Intensions, Types and Existence

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Abstract

This is a dissertation about two questions. The questions are whether the senses of the expressions of a language can be mathematically modelled, and in case they can, how can these senses be mathematically modelled. This is also a dissertation about an answer: intensions, despite being a good mathematization of senses, are not a sufficient means for grasping senses totally. Intensions are not a new concept, they have a long history with roots on a logic of Sense and Denotation which started with Frege, was developed syntactically and axiomatically by Church, and semantically by Montague. Thanks to them, senses and intensions are currently familiar notions for logicians. And although senses continue being a more obscure notion, intensions however are a well defined concept: they are functions from possible worlds to objects.

The present work offers two formal languages with intensions: a First-Order Intensional Hybrid Logic and an Intensional Hybrid Type Theory. Both languages include expressions denoting intensions and also a hybrid machinery for extensionalizing the intensions at given worlds of a model. But they are not purely intensional languages, since they also include expressions for denoting extensions. A powerful type notation is also included in order to differentiate intensional and extensional expressions, intensional and extensional predication, and well formed formulas. The distinction between intensional and extensional predication amounts to claiming the existence of two kinds of concepts of predicates: one intended as a function between concepts and the other as a function between objects and concepts.

The traditional issues in intensional logic—constant and varying domain models and *de dicto* and *de re* readings—are also analyzed from a novel point of view due to the fact that the previous languages do not only include intensional expressions, but also hybrid operators and a disambiguating type notation. The problem of non-denoting terms is studied assuming that intensions are partial functions; and the problem of the identity of senses, although solved for alethic contexts by means of the identity of intensions, needs a more fine-grained solution for epistemic contexts. A more precise answer can be found going beyond intensions to the hyperintensions realm.

Some philosophical notions, as existence and denotation, are also explored from the point of view of our formal languages. Finally, Gödel's proof for the existence of God and Caramuel's argument against the existence of God are analyzed in order to offer two suggestive exercises for intensional logic and even for formal ontology.

Contents

Introduction

What does "intensional" mean?

The first problem we need to face when addressing the topic of intensions is to search for an adequate definition. The origin of intensions can be traced back to the research about the meaning of linguistic expressions. But are intensions and meaning the same thing? We know that *meaning* is a complex concept, a very difficult one to grasp, as shown by the history of logic and other linguistic sciences. "Instead of dealing with such a difficult concept, it should be better to reduce it to a more understandable one," was possibly the thought of the researchers in the field. And, so, *intensions* appeared. While meaning is an elusive notion, intensions can be mathematically studied because they are functions: functions from worlds to entities. These entities can be individuals, truth values, sets of individuals,... The understanding of *intensions* as *functions* makes things so much easier, but the question about the adequacy of the notion of intension as an explanation of the concept of meaning still remains, in our view, open to discussion.

To say that intensions do not exhaust the content of meaning is not the same as saying that intensions cannot provide us with a good way for approaching the notion of meaning. A real understanding does not always imply a complete one. Therefore, we will make use of intensions as the most useful approach to the concept of meaning we have so far.

Furthermore, intensions raise also the question of what really is an intensional logic. Many logics are considered to be intensional: from the basic propositional modal logic to the more complex formal languages of epistemic logic. There are theories which qualify a logic as intensional if the extensionality principles do not apply,[1] other views see intensionality related to expressions such as necessity, possibility,... and some say that intensionality derives from oblique contexts. To this diversity we can add that intensionality is sometimes seen only as a feature of the semantics of the language, while others vindicate intensionality as a characteristic linked to the language of a

[1]Here the extensionality principles I refer to are those that apply in set theory.

logic, where you should find intensional variables, constants and operators. Hence, the answer to the question about what really characterizes a logic as intensional is a difficult task too.

Surely all the aforementioned logics can be considered intensional in a proper way, but what we really are looking for, is an intensional logic in its deeper sense. A logic which can deal with intensions in its semantics and also in its language. Consequently, a logic with only intensional semantic interpretations of expressions of a merely extensional language does not fulfill our objectives. We need a formal language where its expressions may denote intensions and not only express them. It means that our formal language will include different kinds of expressions: some denoting intensions and others denoting extensions, and not a unique kind of expressions interpreted extensionally or intensionally depending only on the context. Furthermore, we want to add to the more common extensional predication—where a predicate is said of an object—a way for predicating of intensions, that is, of the functions as a whole. In order to incorporate this, we will need to add some new machinery to the languages based on classical logic and modal logic. The new machinery, containing mainly a wider range of expressions of different types and hybrid operators, are not easy to incorporate. There are some profound modifications to be done in the syntax and semantics of the formal languages. But it will be worth it.

In order to fully understand the new machinery and because we are in the search of the core of intensionality, we have decided to make a progressive development of our research about this concept.

The Chapters

The present dissertation is divided in three parts, each part corresponding to a word of the title. The first part, characterized by the word *intensions*, pretends to be a historical account about how intensions became increasingly important in the logic of the last century. The second part, represented by the word *types*, is more technical, and focus on the description of formal languages whose expressions, even in first-order modal logic, belong to a given type. The third part, labelled with the word *existence*, is an application of our results about intensions and types to philosophical matters, particularly, to the ontological proofs for and against the existence of God. The different chapters of this dissertation take the previous triple division and make it concrete by means of a detailed study of the historical background of intensions, formal languages or philosophical problems.

Chapter 1, which should be identified with the first part (*intensions*), offers an historical account of intensional logic. It is far from comprehensive, since our dissertation is not mainly an exposition on the History of intensional logic. It covers only three authors of this History: Frege, Church and Montague, whose choice is more than justified. Frege (1892) introduced the idea into contemporary logic that it is possible to differentiate between what a name denotes (*Bedeutung*) and what a name expresses (*Sinn*). Church (1951) offered the first formal language and the first system of axioms for dealing with a logic of Sense (*Sinn*) and Reference (*Bedeutung*). And Montague (1974b) came up with a semantics for an intensional logic in order to link natural language to formal languages. Therefore, we have taken from Frege the idea, from Church the formal language and from Montague the semantics, the three elements which will be the core of the following chapters.

In chapter 2 we take as our start point two works, published more recently, in intensional logic: (Fitting & Mendelsohn, 1998) and (Braüner, 2008). We begin with a presentation of the syntax and semantics of one of the formal languages included in (Fitting & Mendelsohn, 1998) and of the first-order intensional hybrid logic of Braüner (2008). But our presentation has not been a mere copy, it has added a type notation to the expressions of the language, something which will show its importance later. Two different semantic approaches have been presented in this chapter: one with varying domain models and other with constant domain models. Furthermore, we have discussed the formal definitions of two predicates: *denotation* and *existence*, when varying domain models are considered. There are also analyses about the problem of predicating of concepts and not only of objects, about the issue of negative formulas which have non-denoting terms, and about the behavior of predicate abstracts and hybrid operators as two different ways of giving us a successful disambiguation procedure between the readings *de dicto* and *de re* of formulas with modal operators.

After having made a critical analysis identifying the achievements and also the difficulties of (Fitting & Mendelsohn, 1998) and (Braüner, 2008), we have continued the search of a more satisfactory first-order intensional hybrid logic. We have tested some hypothetical claims in order to identify what types must have the concepts of an extensional expression. For example, if a monary extensional predicate is of type $\langle \iota o \rangle$, what is the type of a concept of this predicate? $\langle \iota o \rangle_1$, $\langle \iota_1 o_1 \rangle$, $\langle \iota o_1 \rangle$ or $\langle \iota_1 o \rangle$? We have arrived to the conclusion (unlike Church) that the concepts of functional types, such as $\langle \iota o \rangle_1$, do not have to be only functions on concepts, such as $\langle \iota_1 o_1 \rangle$, but also functions from extensional objects to concepts $\langle \iota o_1 \rangle$. This thesis together with some other results derived from our section 2.4: "Laboratory of Intensions", have provided us with new insights into intensional logic. Based on these insights we have developed a First-Order Intensional Hybrid Logic, which allows us to give a great deal of expressiveness to a formal language together with a successful account of extensional and intensional predication, and also to offer a formalization of the predicates of denotation and existence, within a constant domain model, with intensional terms and hybrid operators.

In chapter 3 we have gone beyond a First-Order Intensional Hybrid Logic and we have presented our Intensional Hybrid Type Theory. A Type Theory that, unlike Montague's Intensional Logic, has not only expressions interpreted intensionally but also expressions, other than variables, interpreted extensionally. We have claimed that, since our logic is not purely intensional, we can use it as a unifier between pure extensional logics and pure intensional ones. Moreover, this chapter centers also in the analysis of Russell's and Frege's Theory of Descriptions, and includes the problematic around the evaluation of formulas that derives from the inclusion of terms which do not designate. The final section of this chapter is dedicated to the identity of senses. This issue was crucial for Frege and Church and, in fact, the roots of intensional logic can be found in the search of a theory which gives a satisfactory account of the identity of two senses. We have shown that identity of intensions is not sufficient for doing this and, apart from the three Alternatives of Church, we have done a brief exposition of the theories of Pavel Tichý and of Yiannis Moschovakis as possible candidates for solving the problem with success. Chapters 2 and 3 correspond with our second part labelled with the word *types*.

Finally, in chapter 4, our third and last part: *existence*, we have taken the ontological arguments of Gödel and Caramuel as practical exercises for intensional logic. The first is an argument for proving the existence of God while the second is an argument against it. Here we have used our previous formal languages as a tool of analysis of notions such as existence, necessary existence, essence, positive property and, even, nothingness. Even though our study of the arguments has not been focused on the demonstration procedure, the formal account of the notions involved in the axioms, definitions and theorems of the arguments provides us with an insight into philosophical problems. A *Formal Ontology*, as a field of research where philosophical problems receive a formal treatment is seen as desirable and still needed nowadays.

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Chapter 1

A Brief History of Intensional Logic

The present chapter is a partial presentation of the history of intensional logic.[1] It is partial in a double sense: it is centered practically in the twentieth century, and it only pays attention to three authors: Frege, Church and Montague. We have chosen these authors because they are three milestones for any contemporary study on intensionality. The idea of dealing with senses from the point of view of a formal language can be found already in Frege. The first language and axiomatic system for a logic of sense and denotation is given by Church. And a possible world semantics for an intensional logic is offered by Montague.

Our exposition follows the order of the publications of three main works in the field of intensional logic: Frege (1892), Church (1951) and Montague (1974b). Since Frege and Montague have been studied more extensively, we have dedicated a more exhaustive analysis to Church, whose intensional logic does not seem to have received the attention of many logicians or linguists, apart from his doctoral students. That is the reason of the irregular length of the different sections that the reader is going to find below.

[1] "Recognition that designating terms have a dual nature is far from recent. The Port-Royal Logic used terminology that translates as 'comprehension' and 'denotation' for this. John Stuart Mill used 'connotation' and 'denotation.' Frege famously used 'Sinn' and 'Bedeutung,' often left untranslated, but when translated, these usually become 'sense' and 'reference.' Carnap settled on 'intension' and 'extension.' However expressed, and with variation from author to author, the essential dichotomy is that between what a term *means*, and what it *denotes*." (Fitting, 2015).

1.1 Frege on Intensionality

1.1.1 Introduction

In Frege there is a philosophy of intensional entities but there is no intensional logic. By absence of intensional logic we mean that there is (in the writings of Frege) no formal logic which develops intensional logic in the same way as extensional logic was developed in *Begriffsschrift* (1879). On the contrary there can be found a philosophy of intensionality under the name of a theory of sense and reference, which is explained in depth in the article *Über Sinn und Bedeutung* (1892). *Sinn* is usually translated as *sense* into english and is *expressed* by a name, while *Bedeutung* is usually translated as *reference* (or denotation) into english and is *indicated* (or denoted, or referred) by a name.

1.1.2 *Having* Sense and Reference

We can say that names **have** both senses (*Sinne*) and references (*Bedeutungen*), but what they really **are** is by far a more difficult question. We can affirm that the expression *Pope Francis* has a sense and has also a reference: its sense is a kind of *mode of presentation* or cognitive content someone has when is thinking about it; its reference is the actual person who rules the Catholic Church. It can be the case that two names have the same reference but different senses, so we can consider the same person to be presented as *Pope Francis* or as *Jorge Mario Bergoglio*. In this case the referent is the same but the sense of the expression "Pope Francis" is different from the sense of "Jorge Mario Bergoglio".

We can assume therefore that names, sentences and other expressions have senses and references. When dealing with the expressions *Pope Francis* and *Jorge Mario Bergoglio* we identify that the referent of them is a person, but what are their senses? We have considered they are different from each other, but what really are they?

1.1.3 *Being* Sense and Reference

For Frege there are two categories of expressions: "complete expressions" and "incomplete expressions". Complete expressions such as terms and sentences have objects as their referent. The particularity of sentences is that, on the one hand, they have as referent one of the truth values, and on the other hand, as sense they have a proposition or *Gedanke* (i.e., a *thought*). However incomplete expressions such as functional expressions or grammatical predicates have no objects as their referent, but functions. The question of the sense of this kind of expressions is much more difficult to solve, and I will henceforth try to go into detail about this in what follows.

It seems to us that it is easy to find a perfect match between Frege's analysis of language and Frege's ontology. It is usually thought[2] that Frege's ontology consists of objects, which are complete entities, and functions, which are incomplete or unsaturated entities. The two categories of linguistic expressions, complete and incomplete, are connected with each entity respectively. The sense and reference of complete expressions are objects, while the sense and reference of incomplete expressions must be some kind of incomplete entities. Incomplete expressions such as "the square root of ( )" (a functional expression) and "( ) is a planet" (a grammatical predicate), have as referent functions, but what do they have as sense? Also functions? Some researchers have answered this question introducing *"sense-functions"*, which would be the sense of functional expressions (Church, 1951). So "sense-functions" would be functions in the realm of sense which are expressed by incomplete expressions. Nevertheless, other philosophers, Dummett, for example,[3] do not agree with this interpretation and understand the sense of incomplete expressions as a peculiar sort of *object* which possesses a certain kind of incompleteness.

In addition to the interpretations of Church and Dummet, we have found compelling the interpretation of Klement who endorses the claim that the sense of function expressions are neither functions, nor objects, but a "particular type of unsaturated entity in the realm of *Sinn*" (Klement, 2002, p. 74). The problem with this interpretation is that it broadens Frege's ontology introducing besides objects and functions, a third type of entity. This new kind of entity "can be understood as packets of descriptive information that are somehow incomplete in the sense of having missing information" (Klement, 2002, p. 76).

Therefore, to sum up, concerning linguistic expressions, we can choose between two possible referents: object and function; and three possible senses: object or sense-function or a third entity which is the "sense of incomplete

[2] "Frege's ontology is usually taken to divide exhaustively between objects and functions; all complete entities are objects, and all incomplete or unsaturated entities are functions. Certainly, there is textual evidence to support this reading. In Frege's own words, 'an object is anything that is not a function' (*CP* 147, cf. *BL* §2)." (Klement, 2002, p. 66).

[3] "Although other authors have pointed out problems with the view that the *Sinne* of incomplete expressions are such sense-functions, Dummett is the only writer on Frege who has seriously challenged it as the true interpretation of what Frege understood the *Sinne* of incomplete expressions to be." (Klement, 2002, p. 67).

expression". Introducing the latter, Klement does not reject the existence of sense-functions, but in case they exist they would not be the sense of incomplete expressions.

1.1.4 Senses Contextualized

A particular problem between sense and reference arises when we have to deal with identity statements. In direct (we can also say extensional) contexts, what really matters is reference, and the truth of identity statements involves only the referents. But in indirect (we can also say intensional) or "oblique contexts", what really matters is sense.

In direct contexts if we say "The best friend of Sancho Panza is Alonso Quijano" and supposing that "Alonso Quijano is Don Quijote de la Mancha", we can see how Leibniz's law of substitutivity of identicals works because there is no difficulty in accepting that "The best friend of Sancho Panza is Don Quijote de la Mancha". However, there is a real challenge to Leibniz's law when we are considering indirect contexts such as "I think that...", "I believe that..." and so on. In these contexts the sentences "Tony believes that Don Quijote is the best friend of Sancho Panza" and "Tony believes that Alonso Quijano is the best friend of Sancho Panza" might not be true at the same time, so they are not logically equivalent and therefore Leibniz's law fails. What is the reason of this failure?

Frege solves the problem saying that there are two kinds of referents: direct (or primary) and indirect (or secondary). Primary referents are the usual referents and are designated by the expressions in direct contexts. Secondary referents are, however, the senses of the expressions, and they are in fact the authentic reference of the expressions in "oblique contexts". So Leibniz's law of substitutivity of identicals fails because the referents of the expressions in indirect contexts are not identical, because they are dealing with their senses, which are not identical either.

If interchangeability of expressions with different senses is allowed without implying a change in the truth value of the whole, then we would be dealing with identity of senses. And hence we can identify the conditions two expressions may accomplish for having the same sense, that is, to be interchangeable in indirect contexts.

1.1.5 What, then, are *Senses*?

After some deliberation, we find the realm of referents more approachable than the realm of senses. In order to clarify this puzzle we are going to try collecting the basic features of sense in Frege:

1. The sense of a name or proposition is an object.

2. The sense of an incomplete expression is either a "sense-function" (Church, 1951) or a "particular type of unsaturated entity" (Klement, 2002, p. 76).

3. Senses exist within a *third realm* apart from physical objects and psychological entities (Frege, 1984, pp. 363-372).

4. The entities within this third realm are "objective but incapable of full causal interaction with the physical world" (Klement, 2002, p. 63).

5. There are expressions that have a sense but do not possess a reference: "the largest natural number" or "the least rapidly convergent series" (Frege, 1984, p. 159).[4]

6. Every sense picks out a unique entity.

7. Identity of senses has to do with interchangeability in indirect contexts.[5]

8. There is an over-existence of senses:

   (a) Every existing entity is picked out by some sense.

   (b) Every entity defined by any set of conditions is presented by a sense.

   (c) Every sense is also picked out by another sense.

1.1.6 Conclusion

With regard to the division between intension and extension currently used in logic, we think it is necessary to make more clear some ideas. After Carnap (1947) and Kripke (1959, 1963), and in many current approaches to intensional logic, intensions are understood as functions from worlds to some kind of entities; extensions nevertheless would be other kind of entities different

[4] "Another defect of ordinary language Frege identifies is that it employs expressions with *Sinne* that present or pick out no object as *Bedeutungen* (*CP* 159, 162-3). That there should be such *Sinne* is not surprising given the reading of *Sinne* given here. A *Sinn* consists of a set of criteria or conditions and picks out an object in virtue of it alone satisfying those conditions. However, certainly, there are conditions or criteria that are not satisfied by any unique object. Frege gives as example the expression 'the least rapidly converging series' (*CP* 159)." (Klement, 2002, p. 62).

[5] "Frege's primary criterion for the identity conditions of *Sinne* is that phrases expressing them should be interchangeable even in all singly oblique contexts." (Klement, 2002, p. 126).

of intensions. Establishing a mere translation between *sense* and *reference*, on the one hand, and *intension* and *extension*, on the other hand, can be quite confusing. Therefore we consider that there is no a parallelism between the current division *intensional/extensional* and the Frege's distinction between *sense/reference*. For Frege sense and reference are not two distinct categories of entities because senses can be also referents, as in the case of indirect contexts. Therefore it could be wrapped up that the current approach to intensional terminology cannot be found in Frege. However, we can arrive to the conclusion that what we currently call intensions originates from Frege's senses, and senses for him are some kind of entities that belong to a third realm (different of the physical and psychological world) and that are expressed, although not exhausted, by linguistic expressions.

Finally, although it is widely known and may sound repetitive, we do not find any formal apparatus in the logical notation of Frege which can deal with senses or oblique contexts. Consequently we have a theory of sense and reference but not a logic of sense and reference.

1.2 Church on Intensionality

1.2.1 Introduction

Logic of Sense and Denotation is how Church[6] called to what we are going to name *Church's Intensional Logic*, a lifelong project revisited again and again for fifty years in some of his papers.

In the first pages[7] of his *Introduction to Mathematical Logic* (1956, pp. 3–9) Church cites Frege's distinction between *sense* and *reference* where he sees an issue that should be studied in depth at another time. In an abstract published in 1946 entitled *A Formulation of the Logic of Sense and Denotation* a well-structured study project is shown (Church, 1946). The development will come in an article, which shares the title with the abstract, published in 1951, wherein we can find the basis of his work on intensionality (Church, 1951).

But Church (1951) is not the end of an intensional logic, it is only the beginning. And it is not a simple beginning. The new *logistic* system, which wants to deal with the sense of the names and not only with their denotation, is systematized in three different alternatives: Alternative (2), Alternative (1) and Alternative (0). These can be differentiated by the conditions that

[6]For an introduction to Church, see (Manzano, 1997)

[7]This pages were written between September, 1947 and June, 1948, as Church himself recognizes on his Preface (1956, p. v).

make two names to have the same sense. The greater the number of the alternative, the weaker the conditions and the easier the study. In (Church, 1951) only the first of the aforementioned alternatives is developed in depth, the second one is only presented briefly, and the third one is not analyzed in any way (Church, 1951, pp. 6–7). The articles written in the 1951-1993 period revisit the 1951 paper from different points of view: dealing with some kind of antinomies, changing some axioms, developing Alternative (2), giving some semantic interpretation in models and so on. These issues will be taken into consideration by Church for the rest of his life: indeed, until 1993, when he wrote his last paper about the logic of sense and denotation (Church, 1993). All this seems to suggest that Church's work on the logic of sense and denotation was never considered complete but a work in progress. But before taking over from Church the unfinished tasks of the logic of sense and denotation, it is necessary to look back to the grounds of his intensional logic. There we find that the logic proposed in 1951 takes into account the system of *A Formulation of the Simple Theory of Types* (Church, 1940). Since it is supposed familiarity with this paper, we would like to remember the main features of his simple theory of types.

1.2.2 Church's Type Theory

If we are going to understand the logic of sense and denotation we need to have at least a general knowledge of Church's notation and its peculiarities (Manzano, 1996, pp. 205–210). First of all, it must be said that his formulation incorporates certain features of the calculus of λ-conversion, introduced in Church (1932) and Church (1941), and takes the concept of function as primitive, since properties and relations can be considered as functions from entities to truth values. He follows also the tradition after Schönfinkel that functions of more than one argument can be represented in terms of functions of one argument whose values are themselves functions (Schönfinkel, 1967). This procedure allows him to consider only functions of one argument at a time.

Definition 1.2.1 (Type symbols)**.** The class of type symbols is determined by the following rules:[8]

1. ι is a type symbol;

2. o is a type symbol;

3. If β and α are type symbols,then $(\beta\alpha)$ is a type symbol.

[8]See (Church, 1940, p. 56).

Comment 1.2.2. Type symbols are written as subscripts upon variables and constants of the formal language indicating the type of the variable or constant (propositional, individual or functional):

1. **Individual type symbol:** The greek letter *iota*, ι, is the type of individuals.

2. **Propositional type symbol:** The greek letter *omicron*, o, is the type of propositions.

3. **Functional type symbol:** If β and α are type symbols, then $(\beta\alpha)$ is a type symbol of functions from elements of type symbol β to elements of type symbol α.

Although Church's reading of the functional type symbols is always from right to left and the omission of parentheses presupposes that association is to the left, we have not followed his customary approach.[9] For reasons of clarity, we have done some modifications in Church's notation in order to render the formulas easier to understand for most current readers. Type symbols, therefore, must be read from left to right and functional type symbols such as $(\beta\alpha)$ should be interpreted as functions from type β to type α. In order to reduce the number of brackets, when we find a typed formula with more than two type symbols we assume that association is to the right. Therefore, from this point on, types written $(\alpha\beta)$ in Church's notation, will be written $(\beta\alpha)$ and interpreted as functions from type β in type α. A type $(o\alpha)\beta$ in Church's notation will be written as $\beta(\alpha o)$ and abbreviated as $\beta\alpha o$. If we find, for example, $g_{\beta\alpha o}$, we can interpret it without ambiguity:[10]

$$\mathbf{g}_{\beta\alpha o} \in D_{\alpha o}^{D_\beta}$$

[9]It is common to read functions from left to right, and it can be, in a certain way, counterintuitive to read the functional types from right to left, but we can find useful Church's reading direction because in such a way the first type symbol in the subscript indicates the type were the function ends, and so we can know more quickly the final range of the function. If we have a sequence of type symbols which starts with o then we know that we have a propositional function, i.e., a function whose values are propositions (truth values, in fact).

[10]Although Church's simple type theory is an attempt to replace set theory as a basis for mathematics, it is possible to translate Church's functional terminology into a set theoretical framework we are more familiar with. In this sense, the properties we usually interpret as sets in our semantics are in Church's terms characteristic functions, and to say that an individual has a property, what we usually translate as the membership of an element in a set, can also be said as the characteristic function having value truth for this element. So the type symbol ιo represents a unary relation, $\iota\iota o$ a binary relation, and so on. If α is any type symbol, αo is intended as a set of elements of type α; if α and β are both type symbols, $\beta\alpha o$ represents a binary relation between elements of type β and elements of type α. On the contrary, a function from individuals to individuals can

that is,

$$\begin{aligned}\mathbf{g}_{\beta\alpha o} : D_\beta &\to D_{\alpha o}\\ \mathbf{x} \mapsto \mathbf{f}_{\alpha o} : D_\alpha &\to D_o\\ \mathbf{y} &\mapsto T, F\end{aligned}$$

Example 1.2.3. For example, ιo will be the abbreviation of (ιo), ooo of $o(oo)$, $(\iota\iota)\iota\iota$ of $((\iota\iota)(\iota\iota))$,... and for a longer expression, if α, β, γ, δ are type symbols, then $\alpha\beta\gamma\delta$ abbreviates $(\alpha(\beta(\gamma\delta)))$, which means the type of functions from elements of type α to elements of type $(\beta(\gamma\delta))$, which are in turn functions from elements of type β to elements of type $(\gamma\delta)$, which are in turn functions from elements of type γ to elements of type δ.

Definition 1.2.4 (Types)**.** The hierarchy of types, intended as a family of domains, is built inductively from the basic types to the functional types:

1. **Basic types:**
   - D_ι, is the type of individuals.
   - $D_o = \{T, F\}$, is the type of the truth values.
2. **Funcional types:**
   - $D_{(\beta\alpha)} = D_\alpha^{D_\beta}$, is the type of all functions from D_β to D_α.

Example 1.2.5. We know that he following type symbol $\iota\iota o$ denotes the functional type from ι to ιo, which is in turn the functional type from ι to o. But how can it be interpreted? It can be seen as the type of propositional functions of two individual variables. For example, "x is the father of y". The functional type symbol ιo, which expresses a function from individuals to truth values, represents suitably a set or a property, so that one element belongs to a set or an individual has a given property if the function representing the set or property maps that element to truth. For example, "y is green".

Syntax

The primitive symbols of the language can be divided into the following two categories:

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be represented by the type symbol $\iota\iota$, etc. We can thus deal with sets, properties and relations, which are here considered as certain kind of functions.

1. **Improper symbols:**
   - the abstraction operator: λ;
   - the parenthesis: (,), $\langle$, $\rangle$;
   - the dot: .
2. **Proper symbols:**
   - *Logical constants:* N_{oo} (negation), A_{ooo} (disjunction), $\Pi_{(\alpha o)o}$ (it is used to express the idea of universal quantification) and $\imath_{(\alpha o)\alpha}$ (it is a selection operator, note that we have left the greek letter *iota*, ι, as the type symbol for individuals and therefore this is the vowel i without the dot and in italics), for each type symbol α;
   - *Variables:* an infinite list of type α, for each type symbol α: $a_\alpha, b_\alpha, \ldots, a'_\alpha, b'_\alpha, \ldots, a''_\alpha, b''_\alpha, \ldots$

Definition 1.2.6 (Formula)**.** A formula is any finite sequence of primitive symbols, some of these sequences are *well-formed* and we call them *well-formed formulas* (*wff*). They are also said to have a certain type according with the following rules:[11]

1. A formula consisting of a single proper symbol (a logical constant or a variable) with a type symbol α is a well-formed formula of type α.
2. If x_β is a variable of type β and $\mathbf{A}_\alpha$ is a wff of type α, then $\langle\lambda x_\beta.\mathbf{A}_\alpha\rangle$ is a wff of type $\beta\alpha$.
3. If $\mathbf{A}_{\beta\alpha}$ and $\mathbf{B}_\beta$ are wff of types $\beta\alpha$ and β respectively, then $(\mathbf{A}_{\beta\alpha}\mathbf{B}_\beta)$ is a wff of type α.

Comment 1.2.7. Following rule 1. of formation of formulas we can consider that N_{oo} is a wff of type oo and a_α is a wff of type α. According with rule 2., we can build wffs by *abstraction*, because λ is called an abstraction operator. This kind of formulas denotes the function whose value on any argument x_β is $\mathbf{A}_\alpha$, therefore its interpretation is in $D_\alpha^{D_\beta}$. Finally, rule 3. allows *juxtaposition* to create wffs. A formula like $(\mathbf{A}_{\beta\alpha}\mathbf{B}_\beta)$ can be interpreted as the value of the function of type $(\beta\alpha)$ denoted by $\mathbf{A}_{\beta\alpha}$ for the argument of type β.

Definition 1.2.8 (Free and bound variable)**.** Like both quantifiers in first order logic: $\forall$ and $\exists$, λ is also a variable binder. An occurrence of a variable x_β in a wff is *bound* if it is an occurrence in a well-formed part of the formula having the form $\langle\lambda x_\beta.\mathbf{A}_\alpha\rangle$, and is *free* if it is not an occurrence in a well-formed part of the formula having the form $\langle\lambda x_\beta.\mathbf{A}_\alpha\rangle$.

[11] Cf. (Church, 1940, p. 57).

Definition 1.2.9 (Abbreviations)**.** Various conventions of abbreviations can also be used:[12]

1. $(\neg \mathbf{A}_o) ::= N_{oo}\mathbf{A}_o$
2. $(\mathbf{A}_o \vee \mathbf{B}_o) ::= A_{ooo}\mathbf{A}_o\mathbf{B}_o$
3. $(\mathbf{A}_o \wedge \mathbf{B}_o) ::= (\neg((\neg \mathbf{A}_o) \vee (\neg \mathbf{B}_o)))$
4. $(\mathbf{A}_o \rightarrow \mathbf{B}_o) ::= ((\neg \mathbf{A}_o) \vee \mathbf{B}_o)$
5. $(\mathbf{A}_o \leftrightarrow \mathbf{B}_o) ::= ((\mathbf{A}_o \rightarrow \mathbf{B}_o) \wedge (\mathbf{B}_o \rightarrow \mathbf{A}_o))$
6. $(\forall x_\alpha \mathbf{A}_o) ::= \Pi_{(\alpha o)o}\langle \lambda x_\alpha . \mathbf{A}_o \rangle$
7. $(\exists x_\alpha \mathbf{A}_o) ::= (\neg \forall x_\alpha \neg \mathbf{A}_o)$
8. $(\imath x_\alpha \mathbf{A}_o) ::= \imath_{(\alpha o)\alpha}\langle \lambda x_\alpha . \mathbf{A}_o \rangle$
9. $Q_{\alpha\alpha o} ::= \langle \lambda x_\alpha . \langle \lambda y_\alpha . \forall f_{\alpha o}(f_{\alpha o} x_\alpha \rightarrow f_{\alpha o} y_\alpha) \rangle \rangle$
10. $(\mathbf{A}_\alpha = \mathbf{B}_\alpha) ::= Q_{\alpha\alpha o}\mathbf{A}_\alpha\mathbf{B}_\alpha$

Comment 1.2.10. These abbreviations (with the usual propositional connectives) have an appearance more familiar to us than the polish notation of the unabbreviated expressions. We can also give a brief account of each abbreviation as follows:

1. The negation function takes a truth value and returns also a truth value, that is the reason why has type *oo*.
2. Disjunction takes two truth values and returns also a truth value.
3. Conjunction corresponds to a binary function from truth values to truth values, so it has type *ooo*.
4. Conditional is also a binary function from truth values to truth values, with type *ooo*.
5. The same applies to the biconditional.

