Lecture notes on "Three Supersymmetric/Topological systems in Quantum Field Theory"

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Abstract

(1+1)-dimensional $\mathcal{N} = 1$ super-symmetric field theory and (3+1)-dimensional $\mathcal{N} = 2$ super-symmetric gauge theory are discussed in a, more or less, unified way, designed to identify the quantum BPS states in both systems. Euclidean 4-dimensional gauge theory with $\mathcal{N} = 2$ twisted super-symmetry is also analized. $\mathbf{C}^\infty$-topological invariants are identified as certain n-point correlation functions in this QFT framework. The twist of the effective dual Abelian gauge theory is briefly described, both from mathematical and physical viewpoints. The physical nature of the topological defects arising in these systems, kinks, BPS and Dirac monopoles, BPST instantons, Liouville and Abrikosov-Nielsen-Olesen selfdual vortices, etcetera, is analyzed,

The thread of the story connecting the QFT systems treated respectively in Sections §.3 and §.4 is the process of TWIST that leads from a conventional extended Supersymetric Gauge Theory to the topological $\mathcal{N} = 2$ SUSY Donaldson QFT. Within Section §.3 the $SL(2, \mathbb{Z}$-dualities establish a link between the weak coupling regime of the original gauge theory and the Wilsonian (abelian) effective gauge theory arising at low energies. We shall also look after the reminiscences of these dualities between the twisted TQFT systems of Section §.4.

Contents

1 Introduction

The MacPherson Manifesto on the IAS/Princeton Quantum Field Theory Program ends with this sentence: "the goal is to develop the sort of intuition common among physicists for those who are used to thought processes stemming from geometry and algebra", see [1].

These Lecture notes are, somehow, conceived in this spirit. To contribute to this goal I have tried to make explicit some concepts which are second nature to physicists whereas are crucial in the realm of the Mathematical Physics, so productive in the last twenty years. In particular, I shall discuss the ubiquitous phenomena of quantum BPS states and anomalies in two supersymmetric field theory models. Quantum BPS states and anomalies in the central charge of the SUSY algebra play an important rôle in both $(1+1)$dimensional $\mathcal{N} = 1$ supersymmetric field theory and $(3+1)$-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory. I shall explain, first, these concepts in the $(1+1)$-D baby model because the complexities that plague the higher dimensional systems are minimized.

The count of the BPS states and the derivation of the anomaly is much more difficult in $\mathcal{N} = 2$ SUSY Yang-Mills, but I will present this topic in a way as close to the $(1+1)$-D case as possible. The electric-magnetic duality transformation is behind the astonishing discovery of the Wilson effective action for any energy scale. The Seiberg-Witten proposal for the Wilson action allows to identify the anomaly in the central charge and elucidate the properties of the BPS states in terms of beautiful mathematics involving elliptic curves, holomorphic differentials and modular forms. The remarkable fact is that the whole construction is based on a few highly plausible physical facts.

Similar structures appear in: (a) supersymmetric sigma models where BPS states are related to hyper-Kahler metrics in 4n-dimensional target manifolds. (b) twisted $\mathcal{N} = 2$ supersymmetric gauge theory leading to Donaldson and Seiberg-Witten invariants of differentiable 4-manifolds. (c) dimensional reduction of supersymmetric string theory. Calabi-Yau and mirror symmetry of 6-dimensional manifolds arise using the same circle of ideas. (d) dimensional reduction of M theory. Short multiplets give rise to 7-dimensional manifolds with either Spin_7 or G_2 holonomy. No wonder on the interest in the magic of these supersymmetric phenomena.

We organize the material in three, non-homogeneous neither in size nor in difficulty, Chapters. In the first Chapter we describe the $\mathcal{N} = 1$ supersymmetric extension of $(1+1)$-dimensional scalar field theory as a warmup. Our construction is standard but we apply the general framework to a particular system with two scalar fields and a rich variety of BPS kinks. In particular we offer a computation of the central charge arising in the Supersymmetric algebra as an anomaly in the $(1+1)$-dimensional QFT system. The second Chapter is devoted to the development of the Seiberg-Witten formulation of $\mathcal{N} = 2$ Supersymmetric Yang-Mills Theory in $(3+1)$ dimensions. The treatment is standard following, besides Seiberg-Witten original paper [12], the comprehensive reviews of Bilal [2] and Alvarez-Gaumé/Hassan [3]. Particular attention is concentrated on the definition of the effective theory that appears at low energies and the description of its structure and topological properties. The last Chapter addresses the construction of topological gauge theories in Euclidean four manifolds. Starting from supersymmetric Yang-Mills systems we end in Euclidean $\mathcal{N} = 2$ SUSY through the process of twisting. We discuss both the high energy regime, where one meets with Donaldson invariants of four manifolds, and the low energy effective theory touching with the Seiberg-Witten equations.

2 $\mathcal{N} = 1$ (1+1)-dimensional Supersymmetric Field Theory

2.1 The classical theory

Points (x^μ, θ) in the physical $\mathbb{R}^{2|2}$ superspace , see [1] , are characterized in terms of local coordinates $x^\mu \in \mathbb{R}^{1,1}$ in Minkowski space-time and $\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$ Grassman Majorana spinors:

$$\theta_\alpha^* = \theta_\alpha, \qquad \alpha = 1, 2 \quad ; \quad \theta_1^2 = \theta_2^2 = \theta_1\theta_2 + \theta_2\theta_1 = 0$$

We choose the metric tensor $x^\mu x_\mu = g^{\mu\nu} x_\mu x_\nu$ as $g = \text{diag}(1, -1)$ and specify the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu}$ in the Majorana representation in terms of the Pauli matrices:

$$\gamma^0 = \sigma^2 \ , \qquad \gamma^1 = i\sigma^1 \ ; \qquad \gamma^5 = \gamma^0 \gamma^1 = \sigma^3$$

The Dirac adjoint is $\bar{\theta} = \theta^t \gamma^0$ and $\theta\bar{\theta} = -2i\theta_1\theta_2$.

If ε is a Grassman Majorana spinor parameter, the "vector field" $Q = \frac{\partial}{\partial \theta} + i\gamma^\mu \theta \partial_\mu$ is the generator of infinitesimal supertranslations in $\mathbb{R}^{2|2}$:

$$\theta \to \theta + i\varepsilon \quad \cong \quad i\bar{Q}\varepsilon\theta = i\varepsilon \qquad x^\mu \to x^\mu + \bar{\theta}\gamma^\mu\varepsilon \quad \cong \quad i\bar{Q}\varepsilon x^\mu = \bar{\theta}\gamma^\mu\varepsilon$$

The components Q_α of the super-charge Q satisfy the anti-commutation relations:

$$\{Q_\alpha, Q_\beta\} = Q_\alpha Q_\beta + Q_\beta Q_\alpha = 2 (\gamma^\mu C)_{\alpha\beta} \ P_\mu \tag{1}$$
$$= -2i \begin{pmatrix} \partial_0 - \partial_1 & 0 \\ 0 & \partial_0 + \partial_1 \end{pmatrix}_{\alpha\beta} \quad ; \qquad C = -\gamma^0$$

Our aim is to build the most general theory for the super-field

$$\vec{\Phi}(x^\mu, \theta) = \sum_{a=1}^{N} \Phi_a(x^\mu, \theta) \, \vec{e}_a \ : \mathbb{R}^{2|2} \to \mathbb{R}^N \quad ; \quad \vec{e}_a \cdot \vec{e}_b = \delta_{ab}$$

invariant with respect to the super-Poincare algebra, i.e. symmetry under the Poincare and the (1) SUSY transformations, the super-Poincare algebra, is required. $\vec{\Phi}(x^\mu, \theta)$ is a map from the super-space to the $\mathbb{R}^N$ "internal" space - $\vec{e}_a, a = 1, 2, \cdots, N$ are ortho-normal vectors in $\mathbb{R}^N$ - that , through the power series expansion

$$\vec{\Phi}(x^\mu, \theta) = \vec{\phi}(x^\mu) + \bar{\theta}\, \vec{\psi}(x^\mu) - \frac{1}{2}\bar{\theta}\theta \, \vec{F}(x^\mu),$$

can be seen as including N scalar fields, $\vec{\phi}(x^\mu) = \sum_{a=1}^{N} \phi_a(x^\mu) \, \vec{e}_a$, N Majorana spinor fields ,

$\vec{\psi}(x^\mu) = \sum_{a=1}^{N} \psi_a(x^\mu) \, \vec{e}_a$, and N auxiliary scalar fields $\vec{F}(x^\mu) = \sum_{a=1}^{N} F_a(x^\mu) \, \vec{e}_a$.

2.1.1 The supersymmetric action

Bearing in mind that the action of the SUSY charges on the superfield is

$$\delta \vec{\Phi} = i\bar{\varepsilon}\, Q\, \vec{\Phi} \Rightarrow \begin{cases} \delta\vec{\phi} = i\bar{\varepsilon}\vec{\psi} \\ \delta\vec{\psi} = \left(\gamma^\mu \partial_\mu \vec{\phi} + \vec{F}\right)\varepsilon \\ \delta\vec{F} = i\bar{\varepsilon}\gamma^\mu \partial_\mu \vec{\psi} \end{cases}$$

and realizing that the covariant derivative $D = \frac{\partial}{\partial\theta} - i\gamma^\mu\theta\partial_\mu$ anti-commutes with Q, it is not difficult to achieve this goal. The dynamics of the $\mathcal{N} = 1$ supersymmetric field theory on $\mathbb{R}^{1,1}$ with no interactions is governed by the action:

$$\begin{aligned} S_0[\vec{\Phi}] &= -\frac{1}{2}\int d^2x d^2\theta\, \bar{D}\vec{\Phi}\, D\vec{\Phi} \\ &= \frac{1}{2}\int d^2x \left\{ \partial_\mu\vec{\phi}\partial^\mu\vec{\phi} + i\vec{\psi}\gamma^\mu\partial_\mu\vec{\psi} - \frac{1}{2}\vec{F}\vec{F} \right\} \end{aligned}$$

The last identity is obtained through Berezin integration along the odd (Grasmann) variables in the superspace. To switch on supersymmetric interactions the recipe is well known: one adds

$$S_I[\vec{\Phi}] = 2\int d^2x d^2\theta\, W[\vec{\Phi}] \quad,$$

where $W[\vec{\Phi}]$ is a unspecified "superpotential", to the free action S_0.

Expanding $W[\vec{\Phi}]$ in a power series in the Grassman variables

$$W[\vec{\Phi}] = W[\vec{\phi}] + \vec{\nabla}W\cdot\bar{\theta}\vec{\psi} - \frac{1}{2}\bar{\theta}\theta\left(i\vec{\nabla}W\cdot\vec{F} + \frac{1}{2}\vec{\psi}\cdot\vec{\Delta}W\cdot\vec{\psi}\right)$$

$$\vec{\nabla}W = \sum_{a=1}^{N}\frac{\partial W}{\partial\phi_a}\cdot\vec{e}_a; \quad \vec{\Delta}W = \vec{\nabla}\otimes\vec{\nabla}W = \sum_{a=1}^{N}\sum_{b=1}^{N}\vec{e}_a\otimes\vec{e}_b\frac{\partial^2 W}{\partial\phi_a\partial\phi_b}$$

$$\vec{\psi}\cdot\vec{\Delta}W\cdot\vec{\psi} = \sum_{a=1}^{N}\sum_{b=1}^{N}\bar{\psi}_a\cdot\frac{\partial^2 W}{\partial\phi_a\partial\phi_b}\cdot\psi_b$$

and performing the Berezin integration, we find

$$S_I[\vec{\Phi}] = -\int d^2x\left(i\vec{\nabla}W\cdot\vec{F} + \frac{1}{2}\vec{\psi}\cdot\vec{\Delta}W\cdot\vec{\psi}\right)$$

as the interacting piece of the $\mathcal{N} = 1$ supersymmetric action $S = S_0 + S_I$.

Solving for the auxiliary fields in the constraint equations $\vec{F} = -i\vec{\nabla}W$ one checks that the "on shell" action reads

$$S = \frac{1}{2}\int d^2x\left\{ \partial_\mu\vec{\phi}\partial^\mu\vec{\phi} + i\vec{\psi}\gamma^\mu\partial_\mu\vec{\psi} - \vec{\nabla}W\cdot\vec{\nabla}W - \vec{\psi}\cdot\vec{\Delta}\cdot\vec{\psi} \right\}$$

so that the field equations are:

$$\Box\vec{\phi} + \vec{\nabla}W\cdot\vec{\Delta}W + \vec{\psi}\cdot\vec{\Delta}W\cdot\vec{\psi} = \vec{0} \tag{2}$$

$$i\,\partial_\mu \vec{\bar{\psi}}\gamma^\mu + \vec{\bar{\psi}} \cdot \vec{\vec{\Delta}}W = \vec{0} \tag{3}$$

The Nöther theorem provides us with the Hamiltonian functions associated to the vector fields Q_1 and Q_2:

$$\tilde{Q} = \int dx \left\{ \gamma^\mu \gamma^0 \vec{\bar{\psi}}\partial_\mu \vec{\phi} + i\gamma^0 \vec{\bar{\psi}}\vec{\nabla}W \right\}$$

$\tilde{Q}_1$ and $\tilde{Q}_2$ induce respectively the flows associated to the super-translations $\theta^1 + i\varepsilon^1$ and $\theta^2 + i\varepsilon^2$ in the co-tangent bundle to the configuration space, the space of initial conditions for the PDE system (2)-(3).

2.1.2 BPS and non-BPS super-solutions

We shall not discuss the super-solutions to the field equations (2)-(3) in full generality, see instead [1]. We describe in this sub-section several types of specially significant super-waves.

A. Homogeneous.

Let us choose choose W such that exist space-time independent scalar field configurations $\vec{\phi}_c$ for which $\vec{\nabla}W\big|_{\vec{\phi}_c} = \vec{0}$. Then,

$$\vec{\phi}(x^\mu) = \vec{\phi}_c \qquad , \qquad \vec{\psi}(x^\mu) = \vec{\psi}_c = \vec{0}$$

are the homogeneous super-solutions to (2)-(3).

B. Plane super-waves.

Considering small fluctuations of the scalar and spinor fields around one homogeneous solution,

$$\vec{\phi}(x,t) = \vec{\phi}_c + \delta\vec{\phi}(x,t) \qquad , \qquad \vec{\psi}(x,t) = \vec{\psi}_c + \delta\vec{\psi}(x,t) \quad ,$$

such that second order effects are negligible - $\mathcal{O}(\delta\Phi)^2$ -, one checks that

$$\delta\vec{\phi}(x,t) = \sum_{j=1}^{N}\sum_{k_j \in \mathbb{Z}} (\vec{a}(k_j)\,e^{i\omega_j t - ik_j x} + \vec{a}^*(k_j)\,e^{-i\omega_j t + ik_j x}) \tag{4}$$

$$\delta\vec{\psi}(x,t) = \sum_{j=1}^{N}\sum_{k_j \in \mathbb{Z}} (\vec{u}(k_j)\,e^{i\omega_j t - ik_j x} + \vec{u}^*(k_j)\,e^{-i\omega_j t + ik_j x}) \tag{5}$$

are the plane super-wave solutions on a large but finite interval of (2)-(3) if and only if:

$$\omega_j^2 = k_j^2 + \lambda_j^2 \quad , \quad \lambda_j^2 \in \mathrm{Spec}\left(\vec{\vec{\Delta}}W\big|_{\vec{\phi}_c} \cdot \vec{\vec{\Delta}}W\big|_{\vec{\phi}_c} \right) \quad .$$

C. Super-solitons.

For time-independent super fields, $\vec{\Phi}[x,\theta] \neq \vec{f}(t)$, the field equations reduce to the ODE system:

$$\frac{d^2\vec{\phi}}{dx^2} = \vec{\nabla}W \cdot \vec{\vec{\Delta}}W + \vec{\bar{\psi}}\vec{\nabla}\vec{\vec{\Delta}}W \cdot \vec{\psi} \tag{6}$$

$$i\frac{d\vec{\psi}}{dx} \cdot \gamma^1 = \vec{\bar{\psi}}\vec{\vec{\Delta}}W \tag{7}$$

The "on shell" energy is,

$$E = \frac{1}{2} \int dx \left(\frac{d\vec{\phi}}{dx} \cdot \frac{d\vec{\phi}}{dx} + \vec{\nabla}W \cdot \vec{\nabla}W + i\vec{\psi}\gamma^1 \frac{d\vec{\psi}}{dx} + \vec{\psi}\vec{\Delta}W\vec{\psi} \right)$$

and the bosonic contribution can be arranged á la Bogomolny.

$$E_B = \frac{1}{2} \int dx \left(\frac{d\vec{\phi}}{dx} \pm \vec{\nabla}W \right) \left(\frac{d\vec{\phi}}{dx} \pm \vec{\nabla}W \right) \mp \int dx \frac{d\vec{\phi}}{dx} \cdot \vec{\nabla}W$$

The Bogomolny bound $E_{\text{BPS}} = \left| \int dW \right|$ is saturated by the solutions of

$$\frac{d\vec{\phi}}{dx} = \mp\vec{\nabla}W \tag{8}$$

and

$$\frac{d\psi_\pm^a}{dx} = -\sum_{b=1}^{N} \frac{\partial^2 W}{\partial\phi^a \partial\phi^b} \cdot \psi_\pm^b \quad, \tag{9}$$

where $\vec{\psi}_\pm = \frac{1}{\sqrt{2}}\left(\vec{\psi}_1 \pm \vec{\psi}_2\right)$ are chiral (Majorana-Weyl) spinors.

Note that the solutions of (8) solve also (6) whereas the system (9) is nothing but (7) diagonalized. Super-solitons are (bosonic) flow lines of $\pm\text{grad}W$ - solutions of (8)- and their fermionic partners, the solutions of (9), which do not contribute to the energy.

A central element in supersymmetric/topological field theory and its application to Geometry and Topology is the concept of BPS state. Why are super-solitons so distinguished? The answer is because they are annihilated by some combination of the super-symmetry generators. Acting on shell static super-fields;,

$$\vec{\Phi}[x,\theta] = \vec{\phi}(x) + \bar{\theta}\vec{\psi}(x) - \frac{1}{2}\bar{\theta}\theta\vec{\nabla}W$$

the effective super-charges are: $Q = \frac{\partial}{\partial\theta} - i\gamma^1\theta\frac{d}{dx}$. Therefore,

$$Q\vec{\Phi} = -i\gamma^1\theta\frac{d\vec{\phi}}{dx} + \vec{\psi} - \theta\vec{\nabla}W \qquad \Leftrightarrow \qquad \begin{cases} \left(Q\vec{\Phi}\right)_1 = \theta_2\frac{d\vec{\phi}}{dx} + \vec{\psi}_1 - \theta_1\vec{\nabla}W \\ \left(Q\vec{\Phi}\right)_2 = \theta_1\frac{d\vec{\phi}}{dx} + \vec{\psi}_2 - \theta_2\vec{\nabla}W \end{cases}$$

The classical BPS states are those for which $\left(Q\vec{\Phi}\right)_1 \pm \left(Q\vec{\Phi}\right)_2 = \vec{0}$ but,

$$\left(Q\vec{\Phi}\right)_1 \pm \left(Q\vec{\Phi}\right)_2 = (\theta_2 \pm \theta_1)\frac{d\vec{\phi}}{dx} + (\vec{\psi}_1 \pm \vec{\psi}_2) - (\theta_1 \pm \theta_2)\vec{\nabla}W \quad.$$

Thus, the super-solitons which satisfy

$$\frac{d\vec{\phi}}{dx} \pm \vec{\nabla}W = \vec{0} \quad , \quad \vec{\psi}_\pm = \vec{0} \quad \text{and} \quad \frac{d\vec{\psi}_\mp}{dx} = -\vec{\Delta}W\Big|_{\vec{\phi}_0} \cdot \vec{\psi}_\mp \tag{10}$$

are classical BPS states. Witten's list [7] -gradient flow lines, holomorphic curves, gauge theory instantons, monopoles , Seiberg-Witten solutions, hyper-Kahler structures, Calabi-Yau metrics, metrics of G_2 and Spin_7 holonomy- of this aristocracy shows the importance, both in Physics and Mathematics, of the BPS states

2.2 A rapid look at the quantum theory

The standard canonical quantization procedure promotes the field configurations to field operators by decreeing that the Poisson super-brackets of the classical theory ought be replaced by super-commutators; if $\vec{\pi}(x) = \dot{\hat{\vec{\phi}}}(x)$ is the momentum,

$$\left[\hat{\vec{\phi}}^a(x), \hat{\vec{\pi}}^b(y)\right] = i\delta^{ab}\delta(x-y) \qquad , \qquad \left\{\hat{\vec{\psi}}_\alpha^a(x), \hat{\vec{\psi}}_\beta^b(y)\right\} = \delta^{ab}\delta_{\alpha\beta}\delta(x-y) \tag{11}$$

in the natural system of units $\hbar = c = 1$.

The quantum SUSY charges

$$\hat{Q} = \int dx \left\{\gamma^\mu\gamma^0\hat{\vec{\psi}}\partial_\mu\hat{\vec{\phi}} + i\gamma^0\hat{\vec{\psi}}\vec{\nabla}\hat{W}\right\}$$

becomes also operators, and the quantum SUSY algebra

$$\left\{\hat{Q}_\alpha, \hat{Q}_\beta\right\} = 2(\gamma^\mu\gamma^0)_{\alpha\beta}\hat{P}_\mu - 2\gamma^1_{\alpha\beta}\hat{T} \tag{12}$$

is given not only in terms of the energy-momentum Nöther invariants,

$$\hat{H} = \hat{P}_0 = \frac{1}{2}\int dx \left\{\hat{\vec{\pi}}\hat{\vec{\pi}} + \frac{d\hat{\vec{\phi}}}{dx}\frac{d\hat{\vec{\phi}}}{dx} + i\hat{\vec{\psi}}^t\gamma^5\frac{d\hat{\vec{\psi}}}{dx} + \vec{\nabla}\hat{W}\vec{\nabla}\hat{W} + \hat{\vec{\psi}}\vec{\Delta}\hat{W}\hat{\vec{\psi}}\right\}$$

$$\hat{P}_1 = \int dx \left\{\hat{\vec{\pi}}\cdot\frac{d\hat{\vec{\phi}}}{dx} + \frac{i}{2}\hat{\vec{\psi}}\frac{d\hat{\vec{\psi}}}{dx}\right\}$$

but also includes the central charge:

$$\hat{T} = \int dx \frac{d\hat{\vec{\phi}}}{dx}\cdot\vec{\nabla}\hat{W} \quad ,$$

and old friend in the disguise of Bogomolny bound.

One can ask a natural question: what is the röle of the classical super-solitons in the quantum theory ? In perturbation theory [1] one expands H around $\vec{\Phi}_c$ and splits the Hamiltonian in "free" and "interaction parts: $\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$,

$$\hat{H}_0 = \frac{1}{2}\int dx \left\{\left[\hat{\vec{\pi}}\hat{\vec{\pi}} + \frac{d\hat{\vec{\phi}}}{dx}\frac{d\hat{\vec{\phi}}}{dx} + \hat{\vec{\phi}}\left(\vec{\Delta}W\Big|_{\vec{\phi}_c}\right)^2\hat{\vec{\phi}}\right] + \left[i\hat{\vec{\psi}}^t\gamma^5\frac{d\hat{\vec{\psi}}}{dx} + \hat{\vec{\psi}}\vec{\Delta}W\Big|_{\vec{\phi}_c}\hat{\vec{\psi}}\right]\right\} \tag{13}$$

The plane super-waves (4)-(5) are compatible with the canonical quantization rules (11) if the Fourier coefficients become non-commuting operators that satisfy:

$$\left[\hat{a}_a^\dagger(k_j), \hat{a}_b(q_l)\right] = \delta_{ab}\delta_{k_j q_l} \qquad , \qquad \left\{\hat{u}_{a\alpha}^\dagger(k_j), \hat{u}_{b\beta}(q_l)\right\} = \delta_{ab}\delta_{\alpha\beta}\delta_{k_j q_l} \tag{14}$$

From (14) one easily derives the spectrum of the operators $\hat{N}_a^B(k_j) = \hat{a}_a^\dagger(k_j)\hat{a}_a(k_j)$ and $\hat{N}_{a\alpha}^F(k_j) = \hat{u}^\dagger_{a\alpha}(k_j)\hat{u}_{a\alpha}(k_j)$:

$$\hat{N}_a^B(k_j)|n_a^B(k_j)\rangle = n_a^B(k_j)|n_a^B(k_j)\rangle \quad , \quad \hat{N}_{a\alpha}^F(k_j)|n_{a\alpha}^F(k_j)\rangle = n_{a\alpha}^F(k_j)|n_{a\alpha}^F(k_j)\rangle$$

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[1]For a lucid analysis of the mathematical meaning of the Fock space and perturbation theory see [8]

where $n_a^B(k_j)$ is a natural number and $n_{a\alpha}^F(k_j) = 0, 1$. Because (after normal-ordering)

$$\hat{H}_0 = \sum_{j=1}^{N} \omega_j \sum_{a,k_j} \left(\hat{N}_a^B(k_j) + \hat{N}_{a\alpha}^F(k_j) \right) \quad ,$$

the ground state of the free theory -the eigen-state with smallest eigenvalue of $\hat{H}_0$- is the vacuum state:

$$|0\rangle = \otimes_{j,\,a,\,k_j} |n_a^B(k_j) = 0\rangle \otimes_{\alpha} |n_{a\alpha}^F(k_j) = 0\rangle$$
$$\hat{a}_a(k_j)|0\rangle = \hat{u}_{a\alpha}(k_j)|0\rangle = 0, \quad \forall j, a, \alpha, k_j$$

Thus, the expectation value of $\hat{\vec{\phi}}(x^\mu)$ at the vacuum is

$$\langle 0|\hat{\vec{\phi}}(x^\mu)|0\rangle = \vec{\phi}_c$$

and there is a one-to-one correspondence between the ground states of the quantum free theory and the homogeneous super-solutions of the classical theory.

The two types of one-particle states

$$\hat{a}_a^\dagger(k_j)|0\rangle = |n_a^B(k_j) = 1\rangle \quad , \quad \hat{u}_{a\alpha}^\dagger(k_j)|0\rangle = |n_{a\alpha}^F(k_j) = 1\rangle$$

are degenerated in energy, which is ω_j. There is a one-to-one correspondence between plane super-waves in the classical theory and the one-particle states with definite momentum in the free quantum theory. Multi-particle states show the statistics of each type of particle: bosonic, the state remains the same under the exchange of two particles or fermionic, the state changes sign under such an exchange.

Before of trying to identify the states in the quantum theory related to the BPS super-solitons there is the need of addressing a very delicate point: we are dealing in fact not with operator-valued functions but with operator-valued distributions. Note the delta-functions in (11). The naïf SUSY algebra relations (12) are therefore non-sense because undefined products of distributions at the same points of space-time are involved. Fortunately, there is a unique method for calculating equal time (super)commutators from the (known) Green's functions of renormalized perturbation theory: the Bjorken-Johnson-Low limit, [9].

