(2)}$ versus $V_{p(1)}$ in Fig. 1 shows. Indeed, the higher the value of an element in $V_{p(1)}$, the higher the corresponding element value of $V_{p(2)}$. So, when one of the expert e_1 or e_2 increases her/his preference valuations, the other expert does the same and in a perfect linear way. Hence, there

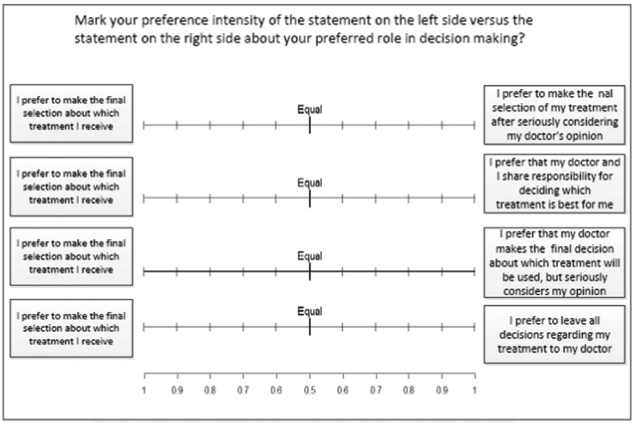

Fig. 2. Questionnaire based on CPS.

Table 3
Patients' essential vector of preference intensities. OCD: Obsessive compulsive disorder. p_{ij} is the intensity of preference of alternative i versus alternative j.

Diagnoses	Patient	Intensities of preferences									
		p_{12}	p_{13}	p_{14}	p_{15}	p_{23}	p_{24}	p_{25}	p_{34}	p_{35}	p_{45}
Schizophrenia	1	0.3	0.2	0.1	0.1	0.4	0.3	0.1	0.5	0.1	0.1
	2	0.4	0.3	0.2	0.1	0.5	0.4	0.1	0.5	0.1	0.1
	3	0.2	0.2	0.2	0.1	0.3	0.2	0.1	0.3	0.1	0.1
	4	0.6	1.0	0.8	0.7	0.9	0.6	0.7	0.3	0.4	0.6
Bipolar disorder	5	0.6	1.0	0.8	0.7	0.9	0.6	0.7	0.3	0.4	0.6
	6	0.8	1.0	0.9	0.8	1.0	0.8	0.8	0.2	0.3	0.5
	7	0.1	0.1	0.1	0.5	0.5	0.5	0.9	0.5	0.9	0.9
	8	0.6	1.0	0.9	0.7	0.9	0.6	0.8	0.3	0.2	0.6
OCD	9	0.1	0.1	0.1	0.5	0.5	0.5	0.9	0.5	0.9	0.9
	10	0.2	0.1	0.3	0.5	0.5	0.5	0.8	0.5	0.7	0.9
	11	0.1	0.2	0.2	0.5	0.6	0.4	0.9	0.5	0.8	0.9
	12	0.1	0.2	0.2	0.5	0.5	0.4	0.8	0.5	0.9	0.9

exists a maximum concordance between these two experts' reciprocal preference relations.

- Regarding experts e_1 and e_3, it is noted that $cor(V_{p(1)}, V_{p(3)}) = -1$ and consequently $\mathcal{CCD}(P^{(1)}, P^{(3)}) = 0$. Thus, the disagreement is maximum. Indeed, when one expert increases his/her preferences the other expert does the opposite and in a perfect linear way. This is reflected in the top right scatter plot of the essential vectors $V_{p(3)}$ versus $V_{p(1)}$ in Fig. 1.
- This example also shows a particular instance of Propositions 1 and 2 where $\mathcal{CCD}(P^{(1)}, P^{(2)}) = 1$. Indeed, Proposition 1 states that it is $V_{p(2)} = a \cdot \mathbf{1} + b \cdot V_{p(1)}$, which in this case results in $a = 0.05,\ b = 1$. The effect is that every essential pairwise intensity of preference is shifted to a new value using a constant amount. The preference relationship between one alternative and the rest of alternatives is essentially the same both experts, and consequently there is no real difference in the degree of agreement between for both experts when considering the set of alternatives as a whole. Indeed, it is worth remarking that the difference of preferences for both experts: $p_{13}^{(1)} - p_{12}^{(1)} = 0.20 - 0.10 = 0.10$ and $p_{13}^{(2)} - p_{12}^{(2)} = 0.25 - 0.15 = 0.10;\quad p_{34}^{(1)} - p_{23}^{(1)} = 0.45 - 0.40 = 0.05$ and $p_{34}^{(2)} - p_{23}^{(2)} = 0.50 - 0.45 = 0.05,$ etc. are the same for all pairs of alternatives compared. Thus, although the fuzzy relations $P^{(1)}$ and $P^{(2)}$ are not coincident, they are in the same tendency vein and they would lead to the same total ordering of the alternatives when the non-dominace degree is applied. As for Proposition 2, it is also true that the correlation consensus degrees between expert e_1 and experts e_3 and e_4 are the same as the correlation consensus degrees between expert e_2 and experts e_3 and e_4, respectively.

6.2. A real application: Concordance among patients' preferences

Recent developments in *Clinical Decision-Making* have led to a new interest on patient autonomy and their active involvement in decision making. Based on empirical evidences it has been tested that patients' choices related to take responsibility about treatment decisions differ among patients. Among others, age, sex, and type of clinical problem have been described as factors that can influence patients' choice. Due to these fats, it could be interesting to understand better patients' preferences in Clinical decision-making and the factors that could influence them (see e.g., De las Cuevas, Peñate and de Rivera [20], Robison and Thomson [54], Rodriguez et al. [56] and Tang et al. [59] among others). Most studies about patients' decision making preferences have been carried out by means of the use of the *Control Preference Scale* (CPS) introduced by Degner [21]. The CPS scale has been validated like an instrument clinically relevant to measure patients' preference roles in health

Table 4
Correlation consensus degree ($\mathcal{CCD}$) between pairs of patients.

	$p^{(1)}$	$p^{(2)}$	$p^{(3)}$	$p^{(4)}$	$p^{(5)}$	$p^{(6)}$	$p^{(7)}$	$p^{(8)}$	$p^{(9)}$	$p^{(10)}$	$p^{(11)}$	$p^{(12)}$
$p^{(1)}$	1.00	0.98	0.95	0.97	0.39	0.30	0.43	0.41	0.36	0.35	0.36	0.33
$p^{(2)}$		1.00	0.97	0.99	0.50	0.39	0.55	0.51	0.26	0.26	0.26	0.23
$p^{(3)}$			1.00	0.93	0.54	0.42	0.56	0.56	0.24	0.25	0.26	0.23
$p^{(4)}$				1.00	0.47	0.39	0.53	0.48	0.32	0.30	0.30	0.28
$p^{(5)}$					1.00	0.95	0.96	0.98	0.28	0.28	0.34	0.29
$p^{(6)}$						1.00	0.96	0.95	0.33	0.35	0.38	0.33
$p^{(7)}$							1.00	0.96	0.23	0.25	0.28	0.22
$p^{(8)}$								1.00	0.26	0.30	0.33	0.27
$p^{(9)}$									1.00	0.98	0.99	0.99
$p^{(10)}$										1.00	0.98	0.97
$p^{(11)}$											1.00	0.99
$p^{(12)}$												1.00

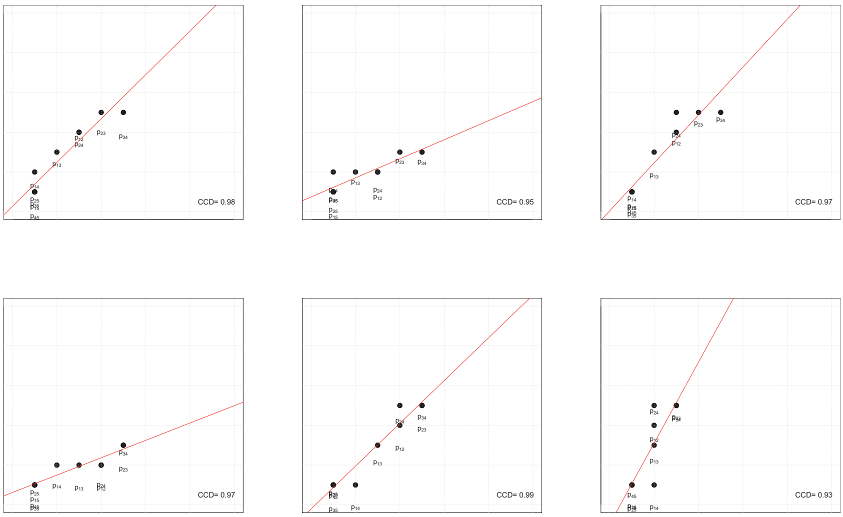

Fig. 3. Plots of essential vectors of intensities of preferences corresponding to patients diagnosed Schizophrenia (Subsection 6.2) and the best adjusted line.

care decision making. This scale gathers the level of control that patients prefer to have in their own medical decisions by selecting one of five possible alternatives, given in Table 2, when questioned "What is the statement that best describe your preferred role in decision making?".

In order to put in practice our proposal for measuring the cohesiveness among a group of agents or experts, the field experiment carried out by De las Cuevas, Peñate and Rivera in [20] was considered. In this study, the authors examined the concordance among psychiatric patients' preferences by means of a statistical approach based on a sample of 507 patients from the Community Mental Health Services on Tenerife Island, Spain. Patients were diagnosed by the psychiatrists using the International Classification of Diseases and the CPS scale was used to gather patients' preferences. For our study, and to facilitate the process and the calculations, 12 patients were considered with 4 of them were diagnosed with *schizophrenia*, another 4 with *bipolar disorder* and other 4 with *obsessive compulsive disorder* (OCD). Each patient filled out a questionnaire based on the CPS scale adapted to our proposal (see Fig. 2). Patients had to mark their degree of preference between pairs of options described in the CPS scale (Table 2 above), which are considered as the alternatives in our preference framework.

Once patients' preferences were gathered (Table 3) we proceed to the apply computation process described in the previous Illustrative Example 6.1. Table 4 shows the correlation consensus degree between all pairs of patients (only the values $i \leq j$ are shown). Finally, the global consensus degree among all studied patients, $\mathcal{CD}(\text{patients})$, which measures the coherence among patients' preferences was:

$$\mathcal{CD}(\text{patients}) = 0.518$$

Taking into account the meaning of this measure as previously discussed, we can deduce that the low degree of coherence among all patients' preferences indicates heterogeneity among them. This fact could well respond to the combination of all patients' preferences without considering their diagnosed disorder. Indeed, when the consensus degree is computed within each collective of patients, i.e. by distinguishing patients according to their disorder, the following values are obtained:

- For patients suffering from schizophrenia: $\mathcal{CD}(\text{schizophrenia}) = 0.963$
- For patients suffering from bipolar disorder: $\mathcal{CD}(\text{bipolar}) = 0.961$

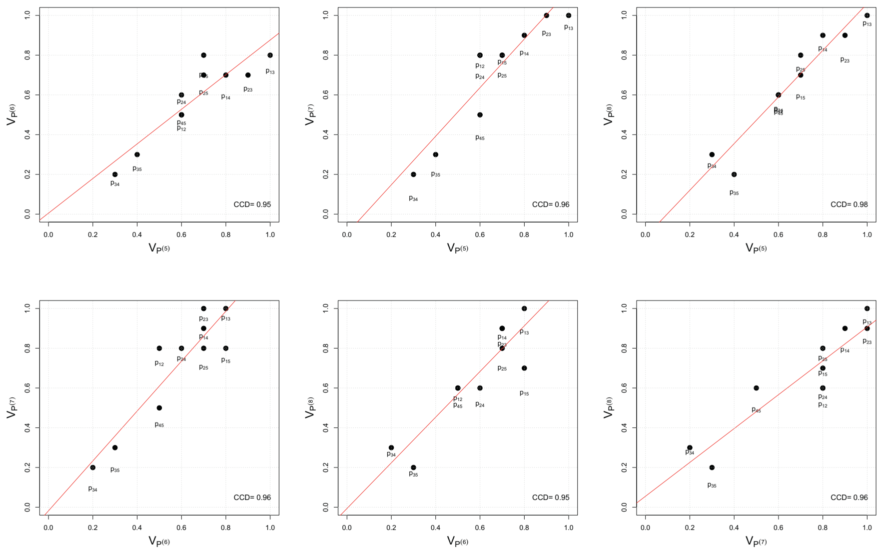

Fig. 4. Scatterplots of essential vectors of intensities of preferences corresponding to patients diagnosed Bipolar disorder (Subsection 6.2) and the best adjusted line.

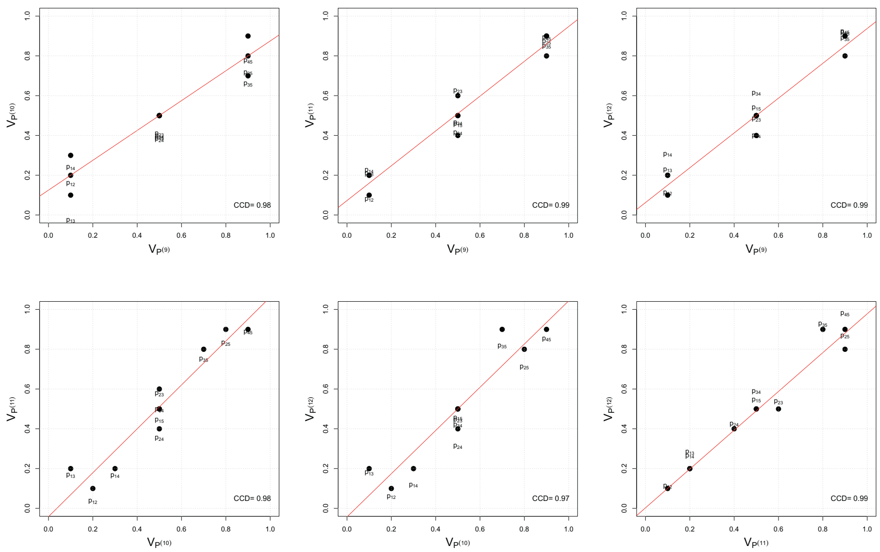

Fig. 5. Plots of essential vectors of intensities of preferences corresponding to patients diagnosed OCD (Subsection 6.2) and the best adjusted line.

- For patients suffering from obsessive compulsive disorder: $\mathcal{CD}(\text{OCD}) = 0.985$

Figs. 3 , 4, Fig. 5 highlight the coherence among preferences inside the same collective of patients, while Fig. 6, 7, and. 8 highlight the disagreement among patients' preferences diagnosed with different disorders. As it was suspected, the coherence among the patients' preferences for each disorder separately is very high. This fact could add to the consideration of the type of disorder as a factor to be taken into account in Clinical Decision-Making.

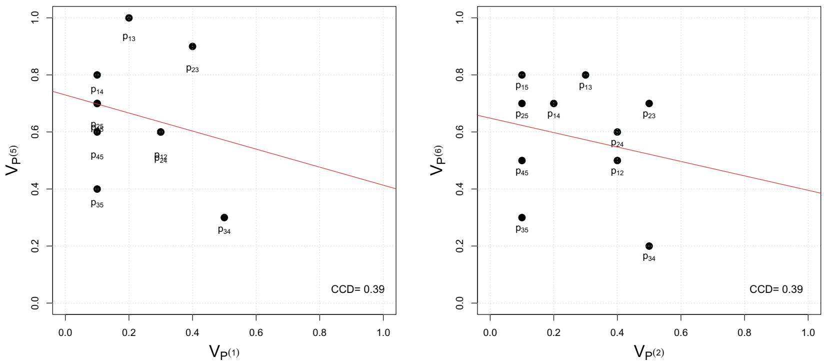

Fig. 6. Plots of essential vectors of intensities of preferences corresponding to patients diagnosed Schizophrenia versus Bipolar disorder (Subsection 6.2) and the best adjusted line.

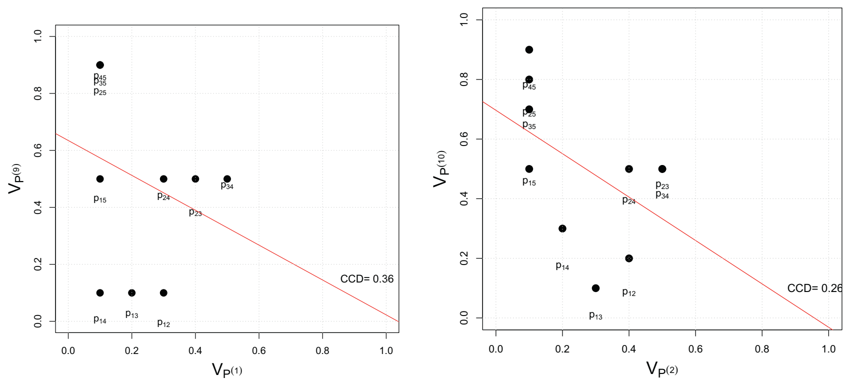

Fig. 7. Plots of essential vectors of intensities of preferences corresponding to patients diagnosed Schizophrenia versus OCD (Subsection 6.2) and the best adjusted line.