[12]Cf. (Church, 1940, p. 58).

6. The universal quantifier is defined by means of the constant $\Pi_{(\alpha o)o}$ and is a propositional function of propositional functions.[13] If the propositional function $\langle \lambda x_\alpha.\mathbf{A}_o \rangle$ maps all elements of type α to truth, then the propositional function $\Pi_{(\alpha o)o}$ maps also the former propositional function to truth.

7. The existential quantifier is defined as usual by means of the universal quantifier.

8. The upside-down iota, $\rotatebox[origin=c]{180}{ι}$, is Russell's definite description operator, which is defined by means of a logical constant, the selection operator ι. It acts as a function of propositional functions.[14]

9. $Q_{\alpha\alpha o}$ is a propositional function and expresses the idea behind Leibniz's identity of two elements having the same properties.

10. This abbreviation takes into account the Leibnizian identity of indiscernibles for dealing with equality. According with this definition two elements x and y are the same if and only if y has every property that x has.

Church gives also the definition of the natural numbers, which have the functional type $(\alpha\alpha)(\alpha\alpha)$, abbreviated α' (although there can also be natural numbers of type α'', i.e., $(\alpha'\alpha')(\alpha'\alpha')$), of the successor function $S_{\alpha'\alpha'}$, of the predecessor function P, of the propositional function which expresses "to be a natural number" N, and some more definitions. We will not dwell on elucidating these abbreviations because they are more interesting for studying the foundations of mathematics than for dealing with intensional logic.

Rules of Inference

Church (1940, p. 60) presents six rules which can be divided into two groups: the first three rules are called rules of λ-*conversion*, and the last three ones are similar to the classical rules of substitution, *modus ponens* and generalization.

[13]In set theoretical terminology, it denotes a property of sets, namely, the property of being universal. The set defined by $\langle \lambda x_\alpha.\mathbf{A}_o \rangle$, which is a formula of type αo, has the property $\Pi_{(\alpha o)o}$ if and only if the set of type αo contains all elements of type α.

[14]In set theoretical terminology, the selection operator selects an element of type α of the set described by the type symbol αo.

- Rules of λ-conversion:

1. *Alphabetic change of bound variable.* To replace any part $\mathbf{M}_\alpha$ of a formula by the result of substituting y_β for x_β throughout $\mathbf{M}_\alpha$, i.e., $\mathbf{M}_\alpha[y_\beta/x_\beta]$, provided that x_β is not a free variable of $\mathbf{M}_\alpha$ and y_β does not occur in $\mathbf{M}_\alpha$.

2. λ-*contraction.* To replace any part $\langle\lambda x_\beta.\mathbf{M}_\alpha\rangle(\mathbf{N}_\beta)$ of a formula by the result of substituting $\mathbf{N}_\beta$ for x_β throughout $\mathbf{M}_\alpha$, i.e., $\mathbf{M}_\alpha[\mathbf{N}_\beta/x_\beta]$ provided that the bound variables of $\mathbf{M}_\alpha$ are distinct both from x_β and from the free variables of $\mathbf{N}_\beta$.

3. λ-*expansion.* Where $\mathbf{A}_\alpha$ is the result of substituting $\mathbf{N}_\beta$ for x_β throughout $\mathbf{M}_\alpha$, to replace any part $\mathbf{A}_\alpha$ of a formula by $\langle\lambda x_\beta.\mathbf{M}_\alpha\rangle(\mathbf{N}_\beta)$, provided that the bound variables of $\mathbf{M}_\alpha$ are distinct both from x_β and from the free variables of $\mathbf{N}_\beta$. I.e., to infer $\langle\lambda x_\beta.\mathbf{M}_\alpha\rangle(\mathbf{N}_\beta)$ from $\mathbf{A}_\alpha$ if $\mathbf{A}_\alpha$ can be inferred from $\langle\lambda x_\beta.\mathbf{M}_\alpha\rangle(\mathbf{N}_\beta)$ by a single application of λ-*contraction*.

- Rules of substitution, *modus ponens* and generalization:

4. *Substitution.* From $\mathbf{F}_{\alpha o}x_\alpha$ to infer $\mathbf{F}_{\alpha o}\mathbf{A}_\alpha$, provided that x_α is not a free variable of $\mathbf{F}_{\alpha o}$.

5. *Modus ponens.* From $\mathbf{A}_o \to \mathbf{B}_o$, and $\mathbf{A}_o$, to infer $\mathbf{B}_o$.

6. *Generalization.* From $\mathbf{F}_{\alpha o}x_\alpha$ to infer $\Pi_{(\alpha o)o}\mathbf{F}_{\alpha o}$, provided that x_α is not a free variable of $\mathbf{F}_{\alpha o}$.

1.2.3 Church's Intensional Logic

The simple theory of types is extended with new symbols (subscripts, a new "intensional" primitive constant) in order to improve the expressive power of the previous language. This new language is located within the framework of Frege's research on intensionality.

Church knows about the theory of sense and denotation that Frege proposes. He even thinks that Frege would have agreed with a formal treatment of these issues within an intensional logic. But Frege did not build any formal system dealing with senses and referents, however he serves as an impulse for building it. That is where Alonzo Church's tentative starts. Church's logic is inspired by Frege but it does not reflect Frege's theory on sense and denotation. Church is aware of having preserved the important features of the theory of Frege, but he is also conscious of having deviated from Frege's views, at least in the introduction of the simple theory of types and perhaps in the abandonment of the notion of a function as something unsaturated.[15]

However, for Church an *expression* or a *name*, which is intended as a closed well formed formula, expresses its *sense* and refers or denotes its *denotation*. The sense of a name presents or picks univocally out its denotation. What the names denote are *entities* and the entities which can be senses of a name are *concepts*. And a concept picks out or is a concept of an entity in a unique manner.

In Church's writings, *sense* and *denotation* translate the words *Sinn* and *Bedeutung* of Frege. However, although the fregean *Begriff* is translated as *concept* in english, for Church *concepts* and *Begriffe* have an entirely different meaning. *Concepts* are intensional entities, anything which can be considered as the sense of a word, as long as *Begriffe* are intended as predicates or properties.[16] On senses and concepts can also be individuated the following features:

- Senses are not univocally connected with a particular language. The same sense can be expressed by names in different languages.
- The existence of concepts is independent of the existence of names. We can suppose the existence of concepts although there are no actual names in any particular language which express these concepts.
- We can assume the existence of more concepts than names, in fact, we can suppose a non-enumerable infinity of concepts.

In order to clarify how Church uses his own terminology and how is linked or differentiated of the vocabulary of Frege, the diagram in Figure 1.1 can be found useful.

Definition 1.2.11 (Intensional type symbols)**.** We add subscripts to the type symbols in definition 1.2.1 in order to obtain the intensional type symbols. We have then:

1. An infinite list of symbols ι_0, ι_1, ι_2,... where ι_0 is written as ι and is the same type symbol ι of the simple theory of types. ι_0 is the type of individuals; and ι_{i+1} is the type of the senses of expressions of type ι_i.

[15] "In favor of a notion according to which the name of a function may be treated in the same manner as any other name." (Church, 1951, p. 4).

[16] "In order to describe what the members of each type are to be, it will be convenient to introduce the term *concept* in a sense which is entirely different from that of Frege's *Begriff* [...] Namely anything which is capable of being the sense of a name of x is called a *concept of x*." (Church, 1951, p. 11).

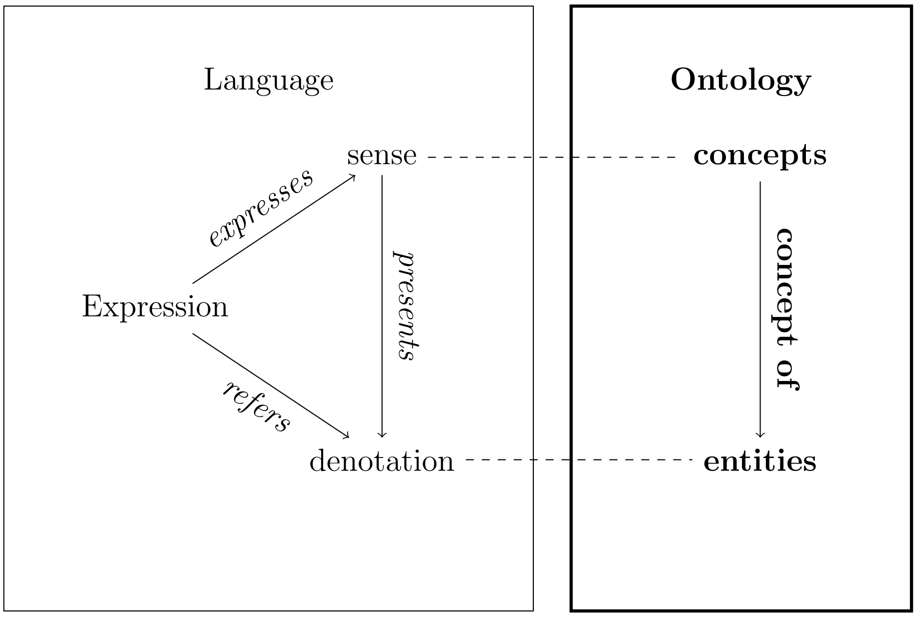

Figure 1.1: Linguistic and ontological terminology in Church's intensional logic.

2. An infinite list of symbols o_0, o_1, o_2,... where o_0 is written as o and is the same type symbol o of the simple theory of types. o_0 is the type of truth values, o_1 is the type of senses of the truth values or the type of propositions, o_{i+1}, is the type of the senses of expressions of type o_i.

3. If α and β are any type symbols, then $(\beta\alpha)$ is a type symbol for functions. If $\mathbf{F}_{(\beta\alpha)}$ and $\mathbf{A}_\beta$ are of type $(\beta\alpha)$ and β respectively, then $\mathbf{F}_{(\beta\alpha)}\mathbf{A}_\beta$ is of type α.

Definition 1.2.12 (Intensional types)**.** We add to definition 1.2.4 on page 14 the following clause:

4. If D_α is a type, then D_{α_1} is a type.

Comment 1.2.13. As in the simple theory of types, greek letters: α, β, γ are used as metavariables for type symbols. Subscript n is used upon such greek letters (α_n) to indicate the result of increasing all subscripts in the type symbol by n, where n is any of the natural numbers $0, 1, 2, ...$ For example, if α is $(\iota o)(\iota_1 o_1)o$, then α_1 is $(\iota_1 o_1)(\iota_2 o_2)o_1$, and α_2 is $(\iota_2 o_2)(\iota_3 o_3)o_2$. The rule for parentheses is the same as before, with the convention that association is to the right.

Definition 1.2.14. For each type symbol α there is a corresponding *domain* which we call the *type* α.

- Type o is the type of the truth values: truth and falsehood. Type o_1 is the type of *concepts* of truth-values, also called *propositions* or *Gedanken*, following Frege. Type o_2 is the type of concepts of propositions or *propositional concepts*. Generally, type o_{i+1} is the type of concepts of the members of the type o_i.

- Type ι is the type of individuals. It can be finite, infinite or empty. Type ι_1 is the type of concepts of individuals or *individual concepts*. Generally, type ι_{i+1} is the type of concepts of the members of the type ι_i.

- If α and β are two types, we have the type $(\beta\alpha)$ of functions, where the argument is a member of type β and the value of the function is a member of type α.

Comment 1.2.15. The family of domains that build the hierarchy of types in Church's intensional logic is simplified by reducing the domains of the conceptual types. As purely extensional types we have: the type of the truth values $D_o = \{T, F\}$; the type of individuals: D_ι; and the functional type $D_{\alpha\beta}$, which consists of all the functions from type α to β. To these types we add the conceptual types: for the types of truth values and individuals we have an intensional hierarchy composed by D_{o_n} and $D_{\iota n}$ for $n = \{1, 2, ...\}$. For the functional type we would expect a type such as $D_{(\alpha\beta)_n}$, to which belong the concepts of functions, but there is no such a kind of type in Church's intensional logic. Instead of creating a new separate type for the concept of functions Church suggests that functions from concepts to concepts can play the part of concepts of functions defined in $D_{\alpha\beta}$. Therefore the concepts of functions are not to be found in a possible separate type $D_{(\alpha\beta)_n}$, but in the type $D_{\alpha_n\beta_n}$, i.e., the type which contains functions from concepts to concepts. The concepts of functions are then considered as functions of concepts, and that is the functional simplification in the type hierarchy which Church adheres to until his last published paper concerning this issue in 1993, where he explains that "a concept of a function of type $\alpha\beta$ is itself a function and is of type $\alpha_1\beta_1$. This assumption is perhaps not unavoidable, but it greatly simplifies the theory and we shall follow it" (Church, 1993, p. 142).

Definition 1.2.16 (Preferred type)**.** Some selected types are considered preferred types, these are types which are necessarily non empty, even if the type of individuals is empty. The preferred types avoid the appearance of denotationless names that cannot be interpreted.

- o_n;
- $\iota\alpha$;
- if β is a preferred type symbol, then $\alpha\beta$ is a preferred type symbol.

Syntax

The improper symbols are the same as before and the proper symbols are:

- *Logical constants:* $C_{o_n o_n o_n}$ (conditional), $\Pi_{(\alpha_n o_n) o_n}$ (the universal predicate), $\iota_{(\beta_n o_n)\beta_n}$ (the selection operator), $\Delta_{\alpha_{n+1}\alpha_n o_n}$ (it expresses the relationship between the sense of a name and its denotation); for all type symbols α, all preferred type symbols β, and all natural numbers n.

- *Variables:* an infinite list of type α, for each type symbol α: $a_\alpha, b_\alpha, \dots,$ $a'_\alpha, b'_\alpha, \dots, a''_\alpha, b''_\alpha, \dots$

The definitions of a formula and of free and bound variable are the same of definition 1.2.6 and of definition 1.2.8.

Definition 1.2.17 (Abbreviations)**.** The following abbreviations can be stated using the logical constants:[17]

1. $\mathbf{A}_{o_n} \to \mathbf{B}_{o_n} := C_{o_n o_n o_n}\mathbf{A}_{o_n}\mathbf{B}_{o_n}$
2. $\forall \mathbf{x}_{\alpha_n}\mathbf{A}_{o_n} := \Pi_{(\alpha_n o_n)o_n}\langle\lambda\mathbf{x}_{\alpha_n}.\mathbf{A}_{o_n}\rangle$
3. $T := \forall a_o(a_o \to a_o)$
4. $T_{o_n} := \forall a_{o_n}(a_{o_n} \to a_{o_n})$
5. $F := \forall a_o a_o$
6. $F_{o_n} := \forall a_{o_n} a_{o_n}$
7. $\neg\mathbf{A}_{o_n} := \mathbf{A}_{o_n} \to F_{o_n}$
8. $\exists \mathbf{x}_{\alpha_n}\mathbf{A}_{o_n} := \neg\forall\mathbf{x}_{\alpha_n}\neg\mathbf{A}_{o_n}$
9. $\iota\mathbf{x}_{\beta_n}\mathbf{A}_{o_n} := \iota_{(\beta_n o_n)\beta_n}\langle\lambda\mathbf{x}_{\beta_n}.\mathbf{A}_{o_n}\rangle$
10. $Q_{\alpha\alpha o_n} := \langle\lambda a_\alpha.\langle\lambda b_\alpha.\forall f_{\alpha o_n}(f_{\alpha o_n} b_\alpha \to f_{\alpha o_n} a_\alpha)\rangle\rangle$

[17] Note that these abbreviations are not the same of definition 1.2.9 because in the simple theory of types the primitive constants were not the same either.

11. $\mathbf{A}_\alpha = \mathbf{B}_\alpha := Q_{\alpha\alpha o}\mathbf{B}_\alpha\mathbf{A}_\alpha$

12. $\mathbf{A}_\alpha =_n \mathbf{B}_\alpha := Q_{\alpha\alpha o_n}\mathbf{B}_\alpha\mathbf{A}_\alpha$

13. $\mathbf{A}_\alpha \neq \mathbf{B}_\alpha := \neg(\mathbf{A}_\alpha = \mathbf{B}_\alpha)$

14. $\mathbf{A}_\alpha \neq_n \mathbf{B}_\alpha := \neg(\mathbf{A}_\alpha =_n \mathbf{B}_\alpha)$

15. $\square_{o_n o_m} := Q_{o_n o_n o_m} T_{o_n}$, where $m \leq n$

The constants: C_{ooo}, $\Pi_{(\alpha o)o}$, T, F, $\neg$, $=$, $\neq$ and $\exists$, amount to the conditional, the universal predicate, truth, falsehood, negation, equality, non-equality and the existential quantifier, respectively. The primitive constant: $\iota_{(\beta o)\beta}$ allows to create names through a description operator. Following Frege's idea, in order to reject the entrance to denotationless names into a formalized language, Church restricts β to be a preferred type, which is necessarily non-empty, wherein there has been selected a particular member of the type called the designated member of that type. This device assures that formulas such as $\iota_{(\beta o)\beta}\mathbf{A}_{\beta o}$ do never lack a denotation. Despite this limitation, Church believed in the possibility of constructing a language where there could exist denotationless names:

> It would also be possible, by recognizing functions having less than an entire type as the range of the argument, to fix the sense of $\iota_{\beta(o\beta)}$ [our $\iota_{(\beta o)\beta}$] in such a way that names of the form $(\iota_{\beta(o\beta)}\mathbf{A}_{o\beta})$ [our $\iota_{(\beta o)\beta}\mathbf{A}_{\beta o}$] would occur that have a sense but no denotation. Frege held that such names do exist in the natural languages, but avoided them in constructing a formalized language. The writer believes that the construction of a formalized language containing denotationless names should also be possible. And it might well be worth while to carry out the construction of such a language in spite of probable complications—if only as a museum piece, to show that the avoidance of denotationless names in a formalized language is a matter of option rather than theoretical necessity. [...] There are sound reasons for the opinion generally held that the meaningfulness of an expression of a formalized language must not be allowed to depend on any question of extralinguistic fact. But these reasons refer to meaningfulness in the sense of having a sense, rather than in the sense of having a denotation. Therefore, tentatively, we shall allow that in some language [...] there may be names which have a sense but no denotation. Hence we also admit concepts that are not concepts of anything. (Church, 1951, p. 14–15, footnote 16).

The abbreviation $\iota \mathbf{x}_\beta$ is therefore used as a description operator. This is purposely differentiated from the description operator of Russell $\imath$, as used in *Principia Mathematica*, which is contextually defined and it is not, strictly speaking, a real description operator but a "mere typographical convenience":

> Strictly speaking, there is no description operator and there are no descriptions in the formalized language of *Principia Mathematica*. For the authors of *Principia* state explicitly that they regard definitions generally as "mere typographical conveniences"—thus not a part of their formalized language, but only a means for its easier interpretation. In the case of an expression—say, a description—introduced by contextual definition, this means that when the longer expression in which it occurs is rewritten in full there is found to be no well-formed part which can be identified with the description. Hence the Fregean analysis of meaning must not be applied to Russell's descriptions. And, in particular, such a description must not be said to denote, except as a manner of speaking, introduced by a contextual definition applied to the meta-language.
> It is even possible that not only the "denotation" but also the "sense" of a Russellian description might be introduced by contextual definition into the metatheory. But until this has actually been done for a particular formalized meta-language, these expressions—especially the "sense"—must be used with a great deal of caution, if at all. (Moreover, it seems unlikely that by such a device the need can be done away with for a direct treatment of intensional notions.) (Church, 1951, p. 15, footnote 17).

Instead of the symbol $\Box$ for expressing necessity, Church has the constant $N_{o_n o_m}$, which should not be confused with the primitive constant for negation: N_{oo}, of the simple theory of types. Subscripts in $\Box$ can be omitted if m is 0 and if all the omitted information can be restored univocally.

Definition 1.2.18 (Concept of)**.** The primitive constant $\Delta_{\alpha_1 \alpha o}$ denotes a binary function whose value is truth in case the first argument is a concept of the second argument and is falsehood in the contrary case. So, in $\Delta_{\alpha_1 \alpha o} \mathbf{A}_{\alpha_1} \mathbf{A}_\alpha$, the expression $\mathbf{A}_{\alpha_1}$ denotes the sense of the expression $\mathbf{A}_\alpha$. A *concept of* x is anything which is capable of being the sense of a name of x.

Comment 1.2.19. In Church's original notation, the primitive constant Δ denotes a binary function which takes as second argument the possible concept of the first argument. We consider this does not make it any easier for the reader to understand the formulas: if you are reading a formula starting with Δ and you are interpreting this symbol as the "concept of" relation, we think it is more simple to begin with the argument which is considered to be the concept and to continue with the argument which is considered to have as concept the first argument. Hence, we have changed the order of the type symbols in Δ from $\Delta_{\alpha\alpha_1 o}$ to $\Delta_{\alpha_1 \alpha o}$. In this way we can say that the expression $\mathbf{A}_{\alpha_1}$ denotes the sense of the expression $\mathbf{A}_{\alpha}$, with the formula $\Delta_{\alpha_1 \alpha o}\mathbf{A}_{\alpha_1}\mathbf{A}_{\alpha}$. With this change we have gained a more understandable interpretation of the formula, which can be read as "$\mathbf{A}_{\alpha_1}$ is a concept of $\mathbf{A}_{\alpha}$".

Definition 1.2.20 (Characterizing function)**.** The function $\phi'_{\beta_1\alpha_1}$ characterizes the function $\phi_{\beta\alpha}$ if the following condition is satisfied, that the value of the function ϕ for an argument ξ is η if and only if the value of the function ϕ' for any concept of ξ as argument is always a concept of η.[18]

Having defined what it is to be a characterizing function, Church makes two assumptions:

1. He identifies each concept σ of a function ϕ with the characterizing function ϕ' of ϕ that is determined by σ.

2. He also assumes that every characterizing function ϕ' of ϕ is to be regarded as a concept of ϕ.

Church is conscious that these assumptions have been made without support on the writings of Frege but however he finds them very useful.

The rules of inference are the same that in the simple theory of types, with the exception that the substitution rule is not included. These are: the λ-conversion rules (alphabetic change of bound variable, λ-contraction and λ-expansion), universal generalization and *modus ponens*.

1.2.4 Heuristic Principles behind Church's Alternatives

Church offers three Alternatives for his logic of sense and denotation. The three alternatives of intensional logic are in close relation with the different conditions you need to consider for speaking of identity of senses. Identity has always been a crucial issue in the debate not only of extensional logic but also in intensional logic. In the very beginning of the "primitive" intensional logic of Frege (1892), the discussion begins with some identity difficulties raised when dealing with different names that are considered to be equal.

[18] This is the exact definition of Church (1951, pp. 15–16).

When we are dealing with the denotation of linguistic expressions, an equality statement poses no problem although we have different names for denoting the same object, because at the core of the equality statement what we have is the identity relation between an object and itself (which is the very same object under two different names, and in extensional logic what we are really doing is operating exclusively with denotations of linguistic expressions and not with senses).

When changing to an intensional logic, the problem of the identity of senses is one of the most important to be addressed. This issue can also be analyzed regarding the synonymity relation between two names. Two names are said to be synonymous when they express the same sense, so with identity of senses we refer to the conditions under which two names express the same sense. These conditions can be weak or can be strong. Church realizes the range of options and formulates three alternatives which can be ordered following the strength of the identity conditions of the senses. In an increasingly order the alternatives are: Alternative (2), Alternative (1) and Alternative (0). The lower the number, the stronger the identity conditions for considering two senses to be equal.

Alternative (2) is presented and just slightly developed in (Church, 1951) and revised in (Church, 1973a) and (Church, 1974) and it has also been studied by (Kaplan, 1964) and (C. Parsons, 1982). Alternative (1) is presented in (Church, 1951) where some axioms are indicated at the end of that article and is developed in (Church, 1993). Alternative (0) is presented in (Church, 1951), is little worked in (Church, 1974) and is mainly developed by (Anderson, 1977) in his dissertation thesis.

Alternative (2)

This alternative refers to a criteria of identity of senses related with proving logical equivalence between two names. In Church's words, Alternative (2) "makes the senses of $\mathbf{A}$ and $\mathbf{B}$ the same whenever $\mathbf{A} = \mathbf{B}$ is logically valid." (1951, p. 5). This alternative, which regards the identity of senses in relation only with the truth-value of $\mathbf{A}$ and $\mathbf{B}$, can be useful in some kinds of intensional logic like modal logic but is of little help in epistemic logic and those logics which take into account propositional attitudes. The problem with this alternative in an epistemic context is that if you believe something that is a logical truth you must believe in all others logical truths, something which does not seem a good solution for dealing with all nuances contained in intensional logic.

Alternative (1)

In this alternative "two names are assumed to have different senses in all cases where it is not already a consequence that the senses are the same." (Church, 1946, p. 31). For Church this is the alternative that is more plausible for working within a fregean framework. And in Church terminology, **A** and **B** would have the same sense if they can be obtained under the application of any of the lambda conversion rules. Therefore the rules of λ-*contraction* and λ-*expansion* along with the rule of α-*conversion*, or alphabetic change of bound variable, preserve the identity of senses.

Alternative (0)

This alternative imposes the strictest conditions concerning the identity of senses and, in this case, two names are said to express the same sense whenever one can be obtained from the other by application of the rule of α-*conversion*, which allows the possibility of doing an innocuous change of a bound variable. This alternative, presented as an amended form of Alternative (1) but without a name in the *addendum* to the abstract (Church, 1946),[19] was called "Alternative (0)" from (Church, 1951) onwards and was considered to correspond to Carnap's notion of intensional structure, "with the one difference that it retains the notion of sense as something to be dealt with in the object language, whereas Carnap's intensional structure is a metatheoretic notion and is dealt with in his meta-language." (Church, 1951, p. 5). Alternative (0) is a more complicated system in which you need to use an infinite list of primitive symbols λ_i. Of this hierarchy of operators, λ_0 is abbreviated by λ and is interpreted as an abstraction operator. The rules of λ-*contraction* and λ-*expansion* are restricted to λ_0, in such a way that there is no lambda conversion that can preserve the sense of two expressions under the application of these rules.

Synonymous Isomorphism

Alternatives (1) and (0) are more closely related to the linguistic expressions that express identical senses than Alternative (2), which is more focused on the equivalence relation between expressions. In that sense, Church uses a variant of Carnap's notion of *intensional isomorphism* (Carnap, 1947) which he calls *synonymous isomorphism* (Church, 1954) for dealing with the identity of senses through synonym replacements in linguistic expressions. As we

[19] The abstract is dated in March, 1946 and the *addendum* in April 29, 1946.

have seen before, synonymous isomorphism refers in Alternative (1) to replacements closed to applications of the λ-conversion rules, and in Alternative (0) to replacement of a bound variable. In order to increase the understanding of the notion of synonymous isomorphism, we would like to consider its application to natural language by means of an example. Following Klement: "In natural language, two propositions A and B are synonymously isomorphic if and only if there is some finite number of synonym replacements that could be made to the expressions in A such that B results." (2002, p. 102). A synonym replacement is the substitution of one expression with another expression that have the same sense in the same contexts. So, for example, the propositions:

1. Some myopic men wear glasses

2. Some short-sighted men wear glasses

would be synonymously isomorphic because you can obtain 2. from 1. through one synonym replacement: "short-sighted" for "myopic", which are different expressions having the same sense in the same contexts. We could also replace "glasses" for "spectacles" and we would obtain from 2.:

3. Some short-sighted men wear spectacles

In this case, 3. would be synonymously isomorphic with 2. but also with 1., although there has been needed two synonym replacements to arrive from 1. to 3. These examples can be very helpful for understanding that two synonymously isomorphic sentences share a common (grammatical) form and there is only replacement between synonym expressions.

Synonymous isomorphism as a criterion for identity of senses can be helpful but it is not a complete one. Although it can explain the identity of sense of two whole propositions, it presupposes some previous synonymies that cannot be explained with this notion. Synonymy between "myopic" and "short-sighted" is simply taken as presupposed and cannot be explained why they are synonymous.[20]

1.2.5 Axiomatization for Church's Alternatives

Church proposes a number of axioms that can be divided into various categories depending on the features of the system we want to construct. Instead of single axioms, he includes axiom schemata, because there exists a version

[20] This short of criticism to the theory of synonymy can be found in Quine (1951, pp. 24–27; 32–36).

of each schema for the different types α and β and type-indices n. Although we usually refer to the axioms in singular, because we find it more manageable, each of them should be understood as a set of axioms or as an axiom scheme.

First of all, some extensional axioms are introduced. These axioms are not specific to intensional logic and they serve for a standard extensional higher-order functional calculus employing the simple theory of types. Axioms 1-7 allow the functioning of a propositional calculus and the laws of quantifiers. Axioms 8-9 are axioms of extensionality. And axiom 10, a consequence of a possible added axiom of choice, is about descriptions.

Intensional Axioms

These axiom schemata were thought to be applied to every system of intensional logic regardless of the alternative which has been taken into account.

11. $\Delta_{(o_{n+1}o_{n+1}o_{n+1})(o_n o_n o_n)o} C_{o_{n+1}o_{n+1}o_{n+1}} C_{o_n o_n o_n}$
12. $\Delta_{((\alpha_{n+1}o_{n+1})o_{n+1})((\alpha_n o_n)o_n)o} \Pi_{((\alpha_{n+1}o_{n+1})o_{n+1})} \Pi_{((\alpha_n o_n)o_n)}$
13. $\Delta_{((\beta_{n+1}o_{n+1})\beta_{n+1})((\beta_n o_n)\beta_n)o} \iota_{((\beta_{n+1}o_{n+1})\beta_{n+1})} \iota_{((\beta_n o_n)\beta_n)}$
14. $\Delta_{(\alpha_{n+1}\alpha_{n+2}o_{n+1})(\alpha_n \alpha_{n+1}o_n)o} \Delta_{\alpha_{n+1}\alpha_{n+2}o_{n+1}} \Delta_{\alpha_n \alpha_{n+1}o_n}$

Axioms 11-14 have the same structure: $\Delta \mathbf{A}_{\alpha_1} \mathbf{A}_{\alpha}$. The main objective of Church is to assure that the *first ascendant* (1993, p. 142)—obtained from the original formula by increasing every subscript in it by 1—of a formula denotes its sense, so he needs these axioms to obtain theorems in which the first ascendant of a formula denotes its sense. Axiom 11 expresses the relation between the constant "$C_{o_n o_n o_n}$" and its *first ascendant*: "$C_{o_{n+1}o_{n+1}o_{n+1}}$", saying that the second constant is the sense of the first constant. Axioms 12, 13 and 14 express something similar to what has been explained about axiom 11, saying that the first ascendants of the constants $\Pi_{((\alpha_n o_n)o_n)}$, $\iota_{((\beta_n o_n)\beta_n)}$ and $\Delta_{\alpha_n \alpha_{n+1} o_n}$ are respectively their senses, or that these constants are presented (as denotation) by their first ascendants (which are their senses).

15. $\forall f_{\beta\alpha} \forall f_{\beta_1\alpha_1} \forall x_\beta \forall x_{\beta_1} [\Delta_{(\beta_1\alpha_1)(\beta\alpha)o} f_{\beta_1\alpha_1} f_{\beta\alpha} \to (\Delta_{\beta_1\beta o} x_{\beta_1} x_\beta$
    $\to \Delta_{\alpha_1\alpha o}(f_{\beta_1\alpha_1} x_{\beta_1})(f_{\beta\alpha} x_\beta))]$
16. $\forall f_{\beta\alpha} \forall f_{\beta_1\alpha_1} [\forall x_\beta \forall x_{\beta_1} (\Delta_{\beta_1\beta o} x_{\beta_1} x_\beta \to \Delta_{\alpha_1\alpha o}(f_{\beta_1\alpha_1} x_{\beta_1})(f_{\beta\alpha} x_\beta))$
    $\to \Delta_{(\beta_1\alpha_1)(\beta\alpha)o} f_{\beta_1\alpha_1} f_{\beta\alpha}]$
17. $\forall x_\alpha \forall y_\alpha \forall x_{\alpha_1} [\Delta_{\alpha_1\alpha o} x_{\alpha_1} x_\alpha \to (\Delta_{\alpha_1\alpha o} x_{\alpha_1} y_\alpha \to (x_\alpha = y_\alpha))]$

Axioms 15, 16 and 17 are at the core of Church's intensional logic, and it is essential to understand their meanings because they not only describe the main features of the heuristic principles behind Church's intensional axioms but also because they have an important role in the difficulties that Church's intensional logic has had to face.