Consider the Fourier transform of the matrix elements of the "chronological" product of two supercharge operators between any two states $|S_1\rangle, |S_2\rangle$ in the Fock space:

$$T(q^\mu) = \int d^2x e^{iq^\mu x_\mu} \langle S_1| T\hat{Q}_\alpha(x^\mu)\hat{Q}_\beta(0)|S_2\rangle \tag{15}$$
$$T\hat{Q}_\alpha(x^\mu)\hat{Q}_\beta(0) = \hat{Q}_\alpha(x^\mu)\hat{Q}_\beta(0) \quad , \quad x^0 > 0$$
$$= \hat{Q}_\beta(0)\hat{Q}_\alpha(x^\mu) \quad , \quad x^0 < 0$$

The BJL definition of such matrix elements is:

$$\lim_{q^0 \to \infty} q^0 T(q^\mu) \underset{\text{def}}{\equiv} i \int dx e^{-iqx} \langle S_1| \{\hat{Q}_\alpha(0,x), \hat{Q}_\beta(0,0)\}|S_2\rangle \tag{16}$$

We also need the Wick's theorem. If,

$$\langle 0| T\hat{\phi}_a(x_1^\mu)\hat{\phi}_a(x_2^\mu)|0\rangle = \Delta_{aa}^B(x_1^\mu - x_2^\mu) \quad , \quad \langle 0| T\hat{\psi}_a^t(x_1^\mu)\hat{\psi}_a(x_2^\mu)|0\rangle = \Delta_{aa}^F(x_1^\mu - x_2^\mu)$$

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are the one-particle Green's functions, the chronological product of a string of n field operators is:

$$TA_1(x_1^\mu)...A_n(x_n^\mu) =: A_1(x_1^\mu)...A_n(x_n^\mu): + \tag{17}$$

$$+\Delta(x_1^\mu - x_2^\mu): A_3(x_3^\mu)...A_n(x_n^\mu): ... + ... + \tag{18}$$

$$+\Delta(x_1^\mu - x_2^\mu)\Delta(x_3^\mu - x_4^\mu)...\Delta(x_{n-1}^\mu - x_n^\mu) \qquad , \tag{19}$$

where $: A_1(x_1^\mu)\cdots A_n(x_n^\mu):$ is the normal-ordered product - all the creation operators are put to the left of all the annihilation operators -. Here, by $A(x^\mu)$ we mean either $\hat{\phi}_a(x^\mu)$ or $\hat{\psi}_{a\alpha}(x^\mu)$ and by $\Delta(x^\mu - y^\mu)$ we denote either $\Delta_{aa}^B(x^\mu - y^\mu)$ or $\Delta_{aa}^F(x^\mu - y^\mu)$.

The procedure for finding the quantum (super)commutator (12) is now clear:

- Wick's theorem applied to $T(q^\mu)$ tells what states $|S_1\rangle, |S_2\rangle$ give a non-zero answer.

- Choose $|S_1\rangle$ and $|S_2\rangle$ accordingly, and compute $\lim_{q^0 \to \infty} q^0 T(q^\mu)$.

- After a long calculation one finds a result compatible with the new relation:

$$\{\hat{Q}_\alpha, \hat{Q}_\beta\} = 2(\gamma^\mu\gamma^0)_{\alpha\beta}\hat{P}_\mu - 2\gamma_{\alpha\beta}^1\hat{T}_R \tag{20}$$

where $\hat{T}_R = \hat{T} + \frac{1}{4\pi}\left|\int dx\frac{d}{dx}(\Delta\hat{W})\right|, \Delta W = \vec{\nabla}\vec{\nabla}W$.

The quantum SUSY algebra is <u>anomalous</u>: the classical and quantum central charges differ in something proportional to the difference between the values of the Laplacian of the super-potential at $x = \pm\infty$. Only if the super-potential is harmonic, the central charge does not receive quantum corrections; but a harmonic super-potential is the necessary condition for the existence of $\mathcal{N} = 2$ super-symmetry.

From (20) one writes the Hamiltonian operator in the form:

$$\hat{P}_0 = (\hat{Q}_1 \pm \hat{Q}_2)^2 + \left|\hat{T}_R\right|$$

Thus, the expectation value of the energy operator at any state $|S\rangle$ satisfies the inequality:

$$\langle S| \hat{P}_0|S\rangle \geq \langle S| |\hat{T}_R| |S\rangle$$

Equality is attained by the quantum BPS states -$\langle BPS| (\hat{Q}_1 \pm \hat{Q}_2)^2|BPS\rangle = 0$ -

$$\langle BPS| \hat{P}_0|BPS\rangle = \langle BPS| |\hat{T}_R||BPS\rangle =$$

$$= \left|W\left(<\hat{\vec{\phi}}>|_\infty\right) - W\left(<\hat{\vec{\phi}}>|_{-\infty}\right)\right| + \frac{1}{4\pi}\left|\Delta W\left(<\hat{\vec{\phi}}>|_\infty\right) - \Delta W\left(<\hat{\vec{\phi}}>|_{-\infty}\right)\right|$$

The quantum BPS states are coherent states such that $\langle BPS|\hat{\vec{\phi}}(x)|BPS\rangle, \langle BPS|\hat{\vec{\psi}}(x)|BPS\rangle$ are the classical super-solitons which solve (10).

2.3   The super-symmetric BNRT model

Let us study a $N = 2 \ \mathcal{N} = 1$ super-symmetric system where the super-potential is.

$$W[\vec{\Phi}] = \sqrt{\lambda}\vec{\Phi} \cdot \vec{e}_1 \left[\frac{4}{3}(\vec{\Phi} \cdot \vec{e}_1)^2 - \frac{m^2}{\lambda} + 2\sigma^2(\vec{\Phi} \cdot \vec{e}_2)^2\right] \qquad ,$$

and λ , m are coupling constants with dimensions of inverse length whereas σ^2 is a non-dimensional parameter. For more details on the so-called BNRT model, see [10].

It is convenient to use non-dimensional super-space and field variables:

$$x^\mu \to \frac{1}{m}x^\mu, \qquad \theta_\alpha \to \frac{1}{\sqrt{m}}\theta_\alpha, \qquad d\theta_\alpha \to \sqrt{m}d\theta_\alpha$$

$$\vec{\Phi} \longrightarrow \frac{m}{\sqrt{\lambda}}\vec{\Phi} \qquad \Leftrightarrow \qquad \{\vec{\phi} \to \frac{m}{\sqrt{\lambda}}\vec{\phi} \ , \quad \vec{\psi} \to m\sqrt{\frac{m}{\lambda}}\vec{\psi} \ , \quad \vec{F} \to \frac{m^2}{\sqrt{\lambda}}\vec{F}\}$$

Thus, $D\vec{\Phi} \to m\sqrt{\frac{m}{\lambda}}D\vec{\Phi}$, $W[\vec{\Phi}] \to \frac{m^3}{\lambda}W[\vec{\Phi}]$ and the $\mathcal{N} = 1$ SUSY action reads:

$$S[\vec{\Phi}] = -\frac{m^2}{\lambda}\left\{\int d^2x d^2\theta \left(\frac{1}{2}\bar{D}\vec{\Phi}D\vec{\Phi} + 2\vec{\Phi}\vec{e}_1[\frac{4}{3}(\vec{\Phi}\vec{e}_1)^2 - 1 + 2\sigma^2(\vec{\Phi}\vec{e}_2)^2]\right)\right\}$$

From $\vec{\nabla}W = [(4\phi_1^2 + 2\sigma^2\phi_2^2 - 1)\vec{e}_1 + 4\sigma^2\phi_1\phi_2\vec{e}_2]$, and

$$\vec{\vec{\Delta}}W = 4[2\phi_1\vec{e}_1 \otimes \vec{e}_1 + \sigma^2\phi_2(\vec{e}_1 \otimes \vec{e}_2 + \vec{e}_2 \otimes \vec{e}_1) + \sigma^2\phi_1\vec{e}_2 \otimes \vec{e}_2] \quad,$$

we obtain the on shell SUSY action:

$$S_B[\vec{\phi}] = \frac{m^2}{2\lambda}\int d^2x \left\{\partial_\mu\vec{\phi}\partial^\mu\vec{\phi} - (4\phi_1^2 + 2\sigma^2\phi_2^2 - 1)^2 - 16\sigma^4\phi_1^2\phi_2^2\right\}; S_F[\vec{\psi}] = \frac{m^2}{2\lambda}\int d^2x \left\{i\vec{\bar{\psi}}\gamma^\mu\partial_\mu\vec{\psi}\right\}$$

$$S_{BF}[\vec{\phi}, \vec{\psi}] = \frac{m^2}{2\lambda}\int d^2x \left\{8\bar{\psi}_1\phi_1\psi_1 + 4\sigma^2(\bar{\psi}_1\phi_2\psi_2 + \bar{\psi}_2\phi_2\psi_1) + \bar{\psi}_2\phi_1\psi_2\right\}$$

Observe that $\Delta W = 8[2 + \sigma^2]\phi_1$ and the super-potential is harmonic only for $\sigma = \pm i\sqrt{2}$, where one finds the celebrated Wess-Zumino model, see [11].

Besides the super-Poincarè symmetry the system is invariant under the discrete group $\mathbb{G} = \mathbb{Z}_2 \times \mathbb{Z}_2$ generated by the internal reflections $\pi_1\vec{\Phi} = -\Phi_1\vec{e}_1 + \Phi_2\vec{e}_2$ and $\pi_2\vec{\Phi} = \Phi_1\vec{e}_1 - \Phi_2\vec{e}_2$. The homogeneous super-solutions are the critical points of W,

$$\vec{\Phi}_{v_\pm}^{(1)} = \pm\frac{1}{2}\vec{e}_1 \qquad\qquad \vec{\Phi}_{v_\pm}^{(2)} = \pm\frac{1}{\sigma\sqrt{2}}\vec{e}_2$$

and the vacuum manifold $\mathcal{V}$ is the union of two orbits of $\mathbb{G}$. Thus, the moduli space of vacua $\mathcal{M} = \frac{\mathcal{V}}{\mathbb{G}} = (\vec{\Phi}_v^{(1)}, \vec{\Phi}_v^{(2)})$ contains two points.

The plane super-waves transmute to the fundamental Bosons and Fermions in the quantum world. Because

$$\vec{\vec{\Delta}}W\left(\vec{\phi}_{v_\pm}^{(1)}\right) = \pm[4\vec{e}_1 \otimes \vec{e}_1 + 2\sigma^2\vec{e}_2 \otimes \vec{e}_2]$$

we identify $\vec{\Phi}_{v_+}^{(1)}$ and $\vec{\Phi}_{v_-}^{(1)}$ respectively as minimum and maximum of W.

As dictated by super-symmetry, we see that there are two Boson/Fermion branches - degenerated in mass - on the $\vec{\Phi}_v^{(1)}$ point in $\mathcal{M}$:

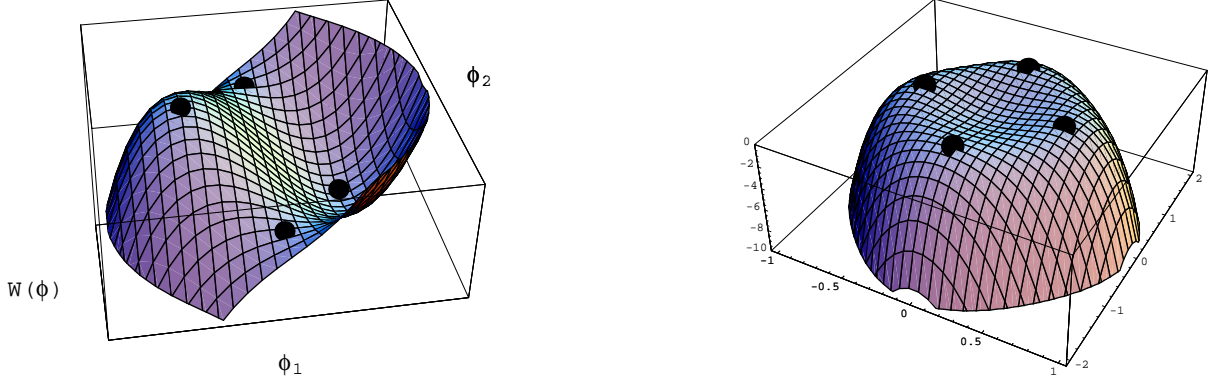|	$\omega_1(k_1 = 0)$	$\omega_2(k_2 = 0)$
Bosons	$4m$	$2\sigma^2 m$
Fermions	$4m$	$2\sigma^2 m$

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Figure 1: Plots of W and U as functions of ϕ_1 and ϕ_2.

We could be tempted also to put the minus sign in the Fermi masses, but the Dirac sea paradigm and the "particle" "hole" identity in the Majorana representation liberate us of such a worry.

Simili modo,

$$\vec{\Delta} W \left(\vec{\phi}_{v_\pm}^{(2)} \right) = \pm 2\sqrt{2}\sigma [\vec{e}_1 \otimes \vec{e}_2 + \vec{e}_2 \otimes \vec{e}_1]$$

tells us that both $\vec{\Phi}_{v_+}^{(2)}$ and $\vec{\Phi}_{v_-}^{(2)}$ are saddle points of W. A $\frac{\pi}{4}$ rotation in $\mathbb{R}^2$ to $2\vec{\Phi} = \Phi_+(\vec{e}_1 + \vec{e}_2) + \Phi_-(\vec{e}_1 - \vec{e}_2)$ allows us to identify the two Boson/Fermion branches on the $\vec{\Phi}_v^{(2)}$ point in $\mathcal{M}$:

	$\omega_+(k_+ = 0)$	$\omega_-(k_- = 0)$
Bosons	$2\sqrt{2}\sigma m$	$2\sqrt{2}\sigma^2 m$
Fermions	$2\sqrt{2}\sigma m$	$2\sqrt{2}\sigma^2 m$

What about the super-solitons that gives rise to extended BPS states in the quantum theory? In this model, the bosonic ODE system (10) becomes

$$\frac{d\phi_1}{dx} = \mp 4\phi_1^2 + 2\sigma^2\phi_2^2 - 1 \qquad , \qquad \frac{d\phi_2}{dx} = \mp 4\sigma^2\phi_1\phi_2 \tag{21}$$

The flows lines of $\pm \mathrm{grad} W$ are the solutions of the ODE:

$$\frac{d\phi_1}{d\phi_2} = \frac{4\phi_1^2 + 2\sigma^2\phi_2^2 - 1}{4\sigma^2\phi_1\phi_2} \tag{22}$$

which admits the integrating factor $|\phi_2|^{-\frac{2}{\sigma}}\phi_2^{-1}$, if $\sigma \neq 1$ and $\sigma \neq 0$, thereby allowing us to find all the flow-lines as the family of curves

$$\phi_1^2 + \frac{\sigma}{2(1-\sigma)}\phi_2^2 = \frac{1}{4} + \frac{c}{2\sigma}|\phi_2|^{\frac{2}{\sigma}} \tag{23}$$

parametrized by the real integration constant c. There is a critical value

$$c^S = \frac{1}{4}\frac{\sigma}{1-\sigma}(2\sigma)^{\frac{\sigma+1}{\sigma}}$$

and the behaviour of a particular curve in the (23) family is described in the following items:

- For $c \in (-\infty, c^S)$, formula (23) describes closed curves in the internal space $\mathbb{R}^2$ that connect the vacua $\vec{\Phi}_{v_+}^{(1)}$ and $\vec{\Phi}_{v_-}^{(1)}$, see Figure 2. Thus, they provide a kink family in the topological sector *(1;1)*. Henceforth, we refer to these kinks as $\mathbf{TK2}^{(1;1)}(c)$. A fixed value of c determines four members in the kink variety related amongst one another by spatial parity and internal reflections. The kink moduli space is defined as the quotient of the kink variety by the action of the symmetry group:
$$\mathcal{M}_{\mathrm{K}} = \frac{\mathcal{V}_{\mathrm{K}}}{\mathbb{P} \times \mathbb{G}} = (-\infty, c^S),$$
the real open half-line parametrized by c.

- In the range $c \in (c^S, \infty)$, equation (23) describes open curves and no vacua are connected. These $\mathrm{grad}\, W$ flow-lines are infinite energy solutions that do not belong to the configuration space $\mathcal{C}$, see Figure 2.

- At the other point of the boundary of $\mathcal{M}_{\mathrm{K}}$, $c = c^S$, we find the separatrices between bounded and unbounded motion and the envelop of all kink orbits in the *(1;1)* topological sector, see Figure 2.

We briefly discuss the $\sigma = 1$ case. The $\sigma = 0$ case is not interesting because the ϕ_2 dependence disappears in the potential: it is a "direct sum" of an $N = 1$ ϕ^4 model and an $N = 1$ free model. Integration of (22) when $\sigma = 1$ gives

$$\phi_1^2 - \phi_2^2 \left(\frac{c}{2} + \log |\phi_2| \right) = \frac{1}{4} \tag{24}$$

where the kink trajectories now appear in the $c \in (-\infty, c^S]$ range, with $c^S = -1 + \ln 2$. The description of the kink orbits is analogous to the description for $\sigma \neq 1$ above.
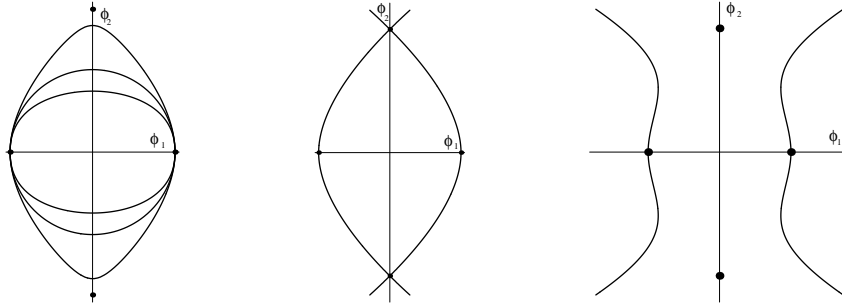
Figure 2: *Flow-lines given by (23): for $c \in (-\infty, c^S)$ (left), $c = c^S$ (middle), and $c \in (c^S, \infty)$ (right).*

With the exception of the one-component topological kinks with $\phi_1 = 0$, there are no other flow lines connecting the vacua $\vec{\Phi}_{v_+}^{(2)}$ and $\vec{\Phi}_{v_-}^{(2)}$, for generic σ^2. If $\sigma^2 < 2$, the spectrum of the second order fluctuation operator around the $\mathrm{TK1}^{(1;1)}$ kinks shows some negative eigen-value and these kinks are unstable -non-BPS-. For the critical values $\sigma^2 = \frac{4}{l(l+1)-4}$, $l = 3, 4, \cdots$, there are Jacobi fields in the spectrum announcing the existence of a whole continuous family of these non-BPS solitons and, probably, the availability of a non-continuously differentiable second superpotential.

If $\sigma^2 = 2$, there exists a "honest" superpotential, where the rôles of Φ_1 and Φ_2 are exchanged; the "vertical" (1; 1) topological sector is plenty of BPS solitons.

3 $\mathcal{N} = 2$ Supersymmetric Gauge Theory

3.1 $\mathcal{N}$=1 SUSY Yang-Mills theory

3.1.1 $\mathbb{R}^{4|4}$ super-space and $\mathcal{N} = 1$ SUSY algebra

We now move to physical $\mathbb{R}^{1,3}$ Minkowski space-time with the following choices of metric tensor and associated Clifford algebra:

$$g^{\mu\nu} = \operatorname{diag}(1, -1, -1, -1)$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \qquad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}; \qquad \gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3$$

$$\sigma^\mu = (\mathbf{1}_2, \sigma^1, \sigma^2, \sigma^3) \qquad , \qquad \bar{\sigma}^\mu = (\mathbf{1}_2, -\sigma^1, -\sigma^2, -\sigma^3)$$

In this representation of the Clifford algebra the Dirac spinors are the direct sum of Weyl spinors $\psi_\alpha \in (\frac{1}{2}, 0)$, $\bar{\chi}^{\dot\alpha} \in (0, \frac{1}{2})$ that belong to the fundamental representations of SO(3, 1), the connected piece to the identity of the Lorentz group:

$$\Psi_D(x^\mu) = \begin{pmatrix} \psi_\alpha(x^\mu) \\ \bar{\chi}^{\dot\alpha}(x^\mu) \end{pmatrix} \in (\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$$

Spinor indices are raised and lowered with the anti-symmetric ε-tensor: $\psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta$, $\bar{\chi}^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\bar{\chi}_{\dot\beta}$. Thus, $\psi\eta = \psi^\alpha\eta_\alpha$, $\bar{\chi}\bar{\xi} = \bar{\chi}_{\dot\alpha}\bar{\xi}^{\dot\alpha}$ are scalar products, whereas $\psi\sigma^\mu\bar{\chi}$, $\bar{\xi}\bar{\sigma}^\mu\eta$ are vector combinations.

Odd coordinates in the $\mathbb{R}^{4|4}$ super-space are Grassman Weyl spinors, θ_α, $\bar{\theta}^{\dot\alpha}$, which satisfy:

$$\theta^2 \equiv \theta\theta = \theta^\alpha\theta_\alpha = -2\theta_1\theta_2, \qquad , \qquad \theta_{\alpha_1}\theta_{\alpha_2} + \theta_{\alpha_2}\theta_{\alpha_1} = \bar{\theta}^{\dot\alpha_1}\bar{\theta}^{\dot\alpha_2} + \bar{\theta}^{\dot\alpha_2}\bar{\theta}^{\dot\alpha_1} = 0$$

If ε and $\bar{\varepsilon}$ are Grassman Weyl spinor parameters, the vector fields

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot\alpha}\bar{\theta}^{\dot\alpha}\partial_\mu \qquad , \qquad \bar{Q}_{\dot\alpha} = -\frac{\partial}{\partial\bar{\theta}^{\dot\alpha}} + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$$

are spinor generators of the super-translation group: $P(x, \varepsilon, \bar{\varepsilon}) = e^{i[-x^\mu P_\mu + \varepsilon Q + \bar{\varepsilon}\bar{Q}]}$. Infinitesimally,

$$\theta_\alpha \to \theta_\alpha + i\varepsilon_\alpha \qquad , \qquad \bar{\theta}^{\dot\alpha} \to \bar{\theta}^{\dot\alpha} - i\bar{\varepsilon}^{\dot\alpha} \qquad , \qquad x^\mu \to x^\mu + \varepsilon^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar{\theta}^{\dot\alpha} + \theta^\alpha\bar{\sigma}^\mu_{\alpha\dot\alpha}\bar{\varepsilon}^{\dot\alpha}$$

The super-charges satisfy the anti-commutation relations:

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \qquad ; \qquad \{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0.$$

The covariant derivatives:

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot\alpha}\bar{\theta}^{\dot\alpha}\partial_\mu \qquad , \qquad \bar{D}_{\dot\alpha} = -\frac{\partial}{\partial\bar{\theta}^{\dot\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$$

which satisfy,

$$\{D_\alpha, Q_\beta\} = \{D_\alpha, \bar{Q}_{\dot\beta}\} = \{\bar{D}_{\dot\alpha}, Q_\beta\} = \{\bar{D}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0$$

and

$$\{D_\alpha, \bar{D}_{\dot\alpha}\} = -2i\sigma^\mu_{\alpha,\dot\alpha}\partial_\mu \qquad ; \qquad \{D_\alpha, D_\beta\} = \{\bar{D}_{\dot\alpha}, \bar{D}_{\dot\beta}\} = 0 \quad ,$$

will also play an important rôle in the formulation of the theory.

We build $\mathcal{N} = 1$ super-symmetric Yang-Mills theory (without matter hyper-multiplets) out of two kinds of super-fields.

3.1.2 The $\mathcal{N} = 1$ super-symmetric action for chiral super-fields

The chiral (anti-chiral) super-fields,

$$\Phi(x^\mu, \theta, \bar{\theta}) : \mathbb{R}^{4|4} \longrightarrow \mathrm{adj}_{\mathbb{C}} \ \mathrm{SU(N)}$$

are maps from the $\mathcal{N} = 1$ super-space to the adjoint representation of the $SU(N)$ group which are constrained by the $\bar{D}_{\dot{\alpha}} \Phi = 0$ chiral ($D_\alpha \Phi^\dagger = 0$ anti-chiral) condition.

The chiral constraint is easily solved in terms of $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$ and θ. Because $\bar{D}_{\dot{\alpha}} y^\mu = \bar{D}_{\dot{\alpha}}\theta = 0$ one finds

$$\Phi(y, \theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y)$$

and a chiral field contains a complex scalar field, $\phi(\equiv \phi^a \frac{\tau^a}{2})$, a Weyl spinor field, $\psi_\alpha \equiv \psi^a_\alpha \frac{\tau^a}{2}$, and a complex auxiliary field, $F(\equiv F^a \frac{\tau^a}{2})$, all of them in the adjoint representation of the $SU(N)$ gauge group - τ^a are the generators of $\mathrm{Lie} SU(N)$-. The action of the super-charges on the chiral superfields is:

$$\Phi \Rightarrow \begin{cases} \delta\phi = i\sqrt{2}\varepsilon\psi \\ \delta\psi = -\sqrt{2}\sigma^\mu\bar{\varepsilon}\partial_\mu\phi + i\sqrt{2}\varepsilon F \\ \delta F = -\sqrt{2}\bar{\varepsilon}\bar{\sigma}^\mu\partial_\mu\psi \end{cases}$$

Because $\delta\Phi = i\varepsilon Q$ and $\delta\Phi^\dagger = i\bar{\varepsilon}\bar{Q}$ are super-derivations, the infinitesimal action of $P(a^\mu, \varepsilon, \bar{\varepsilon})$ on the super-action functional

$$S_0 = \frac{1}{4} \int d^4x \ d^2\theta \ d^2\bar{\theta} \ \mathrm{tr}\Phi^\dagger\Phi \quad ,$$

is a total super-derivative and, thus, S_0 is $\mathcal{N} = 1$ SUSY invariant. The Berezin integral - ($\int d^2\theta \ d^2\bar{\theta} \ \theta^2\bar{\theta}^2 = 4$)- and the power "series" expansion of the chiral super-field:

$$\Phi(y, \theta) = \phi(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) - \frac{1}{4}\theta^2\bar{\theta}^2\Box\phi(x) + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi(x)\sigma^\mu\bar{\theta} + \theta^2 F(x) \qquad (25)$$

allow us to write S_0 in terms of the super-field components:

$$S_0 = \int d^4x \ \mathrm{tr}(\partial_\mu\phi^\dagger\partial^\mu\phi - i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + F^\dagger F) \quad . \qquad (26)$$

S_0 determines a free dynamics but one can switch on interactions compatibles with $\mathcal{N} = 1$ super-symmetry using a chiral super-potential:

$$S_1 = \int d^4x [\int d^2\theta \ \mathcal{W}(\Phi) + \int d^2\bar{\theta} \ \bar{\mathcal{W}}(\Phi^\dagger)].$$

3.1.3 $\mathcal{N} = 1$ supersymmetric action for vector super-fields

The vector super-fields are also maps from the super-space to the adjoint representation of $SU(N)$

$$V(x^\mu, \theta, \bar{\theta}) : \mathbb{R}^{4|4} \longrightarrow \mathrm{adj} \ \mathrm{SU(N)}$$

which satisfy a reality condition: $V = V^\dagger$. In the Wess-Zumino gauge, the power expansion in the odd variables reads:

$$V(x, \theta, \bar{\theta}) = -\theta\sigma^\mu\bar{\theta}A_\mu(x) + i(\theta^2\bar{\theta}\bar{\lambda}(x) - \bar{\theta}^2\theta\lambda(x)) + \frac{1}{2}\theta^2\bar{\theta}^2 D(x)$$

The super-connection V includes the connection $A_\mu \equiv A_\mu^a \frac{\tau^a}{2}$ on a $SU(N)$ bundle over $\mathbb{R}^{1,3}$, its SUSY partner, the Weyl spinor $\lambda \equiv \lambda^a \frac{\tau^a}{2}$, and the real auxiliary field D. Chiral fields Λ give rise to super gauge transformations on V, $V' = V + \Lambda + \Lambda^\dagger$, and the spinor field strength,

$$W_\alpha = \frac{1}{8g} \bar{D}^2 (e^{2gV} D_\alpha e^{-2gV}) ,$$

where g is the coupling constant, transform under super-gauge transformations as:

$$W'_\alpha = e^{-i2g\Lambda} W_\alpha e^{i2g\Lambda} .$$

Note that $\bar{D}_{\dot{\alpha}} W_\alpha = 0$ and W is a chiral spinor field. In the appropriate coordinates the Grassman power expansion of W reads,

$$W_\alpha(y, \theta) = (-i\lambda_\alpha + \theta_\alpha D - i\sigma^{\mu\nu}\theta_\alpha F_{\mu\nu} + \theta^2 \sigma^\mu \nabla_\mu \bar{\lambda}_\alpha)(y)$$

where the following tensors

$$\sigma^{\mu\nu} = \frac{1}{4}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu) \quad ; \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu] \quad ; \quad \nabla_\mu \lambda_\alpha = \partial_\mu \lambda_\alpha + g[A_\mu, \lambda_\alpha]$$

enter.