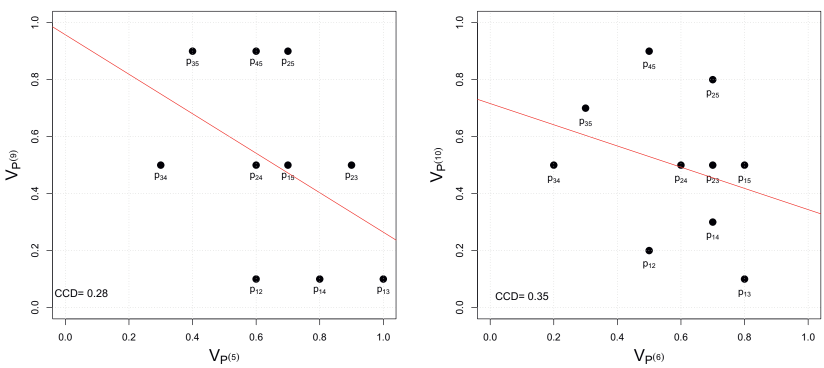

Fig. 8. Plots of essential vectors of intensities of preferences corresponding to patients diagnosed Bipolar disorder versus OCD (Subsection 6.2) and the best adjusted line.

7. Concluding remarks and future research

Research in the area of consensus measurement has advanced mainly in Social Choice Theory and Theory of Decision Making. In this work, a new consensus measure for reciprocal preference relations based on the classical definition of the Pearson correlation coefficient is studied. This new measure, the correlation consensus degree, pursues the measurement of the concordance between the intensities of pairwise preference values given by experts, decision makers or agents. This work open a new avenue to measure consensus. The correlation consensus degree between two reciprocal preference relations is neither a distance function nor a similarity function unlike the traditional consensus measures studied before. Nevertheless, the given correlation consensus degree verifies important properties that are common either to distances and/or similarities measures as well as additional properties that have been described in this work and that are different to traditional consensus measures properties. The novelty of the proposed correlation consensus measure as well as its application is shown with two examples. The first of the examples is used to illustrate the computation process and discussion of the results, while the second example covers a real life Clinical Decision-Making application. Both examples show the versatility and the applicability of the proposed measurement of consensus to a variety of real situations.

A future line of enquiry is the investigation of flexible consensus reaching processes based on the new correlation consensus degree. These processes would allow to produce a consensus solution by an iterative feedback mechanism accommodated to the this specific consensus measurement. We expect to conduct further investigations of these issues and report our findings in the future.

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Chapter 4

Publication III: A new consensus ranking approach for correlated ordinal information based on Mahalanobis distance

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A new consensus ranking approach for correlated ordinal information based on Mahalanobis distance

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We investigate from a global point of view the existence of cohesiveness among experts' opinions. We address this general issue from three basic essentials: the management of experts' opinions when they are expressed by ordinal information; the measurement of the degree of dissensus among such opinions; and the achievement of a group solution that conveys the minimum dissensus to the experts' group.

Accordingly, we propose and characterize a new procedure to codify ordinal information. We also define a new measurement of the degree of dissensus among individual preferences based on the Mahalanobis distance. It is especially designed for the case of possibly correlated alternatives. Finally, we investigate a procedure to obtain a social consensus solution that also includes the possibility of alternatives that are correlated. In addition, we examine the main traits of the dissensus measurement as well as the social solution proposed. The operational character and intuitive interpretation of our approaches are illustrated by an explanatory example.

1. Introduction

A considerable amount of literature has contributed to the research issue of obtaining consensus in group decision making problems. This issue is an active subject in several areas such as Social Choice Theory and Decision Making Theory. From the Social Choice perspective several contributions can be emphasized, e.g., [2,4,9,25,26,49], among others. From the Decision Making Theory, it has been successfully tackled by a great amount of contributions, e.g., [28,32,35,36,58], among others. Besides these main areas, there are some other methodologies that proposed different definitions of the consensus concept. It is worth mentioning the work of González Jaime et al. [38] and López Molina, De Baets and Bustince [47].

Any group decision making problem focused on obtaining consensus involves at least three key pillars. The first one is the way in which experts give their opinions on a set of alternatives and how such an information is managed. Once the opinions of the agents have been gathered it seems natural to measure how much cohesiveness these opinions generate. Thus, the second pillar is to establish a mechanism able to provide such measurements. Apart from determining the degree

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of consensus among experts the main aim of a group decision making problem is to determine a solution. The better solution the greater agreement this solution generates among experts. Consequently, supplying a method to achieve a group consensus solution is the third pillar.

We now briefly review the previous literature related to each basic essentials.

Information formats. Generally speaking, experts can express their opinions by means of ordinal or cardinal information, the former being more extensively used in the research issue addressed in this work. Nonetheless, contributions dealing with cardinal information include the approaches proposed by Herrera-Viedma, Herrera and Chiclana [36], González-Pachón and Romero [30] and González-Arteaga, Alcantud and de Andrés Calle [26]. The representation of ordinal information has been a subject of study for over two centuries for linear orders (see e.g., [3,9]), weak orders(see e.g., [4,16,24]) and fuzzy preferences (see e.g., [12,22,51,56], among others).

Regardless of the experts' information format, it is necessary to manipulate it in order to make suitable computations. In the literature several procedures to codify linear and complete preorders into numerical values can be found (see [7,8,14,25], among others), Borda [8] being the first author to manage ordinal preferences in such way.

Consensus measurement. This topic was initiated by Bosch [9] from the Social Choice perspective. In this vein McMorris and Powers [49] characterized consensus rules defined on hierarchies, while García-Lapresta and Pérez-Román [25] introduced a class of consensus measures based on distances. Subsequently, Alcalde-Unzu and Vorsatz [2,3] proposed and characterized a family of linear and additive consensus measures based on measuring similarity among preferences. From another point of view, Alcantud, de Andrés Calle and Cascón [5,6] introduced the analysis when opinions are dichotomous.

The use of distance and similarity functions has provided interesting insights about cohesiveness measurement. We highlight the role of the Kemeny, Mannhattan, Jacard, Dice and Cosine distance functions (see e.g., [5,13,15,25]). Moreover, it is also possible to apply some association measures to that purpose (see e.g., [14,21,27,33,42,55]).

Group consensus solution. Finding the best option or solution from alternatives is the main aim in group decision making problems. Recently, various approaches have been developed to solve this problem from a variety of science areas: *Operational Research* (see e.g., [17,20]), *Statistical Analysis* (see e.g., [1,23,45]), *Fuzzy Theory* (see e.g., [18,46,59]), and *Computational Analysis* (see e.g., [37,60]).

Traditionally, the achievement of a global solution has been considered as an aggregation problem of experts' opinions in order to obtain a social solution. Different methods have been proposed and analyzed for aggregating agents' opinions (preferences in the case of ordinal information) into a social solution. Borda [8] first examined this problem in a voting context and Kendall [41] subsequently revised Borda's method in a statistical framework.

Other authors also proposed alternative distance-based aggregation rules e.g., Eckert and Klamler [19], Klamler [43,44], Meskanen and Nurmi [50], Ratliff [52,53], and Saari and Merlin [54], even though Kemeny's rule [39] could be considered as a landmark in aggregation procedures based on distances. Following Kemeny's rule, Cook and Seiford [14] established an equivalence between the Borda-Kendall method [40] and their approach. González-Pachón and Romero [28] developed a general framework for distance-based consensus models under the assumption of a generic l_p metric. These authors have recently designed socially optimal decisions in a consensus scenario [31].

Once we have reviewed the related literature we now summarize the main contributions of this paper.

- We focus on group decision making problems where agents or experts provide their opinions on a set of alternatives by complete preorders. In this regard, we propose a new codification procedure to transform the original opinions/preferences of agents into numerical vectors in order to manage them. For the purpose of better understanding this process we investigate exactly which vectors are realizations by a *canonical codification procedure* of generic complete preorders. The characterization of the new codification procedure is a key point because it ensures consistency of our approach and its use in any methodology.
- In order to measure the degree of cohesiveness among agents' preferences, we design an indicator of dissensus for a finite collection of complete preorders on a finite set of alternatives based on the Mahalonobis distance, which is dependent on a positive definite matrix (the parameter) that captures the importance and possible cross-relations of each alternative, namely, the *Mahalanobis dissensus measures*. Any such indicator ranks the profiles of complete preorders (in the form of codified matrices) according to their inherent cohesiveness. The strength of our measurement unlike other aforementioned approaches based on distances is the inclusion of the relationships among alternatives. Then, the new measure incorporates relevant information that in other way is ignored. Moreover, we investigate the main characteristics of the novel measure and prove that a partial order can be naturally induced on the parametric class of all *Mahalanobis dissensus measures*.
- Then we exploit these measures in order to propose a consensus solution especially designed for profiles of preferences on possibly correlated alternatives and to overcome the drawbacks of the aforementioned distance-based methodologies. That solution aggregates individual opinions into a social preference on the alternatives by minimizing dissensus with respect to the original profile of preferences. In order to facilitate the computation of such compromise solution we prove that the problem is equivalent to minimizing the Mahalanobis distance to a single average vector. Whatever the statement of the minimization problem, the objective function is restricted to feasible codified vectors, which emphasizes

the importance of our characterization for the canonical codification procedure. Some properties of our *Mahalanobis consensus solution* are proven and discussed.

In addition, an explanatory example illustrates the operational characteristics and intuitive interpretation of our approaches to find rankings that best agree with the original opinions.

This paper is organized as follows. Section 2 is devoted to the problem of transforming ordinal information about individual preferences into numerical vectors as well as essential notation. Section 3 introduces the basic definition of dissensus measure and the Mahalanobis class of dissensus measures. Here we also explore their main traits too. In Section 4 we set forth the definition of our proposal of Mahalanobis social consensus solutions, prove some of its properties, and solve a visually appealing example. Finally, some concluding remarks are pointed out in Section 5.

2. Ordinal information

Most group decision making problems can usually manage different types of information. In this contribution we focus on the representation of agents' opinions by means of rankings allowing ties since most real situations involve such a kind of information. Dealing with this type of information necessarily entails determining how it is represented. In the specialized literature it is possible to find several approaches or procedures to codify ordinal information into numerical values (see [7,8,14,25], among others).

Due to the importance of the choice of the codification procedure to accomplish any methodology over ordinal information, it should be relevant to dispose of a consistent codification procedure. Accordingly, in this section we provide and characterize a new method to handle ordinal information as well as the basic notation of our proposal.

2.1. Notation

Consider a society of agents or experts $\mathbf{N} = \{1, 2, \ldots, N\}$, $N > 1$. Let $X = \{x_1, \ldots, x_k\}$ be a finite set of k issues, options or alternatives $|X| \geq 2$. Abusing notation, on occasions we refer to issue x_s as issue s for convenience.

Assume experts grade alternatives by means of complete preorders (also known as weak orders). Technically speaking, a complete preorder R on X means a complete and transitive binary relation on X. We write $W(X)$ to denote the set of all complete preorders on X.[1]

Let $R \in W(X)$ be a complete preorder on X, then $x_s \succ_R x_k$ means x_s is strictly preferred to x_k, $x_s \sim_R x_k$ means x_s and x_k are equally preferred and $x_s \succeq_R x_k$ means alternative x_s is at last as good as x_k. For a complete preorder $R \in W(X)$, let R^{-1} be the inverse of R such that $x_s \succ_{R^{-1}} x_k \Leftrightarrow x_k \succ_R x_s$ for all $x_s, x_k \in X$.

A *profile* $\mathcal{P} = (R_1, \ldots, R_N) \in W(X) \times \ldots \times W(X) = W(X)^N$ of the society $\mathbf{N}$ on the set of alternatives X is a collection of N complete preorders, where R_i represents the preferences of the individual i on the k alternatives for each $i = 1, \ldots, N$. Given a profile $\mathcal{P} = (R_1, \ldots, R_N)$, its inverse is denoted by $\mathcal{P}^{-1} = (R_1^{-1}, \ldots, R_N^{-1})$.

Any permutation σ of the agents/experts $\{1, 2, \ldots, N\}$ determines a permutation of $\mathcal{P}$ by $\mathcal{P}^\sigma = (R_{\sigma(1)}, \ldots\ldots, R_{\sigma(N)})$. Analogously, any permutation π of the alternatives $\{1, 2, \ldots, k\}$ determines a permutation of every complete preorder $R \in W(X)$ such that the permuted profile is denoted by ${}^\pi\mathcal{P} = ({}^\pi R_1, \ldots\ldots, {}^\pi R_N)$. We write $P(X) = \cup_{N \geqslant 1} W(X)^N$ to denote the set of all profiles for arbitrary societies.

The codification of preferences by numerical vectors has been used extensively in both theoretical and practical situations. Borda [8] was first to manage ordinal preferences in such way. His method, known as the "method of marks" or "Borda-Kendall method", has been widely disseminated in several areas.

Following the Social Choice tradition, the components of a numerical vector represent the rank or priority assigned to each alternative, or their average in case of ties. This convention has been exemplified by Black [7], Cook and Seiford [14] and García-Lapresta and Pérez-Román [25].

We now introduce notation related to the codification of linear and complete preorders by means of numerical vectors.

Let $R \in W(X)$ be a complete preorder on X, a *codified complete preorder* is a real-valued vector $M_R = (m_1, \ldots, m_k)$ where m_j represents the codification value corresponding to alternative x_j. It relates to R in the sense that $x_i \succeq_R x_j \Leftrightarrow m_i \geq m_j$.

A *codified profile* of $\mathcal{P}$ is a $N \times k$ real-valued matrix

$$M_{\mathcal{P}} = \begin{pmatrix} m_{11} & \ldots & m_{1k} \\ \vdots & \ddots & \vdots \\ m_{N1} & \ldots & m_{Nk} \end{pmatrix}_{N \times k}$$

where m_{ij} is the codification value of expert i over the alternative x_j. We write $\mathbb{M}_{N \times k}$ for the set of all $N \times k$ real-valued matrices. Thus $M_{\mathcal{P}} = (M_{R_1}, \ldots, M_{R_N}) \in \mathbb{M}_{N \times k}$ produces a unique profile $\mathcal{P}$ of complete preorders, although every profile of complete preorders can be associated with infinitely many matrices from $\mathbb{M}_{N \times k}$. For simplicity, on occasions we refer to $M_{\mathcal{P}}$ as M.

[1] It is assumed that a linear order on X is an antisymmetric weak order on X.

Row i of the profile $M_{\mathcal{P}}$ is identified by M_i. It describes the codification preferences of expert i over all alternatives, $M_i = M_{R_i} \in \mathbb{M}_{1\times k}$. Similarly, column j of the codification profile $M_{\mathcal{P}}$ captures the codification of agents' preferences on the alternative j, and it is denoted by $M^j \in \mathbb{M}_{N\times 1}$.

Any permutation σ of the experts $\{1, 2, \ldots, N\}$ determines a codified profile $M^{\sigma} = (M_{\sigma(1)}, \ldots, M_{\sigma(N)}) \in \mathbb{M}_{N\times k}$ by permutation of the rows of M: row i of the profile M^{σ} is row $\sigma(i)$ of the profile $M \in \mathbb{M}_{N\times k}$. Similarly, any permutation π of the alternatives $\{1, 2, \ldots, k\}$ determines a codified profile ${}^{\pi}M \in \mathbb{M}_{N\times k}$ by permutation of the columns of $M \in \mathbb{M}_{N\times k}$: column j of the profile ${}^{\pi}M$ is column $\pi(j)$ of the codification profile M. Notice that $M_{\mathcal{P}^{\sigma}} = (M_{\mathcal{P}})^{\sigma}$ and $M_{{}^{\pi}\mathcal{P}} = {}^{\pi}(M_{\mathcal{P}})$.

2.2. The canonical codification. Definition and characterization

In this subsection we define a new way to represent ordinal preferences by numerical vectors, namely, the*canonical codification*. Moreover, we characterize the new codification procedure to associate every profile of complete preorders with a unique matrix. Therefore, the use of this particular codification procedure is consistence and it could be used in any approach or methodology. Along this section, some illustrative examples are included to put it in practice.

Definition 1. The *canonical codified complete preorder* associated with $R \in W(X)$ is defined by the numerical vector $K_R = (c_1, \ldots, c_k) \in (\{1, \ldots, k\})^k$ where $c_j = |\{q : x_j \succcurlyeq_R x_q\}|$ and therefore c_j accounts for the number of alternatives that are graded at most as good as x_j.

A *canonical codified profile* associated with $\mathcal{P} = (R_1, \ldots, R_N) \in W(X)^N$ is an $N \times k$ real-valued matrix denoted as $K_{\mathcal{P}} = (K_{R_1}, \ldots, K_{R_N}) \in \mathbb{M}_{N\times k}$. Each K_{R_i} is row i in $K_{\mathcal{P}}$ and it corresponds to the canonical codified complete preorder associated with R_i.

Let us now provide an example in order to improve the understanding of our codification proposal.

Example 1. Let R_1, R_2, R_3 be the complete preorders on $\{x_1, x_2, x_3\}$ such that:

$$\begin{aligned}
R_1 &: \quad x_1 \succ_{R_1} x_2 \sim_{R_1} x_3, \\
R_2 &: \quad x_2 \succ_{R_2} x_1 \succ_{R_2} x_3, \\
R_3 &: \quad x_3 \succ_{R_3} x_1 \sim_{R_3} x_2.
\end{aligned}$$

Following Definition 1 their respective canonical codifications are $K_{R_1} = (3, 2, 2)$, $K_{R_2} = (2, 3, 1)$, and $K_{R_3} = (2, 2, 3)$. We consider only for illustration that these complete preorders define a profile, $\mathcal{P} = (R_1, R_2, R_3)$. Then its respective canonical codified profile is

$$K_{\mathcal{P}} = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 2 & 3 \end{pmatrix}.$$

In order to motivate the main result of this section, let us observe that not all vectors of natural values are feasible canonical codified complete preorders. For example, by means of the canonical codification it is not possible to get $K_R = (1, 1, 1)$ with $k = 3$ because if there is a tie among the three alternatives Definition 1 produces (3, 3, 3).

