

# **Logical Systems:** On the Concept, Expressive Power and Expressiveness Characterizations

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**Logical Systems**  
**On the Concept, Expressive Power and Expressiveness**  
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*À memória de Ivan, Deusdete e Etelvina,  
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# Resumen, introducción y contribuciones de la tesis

## Resumen

Esta tesis es una investigación sobre los conceptos principales ocurriendo en los teoremas tipo-Lindström, esto es, el concepto de sistema lógico y el concepto de expresividad. Lindström, entre otros resultados similares, caracterizó la lógica de primer orden como siendo máximamente expresiva entre las lógicas que tienen compacidad y la propiedad de Löwenheim-Skolem. Para tal, él tenía que dar una definición precisa de qué es una lógica y qué es una relación de expresividad. Tales resultados suelen ser usados para extraer conclusiones fuertes sobre la naturaleza de la lógica y, específicamente, de la lógica de primer orden. No obstante, con la excepción de una discusión inicial en el libro sobre lógicas modelo-teóricas editado por Barwise y Feferman, hay pocas discusiones conceptuales de estos resultados de caracterización en la literatura. Específicamente, nos parece problemática la falta de justificación de por qué cierto concepto de sistema lógico y cierto concepto de expresividad fueran elegidos. Nuestro propósito es de contribuir a esta discusión y también proponer un criterio de expresividad más amplio, por medio de traducciones entre lógicas.

**Palabras-clave:** Teoremas tipo-Lindström, sistemas lógicos, nociones de expresividad, traducciones entre lógicas.

## Introducción

Esta tesis nació de una mezcla de espanto y extrañeza con respecto al teorema de Lindström. En líneas generales, el teorema dice que si una lógica  $\mathcal{L}$  es por lo menos tan expresiva que la lógica de primer orden  $\mathcal{FOL}$  y tiene las propiedades  $P_1, P_2, \dots$ , entonces  $\mathcal{L}$  es tan expresiva cuanto  $\mathcal{FOL}$ . Ese teorema es, por lo tanto, una caracterización de  $\mathcal{FOL}$  respecto a sus extensiones, en términos de expresividad y  $P_1, P_2, \dots$ . Así, ese resultado presupone una definición general de lógica y de expresividad. Además, también es necesaria una justificación de por qué  $P_1, P_2, \dots$  son las propiedades correctas para caracterizar una lógica en términos de expresividad.

En la presentación original de su teorema e sus variantes, Lindström define una “lógica de primer-orden generalizada” como una colección de clases de estructuras modelo-teóricas cerradas bajo algunas operaciones [Lin69], estas clases de estructuras que forman una lógica deben ser entendidas como

una colección de clases de modelos de cada formula en la lógica. Una relación de expresividad que emerge de esta definición es sencillamente una inclusión entre clases de estructuras.

Una vez que esas definiciones fueran dadas “solamente por una cuestión de generalidad” sin más explicación, uno inmediatamente empieza a cuestionar qué exactamente está siendo demostrado con esta caracterización. Dos cuestiones principales naturalmente son:

- Por qué esa definición de lógica?
- Por qué esa definición de expresividad relativa?

Posteriormente, el propio Lindström sintió necesidad de ofrecer más explicaciones para sus resultados y publico un artículo de exposición [Lin74], en donde él definió más cuidadosamente el concepto de una lógica abstracta, dando algunos axiomas, y también diciendo explícitamente la noción de expresividad a ser usada. Por aquél tiempo, algunas críticas y mejoras hechas por Barwise estaban ya circulando y fueron publicadas también en 1974 [Bar74]. Así que nuestra extrañeza no es de todo aislada. También podemos verla en la presentación del dicho teorema por Chang y Keisler [CK90, p. 130] (en donde la definición 2.5.1 es una versión formal de la dada anteriormente para una lógica de primer orden generalizada)

Debería ser enfatizado que lógicas en el sentido de la Definición 2.5.1 tratan de la misma clase de modelos que la lógica de primer orden, y solamente las sentencias y la relación de satisfacción puede ser diferente. Esto es una restricción significativa que deja una grande laguna en el teorema de Lindström. Hay muchos ejemplos de lógicas en un sentido generalizado que estudian modelos con estructura adicional y por lo tanto no encajan en ese marco. Estas incluyen lógicas modales, lógicas de la programación y lógicas para modelos con topologías y medidas. La lógica sentencial y la lógica  $\omega$  como descritas en este libro no son ejemplos de lógicas abstractas en el sentido de 2.5.1, porque ellas también tratan de diferentes clases de modelos respecto a la lógica de primer orden.<sup>1</sup>

En particular, la noción referida de lógica era tan ajena a nosotros que decidimos buscar de donde ha venido. También hemos querido investigar la

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<sup>1</sup>Traducción nuestra.



elección de una específica noción de expresividad, que aparece en el teorema de Lindström y en los varios resultados de caracterización por expresividad que vinieran después. Así, esta tesis está organizada como se sigue: el capítulo 2 trata del concepto de sistema lógico, el capítulo 3 expone diferentes conceptos de expresividad adecuados a cada tipo general de sistema lógico. En el último capítulo exponemos y discutimos los diversos resultados de caracterización de la lógica de primer orden y algunos de sus fragmentos. En el apéndice una exposición detallada del teorema de Lindström es dada.

La literatura sobre el concepto de sistema lógico es escasa, específicamente, solo conocemos un libro dedicado a la cuestión, i.e. [Gab94]. Es una colección de 15 artículos sobre el tema de qué es un sistema lógico. Excepto aquellos en donde solamente una definición rápida de una “lógica” es dada, no hay muchos más trabajos en la literatura. De todos modos, el dicho libro da una buena perspectiva de la noción de lógica, no obstante lo vemos como muy fragmentado, como solamente una colección de las visiones de cada autor sobre la cuestión. Dado esto, decidimos dedicar el capítulo 2 a este tema, y intentamos dar un imagen más coherente, naturalmente manteniendo [Gab94] como un guía.

En el capítulo 2 repasamos los orígenes de la visión general de lógica y los separamos en tres perspectivas amplias: una abstracta (i.e. Tarskiana), una prueba-teórica y una modelo-teórica (con semántica Tarskiana). También hay una perspectiva general interesante que surgió de la teoría de las categorías, que da una noción de sistema lógico indexada por una asignatura. Esta clase de sistemas es construida “nativamente” con en propósito de comparar lógicas. Desafortunadamente, ese tópico está fuera del alcance de esta tesis y el lector es referido a [Mes89] y [MGDT07].<sup>2</sup>

Las semánticas juego-teóricas fueran propuestas como capaces de dar una interpretación mas adecuada para los cuantificadores, respecto a las semánticas Tarskianas [Hin88]. Esta semántica se a convertido en un marco muy rico para la definición de lógicas (un ejemplo notable es dado en [HS89]). No obstante, las lógicas extraídas de las semánticas juego-teóricas también están ausentes en esta tesis.

Veremos que la visión modelo-teórica de la lógica empieza a arraigarse ya tarde en el siglo XX, cerca de los años 1960, a pesar de que los métodos modelo-teóricos ya estuviesen disponibles por algunas décadas. Vemos que

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<sup>2</sup>El lector también es referido a [Cal00], done se puede encontrar definiciones de sistemas de consecuencia, sistemas lógicos y de lógicas. Un sistema lógico es definido como cualquier estructura matemática de la cual un sistema de consecuencia puede ser extraído, y es vinculado a una asignatura. Finalmente una lógica es definida como una familia de sistemas lógicos. Para cada uno de ellos, se define también sus respectivos morfismos.

una mudanza importante en la concepción de lógica fue debida al trabajo de Mostowski con los cuantificadores generalizados [Mos57]. De ahí vino la noción utilizada en la prueba de Lindström. Así, el capítulo presenta un panorama razonablemente amplio sobre el concepto de sistema lógico y su evolución.

En el capítulo 3 revisamos la noción de expresividad utilizada en la prueba de Lindström. La siguiente clasificación de la expresividad relativa entre lógicas será ofrecida. Para lógicas modelo-teóricas, tenemos dos marcos: expresividad uni-clase y expresividad multi-clase. El criterio de expresividad preciso utilizado por Lindström (referido aquí como  $\preceq_{EC}$ ) es definido dentro del primer marco. Aquí  $\preceq_{EC}$  puede ser visto como involucrando traducciones de formulas de lógicas modelo-teóricas definidas dentro de una misma clase de estructuras (de ahí el nombre uni-clase). También analizamos algunos criterios más amplios en este marco ( $\preceq_{PC}$  y  $\preceq_{RPC}$ ). Estos últimos aún no sirven para capturar la expresividad relativa respecto a lógicas definidas en clases de estructuras diferentes, una necesidad subyacente en la cita de Chang y Keisler arriba. Para obtener tal capacidad, uno tiene que moverse hacia un marco más amplio, permitiendo traducciones de formulas y también traducciones de estructuras: el marco multi-clase.

Dos criterios formales en el marco multi-clase son analizados ( $\preceq_{gv}$  y  $expressiveness_g$ ). Argumentamos que ambos no son adecuados. Entonces proponemos que moverse para un marco aún más amplio no solamente nos liberaría de los problemas inherentes al marco multi-clase, pero también nos daría un abordaje de la expresividad para las lógicas Tarskianas y prueba-teóricas. Este marco solamente permite traducciones entre formulas que preservan la relación de consecuencia o teoremicidad/validez (en el aso que uno toma una lógica como un conjunto de fórmulas). Llamamos ese marco “expresividad traduccional”.

Que se sepa, en la literatura hay solamente un criterio de expresividad en este marco. El criterio es dado en [MDT09]. Mostraremos que él es aún inadecuado, una vez que cuenta como relaciones de expresividad casos que intuitivamente no lo son. Analizamos la literatura sobre traducciones entre lógicas y presentamos dos clasificaciones que fueran dadas en [Mor16] y [Fre10]. En la secuencia, propondremos algunos criterios de adecuación para expresividad y un criterio formal ( $expressiveness_{gg}$ ) para expresividad traduccional es dado. Argumentamos que  $expressiveness_{gg}$  satisface los criterios de adecuación. Algunas traducciones bien conocidas son presentadas y argumentamos que aquellas que satisfacen  $expressiveness_{gg}$  son razonablemente dichas inducir una relación de expresividad entre las lógicas involucradas.

En el capítulo 4, algunas caracterizaciones de lógicas hechas con respecto a expresividad son discutidas. El trabajo masivo hecho en este ámbito es para lógicas modelo-teóricas y usando  $\preceq_{EC}$ . En la primera sección, algunos trabajos de caracterización usando  $\preceq_{EC}$  son presentados. Comenzamos con el trabajo pionero de Mostowski, caracterizando  $\mathcal{FOL}$ , seguido por Lindström y Tharp. Recientemente, un nuevo impulso fue dado para caracterizaciones de fragmentos of  $\mathcal{FOL}$ . Presentaremos el trabajo de de Rijke y van Benthem et al. sobre teoremas tipo-Linström para lógicas modales y para los fragmentos de  $n$ -variables de  $\mathcal{FOL}$ . La sección termina con una discusión sobre esas caracterizaciones.

La última sección del capítulo no es más que un prospecto para investigación. Ella consiste en sugerir una caracterización de lógicas utilizando una noción de expresividad definida en el marco traduccional. De hecho, ya en [MDT09] una tal caracterización es ofrecida: la lógica proposicional de cláusulas Horn es máximamente expresiva entre las lógicas compactas. No obstante, el criterio de expresividad traduccional utilizado es inadecuado (como mostramos en el capítulo 3). A pesar de ello, esta proposición ejemplifica el tipo de caracterización general que nosotros planteamos.

## Contribuciones de esta tesis

### Capítulo 2

Como hemos dicho, la literatura sobre el concepto de sistema lógico es escasa y el principal libro sobre la cuestión [Gab94], a pesar de ser muy bueno, tiene el problema de ser muy fragmentado. No obstante, el propio Gabbay da una interesante perspectiva evolucionaria de la lógica en [Gab14]. El problema de ese trabajo es que está influenciado quizás demasiado por las demandas de la ciencia de la computación e inteligencia artificial. Además, no sabemos de una presentación razonablemente coherente sobre las diversas propuestas, especialmente en lo que se trata de la discusión del nacimiento de las concepciones de lógica abstracta y lógicas modelo-teóricas. Hemos dividido el capítulo en los tres campos usuales: lógicas abstractas, prueba-teóricas y modelo-teóricas, el propósito fue fornecer una perspectiva mas coherente de los tres marcos.

### Capítulo 3

La literatura está llena de comparaciones de lógicas en términos de sub-lógica, fuerza, inmersiones, interpretaciones, simulaciones, etc. Se necesita urgentemente una limpieza y padronización de las nociones involucradas,

de modo a evitar paradojas (como en la sección 3.4.0.1) y posibilitar la comparación de resultados.

Una contribución del capítulo 3 es un paso hacia la elucidación de la noción de expresividad entre lógicas. Proponemos que todas estas relaciones de sub-lógica, fuerza, etc. sean tratadas como relaciones de expresividad siempre que sean basadas en la siguiente intuición: una lógica  $\mathcal{L}_2$  es por lo menos tan expresiva cuanto  $\mathcal{L}_1$ , si para toda fórmula en  $\mathcal{L}_1$  existe una fórmula en  $\mathcal{L}_2$  con el mismo significado. Entonces proponemos que la expresividad relativa entre lógicas puede ser capturada formalmente en tres marcos: uni-clase, multi-clase y expresividad traduccional.

Esa organización debe situar mejor y elucidar los varios resultados de expresividad y discusiones relacionadas en la literatura. Por ejemplo, ello clarifica la discusión de Shapiro en [Sha91]. Utilizando los conceptos definidos aquí, podemos decir que el autor está argumentando que mismo en el marco estricto de expresividad uni-clase, existen criterios formales distintos y conflictivos (e.g.  $\preceq_{EC}$  y  $\preceq_{PC}$ ), ninguno de ellos siendo caminos reales para expresividad. Esto no parece ser algo ampliamente conocido, y cuya ignorancia puede ser engañadora, e.g. al tratar resultados usando  $\preceq_{PC}$  como si fuesen equivalentes a resultados usando  $\preceq_{EC}$  (esto aparentemente ocurre en [AFFM11]). Como una segunda contribución, mostramos que algunos criterios de expresividad multi-clase son inadecuados. Otra contribución es la propuesta de criterios de adecuación para expresividad y de un criterio formal para el marco de expresividad traduccional.

## Capítulo 4

El libro principal de caracterizaciones en términos de expresividad para extensiones de  $\mathcal{FOL}$  es [BF85]. Todos los trabajos seminales de Mostowski, Lindström y Tharp son expuestos y extendidos en dicho libro. No obstante, muchas de las ideas y consideraciones conceptuales encontradas en los artículos originales y relacionados, son omitidas allí. Además, hay una nueva tendencia de caracterizaciones de fragmentos de  $\mathcal{FOL}$  que no está presente en ese libro.

En [dA13] encontramos una presentación detallada del teorema de Lindström y del nuevo teorema modal tipo-Lindström de van Benthem [vB07]. Sin embargo, no hay una presentación detallada del método de Ehrenfeucht-Fraïssé para caracterizar la equivalencia elemental, y hay algunos errores en la presentación de las fórmulas  $\phi_{\mathfrak{U}, \vec{a}_s}^n$  usadas en dicho método. Además, no conocemos una presentación sencilla de los primeros resultados de caracterización de  $\mathcal{FOL}$  que también exponga los nuevos trabajos hechos respecto

a sus fragmentos. Así que la principal contribución del capítulo 4 es dar una breve presentación de tales resultados, intentando traer de vuelta las consideraciones conceptuales que son tan importante para ellos.

## **Apéndice**

Finalmente, nos parece que los resultados de Linström son aún bastante desconocidos. Ellos ya aparecen en algunos manuales de lógica, como [CK90], [EFT96] y [Hed04]. La presentación de Ebbinghaus et al. nos parece la mejor, visto que es dada paso a paso y de una forma clara. A pesar de ello, su presentación de la caracterización de la equivalencia elemental es muy deficiente. Esa noción es fundamental para los resultados de Lindström, y sin una conocimiento claro de ella, el lector no entenderá de que van estos resultados. Así que la contribución del apéndice es una presentación completa de los teoremas de Lindström (siguiendo las líneas de [EFT96]), incluyendo una explicación paso a paso dos juegos de Ehrenfeucht-Fraïsé y su rol en la caracterización de la equivalencia elemental.