One sees easily that

$$S_2 = -\frac{1}{2} \int d^4x \, d^2\theta \, \text{tr} W^\alpha W_\alpha = \int d^4x \, \text{tr}(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} - i\lambda\sigma^\mu\nabla_\mu\bar{\lambda} + \frac{D^2}{2})$$

is a $\mathcal{N}{=}1$ SUSY and super-gauge invariant action. Moreover, re-scaling $gV \to V$ and defining the complex gauge coupling $\tau = \frac{\Theta}{2\pi} + i\frac{4\pi}{g^2}$ in terms of the instanton angle Θ, one writes S_2 in the form:

$$
\begin{aligned}
S_2 &= \frac{\text{Im}\tau}{16\pi} \int d^4x \, d^2\theta \, \text{tr} W^\alpha W_\alpha \\
&= \frac{1}{g^2} \int d^4x \, \text{tr}(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu\nabla_\mu\bar{\lambda} + \frac{D^2}{2}) + \frac{\Theta}{32\pi^2} \int d^4x \, F_{\mu\nu}\tilde{F}^{\mu\nu}
\end{aligned}
$$

The minimal coupling principle -which dictates the gauge interactions between charged fields by changing derivatives by covariant derivatives- is generalized to the super-symmetric world by replacing S_0 by

$$
\begin{aligned}
S_3 &= \frac{1}{4g^2} \int d^4x \, d^2\theta \, d^2\bar{\theta} \, \text{tr} \, \Phi^\dagger e^{-2V} \Phi = \int d^4x \, \text{tr}(|\nabla_\mu\phi|^2 - i\bar{\psi}\bar{\sigma}^\mu\nabla_\mu\psi + F^\dagger F) \\
&\quad - \int d^4x \, \text{tr}(\phi^\dagger[D, \phi] - \sqrt{2}i(\phi^\dagger\{\lambda, \psi\} - \bar{\psi}[\bar{\lambda}, \phi])
\end{aligned}
$$

and adding S_3 to the S_1 and S_2 actions. Thus, the action of the so called $\mathcal{N} = 1$ SUSY minimal gauge theory is:

$$
\begin{aligned}
S_3 + S_2 + S_1 &= \frac{1}{4g^2} \int d^4x \, d^2\theta \, d^2\bar{\theta} \, \text{tr} \, \Phi^\dagger e^{-2V} \Phi + \frac{\text{Im}\tau}{16\pi} \int d^4x \, d^2\theta \, \text{tr} \, W^\alpha W_\alpha \\
&\quad + \int d^4x \, [\int d^2\theta \mathcal{W}(\Phi) + \int d^2\bar{\theta}\bar{\mathcal{W}}(\Phi^\dagger)]
\end{aligned}
\tag{27}
$$

3.2 $\mathcal{N}$=2 supersymmetric Yang-Mills theory

3.2.1 $\mathcal{N} = 2$ supersymmetry and holomorphy

Four dimensional $\mathcal{N} = 2$ supersymmetric theories are based on the extended super-space $\mathbb{R}^{4|(4,4)}$ where the odd dimensions are parametrized by two sets of Weyl spinor pairs: θ_α^I, $\bar{\theta}_I^{\dot\alpha}$, $I = 1, 2$. Sometimes we shall also use the notation: $\theta_\alpha^1 = \theta_\alpha$, $\theta_\alpha^2 = \tilde{\theta}_\alpha$, $\bar{\theta}_1^{\dot\alpha} = \bar{\theta}^{\dot\alpha}$, $\bar{\theta}_2^{\dot\alpha} = \tilde{\bar{\theta}}^{\dot\alpha}$, in order to compare the $\mathcal{N} = 2$ with the $\mathcal{N} = 1$ theory.

There are accordingly, two pairs of $\mathcal{N} = 2$ super-charges, vector fields that generate super-translations in $\mathbb{R}^{4|(4,4)}$,

$$Q_\alpha^I = \frac{\partial}{\partial\theta_I^\alpha} - i\sigma_{\alpha\dot\alpha}^\mu \bar{\theta}^{\dot\alpha I}\partial_\mu \quad , \qquad \bar{Q}_{\dot\alpha I} = -\frac{\partial}{\partial\bar{\theta}^{\dot\alpha I}} + i\theta_I^\alpha \sigma_{\alpha\dot\alpha}\partial_\mu,$$

and close the super-algebra:

$$\{Q_\alpha^I, \bar{Q}_{\dot\alpha J}\} = 2\sigma_{\alpha\dot\alpha}^\mu \hat{P}_\mu \delta_J^I$$

$$\{Q_\alpha^I, Q_\beta^J\} = 0, \qquad \{\bar{Q}_{I\dot\alpha}, \bar{Q}_{J\dot\beta}\} = 0 \quad .$$

An $\mathcal{N} = 2$ chiral super-field $\Psi(x, \theta^I, \bar{\theta}_I$ is an $\mathcal{N} = 2$ super-field which satisfies the constraints: $\bar{D}_I^{\dot\alpha}\Psi(x, \theta^I, \bar{\theta}_I) = 0$. Because $\bar{D}_I^{\dot\alpha}\tilde{y}^\mu = 0$, one obtains easily the general solution of the constraints as a function of $\tilde{y}^\mu = y^\mu + i\tilde{\theta}\sigma^\mu\tilde{\bar{\theta}}$:

$$\Psi(\tilde{y}, \theta) = \Phi(\tilde{y}, \theta) + \sqrt{2}\tilde{\theta}^\alpha W_\alpha(\tilde{y}, \theta) + \tilde{\theta}^\alpha\tilde{\theta}_\alpha G(\tilde{y}, \theta)$$

Here, $\Phi(\tilde{y}, \theta)$ and $G(\tilde{y}, \theta)$ are $\mathcal{N} = 1$ chiral super-fields whereas $W_\alpha(\tilde{y}, \theta)$ is an $\mathcal{N} = 1$ chiral spinor super-field. We choose for Φ and W_α the $\mathcal{N} = 1$ chiral super-fields of SUSY Yang-Mills theory discussed in the previous subsection. Moreover, we impose the extra reality constraints $D^{I\alpha}D_\alpha^J\Psi = \bar{D}_I^{\dot\alpha}\bar{D}_{J\dot\alpha}\Psi^\dagger$, which allow to solve for the auxiliary field G in terms of Φ and V:

$$G(\tilde{y}, \theta) = -\frac{1}{2}\int d^2\bar{\theta} \, [\Phi(x^\mu + i\tilde{\theta}\sigma^\mu\tilde{\bar{\theta}}, \theta, \bar{\theta})]^\dagger \exp[-2V(x^\mu + i\tilde{\theta}\sigma^\mu\tilde{\bar{\theta}}, \theta, \bar{\theta})]$$

Thus, all the basic field of minimal SUSY Yang-Mills are encompassed in the $\mathcal{N} = 2$ chiral super-field Ψ and shown - together with their spins- in the next array:

	$s = 1$	$s = 1/2$	$s = 0$
W_α	A_μ	λ_α	
Φ		ψ_α	ϕ

Note that the action

$$\begin{aligned} S &= \mathrm{Im}[\frac{\tau}{16\pi}\int d^4x \, d^2\theta \, d^2\tilde{\theta} \, \frac{1}{2}\mathrm{tr}\Psi^2] \\ &= \mathrm{Im}\frac{\tau}{16\pi}\int d^4x \, \mathrm{tr}[\int d^2\theta \, W^\alpha W_\alpha + \int d^2\theta \, d^2\bar{\theta} \, \Phi^\dagger e^{-2V}\Phi] \end{aligned}$$

is invariant under the $\mathcal{N} = 2$ super-translations. Therefore, the choice of equal coefficients for the kinetic terms of the $s = \frac{1}{2}$ fields, $\bar{\lambda}\bar{\sigma}^\mu\nabla_\mu\lambda$ $\&$ $\bar{\psi}\bar{\sigma}^\mu\nabla_\mu\psi$, and the prohibition of superpotentials in the $\mathcal{N} = 1$ minimal SUSY Yang-Mills action leads to the admission of extended $\mathcal{N} = 2$ supersymmetry.

Forgetting about renormalizability we can write the most general action, invariant under $\mathcal{N} = 2$ super-translations, for a chiral $\mathcal{N}=2$ super-field in the form:

$$
\begin{aligned}
S &= \frac{1}{16\pi}\text{Im} \int d^4x \, d^2\theta \, d^2\tilde{\theta} \, \mathcal{F}(\Psi) \\
&= \frac{1}{16\pi}\text{Im} \int d^4x \, [\int d^2\theta \mathcal{F}_{ab}(\Phi)W^{\alpha a}W_\alpha^b + \int d^2\theta d^2\bar{\theta}(\Phi^\dagger e^{-2V})^a \mathcal{F}_a(\Phi)]
\end{aligned}
\tag{28}
$$

The holomorphic functional $\mathcal{F}(\Psi)$ is called the prepotential and we have a $\mathcal{N} = 2$ gauged super-symmetric sigma model where $\mathcal{F}_a \Phi^a = \frac{\partial \mathcal{F}}{\partial \Phi^a}\Phi^a$ is the Kahler potential and $\mathcal{F}_{ab} = \frac{\partial^2 \mathcal{F}}{\partial \Phi^a \Phi^b}W^{\alpha a}W_\alpha^b$ is the gauge kinetic function. Thus, $g_{ab} = \text{Im}\frac{\partial^2 \mathcal{F}}{\partial \Phi^a \Phi^b}$ can be interpreted as a metric -of special Kahler type- in the adjSU(2) target space.

Obviously,

$$
\mathcal{F}_{YM}(\Psi) = \frac{1}{2} \, \text{tr} \, \tau \Psi^2 \qquad \text{(classically)}
$$

is the right prepotential in the weak g-coupling limit, the classical limit for the basic fields above. Seiberg and Witten in an spectacular paper, [12], posed the following question: what is $\mathcal{F}_{YM}^{quantum}(\Psi)$? How looks the prepotential for stronger values of g (low energies) where quantum effects on the basic fields are important? Holomorphy/$\mathcal{N} = 2$ supersymmetry and electric-magnetic duality guided to these authors to provide a most beautiful answer.

3.2.2 Classical vacuum moduli space: super-symmetric Higss mechanism

The search for super-solutions begins by integrating out the auxiliary fields:

$$
S_{aux} = \frac{1}{g^2} \int d^4x \, \text{tr}(\frac{D^2}{2} - \phi^\dagger[D, \phi] + F^\dagger F) = -\int d^4x \, \frac{1}{2g^2}\text{tr}[\phi^\dagger, \phi]^2 \quad .
\tag{29}
$$

The homogeneous super-solutions are super-fields without soul - no dependence on Grassman variables- which are zeroes of the Higgs potential: $U(\phi) = \frac{1}{2}\text{tr}[\phi^\dagger, \phi]^2$. We shall restrict ourselves to the gauge group G = SU(2) case. Then, it is obvious that

$$
\phi_0 = \frac{1}{2}\tau^3 a \qquad , \quad a \in \mathbb{C}
$$

is a continuous set of zeroes of U. The "classical vacuum orbit" $\mathcal{V}$ -the manifold of all the zeroes of U- is:

$$
\begin{aligned}
\mathcal{V} &\equiv ad \, G(\phi_0) \times \mathbb{C}^* = \frac{\text{SU}(2)}{\text{U}(1)}(\simeq \text{S}^2) \times \mathbb{C}^* \quad , \quad \text{if} \quad a \in \mathbb{C}^* = \mathbb{C} - \{0\} \\
\mathcal{V} &\simeq \text{point} \quad , \quad \text{if} \quad a = 0.
\end{aligned}
$$

The classical vacuum moduli space $\mathcal{M}$ is the quotient of $\mathcal{V}$ by the action of adG: $\mathcal{M} \equiv \frac{\mathcal{V}}{\text{adG}} = \mathbb{C}$.

We look now for the plane super-wave solutions becoming fundamental quanta in the quantum perspective. The vacuum degeneracy induces a super-symmetric Higgs mechanism which is read from the quadratic terms in the expansion of the action around a point in $\mathcal{M}$ for the basic fields $A_\mu, \lambda_\alpha, \psi_\alpha$, and ϕ. Small fluctuations of the scalar field of the form $\phi(x) = \phi_0 + \eta(x)$ give three types of quadratic terms in the expansion of the Lagrangian:

- From $\frac{1}{g^2}[A_\mu, \phi^\dagger][A^\mu, \phi] \cong \frac{1}{g^2}[A_\mu, \phi_0^\dagger][A^\mu, \phi_0] + o(\eta)$, we see that the vector field plane wave solutions are of two types: there is no quadratic term for A_μ^3 and the corresponding plane waves propagate at the speed of light: $m_3^2 = 0$. The plane waves of the linear combination of vector fields $W_\mu^\pm = A_\mu^1 \pm i A_\mu^2$ carry energy $m_{W^\pm}^2 = |a|^2 g^2$ - after re-scaling back $\Psi \to g\Psi$- in the zero momentum limit.

- $\frac{1}{2g^2}\mathrm{tr}[\phi^\dagger, \phi]^2 \simeq \frac{1}{2}[\phi_0^\dagger, \eta][\eta^\dagger, \phi_0] + o(\eta^3)$, and the situation is identical for the scalar field. There are two "massive" field combinations, $\eta^\pm = \eta^1 \pm i\eta^2$ with masses $m_{\eta_\pm}^2 = |a|^2 g^2$, and one "massless" scalar field η^3: $m_{\eta^3}^2 = 0$.

- The mass terms for the fermions are more difficult to elucidate, but from

$$\frac{i\sqrt{2}}{g^2}\mathrm{tr}\left[\phi^\dagger\{\psi, \lambda\} - \{\bar{\psi}, \bar{\lambda}\}\phi\right] \cong \frac{i\sqrt{2}|a|}{2g^2}\left[e^{-i\omega}(\psi^1\lambda^2 + \lambda^1\psi^2) - e^{i\omega}(\bar{\psi}^1\bar{\lambda}^2 + \bar{\lambda}^1\bar{\psi}^2)\right]$$

it is not difficult to see that the field combinations $\lambda_\pm = \frac{1}{2}(\psi^2 \pm e^{-i\omega}\lambda^1)$ and $\psi_\pm = \frac{1}{2}(\psi^1 \pm e^{-i\omega}\lambda^2)$ have masses $m_{\lambda_\pm} = m_{\psi_\pm} = |a|g$, whereas λ^3 and ψ^3 are massless: $m_{\lambda^3} = m_{\psi^3} = 0$.

The particle spectrum arising from this analysis is summarized in the following arrays, which show the basic fields and the associated quanta in the physics folklore nomenclature.

A. Massless $\mathcal{N} = 2$ super-multiplet:

$s = 1$	$s = 1/2$	$s = 0$
A_μ^3 (photon)	λ_α^3 (photino)	
	ψ_α^3 (Higgsino)	η^3 (complex Higgs boson)

B. Two massive (but short) $\mathcal{N} = 2$ supermultiplets:

$s = 1$	$s = 1/2$	$s = 0$
$W_\mu^\pm$ (massive vector bosons)	$\lambda_\alpha^\pm$ (gauginos)	
	$\psi_\alpha^\pm$ (goldstino)	$\eta^\pm$ (complex Goldstone bosons)

It is amazing to realize that the would be massive Higgs particle in the usual (bosonic) mechanism remains massles in the super-symmetric version of spontaneous gauge symmetry breaking. Also, the Goldstone (massless) bosons, hidden in the gauge fields grown massive, are themselves massive in the super-symmetric theory.

The choice of a vacuum to quantize leads to a particle spectrum organized in $\mathcal{N} = 2$ super-multiplets and $\mathcal{N} = 2$ super-symmetry is unbroken. The degeneracy in the particle spectrum with respect to the gauge $G = SU(2)$ group is lost after the vacuum's choice and this symmetry is spontaneously broken to $H = U(1)$. There is an important CAVEAT: singularities arise at $a = 0$ and many more particles are massless.

3.2.3 Monopoles and dyons

The search for super-solitons is identical to the search performed in $\mathcal{N} = 1$ supersymmetric (1+1)-dimensional field theory, albeit analytically more involved. The energy for static purely bosonic configurations can also be arranged a la Bogomolny:

$$E_B = \int d^3x \left[\frac{1}{2}\mathrm{tr}\vec{B}\vec{B} + \frac{1}{2}\mathrm{tr}\vec{D}\phi^\dagger\vec{D}\phi + \frac{1}{2}\mathrm{tr}[\phi, \phi^\dagger]\right]$$

$$\geq \int d^3x \frac{1}{2}[\mathrm{tr}\vec{B}\vec{B} + \mathrm{tr}\vec{D}\phi^\dagger\vec{D}\phi] = \frac{1}{2}\int d^3x \, \mathrm{tr}\{(\vec{B} - \vec{D}\phi^\dagger)(\vec{B} - \vec{D}\phi)\} + \frac{a}{2}(Q_m + Q_m^\dagger)$$

where
$$Q_m = \int_{S_\infty^2} \vec{b} \cdot d\vec{S} = \frac{1}{a} \int_{S_\infty^2} \text{tr}(\phi \vec{B}) \cdot d\vec{S} = \frac{1}{a} \int d^3x \, \text{tr}(\vec{B} \cdot \vec{D}\phi)$$

is the field configuration magnetic charge.

If $[\phi, \phi^\dagger] = 0$ the Bogomolny bound is saturated by field configurations which solve the PDE system:

$$\vec{B} = \vec{D}\phi \qquad . \tag{30}$$

There is a fairly complete knowledge of the moduli space of solutions of (30), see [14]-[15]. We simply collect three important results:

1. Every finite energy solution of (30) belong to a topological sector characterized by the winding number of the map from the boundary of space to the vacuum orbit, provided by the behaviour of the Higss field at infinity: $\vec{\phi}|_\infty : S^2 \simeq \partial \mathbb{R}^3 \longrightarrow \mathcal{V} \equiv S^2$.

2. The finite energy solutions are solitons and for topological charge equal to 1 the BPS magnetic monopole is found:

$$\phi_1^a(\vec{x}) \frac{\tau^a}{2} = a_1 f(r) \frac{x^a}{r} \frac{\tau^a}{2}, \quad \phi_2^a(\vec{x}) \frac{\tau^a}{2} = a_2 f(r) \frac{x^a}{r} \frac{\tau^a}{2}, \quad A_i^a(\vec{x}) = \frac{1}{g} A(r) \varepsilon_{ij}^a \frac{x^j}{r^2} \frac{\tau^a}{2}$$

$$f(r) = \frac{\cosh r}{\sinh r} - \frac{1}{r} \qquad ; \qquad A(r) = 1 - \frac{r}{\sinh r} \quad .$$

ϕ_1^a and ϕ_2^a are respectively the real and imaginary part of ϕ^a, $a = a_1 + ia_2$ sets the Higgs field vacuum value, and this solution shows a magnetic monopole centered at the origin. Its orbit by the action of the translation group and the unbroken $U(1)$ subgroup of the gauge group is the moduli space of magnetic monopoles with topological charge equal to 1.

3. In general, for any finite energy solution of (30), the energy in the center of mass is:

$$E_M = M_M = |aQ_m| = \frac{4\pi}{g} |n_m a|$$

$n_m \in \mathbb{Z}$ is the winding number of the $S_\infty^2 \to S^2$ map and, thus, the magnetic charge is quantized.

In fact, it is possible invoke electric-magnetic duality to understand (30) as the $\alpha = 0$ member of the PDE family:

$$\vec{B} = \cos\alpha \vec{D}\phi \qquad \vec{E} = \sin\alpha \vec{D}\phi. \tag{31}$$

The solutions of (31) have also electric charge, although it is not quantized at this level. The dyon mass is

$$E_D = M_D = |a|(|Q_e|^2 + |Q_m|^2)^{\frac{1}{2}}$$

and thus, it is given in terms of the magnetic and electric charges:

$$Q_e = \int_{S_\infty^2} \vec{e} \cdot d\vec{S} = \frac{1}{a} \int_{S_\infty^2} \text{tr}(\phi \vec{E}) \cdot d\vec{S}$$

3.2.4 The lattice of classical BPS charges

In the framework of $\mathcal{N} = 2$ super-symmetry we notice that the Hamiltonian functions associated to the vector fields Q^I are,

$$\tilde{Q}^I = -\frac{i}{g^2} \int d^3x \{\vec{\sigma}\sigma_2 \mathrm{tr}\psi^{\dagger I}(i\vec{E}+\vec{B})+\sqrt{2}\varepsilon^{IJ}\mathrm{tr}\ \psi^J \nabla_0\phi^\dagger + \sqrt{2}\vec{\sigma}\ \mathrm{tr}\ \vec{\nabla}\phi^\dagger \varepsilon^{IJ}\psi^J + i\sigma_2\ \mathrm{tr}\ \psi^{\dagger I}[\phi^\dagger, \phi]\} \quad (32)$$

where we denote by $\psi = \psi^1, \lambda = \psi^2$ the two spinor fields, explicitly showing that ψ and λ form a $SU(2)_I$-doublet. The super-Poisson algebra

$$\{\tilde{Q}^I_\alpha, \bar{\tilde{Q}}_{\dot{\alpha}J}\} = 2\sigma^\mu_{\alpha\dot{\alpha}}\tilde{P}_\mu\delta^I_J \quad , \quad \{\tilde{Q}^I_\alpha, \tilde{Q}^J_\beta\} = \varepsilon_{\alpha\beta}Z^{IJ}, \qquad \{\bar{\tilde{Q}}_{I\dot{\alpha}}, \bar{\tilde{Q}}_{J\dot{\beta}}\} = \varepsilon_{\dot{\alpha}\dot{\beta}}Z^*_{IJ}$$

admits a complex central charge: $Z^{IJ} = 2\ \varepsilon^{IJ}Z = 2\ \varepsilon^{IJ}a(Q_e + iQ_m)$.

Define now

$$a_\alpha = \frac{1}{\sqrt{2}}\ [\tilde{Q}^1_\alpha + \varepsilon_{\alpha\beta}(\tilde{Q}^2_\beta)^\dagger] \quad , \qquad b_\alpha = \frac{1}{\sqrt{2}}\ [\tilde{Q}^1_\alpha - \varepsilon_{\alpha\beta}(\tilde{Q}^2_\beta)^\dagger]$$

In the center of mass reference system $\tilde{P}_0 = M$ and the non-null anti-commutators among the $a_\alpha, a^\dagger_\alpha, b_\alpha, b^\dagger_\alpha$, variables are:

$$\{a_\alpha, a^\dagger_\beta\} = 2\ (M + |Z|)\ \delta_{\alpha\beta} \quad , \quad \{b_\alpha, b^\dagger_\beta\} = 2(M - |Z|)\delta_{\alpha\beta}$$

Therefore, the classical BPS states, satisfying the relation $M = |Z|$, are organized in short multiplets of the super-symmetry algebra.

The classical BPS multiplets are, thus, uniquely characterized by their electric and magnetic charges. The magnetic charge is quantized because of topological reasons. The electric charge should comply with the Dirac-Schwinger quantization condition, see e.g. [16], in a consistent quantum theory with electric and magnetic charges. The electric charge is accordingly also quantized and the effect of the $U(1)$ anomaly produced by the $F \wedge F$- term on a magnetic charge is a net shift of the electric charge by an amount proportional to the instanton angle and the magnetic charge itself, [17]:

$$Q_e = g(n_e - \frac{\Theta}{2\pi}n_m) \quad , \qquad n_e \in \mathbb{Z} \quad .$$

Therefore, there is a lattice of dyonic classical BPS states with masses given by the beautiful formula:

$$\boxed{M_D = g|a||n_e + i\tau n_m|}$$

.

3.3 Low energy effective theory

3.3.1 Wilson effective action: quantum moduli space of vacua

$\mathcal{N} = 2$ SUSY Yang-Mills theory is renormalizable and asymptotically free. This means that the latter theory is the microscopic theory which controls the weak coupling/classical/high energy behaviour. The challenge is to elucidate the nature of the low energy effective theory.

There are two types of effective actions in quantum field theory. One is the standard generating functional $\Gamma[\phi]$ of one-particle irreducible Feynman diagrams. Momentum integration in loop-diagrams are from zero up to a UV-cutoff which is taken to infinity at the end. $\Gamma[\phi] \equiv \Gamma[\mu, \phi]$ also depends on the scale μ used to define the renormalized vertex functions. A quite different object

is the Wilsonian effective action $S_W[\mu, \phi]$. It is defined as $\Gamma[\mu, \phi]$ except that all the loop-momenta are integrated down to μ which serves as infrared cutoff.

Thus, if we choose $\mu^2 \leq g^2|a|^2$ the Wilson effective action $S_W[\mu, \phi]$ depends only on the Abelian fields which emerge from the super-symmetric Higgs mechanism as massless. Denoting these fields as $\phi = \phi^3, \psi = \psi^3, F = F^3, A_\mu = A_\mu^3, \lambda = \lambda^3, D = D^3, f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, we collect them in either (two) $\mathcal{N}=1$,

$$\varphi(y, \theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y)$$

$$v(x, \theta, \bar{\theta}) = -\theta\sigma^\mu\bar{\theta}A_\mu + i(\theta^2\bar{\theta}\bar{\lambda} - \bar{\theta}^2\theta\lambda) + \frac{1}{2}\theta^2\bar{\theta}^2 D; \quad w(y, \theta) = (-i\lambda + \theta D - i\sigma^{\mu\nu}\theta f_{\mu\nu} + \theta^2\bar{\sigma}^\mu\partial_\mu\bar{\lambda})(y)$$

or (one) $\mathcal{N}=2$, Abelian super-multiplets:

$$\chi(\tilde{y}, \tilde{\theta}) = \varphi(\tilde{y}, \theta) + \sqrt{2}\,\tilde{\theta}^\alpha w_\alpha(\tilde{y}, \theta) + \tilde{\theta}^\alpha\tilde{\theta}_\alpha G(\tilde{y}, \theta); \quad G(\tilde{y}, \theta) = -\frac{1}{2}\int d^2\bar{\theta}[\varphi(\tilde{y}, \theta, \bar{\theta})]^\dagger \exp[-2v(\tilde{y}, \theta, \bar{\theta})].$$

Moreover, the scale fixed as infrared cutoff is naturally provided by $\langle\phi\rangle = a$. There is a subtlety here: the action of the $SU(2)$ Weyl group sends a to -a which are thus the same point in $\mathcal{M}$. Therefore a good coordinate in $\mathcal{M}$ is $\langle\phi^2\rangle = a^2$. In any case, S_W depends on which point of the vacuum moduli space we choose to quantize.

The abelianization process leading to the Wilson effective action do not spoil $\mathcal{N} = 2$ super-symmetry. S_W is of the general form:

$$S_W = \frac{1}{16\pi}\text{Im}\int d^4x\, d^2\theta\, d^2\tilde{\theta}\, \mathcal{F}(\chi) = \frac{1}{16\pi}\text{Im}\int d^4x[\int d^2\theta\mathcal{F}''(\varphi)w^\alpha w_\alpha + \int d^2\theta d^2\bar{\theta}(\varphi^\dagger\mathcal{F}'(\varphi)]$$

and the Seiberg-Witten problem is the determination of the prepotential $\mathcal{F}$. One property will be crucial in achieving this goal: $\mathcal{F}$ is holomorphic in the $\mathcal{N} = 2$ chiral super-field χ and, defining the parameters in terms of the expectation value $\langle\chi\rangle$ in the vacuum of this field, is also holomorphic in the parameters of the theory.

The mathematical/physical meaning of $\mathcal{F}$ is unveiled by expanding S_W in component fields:

$$S_W^I[\mu, \varphi, w_\alpha] = \frac{\text{Im}}{4\pi}\int d^4x[\mathcal{F}''(\phi)(-\frac{1}{4})f_{\mu\nu}(f^{\mu\nu} - i\tilde{f}^{\mu\nu}) - i\mathcal{F}''(\phi)\bar{\lambda}\sigma^\mu\partial_\mu\lambda + \cdots]$$

$$S_W^{II}[\mu, \varphi, w_\alpha] = \frac{\text{Im}}{4\pi}\int d^4x[\mathcal{F}''(\phi)\partial_\mu\phi\partial^\mu\phi - i\mathcal{F}''(\phi)\psi\sigma^\mu\partial_\mu\bar{\psi} + \cdots]$$

These formulas show that the effective action is nothing but the action for a (3+1)D gauged non-linear $\mathcal{N} = 2$ SUSY sigma model where the target manifold is the vacuum moduli space: $\mathcal{M} = \mathbb{C}$. $\text{Im}\mathcal{F}''(\phi)$ determines the kinetic terms and plays the role of a metric in configuration space. For constant configurations, it defines the metric in the vacuum moduli space: $ds^2 = \text{Im}\mathcal{F}''(a)dad\bar{a} = \text{Im}\tau(a)dad\bar{a}$.