Considering these limitations, we now proceed to identify exactly which vectors correspond to a canonical codified complete preorder.

Proposition 1. *Given a vector $c = (c_1, \ldots, c_k) \in (\{1, \ldots, k\})^k$, this vector is the canonical codified complete preorder K_R associated with $R \in W(X)$ if and only if the increasingly ordered vector $(c_{(1)}, \ldots, c_{(k)})$ verifies*

(i) $c_{(1)} = t_1$,
(ii) $c_{(j+1)} = c_{(j)} + t_{j+1} \cdot D_{j+1}, \quad j \in \{1, \ldots, k-1\}$,

where t_j is the number of values equal to $c_{(j)}$ among the components of c and

$$D_{j+1} = \begin{cases} 0 & \text{if} \quad c_{(j+1)} = c_{(j)}, \\ 1 & \text{otherwise}. \end{cases}$$

Proof. Let $R \in W(X)$ be a complete preorder whose canonical codification is $K_R = (c_1, \ldots, c_k)$. Given a permutation on the alternatives τ, R^{τ} denotes the permutation τ on the complete preorder $R \in W(X)$ such that $K_{R^{\tau}} = (c_{(1)}, \ldots, c_{(k)})$ and $c_{(1)} \leqslant \ldots \leqslant c_{(k)}$.

Throughout the proof, $t \in \mathbb{N}^k$ stands for the vector containing the number of coincidences for the elements of K_R, $t = (t_1, t_2, \ldots, t_k) = (|T_1|, |T_2|, \ldots, |T_k|)$ where $T_j = \{t \in \{1, \ldots, k\} \mid c_t = c_{(j)}\}$ for $j \in \{1, \ldots, k\}$. Thus $|T_j|$ is the number of ties equal to $c_{(j)}$. The t vector is also called the ties vector of K_R.

Let us first examine *necessity*. Given a canonical codified complete preorder $K_R = (c_1, \ldots, c_k) \in (\{1, \ldots, k\})^k$ of $R \in W(X)$, let us check conditions *(i)* and *(ii)*.

(i) To deduce $c_{(1)} = t_1$, we consider Definition 1

$$c_{(1)} = |\{q : x_{(1)} \succcurlyeq x_q\}|,$$

where $x_{(1)}$ is the alternative associated with $c_{(1)}$. Then, $c_{(1)} = t_1$ due to the fact that $c_{(1)}$ is the number of alternatives equally preferred to $x_{(1)}$.

(ii) To deduce $c_{(j+1)} = c_{(j)} + t_{j+1} \cdot D_{j+1}, \quad j \in \{1, \ldots, k-1\}$, we claim that if alternative $x_{(j+1)}$ is equally preferred to alternative $x_{(j)}$, then $c_{(j+1)} = c_{(j)}$. In other case, by Definition 1

$$c_{(j+1)} = |\{q : x_{(j+1)} \succcurlyeq x_q\}|.$$

Hence, $c_{(j+1)}$ is the sum of the number of the strictly less preferred alternatives to $x_{(j+1)}$ plus the number of equally preferred alternatives to $x_{(j+1)}$. Formally,

$$c_{(j+1)} = |\{q : x_{(j+1)} \succ x_q\}| + |\{q : x_{(j+1)} \sim x_q\}|.$$

Then, $c_{(j+1)} = c_{(j)} + t_{j+1}$.

We now proceed to prove *sufficiency*. Suppose a numerical vector that verifies conditions *(i)* and *(ii)*, $c = (c_1, \ldots, c_k) \in (\{1, \ldots, k\})^k$. We are in a position to build a complete preorder $R \in W(X)$ such that $K_R = c$ as follows.

By ordering in an increasing order the vector c, we obtain an ordered vector, $c_{()} = (c_{(1)}, \ldots, c_{(k)})$ and it is easy to compute its associated ties vector $t = (t_1, \ldots, t_k) = (|T_1|, \ldots, |T_k|)$. Then

$$c_{()} = (c_{(1)}, \ldots, c_{(k)}) = (\overbrace{t_1, \ldots, t_1}^{t_1 \text{ times}}, \overbrace{t_1 + t_2, \ldots, t_1 + t_2}^{t_2 \text{ times}}, \ldots, \overbrace{k, \ldots, k}^{t_k \text{ times}})$$

and consequently, we can deduce the complete preorder R^{τ}:

$$x_{k-t_k+1} \sim \ldots \sim x_k \succ \ldots \succ x_{t_1+1} \sim \ldots \sim x_{t_1+t_2} \succ x_1 \sim \ldots \sim x_{t_1}$$

whose associated canonical codification is $c_{()}$. The proof is completed due to $K_R = c$. $\square$

Now we proceed to exemplify the relevance of this result.

Example 2. In order to verify the necessity of establishing a characterization of the codification procedure, let us check if some numerical vectors can actually represent codified complete preorders for the case of four alternatives.

- Consider the numerical vector $c = (3, 4, 1, 1)$. First, its increasingly ordered vector and its corresponding ties vector are determined, $c_{()} = (1, 1, 3, 4)$ and $(|T_1|, |T_2|, |T_3|, |T_4|) = (2, 2, 1, 1)$, respectively. Second, by Proposition 1, we check that if c represents a canonical codification K_R for some complete preorder $R \in W(X)$ then the first element of K_R should be 2. Therefore, c is not a canonical codified complete preorder.
- We repeat the previous exercise for the numerical vector $c = (2, 3, 3, 1)$. Then, $c_{()} = (1, 2, 3, 3)$ is its increasingly ordered vector and $(|T_1|, |T_2|, |T_3|, |T_4|) = (1, 1, 2, 2)$ is its ties vector. Using Proposition 1, the first, second and third element of K_R should be 1, 2 and $2 + 2 = 4$, respectively. However, the latter is not true since $c_{(3)} = 3$. Thus, the vector c does not represent any complete preorder by the canonical codification.
- Finally, given $c = (4, 2, 2, 1)$ a numerical vector, being its corresponding increasingly ordered vector $c_{()} = (1, 2, 2, 4)$ and its ties vector $(|T_1|, |T_2|, |T_3|, |T_4|) = (1, 2, 2, 1)$. By means of Proposition 1, if c represents a canonical codification K_R for some complete preorder $R \in W(X)$, the first and second element of K_R should be 1 and $1 + 2 = 3$ respectively, but it is not true because $c_{(2)} = 2$. Therefore, c does not represent any complete preorder by the canonical codification.

3. A new dissensus measure for ordinal information: the class of Mahalanobis dissensus measures

A considerable amount of the most cited contributions on consensus measurement have addressed this topic considering functions that assign to every ranking profile a real number *from the unit interval*. Therefore, the higher the assignment, the more coherence among agents' preferences.

In this contribution we focus on the notion of dissensus measurement, concretely, our approach resembles the notion of a "measure of statistical dispersion", in the sense that 0 captures the natural notion of unanimity as total lack of variability, and then increasingly higher numbers mean more disagreement among rankings in the profile. Then, we introduce a new broad class of dissensus measures associated with a reference matrix, namely the *Mahalanobis dissensus measures* that includes the possibility of cross-related alternatives. Moreover, some important properties of the new measurement are included.

Definition 2. A *dissensus measure* is a mapping δ: $W(X)^N \to [0, \infty)$ given by

$$\delta(\mathcal{P}) = \delta^*(M_{\mathcal{P}})$$

for each profile $\mathcal{P} \in W(X)^N$ and its codified profile $M_{\mathcal{P}} \in \mathbb{M}_{N\times k}$, where δ^* is a mapping $\delta^* : \mathbb{M}_{N\times k} \to [0, \infty)$ with the property:

(I) $\delta(\mathcal{P}) = 0$ if and only if $\mathcal{P}$ is unanimous. In other words, $\delta^*(M_{\mathcal{P}}) = 0$ if and only if $M_{\mathcal{P}}$ is unanimous.

Henceforth we also deal with dissensus measures that are normal, in the following sense:

Definition 3. A dissensus measure is *normal* if it further verifies:

(II) *Anonymity*: $\delta(\mathcal{P}^\sigma) = \delta^*((M_\mathcal{P})^\sigma) = \delta^*(M_\mathcal{P}) = \delta(\mathcal{P})$ for each permutation σ of the agents and $M_\mathcal{P} \in \mathbb{M}_{N\times k}$.
(III) *Neutrality*: $\delta(^\pi\mathcal{P}) = \delta^*(^\pi(M_\mathcal{P})) = \delta^*(M_\mathcal{P}) = \delta(\mathcal{P})$ for each permutation π of the alternatives and $M_\mathcal{P} \in \mathbb{M}_{N\times k}$.

Before providing our main definition, we recall the *Mahalanobis distance* [48] on which our measure is based. This distance is a common tool in multivariate statistical analysis, e.g., in regression models. We select it in our proposal because it allows to take into account cross relations among alternatives which is frequent in real situations.

Definition 4. Let $\Sigma \in \mathbb{M}_{k\times k}$ be a positive definite matrix and $x, y \in \mathbb{R}^k$ be two row vectors. The Mahalanobis (squared) distance on $\mathbb{R}^k$ associated with Σ is defined by [2]

$$d_\Sigma(x, y) = (x - y)\Sigma^{-1}(x - y)^t$$

The Mahalanobis distance includes some particular distances such as the (squared) Euclidean distance when Σ is the identity matrix.

Definition 5. Let $\Sigma \in \mathbb{M}_{k\times k}$ be a positive definite matrix and let us fix a codification procedure for profiles of complete preorders, $\mathcal{P} \in W(X)^N$. The *Mahalanobis dissensus measure* associated with Σ is the mapping δ_Σ: $W(X)^N \to [0, \infty)$ given by

$$\delta_\Sigma(\mathcal{P}) = \delta^*_\Sigma(M_\mathcal{P}),$$

for each profile $\mathcal{P} \in W(X)^N$ and its codified profile $M_\mathcal{P} \in \mathbb{M}_{N\times k}$, where δ^*_Σ is the mapping $\delta^*_\Sigma : \mathbb{M}_{N\times k} \to [0, \infty)$ given by

$$\delta^*_\Sigma(M_\mathcal{P}) = \frac{1}{C_N^2} \cdot \sum_{i<j} d_\Sigma(M_i, M_j),$$

and $C_N^2 = \frac{N(N-1)}{2}$ is the number of unordered pairs of the N agents.

Notice that δ^*_Σ is the arithmetic mean of the Mahalanobis distances between each pair of codified complete preorders for each agent following Hays's approach [34].

Remark 1. The Mahalanobis dissensus measure satisfies the assumption of Definition 2 because

$$\delta_\Sigma(\mathcal{P}) = \delta^*_\Sigma(M_\mathcal{P}) = 0$$

if and only if $\mathcal{P}$ is unanimous. This fact is easy to prove since d_Σ is a distance.

Along this contribution we use the codification procedure given in Section 2 even though the Mahalanobis dissensus measure is compatible with different codification procedures.

To emphasize the advantages of our proposal, it could be interesting not only to obtain values but to compare them in order to rank the original profiles attending to their degree of dissensus. In this sense, next definition is provided.

Definition 6. Each dissensus measure δ_Σ, for a positive definite matrix $\Sigma \in \mathbb{M}_{k\times k}$, produces *a ranking of profiles* of complete preorders $\succcurlyeq_{\delta_\Sigma}$ by establishing that

$$\mathcal{P} \succcurlyeq_{\delta_\Sigma} \mathcal{P}' \quad \text{iff} \quad \delta^*_\Sigma(M_{\mathcal{P}'}) \geqslant \delta^*_\Sigma(M_\mathcal{P}),$$

for $\mathcal{P}, \mathcal{P}' \in W(X)^N$ two profiles with codified profiles $M_\mathcal{P}, M_{\mathcal{P}'} \in \mathbb{M}_{N\times k}$.

This is to say, a profile $\mathcal{P}$ conveys at least as much consensus as the profile $\mathcal{P}'$ when the dissensus measure of codified profile of $\mathcal{P}'$ is at least as large as the dissensus measure of the codified profile of $\mathcal{P}$.[3]

By way of illustration, we present the following example.

Example 3. Let Σ be the identity matrix and $\mathcal{P}_1, \mathcal{P}_2 \in W(X)^2$ be two profiles whose numerical codifications are

$$M_{\mathcal{P}_1} = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 2 & 3 \end{pmatrix} \text{ and } M_{\mathcal{P}_2} = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 1 \end{pmatrix}.$$

Their Mahalanobis dissensus measures are computed as:

- $\delta_\Sigma(\mathcal{P}_1) = \delta^*_\Sigma(M_{\mathcal{P}_1}) = (1, 0, -1)\Sigma^{-1}(1, 0, -1)^t = 2.$
- $\delta_\Sigma(\mathcal{P}_2) = \delta^*_\Sigma(M_{\mathcal{P}_2}) = (1, -1, 1)\Sigma^{-1}(1, -1, 1)^t = 3$.

Assuming Definition 6 we can conclude $\mathcal{P}_1 \succ_{\delta_\Sigma} \mathcal{P}_2$.

The major source of uncertainty in the Mahalanobis dissensus measure is the choice of the Σ matrix. In this regard, we propose next definition for establishing a partial order on the set of all Mahalanobis dissensus measures and then to overcome this possible drawback.

[2] Our choice of $d_\Sigma(x, y)$ coincides with Mahalanobis' original definition [48].
[3] As is standard practice, the asymmetric part of the complete preorder $\succcurlyeq_{\delta_\Sigma}$ is denoted by $\succ_{\delta_\Sigma}$.

Definition 7. Let Δ be the set of all Mahalanobis dissensus measures. For any $\delta_{\Sigma_1}, \delta_{\Sigma_2} \in \Delta$ associated with $\Sigma_1, \Sigma_2 \in \mathbb{M}_{k\times k}$, a binary relation R_Δ is defined by

$$\delta_{\Sigma_1} \succcurlyeq_{R_\Delta} \delta_{\Sigma_2} \Leftrightarrow \delta^*_{\Sigma_1}(M) \geq \delta^*_{\Sigma_2}(M),$$

for each N and for all codified profile $M \in \mathbb{M}_{N\times k}$.

This relation verifies the property of reflexivity, antisymmetry and transitivity. Therefore, R_Δ is a partial order in Δ.

In order to analize the properties of the Mahalanobis dissensus measures, it seems reasonable that we initially explore if these measures satisfy anonymity and neutrality, that is, if the Mahalanobis dissensus measures are normal dissensus measures and then the rest of their properties.

Let $\Sigma \in \mathbb{M}_{k\times k}$ be a positive definite matrix and let us fix a codification procedure for profiles of complete preorders, $\mathcal{P} \in W(X)^N$ such that for each profile $\mathcal{P}$ produces its codified profile $M_\mathcal{P} \in \mathbb{M}_{N\times k}$. The Mahalanobis dissensus measures verify:

Anonymity. Given permutation σ of the agents in the profile $\mathcal{P}$, a Mahalanobis dissensus measure δ_Σ verifies *anonymity* since

$$\delta_\Sigma(\mathcal{P}) = \delta^*_\Sigma(M_\mathcal{P}) = \delta^*_\Sigma\left((M_\mathcal{P})^\sigma \right) = \delta_\Sigma(\mathcal{P}^\sigma)$$

for any codified profile $M_\mathcal{P} \in \mathbb{M}_{N\times k}$.

Neutrality. A Mahalanobis dissensus measure δ_Σ verifies *neutrality* if and only if the associated Σ matrix is a diagonal matrix whose diagonal elements have to be equal among them. Formally:

$$\delta_\Sigma(\mathcal{P}) = \delta^*_\Sigma(M_\mathcal{P}) = \delta^*_\Sigma({}^\pi M_\mathcal{P}) = \delta_\Sigma({}^\pi\mathcal{P}),$$

for any codified profile $M_\mathcal{P} \in \mathbb{M}_{N\times k}$ and for any permutation π of $\{1, \dots, k\}$ if and only if $\Sigma = diag\{\lambda, \dots, \lambda\}$ for a value $\lambda > 0$. [4]

Noting the previous result and being critical of our measure, it could be considered as a drawback the fact that *neutrality* is only verified when Σ matrix is so specific. Thinking about it, we can point out that the main contribution of our approach is to allow different roles for alternatives. This fact produces that traditional *neutrality* property is only verified when alternatives are not related and are exchangeable.

In order to overcome this drawback and emphasize the advantages of the Mahalanobis dissensus measures (cross relations among alternatives allowed), we propose to recall the neutrality property. If the alternatives are relabeled, there exists a way to recover the same value of the Mahalanobis dissensus measure, δ_Σ for each profile, as Proposition 2 shows.