These axioms are important because they formalize the previous defined notion of characteristic function (definition 1.2.20 on page 25) that is so crucial in Church's intensional logic:

$$\begin{gathered} f_{\beta_1\alpha_1} \text{ characterizes } f_{\beta\alpha} ::= \\ \forall x_{\beta_1}\forall x_\beta(\Delta_{\beta_1\beta o}x_{\beta_1}x_\beta \to \Delta_{\alpha_1\alpha o}(f_{\beta_1\alpha_1}x_{\beta_1})(f_{\beta\alpha}x_\beta)) \end{gathered} \tag{1.1}$$

In order to simplify the reading we have abbreviated the symbols of the types representing the first ascendant of a name with the same expression with an asterisk. The expressions f and f^* become the abbreviations of $f_{\beta\alpha}$ and $f_{\beta_1\alpha_1}$ respectively; x and x^* of x_β and x_{β_1}. With this new abbreviated notation, we can formalize the notion of characterization by saying that a function f^* *characterizes* a function f if and only if:

$$\forall x^*\forall x[\Delta x^*x \to \Delta f^*x^*fx] \tag{1.2}$$

Axiom 15 says that if f^* is a concept of f then, if x^* is a concept of x, then f^*x^* is a concept of fx. So in terms of characterization we can say that if f^* is a concept of f then f^* characterizes f. Axiom 16 says that if x^* is a concept of x then, if f^*x^* is a concept of fx, then f^* is a concept of f. Therefore, if f^* characterizes f, then f^* is a concept of f.

Axiom 17 says that if two entities have the same concept, then these two entities are equal. Therefore a concept is a concept of at most one thing, and each concept thus individuates univocally a thing.

Intensional Axioms for Alternative 2

The specific axioms for Alternative (2) repeat axioms schemata 1-17 with the addition of the necessity operator $\Box$ and with some changes in the type symbols. Axioms 18-27 copy the schemata from axioms 1-10. Axioms 28-31 are similar to 11-14. Axioms 32, 33, *33 and 34 mirrors axioms 15, 16 and 17. Finally, the last axioms 35-38 have to do with the behavior of the operator $\Box$.

We offer only some examples of these axioms in order to see their structure:

18. $\Box\forall f_{\alpha_1\beta_1 o_1}(\forall x_{\alpha_1}\forall y_{\beta_1} f_{\alpha_1\beta_1 o_1}x_{\alpha_1}y_{\beta_1} \to \forall y_{\beta_1}\forall x_{\alpha_1} f_{\alpha_1\beta_1 o_1}x_{\alpha_1}y_{\beta_1})$

28. $\Box[\Delta_{(o_{n+2}o_{n+2}o_{n+2})(o_{n+1}o_{n+1}o_{n+1})o_1} C_{o_{n+2}o_{n+2}o_{n+2}} C_{o_{n+1}o_{n+1}o_{n+1}}]$

33. $\Box\forall f_{\beta_1\alpha_1}\forall f_{\beta_2\alpha_2}[\forall x_{\beta_1}\forall x_{\beta_2}(\Delta_{\beta_2\beta_1o_1}x_{\beta_2}x_{\beta_1} \to \Delta_{\alpha_2\alpha_1o_1}(f_{\beta_2\alpha_2}x_{\beta_2})(f_{\beta_1\alpha_1}x_{\beta_1})) \to \Delta_{(\beta_2\alpha_2)(\beta_1\alpha_1)o_1} f_{\beta_2\alpha_2} f_{\beta_1\alpha_1}]$

35. $\forall f_{\alpha_1o_1}\forall x_{\alpha}\forall x_{\alpha_1}\forall x_{\alpha_2}[\Delta_{\alpha_1\alpha o}x_{\alpha_1}x_{\alpha} \to (\Delta_{\alpha_2\alpha_1o_1}x_{\alpha_2}x_{\alpha_1} \to (\Box(\Delta_{\alpha_2\alpha_1o_1}x_{\alpha_2}x_{\alpha_1}) \to (\Box(\Pi_{(\alpha_1o_1)o_1}f_{\alpha_1o_1}) \to \Box(f_{\alpha_1o_1}x_{\alpha_1}))))]$

Intensional Axioms for Alternative 1

The axiomatization for Alternative (1) starts with some axioms (39-45) indicating that any two senses expressed by expressions which have a different composition, i.e., having a different primitive constant as main operator, are not identical. Therefore, the sense expressed by a conditional expression is not identical with the sense of an expression whose main operator is a universal quantifier, or a descriptor. The structure of the syntactic expression is essential in Alternative (1), moving away from Alternative (2) where the main point was rooted in the truth value of the expressions and not in its composition. We give axiom 39 as an example:

39. $\forall p_{o_{n+1}}\forall q_{o_{n+1}}\forall f_{\alpha_{n+1}o_{n+1}}(C_{o_{n+1}o_{n+1}o_{n+1}}p_{o_{n+1}}q_{o_{n+1}} \neq \Pi_{(\alpha_{n+1}o_{n+1})o_{n+1}}f_{\alpha_{n+1}o_{n+1}})$

Apart from the previous axioms Church introduces some axioms for dealing with the identity of senses in expressions which have the same composition. Axioms 45 and 46 indicate that the senses of two conditional expressions are the same whenever the senses expressed by their antecedents and consequents are identical.

45. $\forall p_{o_{n+1}}\forall q_{o_{n+1}}\forall r_{o_{n+1}}\forall s_{o_{n+1}}(C_{o_{n+1}o_{n+1}o_{n+1}}p_{o_{n+1}}q_{o_{n+1}} = C_{o_{n+1}o_{n+1}o_{n+1}}r_{o_{n+1}}s_{o_{n+1}} \to (p_{o_{n+1}} = r_{o_{n+1}}))$

46. $\forall p_{o_{n+1}}\forall q_{o_{n+1}}\forall r_{o_{n+1}}\forall s_{o_{n+1}}(C_{o_{n+1}o_{n+1}o_{n+1}}p_{o_{n+1}}q_{o_{n+1}} = C_{o_{n+1}o_{n+1}o_{n+1}}r_{o_{n+1}}s_{o_{n+1}} \to (q_{o_{n+1}} = s_{o_{n+1}}))$

Axioms 47-50 are similar to the previous ones but refer to the primitive constants: Π, ι and Δ. The last three axioms: 51-53, establish that the senses expressed by formulas whose components have different type symbols are not identical. For example:

51. $\forall f_{\beta_{m+n+2}o_{n+1}}\forall f_{\beta_{m+n+2}o_{m+n+2}}(\iota_{(\beta_{m+n+2}o_{n+1})\beta_{m+n+2}}f_{\beta_{m+n+2}o_{n+1}} \neq \iota_{(\beta_{m+n+2}o_{m+n+2})\beta_{m+n+2}}f_{\beta_{m+n+2}o_{m+n+2}})$

As we have seen, Church offers axioms for dealing with the identity of senses of expressions which specifically relates to the primitive constants, but it would have been desirable that Church had offered some axioms for dealing with the identity of senses in a general way, as he actually did in (1974, pp. 149–152) when he gave an axiomatization for Alternative (0).

Intensional Axioms for Alternative 0

Regarding the axioms for Alternative (0), it must be said that they cannot be found in (Church, 1951) but in (Church, 1974). As a new element we find that it is introduced a superscript m in the primitive constant Δ, which is part of Church's Tarskian solution for solving some semantical antinomies. This solution is based on the creation of a hierarchy of Δ, always appearing with a superscript m, that mirrors the Tarskian hierarchy of languages and metalanguages. From all the axioms, it is worth considering axiom 64, an axiom which, along with axioms 10-17, allows the derivation of many other axioms, for example, those for Alternative (1) (Klement, 2002, p. 111).

64. $\forall f_{\beta\alpha} \forall f_{\beta_1\alpha_1} \forall x_\beta \forall x_{\beta_1} \forall y_\beta \forall y_{\beta_1} (\Delta^m_{(\beta_1\alpha_1)(\beta\alpha)o} f_{\beta_1\alpha_1} f_{\beta\alpha} \to (\Delta^m_{\beta_1\beta o} x_{\beta_1} x_\beta \to (\Delta^m_{\beta_1\beta o} y_{\beta_1} y_\beta \to (f_{\beta_1\alpha_1} x_{\beta_1} = f_{\beta_1\alpha_1} y_{\beta_1} \to (x_{\beta_1} = y_{\beta_1})))))$

The meaning of axiom 64 can be easily described if we focus on the last conditional of the axiom:

$$f_{\beta_1\alpha_1} x_{\beta_1} = f_{\beta_1\alpha_1} y_{\beta_1} \to x_{\beta_1} = y_{\beta_1}$$

If the senses of the expressions fx and fy are the same then the senses of the arguments x and y are also identical. This seems a good principle for dealing with synonymy, because it captures the idea that if the values of fx and fy are synonymous, then x and y must also be synonymous. Apart from synonymy, the axiom says that a concept of a function is an injective function (or one-to-one function) on concepts. For Anderson (1980, p. 222), axiom 64 implies that not every entity in certain types can be "concepted". He appreciates the role of this axiom and wants to preserve it in order to construct a consistent logic of sense and denotation.[21]

[21] The reason for postulating axiom 64 is just to have as consequence the principle (F) saying that "if (FA) is synonymous with (FB), then A is synonymous with B." And Anderson concludes "that Principle (F) is central to the Fregean program. Further, since Alternative (0) is the only *systematic* criterion of which we are aware, we should be reluctant to abandon it without compelling reason." (Anderson, 1980, p. 224).

1.2.6 Problems with Church's Intensional Logic

Problems with Paradoxes

As we have seen, Alternatives (1) and (0) deal with senses that are so closely related with the linguistic expressions which express them that they have a high probability to be attacked by some analogue of the traditional semantical antinomies.[22] Church is aware of this additional complications that can be derived from these alternatives and in a footnote he also indicates that:

> The writer is indebted to Leon Henkin for raising the question of the cardinal number of the concepts (senses) of a given type, in connection with the answer to which an antinomy may easily appear, at least unless appropriate caution has been exercised in regard to the assumptions which are made (in the form of axioms and rules). Because of this and other possibilities of self-contradiction, no logistic treatment of sense and denotation can be accepted as more than provisional until its consistency has been thoroughly studied. (1951, p. 6, footnote 6).

This intuition about the possibility of derived problems came true only seven years after Church's article in Myhill (1958), where it was shown that Alternative (1) was formally inconsistent, something that Church himself recognized.[23] As usual, the origin of the antinomy derives from Cantor's theorem concerning the greater cardinality of the number of classes of entities (of a certain type) than the cardinality of the number of entities (of the same type). The analysis of this antinomy will not be done here and we refer the reader to the studies of Anderson (1977, chapter 3), (1980, pp. 221–223) and (1987, pp. 107–108) for a deeper approach. Anderson relates Myhill's antinomy to Russell's paradox and calls this problem "the Russell-Myhill antinomy".

Another problem that attacked Church's original intensional logic stems from a version of the Epimenides paradox which could be formulated in the system, as David Kaplan reported (Anderson, 1987, p. 113, note 6). We also remit to the papers of Anderson referred in the previous paragraph for further study and for technical details about this paradox.

[22] Church realizes about the danger of this proximity: "Indeed, the stronger are the conditions required in order that two names shall express the same sense, or that concepts shall be identical, the more closely will the abstract theory of concepts resemble the more concrete theory of the names themselves–with the relations symbolized by $\Delta_{o\alpha_1\alpha}$ serving as analogues of the relations of denoting in the semantical theory. Hence, the stronger these conditions are, the greater is the danger of antinomies analogous to Richard's or Grelling's or the Epimenides." (1974, p.149).

[23] "In fact *A Formulation* is unsound or faulty in many ways." (Church, 1973a, p. 24).

Traditionally there have been two solutions to these semantical antinomies and paradoxes. The first one is to adopt a *ramified* theory of types, where you distinguish not only between types but also between orders. The second solution comes from Tarski, who draws a distinction between object language and metalanguage; a distinction which has been widely accepted since then and that is now common knowledge. Church, in his subsequent revisions of his original logic of sense and denotation, will make use of both attempts of solution. In his revision of the seventies, Church modified his system by adding a superscript m to the semantic operator Δ, where m can be $l, l+1, l+2$, etc. This hierarchy mirrors the tarskian distinction between language and metalanguage. Unfortunately, although Church introduced these changes for avoiding the semantical paradoxes, Anderson (1977, pp. 22–29) and (1980, pp. 221–223) showed that these paradoxes could appear again in the modified system. This led Church to consider the other attempt, and in his revised formulation of the logic of sense and denotation of the nineties, he adopts a *ramified* type theory (no longer a *simple* one) for avoiding the semantical paradoxes. In between these attempts for eliminating the semantical paradoxes we find the proposal of Anderson, not adopted by Church, of removing axiom 16 in favor of a new axiom schema. This modification would match with the Tarskian hierarchy and it would not be needed to drop out the simple type theory in favor of a fully ramified one. It is possible that Church considers axiom 16 so central to his system that he prefers to abandon his simple type theory than ending up without axiom 16.

Problems with Intensionality

As Church says in (1973a, p. 25) A. F. Bausch discovered a flaw in the axioms for Alternative (2) of (Church, 1951) consisting not in an inconsistency but in a reduction to extensionality. It means that it can be proved in Alternative (2), making use of axiom 16, that there is only one true proposition.[24]

[24]Parsons gives a formal sketch of the proof in (2001, pp. 515–516), here he also gives the following informal account: "The proof exploits the fact that in alternative 2 necessarily equivalent concepts are identical. That is, two concepts that are necessarily concepts of the same thing are identical. The proof proceeds by defining two concepts that can be proved to be concepts of the same function, because they can be proved to characterize the same function f; axiom set 16 thus makes them (provably) concepts of *f*. Thus, in alternative 2 they are identical. However, they are defined in such a way they are *not* identical, because they differ for certain arguments that are not concepts of anything (recall that axiom set 16 ignores what functions do to non-concepts). At least they differ if such arguments exist. And such arguments can be shown to exist whenever there is more than one true proposition. The only escape from the inconsistency then is to conclude that there is not more than one true proposition." (p. 515).

> In effect, the discussion on pages 15-16 of *A Formulation*, leading up to the assumption that φ' *is a concept of a function* φ *if and only if it is a characterizing function of* φ, is incompatible with the principle of Alternative (2), that the senses of names **A** and **B** are the same if and only if the equation **A** = **B** is logically valid. Because this (italicized) assumption is embodied in the axioms as Axioms $15^{\alpha\beta}$ and $16^{\alpha\beta}$, the result is that the axioms lead, not quite to an inconsistency, but to a reduction to extensionality, in the sense that it is a theorem in each type α that two concepts of the same thing are always identical. (Church, 1973a, p. 25).

Problems with Emptiness

Frege rejected from a formalized language names with sense but without denotation. Remember, however, that Church believed that

> the construction of a formalized language containing denotationless names should also be possible. [...] Therefore, tentatively, we shall allow that in some language [...] there may be names which have a sense but no denotation. Hence we also admit concepts that are not concepts of anything; and although no name in this present language has such a concept as its sense, we may wish in the construction of the language to allow for existence of such concepts. (1951, pp. 14–15, footnote 16).

Church explains that there is room for dealing with empty terms and their senses in a formalized language. For doing this, following T. Parsons (2001), we need to make some changes, especially in axiom 16 which should be "incompatible with the existence of empty terms in any language, at least if their senses are included in the ontology" (p. 517).[25] For T. Parsons (2001) the very problem comes from the fact that "axiom set 16 makes functions be senses without looking at what they do to arguments that are not senses" (p. 517). Therefore, Parsons' proposal, based on (Church, 1973a, 1974, 1993) and (Anderson, 1984), is that "a functional sense should be something like a partial function, defined on senses and not defined at all for non-senses" (p. 518).

The semantics for this logic with empty names and empty senses should contain some entity, which is not already in the domain, added to the domain

[25] "And for Alternatives (1) and (2), there is moreover the possibility that Axioms $16^{\alpha\beta}$ may have to be modified in connection with considerations concerning vacuous concepts." (Church, 1951, p. 4, footnote 4).

of a given type. So we should have an extended domain for any given type resulting from adding to the original domain a new entity, called a *zip*. If we consider that a term has intuitively no denotation at all, in this semantics we can stipulate that this term refers to the zip of the respective type. We add *zips*, symbolized as $\otimes$, to every domain of non-functional types: so $\otimes_\alpha$ is the zip for the type α. We also assume that zips cannot be concepts of anything. In this regard, an empty concept of type α_1 would be a concept of $\otimes_\alpha$, and, as a result, a proposition (of type o_1) which is neither true nor false, would be a concept of $\otimes_{o_0}$.

The formal account of this theory is solved by changing Church's notion of characterization of a function, as given in definition 1.2.20 and equation (1.1), by a new one of *supercharacterization* which, taking into account the introduction of zips, is defined as:

Definition 1.2.21 (Supercharacterization)**.** The function $f_{\beta_1\alpha_1}$ *supercharacterizes* the function $f_{\beta\alpha}$ if and only if

$$\forall x_{\beta_1}\forall x_\beta(\Delta_{\beta_1\beta o}x_{\beta_1}x_\beta \rightarrow \Delta_{\alpha_1\alpha o}(f_{\beta_1\alpha_1}x_{\beta_1})(f_{\beta\alpha}x_\beta))\wedge$$
$$\forall x_{\beta_1}(\neg\exists x_\beta\Delta_{\beta_1\beta o}x_{\beta_1}x_\beta \rightarrow f_{\beta_1\alpha_1}x_{\beta_1} = \otimes_{\alpha_1})$$

By replacing this definition by definition 1.2.20 in axioms 15 and 16 Parsons gives us a tool for grasping empty senses in our formalized language. Furthermore, Parsons shows an interpretation of Church's intensional logic which allows us to deal with *de dicto* readings as well as *de re* readings. In natural language there is ambiguity concerning the interpretation of predicates that can receive a *de dicto* or a *de re* reading. In this regard, the *concept of* relation can also have both kinds of readings. Although Church has considered only the *de dicto* reading, there is nothing to prevent Church's intensional logic from doing a *de re* reading. For Church the *concept of* relation relates functions of functional types $\beta_1\alpha_1$ with functions of functional types $\beta\alpha$, but should there be possible a *concept of* relation between functions of type $\beta\alpha_1$ and functions of type $\beta\alpha$? This is just the solution Parsons offers for the problem. Therefore a function of type $\beta\alpha$ can have as concept either a function of type $\beta_1\alpha_1$ or a function of type $\beta\alpha_1$. The first possibility fits a *de dicto* reading and the second one a *de re* reading, which Parsons calls "Fregean" and "Russellian" instances of the *concept of* relation respectively (T. Parsons, 2001, p. 526).

> Note that on this account a concept can be a concept of more than one thing. But this does not really violate the idea that *concept of* is functional, because the types are different; a concept can never be a concept of two different things in the same sense of 'concept of'. (T. Parsons, 2001, p. 528).

Therefore Church's intensional logic is opened to many new possibilities due to its richness. As we have seen, it not only allows the introduction of empty terms but also the duality of *de re* and *de dicto* readings. Both improvements if not directly developed by Church at least seriously suggested.[26]

1.2.7 Conclusion

We can define Church's intensional logic as an attempt to find stronger conditions for dealing with the identity of senses. The question that serves as a major impetus is *What is the criterion of identity for senses?* or *Under what circumstances do two formulas express the same sense?* It is easy to pinpoint the question but, in order to clarify the solution, we are going to consider some answers that mirror Church's attempt of solving this problem. Instead of talking about too abstract senses in the hierarchy of intensional types (what is really a concept of a concept of a concept of a proposition?), in the following points we will consider the more concrete question of what is the criterion of identity for propositions.

1. Two sentences express the same proposition when they are synonymous. But this is only a change of the name of the problem: what is synonymy? This proposal is thus too vague.

2. Two sentences express the same proposition when they are both true or both false, that is, when the two sentences are equivalent. This is simple but reduces intensional logic to extensional logic.

3. Two sentences express the same proposition when they are necessary equivalent. This is Alternative (2) and can be useful in modal contexts but not in contexts related to belief, assertion,...

4. Two sentences express the same proposition when there is an *intensional isomorphism* between then. This is a concept that Church takes from Carnap as the basis of Alternative (0) in (Church, 1951), but later, in (1954), rejected as too weak. The problem lies in the very notion of intensional isomorphism, which ends up confusing synonymy with logical equivalence.

[26]The first of them has been pointed out in section 1.2.6, and the second one is indicated in (Anderson, 1998, p. 155, footnote 63): "Church nowhere in print expresses sympathy with the idea that there might be such a thing of a 'de re' reading of a belief sentence or other sentence apparently involving obliquities. However, in lectures on open problems in intensional logic at U.C.L.A. in 1977 [recorded by Nathan Salmon] he says the idea is 'tenable' but leads to some surprises."

5. Two sentences express the same proposition when there is an *synonymous isomorphism* between then. In Church (1954) intensional isomorphism is replaced by synonymous isomorphism as the basis of Alternative (0) and Alternative (1). Church modified Carnap's criterion by requiring not that the expressions being replaced be necessarily equivalent but that the corresponding simple parts of the expressions be synonymous. But have we returned to the vagueness of point 1? What is synonymy? To solve the problem Church "supposes that it will be given as part of the semantical basis of the language which primitive expressions are stipulated to be synonymous with each other and with complex expressions present in the language." (Anderson, 1998, p. 158).

Summing up, from all the previous exposed possible answers, only two were taken seriously by Church in his three different alternatives for an intensional logic. One of them is the criterion based on logical equivalence which is natural when dealing with modal logic and is developed as Alternative (2). The other criterion refers to synonymous isomorphism which is suitable for the logics of belief and knowledge and is developed in a weaker form as Alternative (1) and in a stronger form as Alternative (0).

The analysis we have done in the present chapter has taken into consideration the main writings of Church into what he called "logic of sense and denotation" or what we have called "Church's intensional logic". The development of this logic cannot be considered fully Fregean—there are many aspects that are strange to a fully Fregean logic (Klement, 2002, pp. 117–124), but is largely inspired by Frege from the very beginning. After almost fifty years of reflection about the logic of sense and denotation, Church finished mixing his criterion of synonymous isomorphism with a ramified theory of types to fulfill a viable intensional logic system. But is it the only possible intensional system for a logic of senses? Although we are not going to enter in detail into the following issue, it must be said that Church (1973b) considered the viability of a second intensional system based on a Russellian intensional logic within a ramified type theory with the addition of a connective expressing propositional identity, called "four-line equality" (Anderson, 1998, pp. 167-168). This is only mentioned to appreciate how much time and effort Church dedicated to the almost never-ending project of consolidating a logic of sense and denotation.

1.3 Montague on Intensionality

1.3.1 Introduction

The work of Richard Montague is focused on giving a model theoretic semantics for natural language. If builds a bridge between two fields that were told apart given its incompatibility: the accuracy of formal languages and the ambiguity of natural language. Montague's enterprise was inspired by the following idea:

> There is in my opinion no important theoretical difference between natural languages and the artificial languages of logicians; indeed, I consider it possible to comprehend the syntax and semantics of both kinds of languages within a single natural and mathematically precise theory. On this point I differ from a number of philosophers, but agree, I believe, with Chomsky and his associates. (Montague, 1974c, p. 222).

Or more categorically: "I reject the contention hat an important theoretical difference exists between formal and natural languages." (Montague, 1974a, p. 188). An idea not very extended at a time where model theoretical semantics were diametrically opposed to natural language:

> It is here that Montague made his biggest contribution. To most logicians (like the first author) trained in model-theoretic semantics, natural language was an anathema, impossibly vague and incoherent. To us, the revolutionary idea in Montague's PTQ paper (and earlier papers) is the claim that natural language is not impossibly incoherent, as his teacher Tarski had led us to believe, but that large portions of its semantics can be treated by combining known tools from logic, tools like functions of finite type, the λ-calculus, generalized quantifiers, tense and modal logic, and all the rest. (Barwise & Cooper, 1981, p. 204).

Together with the model theoretical approach to natural language for constructing "an adequate and comprehensive semantical theory" (Montague, 1974c, p. 222), Montague put the basis of this theory in the Principle of Compositionality which reads: "the meaning of a compound expression is a function of the meanings of its parts and of the way they are syntactically combined." (Partee, 1984, p. 281).

This principle is the link between the syntax and the semantics of natural language and it implies that any syntactic element of a sentence must have a meaning, and that any syntactic rule needs a semantic rule as a counterpart in order to indicate how the meaning of the compound expression is obtained (Janssen, 2016).

Montague's semantics was developed on the basis of the Carnapian distinction between *extension* and *intension*, and made use of the *possible world* semantics of Kripke. In (Montague, 1974b), instead of introducing directly the semantics for fragments of natural language, Montague decided to present the syntax and semantics of the artificial language which is currently known as Montague's Intensional Logic. And then to give a way for translating the natural language into the artificial language:

> We could [...] introduce the semantics of our fragment directly; but it is probably more perspicuous to proceed indirectly, by (1) setting up a certain simple artificial language, that of tensed intensional logic, (2) giving the semantics of that language, and (3) interpreting English indirectly by showing in a rigorous way how to translate it into the artificial language. (Montague, 1974b, p. 256).

In this case the natural language expressions are interpreted indirectly into the semantics, since they have been translated first to the artificial language.[27].

Montague's Intensional Logic is characterized for being a typed intensional language, where functional types play a central role given that except for the expressions of the basic types, the vast majority have a functional type and, therefore, denote functions. The logic has also expressions with *lambda*. The λ operator allows the constructions of functions from other given expressions and is considered such an important tool "that Barbara Partee said: 'lambdas really changed my life'; in fact lambdas changed the lives of all semanticists." (Janssen, 2016). Montague's Intensional Logic has modal operators and its models include a set of possible worlds and, finally, it has also tense operators and models which include a structure of time.

Now we present Montague's Intensional Logic as exposed in his influential article *The Proper Treatment of Quantification in Ordinary English* (1974b), which is usually abbreviated as PTQ in the literature. We have based our presentation in (Dowty, Wall, & Peters, 1981, pp. 154–162) which has a different notation than PTQ. The usual symbols for quantifiers $\forall$ and $\exists$ are used, instead of $\bigwedge$ and $\bigvee$ respectively. Our symbols for variables are $x, y, \ldots$ instead of $u, v, \ldots$ The tense operators $\mathbf{F}$ and $\mathbf{P}$ read as "it will be the case that" and "it has been the case than", respectively, are the respective

[27]Montague also offered a way of giving a model theoretic semantics for natural language directly, without a previous translation to a logical language in (1974a)

substitutes of Montague's W and H. Furthermore, we use w for a possible world and t for a moment of time instead of Montague's i and j respectively (Dowty et al., 1981, p. 177, n. 3).

1.3.2 Syntax

Definition 1.3.1 (Type)**.** Let t, e and s be any fixed objects. Then the set of types (TYPES) is defined recursively as follows:

1. t is a type.
2. e is a type.
3. If a and b are any types, then $\langle a, b\rangle$ is a type.
4. If a is any type, then $\langle s, a\rangle$ is a type.

Note that s is not itself a type, but it is used to form functional types.

Definition 1.3.2 (Basic Expressions)**.** It is employed a denumerably infinite set of variables and non-logical constants:

1. For each natural number n and for each type a, a denumerably infinite set of *non-logical constants* $c_{n,a}$, $d_{n,a}, \ldots$ The set of all constants of type a is called CON_a.
2. For each natural number n and for each type a, a denumerably infinite set of *variables* $x_{n,a}, y_{n,a}, \ldots$ The set of all variables of type a is called VAR_a.

Definition 1.3.3 (Meaningful Expressions)**.** The set of meaningful expressions of type a, ME_a, is defined recursively as follows:

1. Every variable of type a is in ME_a.
2. Every constant of type a is in ME_a.
3. If $\alpha \in \mathsf{ME}_a$ and x is a variable of type b, then $\lambda x\alpha \in \mathsf{ME}_{\langle b,a\rangle}$.
4. If $\alpha \in \mathsf{ME}_{\langle a,b\rangle}$ and $\beta \in \mathsf{ME}_a$, then $\alpha(\beta) \in \mathsf{ME}_b$.
5. If α and β are both in ME_a, then $\alpha = \beta \in \mathsf{ME}_t$.
6. If $\phi \in \mathsf{ME}_t$, then $\neg\phi \in \mathsf{ME}_t$.
7. If $\phi, \psi \in \mathsf{ME}_t$, then $[\phi \vee \psi] \in \mathsf{ME}_t$.

8. If $\phi, \psi \in \mathsf{ME}_t$, then $[\phi \wedge \psi] \in \mathsf{ME}_t$.

9. If $\phi, \psi \in \mathsf{ME}_t$, then $[\phi \rightarrow \psi] \in \mathsf{ME}_t$.

10. If $\phi, \psi \in \mathsf{ME}_t$, then $[\phi \leftrightarrow \psi] \in \mathsf{ME}_t$.

11. If $\phi \in \mathsf{ME}_t$, and x is a variable of any type, then $\forall x \phi \in \mathsf{ME}_t$.

12. If $\phi \in \mathsf{ME}_t$, and x is a variable of any type, then $\exists x \phi \in \mathsf{ME}_t$.

13. If $\phi \in \mathsf{ME}_t$, then $\Box \phi \in \mathsf{ME}_t$.

14. If $\phi \in \mathsf{ME}_t$, then $\mathbf{F}\phi \in \mathsf{ME}_t$.

15. If $\phi \in \mathsf{ME}_t$, then $\mathbf{P}\phi \in \mathsf{ME}_t$.

16. If $\alpha \in \mathsf{ME}_a$, then $\hat{}\alpha \in \mathsf{ME}_{\langle s,a \rangle}$.

17. If $\alpha \in \mathsf{ME}_{\langle s,a \rangle}$, then $\check{}\alpha \in \mathsf{ME}_a$.

Between all the meaningful expressions, the last ones are perhaps the less familiar to the reader:

> The expression $[\hat{}\alpha]$ is regarded as denoting (or having as its *extension*) the *intension* of the expression α. The expression $[\check{}\alpha]$ is meaningful only if α is an expression that denotes an intension or sense; in such a case $[\check{}\alpha]$ denotes the corresponding extension. (Montague, 1974b, p. 257).

1.3.3 Semantics

Definition 1.3.4 (Possible Denotations)**.** The set of possible denotations of type a is defined as follows:

1. $D_e = A$

2. $D_t = \{0, 1\}$

3. $D_{\langle a,b \rangle} = D_b^{D_a}$

4. $D_{\langle s,a \rangle} = D_a^{(W \times T)}$

where a and b are any types, and 0 and 1 are the truth values falsehood and truth respectively.

Definition 1.3.5 (Senses of type a)**.** The set of senses of type a, S_a, is defined as $D_{\langle s,a \rangle}$.

Definition 1.3.6 (Intensional Model)**.** A model for Montague's Intensional Logic is the quintuple

$$\mathcal{M} = \langle A, W, T, <, \mathsf{F} \rangle$$

such that A, W and T are any non-empty sets, $<$ is a linear ordering on the set T, and F is an interpretation function whose domain is defined on the set of all non-logical constants of the language and whose value is in the set of senses of type a, $S_a = D_{\langle s,a \rangle}$.

where A is considered to be the set of individuals, W is the set of possible worlds and T the set of moments of time. With respect to A, the set of individuals, Montague explains:

> Or possible individuals. If there are individuals that are only possible but not actual, A is to contain them; but this is an issue on which it would be unethical for me as a logician (or linguist or grammarian or semanticist, for that matter) to take a stand. (1974b, p. 257, footnote 8).