Thus, we can understand

$$\tau(a) = \frac{\Theta_{eff}(a)}{2\pi} + i\frac{4\pi}{g_{eff}^2(a)} = \mathcal{F}''(a)$$

as the running coupling constant at the $\mu^2 \propto <|\phi|^2> = |a|^2$ squared energy scale. In the classical/high energy regime we have the classical vacuum moduli space

$$\mathcal{M} = \mathbb{C} \qquad ds^2 = \text{Im}\,\tau_{cl}\, dad\bar{a} = \text{Im}\,(\frac{\Theta}{2\pi} + i\frac{4\pi}{g^2})\, dad\bar{a}$$

whereas,

$$\mathcal{M}_{quantum} = \mathbb{C} \qquad ds^2 = \text{Im}\,\mathcal{F}''(a)dad\bar{a}$$

is the quantum moduli space when quantum fluctuations become important.

3.3.2 Weak coupling regime

In a neighborhood of the infinity point in $\mathcal{M}$, the $a \to \infty$ limit, the fluctuation modes with large momentum p dominate S_W. Asymptotic freedom tells us that large momentum amounts to weak coupling and we can rely on perturbation theory in this regime. The evaluation of the perturbation theory contribution to the prepotential can be performed through the analysis of the classical global symmetry of the theory: $U(2) \simeq U(1)_C \otimes SU(2)_I$. The $U(1)_C$ sub-group is a R-transformation with R-character equal to 2:
$U(1)_C : \Phi \longrightarrow e^{2i\alpha}\Phi(e^{-i\alpha}\theta)$. $\theta, \tilde{\theta}$ form a doublet under $SU(2)_I$ -and so do $\psi = \psi^1, \lambda = \psi^2$- but the $U(1)_I$ sub-group has R-character equal to 0: $U(1)_I : \Phi \longrightarrow \Phi(e^{-i\alpha}\theta)$. In perturbation theory of the microscopic Yang-Mills theory the fermionic triangle diagrams show that $\partial_\mu J_5^\mu = -\frac{1}{4\pi}\mathrm{Tr}F_{\mu\nu}\tilde{F}^{\mu\nu}$ and the $U(1)_C$ symmetry is anomalous.

Integration of the fast quantum fluctuations lead to the effective Abelian theory that, accordingly, should include a term of the form :$\delta\mathcal{L}_{eff} = -\frac{\alpha}{4\pi^2}f_{\mu\nu}\tilde{f}^{\mu\nu}\cdots$. Thus,

$$\mathcal{L}_W^I = \frac{1}{16\pi}\mathrm{Im}[\mathcal{F}''(\varphi)(-f_{\mu\nu}f^{\mu\nu} + if_{\mu\nu}\tilde{f}^{\mu\nu})] - \frac{\alpha}{4\pi^2}f_{\mu\nu}\tilde{f}^{\mu\nu} + \cdots \tag{33}$$

where α is the parameter of the $U(1)_C$ transformation. Is this term encoded in the prepotential? The action of a $U(1)_C$ transformation on $\mathcal{L}_W^I$ reads:

$$\mathcal{L}_W^I(e^{2i\alpha}\varphi) = \frac{1}{16\pi}\mathrm{Im}[\mathcal{F}''(e^{2i\alpha}\varphi)(-f_{\mu\nu}f^{\mu\nu} + if_{\mu\nu}\tilde{f}^{\mu\nu})] + \cdots \tag{34}$$

Comparing (33) and (34), we find:

$$\mathcal{F}''(e^{2i\alpha}\varphi) = \mathcal{F}''(\varphi) - \frac{4\alpha}{\pi} \quad \equiv \quad \mathcal{F}'''(\varphi) = \frac{2i}{\pi\varphi} \quad .$$

The integration of the last expression is elementary: in terms of the integration constant Λ^2 we obtain $\mathcal{F}_{pert}(\varphi) = \frac{i}{2\pi}\varphi^2 \ln\frac{\varphi^2}{\Lambda^2}$. Therefore, near the infinity point in $\mathcal{M}$ the prepotential including one-loop effects is:

$$\mathcal{F}_{pert}(\chi) = \frac{i}{2\pi}\chi^2 \ln\frac{\chi^2}{\Lambda^2} \quad .$$

From $\mathcal{F}(a) = \frac{i}{2\pi}a^2 \ln\frac{a^2}{\Lambda^2}$ one easily calculates the effective coupling constant $\tau(a) = \frac{i}{\pi}(\ln\frac{a^2}{\Lambda^2} + 3)$. Then,

$$\frac{4\pi^2}{g^2(a)} = \ln\frac{|a|^2}{\Lambda^2} + 3 \tag{35}$$

and the effective beta function -$\frac{dg(a)}{d\ln|a|} = \beta(g)$- is: $\beta(g) = -\frac{1}{4\pi^2}g^3$. Λ^2 is nothing but the dynamically generated by quantum effects renormalization-invariant scale: defining the renormalization point as $\mu^2 = e^3|a|^2$ we have: $\Lambda^2 = \mu^2 \exp\{-\frac{4\pi^2}{g^2(\mu)}\}$. $|a|^2 = \infty$ is the weak coupling renormalization scale, $\mu^2 = \Lambda^2$ is the strong coupling scale: $g(\Lambda^2) = \infty$.

There are anti-self-dual gauge connections in the Euclidean version of the theory. Solutions of the self-duality equations $F_{\mu\nu}^E = -\tilde{F}_{\mu\nu}^E$ give rise to instantons in the parent Minkowskian theory, which interpolate between pure gauge configurations with different winding number at $t = -\infty$ and $t = \infty$. Standard instanton physics assigns an Euclidean action of :

$$\exp(-\frac{8\pi^2 k}{g^2(a)}) = e^{-6k}(\frac{\Lambda}{a})^{4k} \tag{36}$$

to a self-dual connection with Chern number k. In (36) we have used the (complexified) equation (35) to write the instanton action in terms of Λ and a, and followed Seiberg, [18], in assigning an R-charge of 2 to Λ.

Weyl-Dirac operators acting on sections of second Chern class k=1 spinor bundles have 8 zero modes according to the Atiyah-Singer index theorem. Thus, Berezin integration in Euclidean fermionic correlation functions in the instanton background provides a non-zero answer only for 4 λ- and 4 ψ-spinors. Such a correlator G changes to $e^{i8\alpha}G$ under $U(1)_C$ transformation. This argument survives analytic continuation to Minkowskian correlations and the $U(1)_C$ symmetry is broken to a $\mathbb{Z}_8$ sub-group by tunneling via one instanton. Therefore, the prepotential taking into account this effect must be of the form $\mathcal{F}_{instanton} = \mathcal{F}_1(\frac{\Lambda}{\varphi})^4\varphi^2$ -only the arbitrary constant $\mathcal{F}_1$ coefficient is left to fix-, because, then, $\mathcal{F}_{instanton} = \mathcal{F}_1(\frac{\Lambda}{\varphi})^4\varphi^2 e^{4i\alpha} = e^{4i\alpha}\mathcal{F}_{instanton}(\varphi)$.

One can check that the $\mathbb{Z}_8$ sub-group is also a quantum symmetry of $\mathcal{F}_{pert}$:

$$\mathcal{F}_{pert} \to e^{4i\alpha}(\frac{i}{2\pi}\chi^2 \ln \frac{\chi^2}{\Lambda^2} - \frac{2\alpha}{\pi}\chi^2) \quad .$$

Then,

$$\frac{2\alpha}{\pi}\frac{1}{16}\mathrm{Im}\int d^4x d^2\theta \mathrm{Tr}W^\alpha W_\alpha = -\frac{\alpha}{4\pi^2}\int d^4x \mathrm{Tr}F_{\mu\nu}\tilde{F}^{\mu\nu} = 8\alpha k$$

and the transformation do not alter the quantum action if $\alpha = \frac{2\pi n}{8}, n \in \mathbb{Z}$. In the weak coupling regime the super-potential collects the one-loop perturbative and instanton corrections:

$$\mathcal{F}(\chi) = \frac{i}{2\pi}\chi^2 \ln \frac{\chi^2}{\Lambda^2} + \sum_{n=1}^{\infty}\mathcal{F}_n(\frac{\Lambda}{\chi})^{4n}\chi^2 \quad . \tag{37}$$

It is worthwhile to mention that $\mathcal{N} = 2$ super-symmetry forbids any further corrections to the perturbative β-function.

One might wonder about whether or not the description developed for large values of a -in terms of the Φ, W and $\mathcal{F}$ is also appropriate in other regions of $\mathcal{M}$, e.g. around the origin. Because $\mathcal{F}(a)$ is holomorphic, $\mathrm{Im}\mathcal{F}''(a) = \mathrm{Im}\tau(a)$ is harmonic: $\partial\bar{\partial}\mathrm{Im}\tau(a) = 0$. Then, $\mathrm{Im}\tau(a) > 0$ $\forall a \in \mathcal{M} \equiv \mathbb{C}\cup\{\infty\}$ if and only if $\mathrm{Im}\,\tau(a) = $ constant (as in the classical case). But $\tau(a)$ is non constant, at least near $a = \infty$.

The way out is to allow for different local descriptions: the coordinates $a, \bar{a}$ and $\mathcal{F}$ are appropriate only in a certain region of $\mathcal{M}$. From $\mathrm{Im}\tau(a) = \frac{1}{\pi}(\ln\frac{|a|^2}{\Lambda^2} + 3)$, we see that there seem to be a priori three points in $\mathcal{M}$ where another system of coordinates $\hat{a}, \bar{\hat{a}}$ could be necessary:

- $\mathrm{Im}\tau(a) \to 0$ and the effective action become non-sense if $a = \pm e^{-3/2}\Lambda$. The strong coupling limit provides two singular points.

- $\partial\bar{\partial}\ln\frac{|a|^2}{\Lambda^2} = \delta^{(2)}(a)$ and $\mathrm{Im}\tau(a)$ fails to be harmonic at the origin.

3.3.3 Electric-magnetic duality

Duality will provide a different set of (dual) fields Φ_D and W_D^α that supply an appropriate description in other region of the vacuum moduli space. How to define the duality transformation ? First, how to define a good metric in all $\mathcal{M}$?

One can write the metric in the form

$$ds^2 = \mathrm{Im}\,da_D da = -\frac{i}{2}(da_D d\bar{a} - da d\bar{a}_D)$$

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where the new coordinate is: $a_D \equiv \frac{\partial \mathcal{F}}{\partial a}$. Before of implementing this idea on the super-fields, it is convenient to pause and explain mathematically the situation. Let us introduce a complex space $\mathbb{X} \cong \mathbb{C}^2$ with coordinates a, a_D and endow $\mathbb{X}$ with the type $(1,1)$ symplectic form $w = \frac{i}{2}(da \wedge d\bar{a}_D - da_D \wedge d\bar{a})$ and also with the holomorphic two form $\omega_h = da \wedge da_D$. We describe a map $f : \mathcal{M} \to \mathbb{X}$ by functions $a(u), a_D(u)$ such that $f^*(\omega_h) = 0$. This ensures that locally, if we pick $u = a$, $a_D = \frac{\partial \mathcal{F}}{\partial a}$ with some holomorphic function $\mathcal{F}$. The metric in $\mathcal{M}$ is the one whose Kahler form is $f^*(\omega)$. We must, however, recall that a good physical parameter is provided by the choice $u = \text{Tr}\langle \phi^2 \rangle$ in the high energy regime.

The parallel duality transformation on the $\mathcal{N} = 1$ chiral super-field is easy to write:

$$\varphi_D \equiv \mathcal{F}'(\varphi) = \frac{d\mathcal{F}}{d\varphi} \qquad ; \qquad \mathcal{F}'_D(\varphi_D) \equiv -\varphi \quad .$$

One immediately realizes the invariance of the abelian effective theory

$$\text{Im} \int d^4x \, d^2\theta \, d^2\bar{\theta} \, \varphi^\dagger \mathcal{F}'(\varphi) = \text{Im} \int d^4x \, d^2\theta \, d^2\bar{\theta} \, (-\mathcal{F}'_D(\varphi_D))^\dagger \varphi_D = \text{Im} \int d^4x \, d^2\theta \, d^2\bar{\theta} \, \varphi_D^\dagger \mathcal{F}'_D(\varphi_D)$$

against this ("complex") canonical transformation:

$$\mathcal{F}_D(\varphi_D) = \mathcal{F}(\varphi) - \varphi \varphi_D$$

The duality transformation on W_α is <u>non-local</u>. The Bianchi identity $df = 0$ for the abelian gauge field $f_{\mu\nu}$ is tantamount in the super-symmetric framework to the constraint: $\text{Im}(D_\alpha w^\alpha) = 0$. We can trade integration on the v super-field by integration on the w super-field in the quantum action provided that the constraint is enforced by a Lagrange multiplier super-field v_D:

$$\int \mathcal{D}v \exp[\frac{i}{16\pi}\text{Im} \int d^4x \, d^2\theta \mathcal{F}''(\varphi)w^\alpha w_\alpha]$$

$$\simeq \int \mathcal{D}w \mathcal{D}v_D \exp[\frac{i}{16\pi}\text{Im} \int d^4x \, d^2\theta \mathcal{F}''(\varphi)w^\alpha w_\alpha + \frac{i}{32\pi}\text{Im} \int d^4x \, d^2\theta \, d^2\bar{\theta} \, v_D D_\alpha w^\alpha]$$

Observe that

$$\int d^2\theta \, d^2\bar{\theta} v_D D_\alpha w^\alpha = -\int d^2\theta \, d^2\bar{\theta} D_\alpha v_D w^\alpha + \int d^2\theta \, \bar{D}^2(D_\alpha v_D w^\alpha)$$

$$= \int d^2\theta(\bar{D}^2 D_\alpha v_D)w^\alpha = -4 \int d^2\theta w_{D\alpha} w^\alpha$$

where we used $\bar{D}_{\dot{\beta}} w^\alpha = 0$ and where the dual w_D is defined from v_D as $w_{D\alpha} \equiv -\frac{1}{4}\bar{D}^2 D_\alpha v_D$. Then one can do the functional integral over w_α- a Gaussian- and one obtains:

$$\int \mathcal{D}v_D \exp[\frac{i}{16\pi}\text{Im} \int d^4x \, d^2\theta(-\frac{1}{\mathcal{F}''}(\varphi)w_D^\alpha w_{D\alpha})]$$

This formula re-expresses the $\mathcal{N} = 1$ super-symmetric abelian effective gauge theory in terms of the dual field w_D^α with the effective coupling $\tau(a) = \mathcal{F}''(a)$ replaced by $-\frac{1}{\tau(a)}$. Moreover, because $\mathcal{F}''_D = -\frac{d\varphi}{d\varphi_D} = -\frac{1}{\mathcal{F}''(\varphi)}$, we can define $\tau_D(a_D) \underset{\text{def}}{\equiv} \mathcal{F}''(a_D) = -\frac{1}{\tau(a)}$, and write the effective action in terms of the dual fields in a completely symmetric way:

$$\frac{1}{16\pi}\text{Im} \int d^4x \, d^2\theta \, d^2\tilde{\theta} \, \mathcal{F}_D(\chi_D) = \frac{1}{16\pi}\text{Im} \int d^4x[\int d^2\theta \mathcal{F}''_D(\varphi_D)w_D^\alpha w_{D\alpha} + \int d^2\theta d^2\bar{\theta}(\varphi_D^\dagger \mathcal{F}'(\varphi_D)])$$

(38)

From the Berezin integration of the constraint term

$$\frac{i}{32\pi}\text{Im}\int dx^4 d\theta^2 d\bar\theta^2 v_D D_\alpha w^\alpha = \frac{1}{8\pi}\int dx^4 v_D^\mu \partial^\nu \tilde f_{\mu\nu} = \frac{1}{8\pi}\int dx^4 \tilde f_D^{\mu\nu} f_{\mu\nu}$$

one realizes that v_D couples to magnetic -rather than electric- charge -for a magnetic monopole $\partial_\mu \tilde f^{0\mu} = 8\pi\delta^{(3)}(x)$-. Therefore, v_D is the connection associated to the $U(1)_D$ sub-group of the $SU(2)_D$ dual Lie group to the $SU(2)$ gauge group. The duality transformation is an electric-magnetic transformation analogous to the duality symmetry -$f_{\mu\nu} \to \tilde f_{\mu\nu}$- of the Maxwell equations in the vacuum. To keep EM duality at work when sources are added to the Maxwell equations requires also the transformations $q_e \to -q_m$, $q_m \to q_e$ among electric and magnetic charges. In our case $\tau(a) \to \tau_D(a_D)$ does the job and there is a parallel weak coupling/strong coupling transformation. It seems thus, highly plausible that the right effective action in the strong coupling regions of the moduli space is (38). A warning: unlike electromagnetism, this statement means that there is no symmetry of the abelian effective theories with respect to EM duality. We will come back to this point later.

3.3.4 $SL(2,\mathbb{Z})$ invariance: the lattice of quantum BPS charges

The exploration of such a possibility calls for the analysis of the full duality transformations group of the Abelian effective action that, for this purpose, it is convenient to write it as:

$$S_W = \frac{1}{16\pi}\text{Im}\int d^4x\, d^2\theta\, \frac{d\varphi_D}{d\varphi} w^\alpha w_\alpha + \frac{1}{32\pi i}\int d^4x\, d^2\theta\, d^2\bar\theta\, (\varphi^\dagger \varphi_D - \varphi_D^\dagger \varphi) \tag{39}$$

Besides the S-duality transformation

$$\begin{pmatrix} \varphi_D \\ \varphi \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \varphi_D \\ \varphi \end{pmatrix}$$

there is also a T-duality transformation,

$$\begin{pmatrix} \varphi_D \\ \varphi \end{pmatrix} \to \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \varphi_D \\ \varphi \end{pmatrix} = T(b)\begin{pmatrix} \varphi_D \\ \varphi \end{pmatrix} \quad , \quad b \in \mathbb{Z}$$

which is a true symmetry of the effective action. The second term in (39) is T-invariant because b is real and the first term gets shifted by

$$\frac{b}{16\pi}\text{Im}\int d^4 x d^2\theta\, w^\alpha w_\alpha = -\frac{b}{16\pi}\int d^4x\, F_{\mu\nu}\tilde F^{\mu\nu} = 0$$

because there are no instantons in an abelian theory. The generators of the full group of duality transformations are: $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. A generic element of the $SL(2,\mathbb{Z})$ duality group, $\mathcal{D}(m,n,p,q)$, $m,n,p,q \in \mathbb{Z}$, $mp - qn = 1$, acts in the form:

$$\begin{pmatrix} \varphi_D' \\ \varphi' \end{pmatrix} = \begin{pmatrix} m & n \\ p & q \end{pmatrix}\begin{pmatrix} \varphi_D \\ \varphi \end{pmatrix} \quad .$$

Because of

$$\mathcal{D}(n,m,p,q)[\mathcal{F}''(\varphi)] = \frac{d(m\varphi_D + n\varphi)}{d(p\varphi_D + q\varphi)} = \frac{m\frac{d\varphi_D}{d\varphi} + n}{p\frac{d\varphi_D}{d\varphi} + q} \quad ,$$

the duality action on $\tau(a)$ is a linear fractional transformation: $\tau' = \frac{m\tau+n}{p\tau+q}$.

The key observation of Seiberg and Witten is that the $\mathcal{N} = 2$ SUSY algebra is anomalous and the renormalized central charge is:

$$Z_R(u) = a(u)n_e + a_D(u)n_m = \begin{pmatrix} n_m & n_e \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix} \tag{40}$$

(if the fields and the dual fields are re-scaled by g and g_D). n_e and n_m count the number of electrically and magnetically charged states, whereas $a(u)$ and $a_D(u)$ play the rôle of the renormalized electric and magnetic charge. The rationale behind this crucial hypothesis is electric-magnetic duality: for electrically charged BPS states it is clear that $\mathbb{Z}_R = n_e a$. The dual formula $\mathbb{Z}_R = n_m a_D$ for magnetically charged BPS states may be verified through an argument á la Bogomolny. Consider the full high energy theory in its effective form (28) and examine the bosonic terms in the Hamiltonian of a magnetic monopole:

$$E = \frac{1}{4\pi} \text{Im} \int d^3x \{\tau_{ab}(\nabla_i\phi^a)(\nabla_i\phi^b) + \frac{1}{2}\tau_{ab}\, B_i^a B_i^b\} \quad , \quad \tau_{ab} = \frac{\partial^2 \mathcal{F}}{\partial\phi^a\partial\phi^b}$$

The duality transformation $\phi^a \to \phi_{Da} = \frac{\partial \mathcal{F}}{\partial\phi^a}$ and the Bogomolny splitting implies:

$$\begin{aligned} E &= \frac{1}{4\pi}\text{Im} \int d^3x \{(-\frac{1}{\tau})^{ab}(\nabla_i\phi_D)_a(\nabla_i\phi_D)_b + \frac{1}{2}\tau_{ab}\, B_i^a B_i^b\} \\ &= \frac{1}{4\pi}\text{Im}\{\int d^3x (-\frac{1}{\tau})^{ab}[(\nabla_i\phi_D)_a \pm \frac{1}{\sqrt{2}}\tau_{ac}B_i^c][(\nabla_i\phi_D)_b \pm \frac{1}{\sqrt{2}}\tau_{bd}B_i^d]\} \\ &= \sqrt{2} \int \partial_i(B_i^a\phi_{Da})d^3x \geq |\frac{\sqrt{2}}{4\pi}\int_{\partial\mathbb{R}^3} d^2S_i\, B_i^a\phi_{Da}| = \sqrt{2}|n_m a_D| \end{aligned}$$

The lattice (40) of BPS dyonic states has built in duality symmetry and the quantum mass of these states is: $M_D^2 = 2|Z_R(u)|^2$.

The problem is thus to determine $a(u)$ and $a_D(u)$. In the weak g coupling limit, when $u \to \infty$,

$$a \simeq \sqrt{2u} \;,\; a_D \simeq \frac{i}{\pi}\sqrt{2u}\,(\ln \frac{2u}{\Lambda^2} + 1) \tag{41}$$

and $u = \infty$ is a branching point -a singularity- of both a and a_D. Singularities are characterized by their monodromies: take u around a counter-clockwise contour of very large radius in the complex u-plane, $u \to e^{2\pi i}u$. One has $a \to -a$ and

$$a_D \to -\frac{i}{\pi}\sqrt{2u}\,(\ln \frac{2e^{2\pi i}u}{\Lambda^2} + 1) = -a_D + 2a \quad .$$

This can be written in terms of the monodromy matrix:

$$\begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix} \to M_\infty \begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix} \quad , \quad M_\infty = PT(-2) = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \quad , \quad P = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Multi-valued functions with prescribed singularities and monodromies around them are unique. Thus, to find $a(u)$ and $a_D(u)$ there is the need of identifying all the singularities and their monodromies.

3.3.5 Strong coupling singularities

We start by answering the following question: How many singularities are in the renormalized electric and magnetic charges as functions of u?

To identify the branching points in the lattice of quantum charges when the theory is considered in different points of $\mathcal{M}$ we proceed in three steps.

1. Recall that the prepotential (37) in the weak coupling regime is invariant under a $\mathbb{Z}_8$ sub-group of $U(1)_C$. Under this $\mathbb{Z}_8$-symmetry $\phi \to e^{i\pi \frac{n}{2}}\phi$ and $\phi^2 \to -\phi^2$ if n odd. $u = \langle \mathrm{tr}\phi^2 \rangle$ breaks this $\mathbb{Z}_8$ invariance further to $\mathbb{Z}_4$ and there is only a $\mathbb{Z}_4$-symmetry left on a given point of the vacuum moduli space. Due to this global symmetry $u \to -u$ singularities of $\mathcal{M}$ should come in pairs: if $u = u_0$ is a singularity also $u = -u_0$ is. The only fixed points of $u \to -u$ are $u = \infty$ and $u = 0$. We know that $u = \infty$ is a singular point of $\mathcal{M}$. So if there are two singularities the other must be the fixed point $u = 0$.

   If there are only two singularities $u = \infty$ and $u = 0$ one concludes by contour deformation that: $M_0 = M_\infty$. Then,

   $$\begin{pmatrix} a'_D(0) \\ a'(0) \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a_D(0) \\ a(0) \end{pmatrix} \qquad \Rightarrow \qquad a'(0) = -a(0)$$

   and $a'^2(0) = a^2(0)$ is not affected by any monodromy. Hence, a^2 would be a good coordinate in all the quantum vacuum moduli space, which is not; the $\frac{1}{2}\langle\phi^2\rangle = 0$ point does not belong to $\mathcal{M}_{\mathrm{quantum}}$ and two singularities only cannot work.

2. The most important singularity occurs when a_D is zero. Magnetic monopoles become massless at the (ultra)-strong coupling regime $u = \Lambda^2$ and therefore,

   $$a_D(u) \simeq c_0(u - \Lambda^2) \qquad , \qquad c_0 = \mathrm{constant}$$

   in a neighborhood of this point. In the next sub-section we shall analyze the effective theory in the monopole patch of $\mathcal{M}$, but we advance that

   $$a(u) \simeq a_0 + \frac{i}{\pi}c_0(u - \Lambda^2)\ln(u - \Lambda^2) \qquad , \qquad a_0 = \mathrm{constant} \qquad (42)$$

   which is enough to unveil the monodromy matrix around this singularity:

   $$\begin{pmatrix} a_D \\ a \end{pmatrix} \to M_{\Lambda^2} \begin{pmatrix} a_D \\ a \end{pmatrix} \qquad , \qquad M_{\Lambda^2} = ST(2)S^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \quad .$$

   The monodromy transformation can also be interpreted as changing the magnetic and electric quantum numbers as

   $$(n_m, n_e) \to (n_m, n_e)M \qquad .$$

   The state of vanishing mass responsible for a singularity should be invariant under the monodromy. This is so for the magnetic monopole:

   $$(1,0)\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = (1,0)$$

   is a left eigen-vector of M_{Λ^2} with unit eigenvalue.

3. To obtain the monodromy matrix at $u = -\Lambda^2$ one observes that the counter-clockwise contour around $u = \infty$ can be deformed into a contour encircling Λ^2 and a contour encircling $-\Lambda^2$, both counter-clockwise. The factorization condition $M_\infty = M_{\Lambda^2} M_{-\Lambda^2}$ follows, and, hence

$$M_{-\Lambda^2} = (TS)T(2)(TS)^{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$$

Because $(1, -1)M_{-\Lambda^2} = (1, -1)$, this singularity corresponds to a dyon becoming massless

In summary, there are three patches in the quantum moduli space, centered around the three singularities. The appropriate variables are respectively the fields of a $\mathcal{N} = 2$ supersymmetric effective abelian gauge theory with three different $U(1)$ abelian groups: the maximal torus of the gauge group, the maximal torus of the dual to the gauge group and, the maximal torus of one diagonal subgroup of the direct product of both.

3.3.6 The Seiberg-Witten prepotential

The strategy for finding $a_D(u)$ and $a(u)$ is to consider them as the two linearly independent solutions of the Schrödinger equation

$$[-\frac{d^2}{dz^2} + V(z)]\psi(z) = 0 \quad , V(z) = -\frac{1}{4}[\frac{1-\lambda_1^2}{(z+1)^2} + \frac{1-\lambda_2^2}{(z-1)^2} - \frac{1-\lambda_1^2-\lambda_2^2+\lambda_3^2}{(z+1)(z-1)}] \quad , z = \frac{u}{\Lambda^2}$$

in the complex plane. $V(z)$ is a meromorphic function of z with second order poles at $-1, 1, \infty$ and residues $-\frac{1}{4}(1-\lambda_1^2), -\frac{1}{4}(1-\lambda_2^2), -\frac{1}{4}(1-\lambda_3^2)$. Here, $\lambda_1, \lambda_2, \lambda_3$, are constants -we assume without loss of generality $\lambda_i \geq 0$- to be fixed later according to the asymptotic behaviour of $a(z), a_D(z)$. The transformation

$$\psi(z) = (z+1)^{\frac{1}{2}(1-\lambda_1)}(z-1)^{\frac{1}{2}(1-\lambda_2)}f(\frac{z+1}{2})$$

is very useful because f satisfies the hypergeometric ODE

$$z(1-z)f''(z) + [c - (a+b+1)z]f'(z) - abf(z) = 0) \tag{43}$$

with

$$a = \frac{1}{2}(1 - \lambda_1 - \lambda_2 + \lambda_3); \quad b = \frac{1}{2}(1 - \lambda_1 - \lambda_2 - \lambda_3); \quad c = 1 - \lambda_1 \quad .$$

We choose the two independent solutions of the second-order ODE (43)

$$f_1(z) = (-z)^a {}_1F_2(a, a+1-c, a+1-b; \frac{1}{z}) \quad , \quad f_2(z) = (1-z)^{c-a-b} {}_1F_2(c-a, c-b, c+1-a-b; 1-z)$$

where ${}_1F_2(a, b, c; z)$ is the Gauss hypergeometric function. The reason for this election is that f_1 and f_2 has simple monodromy properties respectively around $z = \infty$ and $z = 1$. Hence, they are good candidates to be identified with $a(z)$ and $a_D(z)$.