Proposition 2 (Weak neutrality). *Let $\Sigma \in \mathbb{M}_{k\times k}$ be a positive definite matrix. For each profile $\mathcal{P} \in W(X)^N$, its codified profile $M \in \mathbb{M}_{N\times k}$ and for each permutation π of the alternatives, it is verified*

$$\delta_\Sigma(\mathcal{P}) = \delta^*_\Sigma(M_\mathcal{P}) = \delta^*_{\Sigma^\pi}({}^\pi M_\mathcal{P}) = \delta_{\Sigma^\pi}({}^\pi\mathcal{P}),$$

where $\Sigma^\pi = \Pi^t_\pi \, \Sigma \, \Pi_\pi$ and $\Pi_\pi \in \mathbb{M}_{k\times k}$ the permutation matrix corresponding to π. [5]

Proof. Proposition 2 proof is similar to analogous result in González-Arteaga, Alcantud and de Andrés Calle [26]. □

Compatibility. Let $\mathcal{P}, \mathcal{P}' \in W(X)^N$ be two profiles and $M_\mathcal{P}, M_{\mathcal{P}'} \in \mathbb{M}_{N\times k}$ be their respective codified profiles. A Mahalanobis dissensus measure δ_Σ is *compatible* with linear transformations of codified profiles if

$$\delta^*_\Sigma(M_{\mathcal{P}'}) \geqslant \delta^*_\Sigma(M_\mathcal{P}) \;\Leftrightarrow\; \delta^*_\Sigma(f(M_{\mathcal{P}'})) \geqslant \delta^*_\Sigma(f(M_\mathcal{P}))$$

where $f(M_\mathcal{P})$, $f(M_{\mathcal{P}'})$ are respective cell-by-cell transformations of the codified profiles $M_\mathcal{P}$ and $M_{\mathcal{P}'}$ by any linear transformation f.

Note compatibility refers to the behavior of the ranking of the profiles previously provided in Definition 6.

Proof. Let f be a linear transformation $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = a + bx$. Using this transformation cell-by-cell on $M_\mathcal{P}$ and $M_{\mathcal{P}'}$, it is obtained $f(M_\mathcal{P}) = a \cdot \mathbf{1}_N + b \cdot M_\mathcal{P}$ and $f(M_{\mathcal{P}'}) = a \cdot \mathbf{1}_N + b \cdot M_{\mathcal{P}'}$, where $\mathbf{1}_N = (1, 1, \dots, 1)$. Then, $(f(M_\mathcal{P}))_i = a \cdot \mathbf{1}_N + b \cdot (M_\mathcal{P})_i = f((M_\mathcal{P})_i)$ and analogously for $M_{\mathcal{P}'}$. This implies

$$f((M_\mathcal{P})_i) - f((M_\mathcal{P})_j) = a \cdot \mathbf{1}_N + b \cdot (M_\mathcal{P})_i - a \cdot \mathbf{1}_N + b \cdot (M_\mathcal{P})_j = b \cdot ((M_\mathcal{P})_i - (M_\mathcal{P})_j).$$

$$\delta^*_\Sigma(f(M_\mathcal{P})) = \frac{1}{C^2_N} \sum_{i<j} d_\Sigma\left[(f(M_\mathcal{P}))_i,\; (f(M_\mathcal{P}))_j \right] =$$

[4] A diagonal matrix Σ with diagonal elements $\{\lambda, \dots, \lambda\}$ is represented as $\Sigma = diag(\lambda, \dots, \lambda)$.

[5] Let π be a permutation of $\{1, 2, \dots, k\}$ and e_i be the i-th vector of the canonical base of $\mathbb{R}^n$, that is, $e_{ij} = 1$ if $i = j$, $e_{ij} = 0$ otherwise. The matrix $\Pi_\pi \in \mathbb{M}_{k\times k}$ whose rows are $e_{\pi(i)}$ is called the permutation matrix associated to π. The rearrangement of the corresponding rows (resp. columns) of a matrix A using π is obtained by left (resp., right) multiplication of Π_π, $\Pi_\pi A$ (resp., $A\,\Pi_\pi$).

$$= \frac{1}{C_N^2} \sum_{i<j} d_\Sigma \Big[f((M_\mathcal{P})_i), \; f((M_\mathcal{P})_j) \Big] =$$
$$= \frac{1}{C_N^2} \sum_{i<j} d_\Sigma \Big(a \cdot \mathbf{1}_N + b \cdot (M_\mathcal{P})_i \,, \; a \cdot \mathbf{1}_N + b \cdot (M_\mathcal{P})_j \Big) =$$
$$= \frac{1}{C_N^2} \sum_{i<j} \Big[\Big(f((M_\mathcal{P})_i) - f((M_\mathcal{P})_j) \Big) \Sigma^{-1} \Big(f((M_\mathcal{P})_i) - f((M_\mathcal{P})_j) \Big)^t \Big] =$$
$$= \frac{1}{C_N^2} \sum_{i<j} \Big[\Big(b \cdot (M_\mathcal{P})_i - b \cdot (M_\mathcal{P})_j \Big) \Sigma^{-1} \Big(b \cdot (M_\mathcal{P})_i - b \cdot (M_\mathcal{P})_j \Big)^t \Big] =$$
$$= \frac{1}{C_N^2} \sum_{i<j} b^2 \Big[\Big((M_\mathcal{P})_i - (M_\mathcal{P})_j \Big) \Sigma^{-1} \Big((M_\mathcal{P})_i - (M_\mathcal{P})_j \Big)^t \Big] =$$
$$= b^2 \frac{1}{C_N^2} \sum_{i<j} d_\Sigma \Big[(M_\mathcal{P})_i, \; (M_\mathcal{P})_j \Big] =$$
$$= b^2 \, \delta_\Sigma^* (M_\mathcal{P}).$$

Therefore, we have $\delta_\Sigma^*(f(M_\mathcal{P})) = b^2 \, \delta_\Sigma^*(M_\mathcal{P})$ and $\delta_\Sigma^*(f(M_{\mathcal{P}'})) = b^2 \, \delta_\Sigma^*(M_{\mathcal{P}'})$.

Now, it is easy to complete the proof. □

Reciprocity. Reciprocity means that if all individual complete preorders are reversed, then the degree of dissensus does not change. A Mahalanobis dissensus measure δ_Σ is *reciprocal* if

$$\delta_\Sigma(\mathcal{P}) = \delta_\Sigma^*(M_\mathcal{P}) = \delta_\Sigma^*(M_{\mathcal{P}^{-1}}) = \delta_\Sigma(\mathcal{P}^{-1})$$

for all $\mathcal{P} = (R_1, \ldots, R_N) \in W(X)^N$ and a codification procedure such that $M_{\mathcal{P}^{-1}} = (k+1) \cdot \mathbf{1}_N - M_\mathcal{P}$ where $\mathbf{1}_N = (1, 1, \ldots, 1)$.

Proof. Let $\mathcal{P} = (R_1, \ldots, R_N) \in W(X)^N$ be a profile whose codified profile is $M_\mathcal{P} \in \mathbb{M}_{N\times k}$. The reverse of the complete preorders produces a new profile $\mathcal{P}^{-1} = (R_1^{-1}, \ldots, R_N^{-1}) \in W(X)^N$ whose codified profile is $M_{\mathcal{P}^{-1}} \in \mathbb{M}_{N\times k}$. The proof is easy from

$$d_\Sigma(M_{R_i^{-1}}, M_{R_j^{-1}}) = d_\Sigma((k+1) \cdot \mathbf{1}_N - M_{R_i}, \; (k+1) \cdot \mathbf{1}_N - M_{R_j}) =$$
$$= d_\Sigma(M_{R_i}, M_{R_j}).$$

□

4. Reaching a social consensus solution based on Mahalanobis distance

The problem of reaching a social consensus solution intends to determine the ranking of alternatives that best agrees with individual preferences, or in other words, the ranking that minimizes the disagreement among individuals.

In this section we present a new proposal to obtain a social consensus solution based on the Mahalanobis distance as well as its properties. The Mahalanobis social consensus solution preserves the advantages of the Mahalanobis distance since it takes into account the correlation among alternatives. In addition, an illustrative example is included to show the graphical interpretation of our proposal.

4.1. Our proposal: the Mahalanobis social consensus solution

Our aim is to determine a complete preorder $\hat{R}$ that provides the best agreement for N rankings taking into account the Mahaloanobis distance. This relation $\hat{R}$ is called the *Mahalanobis social consensus solution*.

Following the traditional approaches and in order to obtain a consensus solution, first of all it is necessary to establish the objective function to optimize. In this contribution, this function is called *Mahalanobis consensus distance function* (MCDF).

Definition 8. Let $\Sigma \in \mathbb{M}_{k\times k}$ be a definite positive matrix and $\mathcal{P} = (R_1, \ldots, R_N) \in W(X)^N$ be a profile of complete preorders. Given a codification procedure, $M_\mathcal{P} = (M_{R_1}, \ldots, M_{R_N}) \in \mathbb{M}_{N\times k}$ is the codified profile of $\mathcal{P}$. The *Mahalanobis consensus distance function (MCDF)* is a mapping $\mathcal{C}_{\Sigma,\mathcal{P}} : \mathbb{M}_{N\times k} \longrightarrow [0, \infty)$ defined by

$$\mathcal{C}_{\Sigma,\mathcal{P}}(M_R) = \sum_{i=1}^{N} d_\Sigma(M_{R_i}, M_R) = \sum_{i=1}^{N} (M_{R_i} - M_R) \Sigma^{-1} (M_{R_i} - M_R)^t$$

and it regards the sum of the Mahalanobis distances from each of the N agent's preferences to a complete preorder R whose codification is M_R.

Once the Mahalanobis consensus distance function has been defined, we proceed to establish our optimization problem:

$$\begin{array}{llcll} \min\limits_{M_R} & \mathcal{C}_{\Sigma,\mathcal{P}}(M_R) & = & \min\limits_{M_R} & \sum\limits_{i=1}^{N} d_\Sigma(M_{R_i}, M_R) \\ \text{s.t.} & M_R \in F & & \text{s.t.} & M_R \in F \end{array}$$

where the feasible set F is the set with elements M_R that are codified complete preorders, so that $M_R = (m_1, \ldots, m_k)$.
Solving the above optimization problem we obtain the following solution, $M_{\hat{R}}$.

Definition 9. A *Mahalanobis consensus solution* is an ordinal ranking of the alternatives obtained by solving

$$\begin{array}{llcll} \min\limits_{M_R} & \mathcal{C}_{\Sigma,\mathcal{P}}(M_R) & = & \min\limits_{M_R} & \sum\limits_{i=1}^{N} d_\Sigma(M_{R_i}, M_R) = \mathcal{C}_{\Sigma,\mathcal{P}}(M_{\hat{R}}) \\ \text{s.t.} & M_R \in F & & \text{s.t.} & M_R \in F \end{array}$$

where $M_{\hat{R}} = (\hat{m}_1, \ldots, \hat{m}_k)$ is a vector which minimizes the Mahalanobis consensus distance function.

The proposed optimization problem can generate complete preorders or linear orders like ranking solutions. If no ties are required, the set of constraints in F has to provide for.

In order to simplify and facilitate the computation of Mahalanobis consensus solutions we present Theorem 1. This new result allows to establish an equivalence between rankings obtained by the method of minimized Mahalanobis consensus distance function (MCDF) and rankings closest to the mean vector $\overline{M}$ defined by the component-wise averages. This theorem makes the method analytically rigorous and provides an intuitively appealing approach.[6]

Theorem 1. *Let $\Sigma \in \mathbb{M}_{k\times k}$ be a positive definite matrix and $M_\mathcal{P} \in \mathbb{M}_{N\times k}$ be a codified profile. The following statements are equivalent:*

1. *$M_{\hat{R}}$ minimizes $\mathcal{C}_{\Sigma,\mathcal{P}}(M_R) = \sum_{i=1}^{N} d_\Sigma(M_{R_i}, M_R)$.*
2. *$M_{\hat{R}}$ minimizes $d_\Sigma(\overline{M_\mathcal{P}}, M_R)$ being*

$$\overline{M_\mathcal{P}} = (\overline{M^1}, \ldots, \overline{M^k}) = \left(\frac{1}{N}\sum_{i=1}^{N} m_{i1}, \ldots, \frac{1}{N}\sum_{i=1}^{N} m_{ik} \right).$$

Proof.

$$\begin{aligned}
\mathcal{C}_{\Sigma,\mathcal{P}}(M_R) &= \sum_{i=1}^{N} d_\Sigma(M_{R_i}, M_R) = \sum_{i=1}^{N} (M_{R_i} - M_R)\Sigma^{-1}(M_{R_i} - M_R)^t = \\
&= \sum_{i=1}^{N} \left(M_{R_i}\Sigma^{-1}M_{R_i}^t - 2M_{R_i}\Sigma^{-1}M_R^t + M_R\Sigma^{-1}M_R^t \right) = \\
&= \left(\sum_{i=1}^{N} M_{R_i}\Sigma^{-1}M_{R_i}^t \right) - 2\left(\sum_{i=1}^{N} M_{R_i}\Sigma^{-1}M_R^t \right) + \left(\sum_{i=1}^{N} M_R\Sigma^{-1}M_R^t \right) = \\
&= \left(\sum_{i=1}^{N} M_{R_i}\Sigma^{-1}M_{R_i}^t \right) - 2\left(\sum_{i=1}^{N} M_{R_i} \right)\Sigma^{-1}M_R^t + NM_R\Sigma^{-1}M_R^t = \\
&= \left(\sum_{i=1}^{N} M_{R_i}\Sigma^{-1}M_{R_i}^t \right) - 2N\overline{M}\,\Sigma^{-1}M_R^t + NM_R\,\Sigma^{-1}M_R^t = \\
&= \left(\sum_{i=1}^{N} M_{R_i}\Sigma^{-1}M_{R_i}^t \right) + N\left(-2\overline{M}\,\Sigma^{-1}M_R^t + M_R\,\Sigma^{-1}M_R^t \right) \\
d_\Sigma(\overline{M_\mathcal{P}}, M_R) &= (\overline{M_\mathcal{P}} - M_R)\Sigma^{-1}(\overline{M_\mathcal{P}} - M_R)^t = \\
&= \overline{M_\mathcal{P}}\,\Sigma^{-1}\,\overline{M_\mathcal{P}}^t + \left(-2\overline{M_\mathcal{P}}\,\Sigma^{-1}M_R^t + M_R\,\Sigma^{-1}M_R^t \right)
\end{aligned}$$

As we can observe the minimization of $\mathcal{C}_{\Sigma,\mathcal{P}}(M_R)$ and $d_\Sigma(\overline{M_\mathcal{P}}, M_R)$ only depends, in both cases, on $-2\overline{M_\mathcal{P}}\,\Sigma^{-1}M_R^t + M_R\,\Sigma^{-1}M_R^t$. Then, both problems are equivalent. □

[6] The degree of computational complexity of our approach is not higher than other related well-known approaches [11]. Nowadays there are several powerful computational tools able to solve this kind of problems for a reasonable size (see e.g., [10] and [57], among others).

A strong evidence of the strength of our proposal to obtain a social consensus solution is given through the consistency between the methodology proposed by Cook and Seiford [14] based on Euclidean distance and ours based on Mahalanobis distance. Concretely, Cook and Seiford [14] formalized the so-called*monotone non-decreasing* property for the case of the*Minimum Variance* method in order to realize the potential of the alignment between the average point and the ranking that minimizes the Euclidean distance. If we apply Cook and Seiford's idea but using a Mahalanobis distance associated with $\Sigma \in \mathbb{M}_{k\times k}$, then their relationship is satisfied when the vectors are expressed in the space of the eigenvectors of the matriz Σ. More precisely, the Mahalanobis consensus solution and the mean vector are linked like the next proposition shows.

Proposition 3. *A Mahalanobis consensus solution $M_{\hat{R}} = (\hat{m}_1, \dots, \hat{m}_k)$ does not reverse preferences given by the average point $\overline{M} = (\overline{M^1}, \dots, \overline{M^k})$ when both are expressed in the basis of the eigenvectors of the matrix Σ, $\overline{M}^e$ and $M^e_{\hat{R}}$, respectively. More precisely:*

$$\overline{M^i}^{\,e} < \overline{M^j}^{\,e} \Longrightarrow \hat{m}^e_i < \hat{m}^e_j \qquad \text{for } i, j \in \{1, \dots, k\},\ \ i \neq j$$

Proof. Consider the spectral decomposition of the matrix $\Sigma = \Gamma^t D_\lambda \Gamma$ where Γ and D_λ contain eigenvectors (by columns) and the corresponding eigenvalues of Σ as diagonal elements, respectively.

Let $E = \Gamma D_\lambda^{-\frac{1}{2}}$ be the matrix that defines the linear transformation in order to change N-dimensional vectors to coordinates of the eigenspace of Σ.

Applying the aforementioned transformation on the vectors $\overline{M}$, and $M_{\hat{R}}$, it yields $\overline{M}^e = \overline{M}E = (\overline{M^1}^{\,e}, \dots, \overline{M^k}^{\,e})$ and $M^e_{\hat{R}} = M_{\hat{R}}E = (\hat{m}^e_1, \dots, \hat{m}^e_k)$, respectively.