## Abstract

This thesis is an investigation of the main concepts appearing in the Lindström-type characterization results, that is, the concept of a logical system and the concept of expressiveness. Lindström, among other similar results, characterized first-order logic as being maximally expressive among the logics having compactness and the Löwenheim-Skolem property. For that matter, he needed to give a precise definition of what counts as a logic, and what is an expressiveness relation. Such results are often used to draw bold conclusions regarding the nature of logic, and specially, of first-order logic. However, with the exception of an initial discussion in the handbook on model-theoretic logics, edited by Barwise and Feferman, there are few conceptual discussions of such characterization results in the literature. Specifically, we found problematic the lack of justification as to why a given concept of logical system and a given concept of relative expressiveness were chosen. Our aim is to contribute to this discussion and also propose a wider criterion for expressiveness based on translations between logics.

**Keywords:** Lindström-type theorems, logical systems, notions of expressiveness, translations between logics.





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# Chapter 1

## Introduction

This thesis grew out of a mixture of astonishment and a certain uneasiness as regards the Lindström theorem. In general lines the theorem says that if a logic  $\mathcal{L}$  is at least as expressive as first-order logic ( $\mathcal{FOL}$ ) and has properties  $P_1, P_2, \dots$ , then it is as expressive as  $\mathcal{FOL}$ . This theorem is, thus, a characterization of  $\mathcal{FOL}$  among its extensions, in terms of expressiveness and  $P_1, P_2, \dots$ . This characterization thus presupposes a general definition of logic and a definition of expressiveness. Besides, one also needs a justification as whether  $P_1, P_2, \dots$  are the right properties to characterize a logic in terms of expressiveness.

In the original presentation of his theorem and its variants, Lindström defines a “generalized first-order logic” as a collection of classes of model-theoretic structures closed under some operations [Lin69], these classes of structures that form a logic are to be understood as a collection of classes of models of each formula in the logic. An expressiveness relation thus issued from this definition is simply an inclusion of classes of structures.

As these definitions were given “just for the sake of generality” without further explanation, one immediately starts to wonder what exactly is being proven by this characterization. The two main issues naturally are:

- Why this definition of logic?
- Why this definition of relative expressiveness?

Later, Lindström himself felt the need to offer further justification for his results and published an expository paper [Lin74], where he defined more carefully the concept of an abstract logic, giving it some general axioms, and the notion of expressiveness to be used. By that time, some critics and

improvements by Barwise were already circulating and were published also in 1974 [Bar74]. Thus, our uneasiness is by no means isolated. We can also see it in Chang and Keisler's presentation of the referred theorem [CK90, p. 130] (where the definition 2.5.1 is a formal version of the one given above of a generalized first-order logic):

It should be emphasized that logics in the sense of Definition 2.5.1 deal with the same class of models as first order logic, and only the sentences and satisfaction relation may be different. This is a significant restriction which leaves a large loophole in Lindström's theorem. There are many examples of logics in a generalized sense which study models with additional structure and thus do not fit within our framework. These include modal logics, programming logics, and logics for models with topologies and measures. Sentential logic and  $\omega$ -logic as described in this book are not examples of abstract logics in the sense of 2.5.1, because they also deal with different classes of models than first-order logic.

In particular, the referred definition of logic was so alien to us that we decided to look where it came from. We also wanted to investigate the choice of the particular notion of expressiveness, appearing in the Lindström theorem and in the various similar expressiveness characterization results that came afterwards. Thus this thesis is organized as follows: chapter 2 deals with the concept of logical system, chapter 3 exposes different concepts of expressiveness adequate to each general kind of logical system. In the last chapter we expose and discuss the diverse expressive characterizations of first-order logic and some of its fragments. In the appendix a detailed presentation of the Lindström theorem is given.

The literature on what is a logical system is by no means extensive, specifically, we know of just one book dedicated to the subject, i.e. [Gab94]. It is a collection of fifteen papers on the issue of what is a logical system. Apart from those where a only quick definition of a "logic" is given, there are no much more works in the literature. Anyway, the referred book gives a good picture of the notion of logic, but we see it as too fragmented, as simply a collection of each author's view of the subject. Given this, we decided to dedicate chapter 2 to this issue, and tried to give a more coherent picture while naturally keeping [Gab94] as the main guide.

In chapter 2 we revise the sources of the general notion of a logic and separate them in three general views: an abstract (a.k.a. Tarskian) view,

a proof-theoretical view and a model-theoretical view (with Tarskian semantics). There is also another interesting general view that grew out of category theory, giving a signature indexed notion of a logical system. This view is “natively” constructed with comparisons of logics in mind. Unfortunately, it is out of the scope of this thesis and the reader is referred to [Mes89] and [MGDT07].<sup>1</sup>

The game-theoretic semantics have been proposed as a more adequate interpretation for quantifiers than Tarskian semantics [Hin88]. This semantics has turned into a rich framework for defining logics (a notable example is given in [HS89]). Nevertheless the logics issuing from game-theoretic semantics are also absent in this thesis.

We shall see that the model-theoretical view of logic starts to settle rather late in the XX century, around the 1960s, despite the model-theoretic methods being available then already for some decades. We see that a main change in the conception of logic was due to the work of Mostowski in generalized quantifiers [Mos57]. From this view came the notion used in Lindström’s proof. Thus the chapter presents a reasonably wide panorama on the concept of logical system and its evolution.

In chapter 3 we revise the notion of expressiveness used in Lindström’s proof. We shall offer a following classification of relative expressiveness between logics. For model-theoretic logics, we have two frameworks of expressiveness: single-class and multi-class expressiveness. The precise criterion for expressiveness used by Lindström (referred to here as  $\preceq_{EC}$ ) is defined within the first framework. Here  $\preceq_{EC}$  can be seen as involving translations of formulas of model-theoretic logics defined within the same class of structures (hence the name “single-class”). We also analyse some wider criteria in this framework ( $\preceq_{PC}$  and  $\preceq_{RPC}$ ). These later still fall short of giving an account of expressiveness for logics defined within different classes of structures, a need underlying Chang and Keisler’s quotation above. To obtain such means, we have to move to the wider framework, allowing besides translations of formulas, also translations of structures: the multi-class framework.

Two formal criteria in the multi-class framework are analysed ( $\preceq_{gv}$  and *expressiveness<sub>g</sub>*). We argue that they are not adequate and it is proposed that moving to a still wider framework might not only free us from the

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<sup>1</sup>The reader is also referred to [Cal00]. There one can find definitions of consequence systems, logic systems and of logic. A logic system is defined as any mathematical structure from which a consequence system can be extracted, and is attached to a signature. Finally a logic is defined a family of logic systems. For each of these, there are also their respective morphisms.

problems inherent in the multi-class framework, but also give an account of expressiveness for Tarskian and proof-theoretic logics. This framework only allows translations between formulas that preserve the consequence relation or theoremhood/validity (in case one takes a logic as a set of formulas). We call this framework “translational expressiveness”.

To the best of our knowledge, in the literature there is only one criterion for expressiveness in this framework. The criterion is given in [MDT09]. We will show that it is still inadequate, as it overgenerates. We analyse the literature on translations between logics and present two classifications that were given in [Mor16] and [Fre10]. In the sequence, we propose some adequacy criteria for expressiveness and a formal criterion ( $expressiveness_{gg}$ ) for translational expressiveness. We argue that  $expressiveness_{gg}$  satisfies the adequacy criteria. Some well-known translations are presented and we argue that the ones satisfying  $expressiveness_{gg}$  are reasonably said to induce an expressiveness relation between the logics involved.

In chapter 4 some characterizations of logics made with respect to expressiveness are discussed. The massive work done in this respect is for model-theoretic logics, and using  $\preceq_{EC}$ . In the first section some characterization works with respect to  $\preceq_{EC}$  are presented. We begin with Mostowki’s pioneer work for characterizing  $\mathcal{FOL}$ , followed by Lindström and Tharp. Recently a new impulse was given to characterizations of fragments of  $\mathcal{FOL}$ . We will present de Rijke and van Benthem et al.’s work on Lindström theorems for modal logics, and the finite-variable fragment of  $\mathcal{FOL}$ . The section ends with a discussion of these characterizations.

The last section of the chapter is no more than a prospect for investigation. It consists in suggesting a characterization of logics using a notion of expressiveness defined in the translational framework. As a matter of fact, already in [MDT09] such a characterization is offered: propositional horn clause logic is maximally expressive among compact logics. Nevertheless, the criterion of translational expressiveness used is inadequate (as we show in chapter 3). Despite this, this proposition exemplifies the sort of general characterization result we envisage.

## 1.1 Contributions of this thesis

### Chapter 2

As we said, the literature on what is a logical system is scarce, and the main book on the subject [Gab94], otherwise very good, has a problem of being too fragmented. Nevertheless, Gabbay himself gives an interesting evolutionary



view of logic in [Gab14]. The problem of this work is that it is influenced perhaps too much by the demands of computer science and AI. Moreover, we do not know of a reasonably coherent presentation of the diverse proposals, specially as regards the discussion on the rising up of abstract logics and model-theoretic logics. Our contribution is the compilation, organization and discussion of the diverse proposals for the concept of logical system. We divided the chapter in the three usual fields of abstract, proof-theoretic and model-theoretic logics, the aim was to provide a more coherent picture of these three frameworks.

### Chapter 3

The literature is filled with comparisons of logics in terms of sub-logic, strength, embeddings, interpretations, simulations, etc. It needs urgently a clean-up and standardization of the involved notions, in order to avoid paradoxes (as in section 3.4.0.1) and to allow the comparison of results.

One contribution of chapter 3 is a step towards the elucidation of the notion of expressiveness between logics. We propose that all these relations of sub-logic, strength, etc. be treated as expressiveness relations as long as they are based in the following intuition: a logic  $\mathcal{L}_2$  is at least as expressive as  $\mathcal{L}_1$  if for every  $\mathcal{L}_1$ -formula, there is an  $\mathcal{L}_2$ -formula with the same meaning. Then we propose that relative expressiveness between logics can be captured formally in three frameworks: single-class, multi-class and translational expressiveness.

This organization shall better situate and elucidate the various expressiveness results and related discussions appearing in the literature. For example, it clarifies Shapiro's discussion [Sha91]. Using the concepts defined here, we can say that the author is arguing to the effect that even in the strict framework of single-class expressiveness, there can be defined different and conflicting criteria (e.g.  $\preceq_{EC}$  and  $\preceq_{PC}$ ), none of them being royal roads to expressiveness. This seems not to be something widely known and whose ignorance may be misleading, e.g. in treating  $\preceq_{PC}$ -results as it were  $\preceq_{EC}$ -results (this apparently occurs in [AFFM11]). As a second contribution, we show that some criteria for multi-class expressiveness are inadequate. Another contribution is the proposal of adequacy criteria for expressiveness and a formal criterion in the framework of translational expressiveness.

## Chapter 4

The handbook for characterizations in terms of expressiveness for extensions of  $\mathcal{FOL}$  is [BF85]. All the seminal works of Mostowski, Tharp and Lindström are exposed and expanded in this book. Nevertheless, many of the insights and conceptual considerations found in the original and related papers are omitted in it. Besides, there is a new trend of characterizing fragments of  $\mathcal{FOL}$  that is not contained there.

In [dA13] we find a detailed presentation of Lindström theorem and of Van Benthem's [vB07] new modal Lindström theorem. Nevertheless, some of the examples given for the formulas  $\phi_{\mathfrak{L}, \vec{a}_s}^n$  used to characterize elementary equivalence are incorrect. Besides, we do not know of a simple presentation of the early characterization theorems of  $\mathcal{FOL}$  that exposes also the new works done with respect to its fragments. Thus the main contribution of chapter 4 is to give a brief presentation of these results, trying to bring back the conceptual considerations that are so important for them.

## Appendix

Finally, we found the Lindström results are still quite unknown. It already appears in some logic manuals such as [CK90], [EFT96] and [Hed04]. Ebbinghaus et al.'s presentation of the Lindström's theorem is the one we find more valuable, it is given in a step by step and illuminating fashion. Nevertheless, their presentation of the characterization of elementary equivalence is very deficient. This notion is fundamental for Lindström theorem, and without a clear grasp of it, the reader will not get "the point" of the theorem. Therefore the contribution of the appendix is a full presentation of the Lindström theorem (along the lines of [EFT96]) including a step-by-step explanation of the Ehrenfeucht-Fraïsé games and its role in the characterization of elementary equivalence.

## Chapter 5

# Conclusions and prospects for future work

Let us summarise what we have seen in this thesis. In chapter 2 the concept of logical system was revised, we presented the first steps towards a general definition of a logic by Hertz and its subsequent seminal generalization made by Tarski. It is interesting to see that even Tarski's general conception of a logic required in the beginnings that the consequence relation were defined by a set of rules of inference. Afterwards, as the role of model-theoretic logics came more and more to the front scene, this condition was lifted. Still, there remained some other conditions that are nowadays considered too restrictive, namely, the finiteness condition and the explosion condition. Moreover, the monotonicity condition now is often lifted, the transitivity has been restricted and even the reflexivity was questioned. So one cannot help feeling that whatever formal system goes as a logic. We saw that application areas are influencing the researches view on what is a logic. It has been stated by Gabbay, for example, that the consequence relation is not even the most important thing in a logic system, as the mechanisms of update and withdrawal of information would have a more important role.

Still, the transference-based and property-based approaches to logical consequence remain marking their distinction on sufficiently expressive systems where they cannot be matched. In these cases, the supporters of the property-based approach (model-theoretic logics) will advocate its priority, given some general set-theoretic criteria for the logicity of their systems; and the supporters of the transference-based approach (proof-theoretic logics) will reject the set-theoretic logicity arguments and stick to their own measures of logicity.

Now by its own nature, the early characterization theorems by Mostowski had initially to blur this distinction in order to reach the conclusion that some systems, more expressive than  $\mathcal{FOL}$ , cannot be defined by (the usual) transference-based approach to logical consequence. The same goes with respect to Lindström's great advances on the early expressiveness characterization results of  $\mathcal{FOL}$ .

Lindström's results characterize  $\mathcal{FOL}$  among its expressive extensions, that is, among the logics defined within the same model-theoretic structures as  $\mathcal{FOL}$ , and with respect to a specific notion of expressiveness. On chapter 3 we investigated the many ways two logics can be compared in terms of expressiveness in the three frameworks: single-class, multi-class and translational expressiveness. We saw for example that even in the framework of single-class expressiveness used in the Lindström's proofs, there can be distinct criteria  $\preceq_{EC}$ ,  $\preceq_{PC}$  and  $\preceq_{RPC}$ . For example we saw that the logics  $\mathcal{L}_{Q_0}$  and  $\mathcal{L}_A$  are equally expressive according to  $\preceq_{PC}$ , but incomparable with respect to  $\preceq_{EC}$ , while weak-second order logic  $\mathcal{L}^{2w}$  is more expressive than both according to  $\preceq_{EC}$ , and equally expressive according to  $\preceq_{RPC}$ .

As diverse as the  $\preceq_{EC}$ ,  $\preceq_{PC}$  and  $\preceq_{RPC}$  are, they still require logics to be defined within the same class of structures, so we investigated possible extensions. We concluded that a broader criterion must be formulated in the wider framework of multi-class expressiveness. There we found two criteria  $\preceq_{gv}$  and  $expressiveness_g$  and we argued that they are not good measures of expressiveness in the multi-class framework. We proposed to lift to an even more abstract measure of expressiveness, in order to avoid some of the problems the former criteria had and allow a greater range of logics for comparison. The later framework was called translational expressiveness. A criterion of translational expressiveness was analysed and we saw it is not adequate. Then some adequacy criteria for expressiveness were presented and discussed, and a formal criterion complying with them was given.

In the final chapter we presented many expressiveness characterization results done with respect to  $\preceq_{EC}$ , beginning with Mostowski, passing by Lindström's seminal work and ending with the recent investigations with respect to fragments of first-order logic. In the sequence there was a discussion of the properties appearing in the characterization, e.g. Compactness, Löwenheim-Skolem, Completeness, and Tarski's Union Property. We saw that the more important the properties appearing in characterizations are, the more relevant the final result is. The importance of the referred properties naturally depend on the sense given to "logic", we distinguished main two senses: as a theory of deduction, as an instrument for characterizing structures of interest. We concluded that the properties appearing in main

expressiveness characterization results are not usually considered relevant for the sense of logic assumed in these characterizations.