Note also that the interpretation function F, which assigns to each non-logical constant of type a a member of the set of senses of type a, S_a, has *only intensional values*, since they are always elements of $D_{\langle s,a \rangle}$.

Definition 1.3.7 (Assignment)**.** An assignment function g is a function whose domain is defined on the set of all variables and which gives as value for each variable of type a a member of D_a.

Note that g assigns an extension to each variable. By contrast, F assigns an intension to each constant.

Definition 1.3.8 (Interpretation of Meaningful Expressions)**.** Let $\mathcal{M}$ be a model for Montague's Intensional Logic, w a possible world such that $w \in W$, t a moment of time such that $t \in T$ and g an assignment function, then given an expression α, we define recursively the extension of α with respect to model $\mathcal{M}$, possible world w, moment of time t and assignment function g, denoted by $[\![\alpha]\!]^{\mathcal{M},w,t,g}$, as follows:

1. If α is a constant, then $[\![\alpha]\!]^{\mathcal{M},w,t,g} = (\mathsf{F}(\alpha))(\langle w, t \rangle)$.
2. If α is a variable, then $[\![\alpha]\!]^{\mathcal{M},w,t,g} = g(\alpha)$.
3. If $\alpha \in \mathsf{ME}_a$ and x is a variable of type b, then $[\![\lambda x \alpha]\!]^{\mathcal{M},w,t,g} = h$ where $h : D_b \to D_a$ is the function defined by $h(\theta) = [\![\alpha]\!]^{\mathcal{M},w,t,g^{\theta}_{x}}$ for any $\theta \in D_b$.

4. If $\alpha \in \mathsf{ME}_{\langle a,b \rangle}$ and $\beta \in \mathsf{ME}_a$, then $[\![\alpha(\beta)]\!]^{\mathcal{M},w,t,g} = [\![\alpha]\!]^{\mathcal{M},w,t,g}([\![\beta]\!]^{\mathcal{M},w,t,g})$.

5. If α and β are both in ME_a, then $[\![\alpha = \beta]\!]^{\mathcal{M},w,t,g} = 1$ iff $[\![\alpha]\!]^{\mathcal{M},w,t,g} = [\![\beta]\!]^{\mathcal{M},w,t,g}$.

6. If $\phi \in \mathsf{ME}_t$, then $[\![\neg\phi]\!]^{\mathcal{M},w,t,g} = 1$ iff $[\![\phi]\!]^{\mathcal{M},w,t,g} = 0$, and $[\![\neg\phi]\!]^{\mathcal{M},w,t,g} = 0$ otherwise.

7. If $\phi, \psi \in \mathsf{ME}_t$, then $[\![[\phi \vee \psi]]\!]^{\mathcal{M},w,t,g} = 1$ iff $[\![\phi]\!]^{\mathcal{M},w,t,g} = 1$ or $[\![\psi]\!]^{\mathcal{M},w,t,g} = 1$.

8. If $\phi, \psi \in \mathsf{ME}_t$, then $[\![[\phi \wedge \psi]]\!]^{\mathcal{M},w,t,g} = 1$ iff $[\![\phi]\!]^{\mathcal{M},w,t,g} = 1$ and $[\![\psi]\!]^{\mathcal{M},w,t,g} = 1$.

9. If $\phi, \psi \in \mathsf{ME}_t$, then $[\![[\phi \to \psi]]\!]^{\mathcal{M},w,t,g} = 1$ iff $[\![\phi]\!]^{\mathcal{M},w,t,g} = 0$ or $[\![\psi]\!]^{\mathcal{M},w,t,g} = 1$.

10. If $\phi, \psi \in \mathsf{ME}_t$, then $[\![[\phi \leftrightarrow \psi]]\!]^{\mathcal{M},w,t,g} = 1$ iff $[\![\phi]\!]^{\mathcal{M},w,t,g} = [\![\psi]\!]^{\mathcal{M},w,t,g}$.

11. If $\phi \in \mathsf{ME}_t$, and x is a variable of any type, then $[\![\forall x\phi]\!]^{\mathcal{M},w,t,g} = 1$ iff $[\![\phi]\!]^{\mathcal{M},w,t,g'} = 1$ for all assignment functions g' exactly like g except possibly for the value assigned to x.

12. If $\phi \in \mathsf{ME}_t$, and x is a variable of any type, then $[\![\exists x\phi]\!]^{\mathcal{M},w,t,g} = 1$ iff $[\![\phi]\!]^{\mathcal{M},w,t,g'} = 1$ for some assignment function g' exactly like g except possibly for the value assigned to x.

13. If $\phi \in \mathsf{ME}_t$, then $[\![\Box\phi]\!]^{\mathcal{M},w,t,g} = 1$ iff $[\![\phi]\!]^{\mathcal{M},w',t',g} = 1$ for all w' in W and all t' in T.

14. If $\phi \in \mathsf{ME}_t$, then $[\![\mathbf{F}\phi]\!]^{\mathcal{M},w,t,g} = 1$ iff $[\![\phi]\!]^{\mathcal{M},w,t',g} = 1$ for some t' in T such that $t < t'$.

15. If $\phi \in \mathsf{ME}_t$, then $[\![\mathbf{P}\phi]\!]^{\mathcal{M},w,t,g} = 1$ iff $[\![\phi]\!]^{\mathcal{M},w,t',g} = 1$ for some t' in T such that $t' < t$.

16. If $\alpha \in \mathsf{ME}_a$, then $[\![\hat{}\alpha]\!]^{\mathcal{M},w,t,g} = h$, where $h : (W \times T) \to D_a$ such that for all $\langle w', t' \rangle$ in $W \times T$, $h(\langle w', t' \rangle) = [\![\alpha]\!]^{\mathcal{M},w',t',g}$.

17. If $\alpha \in \mathsf{ME}_{\langle s,a \rangle}$, then $[\![\check{}\alpha]\!]^{\mathcal{M},w,t,g} = [\![\alpha]\!]^{\mathcal{M},w,t,g}(\langle w, t \rangle)$.

Definition 1.3.9 (Truth of a formula with respect to $\mathcal{M}$ and to $\langle w, t \rangle$)**.** A formula ϕ, such that $\phi \in \mathsf{ME}_t$ is true with respect to $\mathcal{M}$ and to $\langle w, t \rangle$ if and only if $[\![\phi]\!]^{\mathcal{M},w,t,g} = 1$ for all assignment functions g.

Definition 1.3.10 (Intension of α with respect to $\mathcal{M}$ and to g)**.** If α is any expression, then the intension of α with respect to $\mathcal{M}$ and to g, denoted $[\![\alpha]\!]_{\mathfrak{c}}^{\mathcal{M},g}$, is that function h with domain $W \times T$ such that for all $\langle w, t\rangle$ in $W \times T$, $h(\langle w, t\rangle)$ is $[\![\alpha]\!]^{\mathcal{M},w,t,g}$.

1.3.4 Some Meaningful Expressions

Let us now introduce the types of some common expressions in Intensional Logic. We present Montague's notation in comparison with Church's notation, in order to see the similarities and differences.

Montague's notation	Church's notation	What is it?
t	o	T, F
e	ι	individuals
$\langle a, b\rangle$	$\alpha\beta$	functional types
$\langle s, a\rangle$	α_1	intensional types
$\langle s, e\rangle$	ι_1	individual concepts
$\langle e, t\rangle$	ιo	one-place predicates of individuals
$\langle s, \langle e, t\rangle\rangle$	$(\iota o)_1$ or $\iota_1 o_1$	properties of individuals/ concepts of predicates
$\langle s, t\rangle$	o_1	propositions
$\langle e, \langle e, t\rangle\rangle$	$\iota\iota o$	relations between individuals/ two-place predicates
$\langle\langle s, e\rangle, t\rangle$	$\iota_1 o$	sets of individual concepts
$\langle\langle s, t\rangle t\rangle$	$o_1 o$	sets of propositions
$\langle s, \langle\langle s, \langle e, t\rangle\rangle, t\rangle$	$((\iota o)_1 o)_1$	properties of properties of individuals
$\langle s, \langle e, \langle e, t\rangle\rangle\rangle$	$(\iota\iota o)_1$	relations-in-intension between individuals/ concepts of two places predicates

1.3.5 Conclusion

Montague's Intensional Logic has shown influential in logic and in the studies concerning natural language. As we have seen, Montague, unlike Church, has contributed to intensional logic with a semantics for the formal language. A language which, for Montague and also for Church, includes expressions denoting intensions.

With respect to the notions of *sense* and *intension*, it must be said that for Montague they are different notions. While all intensions are senses, not all senses have to be intensions. There can be senses that are not intensions of any expression of the language. “The set of senses of type a is simply the set of ‘possible intensions’ out of which the intensions of expressions of type a are to be chosen. Thus all intensions of expressions will be senses, but not necessarily all senses will be intensions of some expression or other.”(Dowty et al., 1981, p. 157).

The work of Montague was continued by Daniel Gallin. In (Gallin, 1975) an axiom system for Montague’s Intensional Logic is presented and it is established a relation between Montague’s Intensional Logic and a Two-Sorted Type Theory where variables over indices (such as $\langle w, t \rangle$) are included. For Gallin, “the cap operator $\hat{}$ acts as a functional abstractor over indices, although s itself is not a type and no variables ranging over indices are present in IL.” (Gallin, 1975, p. 13). Gallin proves also completeness for his axiomatic formulation with general models based on Henkin (1950). In Two-Sorted Type Theory (Gallin, 1975, pp. 58-63), $\hat{}\alpha$ is explicitly defined in terms of abstraction over indices.

Chapter 2

First-Order Intensional Hybrid Logic

2.1 Introduction

This chapter begins with an approach to first-order intensional logic whose intensional expressions are always interpreted extensionally at a given world of a certain model and goes further introducing intensional expressions which are interpreted as what they really are: intensions. It is important to note that, as we move in a modal framework, the interpretation of formulas is relativized to worlds and so, the classical notions of truth and validity applied to formulas, must be redefined as it can be seen in definitions 2.2.14, 2.2.16, 2.2.17 and 2.2.19. Hence, in all the logics presented in this chapter we differentiate—and the following concepts must be grasped with care—between:

1. Validity in a class of frames: when a formula is valid in every frame of the class.
2. Validity in a frame: when a formula is true in every model based on a given frame.
3. Validity: when a formula is true in every model.
4. Truth in a model: when a formula is true at every world of a model.
5. Truth at a world of a model.

When we talk about expressions which are interpreted extensionally or about extensionalized intensions, we are always referring to expressions which are interpreted, not in general, but at a given world of a model.

2.1.1 Background

We have not started from scratch, and the content of this chapter is indebted to Melvin Fitting, Richard Mendelsohn and Torben Braüner for their work on the field.[1] But the debt to those authors is not only implicit but explicit, mainly in the first sections. Here, we offer two approaches substantially based on the first-order modal logic of Fitting and Mendelsohn (1998) and on the first-order intensional hybrid logic of Braüner (2008). But what follows is not only a copy: it is a translation to a new symbolic language where a type notation has been introduced in order to differentiate extensional from intensional expressions. Despite we know that in first-order logic we dispense with type notation, we have decided to introduce it because type notation itself (written as subscripts in the expressions of the language) plays an important role in our research about intensional logic. As we are using types elsewhere in the logics presented through this dissertation, it is relevant to introduce them from the lowest level, that being so, we can have acquaintance with the symbols for a better understanding on how all the logics relate and change. The use of the same notation throughout all our study, at the beginning appears as an unnecessary complication, but at the end will result in a finer understanding on how intensions can be studied in different formal systems.

2.1.2 The Different Sections

In section 2.2, we present a first-order intensional logic, which is essentially the logic of Fitting and Mendelsohn (1998). This book provides a great insight into intensional logic: it is plenty of discussions about actualist and possibilist quantifiers, varying and constant domain semantics, *de re* and *de dicto* modal readings, scope disambiguation in negative statements (wide and narrow scope), definite descriptions, equality and identity, existence and designation, semantics with partial interpretation functions or *nil* entities when there are non-denoting terms. It has been one of the main influential resources in the present study. However, one of the drawbacks of the book is that, when you try to find what is the first-order logic they propose, you have to revisit practically all the pages of the book. It lacks of a comprehensive presentation of a preferable logic, which synthesizes all the previous discussions, in only a few pages at once. This is what we have ventured to do in section 2.2. The choice we have done takes actualist quantification within a varying domain semantics, uses predicate abstracts—in order to remove modal ambiguities *de dicto* and *de re* and scope ambiguities—and considers

[1] Cf. (Fitting & Mendelsohn, 1998), (Fitting, 2004) and (Braüner, 2011).

definite descriptions as intensional constants. It also deals with the equality relation only between extensional variables, differentiates between existence and designation, and prefers a partial interpretation function for evaluating formulas where non-designating terms appear. Some of the choices done here are not justified neither in this section, nor in this chapter. Others will be discussed in the following chapters. In this section we study with more depth the difference between denotation and existence, which is crucial for understanding how formulas are evaluated. Denotation is linked with the way the assignment function g and the partial interpretation function F proceed, and existence has a close relationship with the *actualist* interpretation of the quantifiers, which range over the domain of each possible world.

The purpose of section 2.3 is to offer an hybridized version of a first-order intensional logic, which is not the logic just presented in section 2.2 but a similar one. The exposition of this logic follows closely Braüner (2008), and little modifications have been made. Braüner (2008), more influenced by Fitting (2004) than for Fitting and Mendelsohn (1998), introduces intensional variables, which can be quantified over and act as arguments to predicates, but dispenses with intensional constants; he also adds the proper hybrid-logical machinery: nominals (the metavariables $a, b, c, \ldots$ range over nominals); two binders, $\forall$, which can also bind nominals, and $\downarrow$; and a satisfaction operator $@_a$, for each nominal a. We have taken Braüner's logic language and we have added our type notation as subscripts of the expressions. λ and predicate abstracts which had an essential role in section 2.2 do not appear at the beginning of this section, the $\downarrow$ binder is introduced instead, and it will play an important role as a disambiguating device. At the end of the section, we have compared the usage of $\downarrow$ and λ in contexts where scope ambiguity poses a problem, just as in negative sentences including non-denoting terms, and in contexts which have to deal with the ambiguity of modal readings *de dicto* and *de re*. We have compared $\downarrow$ and λ in order to see if both are necessary for having an intensional hybrid language with good expressiveness features, or if we can do away with one of them. If the comparison resulted in the acquisition of equivalent expressiveness power, to opt for one or another would be only a matter of preference.

In section 2.4, on the basis of the two accounts of sections 2.2 and 2.3, we have tried to explore new possible ways to improve a first-order intensional logic by means of an innovative interpretation of the type notation and through the use of hybrid machinery. The main point of this section is that we want to show that the hybrid machinery has not to be introduced *ad hoc* over a preexisting intensional logic (only for creating an unpublished logical system), but we try to explain how the intensional logic itself is claiming for some kind of mechanism which allows us to differentiate between predicating of an intension (intended as a function) and predicating of a "devalued" intension, i.e., of an intension which has been used to individuate a concrete entity at a certain world. A predication that should also be respectful with the types of the expressions and should not ignore the coherence of type notation.

Section 2.4 also explores the richness of using predicates which can be interpreted other than functions from worlds to sets—and whose types, as we will see, have the form $\langle\sigma\langle\iota o\rangle\rangle$ or $\langle\iota o\rangle$, both are different notations for the same type—as Fitting and Mendelsohn (1998), and Braüner (2008) do, and considers a new, but at the same time old, way of interpreting predicates, namely, that predicates can also be rigid. We are claiming for a logic having intensions but also extensions in its own right. Extensions do not have to reduce only to be the interpretation of any kind of object terms (extensional terms and extensionalized terms) at a given world, they can also be the interpretation of predicates. An intensional logic can have extensional and intensional variables, extensional and intensional constants and function symbols; but in the same way, an intensional logic can also have intensional and extensional predicates as well. In order to avoid the confusion between an extensional (or rigid) predicate and an intensional one, what it would be needed is a procedure which enables us to distinguish between them. The device we promote is a concrete one: a powerful type notation, the same we have been applying since the beginning of this chapter. Through our type notation, we can indicate when we are coping with an intensional predicate or an extensional one. With an extensional predicate we understand a predicate whose interpretation at every world of a model is unchanged, thus it is a *rigid* predicate, as those we can find between the predicates of mathematics. What we are proposing, in essence, is that we do not need a logic for mathematical statements (the classical first-order logic) and other logic for modal contexts, in principle it is possible to deal with sentences of mathematics and of modal contexts with the same logic without confusing predicates or other expressions. This is the reason behind our research in section 2.4. But what the reader is going to find in this section are not conclusive theses but only some attempts to arrive to a thesis.

Section 2.4 must be contemplated as a sort of laboratory where some experiments with our type notation have being made. We analyze the possible types we can count on, we study their plausibility and test the interpretation of our types in natural language, also in contexts where we have modal ambiguities *de dicto* and *de re*. We also see how the differentiation between extensional and intensional types is so closely related with modal operators, just because of the very nature of extensional and intensional expressions. An extensional statement, like a mathematical one, has in its background a necessity claim of being true at every world in a model, while an intensional statement has, in its background, the possibility of change at each world of a model. As a result, modal operators could be removed in extensional contexts, where mathematical (extensional) expressions are used, and left unchanged in intensional contexts. But given that modal operators qualify our intensional expressions: is it not possible to ask for coherence with regard to the type of the expressions? As we are differentiating between extensional and intensional expressions, we propose that modal operators do not have to qualify extensional expressions but only intensional ones, i.e., modal operators (of type $\langle o_1 o \rangle$) should not go with names of truth values (of type o) but only with names of propositions (of type o_1).

Finally, section 2.5 pretends to take the best of all the previous sections: the intensional machinery of Fitting and Mendelsohn, the hybridization mechanisms of Braüner's hybrid logic and the results of our own exploration in the field of intensional hybrid logic with a powerful type notation. In section 2.5, we develop a syntax and a semantics for a first-order intensional hybrid logic with some of the desired features we have pointed out, but without others, such as having extensional predicates, something we have left for our type theory in the next chapter. We have introduced, however, intensional predication, that is, the possibility of ascribing properties to intensions and not only to intensions extensionalized at a given world. We defend also the coherence of our type notation, specially the types of predicates, in comparison with the classical interpretation. And we also discuss some issues concerning how hybrid formulas can deal with denotation and existence claims. As intensional predication and existential claims are one of the main points of our work, we finish with them the section and the chapter.

As a conclusive remark and before beginning with the other sections, it has to be noted that we have considered object terms, rigid terms and extensional expressions, to be all of extensional type, while non-rigid terms and intensional expressions are of intensional type. To be extensional and to be rigid are then, from our point view, practically synonymous. The only difference is that when we are talking about extensional, we are only referring to the type of the expression, and when we mention rigid, we rather refer to the procedure for arriving to the extensional type, and therefore an expression can be rigid if it is extensional by itself or if it has been extensionalized, through the use of a satisfaction operator, for example.

2.2 A First Approach to First-Order Intensional Logic

This approach is essentially based on Fitting and Mendelsohn's book on *First-Order Modal Logic* (1998). We have done some modifications, though: they do not use any type notation and we have introduced one. Remember that, although it is not customary to introduce types in first-order logic, we have chosen to use types to acquaint ourselves with them, as it will make the comparison between logics more unchallenging. In this section, the notation is mainly used to indicate explicitly what are the types of the expressions, but they are rarely used as subscripts. As we do not have constants of different types, neither variables, nor functors or predicates, we have left them "unsubscripted", as is usual in first-order logic. Functors and predicates are always intensional and they only can vary its n-ariness. In essence, and despite certain changes in the symbols used, the logic of Fitting and Mendelsohn (1998) and the logic presented in this section, can be considered the same.

Concerning our type notation, we have done a mixed use of Montague and Church's notation, although we tend to favor, surely because of the greater familiarity, Church's notation. Note that we presuppose a distinction between types and signature. A *signature* is

> a set of individual constants, predicate symbols and function symbols; each of the predicate symbols and function symbols has an *arity* (for example it is binary if its arity is 2). Each signature K gives rise to a first-order language, by building up formulas from the symbols in the signature together with logical symbols (including =) and punctuation. (Hodges & Scanlon, 2013).

Types can be basic types (as ι and o), or complex functional types: the type of functions of one type to another. Our signature corresponds with our type notation. The elements of a signature have also a *type*, indicated by a *type symbol* as a subscript, which can be the type of individuals or a functional type. The type of a predicate of a signature is determined, for example, by the number (*arity*) and type of its arguments. Any expression of the language is of a type which is indicated by a type symbol. And, in our case, types can be extensional or intensional, so our notation has also extensional and intensional type symbols.

In his type theory, Montague uses the symbols t, e and s, for *truth values*, *entities* and *senses* respectively. We will also make use of a triple division of the basic symbols, but ours are a little bit different. Following Church's notation: the symbol t will be replaced by the greek letter *omicron*, o; the symbol e, by the greek letter *iota*, ι; and s, which does not appear in Church, by σ; in order to have greek letters altogether. The letter σ by itself is not a type and it is used for giving us the type $\langle\sigma\alpha\rangle$, which is the type of expressions naming the intension (or the *sense*) of an expression of type α. If we wanted to follow strictly Church's notation, the type of the sense of an expression of type α should be given by α_1 and, in fact, in many cases, we use α_1 as a simplification of the notation $\langle\sigma\alpha\rangle$, for any type α.

In this logic we consider that constants are intensional, not only those that stand for definite descriptions, such as "the present king of Spain", but also for proper names, such as "Aureliano Buendía", whose designation may vary; function symbols are also intensional —they can, for its part, to have intensional terms as arguments; and, finally, predicates are intensional too — they are interpreted as properties of entities that vary from world to world. On the other hand, variables are interpreted extensionally and we do not have intensional variables. The semantics supposes varying domains for the interpretation of quantifiers, and we have a partial interpretation function for the interpretation of constants of the language.

2.2.1 Types

The types we consider are functional types following the tradition of Church and Montague, which differentiates from Fitting's relational notation exposed in his book *Types, Tableaus and Gödel's God* (2002).[2]

Definition 2.2.1 (Types)**.** The set TYPES of types of first-order intensional logic is defined as follows:

- ι, o are basic types;
- $\langle\sigma\iota\rangle$ is a type;
- $\langle\sigma\langle(t^1\dots t^n)\iota\rangle\rangle$ and $\langle\sigma\langle(t^1\dots t^n)o\rangle\rangle$ are types, where each type $t^1\dots t^n$ can be either of type ι or of type $\langle\sigma\iota\rangle$.

In order to clarify the readability of types with σ, we are going to abbreviate, following Church's notation, the functional types with σ using a subscript 1: $\langle\sigma\alpha\rangle$ can then be expressed as α_1, where α can be ι, $\langle(t^1\dots t^n)\iota\rangle$ or $\langle(t^1\dots t^n)o\rangle$, where each type $t^1\dots t^n$ can be either of type ι or of type $\langle\sigma\iota\rangle$. For example, functional type $\langle\sigma\iota\rangle$ can be written as ι_1 and $\langle\sigma(\iota o)\rangle$ as $\langle\iota o\rangle_1$.

[2] For a clarifying exposition about functional and relational types see (Manzano, 1996, pp. 186–214).

Example 2.2.2. As said before, one of the basic types is the type ι, which is the type of individuals. Type $\langle \iota o \rangle$ is the type of functions from individuals to truth values. This type is, therefore, the type of ordinary monadic predicates, which can be interpreted as characteristic functions from the set of individuals.

If we add intensional machinery to our previous types, we would have new types where σ plays a new role. The intensional type $\langle \sigma\iota \rangle$, would be the type of functions from worlds to individuals, and the type $\langle \sigma(\iota o) \rangle$, would be the type of functions from worlds to properties or to characteristic functions from individuals. Expressions typed with any of these subscripts will be considered intensional and, therefore, world dependent.

2.2.2 Syntax

In order to have and differentiate new intensional expressions from the extensional (classical) ones, we are going to indicate as a subscript the type of the expression:

1. **Individual Variables:** $x, y, z, \ldots$ They all are considered to be extensional, x_ι.
2. **Constant symbols:** $c, d, \ldots$ They are interpreted non-rigidly, being of type $\langle \sigma\iota \rangle$, and can stand for *definite descriptions* or *proper names.*
3. **Function symbols:** $f, h, \ldots$ They can be n-ary and the type $t^1 \ldots t^n$ of each one of their arguments can be the type ι or the type $\langle \sigma\iota \rangle$. Function symbols are considered to be intensional, their type being $\langle \sigma \langle (t^1 \ldots t^n)\iota \rangle \rangle$.
4. **Predicate symbols:** $P, Q, \ldots$ They can also be n-ary and we are going to allow only variables to follow the predicate symbol in order to avoid ambiguities with intensional terms and modal operators (i.e., problems concerning mainly *de re* and *de dicto* readings). Predicate symbols are considered to be intensional. Their type is

$$\langle \sigma \langle (\overbrace{\iota \ldots \iota}^{n}) o \rangle \rangle$$

Definition 2.2.3 (Term)**.** The terms of the language are specified by the following rules:

1. Every variable is a term of type ι.
2. Every constant symbol is a term of intensional type $\langle \sigma\iota \rangle$.

3. $f_{\langle\sigma\langle(t^1\ldots t^n)\iota\rangle\rangle}(\tau^1_{t^1},\ldots,\tau^n_{t^n})$ is a term of intensional type, where f is an n-place function symbol, and $\tau^1,\ldots,\tau^n$ are terms, and $t^i \in \{\iota,\iota_1\}$ for each t^i and $1 \leq i \leq n$.

Definition 2.2.4 (Predicates)**.** The predicates of the language can be:

1. Predicate letters of arity n: $P, Q, R, \ldots$ Their type is: $\langle\sigma\langle(\overbrace{\iota\ldots\iota}^{n})o\rangle\rangle$.

2. Predicate abstracts of arity n: $\langle\lambda x^1_\iota \ldots x^n_\iota.\phi\rangle$, where $x^1_\iota \ldots x^n_\iota$ are variables and ϕ is a formula. Note that in the definition of predicate abstracts we presuppose the definition of formulas of below. The purpose of predicate abstracts is that we can obtain predicates from formulas: given a certain formula, a predicate can be *abstracted* from it. Their type is: $\langle\sigma\langle(\overbrace{\iota\ldots\iota}^{n})o\rangle\rangle$.

Definition 2.2.5 (Formula)**.** Any sequence of symbols of the language is a formula. A well formed formula (wff) is a sequence of symbols of the language which follows the following rules:

1. $P_{\langle\sigma\langle(\underbrace{\iota\ldots\iota}_{n})o\rangle\rangle}(x^1_\iota,\ldots,x^n_\iota)$ is an atomic formula. Note that the predicate letter is only followed for variables and not, as usual, for any kind of terms.

2. $x_\iota = y_\iota$ is an atomic formula.

3. If ϕ is a formula, then $\neg\phi$ is a formula.

4. If ϕ and ψ are formulas, then $\phi \wedge \psi$ is a formula.

5. If ϕ and ψ are formulas, then $\phi \vee \psi$ is a formula.

6. If ϕ and ψ are formulas, then $\phi \rightarrow \psi$ is a formula.

7. If ϕ and ψ are formulas, then $\phi \leftrightarrow \psi$ is a formula.

8. If ϕ is a formula, then $\Box\phi$ is a formula.

9. If ϕ is a formula, then $\Diamond\phi$ is a formula.

10. If ϕ is a formula and x a variable, then $\forall x\phi$ is a formula.

11. If ϕ is a formula and x a variable, then $\exists x\phi$ is a formula.

12. If $\langle\lambda x^1_\iota \ldots x^n_\iota.\phi\rangle$ is a predicate abstract and τ is a term, then $\langle\lambda x^1_\iota \ldots x^n_\iota.\phi\rangle(\tau^1 \ldots \tau^n)$ is a formula.

Definition 2.2.6 (Free variable occurrence)**.** The set of free variable occurrence of a given formula ϕ, $\mathsf{FREE}(\phi)$, is determined by the following rules:

1. Every occurrence of a variable in an atomic formula is a free occurrence.

2. The free variable occurrences of $\neg\phi$ are those of ϕ.

3. The free variable occurrences of $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \rightarrow \psi$, $\phi \leftrightarrow \psi$, are those of ϕ together with those of ψ.

4. The free variable occurrences of $\Box\phi$ and $\Diamond\phi$ are those of ϕ.

5. The free variable occurrences of $\forall x\phi$ and $\exists x\phi$ are those of ϕ, except for occurrences of x.

6. The free variable occurrences of a predicate abstract $\langle\lambda x_\iota^1 \ldots x_\iota^n.\phi\rangle$ are those of ϕ except for occurrences of $x_\iota^1 \ldots x_\iota^n$. The free variable occurrences of $\langle\lambda x_\iota^1 \ldots x_\iota^n.\phi\rangle(\tau^1 \ldots \tau^n)$ are those of the predicate abstract together with all variable occurrences in $\tau^1 \ldots \tau^n$.

Definition 2.2.7 (Sentence)**.** A sentence or a closed formula is a formula with no free variable occurrences.

2.2.3 Semantics

In the semantics we assume varying domain models, where quantifier domains can vary from world to world. We distinguish between the domain of the model D_ι and a domain function δ mapping members of W to non-empty sets. D_ι is the set $\bigcup_{w\in W} \delta(w)$. We refer to $\delta(w)$ as the domain of the world w, and think of it as the set of things that exist at that world. We will use then *actualist quantification* because quantifiers range over the domain of each particular world, where the *actually* existents of each world are. Think of D_ι as the set of things we can talk about at each world w in a significant way.

Given that models are based on frames we begin defining a frame:

Definition 2.2.8 (Frame)**.** A frame $\mathcal{F} = \langle W, R\rangle$, consists of a non-empty set, W, whose members are generally called *possible worlds*, and a binary relation, R, on W, generally called the *accessibility relation*.

w_1 and w_2 are elements of W denoting possible worlds. If w_1 and w_2 are in the relation R, we would write $w_1 R w_2$ or $\langle w_1, w_2\rangle \in R$, and read this as w_2 *is accessible from* w_1.

Definition 2.2.9 (Varying Domain Model)**.**

$$\mathcal{M} = \langle W, R, D_\iota, \langle \delta(w) \rangle_{w \in W}, \mathsf{F} \rangle$$

where

1. $W \neq \emptyset$;

2. R is a binary relation on W;

3. $D_\iota \neq \emptyset$

4. $\langle \delta(w) \rangle_{w \in W}$ is a family of domain functions such that for each world w, $\delta(w) \subseteq D_\iota$. $\delta(w)$ is called the domain of quantification at the world w.

5. F is a partial interpretation function which assigns:

    (a) To each n-place predicate symbol P and to each possible world $w \in W$ a characteristic function of n-tuples on D_ι:

    $$\mathsf{F}(P)(w) \in \{T, F\}^{(D_\iota)^n}$$

    (b) To each constant c of the language (which are always of type $\langle \sigma \iota \rangle$), and to some (possibly no) member $w \in W$, a member of the domain D_ι. For a given constant c, the function F can be defined at a world w or not, therefore we have two possibilities:

    i. If $\mathsf{F}(c)(w)$ is defined, then $\mathsf{F}(c)(w) \in D_\iota$.

    ii. It can be the case that $\mathsf{F}(c)(w)$ is not defined, in this situation we have no value for the function.

    (c) To each n-place function symbol f of the language, and to each $w \in W$, some n-ary function such that $\mathsf{F}(f)(w)$ is a partial function from $D_{t^1} \times \cdots \times D_{t^n}$ into D_ι, where $t^i \in \{\iota, \langle \sigma \iota \rangle\}$ for each t^i and $1 \leq i \leq n$.

Remark. Note that F always give value to predicate symbols. However, F does not always give value to constants. $\mathsf{F}(f)(w)$, moreover, is a partial function.