We must prove that if $\overline{M^i}^{\,e} < \overline{M^j}^{\,e}$ then $\hat{m}^e_i \leq \hat{m}^e_j$ for $i, j \in \{1, \dots, k\}$, $i \neq j$. Suppose $\overline{M^i}^{\,e} < \overline{M^j}^{\,e}$ and $\hat{m}^e_i > \hat{m}^e_j$. Let M' be a vector such that $M'^e = M'E = (m'^e_1, \dots, m'^e_k)$ and its elements are obtained from $M^e_{\hat{R}}$ by interchanging $\hat{m}^e_i$ and $\hat{m}^e_j$, i.e.,

$$m'^e_r = \begin{cases} \hat{m}^e_j & \text{if} \quad r = i, \\ \hat{m}^e_i & \text{if} \quad r = j, \\ \hat{m}^e_r & \text{otherwise.} \end{cases}$$

First, we obtain $d_\Sigma(\overline{M}, M_{\hat{R}})$:

$$\begin{aligned} d_\Sigma(\overline{M}, M_{\hat{R}}) &= (\overline{M} - M_{\hat{R}})\Sigma^{-1}(\overline{M} - M_{\hat{R}})^t \\ &= (\overline{M} - M_{\hat{R}})\Gamma D_\lambda^{-1}\Gamma^t(\overline{M} - M_{\hat{R}})^t \\ &= (\overline{M} - M_{\hat{R}})\Gamma D_\lambda^{-\frac{1}{2}} D_\lambda^{-\frac{1}{2}}\Gamma^t(\overline{M} - M_{\hat{R}})^t \\ &= (\overline{M} - M_{\hat{R}})E E^t(\overline{M} - M_{\hat{R}})^t \\ &= (\overline{M}E - M_{\hat{R}}E)(\overline{M}E - M_{\hat{R}}E)^t \\ &= (\overline{M}^e - M^e_{\hat{R}})(\overline{M}^e - M^e_{\hat{R}})^t \end{aligned}$$

Analogously, we compute $d_\Sigma(\overline{M}, M')$:

$$d_\Sigma(\overline{M}, M') = (\overline{M} - M')\Sigma^{-1}(\overline{M} - M')^t = (\overline{M}^e - M'^e)(\overline{M}^e - M'^e)^t.$$

Next, we must get $d_\Sigma(\overline{M}, M_{\hat{R}}) - d_\Sigma(\overline{M}, M')$:

$$\begin{aligned} & d_\Sigma(\overline{M}, M_{\hat{R}}) - d_\Sigma(\overline{M}, M') \\ &= (\overline{M}^e - M^e_{\hat{R}})(\overline{M}^e - M^e_{\hat{R}})^t - (\overline{M}^e - M'^e)(\overline{M}^e - M'^e)^t \\ &= \left(\overline{M}^e\,\overline{M}^{et} - 2\overline{M}^e M^{et}_{\hat{R}} + M^e_{\hat{R}} M^{et}_{\hat{R}}\right) - \left(\overline{M}^e\,\overline{M}^{et} - 2\overline{M}^e M'^{et} + M'^e M'^{et}\right) \\ &= (\overline{M}^e_1, \dots, \overline{M}^e_i, \dots \overline{M}^e_j, \dots \overline{M}^e_k)\left(m'^e_1, \dots, m'^e_i, \dots, m'^e_j, \dots, m'^e_k\right) \\ &\quad - (\overline{M}^e_1, \dots, \overline{M}^e_i, \dots \overline{M}^e_j, \dots \overline{M}^e_k)\left(\hat{m}^e_1, \dots, \hat{m}^e_i, \dots, \hat{m}^e_j, \dots, \hat{m}^e_k\right) \\ &= (\overline{M}^e_1, \dots, \overline{M}^e_i, \dots \overline{M}^e_j, \dots \overline{M}^e_k)\left(\hat{m}^e_1, \dots, \hat{m}^e_j, \dots, \hat{m}^e_i, \dots, \hat{m}^e_k\right) \\ &\quad - (\overline{M}^e_1, \dots, \overline{M}^e_i, \dots \overline{M}^e_j, \dots \overline{M}^e_k)\left(\hat{m}^e_1, \dots, \hat{m}^e_i, \dots, \hat{m}^e_j, \dots, \hat{m}^e_k\right) \\ &= 2\left(\overline{M}^e M'^{et} - \overline{M}^e M^{et}_{\hat{R}}\right) = 2\left(m^e_i\hat{m}^e_j + m^e_j\hat{m}^e_i - m^e_i\hat{m}^e_i - m^e_j\hat{m}^e_j\right) \\ &= 2(\hat{m}^e_i - \hat{m}^e_j)(m^e_j - m^e_i). \end{aligned}$$

Then, $d_\Sigma(\overline{M}, M_{\hat{R}}) - d_\Sigma(\overline{M}, M') = 2\,(\hat{m}^e_i - \hat{m}^e_j)(m^e_j - m^e_i) > 0$, so $d_\Sigma(\overline{M}, M_{\hat{R}}) > d_\Sigma(\overline{M}, M')$ and, thus $d_\Sigma(\overline{M}, M_{\hat{R}})$ is not minimal and $M_{\hat{R}}$ is not the Mahalanobis consensus solution. In that way, a contradiction is reached. Consequently, the hypothesis $\overline{M^i}^{\,e} < \overline{M^j}^{\,e} \Longrightarrow \hat{m}^e_i < \hat{m}^e_j$ is verified. $\square$

Additionally to the previous results it should be interesting to study if Mahalanobis social consensus solutions satisfy other properties usually claimed in Social Choice Theory. In the next subsection we explore some of them.

4.2. Properties of the Mahalanobis social consensus solution

We now proceed to define and prove the main properties of the Mahalanobis social consensus solution. These properties ensure the suitability and avoid weird behaviors of the new approach. Moreover, these good theoretical properties make it easier to accept the social solution obtained for the group.

- *Anonymity*. Any member's ranking is considered equal in importance to the ranking preferred by any other member. More precisely, given a profile $\mathcal{P} \in W(X)^N$, a Mahalanobis social consensus solution does not change for each permutation σ of the agents. The problem to solve then is

$$\begin{array}{ll} \min\limits_{M_R} & \sum\limits_{i=1}^{N} d_\Sigma(M_{\sigma(i)}, M_R) \\ \text{s.t.} & M_R \in F \end{array}$$

Proof. It is straightforward that a ranking $\hat{R}$ whose codified complete preorder is $M_{\hat{R}}$ given by Definition 9 is also a solution to the above problem since $\mathcal{C}_{\Sigma,\mathcal{P}}(M_R) = \mathcal{C}_{\Sigma,\mathcal{P}^\sigma}(M_R^\sigma)$ for all M_R and σ. □

- *Unanimity*. If all agents show the same preferences on all alternatives, then a Mahalanobis social consensus solution coincides with such common complete preorder.

Proof. This is easily seen since the column means of the codified profile is equal to that common codified complete preorder. That means, it belongs to the feasible set F and Theorem 1 produces the result. □

- *Neutrality*. Generally speaking, this property means all alternatives are treated strictly equal. More precisely, any relabelling of the alternatives or issues induces the corresponding permutation of a Mahalanobis social consensus solution. Due to the fact that our proposal presents a collection of functions MCDFs, relying on Σ matrix, it should be reasonable that the verification of this property depends on Σ.

Consider a Mahalanobis social consensus solution $M_{\hat{R}}$ obtaining by Definition 9. Given π a permutation of the set of alternatives. The MCDF after permuting the alternatives can be written as

$$\mathcal{C}_{\Sigma,{}^\pi\mathcal{P}}(M_R) = \sum_{i=1}^{N} d_\Sigma(M_{{}^\pi R_i}, M_R) = \sum_{i=1}^{N} d_\Sigma({}^\pi M_{R_i}, M_R)$$

Due to the previous reasoning, the property to prove is:

$$\begin{array}{llclcl} \min\limits_{M_R} & \mathcal{C}_{\Sigma,{}^\pi\mathcal{P}}(M_R) & = & \min\limits_{M_R} & \sum\limits_{i=1}^{N} d_\Sigma(M_{{}^\pi R_i}, M_R) & = \mathcal{C}_{\Sigma,{}^\pi\mathcal{P}}(M_{{}^\pi\hat{R}}) \\ \text{s.t.} & M_R \in F & & \text{s.t.} & M_R \in F & \end{array}$$

where $M_{{}^\pi\hat{R}} = {}^\pi M_{\hat{R}} = (\hat{m}_{\pi(1)}, \ldots, \hat{m}_{\pi(k)})$ is a consensus solution for this problem if and only if $\Sigma = diag\{\lambda, \ldots, \lambda\}$ for some $\lambda > 0$.

Proof. We consider the following two problems to solve:

- $$\begin{array}{llcll} \min\limits_{M_R} & \mathcal{C}_{\Sigma,\mathcal{P}}(M_R) & = & \min\limits_{M_R} & \sum\limits_{i=1}^{N} d_\Sigma(M_{R_i}, M_R) \\ \text{s.t.} & M_R \in F & & \text{s.t.} & M_R \in F \end{array}$$

- $$\begin{array}{llcll} \min\limits_{M_R} & \mathcal{C}_{\Sigma,{}^\pi\mathcal{P}}(M_R) & = & \min\limits_{M_R} & \sum\limits_{i=1}^{N} d_\Sigma(M_{{}^\pi R_i}, M_R) \\ \text{s.t.} & M_R \in F & & \text{s.t.} & M_R \in F \end{array}$$

By Theorem 1 the resolution of these problems can be reduced to minimize $d_\Sigma(\overline{M}, M_R)$ and $d_\Sigma({}^\pi\overline{M}, M_R)$, respectively.

$$d_\Sigma(\overline{M}, M_R) = (\overline{M} - M_R)\Sigma^{-1}(\overline{M} - M_R)^t.$$

In order to simplify the notation and due to the equivalence among the set of complete preorders and the set of their permutations, we can write ${}^\pi M_R$ for some M_R. Thus,

$$\begin{aligned} d_\Sigma({}^\pi\overline{M},\ {}^\pi M_R) &= ({}^\pi\overline{M} - {}^\pi M_R)\ \Sigma^{-1}\ ({}^\pi M_i - {}^\pi M_R)^t = \\ &= (\overline{M}\Pi_\pi - M_R\Pi_\pi)\ \Sigma^{-1}\ (\overline{M}\Pi_\pi - M_R\Pi_\pi)^t = \\ &= (\overline{M} - M_R)\ \Pi_\pi\ \Sigma^{-1}\ \Pi_\pi^t\ (\overline{M} - M_R)^t. \end{aligned}$$

Let us first prove *sufficiency*. If $\Sigma = diag\{\lambda, \dots, \lambda\}$ for a value $\lambda > 0$, then $\Pi_\pi\ \Sigma^{-1}\ \Pi_\pi^t = \Sigma^{-1}$ and consequently,

$$d_\Sigma(\overline{M},\ M_R) = d_\Sigma({}^\pi\overline{M},\ {}^\pi M_R)$$

that is, the distance to minimize coincides for both problems and the result is straightforward.[7]

Let us now prove *necessity*. Assuming that given a codified profile $M \in \mathbb{M}_{N\times k}$ and for each π, $d_\Sigma(\overline{M},\ M_R) = d_\Sigma({}^\pi\overline{M},\ {}^\pi M_R)$, therefore $\Pi_\pi\ \Sigma^{-1}\ \Pi_\pi^t = \Sigma^{-1}$, we must prove that $\Sigma = diag\{\lambda, \dots, \lambda\}$.

The proof of this point is similar to the demonstration included in González-Arteaga, Alcantud and de Andrés Calle [26, Appendix A, Proof of Property 1]. □

In the same way that happens to the Mahalanobis dissensus measure, the Mahalanobis social consensus solution preserves the advantages of the Mahalanobis distance, concretely, it takes into account the correlation among alternatives. Therefore, on the question of *neutrality* for the consensus solution, the reasons and the clarifications aforementioned in Section 3 (Property 2) are maintained. We now present a weak version of the neutrality property.

- *Weak neutrality*. Any relabelling of the alternatives or issues induces the corresponding permutation of the Mahalanobis social consensus solution associated to the appropriate permutation on Σ. Formally:

Let $\Sigma \in \mathbb{M}_{k\times k}$ be a positive definite matrix. For each profile $\mathcal{P} \in W(X)^N$ whose codified profile is $M \in \mathbb{M}_{N\times k}$ and for each permutation π of the alternatives, the problem to solve is

$$\begin{array}{llcll} \min\limits_{M_R} & \mathcal{C}_{\Sigma^\pi,\ {}^\pi\mathcal{P}}(M_R) & = & \min\limits_{M_R} & \sum\limits_{i=1}^{N} d_{\Sigma^\pi}({}^\pi M_{R_i}, M_R), \\ \text{s.t.} & M_R \in F & & \text{s.t.} & M_R \in F \end{array}$$

where $\Sigma^\pi = \Pi_\pi^t\ \Sigma\ \Pi_\pi$.

Therefore, the minimization of the MCDF, $\mathcal{C}_{\Sigma^\pi,\ {}^\pi\mathcal{P}}(M_R)$ produces a Mahalanobis social consensus solution ${}^\pi M_{\hat{R}} = M_{\pi\hat{R}}$ obtained from $M_{\hat{R}}$.

Proof. Let us consider the set of codified complete preorders in the form ${}^\pi M_R = M_{\pi R}$ like possible solutions. Since Definition 9, it is sufficient to prove

$$d_{\Sigma^\pi}({}^\pi M_{R_i}, {}^\pi M_R) = d_\Sigma(M_{R_i}, M_R) \quad \text{for} \quad i = 1, \dots, N.$$

Using the fact that the permutation matrix Π_π is orthogonal

$$\begin{aligned} d_{\Sigma^\pi}({}^\pi M_{R_i}, {}^\pi M_R) &= ({}^\pi M_{R_i} - {}^\pi M_R)\ (\Sigma^\pi)^{-1}\ ({}^\pi M_{R_i} - {}^\pi M_R)^t = \\ &= (M_{R_i}\Pi_\pi - M_R\Pi_\pi)\ (\Pi_\pi^t\Sigma\Pi_\pi)^{-1}\ (M_{R_i}\Pi_\pi - M_R\Pi_\pi)^t = \\ &= (M_{R_i} - M_R)\ \Pi_\pi\ \Pi_\pi^t\Sigma^{-1}\Pi_\pi\ \Pi_\pi^t\ (M_{R_i} - M_R)^t = \\ &= (M_{R_i} - M_R)\Sigma^{-1}(M_i - M_R)^t = \\ &= d_\Sigma(M_{R_i}, M_R). \end{aligned}$$

Then, the proof is straightforward. □

- *Consistency*. Given a set of agents divided in two disjoint subcommittees. Suppose that Mahalanobis social consensus solutions obtained for each subcommittee coincide. Then, Mahalanobis social consensus solutions derived from the original set of agents are the same that the obtained for the subcommittees.

Proof. Let $\mathbf{N} = \mathbf{N}^{(1)} \cup \mathbf{N}^{(2)}$ be a partition of the set $\mathbf{N}$ of agents in two disjoint subcommittees. The Mahalanobis social consensus solutions for each subcommittee are $M_{\hat{R}^{(1)}}$ and $M_{\hat{R}^{(2)}}$, respectively. According to the hypothesis: $M_{\hat{R}^{(1)}} = M_{\hat{R}^{(2)}}$. The MCDF for the set of agents $\mathbf{N}$ can be written as

$$\begin{aligned} \mathcal{C}_{\Sigma,\mathcal{P}}(M_R) &= \sum_{i=1}^{N} d_\Sigma(M_{R_i}, M_R) = \\ &= \sum_{i\in\mathbf{N}^{(1)}} d_\Sigma(M_{R_i}, M_R) + \sum_{i\in\mathbf{N}^{(2)}} d_\Sigma(M_{R_i}, M_R) = \\ &= \mathcal{C}_{\Sigma,\mathcal{P}^{(1)}}(M_{R^{(1)}}) + \mathcal{C}_{\Sigma,\mathcal{P}^{(2)}}(M_{R^{(2)}}). \end{aligned}$$

$M_{\hat{R}^{(1)}} = M_{\hat{R}^{(2)}}$ minimizes the first and the second summand. Therefore, the minimum of the first term in the above equality is reached in $M_{\hat{R}^{(1)}} = M_{\hat{R}^{(2)}}$ because both summands are positive. □

[7] Recall Π_π is the permutation matrix corresponding to π.

- *Compatibility*. Let $M^* = a \cdot \mathbf{1}_N + b \cdot M$ the matrix arising from an affine transformation of $M \in \mathbb{M}_{N\times k}$ which represents the codified profile associated with $\mathcal{P} \in W(X)^N$. The Mahalanobis social consensus solution obtained for M^* is $M^*_{\hat{R}} = a \cdot \mathbf{1}_N + b \cdot M_{\hat{R}}$ being $M_{\hat{R}}$ the corresponding Mahalanobis social consensus solution for M.

Proof. The problem to solve is

$$\begin{array}{ll} \min\limits_{M^*_R} & \sum\limits_{i=1}^{N} d_\Sigma(M^*_{R_i}, M^*_R) \\ \text{s.t.} & M^*_R \in F^* \end{array}$$

where F^* is the set of all possible vectors that represent codified complete preorders using the affine transformation.