To make the identification precise, we observe that when $z \to \infty$ one has $V(z) \simeq -\frac{1}{4}\frac{1-\lambda_3^2}{z^2}$. The two independent solutions behave asymptotically as $z^{\frac{1}{2}(1\pm\lambda_3)}$, if $\lambda_3 \neq 0$, and $\sqrt{z}, \sqrt{z}\ln z$, if $\lambda_3 = 0$. Comparison with (41) tells us that the latter case is realized if $a(z)$ is going to be identified with f_1. Simili modo, if $\lambda_3 = 0$, as $z \to 1$ one has $V(z) \simeq -\frac{1}{4}(\frac{1-\lambda_2^2}{(z-1)^2} - \frac{1-\lambda_1^2-\lambda_2^2}{2(z-1)})$, where the sub-leading term is kept.

From the logarithmic asymptotics of $a_D = c_0\Lambda^2(z-1)$, $a = a_0 + \frac{i}{\pi}c_0\Lambda^2(z-1)\ln(z-1)$ when $z \to 1$ one sees that $\lambda_2 = 1$ and $-\frac{\lambda_1^2}{8} = \frac{ic_0}{\pi a_0}$. The $\mathbb{Z}_2$ symmetry $u \to -u$ on the moduli space implies that, as $z \to -1$ the potential does not have a double pole either, so that $\lambda_1 = 1$ also. Therefore,

$$\lambda_1 = \lambda_2 = 1 \quad , \quad \lambda_3 = 0 \quad , \quad V(z) = -\frac{1}{4}\frac{1}{(z+1)(z-1)}$$

and $a = b = -\frac{1}{2}, c = 0$. The two solutions

$$a_D(u) = i\frac{u-\Lambda^2}{2\Lambda^2}F(\frac{1}{2},\frac{1}{2},2;\frac{\Lambda^2-u}{u}) \quad , \quad a(u) = \sqrt{\frac{2}{\Lambda^2}}(u+\Lambda^2)^{\frac{1}{2}}F(-\frac{1}{2},\frac{1}{2},1;\frac{2\Lambda^2}{u+\Lambda^2})$$

have the correct monodromies as well as the correct asymptotics. One can invert the equation on the right to find $u(a)$. To obtain $a_D(a)$ one plugs-in this result into $a_D(u)$. Integration with respect to a yield $\mathcal{F}(a)$ and, hence, the low energy theory.

In the Seiberg and Witten work, the whole picture is related to an elliptic curve. A brief summary is as follows: M_∞, M_{Λ^2} and $M_{-\Lambda^2}$ belong to the $\Gamma(2)$ sub-group of $SL(2,\mathbb{Z})$ matrices with integer mod 2 entries. If $\mathbb{H}$ is the half-plane, the quotient $\frac{\mathbb{H}}{\Gamma(2)}$ is the moduli space of elliptic curves given by the algebraic equation:

$$y^2 = (x-1)(x+1)(x-z) \qquad . \tag{44}$$

The modular forms of weight $\frac{1}{2}$ and level 2 provide a six-to-one map $\alpha : \frac{\mathbb{H}}{\Gamma(2)} \cong \mathcal{M}_q \longrightarrow \mathbb{P}^1$ from this moduli space, isomorphic to the quantum vacuum moduli space $\mathcal{M}_q$, to the projective complex line :

$$\alpha(\tau(u)) = \left(\Theta\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}(0,\tau), \Theta\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}(0,\tau)\right) \quad , \quad z = \frac{\Theta\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}(0,\tau)}{\Theta\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}(0,\tau)} \quad ,$$

see [19]. Here, we refer by τ, not the coupling constant, but the modular parameter of the elliptic curves (44) which have an structure of level 2. Thus, $\mathcal{M}_q$ is a 6-fold covering of $\mathbb{P}^1$.

$\tau = \infty$ is a cusp - a fixed point under the action of $T(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma(2)$-:

$$\alpha(\tau + 1) = \left(\Theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(0,\tau), e^{i\frac{\pi}{4}}\Theta\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}(0,\tau)\right) \quad ,$$

and having in mind that $\lim_{\mathrm{Im}\tau\to\infty} \Theta\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}(0,\tau) = 1$, $\lim_{\mathrm{Im}\tau\to\infty} \Theta\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}(0,\tau) = 0$, this point corresponds to $z = \infty$- a cuspidal curve in the family (44)-.

$N = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is other element in $\Gamma(2)$ which amounts to:

$$\alpha(-\frac{1}{\tau} \pm 1) = \left((-i\tau)^{\frac{1}{2}}\Theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(0,\tau), e^{\pm i\frac{\pi}{4}}(-i\tau)^{\frac{1}{2}}\Theta\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}(0,\tau)\right) \qquad .$$

$N(\tau = \infty) = \pm 1$ and, therefore, $\tau = \pm 1$ - or $z = \pm 1$- are new cusp points, which also characterize nodal curves in (44).

The physical meaning of this construction is related to the lattice of charges $L(u) = \mathbb{Z}a_D(u) \otimes \mathbb{Z}a(u)$. The $E(u)$ elliptic curve (44) is nothing but $E(u) = \frac{\mathbb{C}}{L(u)}$ and singular curves appear at the singularities of the lattice of BPS states. Moreover, if we denote by γ_D and γ the 1-cycles which generates the first homology group $H_1(E(u), \mathbb{Z})$ of the elliptic curve, and pick the linear combination

$$\lambda = \frac{1}{\sqrt{2}\pi} \frac{\sqrt{x-z}}{\sqrt{x^2-1}}.dx = \frac{1}{\sqrt{2}\pi} \frac{(x-z)dx}{y}$$

of the two holomorphic differentials on the curve, we have

$$a_D(u) = \oint_{\gamma_D} \lambda = \frac{\sqrt{2}}{\pi} \int_1^z \frac{dx\sqrt{x-z}}{\sqrt{x^2-1}} \qquad , \qquad a(u) = \oint_\gamma \lambda = \frac{\sqrt{2}}{\pi} \int_{-1}^1 \frac{dx\sqrt{x-z}}{\sqrt{x^2-1}} \quad .$$

Derivation with respect to the coordinate in the moduli shows the effective coupling constant as the modulus of the elliptic curve $E(u)$:

$$\frac{da_D}{dz} = \oint_{\gamma_D} \frac{d\lambda}{dz} \quad , \quad \frac{da}{dz} = \oint_\gamma \frac{d\lambda}{dz} \quad ; \quad \tau(u) = \frac{da_D/dz}{da/dz} = \frac{da_D}{da} \quad .$$

3.4 $\mathcal{N}$=2 dual SUSY QED

3.4.1   The ultra-strong limit: low energy effective theory

We now closely focus on the peculiar scaling limit $a_D(\Lambda^2) = 0$, usually called ultra-strong limit. Magnetically charged states become thus massless and the magnetically charged massless $\mathcal{N} = 2$ hyper-multiplet,

$$M(y,\theta) = \phi_m(y) + \sqrt{2}\theta\psi_m(y) + \theta^2 F_m(y) \qquad , \qquad \tilde{M}(y,\theta) = \tilde{\phi}_m(y) + \sqrt{2}\theta\tilde{\psi}_m(y) + \theta^2 \tilde{F}_m(y),$$

must be "integrated in" the low energy effective theory. The spin of the physical fields, two Weyl spinors ψ_m, $\tilde{\psi}_m$, and, two complex scalars ϕ_m, $\tilde{\phi}_m$, is shown next:

$$
\begin{array}{cccc}
\text{Spin} & 0 & \frac{1}{2} & 0 \\
 & & \psi_m & \\
 & \phi_m & & \tilde{\phi}_m \\
 & & \tilde{\psi}_m &
\end{array}
\tag{45}
$$

All of them are massless and magnetically charged; accordingly, the associated quanta are light dual electrons, dual positrons, magnetic monopoles (dual s-electrons) and magnetic anti-monopoles (dual s-positrons). It is important to note that ϕ_m and $\tilde{\phi}_m$ form one doublet under $SU(2)_I$ whereas ψ_m and $\tilde{\psi}_m$ are singlets.

There is also the $\mathcal{N} = 2$ chiral super-field

$$\chi_D(y,\theta) = \varphi_D(y) + \sqrt{2}\tilde{\theta}W_D(y) + \tilde{\theta}^2 G_D(y).$$

which includes all the dual Abelian fields introduced in sub-Section §3.3.3. The spin of the physical fields v_μ^D, λ_D, ψ_D, ϕ_D is shown next

$$
\begin{array}{ccc}
 & & \text{Spin} \\
 & v_\mu^D & 1 \\
\lambda_D & \psi_D & \frac{1}{2} \\
 & \phi_D & 0
\end{array}
$$

and the corresponding quanta are dual photons , dual photinos , dual Higgsinos, and dual complex Higgs particles.

The $\mathcal{N} = 2$ super-symmetric low energy action in this limit is built from three pieces:

$$
\begin{aligned}
S_M &= \int d^4x \int d^2\theta \int d^2\bar{\theta} \left\{ \tilde{M} e^{2V_D} \tilde{M}^\dagger + M^\dagger e^{-2V_D} M \right\} \\
S_W &= \frac{\text{Im}}{16\pi} \int d^4x \int d^2\theta d^2\tilde{\theta} \tau_D(0) \chi_D \chi_D \quad ; \quad S_Y = \sqrt{2} \int d^4x [\int d^2\theta \varphi_D M \tilde{M} + \text{h.c.}]
\end{aligned}
$$

S_M describes the $U(1)_D$ gauge dynamics of the light magnetically charged fields, S_W is the Seiberg-Witten effective action exactly at the ultra-strong limit, and , S_Y is a Yukawa type action needed to ensure $\mathcal{N} = 2$ super-symmetry through the achievement of $SU(2)_I$ invariance in the total effective action $S_{\text{eff}} = S_W + S_M + S_Y$, see [20].

A better understanding of the low energy effective theory in the ultra-strong limit requires a closer look to the theory in a neighborhood of the $u = \Lambda^2$ point in the quantum moduli space of vacua. One introduces the magnetic coupling g_D as

$$
\tau_D(a_D) = \frac{4\pi i}{g_D^2(a_D)}
$$

because there is no instanton angle in QED and $\Theta_D = 0$. Moreover, from the β-function of $(\text{QED})_D$,

$$
\mu \frac{dg_D}{d\mu} = \frac{g_D^3}{8\pi^2} \quad ,
$$

and because $\mu \propto a_D = c_0(u - \Lambda^2)$ (in the monopole patch) we obtain

$$
a_D \frac{d\tau_D}{da_D} = -\frac{i}{\pi} \quad \Rightarrow \quad \tau_D = -\frac{i}{\pi} \ln a_D \quad .
$$

Now the physical meaning of this limit is clear: $\lim_{a_D \to 0} i\tau_D \simeq -\infty$ means that the ultra-strong limit is the g_D weak coupling limit (the g strong coupling limit, of course). Note also that from $\tau_D = \frac{d(-a)}{da_D}$ we can deduce formula (42) in subsection §. 3.3.5:

$$
a \simeq a_0 + \frac{i}{\pi} a_D \ln a_D = a_0 + \frac{ic_0}{\pi}(u - \Lambda^2) \ln[c_0(u - \Lambda^2)] \quad .
$$

The key observation is that S_W at $a_D = 0$ becomes a Ginzburg-Landau type effective action with all the quantum fluctuations integrated out. Therefore, the effective theory at the ultra-strong limit is a classical field theory and, rescuing the Planck constant, wave-particle duality tells us that the field theoretical coupling g_D is related to the magnetic charge of the dual light quanta in the form: $g_d = \hbar \bar{g}_D$. Thus, one must replace in S_W $\tau_D(0)$ by

$$
\bar{\tau}_D(0) = \hbar^2 \tau_D(0) \quad ,
$$

which remains finite at the ultra-strong limit $\tau_D \to \infty$ only if $\hbar \to 0$, i.e., the g-strong coupling limit is nothing but the classical limit for the magnetically charged quanta !!!.

3.4.2 Extended states in the ultra-strong limit

In order to search for solitons in the ultra-strong limit we perform Berezin integration in the low energy Ginzburg-Landau action to find:

$$S_W = \int d^4x \{ -\frac{1}{4} f^D_{\mu\nu} f^{\mu\nu}_D - i\bar{\lambda}_D \bar{\sigma}^\mu \partial_\mu \lambda_D + \frac{D^2}{2} + + \partial_\mu \phi_D^* \partial^\mu \phi_D - i\bar{\psi}_D \bar{\sigma}^\mu \partial_\mu \psi_D + F_D^* F_D \}$$

$$S_M = \int d^4x \{ (\nabla_\mu \phi_m)^* \nabla^\mu \phi_m + \nabla_\mu \tilde{\phi}_m (\nabla^\mu \tilde{\phi}_m)^* + F_m^* F_m + \tilde{F}_m^* \tilde{F}_m + $$
$$+ \bar{g}_D D(\phi_m^* \phi_m - \tilde{\phi}_m^* \tilde{\phi}_m) - i(\bar{\psi}_m \bar{\sigma}^\mu \nabla_\mu \psi_m + \bar{\tilde{\psi}}_m \bar{\sigma}^\mu \nabla_\mu \tilde{\psi}_m) \}$$

$$S_Y = \sqrt{2}\bar{g}_D \int d^4x [\{ F_D \phi_m \tilde{\phi}_m + 2\psi_D(\tilde{\psi}_m \phi_m + \psi_m \tilde{\phi}_m) + \phi_D(\phi_m \tilde{F}_m + \tilde{\phi}_m F_m) + \text{h.c.} \}$$
$$- \{ \phi_m^* \psi_m \lambda_D - \phi_m \bar{\psi}_m \bar{\lambda}_D + \tilde{\phi}_m \bar{\tilde{\psi}}_m \bar{\lambda}_D - \tilde{\phi}^* \psi_m \lambda_D \}].$$

The covariant derivatives on the spinor and scalar fields of the hypermultiplet are:

$$\nabla_\mu \phi_m = \partial_\mu \phi_m + i\bar{g}_D A^D_\mu \phi_m, \qquad \nabla_\mu \psi_m = \partial_\mu \psi_m + i\bar{g}_D A^D_\mu \psi_m$$
$$\nabla_\mu \tilde{\phi}_m = \partial_\mu \tilde{\phi}_m - i\bar{g}_D A^D_\mu \tilde{\phi}_m \qquad \nabla_\mu \tilde{\psi}_m = \partial_\mu \tilde{\psi}_m - i\bar{g}_D A^D_\mu \tilde{\psi}_m \qquad .$$

After performing integration on the auxiliary fields D and F_m, $\tilde{F}_m$ we obtain the bosonic part of the effective action:

$$S^B_{eff} = \int d^4x \{ -\frac{1}{4} f^D_{\mu\nu} f^{\mu\nu}_D + \partial_\mu \phi_D^* \partial^\mu \phi_D + (\nabla_\mu S)^\dagger (\nabla^\mu S) - \frac{\bar{g}_D^2}{2}(4|\phi_D|^2 + S^\dagger S) S^\dagger S \} \quad . \tag{46}$$

We have introduced the $SU(2)_I$ spinor S:

$$S = \begin{pmatrix} \phi_m \\ \tilde{\phi}_m^* \end{pmatrix} \quad .$$

It is obvious that absolute minima of the energy in the effective theory must satisfy $\phi_D(x) = a_D = 0$. Assuming this condition the energy density for time-independent axially symmetric configurations can be written la Bogomolny:

$$\mathcal{E}^B_{eff}(a_D = 0) = \int d^2x \{ |f^D_{12} \mp \bar{g}_D S^\dagger S|^2 + |(\nabla_1 \pm i\nabla_2)S|^2 \} + |\int d^2x f^D_{12}| \tag{47}$$

Solutions of the first-order equations

$$f^D_{12} = \pm \bar{g}_D S^\dagger S \qquad , \qquad (\nabla_1 \pm i\nabla_2)S = 0 \tag{48}$$

are absolute minima of $\mathcal{E}^B_{eff}(a_D = 0)$. Unfortunately, the sign combination in (48) is such that there are no regular solutions besides the trivial $S = 0, A^D_i = 0$ solution: the system is frustrated.

Nevertheless, an identical analysis in the Poincaré disc, instead $\mathbb{R}^2$, requires to add to the Lagrangian density the piece: $\mathcal{L}_R = -\frac{1}{2}RS^\dagger S$, where R is the constant negative curvature. After some rather trivial re-scaling, a new system of first-order equations appear:

$$f^D_{12} = \mp \frac{4}{(1 - x_1^2 - x_2^2)}(1 - S^\dagger S) \quad , \quad (\nabla_1 \pm i\nabla_2)S = 0 \quad . \tag{49}$$

There are vorticial solutions to the new system of first-order equations (49) of the form:

$$\tilde{\phi}_m^{(n)}(z) = 0 \quad , \quad \phi_m^{(n)}(z) = \frac{2(1-|z|^2)}{1 - f_n^* f_n(z)} \cdot \frac{f_n'(z)}{|f_n'(z)|} \quad , \quad f_n(z) = \prod_{i=1}^{n} \frac{(z - a_i)}{(1 - a_i^* z)} \quad .$$

which support a total electric flux of

$$|\int d^2 x f_{12}^D(x_1, x_2)| = \frac{2\pi}{\bar{g}_D}(n-1)$$

concentrated around the points $z = a_i$ of the complex plane. These solitonic solutions are nothing but "dual" instantons -dual self-dual configurations- with cylindrical symmetry.

The whole picture of the spectrum of $\mathcal{N} = 2$ $SU(2)$ super-symmetric Yang-mills theory is summarized in the following boxes:

weak coupling	strong coupling
light quanta: electrically charged	light quanta: magnetically charged
extended states: magnetically charged	extended states : electrically charged
S^1-invariant instantons	cylindrically-invariant instantons

3.4.3 Soft breaking of $\mathcal{N}=2$ supersymmetry: monopole condensation

Adding a mass term in the original non-abelian theory

$$S_\mu = \mu \int d^4x [\int d^2\theta \operatorname{tr} \Phi^2 + \text{h.c.}] = -\mu \int d^4x \operatorname{tr}(\phi^*\phi + i(\psi\psi + \bar{\psi}\bar{\psi}))$$

$\mathcal{N} = 2$ super-symmetry is "softly" broken to $\mathcal{N} = 1$. This addition affects the low energy effective theory through the trading of $\operatorname{tr}\Phi^2$ by an abelian chiral super-field $U(y, \theta)$ which is a functional of φ_D in the monopole patch,

$$U(\varphi_D) = U(\phi_D) + \sqrt{2}\theta\psi_D U'(\phi_D) + \theta^2 (U'(\phi_D)F_D - \frac{1}{2}U''(\phi_D)\psi_D\psi_D) \quad ,$$

supplying -after Berezin integration- a new term to the effective action:

$$S_\mu = \mu \int d^4x \{U'(\phi_D)F_D - \frac{1}{2}U''(\phi_D)\psi_D\psi_D + \text{h.c.}\}.$$

Taking into account only the contribution of the constant modes, the effective action at $u = \Lambda^2$, reduces to the effective potential:

$$V_{eff} = \sqrt{2}\bar{g}_D\varphi_D M\tilde{M} + \mu U(\varphi_D) + \text{h.c.} \quad .$$

The expectation values of M, $\tilde{M}$ and φ_D

$$\langle M(x)\rangle = \langle \phi_m(x)\rangle = m \quad , \quad \langle \tilde{M}(x)\rangle = \langle \tilde{\phi}_m(x)\rangle = \tilde{m}$$
$$\langle \varphi_D(x)\rangle = \langle \phi_D(x)\rangle = a_D$$

are given by the minima of V_{eff}:

$$\frac{dV_{eff}}{d\varphi_D} = \frac{dV_{eff}}{dM} = \frac{dV_{eff}}{d\tilde{M}} = 0$$
$$\sqrt{2}\bar{g}_D m\tilde{m} + \mu U'(a_D) = 0 \quad , \quad a_D\tilde{m} = a_D m = 0 \quad . \tag{50}$$

There is also the constraint $|m| = |\tilde{m}|$ coming from the integration of the D-terms. When $\mu = 0$, $m = \tilde{m} = 0$, $a_D = 0$ solve (50), and we come back to the quantum moduli space of the $\mathcal{N} = 2$ theory. If $\mu U'(0) \neq 0$ the vacuum manifold is different:

$$a_D = 0 \quad , \quad m\tilde{m} = |m|^2 e^{i(\alpha + \tilde{\alpha})} = -|\frac{\mu}{\bar{g}_D \sqrt{2}} U'(0)|e^{i\beta} \tag{51}$$

is the solution (50). Only the sum of the phases of m and $\tilde{m}$ is fixed, $\beta + \frac{\pi}{2} = \alpha + \tilde{\alpha}$, and there is a circle of gauge-equivalent vacua parametrized by $\alpha - \tilde{\alpha}$.

(Massless) magnetic monopoles, dual s-electrons, condense: spontaneous symmetry breaking of the dual Abelian group arises from identifying the vacuum manifold as the orbit of the $U(1)_D$ action. Note that now the quantum moduli space of vacua is formed by the $u = \pm \Lambda^2$. In each of these two points a $U(1)$ group acts, giving rise to one circle orbit. Expanding around the expectation values of the M and $\tilde{M}$ fields

$$M(y, \theta) = m + h_m(y) + \sqrt{2}\theta\psi_m(y) + \theta^2 F_m(y)$$

$$\tilde{M}(y, \theta) = \tilde{m} + \tilde{h}_m(y) + \sqrt{2}\theta\tilde{\psi}_m(y) + \theta^2 \tilde{F}_m(y)$$

scalar and vector mass terms arise in the Lagrangian density:

$$\mathcal{L} = \bar{g}_D^2 |m|^2 (|h_m|^2 + |\tilde{h}_m|^2 + A_\mu^D A_D^\mu) \quad .$$

The electric Higgs mechanism for the fermions, albeit in a more involved manner also happens.

3.4.4 Stable semi-local vortices

Variational calculus on the effective Bosonic action for the ϕ_d field tells us that topological defects can arise as solutions of the classical field equations such that $\phi_m(x)\tilde{\phi}_m(x) \neq m\tilde{m}$ in some region of the plane if and only if $U''(0) = 0$. But this is indeed the case because when $u \to \Lambda^2$, $U(\varphi_D) = a_0\varphi_D + \Lambda^2$, as one can check from the Seiberg-Witten solution for the prepotential near $u = \Lambda^2$ and $u = \infty$.

The effective action being Abelian, we expect to find planar solitons if $\mu \neq 0$; we look for minima of the bosonic effective energy per unit of length of time-independent axi-symmetric configurations:

$$\mathcal{E}_{eff}^B(a_D = 0) = \int d^2x \{\frac{1}{4} f_{12}^D f_{12}^D + (\nabla_i \phi_m)^* \nabla_i \phi_m + (\nabla_i \tilde{\phi}_m)^* \nabla_i \tilde{\phi}_m +$$

$$+ |\sqrt{2}\bar{g}_D \phi_m \tilde{\phi}_m + C|^2 + \frac{\bar{g}_D^2}{2}(|\phi_m|^2 - |\tilde{\phi}_m|^2)^2\} \quad ,$$

where $C = C_1 + iC_2 = \mu U'(0)$ and $i = 1, 2$.

Following the method of Ref. [21] we restrict the search to the trial surface in the $\mathcal{N} = 2$ I-space $\mathbf{C}^2$ defined by:

$$\tilde{\phi}_m = -e^{i\beta}\phi_m^* \quad , \quad \tan\beta = \frac{C_2}{C_1} \quad .$$

This condition is compatible with the asymptotic behaviour

$$\lim_{r \to \infty} \phi_m(x_1, x_2) = m \quad , \quad \lim_{r \to \infty} \tilde{\phi}_m(x_1, x_2) = \tilde{m}$$

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guaranteeing finiteness of the "string tension" $\mathcal{E}^B_{eff}(a_D = 0)$. The topological conditions for the existence of vortices are met, and, introducing the new variables

$$\phi_m = \frac{1}{2}\phi \qquad f^2 = 2\sqrt{2}\frac{|C|}{\bar{g}_D}$$

we end, for a given β, with the Ginzburg/Landau/Higgs free energy:

$$\mathcal{E}^B_{eff}(a_D = 0, \beta) = \int d^2x\{\frac{1}{2}f^D_{12}f^D_{12} + \frac{1}{2}(\nabla_i\phi)^*\nabla_i\phi + \frac{\bar{g}^2_D}{8}(|\phi|^2 - f^2)^2\}$$

From the Bogomolny splitting

$$\mathcal{E}^B_{eff}(a_D = 0, \beta) = \int d^2x\{[F^D_{12} \mp \frac{\bar{g}_D}{2}(f^2 - |\phi|^2)]^2 + |(\nabla_1 \pm i\nabla_2)\phi|^2\}\pm \\ \pm \int d^2x\{\frac{\bar{g}_D}{2}(f^2 - |\phi|^2)F^D_{12} \pm \frac{i}{2}(\nabla_1\phi^*\nabla_2\phi - \nabla_2\phi^*\nabla_1\phi)\} \tag{52}$$

it is seen that the absolute minima satisfy the first order equations:

$$(\nabla_1 \pm i\nabla_2)\phi = 0 \qquad , \qquad F^D_{12} = \pm\frac{\bar{g}_D}{2}(f^2 - |\phi|^2) \tag{53}$$

The string tension is easy to compute for the self-dual vortex solutions:

$$\mathcal{E}^B_{eff}(a_D = 0) = \frac{\bar{g}_D}{2}f^2|\int d^2xF^D_{12}| \qquad ; \tag{54}$$

it is proportional to the electric flux. Due to the topology of the configuration space of the problem, the "electric" flux of the vortex solutions is quantized in "quanta" of $\frac{2\pi}{\bar{g}_D}$ flux. Therefore, the self-dual solutions are electric flux lines of energy per length unit $\mathcal{E}^B_{eff}[SD] = \pi f^2n$ which arise at the critical point between type I and type II dual superconductivity. The existence of electric flux filaments warrants the confinement of the electric charges at this limit of the super-symmetric system.

A subtle question remains: despite favorable topology it is not clear whether or not these electric vortices are stable because the amplitude of internal space. To address this question we define "natural" fields:

$$\phi_\pm = \tilde{\phi}_m \pm e^{i\beta}\tilde{\phi}_m \qquad .$$

In the new field variables the energy per length unit reads:

$$\mathcal{E}^B_{eff}(a_D = 0) = \frac{1}{2}\int d^2x\left[f^D_{12}f^D_{12} + |\nabla_i\phi_+|^2 + \frac{1}{2}|\nabla_i\phi_-|^2 + \bar{g}^2_D\{\frac{(|\phi_+|^2 + |\phi_-|^2 - f^2)^2}{4} + |\phi_+|^2\}\right] \tag{55}$$

Dropping the last term in (55) the vortices would be in neutral equilibrium with $\mathbb{C}P^1$-lumps, arising semi-local topological defects as a continuum interpolation between both kinds of solitons, see [23]. The $|\phi_+|^2$-term means that the evolution from the vortices towards the $\mathbb{C}P^1$-lumps requires energy; thus, the electric vortices are stable.