Definition 2.2.10 (Assignment)**.** An assignment g is a function that to each variable x, which are always of extensional type, i.e., of type ι, assigns an element of D_ι:

$$g(x) \in D_\iota$$

Definition 2.2.11 (Variant)**.** An x-variant assignment g' of g is an assignment having the same values that the original assignment g except, possibly, for x. That is, $g(y) = g'(y)$ for all y, such that $y \neq x$, and $g'(x) \in D_\iota$.

Definition 2.2.12 (Variant at w)**.** g' is an x-variant assignment at w of g when g' is an x-variant of g and

$$g'(x) \in \delta(w)$$

Definition 2.2.13 (Interpretation of terms in a world w of a model $\mathcal{M}$ using assignment g)**.** The interpretation of every term, τ, of the language is given considering an assignment g, an interpretation function F, and a given world w, such that:

1. If x is a free variable, x designates at w and

$$[\![x]\!]^{\mathcal{M},g,w} = g(x)$$

2. If c is a constant symbol of type $\langle\sigma\iota\rangle$, then:

   (a) if $\mathsf{F}(c)(w)$ is defined, then c designates at w and $[\![c]\!]^{\mathcal{M},g,w} = \mathsf{F}(c)(w)$;

   (b) if $\mathsf{F}(c)(w)$ is not defined, then c does *not* designate at w.

3. If f is an n-place function symbol, and each term $\tau^1, \ldots, \tau^n$ designates at w, and $\langle[\![\tau^1]\!]^{\mathcal{M},g,w}, \ldots, [\![\tau^n]\!]^{\mathcal{M},g,w}\rangle$ is in the domain of the function $\mathsf{F}(f)(w)$, then $f(\tau^1, \ldots, \tau^n)$ designates at w, and

$$[\![f(\tau^1, \ldots, \tau^n)]\!]^{\mathcal{M},g,w} = (\mathsf{F}(f)(w))([\![\tau^1]\!]^{\mathcal{M},g,w}, \ldots, [\![\tau^n]\!]^{\mathcal{M},g,w})$$

   Although difficult to read, the meaning of the previous expression is not difficult to understand. $(\mathsf{F}(f)(w))([\![\tau^1]\!]^{\mathcal{M},g,w}, \ldots, [\![\tau^n]\!]^{\mathcal{M},g,w})$ says that we apply the designation of the function symbol f at w to the designation of every term $\tau^1, \ldots, \tau^n$ at w. So, first we *extensionalize* (identify the designation on D_ι) every term being as arguments of the function symbol, and then apply to them the function which is the designation of the function symbol at w.

Remark. Variables always designate. Constants are said to designate at a world when the value $\mathsf{F}(c)(w)$ is defined, but $\mathsf{F}(c)(w)$ needs not to be a member of the local domain $\delta(w)$. Since $\mathsf{F}(f)(w)$ is a partial function, for a complex term as $f(\tau^1, \ldots, \tau^n)$, designation at w guarantees that all terms $\tau^1, \ldots, \tau^n$ designate plus the partial function $\mathsf{F}(f)(w)$ being defined for these arguments.

Definition 2.2.14 (Truth of a formula in a world w of a model $\mathcal{M}$ under assignment g)**.** Let $\mathcal{M} = \langle W, R, D_\iota, \langle \delta(w) \rangle_{w \in W}, \mathsf{F} \rangle$ be a varying domain first-order modal model. Now we can define the truth of a formula at a world w in model $\mathcal{M}$ using assignment g:

1. $[\![P(x^1, \ldots, x^n)]\!]^{\mathcal{M},g,w} = T$ iff $(\mathsf{F}(P)(w))(g(x^1), \ldots, g(x^n)) = T$
2. $[\![x = y]\!]^{\mathcal{M},g,w} = T$ iff $g(x) = g(y)$
3. $[\![\neg\phi]\!]^{\mathcal{M},g,w} = T$ iff $[\![\phi]\!]^{\mathcal{M},g,w} = F$
4. $[\![\phi \wedge \psi]\!]^{\mathcal{M},g,w} = T$ iff $[\![\phi]\!]^{\mathcal{M},g,w} = T$ and $[\![\psi]\!]^{\mathcal{M},g,w} = T$
5. $[\![\phi \vee \psi]\!]^{\mathcal{M},g,w} = T$ iff $[\![\phi]\!]^{\mathcal{M},g,w} = T$ or $[\![\psi]\!]^{\mathcal{M},g,w} = T$
6. $[\![\phi \rightarrow \psi]\!]^{\mathcal{M},g,w} = T$ iff $[\![\phi]\!]^{\mathcal{M},g,w} = F$ or $[\![\psi]\!]^{\mathcal{M},g,w} = T$
7. $[\![\phi \leftrightarrow \psi]\!]^{\mathcal{M},g,w} = T$ iff $[\![\phi]\!]^{\mathcal{M},g,w} = [\![\psi]\!]^{\mathcal{M},g,w}$
8. $[\![\Box\phi]\!]^{\mathcal{M},g,w} = T$ iff $[\![\phi]\!]^{\mathcal{M},g,w'} = T$ for all $w' \in W$ such that wRw'
9. $[\![\Diamond\phi]\!]^{\mathcal{M},g,w} = T$ iff $[\![\phi]\!]^{\mathcal{M},g,w'} = T$ for some $w' \in W$ and wRw'
10. $[\![\forall x\phi]\!]^{\mathcal{M},g,w} = T$ iff $[\![\phi]\!]^{\mathcal{M},g',w} = T$, for all x-variant g' of g at w
11. $[\![\exists x\phi]\!]^{\mathcal{M},g,w} = T$ iff $[\![\phi]\!]^{\mathcal{M},g',w} = T$, for some x-variant g' of g at w
12. And finally we present the evaluation of formulas with predicate abstracts: $[\![\langle \lambda x^1 \ldots x^n.\phi\rangle(\tau^1 \ldots \tau^n)]\!]^{\mathcal{M},g,w}$. For this evaluation, we consider two cases depending on wether or not all the terms designate at w:

    (a) If each $\tau^1 \ldots \tau^n$ designates at w:

    $$[\![\langle \lambda x^1 \ldots x^n.\phi\rangle(\tau^1 \ldots \tau^n)]\!]^{\mathcal{M},g,w} = T \text{ iff } [\![\phi]\!]^{\mathcal{M},g',w} = T$$

    where g' is the $x^1 \ldots x^n$-variant of g, such that $g'(x^1) = [\![\tau^1]\!]^{\mathcal{M},g,w}$, ..., $g'(x^n) = [\![\tau^n]\!]^{\mathcal{M},g,w}$.

    (b) If any of the $\tau^1 \ldots \tau^n$ does not designate at w:

    $$[\![\langle \lambda x^1 \ldots x^n.\phi\rangle(\tau^1 \ldots \tau^n)]\!]^{\mathcal{M},g,w} = F$$

Comment 2.2.15. The previous definition of evaluation of formulas makes false predicate abstract formulas with non-denoting terms without introducing a third truth value into the logic. It amounts to having two truth values (true and false) and rejecting as false the formulas which tend to apply a predicate (positive or negative) to a non-designating term. Designation is therefore crucial in our model because it determines the boundary where the meaningfulness of expressions is assessed. Note that we are here dealing with two problematic predicates: *existence* and *designation*. This distinction is on the basis of the possibility of speaking with terms denoting an entity in the domain of a model, but which does not exist at the world of evaluation. For instance, although my father is dead and he does not exist currently, I can speak about him because *my father* has a denotation at the current world despite not having actual existence.

Definition 2.2.16 (Truth in a model)**.** A formula ϕ is considered to be true in a model $\mathcal{M}$ if it is true at every world of the model under assignment g.

Definition 2.2.17 (Validity)**.** A formula ϕ is valid if it is true in every model $\mathcal{M}$.

Before introducing the definitions of validity in a frame and validity in a class of frames, let us classify frames depending on the properties that the accessibility relation holds:

Definition 2.2.18. Let $\langle W, R \rangle$ be a frame. We say it is:

- *reflexive*: if $w_1 R w_1$, for every $w_1 \in W$;
- *symmetric*: if $w_1 R w_2$ implies $w_2 R w_1$, for all $w_1, w_2 \in W$;
- *transitive*: if $w_1 R w_2$ and $w_2 R w_3$ imply $w_1 R w_3$, for all $w_1, w_2, w_3 \in W$;
- *serial*: if, for each $w_1 \in W$, there is some $w_2 \in W$ such that $w_1 R w_2$.

Depending on the properties a class of frames has, different modal logics can be characterized as follows:

- modal logic **K**, characterizes by not imposing any condition on the frame;
- **D**: serial;
- **T**: reflexive;
- **B**: reflexive and symmetric;
- **K4**: transitive;
- **S4**: reflexive and transitive;

- **S5**: reflexive, symmetric and transitive.

Definition 2.2.19 (Valid in a frame and **L**-valid)**.** A formula ϕ is *valid in a frame* if it is valid in every model based on that frame. And, if **L** is a collection of frames, ϕ is **L**-valid if ϕ is valid in every frame in **L**.

What characterizes the different modal logics are thus the **L**-valid formulas, for particular classes **L** of frames. The logic **S5** is then characterized by the class of frames being reflexive, symmetric and transitive. When nothing is said on the contrary, we prefer moving within **S5**, which seems more appropriate for dealing with metaphysical necessity.

2.2.4 Existence and Designation

One of the most important ideas we have found in Fitting and Mendelsohn (1998) is the introduction of a formal definition for the predicates "existence" and "denotation". These predicates have sometimes been considered either as non proper predicates or as metalinguistic predicates. In classical logic variables are assumed to designate existent objects, therefore if variables always *designate* and always designate *existent* objects, it would be redundant to indicate something which is always presupposed. In our varying domain first-order model these predicates are neither redundant nor equivalent.

The logical language introduced in this section provides us some mechanisms for defining the predicates of "existence" and "designation". Predicate abstracts will help us to do this. We must remember, however, that our predicates of "existence" and "designation" always refer to a given world of a model. Below are the definitions of both predicates:

1. Existence:
$$\mathbf{E}(\tau) ::= \langle \lambda x.\exists y(y = x)\rangle(\tau)$$

2. Designation:
$$\mathbf{D}(\tau) ::= \langle \lambda x.x = x\rangle(\tau)$$

Comment 2.2.20. The conditions for these previous formulas to be true at w are the following:

1. For the formula $\mathbf{E}(\tau)$ to be true at a world w of a model $\mathcal{M}$ under assignment g, τ needs to designate at w and this designation should belong to $\delta(w)$. We will prove it in Proposition 2.2.22.

2. For the formula $\mathbf{D}(\tau)$ to be true at w of a model $\mathcal{M}$ under assignment g, the only requirement is that τ designates at w. We will prove it in Proposition 2.2.23.

The consideration on the one side, of *existence* as a predicate, and on the other side, the difference of scope in negation statements (as indicated in Russell and Whitehead (1925)), allows us to distinguish between saying of an individual that it has the negative property of *non-existence* and negating that an individual has the property of existence:

1. Non-existence (negative property):

$$\overline{\mathbf{E}}(\tau) ::= \langle \lambda x. \neg \exists y (y = x) \rangle (\tau)$$

2. Non existence (negation of existence):

$$\neg \mathbf{E}(\tau) ::= \neg \langle \lambda x. \exists y (y = x) \rangle (\tau)$$

Comment 2.2.21. The requirements for the previous formulas to be true at w are:

1. Formula $\overline{\mathbf{E}}(\tau)$ is true at world w of a model $\mathcal{M}$ under assignment g, when τ designates at w but this designation is not in $\delta(w)$. We prove this in Proposition 2.2.24.

2. Formula $\neg \mathbf{E}(\tau)$ is true at world w of a model $\mathcal{M}$ under assignment g, when

   (a) either τ does not designate at w;

   (b) or τ designates at w but $[\![\tau]\!]^{\mathcal{M},g,w} \notin \delta(w)$.

   We will prove this in Proposition 2.2.25.

Remark. Note that in the definitions of the predicates of "existence" and "designation", the equality relation symbol appears in the predicate abstract. Although not stated explicitly previously, we would like to underline that the interpretation of the equality symbol at each possible world is the identity relation. Therefore all models are normal.

Proposition 2.2.22. *Let $\mathcal{M} = \langle W, R, D_\iota, \langle \delta(w) \rangle_{w \in W}, \mathsf{F} \rangle$ be a normal, varying domain, first-order modal model, let g an assignment, and w a world in W. For any term τ: $[\![\mathbf{E}(\tau)]\!]^{\mathcal{M},g,w} = T$ if and only if τ designates at w and $[\![\tau]\!]^{\mathcal{M},g,w} \in \delta(w)$.*

Proof. $[\![\mathbf{E}(\tau)]\!]^{\mathcal{M},g,w} = T$,

iff $[\![\langle \lambda x. \exists y (y = x) \rangle (\tau)]\!]^{\mathcal{M},g,w} = T$

iff τ designates at w and $[\![\exists y(y = x)]\!]^{\mathcal{M},g',w} = T,$ where g' is an x-variant of g such that $g'(x) = [\![\tau]\!]^{\mathcal{M},g,w}$

iff τ designates at w and $[\![y = x]\!]^{\mathcal{M},g'',w} = T$ for some y-variant g'' of g' at w (g' is an x-variant of g such that $g'(x) = [\![\tau]\!]^{\mathcal{M},g,w}$)

iff τ designates at w and $[\![y]\!]^{\mathcal{M},g'',w} = [\![x]\!]^{\mathcal{M},g'',w}$,
(g'' being an y-variant of g' at w,
and g' an x-variant of g with $g'(x) = [\![\tau]\!]^{\mathcal{M},g,w}$),

iff τ designates at w and $g''(y) = g''(x)$,
$g''(y) \in \delta(w)$ while $g''(x) = g'(x) = [\![\tau]\!]^{\mathcal{M},g,w}$;
(g'' being a y-variant of g' at w
and g' an x-variant of g, such that $g'(x) = [\![\tau]\!]^{\mathcal{M},g,w}$)

iff τ designates at w and $[\![\tau]\!]^{\mathcal{M},g,w} \in \delta(w)$.

□

Proposition 2.2.23. *Let $\mathcal{M} = \langle W, R, D_\iota, \langle\delta(w)\rangle_{w\in W}, \mathsf{F}\rangle$ be a normal, varying domain, first-order modal model, let g an assignment, and w a world in W. Then for any term τ: $[\![\mathbf{D}(\tau)]\!]^{\mathcal{M},g,w} = T$ if and only if τ designates at w in $\mathcal{M}$ with respect to g.*

Proof. $[\![\mathbf{D}(\tau)]\!]^{\mathcal{M},g,w} = T$

iff $[\![\langle\lambda x.x = x\rangle(\tau)]\!]^{\mathcal{M},g,w} = T$

iff τ designates at w and $[\![x = x]\!]^{\mathcal{M},g',w} = T$
(g' is an x-variant of g such that $g'(x) = [\![\tau]\!]^{\mathcal{M},g,w}$),

iff τ designates at w and $[\![x]\!]^{\mathcal{M},g',w} = [\![x]\!]^{\mathcal{M},g',w}$,
(g' is an x-variant of g such that $g'(x) = [\![\tau]\!]^{\mathcal{M},g,w}$),

iff τ designates at w.

□

Proposition 2.2.24. *Let $\mathcal{M} = \langle W, R, D_\iota, \langle\delta(w)\rangle_{w\in W}, \mathsf{F}\rangle$ be a normal, varying domain, first-order modal model, let g an assignment, and w a world in W. Then for any term τ: $[\![\overline{\mathbf{E}}(\tau)]\!]^{\mathcal{M},g,w} = T$ if and only if τ designates at w and $[\![\tau]\!]^{\mathcal{M},g,w} \notin \delta(w)$.*

Proof. $[\![\overline{\mathbf{E}}(\tau)]\!]^{\mathcal{M},g,w} = T$,

iff $[\![\langle\lambda x.\neg\exists y(y = x)\rangle(\tau)]\!]^{\mathcal{M},g,w} = T$

iff τ designates at w and $[\![\neg\exists y(y = x)]\!]^{\mathcal{M},g',w} = T$, where g' is an x-variant of g such that $g'(x) = [\![\tau]\!]^{\mathcal{M},g,w}$

iff τ designates at w and $[\![y = x]\!]^{\mathcal{M},g'',w} = F$ for all y-variants g'' of g' at w (g' is an x-variant of g such that $g'(x) = [\![\tau]\!]^{\mathcal{M},g,w}$)

iff τ designates at w and $[\![y]\!]^{\mathcal{M},g'',w} \neq [\![x]\!]^{\mathcal{M},g'',w}$, ($g''$ being any y-variant of g' at w, and g' an x-variant of g with $g'(x) = [\![\tau]\!]^{\mathcal{M},g,w}$),

iff τ designates at w and $g''(y) \neq g''(x)$, $g''(y) \in \delta(w)$ while $g''(x) = g'(x) = [\![\tau]\!]^{\mathcal{M},g,w}$; ($g''$ being any y-variant of g' at w and g' an x-variant of g, such that $g'(x) = [\![\tau]\!]^{\mathcal{M},g,w}$)

iff τ designates at w and $[\![\tau]\!]^{\mathcal{M},g,w} \notin \delta(w)$.

□

Proposition 2.2.25. *Let $\mathcal{M} = \langle W, R, D_\iota, \langle\delta(w)\rangle_{w\in W}, \mathsf{F}\rangle$ be a normal, varying domain, first-order modal model, let g an assignment, and w a world in W. Then for any term τ: $[\![\neg\mathbf{E}(\tau)]\!]^{\mathcal{M},g,w} = T$ if and only if:*

1. *τ does not designate at w; or*
2. *τ designates at w and $[\![\tau]\!]^{\mathcal{M},g,w} \notin \delta(w)$.*

Proof. $[\![\neg\mathbf{E}(\tau)]\!]^{\mathcal{M},g,w} = T$

iff $[\![\neg\langle\lambda x.\exists y(y = x)\rangle(\tau)]\!]^{\mathcal{M},g,w} = T$

iff $[\![\langle\lambda x.\exists y(y = x)\rangle(\tau)]\!]^{\mathcal{M},g,w} = F$

iff *1.* τ does not designate at w, or

2. τ designates at w but $[\![\langle\lambda x.\exists y(y = x)\rangle(\tau)]\!]^{\mathcal{M},g,w} = F$

iff τ designates at w but $[\![\exists y(y = x)]\!]^{\mathcal{M},g',w} = F$ where g' is an x-variant of g such that $g'(x) = [\![\tau]\!]^{\mathcal{M},g,w}$

iff τ designates at w and $[\![y = x]\!]^{\mathcal{M},g'',w} = F$ for all y-variants g'' of g' at w (g' is an x-variant of g such that $g'(x) = [\![\tau]\!]^{\mathcal{M},g,w}$)

iff τ designates at w and $[\![y]\!]^{\mathcal{M},g'',w} \neq [\![x]\!]^{\mathcal{M},g'',w}$, ($g''$ being any y-variant of g' at w, and g' an x-variant of g with $g'(x) = [\![\tau]\!]^{\mathcal{M},g,w}$),

iff τ designates at w and $g''(y) \neq g''(x)$,

$g''(y) \in \delta(w)$ while $g''(x) = g'(x) = [\![\tau]\!]^{\mathcal{M},g,w}$;
(g'' being any y-variant of g' at w
and g' an x-variant of g, such that $g'(x) = [\![\tau]\!]^{\mathcal{M},g,w}$)
iff τ designates at w and $[\![\tau]\!]^{\mathcal{M},g,w} \notin \delta(w)$;

iff *1.* τ does not designate at w, or
2. τ designates at w and $[\![\tau]\!]^{\mathcal{M},g,w} \notin \delta(w)$.

□

The terms that designate can refer to existent objects in the world of evaluation or to non existent objects in the same world, therefore we can say of this kind of terms that either they have the property of existence or the non-existence property:

$$\mathbf{D}(\tau) \leftrightarrow \mathbf{E}(\tau) \vee \overline{\mathbf{E}}(\tau)$$

Of non-designating terms we can say that they do not exist:

$$\neg\mathbf{E}(\tau) ::= \neg\langle\lambda x.\exists y(y = x)\rangle(\tau)$$

but not that they have the property of non-existence.

Proposition 2.2.26. *For each term τ, the following is true at each world of a varying domain model:*

$$\mathbf{D}(\tau) \leftrightarrow \mathbf{E}(\tau) \vee \overline{\mathbf{E}}(\tau)$$

Proof. Let $\mathcal{M}$ be any normal, varying domain first-order modal model, g any assignment and $w \in W$. To prove that $[\![\mathbf{D}(\tau) \leftrightarrow \mathbf{E}(\tau) \vee \overline{\mathbf{E}}(\tau)]\!]^{\mathcal{M},g,w} = T$, we will distinguish two cases:

1. τ designates at w;

2. τ does not designate at w.

Case 1. τ designates at w. By proposition 2.2.23, $[\![\mathbf{D}(\tau)]\!]^{\mathcal{M},g,w} = T$. Either $[\![\tau]\!]^{\mathcal{M},g,w} \in \delta(w)$ or $[\![\tau]\!]^{\mathcal{M},g,w} \notin \delta(w)$. In the first case, $[\![\mathbf{E}(\tau)]\!]^{\mathcal{M},g,w} = T$ by proposition 2.2.22. Therefore, $[\![\mathbf{E}(\tau) \vee \overline{\mathbf{E}}(\tau)]\!]^{\mathcal{M},g,w} = T$, and so $[\![\mathbf{D}(\tau) \leftrightarrow \mathbf{E}(\tau) \vee \overline{\mathbf{E}}(\tau)]\!]^{\mathcal{M},g,w} = T$. In the second case, $[\![\overline{\mathbf{E}}(\tau)]\!]^{\mathcal{M},g,w} = T$ by proposition 2.2.24. Therefore, $[\![\mathbf{E}(\tau) \vee \overline{\mathbf{E}}(\tau)]\!]^{\mathcal{M},g,w} = T$, and so $[\![\mathbf{D}(\tau) \leftrightarrow \mathbf{E}(\tau) \vee \overline{\mathbf{E}}(\tau)]\!]^{\mathcal{M},g,w} = T$.

Case 2. τ does not designate at w. Thus, $[\![\mathbf{D}(\tau)]\!]^{\mathcal{M},g,w} = F$, by proposition 2.2.22. Since τ does not designate at w, any formula $\langle\lambda x.\phi\rangle(\tau)$ is false, and in particular $\langle \lambda x.\exists y(y = x)\rangle(\tau)$, which is $\mathbf{E}(\tau)$; and also $\langle \lambda x.\neg\exists y(y = x)\rangle(\tau)$, which is $\overline{\mathbf{E}}(\tau)$. Thus, $[\![\mathbf{E}(\tau)]\!]^{\mathcal{M},g,w} = F = [\![\overline{\mathbf{E}}(\tau)]\!]^{\mathcal{M},g,w} = F$. Therefore, $[\![\mathbf{D}(\tau) \leftrightarrow \mathbf{E}(\tau) \vee \overline{\mathbf{E}}(\tau)]\!]^{\mathcal{M},g,w} = T$.

□

Example 2.2.27. Suppose we have a varying domain model with two worlds w_1 and w_1, where $D_\iota = \{\mathbf{c}, \mathbf{d}, \mathbf{e}\}$, $\delta(w_1) = \{\mathbf{c}, \mathbf{e}\}$ and $\delta(w_2) = \{\mathbf{d}\}$. We take the worlds to be interpreted in a temporal way and the universe to be the universe of composers, where Max Richter is **c**, Arvo Pärt is **e** and Vivaldi is **d**. Our present century being w_1 and the eighteenth century w_2. The accessibility relation $R = \{\langle w_1, w_2\rangle, \langle w_2, w_1\rangle\}$. Our language has three intensional constants: c, d and e which are interpreted in w_1 as **c** (Max Richter), **d** (Vivaldi) and **e** (Arvo Pärt) respectively. We also know that the interpretation of d at w_2 is defined (as **d**) but that the interpretation of e at w_2 is not defined. So we have: $\mathsf{F}(c)(w_1) = \mathbf{c}$, $\mathsf{F}(e)(w_1) = \mathbf{e}$, $\mathsf{F}(d)(w_1) = \mathbf{d}$, $\mathsf{F}(c)(w_2) = \mathbf{c}$, $\mathsf{F}(d)(w_2) = \mathbf{d}$ and $\mathsf{F}(e)(w_2)$ is not defined.
Taking our present century as the world of evaluation, we can say that Max Richter exists and that the name Vivaldi designates although Vivaldi does not exist:

1. $\mathbf{E}(c)$
2. $\mathbf{D}(d)$
3. $\overline{\mathbf{E}}(d)$

Let us see that all three are true at w_1. Let $\mathcal{M}$ be the structure described in our example and g any assignment on D_ι.

1. $\mathbf{E}(c)$ is true at the present century w_1 iff $\langle \lambda x.\exists y(y = x)\rangle(c)$ is true at w_1. As c designates at w_1, then

   $$[\![\langle \lambda x.\exists y(y = x)\rangle(c)]\!]^{\mathcal{M},g,w_1} = T \text{ iff } [\![\exists y(y = x)]\!]^{\mathcal{M},g',w_1} = T$$

   where g' is the x-variant of g, such that $g'(x) = [\![c]\!]^{\mathcal{M},g,w_1} = \mathbf{c}$.

   $$[\![\exists y(y = x)]\!]^{\mathcal{M},g',w_1} = T \text{ iff } [\![y = x]\!]^{\mathcal{M},g'',w_1} = T$$

   for some y-variant g'' of g' at w_1. Take as g'' the y-variant of g' at w_1 such that $g''(y) = \mathbf{c}$.
   The formula is true at w_1 since $\mathbf{c} \in \delta(w_1)$ and

   $$[\![y]\!]^{\mathcal{M},g'',w_1} = g''(y) = \mathbf{c} = [\![c]\!]^{\mathcal{M},g,w_1} = [\![x]\!]^{\mathcal{M},g'',w_1}$$

2. $\mathbf{D}(d)$ is true at w_1 iff $\langle\lambda x.x = x\rangle(d)$ is true at w_1. As d is defined at w_1, we have that

$$[\![\langle\lambda x.x = x\rangle(d)]\!]^{\mathcal{M},g,w_1} = T \text{ iff } [\![x = x]\!]^{\mathcal{M},g',w_1} = T$$

where g' is the x-variant of g, such that $g'(x) = [\![d]\!]^{\mathcal{M},g,w_1} = \mathbf{d}$. Therefore, the formula $\mathbf{D}(d)$ is true as we have that $\mathbf{d} = \mathbf{d}$ true and we do not need $\mathbf{d}$ to be in world w_1.

3. $\overline{\mathbf{E}}(d)$ is true at w_1 iff $\langle\lambda x.\neg\exists y(y = x)\rangle(d)$ is true at w_1. As d is defined at w_1, we have that

$$[\![\langle\lambda x.\neg\exists y(y = x)\rangle(d)]\!]^{\mathcal{M},g,w_1} = T \text{ iff } [\![\neg\exists y(y = x)]\!]^{\mathcal{M},g',w_1} = T$$

where g' is the x-variant of g, such that $g'(x) = [\![d]\!]^{\mathcal{M},g,w_1} = \mathsf{F}(d)(w_1) = \mathbf{d}$.

$$[\![\neg\exists y(y = x)]\!]^{\mathcal{M},g',w_1} = T \text{ iff } [\![y = x]\!]^{\mathcal{M},g'',w_1} = F$$

for all y-variant g'' of g' at w_1. Let g'' be any y-variant of g' at w_1. As $g''(y) \in \delta(w_1)$, we can have that $g''(y) = \mathbf{c}$ or that $g''(y) = \mathbf{e}$ and it is false that $\mathbf{c} = \mathbf{d}$ or that $\mathbf{e} = \mathbf{d}$. And then the formula $\overline{\mathbf{E}}(d)$ is true at w_1.

On the other hand, if we evaluate the previous formulas from the eighteenth century, we have that Max Richter ($\mathbf{c}$) does not exist at w_2, that Vivaldi ($\mathbf{d}$) designates at w_2 and that to say of Vivaldi that he has the negative property of non-existence at w_2 is false:

1. $\mathbf{E}(c)$ is false at w_2.

2. $\mathbf{D}(d)$ is true at w_2.

3. $\overline{\mathbf{E}}(d)$ is false at w_2.

1. $\mathbf{E}(c)$ is true at w_2 iff $\langle\lambda x.\exists y(y = x)\rangle(c)$ is true at w_2. As $\mathsf{F}(c)(w_2)$ is defined, c designates at w_2, then

$$[\![\langle\lambda x.\exists y(y = x)\rangle(c)]\!]^{\mathcal{M},g,w_2} = T \text{ iff } [\![\exists y(y = x)]\!]^{\mathcal{M},g',w_2} = T$$

where g' is the x-variant of g, such that $g'(x) = [\![c]\!]^{\mathcal{M},g,w_2} = \mathbf{c}$.

$$[\![\exists y(y = x)]\!]^{\mathcal{M},g',w_2} = T \text{ iff } [\![y = x]\!]^{\mathcal{M},g'',w_2} = T$$

for some y-variant g'' of g' at w_2. The only possibility we have at w_2 is that $g''(y) = \mathbf{d}$. And then

$$g''(y) = \mathbf{d} \neq \mathbf{c} = g''(x) = g'(x)$$

Consequently the formula $\mathbf{E}(c)$ can not be true at w_2, and hence it is false at w_2.

2. $\mathbf{D}(d)$ is true at w_2 iff $\langle \lambda x.x = x\rangle(d)$ is true at w_2. As d is defined at w_2, we have that

$$[\![\langle \lambda x.x = x\rangle(d)]\!]^{\mathcal{M},g,w_2} = T \text{ iff } [\![x = x]\!]^{\mathcal{M},g',w_2} = T$$

where g' is the x-variant of g, such that $g'(x) = [\![d]\!]^{\mathcal{M},g,w_2} = \mathbf{d}$. Therefore we have that $\mathbf{d} = \mathbf{d}$, which is also true. Ergo, the formula $\mathbf{D}(d)$ is true at w_2.

3. $\overline{\mathbf{E}}(d)$ is true at w_2 iff $\langle \lambda x.\neg\exists y(y = x)\rangle(d)$ is true at w_2. As d is defined at w_2, we have that

$$[\![\langle \lambda x.\neg\exists y(y = x)\rangle(d)]\!]^{\mathcal{M},g,w_2} = T \text{ iff } [\![\neg\exists y(y = x)]\!]^{\mathcal{M},g',w_2} = T$$

where g' is the x-variant of g, such that $g'(x) = [\![d]\!]^{\mathcal{M},g,w_2} = \mathsf{F}(d)(w_2) = \mathbf{d}$.

$$[\![\neg\exists y(y = x)]\!]^{\mathcal{M},g',w_2} = T \text{ iff } [\![y = x]\!]^{\mathcal{M},g'',w_2} = F$$

for all y-variant g'' of g' at w_2. Let g'' be any y-variant of g' at w_2. As $g''(y) \in \delta(w_2)$, we only have that $g''(y) = \mathbf{d}$. But it is true that $g''(y) = \mathbf{d} = \mathbf{d} = g'(x)$, then $[\![y = x]\!]^{\mathcal{M},g'',w_2} = T$. And therefore the formula $\overline{\mathbf{E}}(d)$ is false at w_2.

Consider now the following formulas:

4. $\neg\mathbf{E}(d)$
5. $\neg\mathbf{E}(e)$
6. $\overline{\mathbf{E}}(e)$

We begin evaluating $\neg\mathbf{E}(d)$ at our present century century w_1 and continue with the evaluation of $\neg\mathbf{E}(e)$ and $\overline{\mathbf{E}}(e)$ at the eighteenth century w_2.

