Fuzzy sets from the ethics of social preferences: slides for ESTYLF 2014

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ESTYLF 2014. Zaragoza.

Outline

Presentation of the problem

Social welfare functions and fuzzy sets

- Definition
- Prominent examples
- Ethical fuzzy sets: variations of the concept

Aggregation of utility streams: The framework

$\mathbf{X} \subseteq \mathbb{R}^{\mathbb{N}}$ is a domain of utility sequences or infinite-horizon utility streams.

Usual notation for utility streams: $\mathbf{x} = (x_1, ..., x_n, ...) \in \mathbf{X}$.

Comparing streams

A *social welfare function* (SWF) is a function $\mathbf{W} : \mathbf{X} \longrightarrow \mathbb{R}$.

$\mathbf{W}(\mathbf{x}) \geqslant \mathbf{W}(\mathbf{y})$ means "$\mathbf{x}$ is (socially) at least as good as $\mathbf{y}$"

It induces a *representable* social welfare ordering according to the expression:

$$\mathbf{x} \succcurlyeq \mathbf{y} \text{ if and only if } \mathbf{W}(\mathbf{x}) \geqslant \mathbf{W}(\mathbf{y})$$

Comparing streams

We are concerned with combinations of axioms of different nature for SWRs / SWFs on $\mathbf{X}$.

- Axioms related to efficiency: Strong/Weak/Partial Pareto, Weak Dominance, or Monotonicity.

  Strong Pareto: If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $\mathbf{x} > \mathbf{y}$ then $\mathbf{x} \succ \mathbf{y}$.

- Axioms related to equity: especially Anonymity, others like Pigou-Dalton transfer principle, variations on the Hammond Equity axiom, ...

  Anonymity: Any finite permutation of a utility stream produces a socially indifferent utility stream.

The codomain of SWFs can be restricted to $[0, 1]$

Because there exist strictly increasing mappings $\rho : \mathbb{R} \longrightarrow [0, 1]$, every social welfare function $\mathbf{W} : \mathbf{X} \longrightarrow \mathbb{R}$ can be transformed into a mapping $\mathbf{W}' = \rho \circ \mathbf{W} : \mathbf{X} \longrightarrow [0, 1]$ in such way that $\mathbf{W}(\mathbf{x}) \geqslant \mathbf{W}(\mathbf{y})$ and $\mathbf{W}'(\mathbf{x}) \geqslant \mathbf{W}'(\mathbf{y})$ are equivalent, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$.

The composition with ρ does not affect the fulfilment of the axioms above: $\mathbf{W}$ is SP, resp., AN, others like MON, IP, WP, WD, ... if and only if so is $\mathbf{W}' = \rho \circ \mathbf{W}$.

- ▷ For the purpose of investigating the existence of SWFs with the axioms we have mentioned, we do not lose generality if the codomain is assumed to be $[0, 1]$.

Main definition

Every social welfare function $\mathbf{W} : \mathbf{X} \longrightarrow [0, 1]$ can be identified with a fuzzy subset of $\mathbf{X}$.

Each $\mathbf{W}(\mathbf{x})$ is interpreted as the degree of membership of $\mathbf{x}$ to the subset of 'ethically acceptable' streams in X.

To better fit these interpretations:
when $\mathbf{X} \subseteq [0, 1]^{\mathbb{N}}$ and both $\mathbf{1} = (1, 1, ..., 1, ...) \in \mathbf{X}$ and $\mathbf{0} = (0, 0, ..., 0, ...) \in \mathbf{X}$ hold true, we restrict our analysis to fuzzy subsets that verify $\mathbf{W}(\mathbf{1}) = 1$ and $\mathbf{W}(\mathbf{0}) = 0$.

Example 1: the Rawlsian fuzzy subset of $[0,1]^{\mathbb{N}}$

The Rawlsian subset of $[0,1]^{\mathbb{N}}$:

$$\mu_R(\mathbf{x}) = \inf\{x_1, x_2, ..., x_n, ...\} \quad \text{for all} \quad \mathbf{x} = (x_1, x_2, ...) \in [0,1]^{\mathbb{N}}$$

As requested by our definition, $\mu_R(\mathbf{1}) = 1$ and $\mu_R(\mathbf{0}) = 0$.

Example 2: δ-discounted fuzzy subsets of $[0,1]^{\mathbb{N}}$

Inspired by the most popular criteria for evaluating infinite streams, the δ-discounted fuzzy subset of $[0,1]^{\mathbb{N}}$ associated with $\delta \in (0,1)$ is

$$\mu_\delta(\mathbf{x}) = (1-\delta)\sum_{i=1}^{+\infty} \delta^{i-1} x_i \quad \text{for all} \quad \mathbf{x} = (x_1, x_2, ...)$$

As requested by our definition, $\mu_\delta(\mathbf{1}) = 1$ and $\mu_\delta(\mathbf{0}) = 0$.

Example 3: δ-rank-discounted fuzzy subsets

Let $\bar{\mathbf{X}}$ be the set of allocations of $[0,1]^{\mathbb{N}}$ whose elements can be permuted to obtain non-decreasing streams.

