On the private provision and use of public goods

Marta Faias

Universidade Nova de Lisboa. CMA and FCT.

e-mail: mcm@fct.unl.pt

Emma Moreno-García

Universidad de Salamanca. IME.

e-mail: emmam@usal.es

Abstract. We provide a non-cooperative game for the private provision of collective goods where the distribution of the levels of utilization matters. This approach allows us to deep the analysis of altruistic behaviors, neutrality results, congestions issues and further externalities captured within our framework.

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1 Introduction

In the classical paper "The pure theory of public expenditure", Samuelson (1954) referred to a public good as a collective consumption good and he defined it as follows:

"... *which all enjoy in common in the sense that each individual's consumption of such a good leads to no subtractions from any other individual's consumption of that good...*"

Since then, we say that a public good is characterized by the properties of non-rivalry and non-excludability. However, the fact that it is impossible to exclude any individuals from consuming a good does not imply that all individuals make the same use of it.

We observe that not all the individuals use public goods with the same intensity. We argue that the distribution of end users of a collective good affects the individual preferences and the welfare of the society. For instance, some agents may prefer a swinging pool with few people while others prefer a more crowded ambient. Some parents may prefer for their children a school where all the students have certain characteristics while others are indifferent about it or even prefer diversity in the group. On the other hand, we also argue that when using public goods, congestion issues might arise that can be analyzed from a point of view based on a common and objective perspective, as it is the case of physical space or capacity restrictions; but in addition and at the same time individual and subjective perception may also play a role.

In this paper, the aforementioned ideas lead us to provide a non-cooperative approach to the provision of public goods, where agents decide simultaneously not only their private contribution for the production of the public goods but also a new variable that capture their level of utilization which may differ among individuals.[1] In this way, we extend the original model by Bergstrom, Blume and Varian (1986) to an scenario where individuals are free to modify their own use of a collective consumption commodity.

To be precise, each player in our modified game, as in the standard one, represents a consumer endowed with an amount of private good that is used as an input to get a collective commodity. However, the strategy sets are enlarged in such a way that a strategy for a player is a pair given by her private contribution for the provision of public good, and a parameter that specifies her level of use of the public good. Thus, a strategy profile not only affects the individual preferences implicitly through the specification of the public good that is generated, but also the profile of degrees of utilization affects preferences explicitly. In fact, the payoffs of the game are given by the corresponding utility functions defined over the bundle of private and public goods and the distribution of the level of use of the collective commodity.

After showing existence of equilibrium for our non-cooperative game, we establish how the approach we propose leads us to the study of a variety of issues. Note that within our framework, an agent may contribute privately to the provision of a public good, regardless of the level of her own use. In particular, an agent may contribute to the collective good without using it or may use it with a null private contribution. This fact allows us to give light to an altruistic behavior which differs from the so called pure and impure altruism in the related literature and also to discuss the free riding problem.[2] We say that an individual is altruistic when she is better off by contributing to the public good without the necessity of modifying her own use. Thus, we show that when an agent is not altruistic then her private contribution is null whenever she makes no use of the

[1] Although we focus on profiles of intensity of utilization, as we remark in the concluding section, this new variable may admit other interpretations giving further approaches.

[2] See Andreoni (1990) for the distinction between pure and impure altruism.

collective good, that is, to become a contributor a non-altruistic consumer requires a real own use of the public good provided. Also within our model, we say that the free riding problem arises if there is a player who use the collective good without contributing to it. In this way, we conclude from our game that when individuals' welfare is decreasing with the utilization of the collective good, for instance because of congestion problems or other type of additional externality, then there is no free-rider at equilibrium.

On the other hand, we remark that the distribution of users of a public good we consider in the analysis motivates further study of congestion problems, consideration of nonanonymous crowding concerns and neutrality results within scenarios where collective commodities are privately provided. The neutrality theorem, that goes back to Warr (1983) concerning the provision of a public good in a voluntary Nash equilibrium, shows that if income is redistributed among contributors so that a loser loses no more than her original donation, the equilibrium remains the same. Within our framework, we demonstrate that, in general, a neutrality result cannot be obtained. In spite of this, we go further and find conditions to recover a neutrality result for our model.