We end the chapter by an speculation whether far-reaching expressive-characterization results could be obtained using an expressiveness criterion in the broad framework of translational expressiveness. A work in this direction was made already by Mossakowski et al., but their concept of expressiveness is not reasonable. Thus the investigation on characterizations using translational expressiveness is an exciting prospect for future work, and there are already some recent works which can be seen in this light (see below).

As for the research agenda, there is a strong need for the development of a notion of structure-preserving translation beyond propositional logics. There is some work on this area developed with category-theoretic methods in [MGDT07], our plan is to see what can be done following their path.

Other important theme for work is to test further the material adequacy of our criterion of translational expressiveness, seeing how it behaves with respect to the translations between logics in the literature. Specifically, one important issue is to investigate further whether our condition on preservation of connectives is adequate. An interesting test is Statman's [Sta79] translation of  $\mathcal{IPL}$  into its implication fragment  $\mathcal{IPL}^{\{\rightarrow\}}$ . Since it is not defined inductively through the formation of formulas, it would not comply with our condition. Nevertheless, it also seems to preserve reasonably the meaning of the  $\mathcal{IPL}$ -connectives into  $\mathcal{IPL}^{\{\rightarrow\}}$ : the translation maps molecular formulas into propositional variables and regulates the behaviour of the variables with implicational axioms.<sup>1</sup> We suspected that this translation could be transformed so as to be defined inductively on formulas, and in the final stage of this work we learned this is indeed the case, by an interesting result due to Haeusler [Hae15].

We generalized Statman's result so that any logic that can be formulated within a general framework of introduction and elimination rules, and has the sub-formula property, can be translated into the implicational fragment of minimal logic. The translation works the same way as Statman's, by formulating the inference rules by means of implicational axioms containing variables with indices. The difference is that Herman's translation is defined inductively on formulas and is more regular. It uses one auxiliary mapping in order to give the implicational axioms for each operator in the source logic, thus it is a general-recursive translation. If this is as it seems, we would have

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<sup>1</sup>E.g. conjunctions  $\phi \wedge \psi$  are mapped to implications containing  $x_{\phi \wedge \psi}$ , among formulas of the sort  $x_{\phi} \rightarrow (x_{\psi} \rightarrow x_{\phi \wedge \psi})$ ,  $x_{\phi \wedge \psi} \rightarrow x_{\phi}$ , etc.

a nice characterization using translational expressiveness in terms of the subformula property. Many other similar translations appeared recently in the literature, e.g. [Jeř17] and [AA17]. Therefore, there are many interesting research prospects.

# Chapter 6

## Appendix

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### 6.1 Lindström Theorems

#### Initial Remarks

In this section the two so-called Lindström theorems will be presented:

1. Let  $\mathcal{L}_I$  refer to the first-order logic. If a logical system  $\mathcal{L}$  is more expressive than  $\mathcal{L}_I$ , then or it lacks the compactness theorem, or it lacks the Löwenheim-Skolem theorem.
2. If a logical system  $\mathcal{L}$  is more expressive than  $\mathcal{L}_I$ , then or it lacks the Löwenheim-Skolem theorem or its set of logical validities is not recursively enumerable.

The whole presentation is closely related to that of [EFT96], the differences may appear in the arrangement of the definitions, lemmas and theorems, and in the few gaps that were filled here and there. The main change is that we find the chapter on the algebraic characterization of elementary equivalence not clear at all, so we present the notion of  $m$ -isomorphism between structures by another method, the Ehrenfeucht Games. For this, we take as a basis the presentation in [EF99].

### 6.1.1 Ehrenfeucht Games

A characterization of elementary equivalence will be studied on the basis of the so-called Ehrenfeucht Games. The rules of this game rests on the notion of a partial isomorphism between two  $\tau$ -structures  $\mathfrak{U} : (A, R_1^A, R_2^A, \dots)$  and  $\mathfrak{B} : (B, R_1^B, R_2^B, \dots)$ , with  $R_1, R_2, \dots \in \tau$ . Unless explicitly stated, the symbol set  $\tau$  will be relational, for reasons that will soon become clear.

**Definition 6.1.1.0.1** (Partial isomorphism). *Let  $\mathfrak{U}$  and  $\mathfrak{B}$  be  $\tau$ -structures. From now on, we denote the domains of such structures  $A$  and  $B$ , respectively, and, for a function  $p$ ,  $\text{dom}(p)$  and  $\text{rng}(p)$  refers to the domain of  $f$  and the range of  $f$ , respectively. Let  $p$  be a map with  $\text{dom}(p) \subseteq A$  and  $\text{rng}(p) \subseteq B$ . In the text, the map  $p$  will be identified with its graph  $\{(a, p(a)) \mid a \in \text{dom}(p)\}$ . Then  $p$  is said to be a partial isomorphism from  $\mathfrak{U}$  to  $\mathfrak{B}$  if*

- $p$  is injective,
- for  $n$ -ary  $R \in \tau$  and all  $a_1, \dots, a_n \in \text{dom}(p)$ ,  $R^A a_1, \dots, a_n$  iff  $R^B p(a_1), \dots, p(a_n)$ .

$q \supseteq p$  means that  $q$  is an extension of  $p$ .

Observe that if  $p \neq 0$  is a map with  $\text{dom}(p) \subseteq A$  and  $\text{rng}(p) \subseteq B$ , then  $p$  is a partial isomorphism from  $\mathfrak{U}$  to  $\mathfrak{B}$  iff the substructures induced by  $\text{dom}(p)$  and  $\text{rng}(p)$  in  $\mathfrak{U}$  and  $\mathfrak{B}$  are isomorphic, i.e.  $p : [\text{dom}(p)]^{\mathfrak{U}} \cong [\text{rng}(p)]^{\mathfrak{B}}$ .

Let  $\vec{a}_n = \langle a_0, \dots, a_{n-1} \rangle \in A$  and  $\vec{b}_n = \langle b_0, \dots, b_{n-1} \rangle \in B$ , then the following statements are equivalent

1. The clauses  $p(a_i) = b_i$  for  $i = 0, \dots, n-1$  and  $p(c^A) = c^B$  for  $c \in \tau$  define a map which is a partial isomorphism from  $\mathfrak{U}$  to  $\mathfrak{B}$ ;
2. For all atomic  $\phi(v_0, \dots, v_{n-1})$ :  $\mathfrak{U} \models \phi[\vec{a}_n]$  iff  $\mathfrak{B} \models \phi[\vec{b}_n]$ ;
3. For all quantifier-free  $\phi(v_0, \dots, v_{n-1})$ :  $\mathfrak{U} \models \phi[\vec{a}_n]$  iff  $\mathfrak{B} \models \phi[\vec{b}_n]$ .



The verification of  $1 \rightarrow 2$  follows by the definition of partial isomorphism. For the direction  $2 \rightarrow 1$ , we define for  $a_0, \dots, a_{n-1}$  and  $b_0, \dots, b_{n-1}$  the function  $p(a_i) = b_i$ . Since  $\mathfrak{U} \models v_i = v_j[\vec{a}_n]$  iff  $a_i = a_j$  and  $\mathfrak{B} \models v_i = v_j[\vec{b}_n]$  iff  $b_i = b_j$ , one easily verifies that  $p$  is well defined: if  $b_i \neq b_j$ , then  $\mathfrak{B} \not\models v_i = v_j[\vec{b}_n]$  and (by 2),  $\mathfrak{U} \not\models v_i = v_j[\vec{a}_n]$ , thus  $a_i \neq a_j$ . By a very similar argument, one sees that  $p$  is injective.

Also,  $p$  is a partial isomorphism, since by hypothesis, for any atomic formula  $\mathfrak{U} \models \phi[a_0, \dots, a_{n-1}]$  iff  $\mathfrak{B} \models \phi[b_0, \dots, b_{n-1}]$ . The verification of  $2 \rightarrow 3$  is a simple induction on the degree of the quantifier-free formulas.

The existence of a partial isomorphism in general does not preserve the validity of formulas with quantifiers, let us see an example. Let  $\tau = \{<\}$  and let  $p$  be a function from  $(\mathbb{R}, <)$  to  $(\mathbb{Z}, <)$ , with  $\text{dom}(p) = \{2, 3\}$  such that  $p(2) = 3, p(3) = 4$ .  $p$  is a partial isomorphism and, by the observation above, for any quantifier free formula  $\phi(v_0, v_1)$ ,  $(\mathbb{R}, <) \models \phi[2, 3]$  iff  $(\mathbb{Z}, <) \models \phi[3, 4]$ .

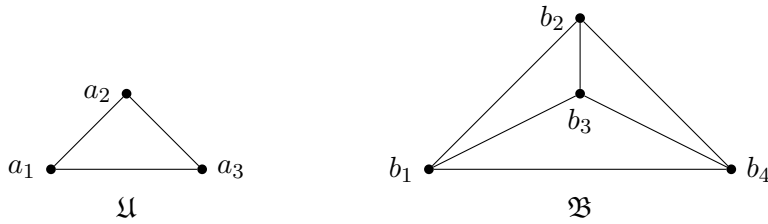
But, this does not hold for a formula with even one quantifier, since

$$(\mathbb{R}, <) \models \exists v_2 (v_0 < v_2 \wedge v_2 < v_1)[2, 3]$$

but it is not the case that

$$(\mathbb{Z}, <) \models \exists v_2 (v_0 < v_2 \wedge v_2 < v_1)[3, 4]$$

For another example, consider the following graphs  $\mathfrak{U} = (\{a_1, a_2, a_3\}, E^A)$  and  $\mathfrak{B} = (\{b_1, b_2, b_3, b_4\}, E^B)$ , with the edge relation  $E^A$  and  $E^B$  as follows.



The function

$$\begin{aligned} p : A &\rightarrow B \\ a_1 &\mapsto b_1 \end{aligned}$$

is a partial isomorphism between  $\mathfrak{U}$  and  $\mathfrak{B}$ . Note that  $p$  can be extended to

$q$  subsequently, to  $q'$ :

$$\begin{aligned} q' : A &\rightarrow B \\ a_1 &\mapsto b_1 \\ a_2 &\mapsto b_2 \\ a_3 &\mapsto b_3 \end{aligned}$$

Thus, if  $\phi$  is a quantifier-free formula, by the result above,

$$\mathfrak{U} \models \phi[a_1, a_2, a_3] \Leftrightarrow \mathfrak{B} \models \phi[q'(a_1), q'(a_2), q'(a_3)].$$

Besides, we have that both  $\mathfrak{U}$  and  $\mathfrak{B}$  satisfies some quantified sentences such as

$$\forall x_1 \exists x_2 x_3 (x_1 \neq x_2 \neq x_3 \wedge E(x_1, x_2) \wedge E(x_1, x_3)).$$

Nevertheless,

$$\mathfrak{B} \models \exists y (y \neq x_1 \neq x_2 \neq x_3) [q'(a_1), q'(a_2), q'(a_3)], \quad (6.1)$$

but

$$\mathfrak{U} \not\models \exists y (y \neq x_1 \neq x_2 \neq x_3) [a_1, a_2, a_3].$$

(1) is the case because there's an element  $b_i \in B$  such that

$$\mathfrak{B} \models (y \neq x_1 \neq x_2 \neq x_3) [b_i, q'(a_1), q'(a_2), q'(a_3)].$$

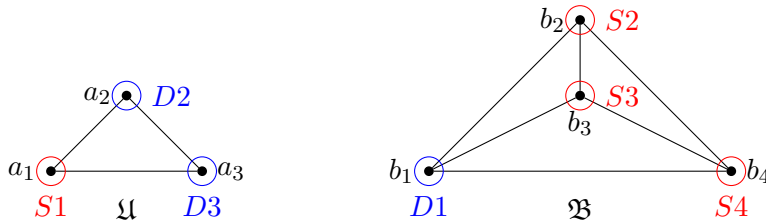
So if the function  $q'$  could be extended in order to have this  $b_i$  in its range,  $\mathfrak{U}$  would also satisfy this sentence. But this can not be so, since  $|A| = 3$ .

We see that the preservation of satisfiability of quantified formulas through partial isomorphisms depends on whether these partial isomorphisms can be extended in certain ways. This is the basic idea under the characterization of the relation of  $\equiv_m$  (m-equivalence) between structures  $\mathfrak{U}$  and  $\mathfrak{B}$ :  $\mathfrak{U} \equiv_m \mathfrak{B}$  holds iff there are partial isomorphisms from  $\mathfrak{U}$  to  $\mathfrak{B}$  that can be extended  $m$  times, adding one element at each step. Let  $\vec{a}_s = \langle a_0, \dots, a_{s-1} \rangle$  for  $\langle a_0, \dots, a_{s-1} \rangle \in A^s$  be a sequence of elements of the domain of  $\mathfrak{U}$ , and similarly for  $\vec{b}_s$ , hereafter we will use  $\vec{a}_s \mapsto \vec{b}_s$  meaning that there is a function  $p$  such that  $p(a_i) = b_i$  for  $(i = 1, \dots, s)$  is a partial isomorphism from  $\mathfrak{U}$  to  $\mathfrak{B}$ . Now we are ready to study the m-equivalence relation from a game-theoretical point of view.

The Ehrenfeucht game  $G_m(\mathfrak{U}, \vec{a}_s, \mathfrak{B}, \vec{b}_s)$  for  $\tau$ -structures  $\mathfrak{U}$  and  $\mathfrak{B}$ ,  $\vec{a}_s \in A^s$ ,  $\vec{b}_s \in B^s$  and  $n \in \mathbb{N}$  consists of two players, the *Spoiler* and the *Duplicator*. Each player make  $m$  moves during the play, and take alternate turns. The Spoiler plays first, and, once he chose an element from a structure, the Duplicator has to chose an element from the other structure. After  $m$  rounds, a sequence of elements  $e_1, \dots, e_m, e'_1, \dots, e'_m$  of  $A$  and  $B$ , respectively, will have been chosen. The Duplicator wins if  $\vec{a}_s, e_1, \dots, e_m \mapsto \vec{b}_s, e'_1, \dots, e'_m$  is a partial isomorphism from  $\mathfrak{U}$  to  $\mathfrak{B}$ , and if  $m = 0$ , it is required that  $\vec{a}_s \mapsto \vec{b}_s$  be a partial isomorphism from  $\mathfrak{U}$  to  $\mathfrak{B}$  (henceforth, this will be abbreviated as  $\vec{a}_s \mapsto \vec{b}_s \in Part(\mathfrak{U}, \mathfrak{B})$ ).

The sequences of elements  $\vec{a}_s$  and  $\vec{b}_s$  in a game  $G_m(\mathfrak{U}, \vec{a}_s, \mathfrak{B}, \vec{b}_s)$  can be understood as a game that starts with some links  $\vec{a}_s \mapsto \vec{b}_s$  between some elements of  $\mathfrak{U}$  and  $\mathfrak{B}$  already given.

The intuitive idea of this game is that the role of the Spoiler is to show that the two structures  $\mathfrak{U}$  and  $\mathfrak{B}$  are as different as possible, and the role of the Duplicator is to show that  $\mathfrak{U}$  and  $\mathfrak{B}$  are as similar as possible. The Spoiler starts playing, and the Duplicator duplicates his moves such that his chosen elements satisfies the same properties as the Spoiler's chosen elements. As an example, consider again the example given before.



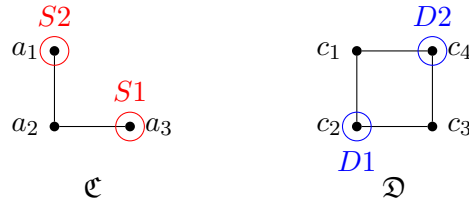
In the picture above, the Spoiler choices will be marked red and the Duplicator's will be marked blue, and will be labelled as  $S_i$  or  $D_i$  if they were chosen in the  $i$ -th round by the Spoiler or Duplicator. Thus, the Duplicator's task is to preserve the edge relation between the chosen elements.

The mapping of elements  $a_1, a_2, a_3 \mapsto b_1, b_2, b_3$  made by the Duplicator forms the function  $q'$  given above. As we know that the function  $q' \in Part(\mathfrak{U}, \mathfrak{B})$ , it follows that the Duplicator wins  $G_3(\mathfrak{U}, \mathfrak{B})$ . Notice that, Duplicator can not win the four-round game  $G_4(\mathfrak{U}, \mathfrak{B})$ , since a choice of Spoiler of the node  $b_4$  could not be matched by Duplicator. That the Duplicator wins  $G_3(\mathfrak{U}, \mathfrak{B})$  means that considering only 3 elements, the structures  $\mathfrak{U}$  and  $\mathfrak{B}$  are isomorphic.