The δ-rank-discounted fuzzy subset of $\bar{\mathbf{X}}$ associated with $\delta \in (0,1)$ is

$$\rho_\delta(\mathbf{x}) = (1-\delta)\sum_{i=1}^{+\infty} \delta^{i-1} x_{\lfloor i \rfloor} \quad \text{for all} \quad \mathbf{x} \in \bar{\mathbf{X}}$$

where $(x_{\lfloor 1 \rfloor}, x_{\lfloor 2 \rfloor}, ...)$ is the non-decreasing infinite stream which is a permutation of $\mathbf{x}$.

As requested by our definition, $\rho_\delta(\mathbf{1}) = 1$ and $\rho_\delta(\mathbf{0}) = 0$.

Ethical fuzzy sets

Combinations of properties of fuzzy subsets of $\mathbf{X}$ yield various concepts of ethical (in the comprehensive sense) fuzzy subsets.

The following definitions refer to **anonymous** fuzzy subsets (of a domain of infinite utility streams $\mathbf{X} \subseteq [0,1]^{\mathbb{N}}$ such that the degree of membership of $\mathbf{1} \in \mathbf{X}$ is 1, resp., of $\mathbf{0} \in \mathbf{X}$ is 0):

- A fuzzy set is anonymous when the degree of membership of any $\mathbf{x} \in \mathbf{X}$ does not change under finite permutations of its coordinates. .

Ethical fuzzy sets: variations of the concept

1. **Ethical**: when **x** allocates more than **y** to some generation, and **x** does not allocate less than **y** to any generation, then **x** has a higher degree of membership than **y**.

2. **Pre-ethical**: when **x** allocates more than **y** to an infinite number of generations, and **x** does not allocate less than **y** to any generation, then **x** has a higher degree of membership than **y**.

3. **Weakly ethical**: when **x** allocates more than **y** to all generations, then **x** has a higher degree of membership than **y**.

Ethical fuzzy sets: variations of the concept

4. **Quasi-ethical**: when $\mathbf{x}$ allocates more than $\mathbf{y}$ to a generation i, and $\mathbf{x}$ and $\mathbf{y}$ allocate the same amount to any generation other than i, then $\mathbf{x}$ has a higher degree of membership than $\mathbf{y}$.

5. **Basically ethical**: when $\mathbf{x}$ does not allocate less than $\mathbf{y}$ to any generation, then $\mathbf{y}$ does not have a higher degree of membership than $\mathbf{x}$.

Ethical fuzzy sets: relationships

Any ethical fuzzy subset of $\mathbf{X}$ is pre-ethical, quasi-ethical, and basically ethical.

Pre-ethical fuzzy subsets of $\mathbf{X}$ are weakly ethical.

Lemma

If a fuzzy subset of $[0,1]^{\mathbb{N}}$ is quasi-ethical and basically ethical then it is ethical.

Results: are there (pre-)ethical fuzzy subsets?

Theorem (Crespo et al., Economic Theory, 2009)

No SWF on $\mathbf{Z} = \{0, 1\}^{\mathbb{N}}$ is Infinite Paretian and anonymous.

Consequence

There do not exist pre-ethical fuzzy subsets of $\mathbf{Z} = \{0, 1\}^{\mathbb{N}}$.

In particular: there do not exist ethical fuzzy subsets of $\mathbf{Z} = \{0, 1\}^{\mathbb{N}}$ (Basu and Mitra, Econometrica, 2003).

Although:

Example 3 (ρ_δ) is an ethical fuzzy subset of $\bar{\mathbf{X}}$ (Zuber and Asheim, Journal of Economic Theory, 2012).

Results: are there weakly ethical fuzzy subsets?

Theorem (Basu and Mitra, 2007)

No SWF on $[0,1]^{\mathbb{N}}$ is Weakly Paretian and anonymous.

Consequence

There do not exist weakly ethical fuzzy subsets of $\mathbf{X} = [0,1]^{\mathbb{N}}$.

Although:

Example 3 (ρ_δ) is a weakly ethical fuzzy subset of $\bar{\mathbf{X}}$.

Results: are there quasi-ethical fuzzy subsets?

- We have mentioned that Example 3 is a quasi-ethical fuzzy subset of $\bar{\mathbf{X}}$.

- In fact, there exist quasi-ethical fuzzy subsets of any $\mathbf{X} \subseteq [0,1]^{\mathbb{N}}$.

  Reason:

Proposition (Basu and Mitra, 2007)

There are SWFs on $\mathbf{X} = [0,1]^{\mathbb{N}}$ that are Weakly Dominant and Anonymous.

Results: are there basically ethical fuzzy subsets?

The answer to this question is affirmative for any $\mathbf{X} \subseteq [0,1]^{\mathbb{N}}$. We just need to check that the *minimax* or Rawlsian fuzzy subset μ_R verifies the requested properties.

Although there are quasi-ethical and also basically ethical fuzzy subsets of $[0,1]^{\mathbb{N}}$, it is remarkable that quasi-ethical fuzzy subsets of $[0,1]^{\mathbb{N}}$ cannot be basically ethical.

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