4. $\neg\mathbf{E}(d)$ is true at w_1 iff $[\![\neg\langle \lambda x.\exists y(y = x)\rangle(d)]\!]^{\mathcal{M},g,w_1} = T$ iff $[\![\langle \lambda x.\exists y(y = x)\rangle(d)]\!]^{\mathcal{M},g,w_1} = F$. This formula can be false when d does not designate at w_1, which is not the case, or if $[\![\exists y(y = x)]\!]^{\mathcal{M},g',w_1} = F$, where g' is the x-variant of g such that

$$g'(x) = [\![d]\!]^{\mathcal{M},g,w_1} = \mathsf{F}(d)(w_1) = \mathbf{d}$$

We have then that $[\![y = x]\!]^{\mathcal{M},g'',w_1} = F$, where g'' is any y-variant of g' at w_1.
As $g''(y) \in \delta(w_1)$, we have two possibilities:

   (a) either $g''(y) = \mathbf{c}$;

(b) or $g''(y) = \mathbf{e}$.

In any case $g''(y) = g'(x)$ is false. Therefore the formula $\neg\mathbf{E}(d)$ is true at w_1.

5. $\neg\mathbf{E}(e)$ is true at w_2 iff $[\![\neg\langle\lambda x.\exists y(y = x)\rangle(e)]\!]^{\mathcal{M},g,w_2} = T$ iff $[\![\langle\lambda x.\exists y(y = x)\rangle(e)]\!]^{\mathcal{M},g,w_2} = F$. Since $\mathsf{F}(e)(w_2)$ is not defined, then $[\![\langle\lambda x.\exists y(y = x)\rangle(e)]\!]^{\mathcal{M},g,w_2}$ is false, and consequently $\neg\mathbf{E}(e)$ is true at w_2.

6. $\overline{\mathbf{E}}(e)$ is true at w_2 iff $[\![\langle\lambda x.\neg\exists y(y = x)\rangle(e)]\!]^{\mathcal{M},g,w_2} = T$. But since $\mathsf{F}(e)(w_2)$ is not defined then, following definition 2.2.14, $[\![\langle\lambda x.\neg\exists y(y = x)\rangle(e)]\!]^{\mathcal{M},g,w_2} = F$.
As a result, the formula $\overline{\mathbf{E}}(e)$ is false at w_2.

Note that for such a denoting term d at w_1, where $[\![d]\!]^{\mathcal{M},g,w_1} \notin \delta(w_1)$, $\neg\mathbf{E}(d)$ and $\overline{\mathbf{E}}(d)$ have the same truth value. Both formulas are true at w_1 and therefore it is equivalent to say of the interpretation of d that it has not the property of existence at w_1 or that it has the negative property of non-existence at w_1.
On the other hand, for the non-denoting term e at w_2, $\neg\mathbf{E}(e)$ and $\overline{\mathbf{E}}(e)$ have different truth values. $\neg\mathbf{E}(e)$ is true at w_2 and $\overline{\mathbf{E}}(e)$ is false at w_2. The reason is that of a non-denoting term we cannot predicate anything, even the non-existence.

In table 2.1 we present a summary of how the formulas about denotation and existence are evaluated in our model of composers. Note the difference in the truth values of $\neg\mathbf{E}(e)$ and $\overline{\mathbf{E}}(e)$ at world w_2, where e does not designate:

Accordingly, we can affirm that for a denoting term τ at each world w of a varying domain model $\mathcal{M}$, the following equivalence is true:

$$\overline{\mathbf{E}}(\tau) \leftrightarrow \neg\mathbf{E}(\tau)$$

while for a non-denoting term τ^* at each world w of a varying domain model $\mathcal{M}$, the previous equivalence is not true. What is true, however, at each world w in a varying domain model $\mathcal{M}$ of a non-denoting term τ^* is:

$$\overline{\mathbf{E}}(\tau^*) \rightarrow \neg\mathbf{E}(\tau^*)$$

In conclusion, with respect to the previous analysis, if we have a term τ, we can face one of the following alternatives concerning designation and existence:[3]

[3]There are alternatives which take into account other options: that τ designates at every world or that τ does not designate at every world, but we are now more interested in designation and existence of terms and truth of formulas in a world of a model and not in every world of a model.

	w_1	w_2
$\mathbf{D}(c)$	T	T
$\mathbf{E}(c)$	T	F
$\neg\mathbf{E}(c)$	F	T
$\overline{\mathbf{E}}(c)$	F	T
$\mathbf{D}(d)$	T	T
$\mathbf{E}(d)$	F	T
$\neg\mathbf{E}(d)$	T	F
$\overline{\mathbf{E}}(d)$	T	F
$\mathbf{D}(e)$	T	F
$\mathbf{E}(e)$	T	F
$\neg\mathbf{E}(e)$	F	T
$\overline{\mathbf{E}}(e)$	F	F

Table 2.1: Evaluation in the model of composers

1. τ designates at a world w_1 and also exists in w_1;
2. τ designates at a world w_1 but does not exist in w_1;
3. τ does not designate at a world w_1 but does exist in other possible world w_2;
4. τ does not designate at a world w_1 and does not exist in any world.

Let us offer some examples in order to understand this classification. Here the interpretation of possible world is given in a temporal way:

1. "Philip VI of Spain" designates at the present time, say w_1, and also exists in w_1.
2. "The discoverer of America" designates at the present time w_1, but does not exist in w_1, although he existed in the past.
3. "The present pope of Avignon" does not designate at w_1 because the papacy of Avignon is not defined at the current time, but there was a time when there were popes in Avignon, and hence there have existed some "present pope of Avignon" in different times of the past.

4. "The wife of Isaac Newton" does not designate at w_1 and, at least in the different states of affairs of the current Universe, there is no time in our History where Newton's wife can be found. Ergo, the wife of Newton does not exist in any world (sure, in the development of our History).

Finally, we conclude saying that designation of a term at a given world can be seen as a necessary condition for a meaningful talking about it. However it is not necessary an existence requirement at the same world, and therefore you can refer to people, places, etc. which do not exist any more at this world. Nonetheless, if a term does not designate at a certain world, all what we can predicate of it (positively or negatively) is false at the same world. This non-denoting term is only like a *flatus vocis*.

2.3 A First Approach to First-Order Intensional Hybrid Logic

If we take a first-order intensional logic and add nominals and satisfaction operators we introduce an hybridization mechanism that results in a first-order intensional hybrid logic. We are following here the work of Torben Braüner (2008).[4] In order to see how hybridization works, we are going to dispense with constants and complex functional expressions. We think that in this way we can understand better the use of nominals, satisfaction operators and $\downarrow$.

2.3.1 Syntax

First of all, to the syntax of first-order intensional logic of Fitting (2004) we add nominals and rigidified terms. We also indicate as a subscript the type of the expression. Types are defined in the same way as in definition 2.2.1 on page 53. Our language has ordinary extensional first-order variables for objects and intensional variables for intensions, predicate symbols and nominals.

1. **Variables:** $x, y, z, \ldots$ They can be extensional, as x_ι, or intensional, as x_{ι_1}.

[4]The same author recognizes that his hybrid system is based on the first-order intensional logic of Fitting as presented in (2004). "In this article we give an intensional version of first-order hybrid logic (which also can be viewed as a hybridized version of Fitting's First-Order Intensional Logic)." (Braüner, 2008, p. 631).

2. **Predicate symbols:** $P, Q, \ldots$ They can be n-ary and they are considered to be intensional. Their type is $\langle(t^1 \ldots t^n)o\rangle_1$, where each of $t^1 \ldots t^n$ can be either ι or ι_1.

3. **Nominals:** $a, b, \ldots$ They are a sort of propositional symbols which are true only at one world.

Definition 2.3.1 (Term)**.** The terms $\tau^1, \ldots, \tau^n$ of the language are specified by the following rules:

1. Every variable (extensional or intensional) is a term either of type ι or of type ι_1.

2. Every rigidified intensional variable $@_a x_{\iota_1}$ is a term of type ι. Where x_{ι_1} is an intensional variable, a is a nominal and $@$ is called a satisfaction operator.

We present a distinction between letters $\tau^1, \ldots, \tau^n$ which range over any sort of terms, of type ι or ι_1, and $\rho^1, \ldots, \rho^n$, which range over terms of type ι. Terms, in general, are variables of any kind. *Object terms*, or extensional terms, however, are extensional variables or rigidified intensional variables. Terms can be of type ι or ι_1, while object terms are only of type ι.

Definition 2.3.2 (Object Term)**.** An object term $\rho^1, \ldots, \rho^n$, also called extensional terms, is either an extensional variable, such as x_ι, or a rigidified (intensional) variable, like $@_a x_{\iota_1}$. Object terms are of type ι.

Any sequence of symbols of the language is a formula. According to clause 1 of definition 2.2.5 on page 55, we did not have permission for building an atomic formula by means of a predicate letter followed by any kind of terms, only extensional variables were allowed. Here, however, atomic formulas can have extensional terms and also intensional terms. The novelty of this approach is that intensional terms do not need to be extensionalized. They can stand for the intensions they denote and, therefore, we can have intensional predication, one of the innovations we have found of the utmost importance.

Definition 2.3.3 (Formula)**.** A well formed formula (wff) is a sequence of symbols of the language following the rules:

1. $P_{\langle(t^1 \ldots t^n)o\rangle_1}(\tau^1_{t^1}, \ldots, \tau^n_{t^n})$ is an atomic formula, where $\tau^1_{t^1}, \ldots, \tau^n_{t^n}$ are terms, extensional or intensional. $t^1 \ldots t^n$ can be either ι or ι_1.

2. $\rho^1_\iota = \rho^2_\iota$ is a formula.

3. $\tau^1_{\iota_1} \equiv \tau^2_{\iota_1}$ is a formula.

4. If ϕ is a formula, then $\neg\phi$ is a formula.

5. If ϕ and ψ are formulas, then $\phi \wedge \psi$ is a formula.

6. If ϕ and ψ are formulas, then $\phi \rightarrow \psi$ is a formula.

7. If ϕ is a formula, then $\Box\phi$ is a formula.

8. If ϕ is a formula and x_ι an extensional variable, then $\forall x_\iota \phi$ is a formula.

9. If ϕ is a formula and x_{ι_1} an intensional variable, then $\forall x_{\iota_1} \phi$ is a formula.

10. If a is a nominal, then a is a formula.

11. If ϕ is a formula and a is a nominal, then $@_a\phi$ is a formula.

12. If ϕ is a formula and a is a nominal, then $\forall a\phi$ is a formula.

13. If ϕ is a formula and a is a nominal, then $\downarrow a\phi$ is a formula.

Remark. For Braüner (2008) the types of predicates are $\langle t^1 \ldots t^n \rangle$, even though the interpretation of P varies from world to world.

$\phi \vee \psi$ abbreviates $\neg(\neg\phi \wedge \neg\psi)$ and $\Diamond\phi$ is an abbreviation for $\neg\Box\neg\phi$. $\exists x_\iota \phi$ abbreviates $\neg\forall x_\iota \neg\phi$ and, analogously, $\exists x_{\iota_1}\phi$ and $\exists a\phi$ are abbreviations for $\neg\forall x_{\iota_1}\neg\phi$ and $\neg\forall a\neg\phi$ respectively.

The notion of free occurrences of variables is defined as usual, analogously to definition 2.2.6 on page 56.

Definition 2.3.4 (Free occurrences of nominals)**.** The notion of free occurrences of nominals is defined as follows:

1. If a is a nominal, then it has one free occurrence of a nominal, namely, itself.

2. If $@_a\phi$ is a formula, then the free nominal occurrences in it are those in ϕ together with the occurrence of a. Satisfaction operators, thus, do not bind nominals.

3. If $\forall a\phi$ is a formula, then the free nominal occurrences in it are those in ϕ except for a.

4. If $\downarrow a\,\phi$ is a formula, then its free nominal occurrences are those of ϕ except for a.

2.3.2 Semantics

In section 2.2, on page 56, we had a semantics with a varying domain model. In Fitting (2004) and Braüner (2008) models with constant domains are considered instead. Our model has two domains: a domain of object quantification and a domain of intension quantification. And both are taken to be constant, that is, the domain of quantification is the same from world to world. Quantification is called *possibilist*, because quantifiers range over every possible existent, including those which actually exist at the world of evaluation. And the domain of intension quantification is not required to be the set of all functions[5] from W to D_ι, but only to be a non-empty subset of partial functions from W to D_ι, that is:

$$D_{\iota_1} \subseteq \{\mathbf{f} \mid \mathbf{f} : H \to D_\iota \text{ and } H \subseteq W\}$$

Definition 2.3.5 (Constant Domain Model)**.**

$$\mathcal{M} = \langle W, R, D_\iota, D_{\iota_1}, \mathsf{F}\rangle$$

where

1. $W \neq \emptyset$;
2. R is a binary relation on W;
3. $D_\iota \neq \emptyset$;
4. D_{ι_1} is a non-empty set of partial functions from W to D_ι;
5. F is an interpretation function which assigns to each n-place predicate symbol P and to each possible world $w \in W$ a characteristic function of n-tuples of D_ι or D_{ι_1}:

   $$\mathsf{F}(P^n)(w) \in \{T, F\}^{(D_{t^1} \times \cdots \times D_{t^n})}$$

   where $t^1 \ldots t^n$ are the types of the arguments of the n-ary predicate and can be ι or ι_1.

[5]For Fitting, to require of the domain of intension quantification to be the set of all functions from W to D_ι, is something that is not desirable. He argues two reasons: "First, not everything reasonably should be considered an intension. There is not much plausibility to an intension that is a wrench in this world, a baby robin in another, and the number 7 in a third. Intensions should have some coherence to them, and though I do not know how to characterize that, clearly not everything mathematically possible will meet a reasonable coherence condition. The second reason for not taking the entire set of functions from $\mathcal{G}$ [our W] to $\mathcal{D}_\mathcal{O}$ [our D_ι] as $\mathcal{D}_\mathcal{I}$ [our D_{ι_1}] is more practical: if we do, a complete proof procedure is almost certainly beyond reach." (2004, pp. 182–183).

Definition 2.3.6 (Assignment)**.** An assignment g is a function that to each extensional variable x_ι assigns an element of D_ι, to each intensional variable assigns an element of D_{ι_1}, and to each nominal a assigns an element of W:

$$g(x_\iota) \in D_\iota$$
$$g(x_{\iota_1}) \in D_{\iota_1}$$
$$g(a) \in W$$

Note that $g(x_{\iota_1})$ has as value a partial function from W to D_ι.

Definition 2.3.7 (Variant)**.** An x_ι-variant assignment g' of g is an assignment having the same values as the original assignment g except possibly for x_ι.

$$g'(x_\iota) \in D_\iota$$

An x_{ι_1}-variant assignment g' of g is an assignment having the same values as the original assignment g except possibly for x_{ι_1}.

$$g'(x_{\iota_1}) \in D_{\iota_1}$$

An a-variant assignment g' of g is an assignment having the same values as the original assignment g except possibly for a:

$$g'(a) \in W$$

Definition 2.3.8 (Denotation of terms in a world w of a model $\mathcal{M}$ using assignment g)**.** Given a model $\mathcal{M} = \langle W, R, D_\iota, D_{\iota_1}, \mathsf{F}\rangle$, the interpretation of every term τ of the language, is given considering an assignment g and a world w:

1. If x_ι is an extensional variable:

$$[\![x_\iota]\!]^{\mathcal{M},g,w} = g(x_\iota)$$

2. If x_{ι_1} is an intensional variable:

$$[\![x_{\iota_1}]\!]^{\mathcal{M},g,w} = g(x_{\iota_1})$$

3. If $@_a x_{\iota_1}$ is a rigidified intensional variable, then:

   (a) if $g(x_{\iota_1})(g(a))$ is defined, that is, if $g(a)$ is in the domain of the partial function $g(x_{\iota_1})$, then $[\![@_a x_{\iota_1}]\!]^{\mathcal{M},g,w} = g(x_{\iota_1})(g(a))$;

   (b) if $g(x_{\iota_1})(g(a))$ is *not* defined, then $@_a x_{\iota_1}$ does *not* designate.

Definition 2.3.9 (Truth of a formula in a world w of a model $\mathcal{M}$ using assignment g)**.** Let $\mathcal{M} = \langle W, R, D_\iota, D_{\iota_1}, \mathsf{F}\rangle$ be a constant domain first-order model. Now we can define the truth of a formula in model $\mathcal{M}$, where F is an interpretation function, g an assignment, w a world, and ϕ a formula.

1. The evaluation of $[\![P(\tau^1, \ldots, \tau^n)]\!]^{\mathcal{M},g,w}$ requires the analysis of two cases depending on whether or not all object terms designate:

   (a) If every $\tau^1, \ldots, \tau^n$ designates at w:

   $$[\![P(\tau^1, \ldots, \tau^n)]\!]^{\mathcal{M},g,w} = T \text{ iff } (\mathsf{F}(P)(w))([\![\tau^1]\!]^{\mathcal{M},g,w}, \ldots, [\![\tau^n]\!]^{\mathcal{M},g,w}) = T$$

   (b) If any of the terms $\tau^1, \ldots, \tau^n$ does not designate at w:

   $$[\![P(\tau^1, \ldots, \tau^n)]\!]^{\mathcal{M},g,w} = F$$

2. The evaluation of $[\![\rho^1 = \rho^2]\!]^{\mathcal{M},g,w}$ requires also two cases:

   (a) If ρ^1 and ρ^2 designate, then:

   $$[\![\rho^1 = \rho^2]\!]^{\mathcal{M},g,w} = T \text{ iff } [\![\rho^1]\!]^{\mathcal{M},g,w} = [\![\rho^2]\!]^{\mathcal{M},g,w}$$

   (b) If ρ^1 or ρ^2 does not designate, then:

   $$[\![\rho^1 = \rho^2]\!]^{\mathcal{M},g,w} = F$$

3. $[\![\tau^1_{\iota_1} \equiv \tau^2_{\iota_1}]\!]^{\mathcal{M},g,w} = T$ iff $[\![\tau^1_{\iota_1}]\!]^{\mathcal{M},g,w} = [\![\tau^2_{\iota_1}]\!]^{\mathcal{M},g,w}$

4. $[\![\neg\phi]\!]^{\mathcal{M},g,w} = T$ iff $[\![\phi]\!]^{\mathcal{M},g,w} = F$

5. $[\![\phi \wedge \psi]\!]^{\mathcal{M},g,w} = T$ iff $[\![\phi]\!]^{\mathcal{M},g,w} = T$ and $[\![\psi]\!]^{\mathcal{M},g,w} = T$

6. $[\![\phi \to \psi]\!]^{\mathcal{M},g,w} = T$ iff $[\![\phi]\!]^{\mathcal{M},g,w} = F$ or $[\![\psi]\!]^{\mathcal{M},g,w} = T$

7. $[\![\Box\phi]\!]^{\mathcal{M},g,w} = T$ iff $[\![\phi]\!]^{\mathcal{M},g,w'} = T$ for all $w' \in W$ such that wRw'

8. $[\![\forall x_\iota \phi]\!]^{\mathcal{M},g,w} = T$ iff $[\![\phi]\!]^{\mathcal{M},g',w} = T$, for all x_ι-variant g' of g

9. $[\![\forall x_{\iota_1} \phi]\!]^{\mathcal{M},g,w} = T$ iff $[\![\phi]\!]^{\mathcal{M},g',w} = T$, for all x_{ι_1}-variant g' of g

10. $[\![a]\!]^{\mathcal{M},g,w} = T$ iff $g(a) = w$

11. $[\![@_a \phi]\!]^{\mathcal{M},g,w} = T$ iff $[\![\phi]\!]^{\mathcal{M},g,g(a)} = T$

12. $[\![\forall a \phi]\!]^{\mathcal{M},g,w} = T$ iff $[\![\phi]\!]^{\mathcal{M},g',w} = T$, for any a-variant g' of g

13. $[\![\downarrow a\phi]\!]^{\mathcal{M},g,w} = T$ iff $[\![\phi]\!]^{\mathcal{M},g',w} = T$, where g' is an a-variant of g and $g'(a) = w$.

Truth in a model and validity are defined as in definitions 2.2.16 and 2.2.19 on pages 60 and 61 respectively.

Remark. Note that although $\downarrow$ and @ are similar because they go with nominals, they are not the same operator. The satisfaction operator @ is used to individuate a certain world where an expression is interpreted, this world is determined by the nominal and can be the present world of evaluation or not. The $\downarrow$ operator, however, fixes the world where the formula has to be interpreted at the present world of evaluation, and the value the assignment function gives to the nominal bound by $\downarrow$, is the world where the expression is being evaluated.

Example 2.3.10. Suppose we have a model with two worlds: w_1, representing the Christian worldview, and w_2, representing the Jihadist worldview. Nominal a names w_1 and nominal b, w_2. The individuals of the domain of the model are Salman Abedi (**c**), the suicide bomber who killed more than 20 people during a concert in Manchester in 2017, and Saint Sebastian (**d**), an early Christian saint. Suppose also that the intensional term x_{ι_1} stands for the description "the man who acted according to his faith", and the intensional predicate P, of type $\langle \iota o \rangle_1$, for "is a martyr". We want to evaluate the formulas $P(@_a x_{\iota_1})$ and $P(@_b x_{\iota_1})$ in the model at each one of the two worlds under assignment g. We also suppose that $@_a x_{\iota_1}$ and $@_b x_{\iota_1}$ are denoting terms, given that the description "the man who acted according to his faith" denotes Saint Sebastian at w_1 and Salman Abedi at w_2. At w_1, $P(@_a x_{\iota_1})$ is read as "Saint Sebastian is a martyr", which is true, since the denotation of $@_a x_{\iota_1}$ at w_1 is given by $g(x_{\iota_1})(g(a)) = g(x_{\iota_1})(w_1) = \mathbf{d}$. On the other hand, $P(@_b x_{\iota_1})$ is evaluated at w_1 as false because it has the reading that "Salman Abedi is a martyr", since the denotation of $@_b x_{\iota_1}$ at w_1 is $g(x_{\iota_1})(g(b)) = g(x_{\iota_1})(w_2) = \mathbf{c}$, i.e., Salman Abedi. At w_2, by contrast, and following a similar procedure, we can conclude that $P(@_a x_{\iota_1})$ is false, while $P(@_b x_{\iota_1})$ is true.

Consider now the formulas $\downarrow a\, P(@_a x_{\iota_1})$ and $\downarrow b\, P(@_b x_{\iota_1})$. If we evaluate $\downarrow b\, P(@_b x_{\iota_1})$ at w_1, what the formula is saying is that the property of being a martyr at w_1 is attributed to the individual denoted by $@_b x_{\iota_1}$ at w_1, once that b has been fixed to w_1. But $@_b x_{\iota_1}$ has now to be interpreted in a different way because $\downarrow b$ has fixed the nominal b at the world of evaluation, and therefore $g'(b) = w_1$. The denotation of $@_b x_{\iota_1}$ is, thus, the result of interpreting x_{ι_1} at the world w_1, which gives **d** as value, and **d** is martyr at w_1, therefore the formula $\downarrow b\, P(@_b x_{\iota_1})$ is true at w_1. The evaluation of $\downarrow a\, P(@_a x_{\iota_1})$ at

w_2 is similar to the previous one. Since the $\downarrow$ operator fixes a to w_2 and so $g'(a) = w_2$, the rigidified term is interpreted as claiming that $g(x_{\iota_1})(g'(a))$ has $\mathbf{c}$ as its value and, therefore, the interpretation of the formula $\downarrow a\, P(@_a x_{\iota_1})$ at w_2 says that $\mathbf{c}$ is a martyr, which is true at w_2, given that Salman Abedi is a martyr within a Jihadist worldview.

Remark. Note also that we have an equality symbol ($=$) for object terms and an equality symbol ($\equiv$) for intensional terms. $=$ denotes a relation between objects and it is interpreted as the genuine identity relation which holds between an object and itself. By contrast, $\equiv$ denotes a relation between functions. $\equiv$ has an stronger interpretation than $=$ because it states the identity between two concepts and, therefore, it has the characteristics of a *synonymy* relation.

2.3.3 Intensional Predication

In section 2.2, although intensions were mainly introduced as the interpretation of constants, and predicates were interpreted as varying from world to world, there was no way of attributing a property to a proper intension, since atomic formulas where formed only with extensional variables, and the only way to introduce intensional terms was through formulas with predicate abstracts, but in this case all intensional terms were always extensionalized. Intensions are a helpful mechanism for individuating concrete individuals at determinate worlds. But intensions are also functions which can be considered by themselves, and they do not need to be extensionalized at a world in order to make them workable. We are then faced with two options when we evaluate an intensional expression at a given world: we can interpret the expression at our world as determining an individual at the world of evaluation, or we can evaluate the expression at our world as determining an intension at the world of evaluation. In the first case we have extensional predication and in the second one we have intensional predication.

Suppose we say that "the winner of the Golden Ball [*Ballon d'Or*] has the most important sport distinction in the world", where the expression "the winner of the Golden Ball" is formalized as the intensional term x_{ι_1} and "has the most important sport distinction in the world" as P, whose type is $\langle \iota_1 o \rangle_1$ —remember that predicates are always interpreted at a world, and that is the reason of the appearance of the last 1 as subscript of the expression between the parentheses. What we pretend to express with the formula $P(x_{\iota_1})$ is that we attribute the property of *having the most important sport distinction in the world* to the intension denoted by "the winner of the Golden Ball", which is an element of the intension domain D_{ι_1} and not of the object domain D_ι.

If we think in different people as possible worlds, the sentence can be true at some worlds: for people which are soccer enthusiasts, and also be false at other worlds, for people which do not profess any sympathy for soccer. The key issue is that the truth value of the formula is not determined by the person individuated by the intensional expression at a particular world, and is independent of whoever is that person at a given world. On the other hand, we can take the same intensional expression x_{ι_1} but as a mere device for picking up a concrete individual at a certain world, as in "the winner of the Golden Ball is handsome", for example. In this case, the intensional expression "the winner of the Golden Ball" determines a concrete individual at the world of evaluation, and the denotation of the predicate "is handsome" (Q) at the same world of evaluation can be applied or not to it, depending on the standards of beauty ruling at the world of evaluation. But, how do we formalize this sentence? As $Q(x_{\iota_1})$? If this were the case, we would never know when an intensional term has to be interpreted as an intension and when it has to be extensionalized. We need some kind of device which allows us to distinguish between the double use of intensional terms, which results in a double interpretation of them.

One of the possible devices, which we are not following here,[6] given item 1 of definition 2.3.3 on page 72, could be to use a type notation in predicates. If a predicate is of type $\langle \iota o \rangle_1$, the expression in the argument place, even an intensional term, should be always interpreted extensionally, since the ι of the type of the predicate commands it. If the predicate is of type $\langle \iota_1 o \rangle_1$, then the predication would be intensional, and the property should be applied to the very same intension. But we do not see the coherence of saying that a predicate of type $\langle \iota o \rangle_1$ is followed by a term of type ι_1. As a result, we have shown preference in using some device to devalue the type of the term from ι_1 to ι.

This device, the one that we follow in definition 2.3.8 on page 75, centers on our satisfaction operator $@_a$. When it is used to rigidify intensional terms of type ι_1, we get an expression, $@_a x_{\iota_1}$, which has type ι. The reason is that the satisfaction operator gives the world where the term has to be interpreted—something that is not necessary for intensions (the functions themselves do not change from world to world)—, doing this, the intension has received as argument the world of evaluation and, therefore, we have as value the object denoted by the term at a world. In this case, if we want to

[6]This device, however, is supposed, in a certain way, in the interpretation of formulas with predicate abstracts which we are going to see in the following pages. The predicate abstract would be of type $\langle \iota o \rangle_1$ while the term in the argument place would be of type ι_1.

have intensional predication we leave the formula unchanged:

$$P_{\langle \iota_1 o \rangle_1}(x_{\iota_1}) \tag{2.1}$$

On the contrary, if we desire extensional predication, the formula is written as:

$$\downarrow aQ_{\langle \iota o \rangle_1}(@_a x_{\iota_1}) \tag{2.2}$$

Formally, the formula $P_{\langle \iota_1 o \rangle_1}(x_{\iota_1})$ is true at a world w if and only if the intension designated by x_{ι_1}—that is, $g(x_{\iota_1})$—belongs to the extension of the predicate P at w:

$$[\![P_{\langle \iota_1 o \rangle_1}(x_{\iota_1})]\!]^{\mathcal{M},g,w} = T \text{ iff } (\mathsf{F}(P)(w))(g(x_{\iota_1})) = T$$

The formula $\downarrow aQ_{\langle \iota o \rangle_1}(@_a x_{\iota_1})$ is true at a world w if and only if the designation of x_{ι_1} at w belongs to the extension of the predicate Q at w. What the $\downarrow$ operator is doing here, is to fix the nominal a to the point of evaluation: $g(a) = w$.

$$\begin{aligned} &[\![\downarrow aQ_{\langle \iota o \rangle_1}(@_a x_{\iota_1})]\!]^{\mathcal{M},g,w} = T \text{ iff } [\![Q(@_a x_{\iota_1})]\!]^{\mathcal{M},g',w} = T, \\ &\quad \text{where } g' \text{ in an } a\text{-variant of } g \text{ and } g'(a) = w, \\ &\text{iff } (\mathsf{F}(Q)(w))([\![@_a x_{\iota_1}]\!]^{\mathcal{M},g',w}) = T \\ &\text{iff } (\mathsf{F}(Q)(w))(g'(x_{\iota_1})(g'(a))) = T \end{aligned}$$

Note that we are assuming that the terms are denoting terms, and so the function $g'(x_{\iota_1})$ is defined at $g'(a) = w$. If $@_a x_{\iota_1}$ were a non-denoting term at w, then the formula $\downarrow aQ(@_a x_{\iota_1})$ would be false.

In Fitting and Mendelsohn (1998) intensional predication did not receive any attention, but in Fitting (2004) it is considered as a proper and interesting way of predication. But the first-order intensional logic of Fitting (2004) is modal but not hybrid, and consequently hybrid operators are not available. There are, however, predicate abstracts, which are shown to be extremely useful in removing certain ambiguities in formal languages. Let us then introduce into the present language formulas with predicate abstracts. This can be done by expanding some of our previous definitions. First of all, we need to indicate that, apart from predicate symbols $P, Q, \ldots$, predicate abstracts are also predicates of the language. If x_ι is an extensional variable and ϕ a formula, then $\langle \lambda x_\iota . \phi \rangle$ is a predicate abstract. Then we expand definition 2.3.3 on page 72 by adding a new item:

14. If $\langle \lambda x_\iota . \phi \rangle$ is a predicate abstract and x_{ι_1} an intensional term, then $\langle \lambda x_\iota . \phi \rangle(x_{\iota_1})$ is a formula.

The free variable occurrences of this formula are those of ϕ except for x_ι, plus the occurrence of x_{ι_1}.
We abbreviate $\langle \lambda x_\iota . \langle \lambda y_\iota . \phi \rangle (y_{\iota_1}) \rangle (x_{\iota_1})$ as $\langle \lambda x_\iota, y_\iota . \phi \rangle (x_{\iota_1}, y_{\iota_1})$. Note also how the predicate abstract acts as a bridge between intensional variables and extensional ones. It acts as if by way of passing through the predicate abstract, intensions were stripped of its abstract category and devalued to a concrete object. Predicate abstracts act, then, as a procedure of "incarnation of intensions", and so intensions can touch the real objects of the world: *Verbum caro factum est*.

And finally, we add to definition 2.3.9 on page 76, the evaluation of formulas with lambda:

14. Let $\langle \lambda x_\iota . \phi \rangle (x_{\iota_1})$ be a formula with a predicate abstract and an intensional variable; we consider two cases depending on wether or not the intensional variable designates at the world of evaluation (remember that $g(x_{\iota_1})$ is a partial function):

    (a) If x_{ι_1} designates at w:

    $$[\![\langle \lambda x_\iota . \phi \rangle (x_{\iota_1})]\!]^{\mathcal{M},g,w} = T \text{ iff } [\![\phi]\!]^{\mathcal{M},g',w} = T$$

    where g' is the x_ι-variant of g, such that $g'(x_\iota) = g(x_{\iota_1})(w)$.