Replacing $M^*_{R_i} = a \cdot \mathbf{1}_N + b \cdot M_{R_i}$ and $M^*_R = a \cdot \mathbf{1}_N + b \cdot M_R$, we obtain:

$$\begin{aligned} \sum_{i=1}^{N} d_\Sigma(M^*_i, M^*_R) &= \sum_{i=1}^{N} (M^*_i - M^*_R)\Sigma^{-1}(M^*_i - M^*_R)^t = \\ &= b^2 \sum_{i=1}^{N} (M_i - M_R)\Sigma^{-1}(M_i - M_R)^t = \\ &= b^2 \sum_{i=1}^{N} d_\Sigma(M_i, M_R). \end{aligned}$$

Therefore, $M^*_{\hat{R}} \in F^*$ if and only if there exists $M_{\hat{R}} \in F$ such that $M^*_{\hat{R}} = a \cdot \mathbf{1}_N + b \cdot M_{\hat{R}}$.
□

- *Reciprocity*. Reciprocity means that if all individual rankings in a profile are reversed, then the consensus solution is obtained by reversing the original solution. This is true for Mahalanobis social consensus solution under a basic condition on the codification procedure. Formally:

Let $\mathcal{P} \in W(X)^N$ be a profile and $\mathcal{P}^{-1} \in W(X)^N$ be its reverse, whose associated codified profiles are $M_{\mathcal{P}}$, $M_{\mathcal{P}^{-1}} \in \mathbb{M}_{N\times k}$, respectively. Fixed a positive definite matrix $\Sigma \in \mathbb{M}_{k\times k}$, the problems to solve are the following:

$$\begin{array}{llllll} (P1) & \min\limits_{M_R} & \mathcal{C}_{\Sigma,\mathcal{P}^{-1}}(M_R) & = & \min\limits_{M_R} & \sum\limits_{i=1}^{N} d_\Sigma((M_{\mathcal{P}^{-1}})_i, M_R) \\ & \text{s.t.} & M_R \in F & & \text{s.t.} & M_R \in F \end{array}$$

$$\begin{array}{llllll} (P2) & \min\limits_{M_R} & \mathcal{C}_{\Sigma,\mathcal{P}}(M_R) & = & \min\limits_{M_R} & \sum\limits_{i=1}^{N} d_\Sigma((M_{\mathcal{P}})_i, M_R) \\ & \text{s.t.} & M_R \in F & & \text{s.t.} & M_R \in F \end{array}$$

Then, the solution of the problem $(P1)$ has to be the reverse of the solution of the problem $(P2)$.

Reciprocity is fulfilled if the codification procedure used on $R \in W(X)$ verifies $M_{R^{-1}} = a \cdot \mathbf{1}_N + b \cdot M_R = (a + bm_1, \ldots, a + bm_k)$ for $a, b \in \mathbb{R}$. Therefore, $(M_{\mathcal{P}^{-1}})_i = a \cdot \mathbf{1}_N + b \cdot (M_{\mathcal{P}})_i$.

Notice that our codification proposal, the canonical codification, satisfies the aforementioned condition since

$$\begin{aligned} M_{R^{-1}} = a \cdot \mathbf{1}_N + b \cdot M_R &= (a + bm_1, \ldots, a + bm_k) = \\ &= (n + 1 - m_1, \ldots, n + 1 - m_k) = \\ &= (n + 1) \cdot \mathbf{1}_N - M_R. \end{aligned}$$

Proof. In order to solve problems $(P1)$ and $(P2)$, Theorem 1 is used. Thus, it is enough to minimize $d_\Sigma(\overline{M}_{\mathcal{P}^{-1}}, M_R)$ and $d_\Sigma(\overline{M}_{\mathcal{P}}, M_R)$ subject to $M_R \in F$, respectively.

Considering that $\overline{M}_{\mathcal{P}^{-1}} = a \cdot \mathbf{1}_N + b \cdot \overline{M}_{\mathcal{P}}$ and being $M_{\hat{R}}$ a solution of problem $(P1)$, it is easy to check that $a \cdot \mathbf{1}_N + b \cdot M_{\hat{R}}$ is a solution of problem $(P2)$ since

$$\begin{aligned} d_\Sigma(\overline{M}_{\mathcal{P}^{-1}},\ a \cdot \mathbf{1}_N + b \cdot M_{\hat{R}}) &= d_\Sigma(a \cdot \mathbf{1}_N + b \cdot \overline{M}_{\mathcal{P}},\ a \cdot \mathbf{1}_N + b \cdot M_{\hat{R}}) = \\ &= b^2\, d_\Sigma(\overline{M}_{\mathcal{P}},\ M_{\hat{R}}). \end{aligned}$$

□

- *Non-dictatorship*. The Mahalanobis social consensus solution is never dictatorial. Recall that in a dictatorship, social choices are based on the preferences of only one expert or agent. Formally, an agent $j \in \mathbf{N}$ exists such that for all alternatives $x_r, x_s \in X$ and for all profiles $\mathcal{P} \in W(X)^N$

$$x_r \succcurlyeq_{R_j} x_s \Longrightarrow x_r \succcurlyeq_{\hat{R}} x_s.$$

Proof. Immediate from Theorem 1. □

Table 1
Formulation of the optimization problem.

Min $d_\Sigma(M_{R_{e_1}}, M_R)+$	$d_\Sigma(M_{R_{e_2}}, M_R)+$	$d_\Sigma(M_{R_{e_3}}, M_R)$	
Subject to M_R belongs to:			
	(3, 3, 3)	(3, 3, 1)	(1, 3, 2)
	(2, 2, 3)	(3, 1, 3)	(2, 3, 1)
	(3, 2, 2)	(1, 3, 3)	(2, 1, 3)
	(2, 3, 2)	(1, 2, 3)	(3, 1, 2)
	(3, 2, 1)		

4.3. Graphical interpretation and discussion: An illustrative example

To clarify and discuss the new approach presented in Section 4.1, we develop an explanatory example.

By way of illustration, we suppose the following group decision making problem: a set of students have to choose the destination of their graduation trip. Students should order destinations offered by a travel agency.

We consider a set of three students of the Faculty of Sciences (experts) $\mathbf{N} = \{e_1, e_2, e_3\}$ and a set of three destinations (alternatives) $X = \{x_1 = \text{Paris}, x_2 = \text{Berlin}, x_3 = \text{Istanbul}\}$. Each student participates in a survey about her/his preferences on the trip destinations where she/he is asked to order them.

Their responses are summarized as follows:

$$\begin{aligned} &\text{Student}\, e_1: && x_3 \succ_{R_{e_1}} x_1 \sim_{R_{e_1}} x_2 \\ &\text{Student}\, e_2: && x_3 \succ_{R_{e_2}} x_1 \sim_{R_{e_2}} x_2 \\ &\text{Student}\, e_3: && x_1 \sim_{R_{e_3}} x_2 \succ_{R_{e_3}} x_3 \end{aligned}$$

The previous complete preorders generate a particular profile $\mathcal{P}$. Applying Definition 1 to each complete preorder the codified profile for $\mathcal{P}$ is

$$M_{\mathcal{P}} = \begin{pmatrix} 2 & 1 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{pmatrix},$$

or also $M_{\mathcal{P}} = (M_{R_{e_1}}, M_{R_{e_2}}, M_{R_{e_3}})$.

In order to obtain a group solution that captures the minimum possible dissensus among students' preferences (i.e., the maximum possible consensus), we must solve the following general optimization problem:

$$\begin{aligned} &\min_{M_R} && \mathcal{C}_{\Sigma,\mathcal{P}}(M_R) \\ &\text{s.t.} && M_R \in F \end{aligned}$$

where F is the feasible set computed by Proposition 1 and

$$\mathcal{C}_{\Sigma,\mathcal{P}}(M_R) = \sum_{i=1}^{N} d_\Sigma(M_{R_i}, M_R) = \sum_{i=1}^{N} (M_{R_i} - M_R)\Sigma^{-1}(M_{R_i} - M_R)^t.$$

This problem adapted to our specific case takes the form gathered in Table 1. Moreover, the corresponding feasible set is displayed in Fig. 1.

This optimization problem can be simplified by means of Theorem 1 hence it boils down to:

$$\text{Min } d_\Sigma(\overline{M}, M_R) = \text{Min } (\overline{M} - M_R)\Sigma^{-1}(\overline{M} - M_R)^t$$

Subject to M_R belongs to:

$$\begin{matrix} (3,3,3) & (3,3,1) & (1,3,2) \\ (2,2,3) & (3,1,3) & (2,3,1) \\ (3,2,2) & (1,3,3) & (2,1,3) \\ (2,3,2) & (1,2,3) & (3,1,2) \\ (3,2,1) & & \end{matrix}$$

where $\overline{M} = (2.34, 1.67, 2.34)$ is the vector of column means of $M_{\mathcal{P}}$.

The solution of this problem hinges on the Σ matrix. We now provide Mahalanobis social consensus solutions under the assumption of three different Σ matrices to enrich the case of study and promote the discussion:

1. Case 1. In the simplest case, the Σ matrix is the identity matrix, $\Sigma = I = diag(1, 1, 1)$. In our example this means that all destinations are equally treated. By solving the corresponding optimization problem the following solutions are obtained (see Table 2):
   - $M_{R_2} = (2, 2, 3)$, that is, $x_3 \succ x_1 \sim x_2$.
   - $M_{R_3} = (3, 2, 2)$, that is, $x_1 \succ x_2 \sim x_3$.

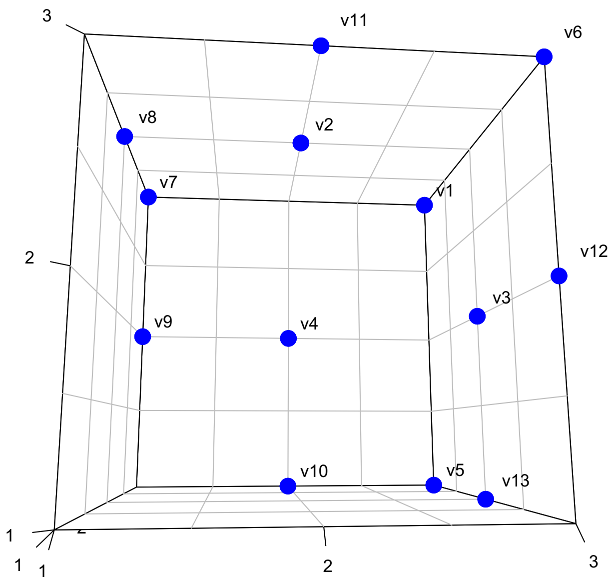

Fig. 1. Graphical display of complete preorders included in Tables 1 and 2. M_{R_i} is labeled by v_i.

Table 2
Mahalanobis distances $d_I(M_{R_i}, \overline{M})$, $d_D(M_{R_i}, \overline{M})$ and $d_{\Sigma_1}(M_{R_i}, \overline{M})$ from codified complete preorders M_{R_i} (elements in F) to the mean point $\overline{M}$.

Complete preorders	Codified complete preorders	Graphic labels	d_I	d_D	d_{Σ_1}
R_1: $x_1 \sim x_2 \sim x_3$	$M_{R_1} = (3, 3, 3)$	v1	2.67	4.07	4.43
R_2: $x_3 \succ x_1 \sim x_2$	$M_{R_2} = (2, 2, 3)$	v2	**0.67**	**0.88**	1.99
R_3: $x_1 \succ x_2 \sim x_3$	$M_{R_3} = (3, 2, 2)$	v3	**0.67**	1.71	2.85
R_4: $x_2 \succ x_1 \sim x_3$	$M_{R_4} = (2, 3, 2)$	v4	2.00	2.69	12.51
R_5: $x_1 \sim x_2 \succ x_1$	$M_{R_5} = (3, 3, 1)$	v5	4.00	5.19	3.07
R_6: $x_1 \sim x_3 \succ x_2$	$M_{R_6} = (3, 1, 3)$	v6	1.33	2.41	14.45
R_7: $x_2 \sim x_3 \succ x_1$	$M_{R_7} = (1, 3, 3)$	v7	4.00	8.52	39.08
R_8: $x_3 \succ x_2 \succ x_1$	$M_{R_8} = (1, 2, 3)$	v8	2.33	6.44	22.04
R_9: $x_2 \succ x_3 \succ x_1$	$M_{R_9} = (1, 3, 2)$	v9	3.67	8.24	47.00
R_{10}: $x_2 \succ x_1 \succ x_3$	$M_{R_{10}} = (2, 3, 1)$	v10	3.67	4.07	18.27
R_{11}: $x_3 \succ x_1 \succ x_2$	$M_{R_{11}} = (2, 1, 3)$	v11	1.00	1.30	**0.77**
R_{12}: $x_1 \succ x_3 \succ x_2$	$M_{R_{12}} = (3, 1, 2)$	v12	1.00	2.13	9.77
R_{13}: $x_1 \succ x_2 \succ x_3$	$M_{R_{13}} = (3, 2, 1)$	v13	2.33	3.10	2.32

About graphical interpretation, on the left of Fig. 2 the elements of the feasible set F are displayed like dots using a color scale. Dots have different colors depending on their Mahalanobis distance, d_I, to the mean point $\overline{M}$ (black triangle). Associated distance values are shown in Table 2.
Additionally, Fig. 3 shows the minimum equidistant surface to $\overline{M}$, that in this case is a blue sphere centered at $\overline{M}$. Moreover, Fig. 3 includes two different perspectives in order to improve the view.
Notice that considering the Σ matrix as the identity matrix is equivalent to using the Euclidean distance ($l_p = l_2$) to compute a solution. The Euclidean distance has been extensively used in other approaches like [14,29,30]. Then, Case 1 could be used to compare our approach with other methods and to show its efficiency. Next cases include the importance and the cross-relations of alternatives by means of several Σ matrices.

2. Case 2. Now we account for a case where alternatives are considered differently by means of a diagonal Σ matrix. In our example this means that all destinations are not equally treated. Suppose for instance $\Sigma = D = diag(0.3, 0.8, 1.2)$ where the third alternative has the biggest significance. In Table 2 we can find the social consensus solution for this particular case:
    - $M_{R_2} = (2, 2, 3)$, that is, $x_3 \succ x_1 \sim x_2$.

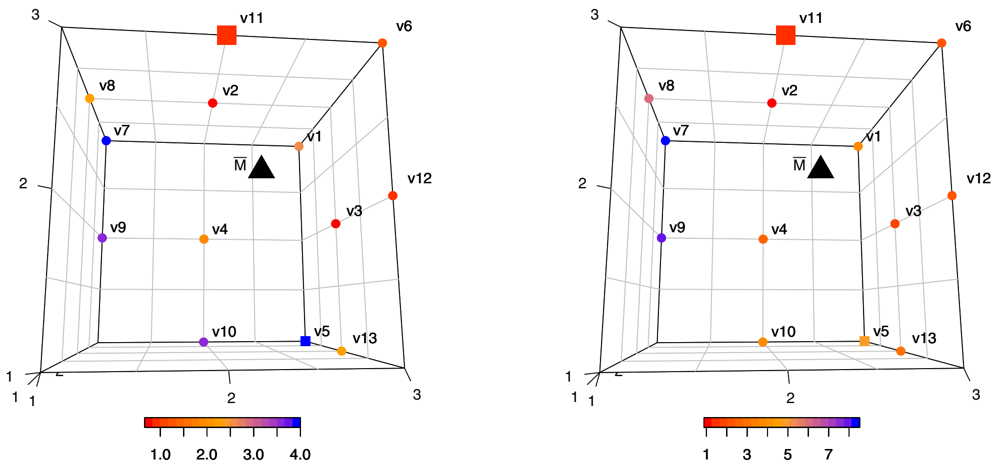

Fig. 2. A display of elements in F with colored dots depending on the distance (on the left d_I and on the right d_D) to the mean point $\overline{M}$ (black triangle). In addition, squares denotes the codified complete preorders (M_{R_5} and $M_{R_{11}}$) included in M.

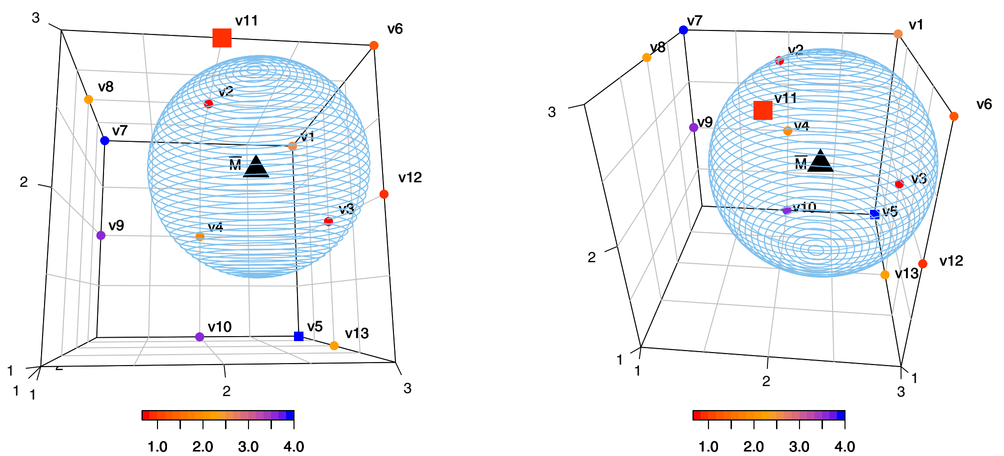

Fig. 3. Graphical interpretation of case 1 in Section 4.3 using d_I.

Analogously to the previous case, on the right of Fig. 2 the elements of the feasible set are shown. The color scale is built for the Mahalanobis distance, d_D, between dots and the mean point $\overline{M}$ (black triangle). Such distance values are also shown in Table 2.
In addition, the aforementioned social solution can be found in Fig. 4 from two perspectives. It shows the minimum equidistant surface to $\overline{M}$, that is a blue ellipsoid centered at $\overline{M}$. On the right, after rotating the ellipsoid, our figure makes clear that dots with labels $v1$ and $v3$ are outside of the ellipsoid, farther away than the dot $v2$.