The remaining of this paper is structured as follows. In Section 2, we present the game on the private provision and use of public goods to be analyzed, and in Section 3, we show an equilibrium existence result for such a game. Section 4 includes some remarks regarding altruism and free rider issues. In Section 5, we study how congestion and crowding externalities may affect optimal contributions. In Section 6, we first present a non-neutrality example and then, under additional requirements we prove a result on neutrality for our game. Finally, in Section 7, we conclude with some remarks.

2 A game on public goods

Consider an economy $\mathcal{E}$ with a finite set $N = \{1, \ldots, n\}$ of individuals who consume a private good and a collective (public) good which may be subject to additional externalities. Each consumer is characterized by external characteristics, endowments of private good and preferences.

In spite of the non-excludability properties of the collective commodity, since its use is not fixed and compulsory, consumers may vary their degree of utilization. Thus, let $\alpha = (\alpha_i, i \in N) \in [0,1]^n$ be a vector that specifies the individual levels of use of the public good.

Each consumer $i \in N$ has an endowment of private good $w_i \in \mathbb{R}_+$ and a preference relation $\succsim_i$ on her consumption of private good, the public good provided and the distribution of users. Each preference $\succsim_i$ is represented by an utility function U_i. That is, given an amount of private good x, public good G, and a vector α, the utility level for a consumer i is given by $U_i(x, G, \alpha)$.

Let us consider a game where the n players are the consumers. Each player i decides both the contribution $g_i \in [0, w_i]$ to the provision of the public good and the degree of use α_i, in percentage terms, of the consumer i. That is, the strategy set for player i is $\Theta_i = [0, w_i] \times [0,1]$. A strategy profile is a vector $(g, \alpha) = (g_i, \alpha_i, i \in N)$. A real function f, defined on $\Theta = \prod_{i \in N} \Theta_i$, specifies the public good that is provided for each strategy profile. Then, the payoff function for player i is given by

$$\pi_i(g, \alpha) = U_i(w_i - g_i, f(g, \alpha), \alpha).$$

We remark that the distribution of the collective commodity users reflects external characteristics of individuals that matter the others and generates externalities that may also affect the public good that is provided. Thus, the way in which α affects the payoffs is tow-fold since each utility function becomes dependent on α both explicitly and implicitly. We argue that α has a *subjective* and an *objective* impact; the former is captured as an explicit argument of each U_i and the latter is reflected as an argument of f. For instance, if we address congestions problems, our model allows us to consider objective congestion levels in the sense that are the same for all consumers, and at the same time subjective congestion issues may appear provided that individuals have different preferences for the distribution of people that use the public good.

Given a profile θ let θ_{-i} be the strategy of every player except i. We have that θ^* is a Nash equilibrium if $\pi_i(\theta^*) \geq \pi_i(\theta^*_{-i}, \theta_i)$ for every $\theta_i \in \Theta_i$ and every $i \in N$. We refer to a Nash equilibrium of this game as a private provision and use equilibrium for the economy $\mathcal{E}$.

Within this scenario, we say that a profile $\theta = (g, \alpha)$ is dominated by $\hat{\theta} = (\hat{g}, \hat{\alpha})$ if $\pi_i(\hat{\theta}) > \pi_i(\theta)$ for every $i \in N$. This approach leads us to give a notion of efficiency by considering the dominance relation. In this way, we say that the vector θ is efficient if it is not dominated by any $\hat{\theta}$.

We remark that different interpretations of the vector α and different specifications of the utility functions and the function f leads to a variety of scenarios as particular cases. For instance, if $U_i(x, G, \alpha) = V_i(x, G)$ and $f(g, \alpha) = \sum_{i=1}^{n} g_i$ we have the model of private provision of public goods provided by Bergstrom, Blume and Varian (1986).

Moreover, our formulation allows also the analysis of the composition of a club as follows. Consider each player i as a representative of a set of consumers of type i and let α_i be interpreted as the measure of individual of type i. A strategy profile $\theta = (g, \alpha)$ defines a club formed by types in $T(\alpha) = \{i \in N | \alpha_i > 0\}$

and size $\sum_{i \in N} \alpha_i$. Individuals in the club provide an amount of local public good given by a function depending on the strategies of the types in the club. For example, $f(g, \alpha) = \sum_{i \in N} \alpha_i g_i = \sum_{i \in T(\alpha)} \alpha_i g_i$.

Note also that within the above scenario α defines the composition of the group of consumers using the public good. When the aggregate size of the end-users of the public good matters for the externalities they confer, but not their characteristics, that is, preferences depend actually on $\sum_{i=1}^{n} \alpha_i$ and not on the distribution given by α, we have the case of "anonymous" crowding as in Buchanan (1965).