It is important to notice that the Duplicator wins a given game on  $n$  rounds if and only if he has a *winning strategy*, that is, he wins no matter what choices are made by the Spoiler, thus there is no place for chance here.

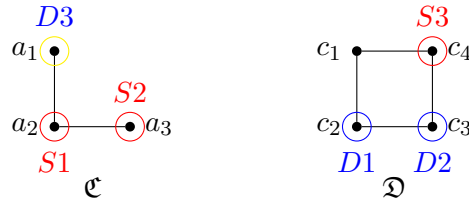
The same goes for the Spoiler, and exactly one player has a winning strategy for some game  $G_n(\mathfrak{C}, \mathfrak{D})$ . Considering the above example, Duplicator has a winning strategy for  $G_3(\mathfrak{U}, \mathfrak{B})$  but Spoiler has a winning strategy for  $G_4(\mathfrak{U}, \mathfrak{B})$ .

Let's see another example.



In the above 2-round game, the Duplicator wins, since no matter which choices of elements from the Spoiler, he can match them, since, if the Spoiler chooses two elements with an edge between them, the Duplicator can reply with elements with an edge between them, or, like in the game illustrated above, the Duplicator was also able to choose two elements with no edge between them.

Looking at the natural continuation of the above game, one could think that the Duplicator would win the 3-round game, since any given node chosen by the Spoiler could be matched. Nevertheless, this apparent win depend on the previous choices by the Spoiler, and this does not configure a legitimate win, since Duplicator has not a winning strategy for it. We can see easily this by the other possible 3-round game in which Spoiler wins:



In this game, Duplicator loses since  $a_2, a_3, a_1 \mapsto c_2, c_3, c_4$  is not a partial isomorphism between  $\mathfrak{C}$  and  $\mathfrak{D}$ , since the chosen elements disagree on the edge relation  $E$  in each structure:  $(a_3, a_1) \notin E^C$  but  $(c_3, c_4) \in E^D$ .

We state some facts about the Ehrenfeucht game that follow easily from its definition. For  $a \in A$ , we abbreviate  $a^s a$  as  $\langle a_0, \dots, a_{s-1}, a \rangle$ .

**Remarks 6.1.1.0.2.** Let  $\mathfrak{U}$  and  $\mathfrak{B}$  be  $\tau$ -structures,  $\vec{a}_s \in A^s$ ,  $\vec{b}_s \in B^s$  and  $m \geq 0$ .

1. The Duplicator wins  $G_0(\mathfrak{U}, \vec{a}_s, \mathfrak{B}, \vec{b}_s)$  iff  $\vec{a}_s \mapsto \vec{b}_s \in Part(\mathfrak{U}, \mathfrak{B})$ ;

2. For  $m > 0$  the following are equivalent:

- (a) The Duplicator wins  $G_m(\mathfrak{U}, \vec{a}_s, \mathfrak{B}, \vec{b}_s)$ ;
- (b) for all  $a \in A$  there is some  $b \in B$  such that the Duplicator wins  $G_{m-1}(\mathfrak{U}, \vec{a}_s a, \mathfrak{B}, \vec{b}_s b)$   
for all  $b \in B$  there is some  $a \in A$  such that the Duplicator wins  $G_{m-1}(\mathfrak{U}, \vec{a}_s a, \mathfrak{B}, \vec{b}_s b)$ ;

3. If the Duplicator wins  $G_m(\mathfrak{U}, \vec{a}_s, \mathfrak{B}, \vec{b}_s)$  and  $n < m$ , then  $G_n(\mathfrak{U}, \vec{a}_s, \mathfrak{B}, \vec{b}_s)$ .

Let  $\tau$  be a finite symbol set, and  $\mathfrak{U}, \mathfrak{B}$  be  $\tau$ -structures. Let  $\vec{a}_s = \langle a_0, \dots, a_{s-1} \rangle \in A^s$  and  $m > 0$ . We will now present a formula  $\phi_{\mathfrak{U}, \vec{a}_s}^m(v_0, \dots, v_{s-1})$  that describes the game-theoretic properties of  $\vec{a}_s$  in any game  $G_m(\mathfrak{U}, \vec{a}_s, \dots)$ . That means that,  $\phi_{\vec{a}_s}^m(v_0, \dots, v_{s-1})$  will be defined in such a way that for any  $\mathfrak{B}$  and  $\vec{b}_s = \langle b_0, \dots, b_{s-1} \rangle \in B^s$ ,

$$\mathfrak{B} \models \phi_{\mathfrak{U}, \vec{a}_s}^m[\vec{b}_s] \text{ iff the Duplicator wins } G_m(\mathfrak{U}, \vec{a}_s, \mathfrak{B}, \vec{b}_s).$$

**Definition 6.1.1.0.3.** Let  $L_s^\tau$  be the set of all  $L$ -formulas on the vocabulary  $\tau$  that has at most  $s$  free variables.

**Definition 6.1.1.0.4.** We define the following set  $\Phi_s$  as

$$\Phi_s = \{\phi \mid \phi \in L_s^\tau \text{ and } \phi \text{ is atomic or negated atomic, with free variables among } v_0, \dots, v_{s-1}\}.$$

**Example 6.1.1.0.5.** For a symbol set  $\tau = P, Q, R$  with, respectively, unary, binary and ternary relation symbols, for  $s \geq 0$  we would have the following sequence of sets for:<sup>1</sup>

$$\Phi_0 = 0 \text{ (since the set is relational and there is no constants to form atomic sentences);}$$

$$\Phi_1 = \{P(v_0), \neg P(v_0)\};$$

$$\Phi_2 = \Phi_1 \cup \{P(v_1), \neg P(v_1), Q(v_0, v_0), Q(v_0, v_1), \dots, Q(v_1, v_1), \neg Q(v_0, v_0), \dots, \neg Q(v_1, v_1)\};$$

$$\Phi_3 = \Phi_2 \cup \{P(v_2), \neg P(v_2), Q(v_0, v_0), \dots, \neg Q(v_2, v_2), R(v_0, v_0, v_0), \dots, \neg R(v_2, v_2, v_2)\};$$

$$[\dots]$$

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<sup>1</sup>This example has been based on [Edgar Almeida- Master's thesis on the first Lindström theorem, Campinas, 2013].

For every finite symbol set  $\tau$  and every natural number  $n$ , we see that the set  $\Phi_n$  is finite.

From the set  $\Phi_s$ , a  $\tau$ -structure  $\mathfrak{U}$  and a sequence  $\vec{a}_s$  of elements of  $\mathfrak{U}$ , we can consider the subset of formulas of  $\Phi_s$  that are satisfied by  $\vec{a}_s$  in  $\mathfrak{U}$ :

$$\Gamma_s^{\vec{a}_s} = \{\phi \mid \phi \in \Phi_s \text{ and } \mathfrak{U} \models \phi[\vec{a}_s]\}.$$

Since  $\Phi_s$  is finite,  $\bigwedge \Gamma_s$  is a first order formula. We now define the sentence that define the game-theoretic properties of  $\vec{a}_s$  in  $\mathfrak{U}$ .

**Definition 6.1.1.0.6.** *We define the first-order  $\tau$ -formula  $\phi_{\mathfrak{U}, \vec{a}_s}^n$  recursively as*

$\phi_{\mathfrak{U}, \vec{a}_s}^0$  is the formula  $\bigwedge \Gamma_s^{\vec{a}_s}$ ;

$\phi_{\mathfrak{U}, \vec{a}_s}^{n+1}$  is the formula  $(\forall v_s \bigvee_{a \in A} \{\phi_{\mathfrak{U}, \vec{a}_s a}^n\}) \wedge (\bigwedge_{a \in A} \{\exists v_s \phi_{\mathfrak{U}, \vec{a}_s a}^n\})$ .

### EXAMPLE

Let's illustrate the construction of the above formulas in an infinite structure. For the symbol set  $\tau$  of the preceding example, consider the structure  $\mathfrak{N} = (\mathbb{N}, P^{\mathbb{N}}, Q^{\mathbb{N}}, R^{\mathbb{N}})$ , where for  $n, m, o \in \mathbb{N}$ ,  $n \in P^{\mathbb{N}}$  iff  $n$  is a prime number;  $(m, n) \in Q^{\mathbb{N}}$  iff  $m$  divides  $n$ ; and  $(m, n, o) \in R^{\mathbb{N}}$  iff  $m$  is between  $n$  and  $o$ .

Let  $\mathcal{I}$  be a variable assignment for  $\mathfrak{N}$ , such that, for  $v_i$  it assigns the  $i^{\text{th}}$  number in the natural sequence of  $\mathbb{N}$ , so that  $\mathcal{I}(v_0) = 0, \mathcal{I}(v_1) = 1, \dots$

With  $\mathcal{I}_{v_0, \dots, v_n}^{a_0, \dots, a_n}$  we denote the assignment that is equal to  $\mathcal{I}$  except that it assigns the variables  $v_0, \dots, v_n$  to the elements  $a_0, \dots, a_n$  of the domain, e.g.  $\mathfrak{N} \mathcal{I}_{v_0, v_1}^{3, 1} \models Q(v_1, v_0)$  iff  $(1, 3) \in Q^{\mathbb{N}}$ . Due to the coincidence lemma in model theory, the assignment function is usually omitted and the notation is abbreviated as  $\mathfrak{N} \models Q(v_1, v_0)[1, 3]$ . We will use the longer notation in some places to avoid confusion in the assignment of free variables.

Let  $s = 2$  and  $\vec{a}_s = \langle 1, 3 \rangle$ . We have that

$$\begin{aligned} \mathfrak{N} \mathcal{I}_{v_0, v_1}^{3, 1} &\models P(v_0) \\ \mathfrak{N} \mathcal{I}_{v_0, v_1}^{3, 1} &\models \neg P(v_1) \\ \mathfrak{N} \mathcal{I}_{v_0, v_1}^{3, 1} &\models \neg Q(v_0, v_1) \\ \mathfrak{N} \mathcal{I}_{v_0, v_1}^{3, 1} &\models Q(v_0, v_0) \\ \mathfrak{N} \mathcal{I}_{v_0, v_1}^{3, 1} &\models Q(v_1, v_0) \\ \mathfrak{N} \mathcal{I}_{v_0, v_1}^{3, 1} &\models Q(v_1, v_1) \end{aligned}$$

Then,

$\Gamma_2^{(1,3)} = \{P(v_0), \neg P(v_1), Q(v_0, v_0), Q(v_1, v_0), \neg Q(v_0, v_1), Q(v_1, v_1)\}$ , and

$\phi_{\mathfrak{N}, \langle 1,3 \rangle}^0$  is the formula  $\bigwedge \Gamma_2^{(1,3)}$ .

$\phi_{\mathfrak{N}, \langle 1,3 \rangle}^1 = (\forall v_0 \bigvee_{a \in \mathbb{N}} \{\phi_{\mathfrak{N}, \langle 1,3,a \rangle}^0\}) \wedge (\bigwedge_{a \in \mathbb{N}} \{\exists v_0 \phi_{\mathfrak{N}, \langle 1,3,a \rangle}^0\})$ .

Suppose  $a = 4$ , then

$$\mathfrak{N}_{v_0, v_1, v_2}^{3,1,4} \models \neg R(v_0, v_0, v_0) \quad (6.2)$$

$$\mathfrak{N}_{v_0, v_1, v_2}^{3,1,4} \models R(v_0, v_1, v_2) \quad (6.3)$$

$$[\dots] \quad (6.4)$$

Thus,

$\phi_{\mathfrak{N}, \langle 1,3,4 \rangle}^0$  is  $\bigwedge \Gamma_3^{(1,3,4)}$  where,

$$\Gamma_3^{(1,3,4)} = \Gamma_2^{(1,3)} \cup \{\neg P(v_2), \dots, \neg R(v_0, v_0, v_0), R(v_0, v_1, v_2), \dots, \neg R(v_2, v_2, v_2)\},$$

for  $(\neg)R(v_i, v_j, v_w)$  with  $i, j, w \leq 2$  such that  $\mathfrak{N}_{v_0, v_1, v_2}^{3,1,4} \models (\neg)R(v_i, v_j, v_w)$ .

For every  $a \in \mathbb{N}$ , we will have in  $\Gamma_3^{(1,3,a)}$  formulas such as  $\neg R(v_0, v_0, v_0)$ ,  $\neg R(v_1, v_1, v_1)$ ,  $\neg R(v_2, v_2, v_2)$ . And, depending on the number  $a$ , we may have some formula  $\psi$  in  $\Phi_3$  such that  $\mathfrak{N}_{v_0, v_1, v_2}^{3,1,a} \models \psi$ , as is the case in (3) above.

Therefore, turning back to the main example,  $\phi_{\mathfrak{N}, \langle 1,3 \rangle}^1$  is the formula:

$$(\forall v_0 \bigvee \{\phi_{\mathfrak{N}, \langle 1,3,0 \rangle}^0, \phi_{\mathfrak{N}, \langle 1,3,1 \rangle}^0, \phi_{\mathfrak{N}, \langle 1,3,2 \rangle}^0, \phi_{\mathfrak{N}, \langle 1,3,3 \rangle}^0, \dots\}) \wedge \quad (6.5)$$

$$\bigwedge \{\exists v_0 (\phi_{\mathfrak{N}, \langle 1,3,0 \rangle}^0), \exists v_0 (\phi_{\mathfrak{N}, \langle 1,3,1 \rangle}^0), \exists v_0 (\phi_{\mathfrak{N}, \langle 1,3,2 \rangle}^0), \exists v_0 (\phi_{\mathfrak{N}, \langle 1,3,3 \rangle}^0), \dots\} \quad (6.6)$$

Since the set of formulas in  $\Phi_3$  is finite, there will be only  $k$  formulas such that,  $\phi_{\mathfrak{N}, \langle 1,3,n \rangle}^1 \neq \phi_{\mathfrak{N}, \langle 1,3,m \rangle}^1$ , for natural numbers  $k, n, m$  (the sets in the formulas 4 and 5 are infinite multisets, but finite sets). Therefore, the disjunctions and conjunctions in (5) and (6) are finite and thus,  $\phi_{\mathfrak{N}, \langle 1,3 \rangle}^1$  is a first-order formula.

Using the same example above, in the case there is no pre-defined sequence  $\vec{a}_s$  of elements of  $\mathbb{N}$  (*i.e.*, when  $s = 0$ ), the formula  $\phi_{\mathfrak{N}}^0 = 0$  and the

formula  $\phi_{\mathfrak{N}}^1$  is

$$\begin{aligned} & (\forall v_0 \bigvee \{\phi_{\mathfrak{N},\langle 0 \rangle}^0, \phi_{\mathfrak{N},\langle 1 \rangle}^0, \phi_{\mathfrak{N},\langle 2 \rangle}^0, \phi_{\mathfrak{N},\langle 3 \rangle}^0, \dots\}) \wedge \\ & \bigwedge \{\exists v_0(\phi_{\mathfrak{N},\langle 0 \rangle}^0), \exists v_0(\phi_{\mathfrak{N},\langle 1 \rangle}^0), \exists v_0(\phi_{\mathfrak{N},\langle 2 \rangle}^0), \exists v_0(\phi_{\mathfrak{N},\langle 3 \rangle}^0), \dots\} \end{aligned}$$

Where, for example,  $\phi_{\mathfrak{N},\langle 0 \rangle}^0 = \neg P(v_0)$ ,  $\phi_{\mathfrak{N},\langle 1 \rangle}^0 = \neg P(v_0)$  and  $\phi_{\mathfrak{N},\langle 3 \rangle}^0 = P(v_0)$ . Then,

$$\phi_{\mathfrak{N}}^1 = \forall v_0(\neg P(v_0) \vee P(v_0)) \wedge \exists v_0 \neg P(v_0) \wedge \exists v_0 P(v_0).$$

**Theorem 6.1.1.0.7.** *For a finite symbol set  $\tau$ , and a  $\tau$ -structure  $\mathfrak{U}$ , the set  $\{\phi_{\mathfrak{U},\vec{a}_s}^n \mid \vec{a}_s \in A^s\}$  is finite.*

*Proof.* By induction on  $n$ . Let  $n = 0$ . As we have seen in the example above, for a finite  $\tau$ , we have a finite  $\Phi_s$  (the set of atomic or negated atomic  $\tau$ -formulas with free variables among  $v_0, \dots, v_{s-1}$ ), and thus the set  $\Gamma_s^{\vec{a}_s} = \{\phi \mid \phi \in \Phi_s \text{ and } \mathfrak{U} \models \phi[\vec{a}_s]\}$  is finite. Thus  $\bigwedge \Gamma_s^{\vec{a}_s} = \phi_{\mathfrak{U},\vec{a}_s}^0$  is finite. To see that  $\{\phi_{\mathfrak{U},\vec{a}_s}^0 \mid \vec{a}_s \in A^s\}$  is finite, observe that, since  $\Phi_s$  will always be finite, there is some  $k$  such that there is at most  $k$  sequences  $\vec{a}_s$  and  $\vec{a}'_s$  of elements of  $A$ , such that  $\phi_{\mathfrak{U},\vec{a}_s}^0 \neq \phi_{\mathfrak{U},\vec{a}'_s}^0$  are pairwise distinct. Therefore,  $\{\phi_{\mathfrak{U},\vec{a}_s}^0 \mid \vec{a}_s \in A^s\}$  is finite.