3. Case 3. Finally, we examine the case of a non-diagonal matrix, which allows to incorporate the interdependence of the alternatives because the role of Σ in the Mahalanobis distance. Let us assume the following particular matrix

$$\Sigma = \Sigma_1 = \begin{pmatrix} 0.30 & 0.37 & -0.36 \\ 0.37 & 0.80 & -0.29 \\ -0.36 & -0.29 & 1.20 \end{pmatrix}.$$

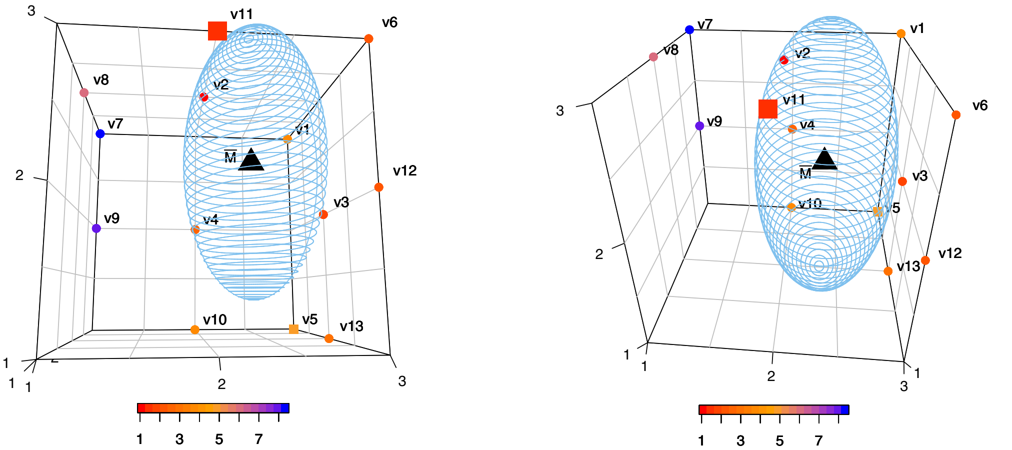

Fig. 4. Graphical interpretation of case 2 in Section 4.3 using d_D.

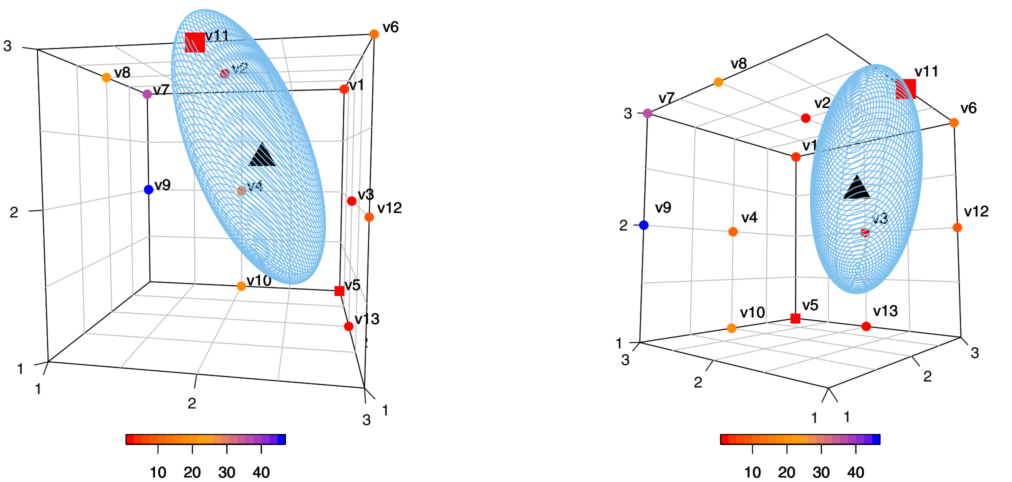

Fig. 5. Graphical interpretation of case 3 in Section 4.3 using d_{Σ_1}.

Since Σ matrix can be considered as a variance-covarince matrix in the Mahalanobis distance, it is easy to compute the corresponding correlation matrix *Corr*, that is, the correlation among the alternatives.[8]

$$Corr = Corr_1 = \begin{pmatrix} 1.00 & 0.75 & -0.60 \\ 0.75 & 1.00 & -0.30 \\ -0.60 & -0.30 & 1.00 \end{pmatrix}.$$

In our example this matrix implies not only that all destinations are not equally treated but they are also correlated. Alternatives x_1 and x_2 are highly positively correlated whereas alternatives x_1 and x_3, and x_2 and x_3, are negatively correlated. Therefore, it is assumed that Paris and Berlin are "positively" correlated destinations. However, the preferences relative to Paris versus Istanbul are more intensively opposite than Berlin versus Istanbul.

[8] The element *ij* of the corelation matrix *Corr* is $\frac{\Sigma_{ij}}{\sqrt{\Sigma_{ii}}\sqrt{\Sigma_{jj}}}$, where Σ_{ij} is the element *ij* of variance-covariance matrix Σ.

In order to solve the optimization problem for this case we observe the corresponding distance values, d_{Σ_1}, contained in Table 2. In this case, we conclude that the solution is:

- $M_{R_{11}} = (2, 1, 3)$, that is, $x_3 \succ x_1 \succ x_2$

Regarding graphical interpretation, Fig. 5 shows the minimum equidistant surface to $\overline{M}$. In this case, it is a blue oriented ellipsoid centered at $\overline{M}$. After a rotation, the graph on the right reveals that dots $v2$ and $v4$ are outside of the ellipsoid, farther than the dot $v11$.

5. Concluding remarks

This study is aimed at proposing a new approach to obtain a group consensus solution under the assumption of ordinal information. A new procedure based on an optimization model has been developed, obtaining a *social consensus solution based on the Mahalanobis distance*. To accomplish such target two new contributions have been developed in addition of the main result: the characterization of a codification procedure for ordinal information, namely, the *canonical codification* and the definition and analysis of a new dissensus measure, namely, the *Mahalanobis dissensus measure*. The use of the Mahalanobis distance as a base of our approaches brings advantage by considering possible cross relations among alternatives. Moreover, the operational character and intuitive interpretation of our approaches have been illustrated by an explanatory example.

The findings of this study have a number of important implications for future practice. Many problems from a variety of fields can be managed by our methods such as the performance of consumers' preferences, Clinical Decision Making problems, allocation of projects, Human Resources Department problems, etc.

Acknowledgments

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Chapter 5

Concluding remarks and future research

In this chapter we present the main results obtained and some concluding remarks drawn along this doctoral thesis as well as some possible future works.

Concluding remarks and obtained results

Group decision making problems have been attaining importance in many real-life application and scientific areas, such as *Economics, Social Sciences, Engineering, Medicine*, so on. In this kind of problems it can be interesting to know the level of agreement among members of a group. In this sense, different authors have proposed in the literature a great variety of approaches to measure and to achieve cohesiveness although they present some drawbacks. Such drawbacks inspired and motivated the objectives pursued in this doctoral thesis. So, the interest of this research was focused on overcoming them by means of a number of proposals which are summed up below.

For one thing, the research studies made during the development of this doctoral thesis take into account possible cross-related alternatives in measurement of cohesiveness. To this end we introduce the Mahalanobis distance that is regularly used in Statistical Analysis and it had hardly been employed in Decision Making.

For another thing, a new via to measure consensus is opened in this doctoral thesis. A novel procedure based on the Pearson correlation coefficient is proposed. This new measure of cohesiveness among experts is neither a distance function nor a similarity function.

From another point of view, in this dissertation, we have provided theoretical definitions of different cohesiveness measures assuming several frameworks. Concretely, *the class of Mahalanobis dissensus measures* is put forth to cardinal evaluations, the *correlation consensus degree* is defined for reciprocal preference relations and *the Mahalanobis dissensus measure* is adopted for ordinal evaluations.

We also propose a new approach based on an optimization problem to achieve a group consensus solution under the assumption of ordinal information (complete preorders), the *Mahalanobis social consensus solution.* The use of the Mahalanobis distance as a base of our approach allows to consider possible cross relations among alternatives. Furthermore, a new outcome is added to that main result to accomplish such target: the characterization of an important codification procedure for ordinal information, namely, the *canonical codification.*

In addition, each one of the proposed measures or procedures for measuring or achieving cohesiveness is accompanied by a detailed study of desirable formal properties in order to show its consistency and applicability.

Finally, it is worth emphasizing that the three supplied contributions contain real and practical applications of the proposed innovative measures and procedures.

As can be seen, all the objectives pursued at the beginning of this research have been successfully achieved by means of the contributions presented in this thesis memory.

Future research

The most straightforward lines of enquiry that could benefit from further study are pointed out below.

One of the challenge of our proposals is that the Mahalanobis dissensus measure requires to have a reference matrix suitable to the problem at hand. Therefore, it is essential to develop procedures to determine such a reference matrix in order to improve our proposals. In this sense, it should be interesting to design methodologies that could use information from the original profile. Therefore, we would be in position to assure that the matrix provided is appropriate to each specific decision making problem. This could be a kind of "endogenous" measure.

From another point of view, it could be appealing to design flexible consensus reaching processes. In our case, we would propose a consensus reaching process based on the correlation consensus degree. This process would allow to produce a consensus solution by an iterative feedback mechanism accommodated to this specific consensus measurement.

Finally, in order to improve and to extend our proposal of Mahalanobis social consensus solutions, we would research on the implementation of the optimization problem used to obtain such solutions, particularly when the number of alternatives is not small.

Appendix A

Annex: Publication quality indicators

This annex encloses some quality indicators of the three scientific journals where the original publications included in this doctoral thesis have been published.

The following tables were taken from Journal Citation Reports (JCR) provided by Thomson Reuters (https://jcr.incites.thomsonreuters.com/)

Publication I: A cardinal dissensus measure based on the Mahalanobis distance. *European Journal of Operational Research*, 251, Issue 2, 575-585. 2016. **WoS-JCR 2015: 2.679, Q1 (9/82)** in Operations Research & Management Science.

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Titles
ISO: Eur. J. Oper. Res.
JCR Abbrev: EUR J OPER RES

Categories
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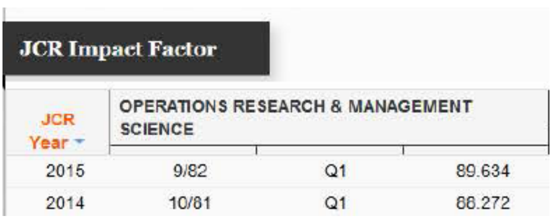
ENGLISH

24 Issues/Year;

Key Indicators

Year	Total Cites	Journal Impact Factor	Impact Factor Without Journal Self Cites	5 Year Impact Factor	Immediacy Index	Citable Items	Cited Half-Life	Citing Half-Life	Eigenfacto Score	Article Influence Score	% Articles in Citable Items	Normalizec Eigenfacto	Average JIF Percentile
2015	31,744	2.679	2.130	3.100	0.575	670	8.8	9.3	0.04714	1.079	96.42	5.37326	89.634
2014	29,872	2.358	1.816	2.911	0.514	658	8.5	9.5	0.04690	0.990	96.81	5.25280	88.272



JCR Impact Factor

JCR Year	OPERATIONS RESEARCH & MANAGEMENT SCIENCE		
2015	9/82	Q1	89.634
2014	10/81	Q1	88.272

Publication II: A new measure of consensus with reciprocal preference relations: The correlation consensus degree. *Knowledge-Based Systems*, 107, 104–116. 2016.
WoS-JCR 2015: 3.325, Q1 (17/130) in Computer Science, Artificial Intelligence.

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Categories
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Languages
ENGLISH

8 Issues/Year;

Key Indicators

Year	Total Cites Graph	Journal Impact Factor Graph	Impact Factor Without Journal Self Cites Graph	5 Year Impact Factor Graph	Immediacy Index Graph	Citable Items Graph	Cited Half-Life Graph	Citing Half-Life Graph	Eigenfacto Score Graph	Article Influence Score Graph	% Articles in Citable Items Graph	Normalizec Eigenfacto Graph	Average JIF Percentile Graph
2015	4,811	3.325	2.664	3.433	0.760	333	3.2	7.4	0.01326	0.819	99.10	1.51143	87.308
2014	3,424	2.947	2.626	3.011	0.410	295	3.0	7.9	0.00923	0.681	99.66	1.03395	87.398

JCR Impact Factor

JCR Year	COMPUTER SCIENCE, ARTIFICIAL INTELLIGENCE		
	Rank	Quartile	JIF Percentile
2015	17/130	Q1	87.308
2014	16/123	Q1	87.398

Publication III: A new consensus ranking approach for correlated ordinal information based on Mahalanobis distance. *Information Sciences*, 372, 546 - 564. 2016.
WoS-JCR 2015: 3,364, Q1 (8/143) in Computer Science, Information Systems.

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36 Issues/Year;

Key Indicators

Year	Total Cites	Journal Impact Factor	Impact Factor Without Journal Self Cites	5 Year Impact Factor	Immediacy Index	Citable Items	Cited Half-Life	Citing Half-Life	Eigenfacto Score	Article Influence Score	% Articles in Citable Items	Normalizec Eigenfacto	Average JIF Percentile
	Graph	Graph	Graph	Graph	Graph	Graph	Graph	Graph	Graph	Graph	Graph	Graph	Graph
2015	16,792	3.364	2.637	3.683	0.855	615	4.7	7.3	0.03697	0.943	99.67	4.21341	94.792
2014	15,616	4.038	2.400	4.221	0.722	859	4.8	7.7	0.02862	0.873	99.65	3.20523	96.043

JCR Impact Factor

JCR Year	COMPUTER SCIENCE, INFORMATION SYSTEMS		
	Rank	Quartile	JIF Percentile
2015	8/144	Q1	94.792
2014	6/139	Q1	96.043

Appendix B

Annex: Spanish summary

This appendix includes an Spanish summary of this dissertation in order to fulfil the requirements in the academic regulations of the University of Salamanca for this specific format of doctoral thesis.

La presente tesis doctoral está elaborada en formato de compendio de publicaciones, según la normativa aprobada por la Comisión de Doctorado y Postgrado de la Universidad de Salamanca el 15 de febrero de 2013. A continuación se especifican las tres publicaciones originales aportadas en esta memoria.

ARTÍCULO I:

Título: *A cardinal dissensus measure based on the Mahalanobis distance.*

Autores: T. González-Arteaga[1], J.C.R. Alcantud[2], R. de Andrés Calle[2].

Revista: European Journal of Operational Research, 251, 575-585, 2016.

DOI: 10.1016/j.ejor.2015.11.019.

WoS-JCR 2015: 2.679, Q1 (9/82) in Operations Research & Management Science.

Afiliaciones:
[1] Dpto. Estadística e Investigación Operativa, Universidad de Valladolid.
[2] Dpto. Economía e Historia Económica, Universidad de Salamanca.

ARTÍCULO II:

Título: *A new measure of consensus with reciprocal preference relations: The correlation consensus degree.*

Autores: T. González-Arteaga[1], R. de Andrés Calle[2], and F. Chiclana[3].

Revista: Knowledge-Based Systems, 107, 104–116, 2016.

DOI: 10.1016/j.knosys.2016.06.002

WoS-JCR 2015: 3.325, Q1 (17/130) in Computer Science, Artificial Intelligence.

Afiliaciones:
[1] Dpto. Estadística e Investigación Operativa, Universidad de Valladolid.

[2] Dpto. Economía e Historia Económica, Universidad de Salamanca.
[3] Centre for Computational Intelligence, Faculty of Technology, De Montfort University, Leicester, Reino Unido.

ARTÍCULO III:

Título: *A new consensus ranking approach for correlated ordinal information based on Mahalanobis distance*

Autores: T. González-Arteaga[1], J.C.R. Alcantud[2], R. de Andrés Calle[2].

Revista: Information Sciences, 372, 546-564, 2016.

DOI: 10.1016/j.ins.2016.08.071.

WoS-JCR 2015: 3,364, Q1 (8/143) in Computer Science, Information Systems.

Afiliaciones:
[1]Dpto. Estadística e Investigación Operativa, Universidad de Valladolid.
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Resumen

El objetivo general de esta tesis doctoral es el desarrollo de enfoques novedosos para la medición de la cohesión/consenso y metodologías que proporcionan soluciones sociales de consenso en problemas de toma de decisiones en grupo, ampliando el campo de investigación de los enfoques tradicionales. Estas cuestiones han sido abordadas y desarrolladas en las tres contribuciones que describimos a continuación.

En la primera contribución se estudia el problema de la medición del grado de consenso/disenso, en un contexto donde los expertos o agentes expresan sus opiniones sobre las alternativas mediante evaluaciones cardinales. La suposición de evaluaciones cardinales en la medición del consenso apenas ha sido examinada previamente en la literatura relativa a la toma de decisiones. Para este objetivo se propone una nueva clase de medidas de consenso basadas en la distancia de Mahalanobis: la *familia de medidas de disenso de Mahalanobis* para perfiles de valoraciones cardinales. La principal ventaja de esta propuesta es que tiene en cuenta los efectos de diferencias en escala y las posibles interrelaciones entre las alternativas. Además, se analizan y se prueban algunas propiedades relevantes de dichas medidas. Para finalizar esta contribución, se presenta y se discute una aplicación de las medidas en un caso empírico real.