An increasing line of research that grows out of the one initiated by Buchanan (1965) defines different general equilibrium models with exchange and club formation with nonanonymous crowding. See, for instance, Conley and Wooders (1997, 2001), Ellickson et al. (1999), Allouch and Wooders (2008) and the references there. Unlike our aim in the current article, the main question addressed in many of these papers is whether it is possible to get an efficient decentralizing price system in which prices can be based on publicly observable characteristics and its relation with the core.

The model we present allows us the consideration of nonanonymous crowding issues within a private provision of public goods setting. Our analysis is based on a non-cooperative approach where players contribute in a voluntary way to the provision of a collective commodity and we do not study market-like outcomes. The formulation and analysis of the impact of nonanonymous crowding on equilibria we propose leads us to address altruistic behavior of individuals in connection with private contributions for providing goods that can be collectively used.

3 Equilibrium vs. efficiency

In this section, first we state sufficient conditions for existence of private provision and use equilibrium and then we present some remarks on efficiency.

Theorem 3.1 *Assume that the following assumptions hold:*

a) For every i utility function U_i is continuous, non-decreasing in G and quasi-concave in (x, G, α_i).

b) f is a continuous function and $f(\theta_{-i}, \cdot)$ is concave for every $\theta_{-i} \in \prod_{j \neq i} \Theta_j$.

Then, the set of Nash equilibrium for our game is non-empty.

Proof. The strategy sets are non-empty, compact and convex. In addition, since f and U_i are continuous functions we have that π_i is continuous as well. Then, it remains to show that π_i is quasi-concave in the strategy chosen by player i.

Let $\theta_i = (g_i, \alpha_i)$ and $\hat{\theta}_i = (\hat{g}_i, \hat{\alpha}_i)$. For each $\lambda \in [0,1]$ let $\theta_i^\lambda = \lambda\theta_i + (1-\lambda)\hat{\theta}_i = (g_i^\lambda, \alpha_i^\lambda)$. Let $G = f(\theta_{-i}, \theta_i)$ and $\hat{G} = f(\theta_{-i}, \hat{\theta}_i)$. Then,

$$\begin{aligned}
\pi_i(\theta_{-i}, \theta_i^\lambda) = \; & U_i(w_i - g_i^\lambda, f(\theta_{-i}, \theta_i^\lambda), \alpha_i^\lambda, \alpha_{-i}) \geq \\
& U_i(w_i - g_i^\lambda, \lambda f(\theta_{-i}, \theta_i) + (1-\lambda) f(\theta_{-i}, \hat{\theta}_i), \alpha_i^\lambda, \alpha_{-i}) \geq \\
& \min\{U_i(w_i - g_i, G, \alpha_i, \alpha_{-i}), U_i(w_i - \hat{g}_i, \hat{G}, \hat{\alpha}_i, \alpha_{-i})\} = \\
& \min\{\pi_i(\theta_{-i}, \theta_i), \pi_i(\theta_{-i}, \hat{\theta}_i)\},
\end{aligned}$$

where the first inequality is due to the concavity properties of f and the fact that U_i is non-decreasing in the amount provided of public good, whereas the second inequality is guaranteed by the quasi-concavity of U_i in (x, G, α_i).

Q.E.D.

To analyze efficiency of equilibrium let us consider the following optimization problem that leads to efficient situations:

$$\max_{(g,\alpha)\in\Theta} \quad \textstyle\sum_{i=1}^{n} U_i(w_i - g_i, G, \alpha)$$
$$s.t. \quad f(g,\alpha) = G$$

We remark that, under differentiability properties of the functions U_i and f, the first order conditions imply $\sum_{i=1}^{n}\left(\frac{\partial U_i}{\partial G}/\frac{\partial U_i}{\partial x}\right) = \frac{\partial f}{\partial g_i}$. This condition is similar to Samuelson's (1954) in that they define the efficiency conditions for public goods as an equality between the sum over all consumers of the marginal rates of substitution between the public and private good and the marginal rate of transformation.