Suppose the theorem holds for  $n$ .

$\Rightarrow$  Let  $a \in A$ ,  $\vec{a}_s a \in A^{s+1}$ , so by the induction hypothesis,

$\Rightarrow \Delta' = \{\phi_{\mathfrak{U},\vec{a}_s a}^n \mid \vec{a}_s a \in A^{s+1}\}$  is finite,

$\Rightarrow$  therefore,  $\forall v_s(\bigvee_{a \in A} \{\phi_{\mathfrak{U},\vec{a}_s a}^n\})$  is finite,

$\Rightarrow$  also  $\bigwedge_{a \in A} \{\exists v_s \phi_{\mathfrak{U},\vec{a}_s a}^n\}$  is finite;

$\Rightarrow$  then,  $\phi_{\mathfrak{U},a^s}^{n+1} = \forall v_s(\bigvee_{a \in A} \{\phi_{\mathfrak{U},\vec{a}_s a}^n\}) \wedge (\bigwedge_{a \in A} \{\exists v_s \phi_{\mathfrak{U},\vec{a}_s a}^n\})$  is finite.

$\Rightarrow$  to see that  $\{\phi_{\mathfrak{U},a^s}^{n+1} \mid a^s \in A^s\}$  is finite one uses an argument similar to the one used in the proof for  $n = 0$ .

□

**Theorem 6.1.1.0.8.** *For a finite and relational  $\tau$ , a  $\tau$ -structure  $\mathfrak{U}$ ,  $\vec{a}_s \in A^s$  and  $n \in \mathbb{N}$ ,  $\phi_{\mathfrak{U},\vec{a}_s}^n$  has at most  $s$  free variables.*



*Proof.* By induction on  $n$ . For  $n = 0$ , it follows from definition, since  $\phi_{\mathfrak{U}, \vec{a}_s}^0 = \bigwedge \Gamma_s^{\vec{a}_s}$  and  $\bigwedge \Gamma_s^{\vec{a}_s}$  by definition has at most  $s$  free variables. Supposing the theorem holds for  $n$ , for every  $a \in A$ , by the definition of  $\phi_{\mathfrak{U}, \vec{a}_s a}^n$  it follows that it has the variable  $v_s$  free. By definition of  $\phi_{\mathfrak{U}, \vec{a}_s}^{n+1}$ , this variable is bounded, thus  $\phi_{\mathfrak{U}, \vec{a}_s}^{n+1}$  has at most  $s$  free variables.  $\square$

**Theorem 6.1.1.0.9.** *With the same conditions as the above theorem, for every  $n \in \mathbb{N}$ ,  $\mathfrak{U} \models \phi_{\mathfrak{U}, \vec{a}_s}^n[\vec{a}_s]$*

*Proof.* By induction on  $n$ . For  $n = 0$  the result follows immediately by the definition. Suppose the theorem holds for  $n$ , then  $\mathfrak{U} \models \phi_{\mathfrak{U}, \vec{a}_s}^n[\vec{a}_s]$ .

By the induction hypothesis, we get (i) that  $\mathfrak{U} \models \phi_{\mathfrak{U}, \vec{a}_s a}^n[\vec{a}_s a]$  for every  $a \in A$ . Then, it holds that  $\mathfrak{U} \models \bigvee_{a \in A} \phi_{\mathfrak{U}, \vec{a}_s a}^n[\vec{a}_s a]$ , since we have proven above that this disjunction is finite. Then, by introduction of universal quantifier  $\mathfrak{U} \models \forall v_s \bigvee_{a \in A} \phi_{\mathfrak{U}, \vec{a}_s a}^n[\vec{a}_s]$ .

Also,  $\mathfrak{U} \models \exists v_s (\phi_{\mathfrak{U}, \vec{a}_s a}^n)[\vec{a}_s]$  and, since this holds for every  $a \in A$  (by i),  $\mathfrak{U} \models \bigwedge_{a \in A} \{\exists v_s (\phi_{\mathfrak{U}, \vec{a}_s a}^n)\}[\vec{a}_s]$ , since this conjunction is finite, as we have proven above.  $\square$

**Definition 6.1.1.0.10.** *The quantifier rank of a formula  $\phi$  or  $qr(\phi)$  is the number of nested quantifiers a given formula has. It is defined inductively as follows:*

- if  $\phi$  is an atomic formula, then  $qr(\phi) = 0$ ;
- if  $\neg\phi$  is a formula, then  $qr(\neg\phi) = qr(\phi)$ ;
- if  $\phi$  and  $\psi$  are formulas, then  $qr(\phi \vee \psi) = \max(qr(\phi), qr(\psi))$ ;
- if  $\exists v_s \phi$  is a formula, then  $qr(\exists v_s \phi) = qr(\phi) + 1$ .

**Theorem 6.1.1.0.11.** *For every  $n \in \mathbb{N}$ ,  $qr(\phi_{\mathfrak{U}, \vec{a}_s}^n) = n$ .*

*Proof.* By induction on  $n$ . For  $n = 0$  the result is immediate. We have that  $\phi_{\mathfrak{U}, \vec{a}_s}^{n+1} = (\forall v_s \bigvee_{a \in A} \{\phi_{\mathfrak{U}, \vec{a}_s a}^n\}) \wedge (\bigwedge_{a \in A} \{\exists v_s \phi_{\mathfrak{U}, \vec{a}_s a}^n\})$ . By the induction hypothesis,  $qr(\phi_{\mathfrak{U}, \vec{a}_s a}^n) = n$ . Therefore,  $qr(\phi_{\mathfrak{U}, \vec{a}_s}^{n+1}) = n + 1$ .  $\square$

Now we connect the above results with the Ehrenfeucht Games presented above.

**Theorem 6.1.1.0.12.** *Ehrenfeucht Theorem.* For a finite relational  $\tau$  and  $\tau$ -structures  $\mathfrak{U}$  and  $\mathfrak{B}$ , sequences  $\vec{a}_s \in A^s$  and  $\vec{b}_s \in B^s$  and  $n \in \mathbb{N}$ , the following are equivalent:

1. Duplicator wins  $G_n(\mathfrak{U}, \vec{a}_s, \mathfrak{B}, \vec{b}_s)$ ;
2. For every  $L(\tau)$ -formula  $\psi(v_0, \dots, v_{s-1})$  with quantifier rank  $\leq n$ ,  $\mathfrak{U} \models \phi[\vec{a}_s]$  iff  $\mathfrak{B} \models \phi[\vec{b}_s]$ ;
3.  $\mathfrak{B} \models \phi_{\mathfrak{U}, \vec{a}_s}^n[\vec{b}_s]$ .

*Proof.* 1  $\leftrightarrow$  3

By induction on  $n$ . For  $n = 0$ : Duplicator wins  $G_0(\mathfrak{U}, \vec{a}_s, \mathfrak{B}, \vec{b}_s)$  iff  $\vec{a}_s \mapsto \vec{b}_s$  is a partial isomorphism from  $\mathfrak{U}$  to  $\mathfrak{B}$  (by the definition of Ehrenfeucht games) iff for every quantifier free sentence  $\phi(\vec{v}_s)$ ,  $\mathfrak{U} \models \phi[\vec{a}_s] \Leftrightarrow \mathfrak{B} \models \phi[\vec{b}_s]$  (by Remark 3). Then,  $\mathfrak{B} \models \bigwedge \Gamma_0^{\mathfrak{U}, \vec{a}_s}[\vec{b}_s]$  ( $\Gamma_0^{\mathfrak{U}, \vec{a}_s}$  is the set of sentences from  $\Phi_s$  that are satisfied by  $\mathfrak{U}$  with  $\vec{a}_s$ ).

Inductive step:

1. Duplicator wins  $G_{n+1}(\mathfrak{U}, \vec{a}_s, \mathfrak{B}, \vec{b}_s)$  iff
2. (i) for every  $a \in A$ , there's a  $b \in B$  such that Duplicator wins  $G_n(\mathfrak{U}, \vec{a}_s a, \mathfrak{B}, \vec{b}_s b)$ ;  
(ii) for every  $b' \in B$ , there's some  $a' \in A$  such that Duplicator wins  $G_n(\mathfrak{U}, \vec{a}_s a', \mathfrak{B}, \vec{b}_s b')$  (by Remark 3) iff
3.  $\mathfrak{B} \models \phi_{\mathfrak{U}, \vec{a}_s a}^n[\vec{b}_s, b]$  (induction hypothesis)
4.  $\mathfrak{B} \models \forall v_s \bigvee_{a \in A} \{\phi_{\mathfrak{U}, \vec{a}_s a}^n\}[\vec{b}_s]$  (by (2.ii))
5.  $\mathfrak{B} \models \bigwedge_{a \in A} \{\exists v_s(\phi_{\mathfrak{U}, \vec{a}_s a}^n)\}[\vec{b}_s]$  (by (2.i))
6.  $\mathfrak{B} \models (\forall v_s \bigvee_{a \in A} \{\phi_{\mathfrak{U}, \vec{a}_s a}^n\} \wedge \bigwedge_{a \in A} \{\exists v_s(\phi_{\mathfrak{U}, \vec{a}_s a}^n)\})[\vec{b}_s]$
7.  $\mathfrak{B} \models \phi_{\mathfrak{U}, \vec{a}_s}^{n+1}[\vec{b}_s]$

1  $\rightarrow$  2

1. For  $n = 0$  it follows by definition by Remark 3 and by definition of partial isomorphism;
2. Suppose Duplicator wins  $G_{n+1}(\mathfrak{U}, \vec{a}_s, \mathfrak{B}, \vec{b}_s)$ ;
3. (i) for every  $a \in A$ , there's a  $b \in B$  such that Duplicator wins  $G_n(\mathfrak{U}, \vec{a}_s a, \mathfrak{B}, \vec{b}_s b)$ ; (ii) for every  $b' \in B$ , there's some  $a' \in A$  such that Duplicator wins  $G_n(\mathfrak{U}, \vec{a}_s a', \mathfrak{B}, \vec{b}_s b')$  (by Remark 3);
4. let  $\psi(v_0, \dots, v_{s-1})$  (or simply  $\psi(\vec{v}_s)$ ) be a formula with quantifier rank  $\leq n + 1$ ;
5.  $\psi(\vec{v}_s) = \neg\psi_1(\vec{v}_s)$ ,  $\psi(\vec{v}_s) = \psi_1(\vec{v}_s) \vee \psi_2(\vec{v}_s)$  or  $\psi(\vec{v}_s) = \exists y\psi_1(\vec{v}_s)$ ;
  - (a) Suppose  $\psi(\vec{v}_s) = \exists y\psi_1(\vec{v}_s)$ ;
  - (b) Then,  $qr(\psi_1(\vec{v}_s, y)) \leq n$ ;
  - (c) Then, (by (2.i) and (2.ii)) the induction hypothesis gives:  $\mathfrak{U} \models \psi_1(\vec{v}_s, y)[\vec{a}_s, a]$  iff  $\mathfrak{U} \models \psi_1(\vec{v}_s, y)[\vec{b}_s, b]$ ;
  - (d) By (c), it follows that  $\mathfrak{U} \models \exists y\psi_1(\vec{v}_s)[\vec{a}_s]$  iff  $\mathfrak{U} \models \exists y\psi_1(\vec{v}_s)[\vec{b}_s]$ ;
6. For the cases when  $\psi(\vec{v}_s) = \neg\psi_1(\vec{v}_s)$  or  $\psi(\vec{v}_s) = \psi_1(\vec{v}_s) \vee \psi_2(\vec{v}_s)$ , one considers a sub-formula  $\psi_n$  of  $\psi(\vec{v}_s)$  such that any sub-formula of  $\psi_n$  has quantifier rank  $\leq n$ ;
7. Then, the proof will go as in the above case.

2  $\rightarrow$  3

1. By theorem 3.9.

□

**Definition 6.1.1.0.13.**  $\mathfrak{U} \equiv_m \mathfrak{B}$

For two structures  $\mathfrak{U}$  and  $\mathfrak{B}$  and  $m \in \mathbb{N}$  we write  $\mathfrak{U} \equiv_m \mathfrak{B}$  if  $\mathfrak{U}$  and  $\mathfrak{B}$  satisfy the same first-order sentences of quantifier rank  $\leq m$ .

The following is a corollary of the Ehrenfeucht theorem:

**Corollary 6.1.1.0.14.** For two structures  $\mathfrak{U}$  and  $\mathfrak{B}$  and  $m \geq 0$  the following are equivalent:

- (a) The Duplicator wins  $G_m(\mathfrak{U}, \mathfrak{B})$ ,
- (b)  $\mathfrak{B} \models \phi_{\mathfrak{U}}^m$ ,
- (c)  $\mathfrak{U} \equiv_m \mathfrak{B}$ .

**Definition 6.1.1.0.15.** Given structures  $\mathfrak{U}$  and  $\mathfrak{B}$ , for  $s \geq 0$  let  $\vec{a}_s \in A^s$ ,  $\vec{b}_s \in B^s$  and  $m \in \mathbb{N}$ , then we define  $W_m(\mathfrak{U}, \mathfrak{B})$  as the set of winning positions for the Duplicator:

$$W_m(\mathfrak{U}, \mathfrak{B}) = \{\vec{a}_s \mapsto \vec{b}_s \mid \text{the Duplicator wins } G_m(\mathfrak{U}, \vec{a}_s, \mathfrak{B}, \vec{b}_s)\}.$$

By the definition of Ehrenfeucht game, the sequence  $(W_n(\mathfrak{U}, \mathfrak{B}))_{n \leq m}$  has the back and forth properties listed below:

**Definition 6.1.1.0.16.** Two structures  $\mathfrak{U}$  and  $\mathfrak{B}$  are said to be  $m$ -isomorphic (in symbols,  $\mathfrak{U} \cong_m \mathfrak{B}$ ), if there is a sequence  $(I_j)_{j \leq m}$  of partial isomorphisms such that:

- (a) Every  $I_j$  is a nonempty set of partial isomorphisms from  $\mathfrak{U}$  to  $\mathfrak{B}$ ;
- (b) (forth property) For every  $j < m$ ,  $p \in I_{j+1}$  and  $a \in A$ , there is a  $q \in I_j$  such that  $q \supseteq p$  and  $a \in \text{dom}(q)$ .
- (c) (back property) For every  $j < m$ ,  $p \in I_{j+1}$  and  $b \in B$ , there is a  $q \in I_j$  such that  $q \supseteq p$  and  $b \in \text{rng}(q)$ .

If a sequence  $(I_j)_{j \leq m}$  has these three properties, we say that  $\mathfrak{U}$  and  $\mathfrak{B}$  are  $m$ -isomorphic via  $(I_j)_{j \leq m}$  or, in symbols,  $(I_j)_{j \leq m} : \mathfrak{U} \cong_m \mathfrak{B}$ .

Now we link the above definition of  $m$ -isomorphism with the other concepts formerly defined.