4 $\mathcal{N} = 2$ twisted SUSY Yang-Mills theory

4.1 $\mathcal{N}$=2 gauge theory on four-manifolds: Wick rotation and twisting

We try now to define a $\mathcal{N} = 2$ super-symmetric gauge theory on a four-dimensional Riemannian manifold, i.e., we replace $\mathbb{R}^{1,3}$ by a manifold X which is locally $\mathbb{R}^4_E$, as the base space. We would expect that the super-space $\mathbb{R}^{4|8}$ should also be replaced by a super-manifold S locally homeomorph to $\mathbb{R}^{4|8}_E$.

The problem is that the fields become sections in associated bundles to $P(\text{Spin}(4), X)$, e.g., scalars, spinors, vectors, etcetera. Thus, X must be spin and this requirement breaks super-symmetry. The only loophole to build a SUSY theory on a 4 manifold locally Euclidean is to perform a process called TWIST. In this Section we shall discuss the seminal papers by Witten Topological Quantum Field Theory, [26], and Monopoles and Four Manifolds, [27], where subtle Donaldson invariants distinguishing between smooth structures in four manifolds are characterized by means of certain expectation values in Topological Quantum Field Theories.

4.1.1   The twist: mixing of internal and external symmetries

We have seen in previous Sections that the group of symmetry of the $\mathcal{N} = 2$ SUSY gauge theory is:

$$H = SU(2)_- \otimes SU(2)_+ \otimes SU(2)_R \otimes U(1)_C \qquad .$$

Here $K = SU(2)_- \otimes SU(2)_+$ is the group of external "rotations", whereas $U(2)_R = SU(2)_R \otimes U(1)_C$ is the internal symmetry group. The idea of the twist is to replace the group K by $\bar{K} = \bar{SU}(2)_- \otimes SU(2)_+$, where $\bar{SU}(2)_-$ is the diagonal sub-group of $SU(2)_- \otimes SU(2)_R$. The description of how this change affects to the super-charges, the super-algebra, the super-manifold and the fields follows.

1. **Twisted super − charges** The "quantum" numbers of the SUSY and twisted SUSY charges with respect to the K and $\bar{K}$ groups are shown in the next Box. The labels are the representations of the $SU(2)$ sub-groups and the $U(1)_C$ charge (the super-script).

SUSY Charge		Twisted SUSY Charge	
Q^I_α	$(\frac{1}{2}, 0, \frac{1}{2})^1$	Q^β_α	$(\frac{1}{2} \otimes \frac{1}{2}, 0)^1$
$\bar{Q}_{I\dot\alpha}$	$(0, \frac{1}{2}, \frac{1}{2})^{-1}$	$\bar{Q}_{\beta\dot\alpha}$	$(\frac{1}{2}, \frac{1}{2})^{-1}$

The explicit expression of the twisted in terms of the non-twisted SUSY charges can be obtained by means of the analogy with one-half spin systems.

- Understanding the super-charges Q^I_α as the states of a system of two spin one-half particles, we identify the following spin arrangements:

$$Q^1_1 \simeq |\uparrow\downarrow\rangle \qquad , \qquad Q^2_1 \simeq |\downarrow\downarrow\rangle \qquad , \qquad Q^1_2 \simeq |\uparrow\uparrow\rangle \qquad , \qquad Q^2_2 \simeq -|\downarrow\uparrow\rangle$$

The interpretation is obvious: the I labels the spin of the first particle , up or down, and the same job does the α for the spin of the second particle. We have chosen to identify $I = 1$ and $\alpha = 2$ with spin up and a $e^{i\pi}$ phase is assigned to the Q^2_2 state to fit

with the fact that $\mathrm{Tr}Q$ is an scalar. The decomposition of the tensor product of two $\frac{1}{2}$ representations in terms of irreducible representations,

$$(\frac{1}{2} \otimes \frac{1}{2}, 0) = (0,0) \oplus (1,0)$$

leads us to introduce an scalar super-charge

$$Q = \frac{1}{2}(Q_1^1 + Q_2^2) \simeq \frac{1}{2}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \qquad,$$

with $(0,0)^1$ quantum numbers. There are also three super-charges,

$$Q^3 = \frac{1}{2}(Q_1^1 - Q_2^2) \quad \simeq \quad \frac{1}{2}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$
$$Q^+ = Q_2^1 \simeq |\uparrow\uparrow\rangle \qquad , \qquad Q^- = Q_1^2 \simeq |\downarrow\downarrow\rangle \quad,$$

which carry $(1,0)$ quantum numbers. Defining $Q^1 = Q^+ + Q^-$, $Q^2 = i(Q^+ - Q^-)$ and using the three complex structures $\eta^a_{\mu\nu}$ ('t Hooft symbols) of the hyper-Khaler manifold $\mathbb{R}^4$ the $(1,0)^1$ representation is organized as a self-dual anti-symmetric tensor:

$$Q_{\mu\nu} = \sum_{a=1}^{3} \eta^a_{\mu\nu} Q^a \qquad , \qquad Q_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} Q^{\rho\sigma}$$

- From the vector of matrices $\sigma^\mu = (i\mathbf{1}_2, \sigma_i)$, where $\sigma_i, i = 1, 2, 3$, are the Pauli matrices, and through the identification,

$$\bar{Q}_{\beta\dot{\alpha}} = (\sigma^\mu Q_\mu)_{\beta\dot{\alpha}} \qquad,$$

a vector super-charge Q_μ with quantum numbers $(\frac{1}{2}, \frac{1}{2})^{-1}$ is obtained in terms of the twisted super-charges $\bar{Q}_{\beta\dot{\alpha}}$:

$$Q_1 = \frac{1}{2}(\bar{Q}_{1\dot{2}} + \bar{Q}_{2\dot{1}}) \qquad , \qquad Q_2 = \frac{i}{2}(\bar{Q}_{1\dot{2}} - \bar{Q}_{2\dot{1}})$$
$$Q_3 = \frac{1}{2}(\bar{Q}_{1\dot{1}} - \bar{Q}_{2\dot{2}}) \qquad , \qquad Q_4 = -\frac{i}{2}(\bar{Q}_{1\dot{1}} + \bar{Q}_{2\dot{2}})$$

2. **Twisted SUSY algebra**

   The SUSY algebra

   $$\{Q^I_\alpha, \bar{Q}_{J\dot{\beta}}\} = \delta^I_J P_{\alpha\dot{\beta}}$$

   becomes in terms of the twisted generators:

   $$Q^2 = Q_\mu^2 = Q_{\mu\nu}^2 = 0 \qquad , \qquad \{Q, Q_\mu\} = P_\mu$$
   $$\{Q, Q_{\mu\nu}\} = 0 \qquad , \qquad \{Q_{\mu\nu}, Q_\rho\} = \varepsilon_{\mu\nu\rho\sigma} P^\sigma + g_{\mu\rho} P_\nu - g_{\nu\rho} P_\mu \qquad (56)$$

3. **Twisted super − manifold**

   The twist prescription also requires the choice of $(x_\mu, \bar{\theta}, \theta_\mu, \bar{\theta}_{\mu\nu})$ as local coordinates in the $\mathbb{R}^{4|8}$ super-manifold. Again, the twisted Grassman variables are built by the same procedure

from the non-twisted Grassman variables, having in mind the behaviour of both kind of odd coordinates with respect to the groups K and $\bar{K}$ shown in the next Box.

Grassman variable	Twisted Grassman variable
$\theta^I_\alpha \qquad (\frac{1}{2}, 0, \frac{1}{2})^1$	$\theta^{\alpha\dot\beta} \qquad (\frac{1}{2}, \frac{1}{2})^1$
$\bar\theta^{\dot\alpha}_I \qquad (0, \frac{1}{2}, \frac{1}{2})^{-1}$	$\bar\theta^{\dot\alpha}_{\dot\beta} \qquad (0, \frac{1}{2} \otimes \frac{1}{2})^{-1}$

Definitions of vector, θ_μ, scalar, $\bar\theta$, and self-dual, $\bar\theta_{\mu\nu}$, Grassman variables immediately follow:

- From $\theta^{\alpha\dot\beta} = (\sigma_\mu \theta_\mu)^{\alpha\dot\beta}$ we find

$$\theta_1 = \frac{1}{2}(\theta^{1\dot2} + \theta^{2\dot1}) \qquad , \qquad \theta_2 = \frac{i}{2}(\theta^{1\dot2} - \theta^{2\dot1})$$
$$\theta_3 = \frac{1}{2}(\theta^{1\dot1} - \theta^{2\dot2}) \qquad , \qquad \theta_4 = -\frac{i}{2}(\theta^{1\dot1} + \theta^{2\dot2})$$

  as the components of θ_μ, which belong to the $(\frac{1}{2}, \frac{1}{2})^1$ representation of $SU(2)_- \otimes \bar{SU}(2)_+ \otimes U(1)_C$.

- The scalar Grassman variable $\bar\theta = \frac{1}{2}(\bar\theta^{\dot1}_{\dot1} + \bar\theta^{\dot2}_{\dot2})$ carries the quantum numbers $(0, 0)^{-1}$ with respect to the same group.

- The three Grassman variables

$$\bar\theta^1 = \bar\theta^{\dot1}_{\dot2} + \bar\theta^{\dot2}_{\dot1} \quad , \quad \bar\theta^2 = i(\bar\theta^{\dot1}_{\dot2} - \bar\theta^{\dot2}_{\dot1}) \quad , \quad \bar\theta^3 = \frac{1}{2}(\bar\theta^{\dot1}_{\dot1} - \bar\theta^{\dot2}_{\dot2})$$

  are assembled in the $(0, 1)^{-1}$ self-dual combination $\bar\theta_{\mu\nu} = \sum_{a=1}^{3} \eta^a_{\mu\nu} \bar\theta^a$.

Thus, it is easy to check that

$$Q = \frac{\partial}{\partial\bar\theta} \quad , \quad Q_\mu = \frac{\partial}{\partial\theta_\mu} - i\bar\theta\partial_\mu \quad , \quad Q_{\mu\nu} = \frac{\partial}{\partial\bar\theta_{\mu\nu}} - i(\varepsilon_{\mu\nu\rho\sigma}\theta^\rho\partial^\sigma + \theta^\mu\partial^\nu - \theta^\nu\partial^\mu)$$

satisfy the twisted SUSY algebra (56), and, therefore, the transformations of the super-Poincare group $P(x_\mu, \varepsilon_\mu, \bar\varepsilon, \bar\varepsilon_{\mu\nu}) = e^{i(-x^\mu P_\mu + \varepsilon^\mu Q_\mu + \bar\varepsilon Q + \bar\varepsilon^{\mu\nu} Q_{\mu\nu})}$ act infinitesimally on the $\mathbb{R}^{4|8}_E$ as:

$$x^\mu \to x^\mu - i\varepsilon^\mu\bar\theta \quad , \quad \theta^\mu \to \theta^\mu + \varepsilon^\mu \quad , \quad \bar\theta \to \bar\theta + \bar\varepsilon \quad , \quad \bar\theta^{\mu\nu} \to \bar\theta^{\mu\nu} + \bar\varepsilon^{\mu\nu} \quad .$$

4. **Twisted fields**

The twist works exactly in the same way for the spinor fields. Denoting by $\psi_{I\alpha}$ -I=1, the ψ (Higgsino), I=2, the λ (gluino)- the spinor fields of the $\mathcal{N} = 2$ chiral multiplet, we display the quantum numbers of both the non-twisted and twisted version in the following Box.

Spinor field	Twisted Spinor field
$\psi_{I\alpha} \qquad (\frac{1}{2}, 0, \frac{1}{2})^{-1}$	$\psi_{\beta\alpha} \qquad (\frac{1}{2} \otimes \frac{1}{2}, 0)^{-1}$
$\bar\psi^I_{\dot\alpha} \qquad (0, \frac{1}{2}, \frac{1}{2})^1$	$\bar\psi^\beta_{\dot\alpha} \qquad (\frac{1}{2}, \frac{1}{2})^1$

Again the twist process leads to deal with vector, ψ_μ, scalar, η, and, self-dual tensor, $\chi_{\mu\nu}$, fields:

- From $\bar\psi_{\dot\alpha}^{\beta} = (\sigma^\mu \psi_\mu)_{\dot\alpha}^{\beta}$ we find

$$\psi_1 = \frac{1}{2}(\psi^{1\dot2} + \psi^{2\dot1}) \qquad , \qquad \psi_2 = \frac{i}{2}(\psi^{1\dot2} - \psi^{2\dot1})$$

$$\psi_3 = \frac{1}{2}(\psi^{1\dot1} - \psi^{2\dot2}) \qquad , \qquad \psi_4 = -\frac{i}{2}(\psi^{1\dot1} + \psi^{2\dot2})$$

as the components of field ψ_μ, which belong to the $(\frac{1}{2}, \frac{1}{2})^1$ representation of $SU(2)_- \otimes \bar{SU}(2)_+ \otimes U(1)_C$.

- The scalar field $\eta = \frac{1}{2}(\psi_{11} + \psi_{22})$ carries the quantum numbers $(0,0)^{-1}$ with respect to the same group.

- The three combinations

$$\chi^1 = \psi_{12} + \psi_{21} \quad , \quad \chi^2 = i(\psi_{12} - \psi_{21}) \quad , \quad \chi^3 = \frac{1}{2}(\psi_{11} - \psi_{22})$$

are assembled in the $(0,1)^{-1}$ self-dual antisymmetric tensor field $\chi_{\mu\nu} = \sum_{a=1}^{3} \eta^a_{\mu\nu} \chi^a$.

The vector field A_μ trades the quantum numbers $(\frac{1}{2}, \frac{1}{2}, 0)^0$ by $(\frac{1}{2}, \frac{1}{2})^0$ after twisting, keeping its vector nature. The Higgs field ϕ and its adjoint $\phi^\dagger$, respectively with quantum numbers $(0,0,0)^{-2}$ and $(0,0,0)^2$ are twisted to two real scalar fields, λ and ϕ, labeled by $(0,0)^{-2}$ and $(0,0)^2$ as irreducible representations of $\bar{SU}(2)_- \otimes U(1)_C$. Finally, a word about the auxiliary fields that we shall not consider in the sequel. Before twisting, D and F belong to a vector representation of $SU(2)_R$: the quantum numbers of $\vec{D} = D\vec{e}_1 + \mathrm{Re}F\vec{e}_2 + \mathrm{Im}F\vec{e}_3$ are $(0,0,1)^1$. After twisting $\vec{D}$ becomes a neutral self-dual tensor $G_{\mu\nu}$ with $(1,0)^0$.

A last remark: all the twisted Grassman variables and fields belong to real representations of the symmetry group. Therefore, super-symmetry on Riemannian four manifolds is possible for real fields.

4.1.2 $\mathcal{N} = 2$ super-symmetric Euclidean action: Q-cohomology

We have collected the following data: a compact oriented 4-manifold X endowed with a Riemannian metric $h_{\mu\nu}$ and a principal fiber bundle P with a simple and compact structural group G. A_μ belongs to the space of G-connections $\mathcal{A}$ in P. $G_{\mu\nu}, \chi_{\mu\nu} \in \Omega^{2+}(X, \mathrm{Lie}G)$ are self-dual two forms, $\psi_\mu \in \Omega^1(X, \mathrm{Lie}G)$ is a one-form and $\eta, \lambda, \phi \in \Omega^0(X, \mathrm{Lie}G)$ are zero forms.

From these ingredients we build a $\mathcal{N} = 2$ super-symmetric action as a Berezin integral in the $\mathbb{R}_E^{4|4}$ super-space:

$$S = \frac{1}{4} \int d^4x \, d^4\theta \, \sqrt{\det h} \, \mathrm{tr} \left\{ \Phi^2 + \Phi_\mu^* \Phi^\mu + \frac{1}{48}\Phi_{\mu\nu}\Phi^{\mu\nu} + \frac{1}{12}\Phi_{\mu\nu\rho}^* \Phi^{\mu\nu\rho} + \frac{1}{12}\sum_{a=1}^{2} \Phi_{\mu\nu\rho\sigma}^a \Phi_a^{\mu\nu\rho\sigma} \right\} \quad . \tag{57}$$

From the self-dual part of the curvature of the connection A_μ,

$$F_{\mu\nu}^+ = \frac{1}{2}(F_{\mu\nu} + \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}) \quad ,$$

and the covariant derivatives of the ϕ and ψ_μ fields,

$$D_\mu \psi_\nu = \nabla_\mu \psi_\nu - \Gamma_{\mu\nu}^\lambda \psi_\lambda \qquad , \qquad D_\mu \phi = \nabla_\mu \phi \quad ,$$

where $\Gamma^\lambda_{\mu\nu}$ are the Christoffel symbols of the metric h and ∇_μ is the covariant derivative given by the A_μ connection, we have defined the following $\mathcal{N}=1$ tensor super-fields:

$$\Phi(x,\theta_\mu) = \phi(x) + \theta_\mu\psi^\mu(x) + \theta_\mu\theta_\nu F_+^{\mu\nu} \quad , \quad \Phi_\mu = iD_\mu\phi(x) - i\theta_\sigma D_\mu\psi^\sigma + \theta_\mu\theta_\nu\theta_\rho\chi^{\nu\rho}(x)$$

$$\Phi_{\mu\nu}(x,\theta_\rho) = i(\theta_\mu\theta_\nu + \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\theta^\rho\theta^\sigma)[\lambda,\phi] \quad , \quad \Phi_{\mu\nu\rho}(x,\theta_\sigma) = -i\varepsilon_{\mu\nu\rho\gamma}\theta^\gamma D_\delta\psi^\delta + \theta_\mu\theta_\nu\theta_\rho\,\eta$$

$$\Phi^1_{\mu\nu\rho\sigma}(x,\theta_\gamma) = \varepsilon_{\mu\nu\rho\sigma}(\Box\phi - \frac{i}{2}[\psi_\gamma,\psi^\gamma]) + \theta_\mu\theta_\nu\theta_\rho\theta_\sigma\,\lambda$$

$$\Phi^2_{\mu\nu\rho\sigma}(x,\theta_\gamma) = \varepsilon_{\mu\nu\rho\sigma}\phi - \frac{i}{4}\theta_\mu\theta_\nu\theta_\rho\theta_\sigma\,([\chi_{\gamma\delta},\chi^{\gamma\delta}] + \frac{1}{4}[\eta,\eta])$$

All the $\Phi_{\mu\dots}$ fields have $U(1)_C$ charge equal to two and the normalization is chosen in such a way that Berezin integration in (57) gives the famous Witten topological action:

$$S = \int d^4x \quad \sqrt{\det h}\ \mathrm{tr}\{\frac{1}{2}F^+_{\mu\nu}F_+^{\mu\nu} + i\chi^{\mu\nu}D_\mu\psi_\nu - i\eta D^\mu\psi_\mu - \frac{i}{8}\phi[\chi_{\mu\nu},\chi^{\mu\nu}]$$

$$- \frac{i}{2}\lambda[\psi_\mu,\psi^\mu] + \frac{1}{2}\lambda D_\mu D^\mu\phi - \frac{i}{2}\phi[\eta,\eta] - \frac{1}{8}[\lambda,\phi]^2\} \tag{58}$$

To test the invariance of the Euclidean action (58) under twisted super-symmetry we must know how the Q-generator acts on the fields. This is easily read from the super-fields:

$$A_\mu(x,\bar\theta) = A_\mu(x) + \bar\theta\psi_\mu(x) \quad , \quad \psi_\mu(x,\bar\theta) = \psi_\mu(x) + \bar\theta D_\mu\phi(x) \quad , \quad \phi(x,\bar\theta) = \phi(x)$$

$$\chi_{\mu\nu}(x,\bar\theta) = \chi_{\mu\nu}(x) + \bar\theta F^+_{\mu\nu}(x) \quad , \quad \eta(x,\bar\theta) = \eta(x) + i\bar\theta[\lambda(x),\phi(x)] \quad , \quad \lambda(x,\bar\theta) = \lambda(x),$$

of $U(1)_C$ charges 0,1,2,-1,-1,-2. We see that:

$$\begin{array}{lll} QA_\mu(x,\bar\theta) = \psi_\mu & Q\phi(x) = 0 & Q\chi_{\mu\nu}(x,\bar\theta) = F^+_{\mu\nu} \\ Q\psi_\mu(x,\bar\theta) = D_\mu\phi & Q\eta(x,\bar\theta) = i[\lambda,\phi] & Q\lambda(x) = 0 \end{array} \quad .$$

From this, one immediately checks that

$$S = \int dx^4\ \sqrt{\det h}\ Q\cdot V(x,\bar\theta) \quad ,$$

where

$$V = \frac{1}{4}\mathrm{tr}\ F_{\mu\nu}\chi^{\mu\nu}(x,\bar\theta) + \frac{1}{2}\mathrm{tr}\ \psi_\mu(x,\bar\theta)D^\mu\lambda - \frac{1}{4}\mathrm{tr}\ \eta(x,\bar\theta[\phi,\lambda] \quad .$$

Therefore, S is Q-exact and the Witten action is invariant under Q: $Q\cdot S = 0$.

Moreover, the energy-momentum tensor, defined through the variation of the energy density $S = \int dx^4 \mathcal{L}$ with respect to the metric tensor,

$$T_{\mu\nu} = \frac{\delta\mathcal{L}}{\delta h_{\mu\nu}} \quad , \quad D_\mu T^{\mu\nu} = 0$$

besides of being divergenceless in the covariant sense, is also Q-exact. $T_{\mu\nu} = Q\Lambda_{\mu\nu}(x,\bar\theta)$, where

$$\Lambda_{\mu\nu}(x,\bar\theta) = \frac{1}{2}\mathrm{tr}\ (F_{\mu\sigma}\chi^\sigma_\nu(x,\bar\theta) + F_{\nu\sigma}\chi^\sigma_\mu(x,\bar\theta) - \frac{1}{2}h_{\mu\nu}F_{\sigma\rho}\chi^{\sigma\rho}(x,\bar\theta) + \psi_\mu(x,\bar\theta)D_\nu\lambda$$

$$+ \psi_\nu(x,\bar\theta)D_\mu\lambda - h_{\mu\nu}\psi_\sigma(x,\bar\theta)D^\sigma\lambda + \frac{1}{4}h_{\mu\nu}\eta(x,\bar\theta)[\phi,\lambda]) \quad .$$

As we shall show, this is a crucial point to establish the topological meaning of the Witten action (58).

4.2 Topological field theory

In this sub-Section we proceed to quantize the classical field theory described in the sub-Section §4.1 . We choose functional integration as (Feynman) quantization procedure and the main result is that expectation values of certain operators are $\mathbf{C}^{\infty}$-topological invariants of four-manifolds. Studies of moduli spaces of self-dual connections in G-bundles on compact Riemannian manifolds by Atiyah, Hitchin, and Donaldson unveiled the lack of dipheomormisms between four-manifolds which, nevertheless, are homeomorphic with respect to each other. This happened at the beginning of the eighties in the past century and, slightly later, polynomial invariants were discovered by Donaldson distinguishing between non-dipheomorphic but homeomorphic four-manifolds. The idea comes from the analysis of the homology of the moduli space of self-dual connections. Following Witten, we shall describe next how the Donaldson invariants arise in the topological field theory proposed above.

4.2.1 Functional integral representation of topological invariants

Let us suppose that there exists an integration measure (whatever that means)

$$\mathcal{DF} \simeq \mathcal{DA} \, \mathcal{D}\phi \, \mathcal{D}\lambda \, \mathcal{D}\eta \, \mathcal{D}\psi \, \mathcal{D}\chi$$

in the space of all the $(A, \phi, \lambda, \eta, \psi, \chi)$ fields. The Feynman principle guides us to define the expectation value in the ground state of a polynomial function of the fields W as the functional integral:

$$Z(W) = \int_{\mathcal{A}/\mathcal{G}} \mathcal{DF} \exp(-\frac{1}{g^2}S) \cdot W = < W > \qquad .$$

Note that the integration domain is the space of field orbits of the gauge group; we take quotient by the action of the gauge group.

Super-symmetry at the quantum level requires:

$$\exp(\varepsilon Q)\mathcal{DF} \exp(-\frac{S}{g^2}) \equiv \mathcal{DF} \exp(-\frac{S}{g^2}) \quad \Rightarrow \quad Z_\varepsilon(W) \neq f(\varepsilon) \, ,$$

i.e., super-symmetry is non-anomalous because both the classical action and the integration measure are invariant with respect to the super-symmetry generator. Moreover, the expectation value of anti-commutators of Q - in the sequel, we consider Q as the vector field acting on the space of fields through super-Poisson brackets- with any functional $\mathcal{O}$ of the fields is zero:

$$Z_\varepsilon(\mathcal{O}) = \int \mathcal{DF} \exp(\varepsilon Q) \cdot \{\exp(-\frac{S}{g^2}) \cdot \mathcal{O}\} = \int \mathcal{DF} \exp(-\frac{S}{g^2})(\mathcal{O} + \varepsilon\{Q, \mathcal{O}\})$$

$$\Rightarrow \quad 0 = < \{Q, \mathcal{O}\} > = \int \mathcal{DF} \exp(-\frac{S}{e^2}) \cdot \{Q, \mathcal{O}\} \qquad .$$

The partition function

$$Z = \int_{\mathcal{A}/\mathcal{G}} \mathcal{DF} \exp(-\frac{1}{g^2}S) \tag{59}$$

is only built from Q-closed states: $Z_\varepsilon = \exp(\varepsilon Q)Z = Z$. We show next that it is the simplest topological invariant, by the simple idea of testing the dependence of Z on the Riemannian metric. Because the variation of $\exp[-\frac{S}{g^2}]$ with respect to the metric is Q-exact, assuming that the integration measure is independent of $h^{\mu\nu}$ (there are no gravitational anomalies), and taking into account

that $< \{Q, \mathcal{O}\} >= 0$ we see that the partition function Z is a topological invariant (independent of the metric):

$$
\begin{aligned}
\delta Z &= \int \mathcal{DF} \exp[-\frac{S}{g^2}] \cdot (-\frac{\delta S}{g^2}) \\
&= -\frac{1}{g^2} \int \mathcal{DF} \exp[-\frac{S}{g^2}] \cdot \{Q, \int dx^4 \sqrt{\det h}\, \delta h^{\mu\nu} \Lambda_{\mu\nu}\} = 0 \qquad .
\end{aligned}
$$

A semi-classical computation of Z gives an exact answer for the topological invariant; one easily shows that the partition function is also independent of the gauge coupling g:

$$
\begin{aligned}
\delta Z &= \delta(-\frac{1}{g^2}) \int \mathcal{DF} \exp[-\frac{S}{g^2}] \cdot S \\
&= \delta(-\frac{1}{g^2}) \int \mathcal{DF} \exp[-\frac{S}{g^2}] \cdot \{Q, V\} = 0 \qquad .
\end{aligned}
$$

4.2.2 Instantons and the $U(1)_C$ anomaly

An infinite dimensional generalization of the steepest descent method for computing integrals of functions of the form $\exp[\frac{1}{g^2} f(x)]$ where g^2 is a very small parameter tells us that the main contribution to the integral is localized around the classical minima of S. It is well known that the moduli spaces of anti-self-dual connections,

$$
F_{\mu\nu} = - * F_{\mu\nu} \qquad \Longleftrightarrow \qquad F_{\mu\nu}^+ = 0 \qquad , \tag{60}
$$

called instantons in physicist's folklore, form such varieties of absolute minima of Witten's topological action S (58).

Calling E the bundle where the fields of the theory are sections, the formal dimension of the moduli space $\mathcal{M}_I$ of anti-self-dual connections in the associated $G = SU(2)$ P-bundle over X is given by the topological formula

$$
d(\mathcal{M}_I) = 8p_1(E) - \frac{3}{2}(\chi(X) + \sigma(X)) = 8p_1(E) - 3(1 + b_+^2) \tag{61}
$$

where

$$
p_1(E) = -\frac{1}{8\pi^2} \int_X \mathrm{tr}\,(F \wedge F)
$$

is the first Pontryagin number of E, F is the curvature of the Yang-Mills connection, and $\chi(X) = b^0 + b^2 + b^4$ and $\sigma(X) = b_+^2 - b_-^2$ are respectively the Euler characteristic and the signature of the 4-manifold X.