In this model, the equilibria may be inefficient. To see this, consider the following example of the particular game stated by Bergstrom, Blume and Varian (1986). There are two consumer, each of them endowed with one unit of private good. Preferences for consumers 1 and 2 are represented by the utility functions $U_1(x,G) = xG^2$ and $U_2(x,G) = xG$, respectively, where x denotes consumption of the private commodity and G is the amount of public good. The Nash equilibrium, given by the contributions $g_1 = 3/5$ and $g_2 = 1/5$, is not efficient. In fact, note that the above necessary condition does not hold

4 Remarks on altruism and free-riders

A remark on altruism. Assume that the partial derivatives of the utility functions and the function f are well-defined. For each i, let us consider the partial derivative of the payoff function π_i with respect the contribution of player

i to the provision of the public good, that is given by

$$\gamma_i(\theta) = \frac{\partial \pi_i}{\partial g_i}(\theta) = -\frac{\partial U_i}{\partial x}(x, G, \alpha) + \frac{\partial U_i}{\partial G}(x, G, \alpha)\frac{\partial f}{\partial g_i}(\theta),$$

being $x = w_i - g_i$ and $G = f(\theta)$. Note that $\gamma_i \geq 0$ if and only if $\frac{\partial f}{\partial g_i} \geq \frac{\partial U_i}{\partial x} / \frac{\partial U_i}{\partial G} = MRS_i$, where MRS_i is the marginal rate of substitution between private and public good for individual i.

When $\gamma_i(\theta) > 0$, the player i prefers to increase her contribution to public good, without altering her utilization, by decreasing her private consumption. In contrast, when $\gamma_i(\theta) < 0$, if player i increases her contribution to the provision of public good it becomes necessary to modify also her utilization level in order to obtain a larger benefit. Thus, we may argue that if $\gamma_i(\theta) > 0$ then player i is altruistic at θ whereas player i is not altruistic at θ whenever $\gamma_i(\theta) < 0$.

Next result formalizes the intuition that if at equilibrium an individual is a contributor to the private provision of a public good without using it, then such an individual is altruistic.

Proposition 4.1 *Let $\theta^* = (g^*, \alpha^*)$ be a Nash equilibrium. Assume that player i is not altruistic whenever $\alpha_i = 0$. Then $\alpha_i^* = 0$ implies $g_i^* = 0$.*

Proof. It is enough to note that if $g_i^* > 0$, then player i is able to improve by choosing a lower contribution to the provision of public good because she is not altruistic.

Q.E.D.

A remark on free-riders. We say that a player is a free-rider at a strategy profile whenever she is not a contributor to the provision of the public good but she uses it. That is, i is a free rider at $\theta = (g_i, \alpha_i, i \in N)$ iff $f(\theta) > 0, g_i = 0$ and $\alpha_i > 0$.

In which follows we state conditions that avoid the free-rider problem at equilibrium.

Proposition 4.2 *Let* $\theta^* = (g^*, \alpha^*)$ *be a Nash equilibrium such that* $f(\theta^*) > 0$. *Assume that* $f(g_{-i}, \alpha_{-i}, 0, \alpha_i)$ *is constant for every* α_i *and* U_i *is decreasing in* α_i. *If* $g_i^* = 0$ *then* $\alpha_i^* = 0$.

Proof. Assume $\alpha_i^* > 0$. Then, player i has incentives to deviate by selecting a strategy (g_i^*, α_i), with $\alpha_i < \alpha_i^*$, since the amount of public good does not change and her payoff increases.

Q.E.D.

5 On congestion and crowding externalities

In this section, we focus on special situations where the profile of degrees of utilization of the collective good affects the individual preferences for the consumption of such a commodity and in this way crowding externalities are generated. In particular, we address scenarios where congestion problems may appear.

We have argued that the use of a collective good may differ from an individual to another and, it may be a variable that is included in the strategies of games where the private provision of public goods is involved. In spite of this, we remark that we can also consider situations where the utilization of a public good is somehow given for a set of individuals under some circumstances. For instance, we may think about the use of public hospitals by persons with a weak health or the use of public schools by a family with many children, or the use of a highway by people who necessarily have to use it when they go to work. In this way we provide a variant of the game we have defined in Section 2.