**Theorem 6.1.1.0.17.** For structures  $\mathfrak{U}$  and  $\mathfrak{B}$ , for  $s \geq 0$  let  $\vec{a}_s \in A^s$ ,  $\vec{b}_s \in B^s$  and  $m \in \mathbb{N}$ , the following are equivalent:

1. The Duplicator wins  $G_m(\mathfrak{U}, \vec{a}_s, \mathfrak{B}, \vec{b}_s)$ ;
2.  $\vec{a}_s \mapsto \vec{b}_s \in W_m(\mathfrak{U}, \mathfrak{B})$  and  $(W_j(\mathfrak{U}, \mathfrak{B}))_{j \leq m} : \mathfrak{U} \cong_m \mathfrak{B}$ ;
3. There is a sequence  $(I_j)_{j \leq m}$  with  $\vec{a}_s \mapsto \vec{b}_s \in I_m$ , such that  $(I_j)_{j \leq m} : \mathfrak{U} \cong_m \mathfrak{B}$ ;
4.  $\mathfrak{B} \models \phi_{\mathfrak{U}, \vec{a}_s}^m[\vec{b}_s]$ ;
5.  $\vec{a}_s$  satisfies in  $\mathfrak{U}$  the same formulas of quantifier rank  $\leq m$  as  $\vec{b}_s$  in  $\mathfrak{B}$ .

*Proof.* (2  $\rightarrow$  3): As  $(W_j(\mathfrak{U}, \mathfrak{B}))_{j \leq m}$  is a sequence of partial isomorphisms that configures winning positions for the Duplicator in each game from 0 to  $m$ , then, by the definition of winning strategy for the Duplicator, the sequence have the back and forth property. So, it is the same as a sequence  $(I_j)_{j \leq m} : \mathfrak{U} \cong_m \mathfrak{B}$ .

(1  $\leftrightarrow$  2): By definition.

(3  $\rightarrow$  1): Suppose  $(I_j)_{j \leq m} : \mathfrak{U} \cong_m \mathfrak{B}$ , and  $\vec{a}_s \mapsto \vec{b}_s \in I_m$ . It is easy to describe a winning strategy for the Duplicator based on this sequence. Without loss of generality, let it be that on the  $i^{\text{th}}$ -move the Spoiler chooses and element  $a_i$  of  $A$ , then  $a_i$  is in the domain of some partial isomorphism  $p \in I_n$  (by the forth property), then, the Duplicator answers with the  $b_i \in B$  such that  $p(a_i) = b_i$ .

(1  $\leftrightarrow$  4  $\leftrightarrow$  5) by the previous theorem.  $\square$

Analogous to the previous theorem, now we have a corollary for  $s = 0$ , the equivalence of items iii and iv are known as the Fraïssé's Theorem.

**Corollary 6.1.1.0.18.** *For structures  $\mathfrak{U}$  and  $\mathfrak{B}$  and  $m \geq 0$ , the following are equivalent:*

- (i) *The Duplicator wins  $G_m(\mathfrak{U}, \mathfrak{B})$*
- (ii)  $(W_j(\mathfrak{U}, \mathfrak{B}))_{j \leq m} : \mathfrak{U} \cong_m \mathfrak{B}$
- (iii)  $\mathfrak{U} \cong_m \mathfrak{B}$
- (iv)  $\mathfrak{U} \equiv_m \mathfrak{B}$
- (v)  $\mathfrak{B} \models \phi_{\mathfrak{U}}^m$ .

**Definition 6.1.1.0.19** (Partially isomorphic structures).  *$\mathfrak{U}$  and  $\mathfrak{B}$  are said to be partially isomorphic (written as  $\mathfrak{U} \cong_p \mathfrak{B}$ ) iff there is a non-empty set  $I$  of partial isomorphisms from  $\mathfrak{U}$  to  $\mathfrak{B}$  that satisfy the back and forth properties.*

The following theorem about partially isomorphic sentences play an important role in the proof of Lindström's Theorem to follow. It is an abstract version of Cantor's theorem that any two countable dense orderings without endpoints are isomorphic.

**Theorem 6.1.1.0.20.** *If  $\mathfrak{U} \cong_p \mathfrak{B}$  and  $\mathfrak{U}$  and  $\mathfrak{B}$  are at most countable, then  $\mathfrak{U} \cong \mathfrak{B}$ .*

*Proof.* (idea of the proof)

From the fact that both domains are countable, one can construct a sequence  $(p_n)_{n \in \mathbb{N}}$  of partial isomorphisms from  $\mathfrak{U}$  to  $\mathfrak{B}$  and obtain a function  $p$  that is an isomorphism from  $\mathfrak{U}$  to  $\mathfrak{B}$  by  $p = \bigcup_{n \in \mathbb{N}} p_n$ .  $\square$

## 6.1.2 Lindström's Theorems

Having the required tool at hand, we can enter the proper terrain of the theorems. As the theorems talk about logical systems being more or less expressive than others, one has to define exactly what is meant by these

concepts. A curious thing is that Lindström did not bother to define carefully what he meant by logical system, or, avoiding troublesome conceptual analyses, he straightforwardly defined logical systems as collection of elementary classes of structures. As for relative expressiveness, he used the definition that a system  $A$  is at least as expressive as another  $B$  if every class of structures that is elementary in  $B$  is also elementary in  $A$ .

Afterwards, others (e.g. Barwise [Bar74], and Ebbinghaus et al [EFT96]) have tried to fill the gap and presented more detailed definitions of logical systems, giving axioms and structural restrictions that a candidate for a logical system must comply. Below, we will recall the definition of (abstract) logical system and expressiveness, state and prove Lindström's theorems, cited just below to refresh the memory:

- Any logical system  $\mathcal{L}$  that is more expressive than first-order logic ( $\mathcal{L}_{\mathcal{I}}$ ) lacks either the compactness or the Löwenheim-Skolem theorem.
- Any logical system  $\mathcal{L}$  that is more expressive than  $\mathcal{L}_{\mathcal{I}}$  either lacks Löwenheim-Skolem theorem or the set of its validities is not recursively enumerable, that is, there can be no completeness theorem for it.

### 6.1.2.1 Abstract Logical Systems

In the original definition of Lindström [Lin69], the term used is “generalized first-order logic”, and it is defined as a sequence of a set of formulas and a binary relation on the class of appropriate structures and the class of formulas, *i.e.* a satisfiability relation. He then defines the notion of an elementary class of structures, *i.e.* the class of structures that are models of a given sentence. This class of structures is then required to satisfy certain properties, such as being closed under isomorphism, etc.

Barwise [Bar74, p. 259] criticizes Lindström approach because Lindström directly deals with classes of structures, and not with sentences that define them, therefore avoiding any kind of syntactic consideration. He says [*ibid*]:

We find this approach unsatisfying on two grounds. In the first place, it seems contrary to the very spirit of model theory where *the primary object of study is the relationship between syntactic objects and the structures they define*. Secondly, it fails to make explicit that *the closure conditions on the classes of structures* (like the formation of indexed unions and its inverse) *arise out of natural syntactic considerations*, considera-

tions which seem implicit in the very idea of a model-theoretic language.

The presentation of Lindström theorems in [EFT96] adheres to Barwise's suggestion, and some syntactic content of logical systems are taken into account. As we have said in the beginning, we will basically follow that excellent book, filling the gaps we find not clear, modifying the manner and order of presentation of some items whenever we find it more illuminating for the non-expert reader.

**Definition 6.1.2.1.1** (Abstract Logical System). *An **abstract logical system**  $\mathcal{L}$  is composed by a function  $L$  and a binary relation  $\models_{\mathcal{L}}$ .  $L$  associates with a given symbol set  $\tau$  a set  $L(\tau)$  of  $\tau$ -sentences of  $\mathcal{L}$ . The function  $L$  and the relation  $\models_{\mathcal{L}}$  are required to satisfy:*

1. *If  $\tau_0 \subseteq \tau_1$  then  $L(\tau_0) \subseteq L(\tau_1)$ ;*
2. *If  $\mathfrak{U} \models_{\mathcal{L}} \phi$ , then, for some  $\tau$ ,  $\mathfrak{U}$  is a  $\tau$ -structure and  $\phi \in L(\tau)$  (that is, the relation  $\models_{\mathcal{L}}$  holds only for structures and sentences of the same type);*
3. *(**Isomorphism property**) If  $\mathfrak{U} \models_{\mathcal{L}} \phi$  and  $\mathfrak{U} \cong \mathfrak{B}$ , then  $\mathfrak{B} \models_{\mathcal{L}} \phi$ ;*
4. *(**Reduct property**) If  $\tau_0 \subset \tau_1$ ,  $\phi \in L(\tau_0)$ , and  $\mathfrak{U}$  is an  $\tau_1$ -structure, then*

$$\mathfrak{U} \models_{\mathcal{L}} \phi \text{ iff } \mathfrak{U}|_{\tau_0} \models_{\mathcal{L}} \phi$$

*( $\mathfrak{U}|_{\tau_0}$  is the restriction of the structure  $\mathfrak{U}$  to the symbol set  $\tau_0$ ).*

Attached to this definition of abstract logic is the model-theoretic definition of the meaning of  $L(\tau)$ -sentences:

**Definition 6.1.2.1.2.** *If  $\phi \in L(\tau)$  and  $\mathcal{L}$ , then the **meaning** of  $\phi$  is given by:*

$$\text{Mod}_{\mathcal{L}}^{\tau}(\phi) = \{\mathfrak{U} \mid \mathfrak{U} \text{ is a } \tau\text{-structure and } \mathfrak{U} \models_{\mathcal{L}} \phi\}.$$

From the definition of the meaning of sentences follows the definition of relative expressiveness between logical systems: a logical system  $\mathcal{L}'$  is *at least as expressive* as  $\mathcal{L}$  iff for every sentence of  $\mathcal{L}$  there is a sentence of  $\mathcal{L}'$  with the **same meaning**.

**Definition 6.1.2.1.3** (Relative Expressiveness). *Let  $\mathcal{L}$ ,  $\mathcal{L}'$  be logical systems:*

- *Let  $\tau$  be a symbol set,  $\phi \in L(\tau)$ ,  $\psi \in L'(\tau)$ .  $\phi$  and  $\psi$  are said to be logically equivalent iff  $Mod_{\mathcal{L}}^{\tau}(\phi) = Mod_{\mathcal{L}'}^{\tau}(\psi)$ .*
- *$\mathcal{L}'$  is at least as expressive as  $\mathcal{L}$  (in symbols  $\mathcal{L} \leq \mathcal{L}'$ ) iff for every  $L(\tau)$ -sentence  $\phi$ , there is an  $L'(\tau)$ -sentence  $\psi$  such that  $\psi$  and  $\phi$  are logically equivalent.*
- *$\mathcal{L}$  and  $\mathcal{L}'$  are equally strong (in symbols  $\mathcal{L} \sim \mathcal{L}'$ ) iff  $\mathcal{L} \leq \mathcal{L}'$  and  $\mathcal{L}' \leq \mathcal{L}$ .*

Now some criteria on the logical systems will be stated in order to restrict them to what we will call *regular logical systems*.

**Definition 6.1.2.1.4.** ***Boole**( $\mathcal{L}$ ), or  $\mathcal{L}$  is able to express the propositional Boolean connectives iff*

1. *Given a  $\tau$  and  $\phi \in L(\tau)$ , there is a  $\chi \in L(\tau)$  such that for every  $\tau$ -structure  $\mathfrak{U}$ ,  $\mathfrak{U} \models_{\mathcal{L}} \phi$  iff  $\mathfrak{U} \not\models_{\mathcal{L}} \chi$ .*
2. *Given a  $\tau$  and  $\phi, \psi \in L(\tau)$ , there is a  $\chi \in L(\tau)$  such that for every  $\tau$ -structure  $\mathfrak{U}$ ,  $\mathfrak{U} \models_{\mathcal{L}} \chi$  iff  $\mathfrak{U} \models_{\mathcal{L}} \phi$  or  $\mathfrak{U} \models_{\mathcal{L}} \psi$ .*

In the course of the proofs bellow, for some abstract logical system  $\mathcal{L}$ , if **Boole**( $\mathcal{L}$ ) holds,  $\neg\phi$  and  $\phi \vee \psi$  are to stand for the formulas  $\chi$  in (1) and (2) respectively.

**Definition 6.1.2.1.5.** ***Rel**( $\mathcal{L}$ ), or  $\mathcal{L}$  permits relativization iff*

*for  $\phi \in L(\tau)$  and unary  $U$ , there's a  $\psi \in L(\tau \cup \{U\})$  such that*

$$(\mathfrak{U}, U^A) \models_{\mathcal{L}} \psi \text{ iff } [U^A]^{\mathfrak{U}} \models_{\mathcal{L}} \phi$$

Where  $[U^A]^{\mathfrak{U}}$  is the substructure of  $\mathfrak{U}$  with domain  $U^A$ . If **Rel**( $\mathcal{L}$ ), then we refer to such  $\psi$  as  $\phi^U$ .

In case there are function symbols in  $\tau$ ,  $[U^A]^{\mathfrak{U}}$  will not be defined if  $U^A$  is not  $\tau$ -closed. It is important then to assure that  $\tau$  is relational. This is done by the following property.

For a symbol set  $\tau$ , one obtains the relational version  $\tau^r$  of  $\tau$  by replacing each  $n$ -ary function symbol by an  $n + 1$ -ary relation symbol, and each constant by an unary predicate. The  $\tau^r$ -structure  $\mathfrak{U}^r$  is obtained by  $\mathfrak{U}$  by interpreting each new  $n + 1$ -ary relation symbol by the graph of the function in  $\mathfrak{U}$  and each new unary predicate will contain only the element of  $A$  that corresponds to the constant being substituted.



**Definition 6.1.2.1.6.**  *$\mathbf{Repl}(\mathcal{L})$ , or  $\mathcal{L}$  permits replacement of function symbols and constants by relation symbols iff*

*If  $\tau$  is a symbol set, and  $\tau^r$  is the relational version of  $\tau$ , then for any  $\phi \in L(\tau)$ , there's a  $\psi \in L(\tau^r)$  such that*

$$\mathfrak{U} \models \phi \text{ iff } \mathfrak{U}^r \models \psi.$$

Thus, if  $\mathbf{Repl}(\mathcal{L})$  for any  $\tau$ , we may assume that  $\tau$  is relational when dealing with  $\mathcal{L}$ -sentences.

**Definition 6.1.2.1.7** (Regular Logical System). *A logical system  $\mathcal{L}$  is said to be **regular** if it is the case that  $\mathbf{Boole}(\mathcal{L})$ ,  $\mathbf{Repl}(\mathcal{L})$  and  $\mathbf{Rel}(\mathcal{L})$ .*

Other properties of logical systems that have a crucial role in the following characterization theorems are given below.

**LöSko**( $\mathcal{L}$ ), or *the Löwenheim-Skolem theorem holds for  $\mathcal{L}$ :*

*If  $\phi \in L(\tau)$  is satisfiable, then there is a model of  $\phi$  whose domain is at most countable.*

**Compact**( $\mathcal{L}$ ), or *the compactness theorem holds for  $\mathcal{L}$ :*

*A set of formulas  $\Phi \subset L(\tau)$  has a model if every finite subset of  $\Phi$  has a model.*

### 6.1.2.2 Compact logical systems

We will now prove a theorem for compact logical systems that give a good insight on the role of compactness in the meaning of sentences in these: if **Compact**( $\mathcal{L}$ ) holds, then the meaning of any  $L(\tau)$ -sentence depends only on finitely many symbols from  $\tau$ . We assume that  $\mathcal{L}$  is at least as expressive as  $\mathcal{L}_I$ , the first-order logic, *i.e.*  $\mathcal{L}_I \leq \mathcal{L}$ . Then, since for every first-order sentence  $\phi$  there's a logically equivalent  $\psi$  in  $\mathcal{L}$ , we will refer to this last sentence as  $\phi^*$ , in the following theorems, and move back and forth from  $\phi$  and  $\phi^*$  referring to a first order sentence, and its logically equivalent counterpart  $\phi^*$  in  $\mathcal{L}$ .