Formula (61) is derived from deformation theory: if A is a solution of (60) $A + \delta A$ is also a solution of the anti-self-duality equations if and only if:

$$
D_\mu\, \delta A_\nu - D_\nu\, \delta A_\mu + \varepsilon_{\mu\nu\rho\sigma} D^\rho \delta A^\sigma = 0 \tag{62}
$$

To count only deformations orthogonal to the orbits of the gauge group we choose the "background"gauge condition:

$$
D_\mu\, \delta A^\mu = 0 \qquad . \tag{63}
$$

The dimension $d(\mathcal{M}_I) = n$ of the space of solutions of (62)-(63) is formally the dimension of the moduli space of anti-self-dual connections (the real dimension if there are no reducible connections).

Looking at formula (58) for Witten's action one immediately checks that (62) is the field equation coming from variations with respect to χ

$$D_\mu \, \psi_\nu - D_\nu \, \psi_\mu + \varepsilon_{\mu\nu\rho\sigma} D^\rho \psi^\sigma = 0 \quad , \tag{64}$$

while variations with respect to η give

$$D_\mu \, \psi^\mu = 0 \quad . \tag{65}$$

Therefore, the dimension of the space of fermionic zero modes in the instanton field is equal to the dimension of the moduli space of anti-self-dual connections. If the gauge group is $SU(2)$ there are no η- and χ-zero modes of fluctuation around anti-self-dual instantons. An index theorem tells us that, generically, $n_+ - n_- = d(\mathcal{M}_I)$, where n_+ is the number of linearly independent ψ_μ zero modes (recall that the $U(1)_C$ charge of ψ is +1) and n_- is the number of linearly independent η and χ zero modes (recall that the $U(1)_C$ carge of η and χ is -1). Physicists read the index theorem as the anomaly in the $U(1)_C$ symmetry induced by instantons at the semi-classical level:

$$d(\mathcal{M}_I) = \Delta U(1)_C = n_+ - n_- \quad ,$$

i.e., the integration measure $\mathcal{DF}$ is not invariant under $U(1)_C$ but transform with a weight $-d(\mathcal{M})$.

4.2.3   k-point correlation functions

Due to the existence of fermionic zero modes between the fluctuations around the instanton field and the properties of Berezin integration measures the partition function - zero-point correlation function- Z vanishes unless X , G and E are such that: $d(\mathcal{M}_I) = 0$. We address first this case, assuming that the real dimension of the moduli space is really 0: there is a finite discrete set of isolated instantons.

Assembling the Bose $\Phi = (A, \phi, \lambda)$ and Fermi $\Psi = (\eta, \psi, \chi)$ fields of the theory under the labels Φ and Ψ the expansion of the action around one of these instantons up to quadratic order reads:

$$S_{(2)} = \int_X \sqrt{\det h} \, (\Phi \Delta_B \Phi + i \Psi D_F \Psi)$$

Here Δ_B is a second-order matrix differential operator that rules the Bosonic small fluctuations around the instanton field. D_F is a real skew-symmetric matrix first-order operator governing the Fermionic small deformations of the instanton and supersymmetry establishes a link between the spectra of these two operators:

$$i D_F \Psi = \omega \Psi \quad \leftrightarrow \quad \Delta_B \Psi = \omega^2 \Psi \quad .$$

The eigenvalues of D_F are purely imaginary and come in pairs. Therefore, the contribution of one instanton to the partition function in the weak coupling limit is

$$\frac{\mathrm{Pffaf} D_F}{\sqrt{\det \Delta_B}} = \pm \prod_n \frac{\omega_n}{\sqrt{\omega_n^2}}$$

because we only need to evaluate the integration of Bosonic and Fermionic Gaussians of width given by the eigenvalues of Δ_B and D_F. There is no problem in the regularization of the infinite product of ratios of eigenvalues but there is no way in solving the ambiguity in sign. The convention is set in three steps:

1. Choose a sign, e.g. $+$, for a given instanton $A_\mu^{I_1}$.

2. Consider a second instanton $A_\mu^{I_2}$ connected to the first instanton via the family of connections

$$A_\mu^t = tA_\mu^{I_2} + (1-t)A_\mu^{I_1} \qquad .$$

   Assign to $A_\mu^{I_2}$ a minus sign if the spectral flow of D_F along the A_μ^t family crosses an odd number of zeroes.

3. This process is independent of the path chosen in the space of connections $\mathcal{A}/\mathcal{G}$: this is equivalent to say that there are no global anomalies and the "Pfaffian" line bundle on $\mathcal{A}/\mathcal{G}$ exists and is trivial. Because the moduli space of (irreducible) anti-self-dual connections is a finite subset of the space of gauge connections - $\mathcal{M} \subset \mathcal{A}/\mathcal{G}$- the above statement is tantamount to the orientability of $\mathcal{M}$.

Adding the contributions to the partition function we find in the weak coupling limit the formula:

$$Z \propto \sum_i \frac{\text{Pfaff}(D_F^{(i)})}{\sqrt{\det \Delta_B^{(i)}}} = \sum_i \left(\pm \prod_{n_i} \frac{\omega_{n_i}}{\sqrt{|\omega_{n_i}|^2}} \right) = \sum_i (-1)^{N_i} \qquad , \qquad N_i = 0, 1 \qquad , \qquad (66)$$

where the N_i give the signs as explained above and the discrete index i runs over the number of instantons. The result is independent of $h_{\mu\nu}$ and g: the partition function Z is the first Donaldson C^∞-topological invariant for four-manifolds.

When $d(\mathcal{M}) > 0$, the non-vanishing correlation functions are path integrals of the form:

$$Z(\mathcal{O}) = \int \mathcal{DF}\, \mathcal{O} \cdot \exp\{-\frac{S}{g^2}\} \qquad , \qquad (67)$$

where $\mathcal{O}$ is any function of the fields such that the $U(1)_C$ charge of $\mathcal{O}$ is equal to $d(\mathcal{M})$; $\mathcal{O}$ supplies the right number of Grassman variables to compensate the fermionic zero modes in the Berezin measure and obtain a non-zero result.

Is $Z(\mathcal{O})$ a topological invariant? The variation of (67) under a change of the metric is:

$$
\begin{aligned}
\delta Z(\mathcal{O}) &= \int \mathcal{DF} \exp(-\frac{S}{g^2}) \cdot (-\frac{\delta S}{g^2}\mathcal{O} + \delta\mathcal{O}) \\
&= \int \mathcal{DF} \exp(-\frac{S}{g^2}) \cdot (-\frac{1}{2g^2}\{Q, \int \sqrt{\det h}\, \delta h^{\mu\nu}\Lambda_{\mu\nu}\}\mathcal{O} + \delta\mathcal{O}) \qquad , \qquad (68)
\end{aligned}
$$

where $\delta\mathcal{O}$ is the variation of $\mathcal{O}$ with respect to the metric $h_{\mu\nu}$. If $\{Q, \mathcal{O}\} = 0$

$$\int \mathcal{DF} \exp(-\frac{S}{g^2})(-\frac{1}{g^2}\{Q, \int \sqrt{\det h}\, \delta h^{\mu\nu}\Lambda_{\mu\nu}\}\mathcal{O}) = -\frac{1}{g^2} < \{Q, \int \sqrt{\det h}\, \delta h^{\mu\nu}\Lambda_{\mu\nu}\}\mathcal{O} >= 0$$

and the first term in (68) is zero. Therefore, the only remaining condition on $\mathcal{O}$ for topological invariance of $Z(\mathcal{O})$ is independence of the metric: $\delta\mathcal{O} = 0$. Thus, we choose $\mathcal{O}$ such that:

$$1) \quad \{Q, \mathcal{O}\} = 0 \qquad \quad 2) \quad \mathcal{O} \neq \{Q, \rho\} \qquad \quad 3) \quad \delta\mathcal{O} = 0 \qquad .$$

Note that $Z(\mathcal{O}) = 0$ if $\mathcal{O} = \{Q, \rho\}$ for any ρ and this justifies the condition 2).

The only field complying with these conditions is the spin zero scalar field ϕ: ϕ belongs to the kernel of Q, does not depend on the metric, and there is no ρ such that $\phi = \{Q, \rho\}$. Therefore,

$\mathcal{O}$ must be defined in terms of the G-invariant polynomials of ϕ; for $G = SU(2)$ there is only one independent G-invariant quadratic polynomial:

$$\mathcal{O} = \text{tr } \phi^2 \qquad .$$

To emphasize that it is a local operator that depends on a point P it is usually written as:

$$W_0(P) = \frac{1}{2}\text{tr } \phi^2(P) \qquad .$$

Because the $U(1)_C$ charge of $W_0(P)$ is 4, when the bundle E and the 4 manifold X are such that $d(\mathcal{M}_I) = 4k$, $k \in \mathbb{Z}$, the k-point correlation function

$$Z(k) = \int \mathcal{DF} \exp(-\frac{S}{g^2}) \prod_{i=1}^{k} W_0(P_i) = < W_0(P_1) \cdots W_0(P_k) > \qquad (69)$$

is a C^∞-topological invariant of the four manifold X, the simplest type of Donaldson polynomial expressed as a vacuum expectation value of field operators in Witten's topological quantum field theory.

4.2.4   Donaldson polynomials

In particular, $Z(k)$ is independent of the choice of the $P_1, P_2, \ldots, P_k$ points in X because it is independent of the metric. To test explicitly the last proposition we compute

$$\frac{\partial}{\partial x^\mu} W_0 = \text{tr } \phi D_\mu \phi = i\{Q, \text{tr } \phi\psi_\mu\}$$

to find a Q-exact answer. Thus, we write

$$W_0(P) - W_0(P') = \int_{P'}^{P} \frac{\partial W_0}{\partial x^\mu} dx^\mu = i\{Q, \int_{P'} W_1\} \qquad ,$$

where $W_1 = \text{tr } (\phi\psi_\mu)dx^\mu = \text{tr } (\phi \wedge \psi)$ is a one-form on X which gives the derivative of W_0 as a BRST commutator. Therefore,

$$\langle(W_0(P) - W_0(P')) \cdot \prod_j W_0(P_j) >= \langle\{Q, i \int_{P'}^{P} W_1 \cdot \prod_j W_0(P_j)\rangle = 0$$

and the expectation value at P is equal to the expectation value at P'.

This process can be iterated recursively

$$\begin{aligned}
0 &= i\{Q, W_0\} & dW_0 &= i\{Q, W_1\} & dW_1 &= i\{Q, W_2\} \\
dW_2 &= i\{Q, W_3\} & dW_3 &= i\{Q, W_4\} & dW_4 &= 0
\end{aligned}$$

$$W_2 = \text{tr}(\frac{1}{2}\psi \wedge \psi + i\phi \wedge F) \quad , \qquad W_3 = i \text{ tr } (\psi \wedge F) \quad , \qquad W_4 = -\frac{1}{2}\text{tr } (F \wedge F) \qquad .$$

Recall that ϕ, ψ and F are respectively zero, one, and two forms in X. Note also that the $U(1)_C$ charge of every W_k is $4 - k$.

Given a j-dimensional homology cycle γ on X the integral $I(\gamma) = \int_\gamma W_j$ is BRST invariant:

$$\{Q, I\} = \int_\gamma \{Q, W_j\} = -i \int_\gamma dW_{j-1} = 0$$

Moreover, I depends only on the homology class of γ. If $\gamma = \partial\beta$ is a boundary, Stoke's theorem shows that $I(\gamma)$ is Q-exact:

$$I(\gamma) = \int_\gamma W_j = \int_\beta dW_j = i \int_\beta \{Q, W_{j+1}\} = i\{Q, \int_\beta W_{j+1}\} \qquad .$$

The quantum field theory formulas for the Donaldson invariants are expectation values of products of $I(\gamma)$ integrals. Suppose that X, G, and E are such that $d(\mathcal{M}) \geq 0$. Let $\gamma_1, \gamma_2, \cdots, \gamma_r$ be homology cycles on X of dimension $j_1, j_2, \cdots, j_r$ chosen such that $d(\mathcal{M}) = \sum_{i=1}^r (4-j_r)$. Clearly, $\prod_{i=1}^r W_{j_i}$ is an operator with $U(1)_C$ charge equal to $d(\mathcal{M})$. The vacuum expectation value:

$$Z(\gamma_1, \ldots, \gamma_r) = \int \mathcal{DF} \exp(-\frac{S}{g^2}) \prod_{i=1}^r I(\gamma_i) = \langle \int_{\gamma_1} W_{j_1} \int_{\gamma_2} W_{j_2} \cdots \int_{\gamma_r} W_{j_r} \rangle \qquad (70)$$

is the TQFT "Donaldson polynomial". In the left-hand member of (70) the explicit dependence in the homology of X is shown:

$$(\gamma_1, \gamma_2, \cdots, \gamma_r) \in H_{j_1}(X) \times H_{j_2}(X) \times \cdots \times H_{j_r}(X) \qquad .$$

The Feynman integral in the right-hand side of (70) reduces to an integration over the Grassman zero modes that span the cotangent bundle to $\mathcal{M}$. Therefore, the Donaldson invariants map the $H_{j_i}(X)$ homology group of X in the $H^{4-j_i}(\mathcal{M})$ cohomology group of the moduli space of instantons $\mathcal{M}$.

4.3 Twist of the low-energy theory: the Seiberg-Witten equations

We now perform the twist of the low energy $\mathcal{N} = 2$ SUSY QED described in the Section §.3.4. The fields forming the $\mathcal{N} = 2$ multiplet, although they are Abelian, present no novelties regarding the process of twisting with respect to the twisting at high energy. We only need to define the twist of the matter fields M and $\tilde{M}$ entering in the monopole patch that play such an importantant rôle in the soft breaking of $\mathcal{N} = 2$ SUSY giving rise to a confinement phase of the system. The magnetically charged matter fields shown in the diamond (45) are twisted through the transmutation of the $SU(2)_C$ doublet of scalar fields $\phi_m^1 = \phi_m$ and $\phi_m^2 = \tilde{\phi}_m$ in a right-handed Weyl spinor:

Scalar fields		Twisted Spinor field	
ϕ_m^I , $I = 1, 2$	$(0, 0, \frac{1}{2})^0$	$S_+ = \begin{pmatrix} 0^\alpha \\ M^\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \phi_m^1 \\ \phi_m^2 \end{pmatrix}$	$(\frac{1}{2}, 0)^0$

The $SU(2)_C$-singlet Weyl spinors in the diamond do not change their Weyl spinor character under twist:

Spinor fields	Twisted Spinor fields	
$\psi_{m\alpha}$ $\quad (\frac{1}{2},0,0)^1$	$\begin{pmatrix} 0_\alpha \\ \mu_\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \psi_{m1} \\ \psi_{m2} \end{pmatrix}$	$(\frac{1}{2},0)^1$
$\tilde{\bar{\psi}}_{m\dot\alpha}$ $\quad (0,\frac{1}{2},0)^{-1}$	$\begin{pmatrix} 0_{\dot\alpha} & \bar\nu_{\dot\alpha} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\tilde{\bar\psi}_{m\dot1} & -\tilde{\bar\psi}_{m\dot2} \end{pmatrix}$	$(0,\frac{1}{2})^{-1}$

Thus, the twisted spinor fields M^α and μ_α belong to the space of sections of a Spin-complex bundle over a four-dimensional Riemannian manifold:

$$M_\alpha,\ \mu_\alpha \in \Gamma(S^+ \otimes L) \qquad , \qquad \bar\nu_{\dot\alpha} \in \Gamma(S^- \otimes L)\ .$$

In order to define the Dirac operator acting on these spinorial sections we start from the Euclidean Clifford algebra

$$\gamma_a = i \begin{pmatrix} 0 & \sigma_a \\ \bar\sigma_a & 0 \end{pmatrix} \quad , \quad \begin{array}{l} \sigma_a = (\vec\sigma, i\mathbf{1}_2) \\ \bar\sigma_a = (-\vec\sigma, i\mathbf{1}_2,) \end{array} \quad , \quad \{\gamma_a, \gamma_b\} = 2\delta_{ab} \quad , \quad \gamma^5 = \begin{pmatrix} -\mathbf{1}_2 & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix}$$

where $a,b = 1,2,3,4$ and $\vec\sigma \equiv (\sigma^1, \sigma^2, \sigma^3)$ are the Pauli matrices. If the equations

$$\delta_{ab} e^{\mu a} e^{\nu b} = h^{\mu\nu} \quad , \quad h^{\mu\nu} e_{a\mu} e_{b\nu} = \delta_{ab} \ , \quad \mu,\nu = 1,2,3,4$$

determine the vier-bein $e^{\mu a}$, the "square root" of the metric $h^{\mu\nu}$ in an oriented Riemannian four-manifold X, the Clifford algebra on the curved space X is defined as follows:

$$\gamma^\mu = e^{\mu a}\gamma_a \quad \Rightarrow \quad \{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2h^{\mu\nu} \quad .$$

The Dirac operator $\slashed{D}_A = \gamma^\mu D_\mu$ acts on the space of sections of the Spin$_\mathbb{C}$ bundle $\Gamma\left((S^+ \oplus S^-) \otimes L\right)$ over X. The covariant derivative D_μ is defined in terms of the spin connection $\omega_{\mu ab}$ and the $U(1)$ connection A_μ.

$$D_\mu = \nabla_\mu + iA_\mu \quad , \quad \nabla_\mu = \partial_\mu + \omega_{\mu ab}[\gamma_a, \gamma_b] \quad .$$

The twisted Seiberg-Witten effective action at low energy involves two contributions:

1. The action including the twisted matter fields together with their Yukawa couplings to the Abelianized fields in the $\mathcal{N} = 2$ supermultiplet:

$$\begin{aligned}
S_M^{\text{eff}} =\ & \int_X \sqrt{\det h}[h^{\mu\nu} D_\mu S^\dagger D_\nu S + \frac{1}{4}RS^\dagger S - \frac{1}{16}S^\dagger \Sigma_{\mu\nu} SS^\dagger \Sigma^{\mu\nu} S] \\
& + \int_X \sqrt{\det h}\{i\phi\lambda S^\dagger S + \frac{1}{2\sqrt{2}}\chi^{\mu\nu}(S^\dagger \Sigma_{\mu\nu}\mu + \mu^\dagger \Sigma_{\mu\nu} S) \\
& - \frac{i}{2}(\nu^\dagger \slashed{D}_A\mu - \mu^\dagger \slashed{D}_A\nu) + \frac{1}{2}(S^\dagger \gamma^\mu \psi_\mu \nu - \nu^\dagger \gamma^\mu \psi_\mu S) & (71) \\
& + \frac{1}{2}\eta(\mu^\dagger S - S^\dagger \mu) + \frac{i}{4}(\phi\nu^\dagger \nu - \lambda\mu^\dagger \mu)\} & (72)
\end{aligned}$$

Here $h^{\mu\nu}$ is the metric tensor in X, R is the corresponding scalar curvature and $\Sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$.

2. The abelianization of the Donaldson action (58) reads

$$S^{\text{eff}} = \int d^4x \ \sqrt{\det h} \ \{\frac{1}{2}f^+_{\mu\nu}f^{\mu\nu}_+ + i\chi^{\mu\nu}D_\mu\psi_\nu - i\eta D^\mu\psi_\mu + \frac{1}{2}\lambda D_\mu D^\mu\phi\} \qquad (73)$$

where $f^+_{\mu\nu}$ is the self-dual Abelian gauge field tensor and the other fields are the Abelian counterparts of the fields entering in the twisted Yang-Mills action (58).

Having in mind that

$$h^{\mu\nu}D_\mu S^\dagger D_\nu S = \frac{1}{2}D_\mu S^\dagger\{\gamma^\mu,\gamma^\nu\}D_\nu S = \slashed{D}_A S^\dagger \ \slashed{D}_A S + D_\mu S^\dagger\Sigma^{\mu\nu}D_\nu S \quad .$$

we select in the effective action $S^{\text{eff}} + S^{\text{eff}}_M$ two definite positive terms involving the spinorial field (after twisting) S:

$$S^{\text{eff}}_{SW} = \int_X \sqrt{\det h} \ \left[(f^+_{\mu\nu} + \frac{i}{2}S^\dagger\Sigma_{\mu\nu}S)(f^{\mu\nu+} + \frac{i}{2}S^\dagger\Sigma^{\mu\nu}S) + \slashed{D}_A S^\dagger \ \slashed{D}_A S + \cdots \right] \qquad (74)$$

We observe that the positive perfect square terms in S^{eff}_{SW} are zero if the following coupled system of non-linear PDE's

$$(1) \ \ \slashed{D}_A S = 0 \quad , \qquad\qquad (2) \ \ f^+_{\mu\nu} = -\frac{i}{2}S^\dagger\Sigma_{\mu\nu}S \qquad (75)$$

holds. Henceforth, the solutions of the PDE system (75), referred to as the non-linear Seiberg-Witten equations, are the absolute minima of S^{eff}_{SW}. These equations describe the obstruction to self-duality of a $U(1)$-gauge field tensor on a four-manifold due to the anomalous magnetic momentum induced by an harmonic right-handed spinor via the coupling $f^+_{\mu\nu}S^\dagger\Sigma_{\mu\nu}S$. Contrarily to the moduli space of instantons the moduli space of the Seiberg-Witten solutions is compact [2] and presents no singularities due to reducible connections. Therefore, diffeomorphism invariants of four-manifolds are more accessible through the topological invariants of the moduli spaces of Seiberg-Witten solutions than via the topological structures of moduli spaces of instantons, see [27], [29].

The dimension of the moduli space of Seiberg-Witten solutions is envisaged from the linearization of the (75) system of PDE equations. Up to first-order in the perturbations $A^{\text{SW}}_\mu(x) + t a_\mu(x)$, $S_{\text{SW}}(x) + s\Psi(X)$ around a given SW solution $(A^{\text{SW}}_\mu, S_{\text{SW}})$, we find the linearized Seiberg-Witten equations

$$\gamma_\mu\left(\nabla_\mu + i a_\mu(x)\right)S_{\text{SW}}(x) \ + \ \gamma_\mu\left(\nabla_\mu + iA^{\text{SW}}_\mu(x)\right)\Psi(x) = 0 \qquad (76)$$

$$\nabla_\mu a_\nu(x) - \nabla_\nu a_\mu(x) \ + \ \frac{\sqrt{|g|}}{2}\varepsilon_{\mu\nu\rho\delta}\left(\nabla_\rho a_\delta(x) - \nabla_\delta a_\rho(x)\right) =$$

$$= \ \frac{i}{4}\left(\Psi^\dagger(x)[\gamma_\mu,\gamma_\nu]S_{\text{SW}}(x) + S^\dagger_{\text{SW}}(x)[\gamma_\mu,\gamma_\nu]\Psi(x)\right) \qquad (77)$$

where $|g|$ is the determinant of the tensor metric in X and ∇_μ denotes the covariant derivative acting on spinorial and/or tensorial sections. Setting, e.g., the background gauge $\left(\nabla_\mu + iA^{\text{SW}}_\mu\right)a_\mu = 0$

_____

[2]The reason is that there are no solutions of the PDE system (75) which shrink to zero size because there are no L^2 solutions in $\mathbb{R}^4$.

to avoid pure gauge perturbations one may identify the dimension of the moduli space of Seiberg-Witten solutions by an index theorem argument. The number of zero modes arising as normalizable solutions (a_μ, Ψ) of equation (76) is essentially captured by [3] the index of the Dirac operator acting on sections of the Spin^C bundle $S_+ \otimes L$: $c_1(L)^2 - \frac{\sigma(X)}{4}$, i.e., it is determined in terms of the first Chern class of the complex line bundle L and the signature of the four manifold X. Simili modo the number of zero modes coming from the solutions of (77) is established from the index of the Hodge operator $d + d^*$: $-\chi(X)/2 - \sigma(X)/2$ where now the Euler characteristic $\chi(X)$ enters. All together the dimension of the moduli space of solutions of the Seiberg-Witten equations is found to be:

$$d(\mathcal{M}_{\mathrm{SW}}) = c_1(L)^2 - \frac{2\chi(X) + 3\sigma(X)}{4} \quad . \tag{78}$$

4.4 Kronheimer-Mrowka basic classes and Seiberg-Witten invariants

We now briefly describe the differential invariants of smooth four manifolds derived from the topological structures arising in the moduli space of solutions of the Seiberg-Witten equations. A good physics flavoured treatment of this topic is offered in Labastida Lectures, see [4]. In Reference [5] a deep and extremely condensed explanation can be found about how the basic classes of Kronheimer and Mrowka, [6], prompt zero dimensional moduli spaces of Seiberg-Witten solutions, which in turn provide a very effective procedure of computation of the Seiberg-Witten invariants. Following Reference [27] we asign to an homology two-cycle in the four manifold X, $\gamma \in H_2(X)$, Donaldson polynomials of the form, recall the formula (70),

$$p_{s,r}(\gamma, X) = \int \mathcal{D}\mathcal{F}\, e^{-S/g^2} \prod_{j=1}^{r} W_0(P_j) \left(\int_\gamma W_2 \right)^s = \langle \prod_{j=1}^{r} W_0(P_j) \left(\int_\gamma W_2 \right)^s \rangle \quad ,$$

which are non null only if the dimension of the moduli space of anti-self-dual instantons is such that : $d(\mathcal{M}) = 4r + 2s$, equivalent to $b_2^+ = $ odd.

Suppose that the four manifold X is such that $p_{s,r+2}(\gamma, X) = 4p_{s,r}(\gamma, X)$. Manifolds enjoying this property are called of simple type. The primitive Donaldson polynomials arising in this situation

$$q_s(\gamma, X) = \begin{cases} p_{s,0}(\gamma, X) \text{ if } s = 1 + b_2^+ \bmod 2 \\[2ex] p_{s,1}(\gamma, X) \text{ if } s = b_2^+ \bmod 2 \end{cases}$$

are assembled in a generating function: $q(\gamma, X) = \sum_{s=0}^{\infty} \frac{1}{s!} q_s(\gamma, X)$. The Kronheimer-Mrowka formula for this generating function is:

$$q_s(\gamma, X) = \exp[\frac{\gamma \cdot \gamma}{2}] \sum_{I=1}^{N} a_I \exp[\kappa_I \cdot \gamma] \quad . \tag{79}$$

In this expansion $a_1, a_2, \cdots, a_N$ are non null rational numbers which define the Seiberg-Witten invariants. $\kappa_I \in H^2(X, \mathbb{Z})$ are the Kronheimer-Mrowka <u>basic</u> classes. A 2D cohomology class, e.g. κ_I, defines a line bundle over a 2D surface through the first Chern class, i.e. $c_1(L_I^2) = \kappa_I$ such that KM classes are related to the square of the line bundle L_I: $c_1(L_I) = c_1(L_I^2/2)$. The basic classes are characterized by the property: $c_1^2(L_I^2) = \kappa_I^2 = 2\chi(X) + 3\sigma(X)$, which is thus an integer number $\kappa_I^2 = 5b_2^+ - b_2^-$.

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[3]In fact the line bundle L is accounted for as a real bundle in such a way that we count twice the index of the Dirac operator.