To be precise, let A be a subset of players for which the strategic behavior is just to choose their private contribution to the public good provision provided that their utilization levels of the public good are given by $\hat{\alpha}_A = (\hat{\alpha}_i, i \in A)$. That is, the strategy set for a player $i \in A \subset N$ is restricted to be $\hat{\Theta}_i = [0, w_i]$ whereas the strategy set for player $i \notin A$ is $\Theta_i = [0, w_i] \times [0, 1]$. A strategy profile is a vector $(g_A, \theta_{-A}) \in \hat{\Theta}$, where $g_A = (g_i, i \in A) \in \prod_{i \in A}[0, w_i]$ and $\theta_{-A} = (g_i, \alpha_i, i \notin A) \in \prod_{i \notin A} \Theta_i$. Thus, $\hat{\Theta} = \prod_{i \in A} \hat{\Theta}_i \times \prod_{i \notin A} \Theta_i$ is the set of strategy profiles. Now, given $\hat{\alpha}_A = (\hat{\alpha}_i, i \in A)$, the real function $\hat{f}$, defined on $\hat{\Theta}$ that specifies the amount of public good that is provided for each strategy profile is given by $\hat{f}(g_A, \theta_{-A}) = f(\theta_A, \theta_{-A})$, being $\theta_A = (g_A, \hat{\alpha}_A)$. In this way, the payoff function for player i is given by

$$\hat{\pi}_i(g_A, \theta_{-A}) = \pi_i(\theta_A, \theta_{-A}) = \pi_i(g_A, \hat{\alpha}_A, \theta_{-A}).$$

We study the behavior of the private contribution best reply functions of individuals in A with respect to the corresponding variables determining the utilization levels of the public good that become crucial in the analysis. To do this, we restrict the analysis to separated utility functions of the form $U_i(x, G, \alpha) = u_i(x) + V_i(G, \alpha)$.[3] That is, individual preferences for the private good consumption are separated from the preferences for the consumption of public good and the profile of its use. To simplify, we also assume that the provision of the public good depends only on the private contributions profile, that is, f depends only on g and it is independent of the profile of utilization degrees. Then

$$\hat{\pi}_i(g_A, \theta_{-A}) = u_i(w_i - g_i) + V_i(f(g), \hat{\alpha}_A, \alpha_{-A}),$$

[3] We state examples for which $V_i(G, \alpha) = h_i(\alpha) v_i(G)$.

with $g = (g_A, g_{-A}), g_{-A} = (g_i, i \notin A)$ and $\alpha_{-A} = (\alpha_i, i \notin A)$. Therefore,

$$\frac{\partial \hat{\pi}_i}{\partial g_i}(g_A, \theta_{-A}) = -\frac{du_i}{dx}(w_i - g_i) + \frac{\partial V_i}{\partial G}(f(g), \alpha)\frac{\partial f}{\partial g_i}(g),$$

where $\alpha = (\hat{\alpha}_A, \alpha_{-A})$.

For each $i \in A$, let Γ_i denote the best reply function for individual i when it is well-defined and interior. When the reaction functions are attained at interior points, we can analyze how contributions to the public good are modified as functions of the degrees of utilization of the public good. Next we show that the sign of the impact of the use of the public good on the contribution of an agent i in A is given by the sign of the derivative of $\frac{\partial V_i}{\partial G}$ with respect the the utilization level.

Theorem 5.1 *Assume that the function f is non-decreasing, concave and continuously differentiable. Assume also that for each player $i \in A$ the functions u_i and V_i are strictly concave and twice continuously differentiable. Moreover, every V_i is non-decreasing on G. Then, the best private contribution reply Γ_i of player $i \in A$ is well-defined as function of α and the sign of $\frac{\partial^2 V_i}{\partial G \partial \alpha_k}$ determines the sign of the partial derivatives of Γ_i with respect to α_k, for every players $i \in A$, and $k \in N$.*

Proof. Let a profile of strategies $\theta = (g_A, \theta_{-A})$ such that $\frac{\partial \hat{\pi}_i}{\partial g_i}(\theta) = 0$. Since u_i and V_i are C^2 and f is C^1, then applying the implicit function theorem, each contribution g_i can be written as a function depending on the parameters $\theta_{-i} = (g_{-i}.\theta_{-A})$, where $g_{-i} = (g_j, j \in A \setminus \{i\})$, and in addition:

$$\frac{\partial \Gamma_i}{\partial \alpha_k}(\theta_{-i}) = -\frac{\frac{\partial^2 V_i}{\partial G \partial \alpha_k}(f(g), \alpha)\frac{\partial f}{\partial g_i}(g)}{\frac{d^2 u_i}{dx^2}(1 - g_i) + \frac{\partial^2 V_i}{\partial G^2}(f(g), \alpha)\left(\frac{\partial f}{\partial g_i}(g)\right)^2 + \frac{\partial V_i}{\partial G}(f(g), \alpha)\frac{\partial^2 f}{\partial g_i^2}(g)}$$

if the denominator is different from zero.