**Theorem 6.1.2.2.1.** *Let it be that **Compact**( $\mathcal{L}$ ) and  $\psi \in L(\tau)$ . Then there is a finite  $\tau_0 \subseteq \tau$  such that for all  $\tau$ -structures  $\mathfrak{U}$  and  $\mathfrak{B}$*

$$\text{if } \mathfrak{U} \upharpoonright \tau_0 \cong \mathfrak{B} \upharpoonright \tau_0, \text{ then } (\mathfrak{U} \models_{\mathcal{L}} \psi \text{ iff } \mathfrak{B} \models_{\mathcal{L}} \psi)$$

*Proof.* Let  $\Phi$  be the following set of  $\tau \cup \{U, V, f\}$ -sentences, for unary relations  $U, V$  and unary function  $f$ .  $\Phi$  is intended to describe the isomorphism between two  $\tau$ -structures  $\mathfrak{A}$  and  $\mathfrak{B}$ :

$$\begin{aligned} & \exists x U(x); \exists x V(x); \\ & \forall x (U(x) \rightarrow \exists y (V(y) \wedge f(x) = y)); \\ & \forall xy (U(x) \wedge U(y) \wedge f(x) = f(y) \rightarrow x = y); \\ & \forall xy (U(x) \wedge U(y) \wedge f(x) \neq f(y) \rightarrow x \neq y); \\ & \forall x (V(x) \rightarrow \exists y (U(y) \wedge f(y) = x)) \text{ and, for every } n\text{-ary } R \in \tau, \\ & \forall x_1, \dots, x_n (U(x_1) \wedge \dots \wedge U(x_n) \rightarrow (R(x_1, \dots, x_n) \leftrightarrow R(f(x_1), \dots, f(x_n))))). \end{aligned}$$

Let  $\Phi^*$  be the set of  $\mathcal{L}$  sentences logically equivalent with the  $L_I$  sentences of  $\Phi$  (remember that  $\mathcal{L}_I \leq \mathcal{L}$ ), and let  $(\mathfrak{A}, U^A, V^A, f^A)$  be a model of  $\Phi^*$ . Then,  $U^A, V^A$  are non-empty and  $f^A$  is an isomorphism from  $[U^A]^\mathfrak{A}$  to  $[V^A]^\mathfrak{A}$ . Then

$$\begin{aligned} & \Rightarrow [U^A]^\mathfrak{A} \cong [V^A]^\mathfrak{A}; \\ & \Rightarrow \text{and, for any } \psi \in L(\tau), [U^A]^\mathfrak{A} \models_{\mathcal{L}} \psi \text{ iff } [V^A]^\mathfrak{A} \models_{\mathcal{L}} \psi, \text{ by the isomorphism property;} \\ & \Rightarrow \text{thus, } (\mathfrak{A}, U^A, V^A, f^A) \models_{\mathcal{L}} \psi^U \text{ iff } (\mathfrak{A}, U^A, V^A, f^A) \models_{\mathcal{L}} \psi^V, \text{ by } \mathbf{Rel}(\mathcal{L}); \\ & \Rightarrow \text{then } (\mathfrak{A}, U^A, V^A, f^A) \models_{\mathcal{L}} \psi^U \leftrightarrow \psi^V, \text{ by } \mathbf{Boole}(\mathcal{L}); \\ & \Rightarrow \text{therefore, } \Phi^* \models_{\mathcal{L}} \psi^U \leftrightarrow \psi^V; \\ & \Rightarrow \text{then, for some finite } \Phi_0^* \subseteq \Phi^*, \Phi_0^* \models_{\mathcal{L}} \psi^U \leftrightarrow \psi^V, \text{ by } \mathbf{Compact}(\mathcal{L}). \end{aligned}$$

Let  $\tau_0$  be the symbol set of the sentences in  $\Phi_0^*$ .

$$\Rightarrow \text{Suppose that } \mathfrak{A} \upharpoonright \tau_0 \cong^{\pi} \mathfrak{B} \upharpoonright \tau_0.$$

Let  $\tau_1 = \tau \cup \{U, V, f\}$ , being the symbols  $U, V, f$ , the same as above. By the isomorphism property, we may assume that the domains of  $\mathfrak{A}$  and  $\mathfrak{B}$  are disjoint, so  $A \cap B = \emptyset$ . We define now a  $\tau_1$  structure  $\mathfrak{C}$  as follows:  $C = A \cup B$ ,  $U^C = A$ ,  $V^C = B$ ,  $R_i^C = R_i^A \cup R_i^B$ , for every  $R_i \in \tau$ , and  $f^C$  is any function such that  $f^C \upharpoonright A = \pi$ .

$$\begin{aligned} & \Rightarrow \text{thus, by construction, } \mathfrak{C} \models_{\mathcal{L}} \Phi_0^*, \text{ and then } \mathfrak{C} \models_{\mathcal{L}} \psi^U \leftrightarrow \psi^V; \\ & \Rightarrow \text{then, } [U^C]^\mathfrak{C} \models_{\mathcal{L}} \psi \text{ iff } [V^C]^\mathfrak{C} \models_{\mathcal{L}} \psi \text{ by } \mathbf{Boole}(\mathcal{L}) \text{ and } \mathbf{Rel}(\mathcal{L}); \\ & \Rightarrow \text{then, } [U^C]^\mathfrak{C} \upharpoonright \tau \models_{\mathcal{L}} \psi \text{ iff } [V^C]^\mathfrak{C} \upharpoonright \tau \models_{\mathcal{L}} \psi \text{ by the reduct property;} \end{aligned}$$

$\Rightarrow$  as  $[U^C]^{e|\tau} = \mathfrak{U}$  and  $[V^C]^{e|\tau} = \mathfrak{B}$ ,

$\Rightarrow$  it follows that  $\mathfrak{U} \models_{\mathcal{L}} \psi$  iff  $\mathfrak{B} \models_{\mathcal{L}} \psi$ .

□

### 6.1.2.3 Lindström's First theorem

**Theorem 6.1.2.3.1.** *If  $\mathcal{L}$  is a regular logic that extends  $\mathcal{L}_I$ , but such that there's a  $\tau$ -sentence  $\psi$  that is not logically equivalent to any  $\mathcal{L}_I$ -sentence, then, there are  $\tau$ -structures  $\mathfrak{U}$  and  $\mathfrak{B}$  s.t. for every  $n \in \mathbb{N}$ , and for every finite  $\tau_0 \subseteq \tau$  and such that*

$$\mathfrak{U} \upharpoonright \tau_0 \cong_n \mathfrak{B} \upharpoonright \tau_0, \mathfrak{U} \models_{\mathcal{L}} \psi \text{ and } \mathfrak{B} \models_{\mathcal{L}} \neg\psi.$$

let  $\phi = \bigvee \{ \phi_{\mathfrak{U} \upharpoonright \tau_0}^n \mid \mathfrak{U} \text{ is a } \tau\text{-structure and } \mathfrak{U} \models_{\mathcal{L}} \psi \}$

$\Rightarrow$  as  $\tau_0$  is finite,  $\phi$  is a first-order sentence, by theorem-6.1.1.0.7;

$\Rightarrow$  by hypothesis,  $\psi$  is not logically equivalent with  $\phi$ , therefore, it is neither so with  $\phi^*$ ;

$\Rightarrow$  then, there is a  $\tau$ -structure  $\mathfrak{B}$  such that  $\mathfrak{B} \models_{\mathcal{L}} \phi^*$  and  $\mathfrak{B} \models_{\mathcal{L}} \neg\psi$ ;

$\Rightarrow$  thus, it is also the case that  $\mathfrak{B} \models \phi$ ;

$\Rightarrow$  by the definition of  $\phi$  it follows that there is a  $\tau$ -structure  $\mathfrak{U}$  such that  $\mathfrak{U} \models_{\mathcal{L}} \psi$  and  $\mathfrak{B} \models \phi_{\mathfrak{U} \upharpoonright \tau_0}^n$ ;

$\Rightarrow$  by the Corollary-3.16 of Ehrenfeucht theorem,  $\mathfrak{U} \upharpoonright \tau_0 \cong_n \mathfrak{B} \upharpoonright \tau_0$ .

This is the main lemma for the Lindström theorem:

**Lemma 6.1.2.3.2 (Main Lemma).** *If  $\mathcal{L}$  is a regular logical system such that  $\mathbf{L\ddot{o}Sko}(\mathcal{L})$ ,  $\mathcal{L}_I \leq \mathcal{L}$ , and for some  $\tau$  there is an  $\mathcal{L}(\tau)$ -sentence  $\psi$  not logically equivalent to any first order sentence, then one of the following holds:*

(a) *There are  $\tau$ -structures  $\mathfrak{U}$  and  $\mathfrak{B}$  such that, for every finite  $\tau_0 \subseteq \tau$ ,*  
 $\mathfrak{U} \upharpoonright \tau_0 \cong \mathfrak{B} \upharpoonright \tau_0$ ,  $\mathfrak{U} \models_{\mathcal{L}} \psi$  and  $\mathfrak{B} \vdash_{\mathcal{L}} \neg\psi$ .

(b) *For a unary relation symbol  $W$  and some specific symbol set  $\tau^+$ , such that  $\tau \cup \{W\} \subset \tau^+$ , there is an  $\mathcal{L}(\tau^+)$ -sentence  $\chi^*$  such that*

(i) *In every model  $\mathfrak{D}$  of  $\chi^*$ ,  $W^D$  is finite and nonempty;*

(ii) (Remark-5) For every  $n \in \mathbb{N}$ , there's a model  $\mathfrak{D}$  of  $\chi^*$  such that  $|W^{\mathfrak{D}}| = n$ .

**Theorem 6.1.2.3.3** (Lindström's First Theorem). *If  $\mathcal{L}$  is a regular logical system such that  $\mathcal{L}_I \leq \mathcal{L}$  and there's a  $L(\tau)$ -sentence  $\psi$  that is not logically equivalent to any first order sentence, then either **Compact**( $\mathcal{L}$ ) fails or **LöSko**( $\mathcal{L}$ ) fails.*

*Proof.* Let  $\mathcal{L}$  be as in the hypothesis of the theorem. Suppose **Compact**( $\mathcal{L}$ ) and **LöSko**( $\mathcal{L}$ ) holds.

Suppose (a) of Main Lemma holds.

$\Rightarrow$  Then there are  $\tau$ -structures  $\mathfrak{U}$  and  $\mathfrak{B}$  such that, for every finite  $\tau_0 \subseteq \tau$ ,  $\mathfrak{U} \upharpoonright \tau_0 \cong \mathfrak{B} \upharpoonright \tau_0$ ,  $\mathfrak{U} \models_{\mathcal{L}} \psi$  and  $\mathfrak{B} \models_{\mathcal{L}} \neg\psi$ .

$\Rightarrow$  By theorem-6.1.2.2.1, there's a finite  $\tau_0 \subseteq \tau$ , such that if  $\mathfrak{U} \upharpoonright \tau_0 \cong \mathfrak{B} \upharpoonright \tau_0$ , then  $(\mathfrak{U} \models_{\mathcal{L}} \psi \text{ iff } \mathfrak{B} \models_{\mathcal{L}} \psi)$ .

$\Rightarrow$  Therefore, we get a contradiction that a structure  $\mathfrak{B}$  satisfies both  $\psi$  and  $\neg\psi$ .

Then, (b) must be the case.

$\Rightarrow$  Consider the following set of sentences:

$$\Sigma = \{\chi^*\} \cup \{\exists x_1 \dots x_n \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \wedge W(x_i) \mid n \in \mathbb{N}\}$$

$\Rightarrow$  By (b.ii) of the Main Lemma, every finite subset of  $\Sigma$  have a model. By **Compact**( $\mathcal{L}$ ),  $\Sigma$  has a model  $\mathfrak{M}$ . But  $W^{\mathfrak{M}}$  is infinite, contradicting (b.i) of Main Lemma.

Therefore, our initial assumption is false, and either **Compact**( $\mathcal{L}$ ) fails or **LöSko**( $\mathcal{L}$ ) fails.  $\square$

Now we proceed to the proof of main lemma. The basic idea is to formulate the following statement from theorem-6.1.2.3.1 in the language of  $\mathcal{L}$ :

$$\mathfrak{U} \upharpoonright \tau_0 \cong_n \mathfrak{B} \upharpoonright \tau_0, \mathfrak{U} \models_{\mathcal{L}} \psi \text{ and } \mathfrak{B} \models_{\mathcal{L}} \neg\psi. \quad (6.7)$$

To do so, we will describe by a conjunction  $\chi$  of first-order sentences saying exactly what that theorem says, and whenever a structure is a model of  $\phi$ , the assertions of the main lemma will be verified. In this formulation,

the terminology of partial isomorphisms  $(I_n)_{n \leq m}$  of the last section will be used. Choose  $\mathfrak{U}$ ,  $\mathfrak{B}$ ,  $\tau_0$  and  $(I_n)_{n \leq m}$  such that  $(I_n)_{n \leq m} : \mathfrak{U} \upharpoonright \tau_0 \cong_m \mathfrak{B} \upharpoonright \tau_0$ ,  $\mathfrak{U} \models_{\mathcal{L}} \psi$  and  $\mathfrak{B} \models_{\mathcal{L}} \neg\psi$ . By the isomorphism property, we may assume that  $A$  and  $B$  are disjoint.

A structure  $\mathfrak{C}$  will be defined which contains the structures  $\mathfrak{U}$  and  $\mathfrak{B}$  and which allows to describe the  $m$ -isomorphism property  $(I_n)_{n \leq m} : \mathfrak{U} \upharpoonright \tau_0 \cong_m \mathfrak{B} \upharpoonright \tau_0$ , and this will be accomplished by including the partial isomorphisms from  $I_n$  as elements of the domain of  $\mathfrak{C}$ .

For this task, we will need new relation symbols to describe completely the theorem 6.1.2.3.1. Let  $\tau^+ = \tau \cup \{U, V, W, P, <, I, G, f, c\}$  where,  $c$  is a constant symbol,  $f$  is a unary function symbol,  $U, V, W, P$  are unary relations,  $<, I$  are binary relations and  $G$  is a ternary relation.  $\mathfrak{C}$  is a  $\tau^+$  structure, its relation and function symbols are interpreted as:

- (a)  $C = A \cup B \cup \{0, \dots, m\} \cup \bigcup_{n \leq m} I_n$ ;
- (b)  $U^C = A$  and  $[U^C]^{\mathfrak{C} \upharpoonright \tau} = \mathfrak{U}$ ;
- (c)  $V^C = B$  and  $[V^C]^{\mathfrak{C} \upharpoonright \tau} = \mathfrak{B}$ ; remember that (b) and (c) are possible since  $A \cap B = \emptyset$  and  $\tau$  is relational;
- (d)  $W^C = \{0, \dots, m\}$ ,  $<^C$  is the natural ordering on  $W^C$ ,  $c^C = m$  and  $f \upharpoonright W^C$  is the predecessor function on  $W^C$ , i.e.  $f(n+1) = n$ ;
- (e)  $P^C = \bigcup_{n \leq m} I_n$ ;
- (f)  $I^C = \{(n, p) \mid n \leq m \text{ and } p \in I_n\}$ ;
- (g)  $G^C = \{(p, a, b) \mid p \in P^C, a \in \text{dom}(p) \text{ and } p(a) = b\}$ .

#### 6.1.2.3.4 The sentence $\chi$ - encoding theorem 6.1.2.3.1 in $\mathcal{FOL}$

Then,  $\mathfrak{C}$  satisfies the following set of  $\tau^+$ -sentences whose conjunction is named  $\chi$ .

For a partial isomorphism  $p \in P$ ,  $Gpxy$  describes the graph of the function  $p$  from the  $\tau_0$ -substructure induced on  $U^C$  to the  $\tau_0$ -substructure induced on  $V^C$ .