En la segunda contribución se estudia un nuevo enfoque para la medición de consenso basado en el coeficiente de correlación de Pearson, bajo la suposición de que las opiniones de los expertos se modelan mediante relaciones de preferencia recíprocas. El nuevo *grado de consenso basado en la correlación* mide la concordancia entre las intensidades de preferencia de pares de alternativas. Aunque un estudio detallado de las propiedades formales de la nueva medida propuesta muestra que verifica propiedades deseables y relevantes que son comunes, o bien a funciones de distancia o bien de similaridad, también se prueba que es diferente a las medidas de consenso tradicionales. Para reforzar la novedad de este trabajo

se presenta una aplicación de la medida en el ámbito de la toma de decisiones clínicas compartidas.

En la tercera contribución se abordan tres pilares fundamentales para el estudio del consenso: el tratamiento de las opiniones de los expertos cuando se expresan mediante información ordinal, la medición del grado de consenso/disenso entre tales opiniones y la obtención de una solución colectiva que recoja el mínimo disenso en el grupo de expertos.
Primero se caracteriza un nuevo procedimiento para codificar información ordinal. Posteriormente se diseña una nueva medida de disenso entre preferencias individuales basada en la distancia de Mahalanobis, una medida especialmente indicada en el caso de alternativas posiblemente correlacionadas. Por último, se propone un procedimiento para la obtención de una solución social de consenso que incluya la posibilidad de alternativas interrelacionadas.
Además, se examinan las principales características de la medida de disenso y de la solución social propuestas. El carácter operacional y la interpretación intuitiva de estos enfoques se ilustran mediante un ejemplo explicativo.

Introducción

Todos los días se toman multitud de decisiones y, aunque la mayoría de las veces sucede de forma inconsciente, en otras ocasiones los individuos juegan un papel activo. Sin embargo, muchas de estas decisiones no se toman de manera individual sino que se toman en grupo, y los procesos implicados en estas ocasiones son complejos; alcanzar decisiones consistentes en este contexto puede ser complicado.

Como consecuencia de esta dificultad, en los últimos años han surgido diferentes tópicos de investigación en varias áreas científicas tales como *Psicología*, las *Ciencias políticas*, la *Economía*, etc. para tratar de abordar los problemas que aparecen en este escenario.

Entre los muchos aspectos que podrían ser objeto de investigación en este contexto, esta tesis se centra en el análisis del consenso en la toma de decisiones en grupo.

El concepto de consenso ó cohesión se ha estudiado ampliamente en *Psicología* y *Sociología*. Gran parte del interés que suscita este tema se debe a la creencia de que mantener la unidad del grupo es importante para conseguir un rendimiento exitoso del mismo. Debemos notar que, por un lado, las posibilidades de alcanzar altos niveles de cohesión dependen en gran parte de ciertas condiciones sociales (Braaten (1991)) y. por otro lado, hay varios aspectos interpersonales que afectan a la cohesión de los grupos tales como la cercanía de sus miembros, el tamaño del grupo, la dificultad de entrada en el mismo, etc. (Eisenberg (2007)). Todos estos elementos sobre la cohesión, sociales o interpersonales, no son objeto de estudio en esta tesis doctoral.

Desde otro punto de vista, *consenso* es un término multifacético, como puede verse en el trabajo recopilatorio de Martínez-Panero (Martínez-Panero (2011)). En dicho trabajo se presentan varias perspectivas sobre la noción de consenso y cómo este concepto se utiliza en desarrollos formales tales como medidas de consenso en la Teoría de la Elección Social, en la Teoría de la Toma de Decisiones y en aplicaciones de Biomatemáticas.

En esta memoria el término cohesión/consenso se refiere al grado de acuerdo existente entre las opiniones individuales emitidas por un grupo de expertos o agentes (una sociedad) sobre un conjunto de alternativas. Así, el término *consenso* se considera de la misma forma que en el trabajo de Martínez-Panero.

Este trabajo se enmarca entre la Teoría de la Elección Social y la Teoría de la Toma de Decisiones. A este respecto, cabe destacar que este área temática tiene impacto en distintos campos científicos en Economía, Ciencias de Computación y Ciencias de la Salud.

Desde el punto de vista de la Teoría de la Elección Social, la medición de la cohesión de un grupo fue introducida en primer lugar por Bosch (2005) con la noción de *medida de consenso*. Alcalde-Unzu and Vorsatz (2013), García-Lapresta and Pérez-Román (2011) y Alcantud et al. (2013b), entre otros, extendieron y desarrollaron soporte axiomático para varias medidas de consenso en el sentido introducido por Bosch.

Desde la perspectiva de la Terоría de Toma de Decisiones y sus aplicaciones, la medición del consenso y su obtención en un grupo de expertos son áreas de investigación destacadas y muy activas. Se pueden señalar, por ejemplo, los trabajos de Kacprzyk and Fedrizzi (1988) y Herrera-Viedma et al. (2014), entre otros.

Aunque existe una variada y amplia literatura sobre metodologías para medir y alcanzar consenso, todavía existen algunos retos de los cuales apenas se ocupan los enfoques existentes. En este sentido, podemos mencionar como retos la medición de la cohesión/consenso cuando las opiniones de los expertos se expresan mediante evaluaciones cardinales, así como la consideración de alternativas con relaciones cruzadas. Desde otra perspectiva, debe resaltarse que la mayoría de las metodologías mencionadas se basan en funciones de distancia o de similaridad, lo que puede suponer una limitación en algunos casos.

La investigación realizada en esta tesis se dirige a la superación de los retos y limitaciones anteriormente expuestos. A continuación, se enumeran los objetivos concretos que han sido tratados en esta tesis docotoral.

Objetivos

El objetivo general del trabajo presentado en esta tesis doctoral es el desarrollo de nuevos enfoques para la medición de la cohesión/consenso y para la obtención de soluciones sociales de consenso en problemas de toma de decisiones en grupo, ampliando el campo de las propuestas tradicionales.

Este objetivo general puede desglosarse en los siguientes particulares:

- Desarrollar nuevas medidas de cohesión, prestando atención explícitamente a la posibilidad de que las alternativas tengan relaciones cruzadas con la intención de cubrir esta laguna, y complementar la literatura existente sobre la medición y el logro del consenso.

- Construir medidas de consenso/disenso desde un punto de vista teórico asumiendo un marco de trabajo donde los expertos o agentes expresen sus evaluaciones sobre las alternativas mediante los siguientes formatos: evaluaciones ordinales, evaluaciones cardinales y relaciones de preferencia recíproca.

- Estudiar detalladamente las propiedades teóricas de las medidas y metodologías propuestas.

- Definir procedimientos novedosos para obtener soluciones sociales de consenso que contemplen la posibilidad de que existan relaciones cruzadas entre las alternativas. Además, dichos procedimientos deben satisfacer propiedades deseables.

- Mostrar aplicaciones prácticas reales de las nuevas metodologías, con el objeto de acreditar su aplicabilidad.

Resumen de los resultados

En esta sección presentamos un resumen de cada artículo que integra esta tesis doctoral en formato de compendio de publicaciones científicas. Este resumen incluye objetivos, medodología, resultados y conclusiones para cada publicación.

Artículo I:

T. González-Arteaga, J.C.R. Alcantud, R. de Andrés Calle. *A cardinal dissensus measure based on the Mahalanobis distance*. European Journal of Operational Research, 251, Issue 2, 575-585. 2016. DOI: 10.1016/j.ejor.2015.11.019.

El objetivo principal de esta contribución es construir nuevas medidas de cohesión/consenso, o de ausencia de la misma, en problemas de toma de decisiones en grupo donde los expertos o agentes expresan sus opiniones sobre un conjunto de alternativas mediante evaluaciones cardinales. Existe un especial interés en que las medidas resultantes contemplen la posibilidad de que las alternativas se encuentren interrelacionadas.

Como metodología para la construcción de dichas medidas se adopta el enfoque de consenso basado en funciones de distancia. Este trabajo utiliza como herramienta central la *distancia de Mahalanobis* (Mahalanobis (1936)) ampliamente reconocida en Estadística.

Como resultado de examinar el problema de medición del grado de cohesión/consenso planteado y teniendo en cuenta las características peculiares de las evaluaciones expresadas mediante valores cardinales, en este artículo se introduce en primer lugar una definición general de *medida de disenso* para perfiles de evaluaciones cardinales. Posteriormente se presenta la *clase de medidas de disenso de Mahalanobis* que contiene todas las medidas de disenso de Mahalanois asociadas a matrices definidas no negativas, denominadas *matrices de referencia.* A través de estas matrices es posible incorporar las interrelaciones entre las alternativas. Así mismo, se puede destacar que se obtuvieron resultados formales que demuestran la consistencia de las medidas propuestas.

Además, en esta publicación se demuestra el cumplimiento de propiedades deseables y resultados teóricos relativos a las medidas de la clase propuesta tales como neutralidad, invariancia frente a transformaciones lineales, monotonía en la replicación del perfil, comportamiento ante la partición del conjunto de alternativas, adición de alternativas y adición de agentes a la sociedad.

Finalmente, el trabajo incluye una aplicación de la medida propuesta a las previsiones realizadas por diferentes instituciones y organizaciones de varias magnitudes económicas relativas a la economía española.

En esta contribución se concluye que la clase de medidas de disenso de Mahalanobis resulta adecuada para la medición del grado de cohesión en un escenario donde los expertos o agentes expresan su opinión sobre las alternativas con evaluaciones cardinales dado que posee buenas propiedades. Así imismo, cabe destacar que estas medidas pueden incorporar relaciones cruzadas y diferencias de escalas en las evaluaciones de las alternativas.

Artículo II:

T. González-Arteaga, R. de Andrés Calle, and F.Chiclana. *A new measure of consensus with reciprocal preference relations: The correlation consensus degree*. Knowledge-Based Systems, 107, 104–116. 2016. DOI: 10.1016/j.knosys.2016.06.002.

El objetivo de este trabajo es el desarrollo de un enfoque novedoso en la medición del consenso bajo la premisa de que los expertos expresan sus valoraciones mediante relaciones de preferencia recíprocas.

Esta nueva medida, a diferencia de los enfoques tradicionales, no se fundamenta ni en funciones de distancia ni en funciones de similaridad. La metodología en este trabajo se aleja de la usual en este escenario ya que no utiliza procedimientos basados en dichas funciones, sino que usa como herramienta básica el c*oeficiente de correlación de Pearson*.

En este trabajo se define una nueva medida de cohesión entre dos expertos, denominada *grado de consenso basado en la correlación*, que utiliza como elemeto central el coeficiente de correlación de Pearson y produce valores en el intervalo unitario, como es tradiccional en la Teoría de la Elección Social. Así mismo, se incluye una extensión de la medida a un grupo de expertos denominada *grado de consenso de un grupo*. También se prueban varios resultados teóricos y propiedades de la medida tales como reflexividad, simetría, reversibilidad y transitividad bajo el máximo. Finalmente, en este trabajo se presenta una aplicación de la medida propuesta a la Toma de Decisiones Clínicas Compartidas para mostrar la aplicabilidad y versatilidad de la medida propuesta.

Como conclusión señalamos que se presenta una nueva medida de consenso construída para medir la concordancia entre las intensidades de preferencia para pares de alternativas dadas por dos expertos o agentes, así como una extensión de la misma para un grupo de expertos o agentes. A diferencia de otras medidas de consenso, la nueva medida presentada no se apoya en funciones de distancia ni funciones de similaridad y abre una nueva vía para la medición del consenso.

Artículo III:

T. González-Arteaga, J.C.R. Alcantud, R. de Andrés Calle. *A new consensus ranking approach for correlated ordinal information based on Mahalanobis distance*. Information Sciences, 372, 546 - 564. 2016. DOI: 10.1016/j.ins.2016.08.071

El objetivo principal de este trabajo consiste en desarrollar de manera teórica medidas de cohesión y soluciones sociales de consenso para un marco de trabajo donde los expertos manifiestan sus valoraciones con evaluaciones ordinales (preórdenes completos) contemplando la posibilidad de interrelación entre las alternativas.

En esta contribución se ha utilizado el procedimiento de construcción de medidas de consenso basadas en funciones de distancia. La herramienta empleada para incorporar la posible interrelación de las alternativas es la *distancia de Mahalanobis*.

En este trabajo se define y se caracteriza un nuevo sistema de codificación de preórdenes completos mediante vectores numéricos. Este resultado nos proporciona un procedimiento que hace posible comprobar si, dado un vector numérico determinado, éste se corresponde con algún preorden completo y en su caso, qué ordenación concreta de las alternativas representa, es decir, la ordenación de las alternativas mostrada en las preferencias del experto. Para conseguir este objetivo se formula la clase de *medidas de disenso de Mahalanobis* y se prueba que se verifican algunas propiedades formales deseables tales como el anonimato, neutralidad débil, compatibilidad y reciprocidad.

Como resultado fundamental de esta contribución se establece un procedimiento para la obtención de una solución social de consenso denominada *solución social de consenso de Mahalanobis*, para perfiles de preferencias, que posee la originalidad de haberse diseñado especialmente para contemplar posibles interrelaciones entre las alternativas. Además, se demuestra que dicha solución verifica propiedades deseables tales como anonimato, unanimidad, neutralidad débil, consistencia, compatibilidad, reciprocidad y la imposibilidad de ser dictatoriales.

Para concluir resaltamos que en este trabajo se ha conseguido definir formalmente una clase de medidas de cohesión bajo la suposición de información ordinal (perfiles de preórdenes completos) que contempla la posibilidad de alternativas interrelacionas. Así mismo, se construyen soluciones sociales de consenso para las situaciones señaladas explotando las características de la distancia de Mahalanobis. Tanto las nuevas medidas como las soluciones sociales de consenso verifican propiedades de interés para su aplicación a situaciones reales.

Conclusiones y trabajos futuros

A continuación se recogen los principales resultados obtenidos en esta memoria. Del mismo modo, se exponen posibles líneas de trabajo futuro que se derivan de los resultados obtenidos.

Conclusiones y resultados obtenidos

Los problemas de toma de decisiones en grupo han cobrado importancia en muchas áreas de investigación tales como *Economía*, *Ciencias Sociales*, *Ingeniería* o *Ciencias de la Salud*, etc. En esta clase de problemas es interesante conocer el grado de acuerdo entre los miembros del grupo. En este ámbito, distintos autores han realizado diversas contribuciones sobre cómo medir y logar la cohesión en los grupos.

A pesar de la gran variedad de enfoques disponibles, existen algunas limitaciones o problemas que inspiraron y motivaron los objetivos perseguidos en esta tesis doctoral. Así, el interés de esta investigación se focalizó en superarlos mediante algunas propuestas cuyos resultados y conclusiones se resumen a continuación.

Por un lado, en la investigación realizada durante el desarrollo de esta tesis relativa a la medición de la cohesión se tuvo en cuenta la posible relación cruzada entre las alternativas. Para este objetivo se introdujo la distancia de Mahalanobis que se usa habitualmente en Estadística y que apenas ha sido empleada en la

Teoría de Toma de Decisiones.

Por otro lado, se abre un nuevo enfoque en la medición del consenso basado en el coeficiente de correlación de Pearson a diferencia de los tradicionales basados en funciones de distancia o de similaridad.

Desde otra perspectiva, en esta memoria se proporcionan definiciones teóricas de diferentes medidas de cohesión, asumiendo varios marcos de trabajo donde los agentes o expertos miembros del grupo pueden expresar sus evaluaciones sobre las alternativas. Concretamente, se define la clase de *medidas de disenso de Mahalanobis* para evaluaciones cardinales, el *grado de consenso basado en la correlación* para relaciones de preferencia recíprocas y las *medidas de disenso de Mahalanobis* para evaluaciones ordinales.

También se propone una nueva metodología basada en problemas de optimización para lograr una solución social de consenso bajo la suposición de información ordinal (preórdenes completos) denominada *solución social de consenso basada en Mahalanobis*. La utilización de la distancia de Mahalanobis como fundamento de nuestra propuesta permite considerar posibles relaciones cruzadas entre las alternativas. Así mismo, un nuevo resultado necesario para alcanzar tal solución acompaña al principal: la caracterización de un procedimiento de codificación para la información ordinal, concretamente, la *codificación canónica*.

Además, cada uno de los procedimientos propuestos para medir o alcanzar la cohesión en problemas de toma de decisiones en grupo se acompaña con un estudio detallado de sus propiedades formales con el objeto de mostrar su consistencia.

Finalmente, hay que destacar que las tres contribuciones aportadas en esta tesis contienen una aplicación práctica y real de las medidas y procedimientos novedosos propuestos.

Trabajos futuros

A continuación se señalan las líneas de trabajo más directas que se beneficiarían de futuras investigaciones.

Uno de los retos de las propuestas presentadas en esta memoria supone que la medida de disenso de Mahalanobis requiere disponer de una matriz de referencia adecuada al problema que se esté tratando. Por tanto, para mejorarla es preciso desarrollar procedimientos que permitan determinar tales matrices de referencia. En esta línea, sería interesante diseñar metodologías que puedan utilizar información del perfil original. De ese modo, se podría garantizar que la matriz proporcionada resulta apropiada a cada problema específico de toma de decisiones, lo que podría ser considerado una medida "endógena" de consenso.

Desde otro punto de vista, se podrían diseñar procesos de obtención de consenso flexibles. En este caso, se propondría plantear un proceso de obtención de consenso fundamentado en el grado de consenso basado en la correlación. Este proceso permitiría producir una solución de consenso mediante un mecanismo iterativo de retroalimentación adaptado a esta medida.

Finalmente, para mejorar y extender la propuesta de soluciones sociales de consenso de Mahalanobis, sería interesante indagar sobre la implementación del problema de optimización utilizado para obtener tales soluciones, especialmente cuando el número de alternativas resulte elevado.

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