    (b) If x_{ι_1} does not designate at w:

    $$[\![\langle \lambda x_\iota . \phi \rangle (x_{\iota_1})]\!]^{\mathcal{M},g,w} = F$$

The idea behind the interpretation of $\langle \lambda x_\iota . \phi \rangle (x_{\iota_1})$ at a given world w, is that if x_{ι_1} designates at w, then the object designated by x_{ι_1} at w has the property specified by ϕ at w; and if x_{ι_1} does not designate at w, then it cannot be predicated anything about it at w, and therefore, to say that x_{ι_1} has the property specified by ϕ at w is false.

Let us revisit our previous examples to analyze how helpful can be predicate abstracts. The example of intensional predication is formalized in the same way as before: "the winner of the Golden Ball has the most important sport distinction in the world" is left as $P(x_{\iota_1})$. But, the example of extensional predication: "the winner of the Golden Ball is handsome" is translated as:

$$\langle \lambda x_\iota . Q(x_\iota) \rangle (x_{\iota_1}) \tag{2.3}$$

We can evaluate the previous formula following a pattern like this: first, we must take the intensional expression "the winner of the Golden Ball", and check if it designates at w; if it does, we take the object determined by the intension at w, and check if the property Q interpreted at w has the previous object as an element. If it is, the formula is true at w. If it is not, the formula is false at w. And if "the winner of the Golden Ball" does not designate at w, then the formula is false at w.

Example 2.3.11. Think about a model where the elements of the domain of individuals are soccer players. In this soccer-model we have three worlds: w_1, w_2 and w_3. w_1 represents a person in 2016 who is a soccer fanatic loving Ronaldo's physique but not Messi's physique; w_2 represents a person in 2009 who does not feel any sympathy for soccer and which thinks that both Ronaldo and Messi are good-looking men; and w_3 represents a person in 1950 who enjoys playing soccer and thinks that Alfredo Di Stéfano is attractive.

At w_1: $P(x_{\iota_1})$, which translates "the winner of the Golden Ball has the most important sport distinction in the world", is true—remember that w_1 is a fanatic. Its truth value does not depend on who is the winner of the Golden Ball at a particular year, but it is the *concept* of the award itself which is the most important sport distinction in the world. At w_2, the formula $P(x_{\iota_1})$ is evaluated to false given that, for w_2, to be awarded with a soccer prize can be important but not the most important sport distinction. At w_3, although the Golden Ball prize had not been awarded yet in 1950, it is possible, however, to refer to the concept denoted by x_{ι_1} and, if enjoying playing soccer, as w_3 does, to evaluate the formula $P(x_{\iota_1})$ as true. The formula $P(x_{\iota_1})$ is, thus, true at w_3.

We continue now with the evaluation of the formula $\langle\lambda x_\iota.Q(x_\iota)\rangle(x_{\iota_1})$, translating "the winner of the Golden Ball is handsome".

At w_1, the description "the winner of the Golden Ball" designates Cristiano Ronaldo and therefore:

$$[\![\langle\lambda x_\iota.Q(x_\iota)\rangle(x_{\iota_1})]\!]^{\mathcal{M},g,w_1} = T \text{ iff } [\![Q(x_\iota)]\!]^{\mathcal{M},g',w_1} = T$$

where g' is the x_ι-variant of g, such that $g'(x_\iota) = g(x_{\iota_1})(w_1)$. And then,

$$\begin{aligned} &[\![Q(x_\iota)]\!]^{\mathcal{M},g',w_1} = T \\ \text{iff } &(\mathsf{F}(Q)(w_1))([\![x_\iota]\!]^{\mathcal{M},g',w_1}) = T \\ \text{iff } &(\mathsf{F}(Q)(w_1))(g(x_{\iota_1})(w_1)) = T \end{aligned}$$

Then the formula is true supposing that Cristiano Ronaldo, $g(x_{\iota_1})(w_1)$, fits the criteria of beauty of w_1, $\mathsf{F}(Q)(w_1)$, which does, and so $\langle\lambda x_\iota.Q(x_\iota)\rangle(x_{\iota_1})$ is true at w_1.

At w_2, x_{ι_1} designates Lionel Messi, and so we have that:

$$[\![\langle\lambda x_\iota.Q(x_\iota)\rangle(x_{\iota_1})]\!]^{\mathcal{M},g,w_2} = T \text{ iff } [\![Q(x_\iota)]\!]^{\mathcal{M},g',w_2} = T$$

where g' is the x_ι-variant of g, such that $g'(x_\iota) = g(x_{\iota_1})(w_2)$. The denotation of x_{ι_1} at w_2 is $g(x_{\iota_1})(w_2)$, which determines Lionel Messi as the winner of the Golden Ball in 2009. If the beauty standards at w_2, that is, $\mathsf{F}(Q)(w_1)$, can be applied to Lionel Messi, then the formula $\langle\lambda x_\iota.Q(x_\iota)\rangle(x_{\iota_1})$ would be true at w_2. If Lionel Messi does not fulfill the idea of a handsome man at w_2, then sentence $\langle\lambda x_\iota.Q(x_\iota)\rangle(x_{\iota_1})$ would be false. As Messi fits the criteria of beauty at w_2, then $\langle\lambda x_\iota.Q(x_\iota)\rangle(x_{\iota_1})$ is true at w_2.

At w_3, we find that the intensional term x_{ι_1} does not have any denotation, given that the Golden Ball prize was not awarded yet. (Remember that $g(x_{\iota_1})$ is a partial function not defined in w_3.) And, therefore, as we can not predicate anything true of non-denoting terms we conclude that:

$$[\![\langle\lambda x_\iota.Q(x_\iota)\rangle(x_{\iota_1})]\!]^{\mathcal{M},g,w_3} = F$$

It is false thus at w_3 that "the winner of the Golden Ball is handsome", since ""the winner of the Golden Ball" does not denote any individual in 1950. By contrast, "the winner of the Golden Ball has the most important sport distinction in the world" is true at w_3, even if there is no winner of the Golden Ball in 1950, since you can denote an intension even at a world where it has no individual object as value.

2.3.4 Scope distinction

As we have seen before, when we have formulas where a predicate is said of a non-denoting term, the formulas are false and "that's it!". But when formulas with negation are taken into account, it results in some complications. In this section we revisit—remember section 2.2.4 on page 61— the inconsistencies with the interpretation of negative statements which have non-denoting terms. The novelty is that here we are going to compare predicate abstracts and hybrid machinery, to see how they work independently. Consider, for example, the proposition expressed by: "the Spanish winner of the current Eurovision contest is famous worldwide", evaluated at the present world $w = g(a)$, year 2017. Suppose that y_{ι_1} stands for "the Spanish winner of the current Eurovision contest" and R for "being famous worldwide". The previous sentence could be formalized in two ways:

1. $\downarrow a R(@_a y_{\iota_1})$, or
2. $\langle\lambda y_\iota.R(y_\iota)\rangle(y_{\iota_1})$

$\downarrow a R(@_a y_{\iota_1})$ is evaluated according to rules 13 and 1 of definition 2.3.9 on page 76. As we know that $@_a y_{\iota_1}$ is a non-designating term at w, it follows that

$$[\![\downarrow aR(@_a y_{\iota_1})]\!]^{\mathcal{M},g,w} = F \tag{2.4}$$

And, according to case (b) of rule 14 of definition 2.3.9 on page 81:

$$[\![\langle \lambda y_\iota.R(y_\iota)\rangle(y_{\iota_1})]\!]^{\mathcal{M},g,w} = F \tag{2.5}$$

In line with evaluations (2.4) and (2.5), it seems there is no problem with the truth values that are assigned to formulas with non-designating terms. But what does it happen when we consider a proposition which is the negation of the previous one, that is, "the Spanish winner of the current Eurovision contest is not famous worldwide"? We can formalize it in different ways depending on the reading we do: either (A) "it is not the case that the Spanish winner of the current Eurovision contest is famous worldwide", or (B) "the Spanish winner of the current Eurovision contest is *not-famous* worldwide". In the first case we are negating the attribution of a property to the object determined by an intension at a given world. In the second case we are saying of the object determined by an intension at a given world that it has a negative property, that is, the property of *being not-famous worldwide*. We know from section 2.2.4 on page 61, that these nuances are captured by means of the use of predicate abstracts. And, in fact, the truth value of (A) and (B) are different when non-denoting terms are considered. Let us remember it.

(A^λ) $\neg\langle \lambda y_\iota.R(y_\iota)\rangle(y_{\iota_1})$ is true at our present world w, while

(B^λ) $\langle \lambda y_\iota.\neg R(y_\iota)\rangle(y_{\iota_1})$ is false at our present world w.

We ask now if the same distinction concerning the scope of the negation symbol can also be captured by the hybrid operators. We try it by interchanging the negation symbol and $\downarrow$. The result is as follows:

($A^\downarrow$) $\neg{\downarrow}aR(@_a y_{\iota_1})$ is true at w. The evaluation of this formula at $w = g(a)$ gives us that, as $g(y_{\iota_1})(w)$ is not defined, then $@_a y_{\iota_1}$ does not designate at $w = g(a)$, and therefore $R(@_a y_{\iota_1})$ is false at w. Consequently $\neg{\downarrow}aR(@_a y_{\iota_1})$ is true at the present world 2017.

($B^\downarrow$) ${\downarrow}a\neg R(@_a y_{\iota_1})$ is true at w. We interpret the predicate the predicate at w and the rigidified intensional variable $@_a y_{\iota_1}$ at the same world $g(a) = w$. The term $@_a y_{\iota_1}$ does not designate at w as we know that $g(y_{\iota_1})(w)$ is not defined. Accordingly, $R(@_a y_{\iota_1})$ is false at w, and $\neg R(@_a y_{\iota_1})$ is true at the present world 2017.

Both cases ($A^{\downarrow}$) and ($B^{\downarrow}$) have the same truth value so it seems that $\neg{\downarrow}aR(@_a y_{\iota_1})$ and ${\downarrow}a\neg R(@_a y_{\iota_1})$ are equivalent formulas. They are also equivalent with $\neg\langle\lambda y_\iota.R(y_\iota)\rangle(y_{\iota_1})$. But, as we have seen, the nuances expressed through $\langle\lambda y_\iota.\neg R(y_\iota)\rangle(y_{\iota_1})$ have not been captured by hybrid operators, and hence there is no way of expressing the attribution of a negative property to an object with only the use of $\downarrow$ and @. When we have denoting terms there is no problem at all: (A^{λ}), (B^{λ}), ($A^{\downarrow}$) and ($B^{\downarrow}$), have always the same truth value. But if non-denoting terms come into play, (B^{λ}) takes a different truth value with respect to the other alternatives.

One way of solving this problem is to dispense with predicate abstracts, and to give a unique formalization to (A) and (B), making unnecessary any difference in scope. In this case, (A) and (B) would express the same proposition which would be formalized as, for example, $\neg{\downarrow}aR(@_a y_{\iota_1})$.

We can, however, combine predicate abstracts and hybrid operators in the same language without having the problem of inconsistencies in the interpretation of the language, provided we restrict the formalization of sentences where a negative property is applied to an object to formulas with predicate abstracts. In this case when we formalize negative statements what, first of all, we need to clarify is if they have the form of a "It is not the case that x is P" sentence or if they have the form of a sentence like "x has the negative property of non-being P".

2.3.5 *De dicto* and *De re*

We are concerned with the distinction between *de dicto* and *de re*, because there are certain ambiguities which result from the combination of intensional terms with modal operators. Imagine we say that "it is necessary that the winner of the current Eurovision contest is Portuguese". Suppose S stands for "is Portuguese" and z_{ι_1} for "the winner of the current Eurovision contest". And so, we get the formula $\Box S(z_{\iota_1})$ from the previous statement. The sentence can be read as saying that *the winner of the current Eurovision contest* at each possible world—where possible worlds are interpreted as time instances—*is Portuguese* at the same worlds; or it can be understood as that *the winner of the current Eurovision contest* at the present world of evaluation *is Portuguese* at every possible world. Is only one of both interpretations correct? We think it is plausible to have both interpretations. In the first case we have a *de dicto* reading, which is the straightforward interpretation in a sentence like "it is necessary that the winner of the current Eurovision contest is Portuguese"; in the second case, however, we have a *de re* reading, which is more easily understood if we paraphrase the sentence as "the winner of the current Eurovision contest is necessarily Portuguese". Intuitively, in the first case we have a false sentence at the present world of 2017, because it has never been necessary in the history of Eurovision contests to be Portuguese for winning the competition; while, in the second case, we can have a true sentence in 2017 because it can be said at every possible world of Salvador Sobral that he is Portuguese.

Traditionally, a *de dicto* reading is a modal interpretation of the predication of terms, where the designation of the term has to be found at every possible world where the formula is evaluated; the designation of the terms, thus, can change from one world to another. The predication is done of the *dictum*, i.e., of the expression, in our case of "the winner of the current Eurovision contest", whose denotation is different depending on the world. However, a *de re* reading takes, first of all, the denotation of the term at the world of evaluation, individuating an entity and, then, predicating about this concrete object, or *rem*, at possibly different worlds. In our case we had to interpret "the winner of the current Eurovision contest" at our world of evaluation (the present time) and then to check if this concrete individual can receive the property of "being Portuguese" at different worlds. In a *de dicto* reading we have to check if it is true *that* "the winner of the current Eurovision contest is Portuguese" at certain worlds; while in a *de re* reading, we check the truthfulness at certain worlds if the "being Portuguese" predicate can be applied *of* "the winner of the current Eurovision contest" as individuated at the world of evaluation (Salvador Sobral in 2017).

We know (from Fitting and Mendelsohn (1998)) that predicate abstracts allow us to differentiate between *de dicto* and *de re* readings. If we take the previous example, the *de dicto* reading would be formalized as:

(λ-*dictum*) $\Box\langle\lambda z_\iota.S(z_\iota)\rangle(z_{\iota_1})$.

And the *de re* reading as:

(λ-*rem*) $\langle\lambda z_\iota.\Box S(z_\iota)\rangle(z_{\iota_1})$.

Let us see how these formulas are evaluated:

(λ-*dictum*) Formula $\Box\langle\lambda z_\iota.S(z_\iota)\rangle(z_{\iota_1})$ is true at world w of a model $\mathcal{M}$ using assignment g if and only if $\langle\lambda z_\iota.S(z_\iota)\rangle(z_{\iota_1})$ is true at any world w' of a model $\mathcal{M}$ accessible from w and, if this is the case, if the object denoted by z_{ι_1} at each w' belongs to the interpretation of S at the same w'. Therefore, we first go to each of the worlds where the formula has to be evaluated and, there, we pick up the extensions of both, the intensional term and the predicate and, finally, check if the predicate applies to the term.

(λ-*rem*) For the formula $\langle \lambda z_\iota . \Box S(z_\iota)\rangle(z_{\iota_1})$ to be true at w, we begin individuating the object denoted by z_{ι_1} at w, and once we have determined the thing (*rem*), we check if at any w' accessible from w the property S at w' can be attributed to the object individuated at the beginning of the evaluation.

In the particular case of our example about the necessity that "the winner of the current Eurovision contest is Portuguese", the formula $\Box\langle \lambda z_\iota . S(z_\iota)\rangle(z_{\iota_1})$ is false at our world of 2017, while $\langle \lambda z_\iota . \Box S(z_\iota)\rangle(z_{\iota_1})$ is true at our present world of 2017.

We ask now, if hybrid operators are also adequate as a new way for solving the problem of the ambiguity between *de dicto* and *de re* readings, in line with the performance of predicate abstracts, without losing any kind of expressiveness. In this case, the use of rigidified terms and the $\downarrow$ operator adds up to remove the ambiguities. In the first place, we present the hybridized version of a the *de dicto* reading:

($\downarrow$-*dictum*) $\Box{\downarrow}aS(@_a z_{\iota_1})$.

And, in the second place, a version of the *de re* reading:

($\downarrow$-*rem*) ${\downarrow}a\Box S(@_a z_{\iota_1})$.

Now, we explain, informally, how the evaluation works:

($\downarrow$-*dictum*) The formula $\Box{\downarrow}aS(@_a z_{\iota_1})$ is true at w if and only if the formula ${\downarrow}aS(@_a z_{\iota_1})$ is true at any w' accessible from w. To evaluate ${\downarrow}aS(@_a z_{\iota_1})$ at each w', we first concentrate in the $\downarrow$ operator. This operator fixes the nominal which bounds to the world where the formula is being evaluated, suppose w', then ${\downarrow}a$ says that the nominal a has to be interpreted not as the world it names in "normal" circumstances, but as w'. Consequently, the individual determined by $@_a z_{\iota_1}$ is the value of the intension denoted by z_{ι_1} when it takes as argument w'. And, finally, as a result, the object individuated by the extensionalized intension at each w' has to belong to the extension of the predicate at each w' for making the formula true.

($\downarrow$-*rem*) The formula ${\downarrow}a\Box S(@_a z_{\iota_1})$ is true at w, when the object determined by z_{ι_1} at w belongs to each of the interpretations of S at any w' accessible from w

In the particular case of our example about the Eurovision contest, the formula $\Box{\downarrow}aS(@_a z_{\iota_1})$ is false at our present world, but ${\downarrow}a\Box S(@_a z_{\iota_1})$ is true at our world of 2017.

Example 2.3.12. Let us now consider a traditional example like this: "the number of planets is necessarily greater than 7", and see how predicate abstraction with λ, on the one side, and the $\downarrow$ operator with nominals, on the other side, work in order to distinguish a *de dicto* reading from a *de re* one.

Via predicate abstracts, the proposition "the number of planets is necessarily greater than 7" is disambiguated in the following way:

- *De dicto*

$$\Box\langle\lambda x_\iota.P(x_\iota)\rangle(x_{\iota_1})$$

- *De re*

$$\langle\lambda x_\iota.\Box P(x_\iota)\rangle(x_{\iota_1})$$

As we have seen again, predicate abstracts and (intensional) terms together are very useful for removing ambiguities. In Fitting and Mendelsohn (1998) this procedure is explained thoroughly and it is also considered by Fitting (2004). But we do not necessarily need this machinery to distinguish formally between such readings.

We can also use rigidified terms and the $\downarrow$ operator, as in Braüner (2008). This hybrid path shows us that we can also get a successful disambiguation mechanism through it:

- *De dicto*

$$\Box{\downarrow}aP(@_a x_{\iota_1})$$

- *De re*

$${\downarrow}a\Box P(@_a x_{\iota_1})$$

$\Box{\downarrow}aP(@_a x_{\iota_1})$ could be translated by "it is necessary that the number of planet is greater than 7" and ${\downarrow}a\Box P(@_a x_{\iota_1})$ by "the number designated by the term 'the number of planets' is necessarily greater than 7". In the first case we go to every world, then determine the designation of the term "the number of planets" there and then evaluate the application of the interpreted predicate "is greater than 7" to the designated object. That shows that the sentence "it is necessary that the number of planet is greater than 7" is false, because there can be worlds where the term "the number of planets" designate a number lower than 7 or nothing at all. In the second case, we try to find the object designated by x_{ι_1} at the world of evaluation and, then, we carry the same value to all the worlds and, in all of them, we check if the interpreted predicate "is greater than 7" can be applied successfully to the designated object. Under this reading the proposition would be true because 8 or 9 (it depends on the last published paper on astrophysics) is necessarily greater than 7.

We proceed now to give a more formal account of the evaluation of the previous formulas at a world w of a given model under an assignment g. Let us suppose that we have a planetary model where the domain contains the planets of the Solar System. We have also two worlds: w_1, the telescopic-world, which is supposed to be our current world where we have telescopes for studying the celestial bodies; and w_2, the naked-eye-world, which is the ancient Babylonian world where celestial bodies were identified only by means of the sense of sight. At w_1, the planets are considered to be 8. At w_2, the planets are 5, provided that the Sun and the Moon are not taken into account. Moreover, the accessibility relation between the two worlds is an equivalence relation; and the description "the number of planets" designates at each world.

The evaluation of $\Box\langle\lambda x_\iota.P(x_\iota)\rangle(x_{\iota_1})$ at w_1 is as follows:

$$[\![\Box\langle\lambda x_\iota.P(x_\iota)\rangle(x_{\iota_1})]\!]^{\mathcal{M},g,w_1} = T \text{ if } [\![\langle\lambda x_\iota.P(x_\iota)\rangle(x_{\iota_1})]\!]^{\mathcal{M},g,w'} = T$$

for all $w' \in W$ such that $w_1 R w'$. In this case w' can be w_1 or w_2. At w_1, $[\![\langle\lambda x_\iota.P(x_\iota)\rangle(x_{\iota_1})]\!]^{\mathcal{M},g,w_1} = T$, provided $[\![P(x_\iota)]\!]^{\mathcal{M},g',w_1} = T$, where g' is the x_ι-variant of g, such that $g'(x_\iota) = g(x_{\iota_1})(w_1)$, and that is number 8. As the interpretation of the predicate P at w_1 is the set of numbers greater than 7, and $8 > 7$, then $\langle\lambda x_\iota.P(x_\iota)\rangle(x_{\iota_1})$ is true at w_1. At w_2, $[\![\langle\lambda x_\iota.P(x_\iota)\rangle(x_{\iota_1})]\!]^{\mathcal{M},g,w_2} = T$, provided $[\![P(x_\iota)]\!]^{\mathcal{M},g',w_2} = T$, where g' is the x_ι-variant of g, such that $g'(x_\iota) = g(x_{\iota_1})(w_2)$, which is the number 5. As the interpretation of the predicate P at w_2 is the same set we have at w_1 formed by the numbers greater than 7, and 5 is not greater than 7, then $\langle\lambda x_\iota.P(x_\iota)\rangle(x_{\iota_1})$ is false at w_2. And therefore $\Box\langle\lambda x_\iota.P(x_\iota)\rangle(x_{\iota_1})$ is false at w_1. The evaluation at w_2 is similar to the previous one and gives us also that $\Box\langle\lambda x_\iota.P(x_\iota)\rangle(x_{\iota_1})$ is false at w_2.

The evaluation of $\Box{\downarrow}aP(@_a x_{\iota_1})$ at w_1 is as follows:

$$[\![\Box{\downarrow}aP(@_a x_{\iota_1})]\!]^{\mathcal{M},g,w_1} = T \text{ if } [\![{\downarrow}aP(@_a x_{\iota_1})]\!]^{\mathcal{M},g,w'} = T$$

for all $w' \in W$ such that $w_1 R w'$. In this case w' can be w_1 or w_2. At w_2, $[\![{\downarrow}\, aP(@_a x_{\iota_1})]\!]^{\mathcal{M},g,w_2} = T$ if $[\![P(@_a x_{\iota_1})]\!]^{\mathcal{M},g',w_2} = T$, where g' is an a-variant of g and $g'(a) = w_2$; if $(\mathsf{F}(P)(w_2))([\![@_a x_{\iota_1}]\!]^{\mathcal{M},g',w_2}) = T$. Given that $[\![@_a x_{\iota_1}]\!]^{\mathcal{M},g',w_2} = g'(x_{\iota_1})(g'(a)) = g(x_{\iota_1})(w_2)$, we have that $@_a x_{\iota_1}$ designates the number 5 at w_2. The interpretation of the predicate P at w_2, $\mathsf{F}(P)(w_2)$, is the set of the numbers greater than 7. But as it is false that 5 is greater than 7, then ${\downarrow}aP(@_a x_{\iota_1})$ is false at w_2. And therefore $\Box{\downarrow}aP(@_a x_{\iota_1})$ is false at w_1.

We evaluate now $\langle\lambda x_\iota.\Box P(x_\iota)\rangle(x_{\iota_1})$ at w_1:

$$[\![\langle\lambda x_\iota.\Box P(x_\iota)\rangle(x_{\iota_1})]\!]^{\mathcal{M},g,w_1} = T \text{ if } [\![\Box P(x_\iota)]\!]^{\mathcal{M},g',w_1} = T$$

where g' is the x_ι-variant of g, such that $g'(x_\iota) = g(x_{\iota_1})(w_1)$. We know that the value of $g'(x_\iota) = g(x_{\iota_1})(w_1)$ and it is fixed to be 8. Then $[\![\Box P(x_\iota)]\!]^{\mathcal{M},g',w_1} = T$ if $[\![P(x_\iota)]\!]^{\mathcal{M},g',w'} = T$, for all $w' \in W$ such that $w_1 R w'$. w' can be w_1 or w_2. If it is w_1, then $[\![P(x_\iota)]\!]^{\mathcal{M},g',w_1} = T$, provided $(\mathsf{F}(P)(w_1))(g'(x_\iota)) = T$. The interpretation of P at w_1 is $\mathsf{F}(P)(w_1)$ which denotes the set of the numbers greater than 7. We know also the value of $g'(x_\iota)$: it is 8. As the property of being greater than 7 applies to 8, then $P(x_\iota)$ is true at w_1. If w' is w_2, then $[\![P(x_\iota)]\!]^{\mathcal{M},g',w_2} = T$, and so $(\mathsf{F}(P)(w_2))(g'(x_\iota)) = T$. The interpretation of P at w_2 is $\mathsf{F}(P)(w_2)$ which denotes the set of the numbers greater than 7. We know also the value of $g'(x_\iota)$: it is 8. As the property of being greater than 7 applies to 8, then $P(x_\iota)$ is true at w_2. Consequently, $\langle \lambda x_\iota . \Box P(x_\iota)\rangle(x_{\iota_1})$ is true at w_1.

Finally, we evaluate $\downarrow a \Box P(@_a x_{\iota_1})$ at w_1:

$$[\![\downarrow a \Box P(@_a x_{\iota_1})]\!]^{\mathcal{M},g,w_1} = T \text{ if } [\![\Box P(@_a x_{\iota_1})]\!]^{\mathcal{M},g',w_1} = T$$

where g' is an a-variant of g and $g'(a) = w_1$. And so the value of the nominal a is fixed to be the name of w_1. Then, $[\![\Box P(@_a x_{\iota_1})]\!]^{\mathcal{M},g',w_1} = T$ provided $[\![P(@_a x_{\iota_1})]\!]^{\mathcal{M},g',w'} = T$, for all $w' \in W$ such that $w_1 R w'$. w' can be w_1 or w_2. If it is w_1, then $[\![P(@_a x_{\iota_1})]\!]^{\mathcal{M},g',w_1} = T$, provided $(\mathsf{F}(P)(w_1))([\![@_a x_{\iota_1}]\!]^{\mathcal{M},g',w_1}) = T$. We have that $[\![@_a x_{\iota_1}]\!]^{\mathcal{M},g',w_1} = g'(x_{\iota_1})(g'(a)) = g(x_{\iota_1})(w_1)$, which has as value 8. As it is true at w_1 that 8 is greater than 7, then the formula $P(@_a x_{\iota_1})$ is true at w_1. If w' is w_2, then $[\![P(@_a x_{\iota_1})]\!]^{\mathcal{M},g',w_2} = T$, provided $(\mathsf{F}(P)(w_2))([\![@_a x_{\iota_1}]\!]^{\mathcal{M},g',w_2}) = T$. We have that $[\![@_a x_{\iota_1}]\!]^{\mathcal{M},g',w_2} = g'(x_{\iota_1})(g'(a)) = g(x_{\iota_1})(w_1)$, which has as value 8. As it is true at w_2 that 8—the denotation of the number of planets at w_1—is greater than 7—the extension of P at w_2, then the formula $P(@_a x_{\iota_1})$ is true at w_2. Consequently, $\downarrow a \Box P(@_a x_{\iota_1})$ is true at w_1.
To sum up, in each one of the cases, either with predicate abstracts or with hybrid operators, we have got the same result: the *de dicto* versions of the formulas are false at w_1 and the *de re* readings of the formulas are true at w_1.

In conclusion, as we have seen, predicate abstracts and hybrid operators are both good mechanisms in order to differentiate between readings *de dicto* and *de re*. Both are also powerful enough for offering intensional predication, but we have only got a fully understanding of the difference in scope with respect to negation through the use of predicate abstracts.

Chapter 3

Intensional Hybrid Type Theory

3.1 Introduction

In the present chapter we present an Intensional Hybrid Type Theory.[1] We study the syntax and the semantics, and we offer also a selection of those types which can be considered more interesting for formalizing the expressions of natural language (section 3.2). Furthermore, our Intensional Hybrid Type Theory, since it includes not only intensional types but also extensional types, can be used as a common framework for the study of contexts where intensional expressions are the essential ones (empirical contexts) and also for the study of those contexts (non-empirical contexts) where extensional expressions, such as mathematical statements, are the basics (section 3.3).

Our chapter also includes two miscellaneous discussions that we have not been able to examine before: descriptions and identity of senses, both crucial in any significant analysis about intensional logic. In section 3.4 we focus on the two main theories of descriptions: Russell's and Frege's theory. We present our rationale for choosing Frege's theory and offer a solution to the problem of non-denoting terms based on partial functions within a bivalent logic. In our last section (section 3.5) we establish that, although identity of intensions, intended as functions from worlds to objects, is a necessary condition for the identity of senses, it is not a sufficient condition. In epistemic contexts identity of intensions is too loose. We need a more fine-grained condition which allow us to go beyond intensions and, so, we grasp a new notion: *hyperintensions*. Synonymous isomorphism (Church), constructions

[1]See a previous formulation of a Hybrid Type Theory in (Areces, Blackburn, Huertas, & Manzano, 2014).

(Tichý) and algorithms (Moschovakis) are the possible candidates we present for giving a successful account of these hyperintensions.

3.2 Intensional Hybrid Type Theory

3.2.1 Syntax

Types

In the previous chapter we have occasionally used for the types the symbols ι, o and, σ. They remember respectively the symbols e (for *entity*), t (for *truth value*) and s (for *sense*) of Montague's type theory. In our intensional hybrid type theory, we give preference to the notation defined in the previous chapter where the symbols ι, o and, the subscript 1—instead of σ—were used.

Definition 3.2.1 (Type)**.** The notion of a type, either extensional or intensional, is defined inductively by the following rules:

1. Functional Types:
$$\iota \mid \iota_1 \mid \langle (t^1 \dots t^n)t \rangle \mid r_1$$
t is either ι or ι_1; $t^1, \dots, t^n, r$ are functional types. Intensional types end in ι_1 and extensional types end in ι.

2. Propositional types:
$$o \mid o_1 \mid \langle p^1 p^2 \rangle$$
p^1, p^2, are propositional types. They are intensional if they end in o_1, and extensional if they end in o.

3. Predicative Types:
$$\langle tp \rangle \mid \langle \pi p \rangle$$
t is a functional type; p is a basic propositional type (o or o_1); π is a predicative type.

In order to avoid equivocal interpretations of how intensional types are formed, mainly with respect to complex functional types, we follow the convention explained in section 2.4 on page 95. As a result, the intension of a functional type is internalized and remains with the symbol situated more to the right.

Definition 3.2.2 (Abbreviation of Functional Intensional Types)**.** If $\langle (t^1 \dots t^n)t \rangle$ is a functional type, then the intensional type r_1 is $\langle (t^1 \dots t^n)t \rangle_1$, which is abbreviated as $\langle (t^1 \dots t^n)t_1 \rangle$.

Chapter 4

Existence

4.1 Introduction

In the previous chapters we have been creating tools, materialized as formal languages, for dealing with a great amount of expressions of natural language. In the present chapter we want to put into practice these tools for shedding light on some philosophical problems. Of all the philosophical problems, we have chosen one that has been linked to formal logic in a very special way. We are referring to the ontological arguments. Much has been said in the field of logic about classical ontological arguments across the twentieth century:[1] there have been formal accounts mainly of the ontological argument(s) of St. Anselm of Canterbury[2] but also of those of Descartes and Spinoza.[3] But in the last quarter of the past century and in our current century one argument has attracted increasing attention: the ontological argument of Kurt Gödel[4] (section 4.3).

As every ontological argument, the purpose of Gödel's proof is to arrive to a demonstration of the existence of God starting only from *a priori* premises, that is, from premises which do not suppose the existence of the world and of the properties related to it. But just as in every argument, although the rules are sound, it is possible to discuss the premises and the definitions to see if they can be considered true or if they assume too much with respect to the conclusion. Our aim for analyzing Gödel's argument has not been

[1]See, for example, our exposition in (Manzano & Moreno, 2010).