- (i)  $\forall p(Pp \rightarrow \forall xy(Gpxy \rightarrow (Ux \wedge Vy)))$ ;
- (ii)  $\forall p(Pp \rightarrow \forall x'yy'((Gpxy \wedge Gpx'y') \rightarrow (x = x' \leftrightarrow y = y')))$ ;

(iii) For every  $n$ -ary  $R \in \tau_0$ :

$$\forall p(Pp \rightarrow \forall x_1 \dots x_n y_1 \dots y_n((Gpx_1 y_1 \wedge \dots \wedge Gpx_n y_n) \rightarrow (Rx_1 \dots x_n \leftrightarrow Ry_1 \dots y_n))).$$

The following are the axioms to assure that  $<$  is a total ordering and that  $W^C$  is its field:

$$(iv) \exists xy(x < y), \forall x \neg(x < x), \forall xyz((x < y \wedge y < z) \rightarrow x < z);$$

$$(v) \forall xy(\exists u(x < u \vee u < x) \wedge \exists v(y < v \vee v < y) \rightarrow (x < y \vee x = y \vee y < x));$$

$$(vi) \forall x(Wx \leftrightarrow (x = c \vee \exists y(y < x \vee x < y))) \wedge \forall x(Wx \rightarrow (x < c \vee x = c)).$$

If  $x$  is in the field of  $<$ , then  $I_x = \{p \mid Pp \wedge Ixp\}$  is a non-empty set:

$$(vii) \forall x(Wx \rightarrow \exists p(Pp \wedge Ixp)).$$

$f$  is the predecessor function:

$$(viii) \forall x(\exists y(y < x) \rightarrow (fx < x \wedge \neg \exists z(fx < z \wedge z < x)))$$

The axioms bellow describe the back and forth properties of sequence  $(I_n)_{n \leq m}$ :

$$(ix) \forall xpu((fx < x \wedge Ixp \wedge Uu) \rightarrow \exists qv(Ifxq \wedge Gquv \wedge \forall x'y'(Gpx'y' \rightarrow Gqx'y')) \text{ (the forth property)});$$

$$(x) \forall xpv((fx < x \wedge Ixp \wedge Vv) \rightarrow \exists qu(Ifxq \wedge Gquv \wedge \forall x'y'(Gpx'y' \rightarrow Gqx'y')) \text{ (the back property)}).$$

And now a sentence that express that  $\mathfrak{U} \models_{\mathcal{L}} \psi$  and  $\mathfrak{B} \models_{\mathcal{L}} \neg\psi$ , remembering that  $U^C = A$  and  $V^C = B$ :

$$(xi) \exists xUx \wedge \exists yVy \wedge \psi^U \wedge \neg\psi^V.$$

**Remark 6.1.2.3.5.** Observe that, for any  $n \in \mathbb{N}$ , there is a model  $\mathfrak{C}$  of  $\chi$  in which the field  $W^C$  of  $<^C$  has exactly  $n + 1$  elements. This is guaranteed by theorem-6.1.2.3.1, since, if the hypothesis of the theorem holds, for every  $n \in \mathbb{N}$  there are structures  $\mathfrak{U}$  and  $\mathfrak{B}$  such that for every finite  $\tau_0$ ,  $\mathfrak{U} \upharpoonright \tau_0 \cong_n \mathfrak{B} \upharpoonright \tau_0$ .

**Lemma 6.1.2.3.6.** If some  $\tau^+$  structure  $\mathfrak{D}$  is a model of  $\chi$ , in which the field  $W^D$  of  $<^D$  is infinite, then the  $U$ -part and  $V$ -part of  $\mathfrak{D}$  are domains of substructures  $\mathfrak{U} = [U^D]^{\mathfrak{D} \upharpoonright \tau}$  and  $\mathfrak{B} = [V^D]^{\mathfrak{D} \upharpoonright \tau}$  such that,

$$\mathfrak{U} \models_{\mathcal{L}} \psi, \mathfrak{B} \models_{\mathcal{L}} \neg\psi \text{ and } \mathfrak{U} \upharpoonright \tau_0 \cong_p \mathfrak{B} \upharpoonright \tau_0. \text{ } ^2$$

*Proof.*  $\mathfrak{D}$  is a model of  $\chi$ , then  $U^D \neq \emptyset$  and  $V^D \neq \emptyset$  and, since  $\tau$  is relational,  $U^D$  and  $V^D$  are domains of  $\tau$ -substructures of  $\mathfrak{D}$ . By (xi), we have that  $\mathfrak{D} \models_{\mathcal{L}} \psi^U$  and  $\mathfrak{D} \models_{\mathcal{L}} \neg\psi^V$ , therefore, by **Rel**( $\mathcal{L}$ ) and the reduct property,  $[U^D]^{\mathfrak{D} \upharpoonright \tau} \models_{\mathcal{L}} \psi$  and  $[V^D]^{\mathfrak{D} \upharpoonright \tau} \models_{\mathcal{L}} \neg\psi$ , that is,

$$\mathfrak{U} \models_{\mathcal{L}} \psi \text{ and } \mathfrak{B} \models_{\mathcal{L}} \neg\psi$$

Every  $p \in P^D$  is a partial isomorphism from  $\mathfrak{U} \upharpoonright \tau_0$  to  $\mathfrak{B} \upharpoonright \tau_0$ , once  $\mathfrak{D}$  satisfies (i),(ii),(iii). Given that  $W^D$  is infinite and  $c^D$  is the greatest element of  $<^D$  (by (vi)),  $<^D$  has an infinite descending chain

$$\dots <^D (ffc)^D <^D (fc)^D <^D c^D.$$

Let us abbreviate  $f^0c$  as  $c$ ,  $fc$  as  $f^1c$ ,  $ffc$  as  $f^2c$  and successively  $f^n c$  for  $n \in \mathbb{N}$ . Now we will extract from the set of partial isomorphisms  $P^D$  a subset  $I$ : the set of  $p \in P^D$  such that there is some predecessor  $k$  of  $c^D$  such that  $p$  is in  $I_k$ :

$$I = \{p \mid \text{there is an } n \text{ with } I^D(f^n c)^D p\}.$$

Now, by the partial isomorphisms in  $I$ , one can conclude that  $I : \mathfrak{U} \upharpoonright \tau_0 \cong_p \mathfrak{B} \upharpoonright \tau_0$ .

This is so, because the set  $I$  is non empty (by (vii)) and because  $I$  satisfies the back and forth property, since (ix,x) are satisfied. For example, given that (vii) is satisfied, for some  $(f^n c)^D$  in  $W^D$ , there is a  $p$  such that  $I^D(f^n c)^D p$ , therefore,  $p \in I$ , and for some  $a \in A$ , (by viii) there is a  $q$  with  $I^D(f^{n+1} c)^D q$  that extends  $p$  such that  $a \in \text{dom}(q)$ , and, by definition of  $I$ ,  $q \in I$ .

□

**Lemma 6.1.2.3.7.** *Assume that **LöSko**( $\mathcal{L}$ ). Then one of the following conditions (a) or (b) bellow holds:*

(a) *There are  $\tau$ -structures  $\mathfrak{U}$  and  $\mathfrak{B}$  such that*

$$\mathfrak{U} \models_{\mathcal{L}} \psi, \mathfrak{B} \models_{\mathcal{L}} \neg\psi \text{ and } \mathfrak{U} \upharpoonright \tau_0 \cong \mathfrak{B} \upharpoonright \tau_0;$$

(b) *In all models  $\mathfrak{D}$  of  $\chi$ , the field  $W^D$  of  $<^D$  is finite.*

<sup>2</sup>For partially isomorphic structures, see Definition-6.1.1.0.19.

*Proof.* The proof will be rather straightforward, and will show that  $\neg(b) \rightarrow (a)$ .

Suppose that there is some model  $\mathfrak{D}$  of  $\chi$  such that the field  $W^D$  of  $<^D$  is infinite. Then by the lemma-6.1.2.3.6,

$$\mathfrak{U} \models_{\mathcal{L}} \psi, \mathfrak{B} \models_{\mathcal{L}} \neg\psi \text{ and } \mathfrak{U} \upharpoonright \tau_0 \cong_p \mathfrak{B} \upharpoonright \tau_0.$$

Now, since **LöSko**( $\mathcal{L}$ ), we may assume that  $\mathfrak{D}$  is countable. Therefore, by theorem-6.1.1.0.20, we have that there is an isomorphism between  $\mathfrak{U} \upharpoonright \tau_0$  and  $\mathfrak{B} \upharpoonright \tau_0$ , so part (a) is satisfied.  $\square$

#### 6.1.2.4 Lindström's Second Theorem

The second Lindström theorem deals with a more restricted class of logical systems, called *effective logical systems*. If  $\tau$  is a decidable symbol set, then

**Definition 6.1.2.4.1.**  $\mathcal{L}$  is an effective logical system iff for every decidable  $\tau$ ,  $L(\tau)$  is a decidable set, and, for every  $\phi \in L(\tau)$  there is a finite  $\tau_0$  such that  $\phi \in L(\tau_0)$ .

Besides this,

**Definition 6.1.2.4.2.** A logical system  $\mathcal{L}$  is an effectively regular logical system, iff **Boole**( $\mathcal{L}$ ), **Rel**( $\mathcal{L}$ ) and **Repl**( $\mathcal{L}$ ), and all this properties are effective:

- in the case of **Boole**( $\mathcal{L}$ ), it is effective iff for every  $\phi \in L(\tau)$  there is a computable function to generate a  $\psi \in L(\tau)$  such that  $\mathfrak{U} \models \phi$  iff  $\mathfrak{U} \not\models \psi$ . The same goes for disjunction.
- in the case of **Rel** and **Repl** also there must be computable functions to generate the relativized formulas and to obtain relational symbol sets, respectively.

Also, for effective logical systems  $\mathcal{L}$  and  $\mathcal{L}'$ ,  $\mathcal{L}'$  is effectively as expressible as  $\mathcal{L}$  ( $\mathcal{L} \leq_{eff} \mathcal{L}'$ ) iff for every decidable  $\tau$  there is a computable function  $*$  that associates every  $\phi \in L(\tau)$  a  $\phi^* \in L'(\tau)$  such that  $Mod_{\mathcal{L}}^{\tau}(\phi) = Mod_{\mathcal{L}'}^{\tau}(\phi^*)$ . The same goes when  $\mathcal{L}$  and  $\mathcal{L}'$  are effectively equally strong ( $\mathcal{L} \sim_{eff} \mathcal{L}'$ ).

In the following proof, we need the Trahtenbrot's theorem

**Theorem 6.1.2.4.3** (Trahtenbrot's Theorem). *The set of finitely valid first order sentences is non-enumerable.*<sup>3</sup>

<sup>3</sup>A first-order  $\tau$ -sentence is finitely valid if every finite  $\tau$ -structure satisfies it.



Now we are ready to state and prove the sought theorem.

**Theorem 6.1.2.4.4** (Lindström's Second Theorem). *Let  $\mathcal{L}$  be an effective regular logical system such that  $(\mathcal{L}_I \leq_{\text{eff}} \mathcal{L})$ . Then, if  $\mathbf{L\ddot{o}Sko}(\mathcal{L})$ , and the set of valid sentences of  $\mathcal{L}$  is enumerable, then  $\mathcal{L}_I \sim_{\text{eff}} \mathcal{L}$ .*

*Proof.* Let  $\mathcal{L}$  satisfy the hypothesis of the theorem. Then we have to prove that  $\mathcal{L} \leq_{\text{eff}} \mathcal{L}_I$ , that is, for a decidable  $\tau$ , for every sentence  $\psi \in L(\tau)$  there's an effective procedure to obtain a logically equivalent  $\phi \in L_I(\tau)$ . The proof is divided in two parts, first we prove that  $\mathcal{L} \leq \mathcal{L}_I$  for every decidable symbol set  $\tau$  and then we give the effective procedure, so that  $\mathcal{L} \leq_{\text{eff}} \mathcal{L}_I$ .

$$\mathcal{L} \leq \mathcal{L}_I \text{ for every decidable } \tau \quad (+)$$

The proof is by absurd. Suppose, for a decidable  $\tau$ , there is a  $\psi \in L(\tau)$  such that  $\psi$  is not logically equivalent to any first order sentence. Since  $\mathcal{L}$  is an effective system, by definition-6.1 above, we can consider only a finite  $\tau_0 \subseteq \tau$  such that  $\psi \in L(\tau_0)$ .

Then, since  $\tau_0$  is finite and  $\mathbf{L\ddot{o}Sko}(\mathcal{L})$ , lemma-6.1.2.3.2 applies. Suppose (a) of lemma-6.1.2.3.2 holds, then there are  $\tau_0$  structures  $\mathfrak{U}$  and  $\mathfrak{B}$  such that  $\mathfrak{U} \models_{\mathcal{L}} \psi$ ,  $\mathfrak{B} \models_{\mathcal{L}} \neg\psi$  and  $\mathfrak{U} \cong \mathfrak{B}$ , which contradicts the isomorphism property.

Therefore, (b.i) and (b.ii) of lemma-6.1.2.3.2 must hold. Then, there is a  $L(\tau^+)$ -sentence  $\chi^*$ , for  $\tau^+ = \tau \cup \{U, V, W, P, <, I, G, f, c\}$  such that for every model  $\mathfrak{C}$  of  $\chi^*$ ,  $W^{\mathfrak{C}}$  is finite.

To go ahead, one needs first to prove the following statement:

$$\phi \text{ is finitely valid iff } \models_{\mathcal{L}} \chi^* \rightarrow (\phi^*)^W. \quad (\dagger)$$

*Proof.* ( $\Rightarrow$ ) Suppose  $\phi$  is finitely valid  $L_I(\tau)$ -sentence. Let  $\mathfrak{C}$  a  $\tau^+$  structure that is a model of  $\chi^*$ , then by the conclusion above and (b.i) of lemma-6.1.2.3.2,  $[W^{\mathfrak{C}}]^{\mathfrak{C} \upharpoonright \tau}$  is a finite substructure of  $\mathfrak{C}$ , therefore, by the hypothesis,  $[W^{\mathfrak{C}}]^{\mathfrak{C} \upharpoonright \tau} \models_{\mathcal{L}} \phi^*$ , since  $\phi^*$  is logically equivalent with  $\phi$ . Therefore  $\mathfrak{C}$  is a model of  $(\phi^*)^W$ , by the reduct property and  $\mathbf{Rel}(\mathcal{L})$ .

( $\Leftarrow$ ) Suppose  $\models_{\mathcal{L}} \chi^* \rightarrow (\phi^*)^W$ . Let  $\mathfrak{D}$  be a finite  $\tau$ -structure. By the isomorphism property, we can assume that the domain  $D$  of  $\mathfrak{D}$  is  $\{0, \dots, m-1\}$ . Expand  $\mathfrak{D}$  to a  $\tau^+$ -structure  $\mathfrak{D}^+$  that is a model of  $\chi$  and that  $W^{\mathfrak{D}^+} = \{0, \dots, m-1\}$  (by b.ii of lemma-6.1.2.3.2). Recall that  $\tau \cap \{U, V, W, P, <, I, G, f, c\} = \emptyset$ , therefore  $[W^{\mathfrak{D}^+}]^{\mathfrak{D}^+ \upharpoonright \tau} = \mathfrak{D}$ . Since by construction  $\mathfrak{D}^+ \models_{\mathcal{L}} \chi^*$ , by hypothesis it follows that  $\mathfrak{D}^+ \models_{\mathcal{L}} (\phi^*)^W$ , therefore,  $[W^{\mathfrak{D}^+}]^{\mathfrak{D}^+} \models_{\mathcal{L}} \phi^*$ , since  $\phi^* \in L(\tau)$ , by the reduct property,  $[W^{\mathfrak{D}^+}]^{\mathfrak{D}^+ \upharpoonright \tau} \models_{\mathcal{L}} \phi^*$ , so  $\mathfrak{D} \models_{\mathcal{L}} \phi^*$ , and finally  $\mathfrak{D} \models_{\mathcal{L}} \phi$ .  $\square$

Now, given the enumeration of  $\mathcal{L}$ -valid sentences  $\alpha_1, \alpha_2, \alpha_3, \dots$ , check for each  $i \in \mathbb{N}$  whether  $\alpha_i$  has the form  $\chi^* \rightarrow (\phi^*)^W$ , such that  $\phi$  is a first-order sentence. If for some  $i$  this is the case, put  $\phi$  on the list  $\Phi_{FV}$ .

By ( $\dagger$ ), every sentence in  $\Phi_{FV}$  is finitely valid, and this procedure yields an enumeration of  $\Phi_{FV}$ , contradicting Trahtenbrot's Theorem. Therefore, it is false that there is an  $\mathcal{L}$ -sentence that is not logically equivalent to any first-order sentence and (+) is proved.

Now we must prove that

$$\mathcal{L} \leq_{\text{eff}} \mathcal{L}_I.$$

Now one must supply an effective procedure which associates with every  $L(\tau)$  sentence  $\gamma$  a sentence  $\phi \in \mathcal{L}_I(\tau)$  with the same models.

Let  $\alpha_1, \alpha_2, \dots$ , be an enumeration of the set of valid  $L(\tau)$ -sentences and \* a computable function that assigns for every  $\phi \in L_I(\tau)$  a sentence  $\phi^* \in L(\tau)$  with the same models (the existence of this function is granted by the hypothesis that  $\mathcal{L}_I \leq_{\text{eff}} \mathcal{L}$ ).

Take a sentence  $\gamma \in L(\tau)$ . Start listing the valid  $L(\tau)$ -sentences  $\alpha_1, \alpha_2, \dots$ , and for every  $i \in \mathbb{N}$ , check whether  $\alpha_i$  has the form  $\gamma \leftrightarrow \phi^*$ , where  $\phi$  is a first-order sentence. If for some  $i$  this is the case, then  $\phi$  is the sought first-order sentence logically equivalent with  $\gamma$ . That for every  $\gamma$  there is such an  $\alpha_i$  is granted by (+).

□

## Chapter 7

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