Evaluation of the a_I coefficients was achieved by Witten in Reference [27]. The pillars of his calculation were settle down in the effective twisted Abelian $\mathcal{N} = 2$ SUSY gauge theory governed by the action $S_M^{\text{eff}} + S^{\text{eff}} = S_{\text{SW}}^{\text{eff}}$, see (72), (73) and (74). Instead of looking at vacuum expectation values of the microscopic $\mathcal{N} = 2$ twisted SUSY non-Abelian gauge theory on X in the high energy regime computations are performed in the infrared (strong coupling) domain where the macroscopic Abelian effective theory emerges. Recall that the partition function Z is independent of the coupling constant g. Thus, identical expectation values can be equivalently calculated either at weak g-coupling, where the action (58) and the partition function (59) are the main instruments, or, at strong g-coupling, where computations are based on the action (73) and the partition function (72) in the monopole patch of the moduli space of vacua [4].The steepest descent approximation applied to the Feynman path integral in this patch shows that it is localized near the saddle points of $S_{\text{SW}}^{\text{eff}}$, i.e., the solutions of the Seiberg-Witten equations (75), rather than around anti-self-dual instantons. Because formula (78) tells us that

$$d(\mathcal{M}_{\text{SW}}) = \frac{1}{4}\left(c_1^2(L^2) - 2\chi(X) - 3\sigma(X)\right)$$

the Kronheimer-Mrowka basic classes are those for which the dimension of the moduli space of solutions of the Seiberg-Witten equations is zero: $d(\mathcal{M}_{\text{SW}}) = 0$. Therefore, there is a finite number of points ν in $\mathcal{M}_{\text{SW}}$ and the Seiberg-Witten partition function collects the number of SW-solutions weighted with their signs, a formula completely analogous to (66):

$$Z_{\text{SW}} \propto n_L = \sum_{i=1}^{\nu} \varepsilon_i = \sum_{i=1}^{\nu} (-1)^{N_i} \ , \ \ N_i = 0, 1 \quad .$$

Again, $N_i = 0, 1$ counts the number of zeroes $\mod 2$ crossed by the spectrum of D_F^t when the fermionic fluctuation operator D_F varies through a family of fields joining two solutions of the Seiberg-Witten equations (recall the analysis just before (66)). The generating function of the Donaldson polynomials of a four manifold X of simple type is then obtained by summing all the contributions of this kind for the N basic classes:

$$q(\gamma, X) = 2^{(1+d(X))}\exp[\frac{\gamma \cdot \gamma}{2}] \sum_{I=1}^{N} n_{L_I} \cdot \exp[c_1(L_I^2) \cdot \gamma] \quad . \tag{80}$$

The factor of 2 is due to the fact that, even though instanton moduli spaces are invariant with respect to the center of $SU(2)$, the Donaldson invariants are defined without dividing by two. The critical exponent $d(X) = \frac{1}{4}(7\chi(X) + 11\sigma(X))$ is a c-renormalization factor which appear when one compares the expectation values of Donaldson polynomials computed in the microscopic $SU(2$ non-Abelian theory with the outcome in the effective Abelian theory of massless monopoles, see [27]-[28]. To set the values of the coefficients 7/4, 11/4 the argument runs as follows: there is no perfect duality invariance when one consider the theory in any pont u of the vacuum moduli space. On a curved 4-manifold gravitational anomalies in the Fermionic fields integration measure of the form:

$$d\mu^F = \exp\left[b(u)\chi(X) + c(u)\sigma(X)\right]d\mu_D^F, \ \ \chi(X) = \frac{1}{24\pi^2}\int_X R \wedge \widetilde{R}, \ \ \sigma(X) = \frac{1}{24\pi^2}\int_X R \wedge R$$

$$R = R_\mu{}^\alpha{}_{\nu\alpha}dx^\mu \wedge dx^\nu \ , \qquad \widetilde{R} = \frac{\sqrt{|g|}}{2}\varepsilon_{\rho\sigma}{}^{\alpha\beta}R_\alpha{}^\gamma{}_{\beta\gamma}dx^\rho \wedge dx^\sigma \tag{81}$$

---

[4]A completely equivalent treatment at strong g-coupling is possible near the $u = -\Lambda^2$ singularity where massless dyons replace massless monopoles as new particles in the spectrum.

arise. Here R and $\widetilde{R}$ are respectively the Ricci tensor and its dual of the Riemannian manifold X. $\chi(X)$, the Euler number, and the signature, $\sigma(X)$, are the only observables of dimension four which are topological invariants and may arise in the topological, twisted, $\mathcal{N} = 2$ Supersymmetric Gauge Theory. Thus a factor $\exp\left[b(u)\chi(X) + c(u)\sigma(X)\right]$ must be included in the twisted action. Witten's cunning strategy was to identify in the weak coupling limit $b(u \to \infty)$ as $b = \frac{7}{4}\log 2$ and $c(u \to \infty)$ as $c = \frac{11}{4}\log 2$ to fit with the Donaldson invariants of known four manifolds as $K3$ surfaces and/or manifolds of simple type. Because topological invariance these values must be constant all over the moduli space and the coefficient $2^{d(X)}$ arise this way also in the strong coupling $u = \pm\Lambda^2$ regimes. It is astonishing how an extremely subtle renormalization coefficient due to the physics of gravitational, and perhaps modular, anomalies in $\mathcal{N} = 2$ SUSY gauge theory may be derived from purely mathematical information about the differential structures of some specific four manifolds.

4.5 Hidden physics behind the low-energy twisted Seiberg-Witten action

Understanding of its physical meaning suggests to scrutinize the two-component theory, see Reference [31], which is akin to the second-order dual QED governed by the twisted SW action on the Euclidean $\mathbb{R}^4$ space-time. The action is:

$$
\begin{aligned}
S_E^{\text{SW}} &= \int dx^4 \left\{ \frac{1}{4}\left(f_{\mu\nu}f_{\mu\nu} + \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}f_{\mu\nu}f_{\rho\sigma}\right) + \frac{1}{2}D_\mu S_+^\dagger D_\mu S_+ + D_\mu S_+^\dagger \Sigma_{\mu\nu} D_\nu S_+ \right\} \\
&+ \int dx^4 \left\{ -i\frac{\lambda}{2}S_+^\dagger \Sigma_{\mu\nu} S_+ \cdot f_{\mu\nu}^+ + \frac{\lambda^2}{8}S_+^\dagger \Sigma_{\mu\nu} S_+ \cdot S^\dagger \Sigma_{\mu\nu} S_+ \right\} \ .
\end{aligned}
\tag{82}
$$

It is convenient to write explicitly the quantities involving the two-component spinor S:

$$
S_+^\dagger(x) = (0 \ \ 0 \ \ \phi_1^*(x) \ \ \phi_2^*(x)) \ \ , \ \ D_\mu = \frac{\partial}{\partial x_\mu} + igA_\mu(x) \ \ , \ \ S_+(x) = \begin{pmatrix} 0 \\ 0 \\ \phi_1(x) \\ \phi_2(x) \end{pmatrix}
$$

$$
\Sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu] = \frac{1}{2}\begin{pmatrix} \sigma_\mu\bar\sigma_\nu - \sigma_\nu\bar\sigma_\mu & 0 \\ 0 & \bar\sigma_\mu\sigma_\nu - \bar\sigma_\nu\sigma_\mu \end{pmatrix}
$$

where $g = \frac{4\pi}{e}$ is the magnetic charge dual to the electric charge e and λ is a non-dimensional coupling which sets the strength of the anomalous magnetic momentum of the spinorial particle.

Besides the conventional photon propagator due to the Maxwell term in the action (82) there is a propagator of a spin one-half particle of the form:

$$
P^{-1} = \left(p_\mu \left[\delta_{\mu\nu}\mathbf{1}_2 + \bar\sigma_\mu\sigma_\nu - \bar\sigma_\nu\sigma_\mu\right] p_\nu \right)^{-1} \ .
\tag{83}
$$

Note that

$$
\det P = p_\mu p_\mu - \frac{1}{12}\varepsilon_{\mu\nu\rho\sigma}p_\mu p_\nu p_\rho p_\sigma - \frac{1}{4}\sum_{a=1}^{3} \eta_{\mu\nu}^a p_\mu p_\nu \cdot \eta_{\rho\sigma}^a p_\rho p_\sigma \ ,
$$

where $\eta_{\mu\nu}^a$ are the three complex structures in R^4 or 't Hooft symbols, and the P-matrix is invertible. There is also a (ineffective) toological term in the SW action: the Abelian second Chern

class. More interesting; six types of electro-magnetic couplings give rise to trivalent or fourvalent vertices:

$$(1) - i\frac{g}{2}A_\mu\left(S_+^\dagger\partial_\mu S_+ - \partial_\mu S_+^\dagger S_+\right) \qquad , \quad (2)\ \frac{g^2}{2}A_\mu A_\mu S_+^\dagger S_+ \tag{84}$$

$$(3) - ig\left(A_\mu S_+^\dagger\Sigma_{\mu\nu}\partial_\nu S_+ - \partial_\mu S_+^\dagger\Sigma_{\mu\nu}A_\nu S_+\right) \quad , \quad (4)\ -g^2 A_\mu S_+^\dagger\Sigma_{\mu\nu}A_\nu S_+ \tag{85}$$

$$(5) - i\frac{\lambda}{2}S_+^\dagger\Sigma_{\mu\nu}S_+ \cdot f_{\mu\nu}^+ \qquad , \quad (6)\ \frac{\lambda^2}{8}S_+^\dagger\Sigma_{\mu\nu}S_+ \cdot S^\dagger\Sigma_{\mu\nu}S_+ \ . \tag{86}$$

These strange Feynman rules are akin to those emerging in Veltman Two component theory and electron magnetic moment, see [31] The Veltman second-order QED Feynman rules come from the Lagrangian obtained from the QED Lagrangian through multiplication of the electron spinor field by the Dirac operator:

$$\psi \longrightarrow \left(-\gamma_\mu D_\mu + m\right)\psi$$

and afterwards projecting to two-component (e.g. only positron) spinors. The twisted Seiberg-Witten action is a dual version of this gauge theory. There are more vertices than in Veltman theory where only the vertices of type (1), (2), and (3) arise. The dual electron propagator becomes also more complex in the Seiberg-Witten framework because the anomalous dual magnetic moment enter. Nevertheless, the main virtue, the separation between electric and magnetic couplings, is shared by the dual Seiberg-Witten twisted QED and Veltman second-order QED, whereas both Lagrangians are non-hermitian.

If $g = 1 = \lambda$ the twisted dual SW effective action can be written in the form

$$= \int dx^4 \left[\frac{1}{2}\left(f_{\mu\nu}^+ + \frac{i}{2}S_+^\dagger\Sigma_{\mu\nu}S_+\right)\left(f_{\mu\nu}^+ + \frac{i}{2}S_+^\dagger\Sigma_{\mu\nu}S_+\right) + (\gamma_\mu D_\mu S_+)^\dagger\gamma_\nu D_\nu S_+\right] \quad , \tag{87}$$

plus "topological" terms.

Solutions of the first-order Seiberg-Witten PDE system (75) in $\mathbb{R}^4$

$$f_{\mu\nu}^+(x) = -\frac{i}{2}S^\dagger(x)\Sigma_{\mu\nu}S(x) \qquad , \qquad \gamma_\mu D_\mu S(x) = 0 \tag{88}$$

are thus absolute minima of S_E^{W}. We know that there are no L^2-integrable solutions of of this PDE system in Euclidean 4D space. Nevertheless, it is possible to obtain non-L^2 solutions by means of several dimensional reduction procedures that help to grasp the physical meaning of the Seiberg-Witten equations.

4.6 Low dimensional Freund and Seiberg-Witten solutions

We devote this subsection to analyze dimensionally reduced solutions to the Seiberg-Witten equations [32]-[33] in order to elucidate the topological nature of the non perfect square terms in the action (87) as well as the hidden physics in the twisted effective dual second-order QED arising at the ultrastrong coupling.

4.6.1 Three-dimensional Freund solutions

We write the Freund equations, the Seiberg-Witten equations with a flip in the spinor term sign, in components:

$$f_{12}^+(x) = f_{34}^+(x) = -\frac{1}{2}\left(|\phi_1(x)|^2 - |\phi_2(x)|^2\right) \tag{89}$$

$$f_{31}^+(x) = f_{24}^+(x) = \frac{i}{2}\Big(\phi_1^*(x)\phi_2(x) - \phi_2^*(x)\phi_1(x)\Big) \tag{90}$$

$$f_{23}^+(x) = f_{14}^+(x) = -\frac{1}{2}\Big(\phi_1^*(x)\phi_2(x) + \phi_2^*(x)\phi_1(x)\Big) \tag{91}$$

This PDE system is nothing but the equation in the left of formula (88) with a plus sign in the right-hand fellow and must be solved together with the two-component Dirac equation:

$$\begin{pmatrix} iD_3 - D_4 & iD_1 + D_2 \\ iD_1 - D_2 & -iD_3 - D_4 \end{pmatrix} \cdot \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad . \tag{92}$$

First, we address dimensional reduction to $\mathbb{R}^3$, i.e. spinors and gauge potentials are x_4-independent and choose $A_4 = 0$. One can check that the Dirac monopole singular potential, in the axial gauge $A_3(x_1, x_2, x_3) = 0$,

$$A_1(x_1, x_2, x_3) = -\frac{x_2}{2r(r - x_3)} \quad , \quad A_2(x_1, x_2, x_3) = \frac{x_1}{2r(r - x_3)} \tag{93}$$

together with the harmonic monopole spinors

$$\phi_1(x_1, x_2, x_3) = \frac{x_1 - ix_2}{2r\sqrt{r(r - x_3)}} \quad , \quad \phi_2(x_1, x_2, x_3) = \frac{1}{2r} \cdot \sqrt{\frac{r - x_3}{r}} \tag{94}$$

satisfy the 3D Freund equations:

$$f_{12}^+(x_1, x_2, x_3) = -\frac{x_3}{r^3} = -\frac{1}{2}\Big(|\phi_1(x)|^2 - |\phi_2(x)|^2\Big)$$

$$f_{31}^+(x_1, x_2, x_3) = -\frac{x_2}{4r^3} = \frac{i}{2}\Big(\phi_1^*(x)\phi_2(x) - \phi_2^*(x)\phi_1(x)\Big)$$

$$f_{23}^+(x_1, x_2, x_3) = -\frac{x_1}{4r^3} = -\frac{1}{2}\Big(\phi_1^*(x)\phi_2(x) + \phi_2^*(x)\phi_1(x)\Big) \quad .$$

A little more work is needed to show that the spinor (94) satisfies the Dirac equation (92), reduced to three dimensions, in the monopole background. The physical meaning of this Freund solution is clear. The gauge potential is singular at $r = x_3$ positive half-axis where the famous Dirac string is located. The harmonic spinors are even more singular at the Dirac string location because they have branching points along that curve. Nevertheless, there are other quantities which are less singular, e.g., the dual magnetic (electric) field only has a pole at the origin:

$$b_1(x_1, x_2, x_3) = f_{23}(x_1, x_2, x_3) = -\frac{x_1}{2r^3} \quad , \quad b_2(x_1, x_2, x_3) = f_{31}(x_1, x_2, x_3) = -\frac{x_2}{2r^3}$$

$$b_3(x_1, x_2, x_3) = f_{12}(x_1, x_2, x_3) = -\frac{x_3}{2r^3} \quad .$$

The "electric" flux through an sphere of radius R centered at the origin is:

$$\Phi = \int_{S_R^2} (b_1(x_1, x_2, x_3)dx_2 \wedge dx_3 + b_2(x_1, x_2, x_3)dx_3 \wedge dx_1 + b_3(x_1, x_2, x_3)dx_1 \wedge dx_2) = 2\pi \ ,$$

i.e., the monopolar electric charge carried by the Freund solution is the minimal compatible with the Dirac quantization condition.

4.6.2 Planar Seiberg-Witten solutions

Finally, we consider the true Seiberg-Witten equations

$$f_{12}^+(x) = f_{34}^+(x) = \frac{1}{2}\Big(|\phi_1(x)|^2 - |\phi_2(x)|^2\Big) \tag{95}$$

$$f_{31}^+(x) = f_{24}^+(x) = -\frac{i}{2}\Big(\phi_1^*(x)\phi_2(x) - \phi_2^*(x)\phi_1(x)\Big) \tag{96}$$

$$f_{23}^+(x) = f_{14}^+(x) = \frac{1}{2}\Big(\phi_1^*(x)\phi_2(x) + \phi_2^*(x)\phi_1(x)\Big) \tag{97}$$

together with the Dirac equation (92):

$$(D_3 + iD_4)\phi_1(x) + (D_1 - iD_2)\phi_2(x) = 0 \tag{98}$$

$$(D_1 + iD_2)\phi_1(x) - (D_3 - iD_4)\phi_2(x) = 0 . \tag{99}$$

Multiplying (98) by $D_1 + iD_2$, (99) by $D_3 + iD_4$, subtracting the second equation from the first, and using (96)-(97) we find:

$$- 2|\phi_1|^2(x)\phi_2(x) + \bar{\partial}_{A_{12}}\partial_{A_{12}}\phi_2(x) + \bar{\partial}_{A_{34}}\partial_{A_{34}}\phi_2(x) = 0 , \tag{100}$$

where we have defined:

$$\bar{\partial}_{A_{12}}\partial_{A_{12}} = (D_1 + iD_2)(D_1 - iD_2) \quad , \quad \bar{\partial}_{A_{34}}\partial_{A_{34}} = (D_3 + iD_4)(D_3 - iD_4) .$$

Multiplication of (100) by $\phi_2^*(x)$, and integration over all $\mathbb{R}^4$ leads to the identity

$$\int_{\mathbb{R}^4} d^4x \ \Big\{|\phi_1|^2(x)\cdot|\phi_2|^2(x) + |\partial_{A_{12}}\phi_2(x)|^2 + |\partial_{A_{34}}\phi_2(x)|^2\Big\} = 0 , \tag{101}$$

after a partial integration, to be satisfied by the Seiberg-Witten solutions.

There are two possibilities:

- - A. $\phi_2 = 0$. On configurations with "electric" spin 1/2, $\sigma_3 \begin{pmatrix} \phi_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \phi_1 \\ 0 \end{pmatrix}$, we consider the planar ansatz:

$$\phi_1(x) = \phi_1(x_1, x_2), \quad \phi_2 = A_3 = A_4 = 0, \quad A_1(x) = A_1(x_1, x_2) \quad A_2(x) = A_2(x_1, x_2).$$

The Seiberg-Witten equations (95-96-97-98-99) reduce to the first-order PDE system:

$$f_{12}(x_1, x_2) = \frac{1}{2}|\phi_1(x_1, x_2)|^2 \quad , \quad (D_1 + iD_2)\phi_1(x_1, x_2) = 0 . \tag{102}$$

The solution of the covariant analyticity condition in (102) is:

$$A_{\bar{z}} = \frac{1}{2}(A_1 + iA_2) = -i\partial_{\bar{z}}\log\phi_1 = \frac{1}{2}(\partial_1 + i\partial_2)\log\phi_1 ,$$

which, because $f_{12} = -2i(\partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z)$, converts the vortex equation (102-left) into the Liouville equation:

$$2\partial_{\bar{z}}\partial_z \log\phi_1 + \phi_1^*\phi_1 = 0 . \tag{103}$$

The general solution of (103) such that $\lim_{x_1^2+x_2^2\to\infty}\phi_1(x_1,x_2)=0$, guaranteeing finite energy density in the x_1-x_2-plane, is:

$$\phi_1^{(k)}(z)=\frac{2f'(z)V^2(z)}{|V(z)|^2+|f(z)V(z)|^2}\quad,\quad z=x_1+ix_2 \tag{104}$$

where the following choice of the V and f functions as

$$V(z)=\prod_{j=1}^{k}(z-z^{(j)})\quad,k\in\mathbb{N}^*\ ,\qquad f(z)=f_0+\sum_{j=1}^{k}\frac{c_k}{z-z^{(j)}}$$

gives rise to non-singular solutions supporting quantized electric flux:

$$\Phi_E=\int\int dx_1dx_2\, f_{12}(x_1,x_2)=\frac{1}{2}\int\int dx_1dx_2\,|\phi_1^{(k)}(z)|^2=\frac{2\pi}{g}k$$

where the g-coupling constant has been re-surfaced.

Observe that k is positive and the flux is located around $z^{(j)}$, the zeroes of $\phi_1(z)$. It spreads out from these zeroes, however, with $|c_j|$, the length scale of the solution, which is a free parameter due to the breaking by the flux tube of the scale invariance of the theory. There is also freedom in choosing $\arg c_j$ because the $U(1)_d$ symmetry and the moduli space of solutions is $\mathbb{C}^{2k}$; the moduli space parameters are the centers of the solitons $z^{(j)}$ and the modulus and phase of c_j determining the scale and phase of each individual soliton.

- -B. $\phi_1=0$, $|\partial_{A_{12}}\phi_2|=|\partial_{A_{34}}\phi_2|=0$.

On configurations with "electric" spin $-1/2$, $\sigma_3\begin{pmatrix}0\\\phi_2\end{pmatrix}=-\begin{pmatrix}0\\\phi_2\end{pmatrix}$, the appropriate planar ansatz reads:

$$\phi_2(x)=\phi_2(x_3,x_4),\ \ \phi_1=A_1=A_2=0,\ \ A_3(x)=A_3(x_3,x_4)\ \ A_4(x)=A_4(x_3,x_4)\,.$$

The Seiberg-Witten equations (95-96-97-98-99) reduce to the first-order PDE system:

$$f_{34}(x_3,x_4)=-\frac{1}{2}|\phi_2(x_3,x_4)|^2\quad,\quad (D_3-iD_4)\phi_2(x_3,x_4)=0\ . \tag{105}$$

The solution of the covariant anti-analyticity condition in (105) is:

$$A_w=\frac{1}{2}(A_3-iA_4)=i\partial_w\log\phi_2=\frac{1}{2}(\partial_3-i\partial_4)\log\phi_2\ ,$$

which, because $f_{34}=-2i(\partial_w A_{\bar w}-\partial_{\bar w}A_w)$, converts the vortex equation (105-left) into the Liouville equation:

$$2\partial_{\bar w}\partial_w\log\phi_2-\phi_2^*\phi_2=0\ . \tag{106}$$

The general solution of (103) such that $\lim_{x_3^2+x_4^2\to\infty}\phi_2(x_3,x_4)=0$, guaranteeing finite energy density in the x_3-x_4-plane, is:

$$\phi_2^{(k)}(\bar w)=\frac{2f'(\bar w)V^2(\bar w)}{|V(\bar w)|^2+|f(\bar w)V(\bar w)|^2}\quad,\quad \bar w=x_3-ix_4 \tag{107}$$

where the following choice of the V and f functions as

$$V(\bar{w}) = \prod_{j=1}^{k}(\bar{w} - \bar{w}^{(j)}) \quad , k \in \mathbb{N}^* , \qquad f(\bar{w}) = f_0 + \sum_{j=1}^{k} \frac{c_k}{\bar{w} - \bar{w}^{(j)}}$$

gives rise to non-singular solutions supporting quantized "magnetic" flux:

$$\Phi_M = \int \int dx_3 dx_4 \, f_{34}(x_3, x_4) = \frac{1}{2} \int \int dx_3 dx_4 \, |\phi_2^{(k)}(\bar{w})|^2 = -\frac{2\pi}{g}k$$

The moduli space of these Type B planar solutions of the Seiberg-Witten equations present analogous features to those of the Type A moduli space described above.

There are also pairs of Type A-Type B planar solutions with the same characteristics- obeying similar ansatzes-in the x_2-x_3-/ x_4-x_1-planes as well as in the x_3-x_1-/x_2-x_4-planes, to wit:

- -A. $\begin{pmatrix} \psi_2 \\ -\psi_2 \end{pmatrix} = 0$. On configurations such that $\sigma_1 \left[\frac{1}{\sqrt{2}} \begin{pmatrix} \psi_1 \\ \psi_1 \end{pmatrix} \right] = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_1 \\ \psi_1 \end{pmatrix}$, we consider the planar ansatz:

  $$\psi_1(x) = \psi_1(x_2, x_3), \quad \psi_2 = A_1 = A_4 = 0, \quad A_2(x) = A_2(x_2, x_3) \quad A_3(x) = A_2(x_2, x_3).$$

  Because $\phi_1 = \phi_2 = \frac{1}{\sqrt{2}}\psi_1$ the Seiberg-Witten equations (95-96-97-98-99) reduce to the first-order PDE system:

  $$f_{23}(x_2, x_3) = \frac{1}{2}|\psi_1(x_2, x_3)|^2 \quad , \quad (D_2 + iD_3)\psi_1(x_2, x_3) = 0 \ , \tag{108}$$

  allowing for an identical moduli space of solutions as (102).

- -B. $\begin{pmatrix} \psi_1 \\ \psi_1 \end{pmatrix} = 0$. On configurations such that $\sigma_1 \left[\frac{1}{\sqrt{2}} \begin{pmatrix} \psi_2 \\ -\psi_2 \end{pmatrix} \right] = -\frac{1}{\sqrt{2}} \begin{pmatrix} \psi_2 \\ -\psi_2 \end{pmatrix}$, we consider the planar ansatz:

  $$\psi_2(x) = \psi_2(x_1, x_4), \quad \psi_1 = A_2 = A_3 = 0, \quad A_1(x) = A_1(x_1, x_4) \quad A_4(x) = A_2(x_1, x_4).$$

  Because $\phi_1 = -\phi_2 = \frac{1}{\sqrt{2}}\psi_2$ the Seiberg-Witten equations (95-96-97-98-99) reduce to the first-order PDE system:

  $$f_{14}(x_1, x_4) = -\frac{1}{2}|\psi_2(x_1, x_4)|^2 \quad , \quad (D_1 - iD_4)\psi_2(x_1, x_4) = 0 \ , \tag{109}$$

  giving rise to the same moduli space of solutions as (105).

- -A. $\begin{pmatrix} \psi_2 \\ -i\psi_2 \end{pmatrix} = 0$. On configurations such that $\sigma_2 \left[\frac{1}{\sqrt{2}} \begin{pmatrix} \psi_1 \\ i\psi_1 \end{pmatrix} \right] = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_1 \\ i\psi_1 \end{pmatrix}$, we consider the planar ansatz:

  $$\psi_1(x) = \psi_1(x_1, x_3), \quad \psi_2 = A_2 = A_4 = 0, \quad A_1(x) = A_2(x_1, x_3) \quad A_3(x) = A_2(x_1, x_3).$$

  Because $\phi_1 = -i\phi_2 = \frac{1}{\sqrt{2}}\psi_1$ the Seiberg-Witten equations (95-96-97-98-99) reduce to the first-order PDE system:

  $$f_{31}(x_2, x_3) = \frac{1}{2}|\psi_1(x_1, x_3)|^2 \quad , \quad (D_1 + iD_3)\psi_1(x_1, x_3) = 0 \ , \tag{110}$$

  allowing for an identical moduli space of solutions as (102).

- -B. $\begin{pmatrix} \psi_1 \\ i\psi_1 \end{pmatrix} = 0$. On configurations such that $\sigma_2\left[\frac{1}{\sqrt{2}}\begin{pmatrix} \psi_2 \\ -i\psi_2 \end{pmatrix}\right] = -\frac{1}{\sqrt{2}}\begin{pmatrix} \psi_2 \\ -i\psi_2 \end{pmatrix}$, we consider the planar ansatz:

$$\psi_2(x) = \psi_2(x_2, x_4)\,, \quad \psi_1 = A_1 = A_3 = 0\,, \quad A_2(x) = A_1(x_2, x_4) \quad A_4(x) = A_2(x_2, x_4)\,.$$

Because $\phi_1 = -i\phi_2 = \frac{1}{\sqrt{2}}\psi_2$ the Seiberg-Witten equations (95-96-97-98-99) reduce to the first-order PDE system:

$$f_{24}(x_1, x_4) = -\frac{1}{2}|\psi_2(x_2, x_4)|^2 \quad , \quad (D_2 - iD_4)\psi_2(x_1, x_4) = 0 \ , \tag{111}$$

giving rise to the same moduli space of solutions as (105).

Finally, we mention that adding a mass term for the spinors that explicitly breaks the scale invariance and spontaneously breaks the $U(1)_D$-gauge symmetry the left equation in (102) is perturbed to

$$f_{12}(x_1, x_2) = \frac{1}{2}\Big(|\phi_1(x_1, x_2)|^2 - 1\Big) \ , \tag{112}$$

and becomes the self-dual or BPS vortex equation, see [34]. Thus, the solutions are the celebrated BPS or self-dual vortices and the real dimension of the moduli space, now $\mathbb{C}^k$, diminishes to $2k$ responding to the freedom of motion of the centers of the quantized "electric" flux tubes. Of course there are analogous electric flux tubes in the x_1-x_3- and x_2-x_3-planes. In the planes containing x_4, however, the flux is "magnetic" but the interpretation of these tubular solutions is doubtful because they should be properly recognized as "instantons" when x_4 refers to "Euclidean" time.

5 A very brief epilogue

This essay has been elaborated by expanding the lecture notes of a short Course that I taught in the Workshop on Geometry and Physics in Miraflores de la Sierra, Spain, September 2001, to a mixed audience of Geometers and Theoretical Physicists. The main theme was the description of the impact in Geometry of special concepts and techniques developed in non-perturbative Quantum Field Theory. This subject achieved impressive success during the fourth quarter of the last Century establishing deep links between Quantum Physics and Algebraic Topology and Geometry, in contrast with Classical Physics more tied to Differential Topology and Geometry. The presentation here is strongly inclined towards the side of Physics.

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