[2]See, for example, (Hartshorne, 1965), (Malcolm, 1960), (Plantinga, 1967, 1974), (Lewis, 1970), (Adams, 1971), (Oppenheimer & Zalta, 1991, 2011) and (Dombrowski, 2006).

[3]For both arguments see (Sobel, 2004, pp. 29–80).

[4]See, for example, (Anderson, 1990), (Hazen, 1998), (Oppy, 1996), (Sobel, 1987, 2004), and (Fitting, 2002).

to study the proof theory behind it, but to look into its philosophical notions: positive property, Godlike property, essence and necessary existence. Gödel's axioms and definitions provide us with a rigorous approach to this philosophical notions, and we have found this approach tremendously appealing. Apart from examining these notions in the argument of Gödel, we have also reconstructed it from a novel point of view which considers god not as an individual object but as an individual concept. As a result, we have recomposed all the properties in the argument, giving to them the option of offering intensional predication, i.e., to predicate of intensions, and not of the objects designated by the intensions. The conclusion of the proof is not so powerful, but it is more credible: we do not conclude that it is necessary that exists a being which is Godlike. We humbly conclude that it is necessary that god is the being which has all positive properties.

In order to achieve a balance between theists and atheists, we have also presented a peculiar argument from a Bishop of the seventeenth century, Juan Caramuel, who offered an argument against the existence of God (section 4.4). As the premises are *a priori*, we have considered it as an ontological argument, in spite of the fact that the expression "ontological argument" is reserved, from Kant onwards, to the arguments for, and not against, the existence of God. Although the deduction rules of the proof are simple, we have offered a more complicated formalization of the axioms of the argument as a way of giving a philosophical use to our intensional hybrid language. The conclusion is very strong, it says that it is impossible that god, an individual concept, designates at any world. But there is some tricky axiom, denying god's contingency, that assumes too much.

In any case, our purpose by offering an analysis of the ontological arguments, has not been to change the convictions of anybody. Unfortunately, they are not going to change the life style and the way of thinking of the reader. Our main objective has been to deal with philosophical problems in a formal way. We claim that the intensional hybrid languages we have presented give us enough machinery for making a novel approach to philosophy. That is the main goal of this chapter. To build a bridge for dealing with philosophical notions with a formal method. This can be considered a contribution to a *formal ontology*, where rigorous definitions of concepts such as denotation or existence are given. And that is just what we try to do at the beginning of this chapter (section 4.2).

4.3 Gödel's Ontological Proof

The first traces of an interest of Gödel for the ontological proof date back to 1944 (Adams, 1995, p. 388), although it was not until 1970 (Gödel, 1995, pp. 403–404) that Gödel showed privately to Dana Scott his work on a proof. Scott wrote a copy of it and this copy was around in private until it was published by Sobel (1987). The proof, as can be seen in Gödel (1995, 403–404), is schematic and written in a formal notation, unlike the classical ontological arguments that were written in an informal way. In its notation Gödel makes use of quantification of properties, of predicate constants which are properties of properties, and he also uses modal operators; therefore, the formalization assumes a higher-order modal logic.

Now, we are going to put forward the proof of Gödel and present each of the axioms, definitions and propositions in a three-level structure: firstly, we begin giving an exposition in ordinary language; secondly, we present a formalization which is close to the notes of Gödel (this is item (a), where an actualist quantification is assumed); thirdly, we give a formal version in the language of our Intensional Hybrid Type Theory (this is item (b)). We have taken the first two levels from Fitting (2002, pp. 140–161).

4.3.1 Axioms

Gödel's ontological proof is based on the following five axioms:

Axiom 4.3.1. Either a given property is positive or its contrary is, that is, exactly one of a property or its complement is positive.

(a) $\forall X(\mathcal{P}(\neg X) \leftrightarrow \neg\mathcal{P}(X))$

(b) $\forall X(\mathcal{P}(\langle\lambda x.\neg X(x)\rangle) \leftrightarrow \neg\mathcal{P}(\langle\lambda x.X(x)\rangle))$
where x is an individual variable of type ι, X is a predicate variable of type $\langle\iota o_1\rangle$, $\langle\lambda x.\neg X(x)\rangle$ and $\langle\lambda x.X(x)\rangle$ are predicate abstracts of type $\langle\iota o_1\rangle$, and $\mathcal{P}$ is a predicate of predicates of type $\langle\langle\iota o_1\rangle o_1\rangle$. $\mathcal{P}$ is a primitive predicate representing *positiveness* and it is an intensional predicate of intensional predicates.

Axiom 4.3.2. Any property entailed by a positive property is positive.

(a) $\forall X\forall Y[\mathcal{P}(X) \wedge \Box\forall x[X(x) \rightarrow Y(x)] \rightarrow \mathcal{P}(Y)]$

(b) $\forall X\forall Y[\mathcal{P}(X) \wedge \Box\forall^{\mathbf{E}} x[X(x) \rightarrow Y(x)] \rightarrow \mathcal{P}(Y)]$
where Y is a predicate variable of type $\langle\iota o_1\rangle$ and $\forall^{\mathbf{E}} x[X(x) \rightarrow Y(x)]$ is an abbreviation for the formula $\forall x[\mathbf{E}(x) \rightarrow (X(x) \rightarrow Y(x))]$. $\forall^{\mathbf{E}} x$

is a relativized quantifier which range over the objects located at the world where the formula is evaluated. It has been introduced in order to express actualist quantification, which is the quantification we presume Gödel had, in a constant domain model as ours. Remenber that $\mathbf{E}(x)$ is a primitive predicate of type $\langle \iota o_1 \rangle$ which is true at a world w of the objects actually located at w.

Axiom 4.3.3. The conjunction of any collection of positive properties is positive.

(a) $\forall X \forall Y[\mathcal{P}(X) \land \mathcal{P}(Y) \to \mathcal{P}(X \land Y)]$

(b) $\forall X \forall Y[\mathcal{P}(X) \land \mathcal{P}(Y) \to \mathcal{P}(\langle \lambda x.(X(x) \land Y(x)) \rangle)]$
where $\langle \lambda x.(X(x) \land Y(x)) \rangle$ is a predicate abstract of type $\langle \iota o_1 \rangle$.

Axiom 4.3.4. Any positive property is necessarily positive.

(a) $\forall X[\mathcal{P}(X) \to \Box \mathcal{P}(X)]$

(b) $\forall X[\mathcal{P}(X) \to \Box \mathcal{P}(X)]$
in this case we have not done any modifications.

Axiom 4.3.5. Necessary existence is a positive property.

(a) $\mathcal{P}(N)$

(b) $\mathcal{P}(N)$
where N is a property of individuals of type $\langle \iota o_1 \rangle$ which stands for necessary existence.

4.3.2 Definitions

In mathematics, axioms and theorems are considered to be essential for the advance of knowledge, but theorems sometimes appear as mechanical procedures for arriving to a conclusion. Axioms are always considered the foundations of a theory but: what about definitions? A lot of creativity is put in definitions, they provide us with the basic elements that can be manipulated in proofs, and you have to choose well your definitions in order to get proofs which are informative enough to get an interesting result. Let us see what are these creative elements of Gödel's proof.

Definition 4.3.6 (Godlike)**.** A Godlike being is any being that has every positive property.

(a) $\forall x[G(x) \leftrightarrow \forall X[\mathcal{P}(X) \to X(x)]]$

(b) G is the abbreviation of the following term:

$$G := \langle \lambda x. \forall X(\mathcal{P}(X) \to X(x)) \rangle$$

"Godlike" is considered to be a predicate of individuals and it has type $\langle \iota o_1 \rangle$, and it is defined as a being which has every positive property. A definition which is very similar to those definitions of Descartes and Leibniz although they preferred *perfections* instead of positive properties.

Definition 4.3.7 (Essence)**.** The essence of an object is a property that the object possesses and that entails every property that the object has.

(a) $\forall X \forall x[X \text{ Ess } x \leftrightarrow \forall Y(Y(x) \to \Box \forall y[X(y) \to Y(y)])]$

(b) Σ is the abbreviation of the following term:

$$\Sigma := \langle \lambda X, x.(X(x) \wedge \forall Y[Y(x) \to \Box \forall^{\mathbf{E}} y(X(y) \to Y(y))]) \rangle$$

Σ is a binary predicate of type $\langle (\langle \iota o_1 \rangle \iota) o_1 \rangle$ which takes two arguments: the first argument is a predicate of type $\langle \iota o_1 \rangle$ and the second is an individual of type ι. Then, $\Sigma(X, x)$ is read as X is the essence of x. Note that in (a) does not appear the condition that an object has to possess its own essential property, $X(x)$, the reason is that in Gödel's notes it does not appear either. Scott thinks that it was an slip and adds in his notes the formula $X(x)$ to the definition of essence: $\forall X \forall x[X \text{ Ess } x \leftrightarrow X(x) \wedge \forall Y(Y(x) \to \Box \forall y[X(y) \to Y(y)])]$ (Sobel, 1987, pp. 256–257).

Definition 4.3.8 (Necessary Existence)**.** An object has the property of necessarily existing if its essence is necessarily instantiated.

(a) $\forall x[N(x) \leftrightarrow \forall X[X \text{ Ess } x \to \Box \exists x X(x)]]$

(b) N is the abbreviation of the following term:

$$N := \langle \lambda x. \forall X[\Sigma(X, x) \to \Box \exists^{\mathbf{E}} x X(x)] \rangle$$

N is a predicate of type $\langle \iota o_1 \rangle$ and applies to individual objects. $\exists^{\mathbf{E}} x X(x)$ is a formula with a relativized quantifier which is an abbreviation for $\exists x(\mathbf{E}(x) \wedge X(x))$.

4.3.3 Propositions

From the previous axioms and definitions we can get the following propositions:

Proposition 4.3.9. *From axioms 4.3.1 and 4.3.2 it can be proved that any positive property is possibly instantiated.*

(a) $\forall X[\mathcal{P}(X) \rightarrow \Diamond\exists x X(x)]$

(b) $\forall X[\mathcal{P}(X) \rightarrow \Diamond\exists^{\mathbf{E}} x X(x)]$

Proposition 4.3.10. *From axiom 4.3.3 and definition 4.3.6 we get that to be a Godlike being is a positive property.*

(a) $\mathcal{P}(G)$

(b) $\mathcal{P}(G)$

Proposition 4.3.11. *From axioms 4.3.1, 4.3.2 and 4.3.3, and definition 4.3.6, we obtain that it is possible that a Godlike being exists.*

(a) $\Diamond\exists x G(x)$

(b) $\Diamond\exists^{\mathbf{E}} x G(x)$

Proposition 4.3.12. *From axiom 4.3.1, axiom 4.3.4 and definition 4.3.7 it can be proved that any being which is Godlike has as its essence the property of being a Godlike being.*

(a) $\forall x[G(x) \rightarrow G \text{ } Ess \text{ } x]$

(b) $\forall^{\mathbf{E}} x[G(x) \rightarrow \Sigma(G, x)]$

Proposition 4.3.13. *From axioms 4.3.1, 4.3.4 and 4.3.5 we get, assuming the logic is* **K** *logic, that if there is at least one being which is Godlike, then this being exists necessarily.*

(a) $\exists x G(x) \rightarrow \Box\exists x G(x)$

(b) $\exists x G(x) \rightarrow \Box\exists^{\mathbf{E}} x G(x)$

Assume the previous axioms, and assume also the logic is **S5***, then it can be proved that if it is possible that there is at least one being which is Godlike, then this being exists necessarily.*

(a) $\Diamond\exists x G(x) \rightarrow \Box\exists x G(x)$

(b) $\Diamond \exists x G(x) \rightarrow \Box \exists^{\mathbf{E}} x G(x)$

Note that the quantifiers in the antecedent is not relativized. The reason is that the necessary actualist existence of a being which is Godlike follows from his possibilist existence.

Proposition 4.3.14. *Assume the logic is* **S5***, then from propositions 4.3.11 and 4.3.13, we conclude that it is necessary that there is at least one being which is Godlike.*

(a) $\Box \exists x G(x)$

(b) $\Box \exists^{\mathbf{E}} x G(x)$

Here finishes Gödel's ontological proof.

4.3.4 Favorable Consequences of Gödel's Proof

Assuming the previous axioms, definitions and propositions we can arrive to other results:

Proposition 4.3.15. *Any individual can only have one essence.*

$$\forall X \forall Y \forall z [\Sigma(X, z) \wedge \Sigma(Y, z) \rightarrow \Box \forall^{\mathbf{E}} w [X(w) \leftrightarrow Y(w)]]$$

Proposition 4.3.16. *An essence is a complete characterization of an individual, that is, one individual can not have the essence of another individual as property.*

$$\forall X \forall y [\Sigma(X, y) \rightarrow \Box \forall^{\mathbf{E}} z (X(z) \rightarrow y = z)]$$

Propositions 4.3.15 and 4.3.16 can be found on the notes of Dana Scott about the proof (Sobel, 2004, p. 146). They can be proved assuming the logic is **K**. From them we can also get that it is not possible that there are two Godlike beings, that is, two distinct objects having the property of being Godlike.

Proposition 4.3.17. *There is exactly one Godlike being.*

$$\exists x \forall y (G(y) \leftrightarrow y = x)$$

Consequently, monotheism is guaranteed by Gödel's proof.

4.3.5 Inconvenient Consequences of Gödel's Proof

Sobel presents some results which can be derived from Gödel's proof which turn out to be problematic for the concept of God (Sobel, 2004, pp. 128–135). The objections raised by Sobel on Gödel's proof are based on a generous interpretation of properties. Given that our language has predicate abstract terms, it is possible to build complex predicates through them. And the possibility of constructing these complex predicates is fundamental for arriving to the objections.

Sobel shows that from proposition 4.3.14 we can get the following result:

Proposition 4.3.18. *Any positive property is necessarily instantiated.*

$$\forall X[\mathcal{P}(X) \to \Box\exists^{\mathbf{E}} xX(x)]$$

Sobel also shows that

Proposition 4.3.19. *Any property which a Godlike being has is necessarily instantiated by a Godlike being:*

$$\forall^{\mathbf{E}} x[G(x) \to \forall X(X(x) \to \Box\exists^{\mathbf{E}} x[G(x) \wedge X(x)])]$$

And now, assuming also that any individual has an essence:

$$\forall^{E} x\exists X\Sigma(X, x)$$

we can arrive to this problematic conclusion:

Proposition 4.3.20. *All what exists, exists necessarily.*

$$\forall^{\mathbf{E}} xN(x)$$

The informal idea of the proof is that for every existent different of a Godlike being g, g would have the property that *there is an object x different of g which has an essence E*. Given that this complex property, as every other property of a Godlike being, is necessarily instantiated, then the essence E is necessarily instantiated. And so, x exists necessarily.

Proposition 4.3.21. *All what is the case, is necessarily the case.*

$$\phi \to \Box\phi$$

Informally, the demonstration of this theorem is based on that for any truth ϕ, a Godlike being would have the property of *being identical with itself in presence of the truth of ϕ*. This property, as any property of a

Godlike being, would be necessarily instantiated. And, given that there exists necessarily a Godlike being, then every truth is a necessary truth.

From proposition 4.3.21 we can conclude that in a system where the axioms of Gödel are true, the modalities collapse, and so actual existence, possibility and necessity are equivalent.

Proposition 4.3.22.

$$(\phi \leftrightarrow \Diamond\phi) \wedge (\Diamond\phi \leftrightarrow \Box\phi) \wedge (\phi \leftrightarrow \Box\phi)$$

Given that $\phi \rightarrow \Diamond\phi$, $\Box\phi \rightarrow \Diamond\phi$, and $\Box\phi \rightarrow \phi$, are theorems of **S5**, and $\phi \rightarrow \Box\phi$ is just proposition 4.3.21, we would only need to prove $\Diamond\phi \rightarrow \phi$ and $\Diamond\phi \rightarrow \Box\phi$.

4.3.6 Open Questions

Gödel's proof is a correct proof and it has been developed by Fitting (2002) in a tableaux calculus and by Sobel (2004) in a natural deduction calculus. Our main question then is not about the correctness of Gödel's proof, what we pretend to discuss below is how Gödel's axioms and definitions can be understood from our own point of view concerning intensions, positive properties, existence and essences. Let us remember that to have a correct demonstration is not a sufficient condition for accepting a conclusion, given that the axioms should also be accepted.

In what follows we will analyze each basic notion of Gödel's axioms and definitions from a critical point of view. In particular the properties of positiveness, Godlike, essence and necessary existence. We conclude that the meaning of the axioms and definitions is much more coherent when variables are considered to be intensional and predicates offer an intensional predication (that is, a predication of intensions).

Positive Properties

In the language of the proof we had a predicate of predicates, $\mathcal{P}$, which has been used to represent the concept of "positive" or "positiveness". Leibniz in his work *Quod ens perfectissimum existit* of 1676 (Gerhardt, 1978, pp. 261–262) used the concept of "perfection" in his proof, Gödel, however, prefers to talk about *positive property* (1995, pp. 403–404). Gödel does not offer a detailed analysis about what a positive property means, he only refers in a note that positive has to be intended in "the moral aesthetic sense (independently of the accidental structure of the world)". He also indicates that it would be possible another reading of positive, meaning "pure 'attribution' as opposed

Conclusion

Intensional Predication

The main result of the present dissertation has been the differentiation between two ways of predication based on a same intensional predicate. We have claimed that an intensional predicate, which is supposed to change possibly its extension at any world of a given model, can be said of extensional terms, but it also can be applied to intensional terms. Predication has been centered, more often than not, in extensional terms and intensional predication has not received the attention it merits. With intensional predication we do not refer to predication of intensional terms that are extensionalized later, as when we use an intensional term with an intensional predicate in a sentence like "the Pope is a nice man". In this case, the intensional term "the Pope" is not used in order to designate the individual concept of the Pope—which is a function from worlds to individuals—, but it is used as a way of determining an individual person at a concrete world. The property of being a nice man is then attributed to a particular person at a given world, and not to a concept. The predication, thus, is extensional even though there is an intensional term in the sentence.

We have discovered that in order to have a proper intensional predication it is not sufficient to have intensional terms and intensional predicates in a sentence. It is also necessary to indicate when the intensional term denotes an intension and not an extensionalized intension at a given world. In natural language we sometimes recognize this kind of predication straightforwardly. In the sentence "the Pope is a religious concept", we can identify how the property of being a religious concept is applied to the individual concept the Pope. But suppose that a term of a formal language τ stands for "the Pope", the predicate symbol P stands for "is a religious concept" and the predicate symbol Q for "is a nice man". Then, if nothing more is indicated symbolically, in the formulas $Q(\tau)$ and $P(\tau)$ we have no way of distinguishing when the predication is extensional or intensional, and we have no means of telling apart when the intension is used to denote a function or an extensional object.

Type Notation

Notwithstanding, we can not allow any kind of ambiguity between the previous formulas $Q(\tau)$ and $P(\tau)$: neither in the use done of the intension nor in the way that predication is executed. We have introduced a powerful type notation, even in first-order intensional expressions, inspired by the work of Church in the field of intensional logic (Church, 1951), which provides us with the necessary means for removing any ambiguity with regard to the interpretation of intensional terms and to the different readings which derive from the two ways of predication. Predicates are then not only formalized as predicate symbols. They are formalized as predicate symbols of a particular type. They can be either of type $\langle to_1 \rangle$, where t is an extensional type, or of type $\langle t_1 o_1 \rangle$, where t_1 is an intensional type. An intensional first-order monary predicate, for example, can be of type $\langle \iota o_1 \rangle$ or of type $\langle \iota_1 o_1 \rangle$. In the first case the predicate is applied to an extensional term, while in the second case it is applied to an intensional term; and what is behind these two types are extensional predication and intensional predication respectively. In the case of terms, they are not allowed either to be formalized only by a symbol τ, since it would result vague. An individual term τ can be of extensional type ι or of intensional type ι_1. As a result, extensional terms can combine with (monary) predicates of type $\langle \iota o_1 \rangle$ to give an atomic formula where extensional predication is exemplified; and intensional terms can combine with predicates of type $\langle \iota_1 o_1 \rangle$ for obtaining a formula with intensional predication.

Expressions denoting Intensions

Consequently, our intensional languages do not include formal expressions which are *only interpreted* in an intensional way, where the object language is the same as an extensional language even though the semantic rules are not. This is usually done in first-order modal logic where the same symbols of first-order classical logic are taken without any modification and are interpreted in a different way. The same predicate symbol P which had only one extension in first-order logic, suddenly has different extensions at the different worlds of a first-order modal model. In this case, the semantics changes but the formal language, although expanded with new operators, does not. By contrast, we have constructed some intensional languages including expressions which denote intensions, and whose expressions are not only interpreted as intensions. Our First-Order Intensional Hybrid Logic and our Intensional Hybrid Type Theory have expressions that denote intensions as well as extensions. And so, both languages include expressions having as their extension the intensions of other expressions. Therefore, our intensional logic has not only an intensional semantics but also an intensional syntax. Our type notation enables the expressions of our languages to denote extensions as well as intensions.

Double Use of Intensional Terms

Among these expressions denoting intensions we have the terms of our language. Our terms can have as denotation an extension or an intension. A first-order term, for example, can denote an individual object, in case its type were ι, or it can denote an individual concept (an intension), in case its type were ι_1. But a formal language where intensional terms only denote intensions has a limited application. In natural language intensional terms are also used to pick up concrete individuals in different contexts. Therefore we have been looking for a method for "extensionalizing" the intensions at particular situations which were not only a semantic rule but an element of the syntax too. As a result, we have been able to make a double use of the intensional terms of our language with a precise *modus operandi*: as denoting directly an intension or as denoting indirectly an extensional object. This double use of intensional terms in our logic reflects the double use in natural language of expressions such as "the president of the USA", which can denote either a concept or a concrete individual at a given instance of time.

Hybridization for Intensions

We have found in hybrid operators the solution to the problem of extensionalizing the intensions of our language. As a consequence, we have created a First-Order Intensional Hybrid Logic and an Intensional Hybrid Type Theory where we have added hybrid machinery to the classical extensional expressions and to the new intensional ones. Nominals: $a, b, \ldots$, satisfaction operators $@_a$ and the $\downarrow$ operator, make up our hybrid machinery. It has allowed us not only to extensionalize the intensional expressions, but also to increase the expressivity of the languages. The nominals have supplied us with names for any world of the model, the satisfaction operators have given us the context where a certain intensional expression has to be evaluated, and the $\downarrow$ operator has allowed us to fix the nominals of a formula to the world of evaluation. The intensional term, c_{ι_1}, for "the Pope", which denotes the concept of the Pope, if preceded by a satisfaction operator $@_a c_{\iota_1}$ gives us a new term with a new type—it has been metamorphosed in an extensional term (also called a rigidified term) with type ι—which denotes the person who is the Pope at the world named by a.

If we want to evaluate the formula $P_{\langle \iota o_1 \rangle}(@_a c_{\iota_1})$, "the Pope is a nice man", at our present world w_0, and we want also to refer to the actual Pope, we can fix a to our present world by means of $\downarrow$, getting so the formula:

$$\downarrow a\, P_{\langle \iota o_1 \rangle}(@_a c_{\iota_1})$$

De Re **and** ***De Dicto***

The distinction between intensional and extensional predication mirrors the traditional distinction between *de dicto* and *de re* readings. If we understand a *dictum* as a concept and a *rem* as an object, in intensional predication, when a property applies to a concept, we would have a *de dicto* reading and, in extensional predication, when a property applies to an object, we would have a *de re* reading.

The distinction *de dicto* and *de re* has been raised, however, in modal contexts mainly with respect to how modal operators behave in connection with intensional terms. If $P_{\langle \iota o_1 \rangle}$ is a predicate symbol for "is Argentinian", and c_{ι_1} is an intensional individual constant for "the Pope", how should the sentence "necessarily the Pope is Argentinian" be interpreted? Does it state that the actual person ruling the Catholic Church is Argentinian at all possible worlds? Is it maybe saying that at any possible world the person who is the Pope at each of them, has to come from Argentina? These two possible interpretations in modal contexts have motivated our study of the readings *de dicto* and *de re*, and we have shown how predicate abstracts, on the one side, and hybrid operators, on the other side, are different formal mechanisms to solve the problem.

Coherence with the Types

Even though predicate abstracts have success in solving the ambiguity between *de dicto* and *de re* interpretations, however, they have given rise to some problems concerning our type notation in the construction of formulas. Formulas with predicate abstracts in (Fitting & Mendelsohn, 1998) and (Fitting, 2004), when analyzed from the point of view of our type notation, do not fulfill the coherence criteria for a well formed formula, since we have predicate abstracts of type $\langle \iota o_1 \rangle$ applied to intensional constants of type ι_1. It can be said that predicate abstracts are the interface between intensional terms and their extensional counterparts but, in any case, they should build well formed formulas. We have solved the issue with the addition of hybrid machinery which allows to extensionalize the intensional term, devaluing its type to ι, and so the predicate abstract of type $\langle \iota o_1 \rangle$ can be applied to it adequately.

Other incoherences with respect to our type notation have been analyzed too, mainly the one dealing with modal operators. It is common to say that modalities qualify propositions. However, it is not rare to find modal sentences constructed with a modal operator and a first-order formula of extensional type. We maintain that first-order formulas of type o should not be combined with modal operators which require propositions. And propositions, being the intensions of first-order sentences, are not of type o but of type o_1. Consequently, we have observed the rule that, since modal operators qualify propositions, a formula in which modal operators qualify sentences of type o, should not be considered a well formed formula. (Unless a formula of extensional type o is intended as a rigid formula of intensional type o_1.)

Extensional and Intensional Expressions

Anyway, in spite of the problem of having modal operators with extensional expressions, we do not have renounced to having extensional expressions in our formal language. And although in our First-Order Intensional Hybrid Logic all the formulas are of intensional type, in our Intensional Hybrid Type Theory we have intensional and extensional formulas. Both formal languages include extensional and intensional terms, but only the second one (our Intensional Hybrid Type Theory) include extensional predicates as well as intensional predicates. The great variety of expressions in our Intensional Hybrid Type Theory makes it a good candidate for unifying extensional logics, which are proper of non-empirical contexts such as mathematics, and intensional logics, which are more appropriate for the empirical statements that natural language conveys. Our Intensional Hybrid Type Theory is not a purely intensional logic where extensions have to be handled indirectly, it is a kind of unifier between extensional and intensional logic where mathematical statements and natural language statements can be formalized and interpreted.

Extensional and Intensional Terms

We have said before that our First-Order Intensional Hybrid Logic and our Intensional Hybrid Type Theory include extensional and intensional terms. We have considered that extensional or rigid terms, whose interpretation do not vary with the context, are a precise way for formalizing proper names, and that intensional terms are the formal counterpart of definite descriptions. We are aware that there are many problems which can not be solved with this assumption, which, though justified, raises many problem when the possible worlds are interpreted within an epistemic framework and not in an alethic context (Kripke, 1980).

Neither extensional terms nor intensional terms—when denoting intensions—raise denotation problems (with respect to the object domain of the model) . But when intensional terms are extensionalized at a given world through hybrid operators, the problem of non-denoting terms appears. A problem that we have had to solve, since our position regarding definite descriptions has been to take Frege's side—descriptions are proper terms—against Russell's contextual account of descriptions. We have dealt with the interpretation of non-denoting terms through partial functions. We have considered intensions as partial functions which are defined possibly at some worlds and possibly not defined at others. And all the atomic formulas containing a non-denoting terms have been evaluated as false, since we have not allowed to predicate anything true of non-denoting terms.

Furthermore, it turns out that some ambiguities, in connection with the negation of formulas containing non-denoting terms, can appear. Following (Russell & Whitehead, 1925), we have used predicate abstracts as a successful way of differentiating between the range of the scope of the negation symbol in those formulas.

Constant and Varying Domains

Moreover, we have dealt with a common problem in modal logic which has to do with quantification. Although we have studied actualist quantification at the beginning of our dissertation, we have mainly assumed in our formal languages that the quantification is possibilist. The rationale has been that it is easier to manage constants domains instead of varying domains, and that actualist quantification can be defined in constant domain models by means of a primitive predicate $\mathbf{E}$ which is true at a world w of the objects that are actually at w.

Identity of Senses

We have also considered the problem of the identity of the senses of two different expressions. We have studied the rules that classically applied to the relation of identity and analyzed the problems which raise when terms other than variables are taken into account. Our conclusion has been that while identity of intensions provide us with a successful account of *synonymy* in alethic contexts, however it is not successful for dealing with the relation of *synonymy* in epistemic contexts. We have called the latter contexts: hyperintensional. In these hyperintensional contexts a more fine-grained notion than intension is needed. It is not sufficient to reduce senses to functions interpreted in a set theoretical way, where you only have a correspondence between arguments and values. A deeper approach which unveils the procedure of how a function gives a value to a particular argument is needed: the *constructions* of Pavel Tichý or the *algorithms* of Yiannis Moschovakis, are the theories we have put forward. Another theory, which does not reduce senses to intensions, but claiming for an stronger relation for the identity of two senses than the mere logical equivalence has also been presented: it is Church's Alternative (0) with its criterion of *synonymous isomorphism*.

Formal Ontology

In our last chapter our objective has been to "extensionalize" intensional logic, i.e., to apply our abstract formal languages to concrete philosophical problems. We have chosen ontological arguments as a good example to start with. Ontological arguments have had a close relationship with logic in the course of the twentieth century and in our current time. From all the ontological arguments at our disposal we have chosen two: one for proving the existence of God (Gödel's ontological proof), and other against the existence of God (Caramuel's ontological proof). But we have not only presented the arguments as a mere mechanical formalization, we have also reconstructed them by means of two main assumptions: that *god* is an individual concept, not an individual object, and that the predicate of existence applies significantly to individual concepts and not to individual objects. Existence, when applied to objects at a world in a model, is therefore reduced to be a predicate of location, which gives only information about the world(s) where the object is situated.

Denotation issues have also been clarified at different moments of the dissertation, either by means of a monary predicate which applies to intensional terms (section 2.2.4 and section 2.5.6) or by means of a binary relation which relates at a given world an intensional term with its extension (section 4.2.2). Moreover we have given a formal account of the notions of existence and denotation in constant domain models and in varying domain models. In varying domain models, we have formalized the denotation predicate as a monary predicate and the existence predicate as a predicate which can be defined with actualist quantifiers (section 2.2.4). In constant domain models, we have formalized the denotation predicate either as a monary predicate (section 2.5.6) or as a binary relation (section 4.2.2), and the existence predicate as a predicate applied to intensions which needs a primitive predicate of location to be formalized (section 4.2.2).

Through the analysis of ontological arguments we have also arrived to the conclusion that there are many philosophical notions which deserve a precise formal account in order to clarify them. We have shown that this can be done with the help of our formal languages. They are powerful enough to formalize a statement such as "there is nothing".

The formal analysis of philosophical problems has opened our field of study to a great amount of philosophical problems which have not been studied with mathematical methods. The enterprise of building a *formal ontology* is still opened and it would deserve much more attention. At the time when Montague presented his work on a model theoretical semantics for natural languages many logicians saw his attempt as an anathema. Maybe the same can be said about the development of a formal ontology. Philosophy and logic have been looking at each other with suspicion. By contrast, we have offered the analysis of ontological arguments as a link for a mutual approach.

Final Claim

In conclusion, if we had to answer briefly to the question: what are then the conclusions of your dissertation? We would reply that in the present work we claim that the concepts of functions are not only functions of concepts, as Church (1951) supposed. The concepts of predicates of type $\langle \iota o \rangle$ are not only intensional predicates of type $\langle \iota_1 o_1 \rangle$ but also intensional predicates of type $\langle \iota o_1 \rangle$. This simple, but fertile assumption, is at the root of many philosophical problems. Predication has usually been reduced to be a predication of objects and not of intensions, and we have not had at our disposal a type notation for telling apart between them. Thanks to the distinction between intensional and extensional predication some philosophical problems, such as the definition of the predicate of existence or the analysis of the ontological arguments, can be solved with success.

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