By the concavity properties of f, u_i and V_i and since $\frac{\partial V_i}{\partial G} \geq 0$ and $\frac{\partial f}{\partial g_i} > 0$, we deduce that $\dfrac{\partial g_i}{\partial \alpha_k} \geq 0$ (resp. ≤ 0) if and only if $\dfrac{\partial^2 V_i}{\partial G \partial \alpha_k} \geq 0$ (resp. ≤ 0).

Q.E.D.

We remark that $\frac{\partial^2 V_i}{\partial G \partial \alpha_k} \geq 0$ can be interpreted as a supermodularity property of the function $V_i(\cdot, \cdot, \alpha_{-i})$. In fact, if the cross-partial derivatives $\frac{\partial^2 V_i}{\partial G \partial \alpha_i}$ are non-negative, then we have a notion of complementarity in the sense that the marginal utility provided by the provision of public goof is increasing in its consumption. In particular, Theorem 5.1 states that, under such a complementarity or supermodularity condition, the larger the utilization of the public good the larger the contribution determined by the best reply functions.

6 Neutrality

In a pure public good scenario, which corresponds to the particular case of $U_i(x, G, \alpha) = V_i(x, G)$ and $f(g, \alpha) = \sum_{i=1}^{n} g_i$, the invariance result of Warr (1983) and Bergstrom, Blume, and Varian (1986), the so-called neutrality result, shows that income redistributions among contributors that leaves the set of contributors unchanged will induce a new equilibrium with the same total public good provision and each consumer has precisely the same individual consumption as she had before.

The next example shows that the neutrality result cannot be extended to our model without additional assumption.

Non-neutrality example. Consider two agents, 1 and 2. Agent 1 is endowed with $w_1 = 1$ and agent 2 has endowments $w_2 = 2$. The utility functions for agent 1 and 2 are given by $U_1(x, G, \alpha) = xG(2 - \alpha_2)$ and $U_2(x, G, \alpha) = xG(2 - \alpha_1)$

respectively. Let $f(g, \alpha) = \alpha_1 g_1 + 0.5\alpha_2 g_2$. Some calculations allow us to assure that there is a Nash equilibrium for the associated game given by:

$$(g_1, g_2, \alpha_1, \alpha_2) = (1/3, 2/3, 1, 1).$$

This equilibrium leads to a level 2/3 of public good and payoffs $(4/9, 8/9)$.

Let us consider a redistribution of the endowments in such a way that each agent no losses more income than its original private contribution. For instance, let $\hat{w}_1 = \hat{w}_2 = 3/2$. The Nash equilibrium is now given by,

$$(\hat{g}_1, \hat{g}_2, \hat{\alpha}_1, \hat{\alpha}_2) = (3/4, 0, 1, 1).$$

The associated levels of public good and payoffs are 3/4 and $(9/16, 9/8)$.

Therefore after the redistribution the level of public good increases. Although the participation levels are always the same and equal to 1, both players obtain a higher payoff after the redistribution and the neutrality result does not hold.

In spite of this non-neutrality result, we state below additional assumptions that allow us to extend the neutrality results by Bergstrom, Blume and Varian (1986) to our framework.

Theorem 6.1 *Let θ^* be a private provision and use equilibrium for the economy $\mathcal{E}$. Assume that f depends only on the aggregate private contributions and on the vector of utilization level , i.e., there is a function F such that $f(g, \alpha) = F(\sum_i^n g_i, \alpha)$. Assume also that given any $\alpha_{-i} \in [0,1]^{n-1}$ the function $F(\cdot, \alpha_{-i}, \cdot)$ is concave. Moreover, for every i the utility function U_i is non-decreasing in G and quasi-concave in (x, G, α_i). Let $\hat{w}$ be a redistribution of endowments such that $\hat{w}_i = w_i$ for all non-contributing players and $g_i^* \geq w_i - \hat{w}$ for every contributing player i. Then there exists an equilibrium $\hat{\theta}$ for the economy with endowments $\hat{w}$*

such that $f(\theta^*) = f(\hat{\theta})$ *and* $\pi_i(\theta^*) = \pi_i(\hat{\theta})$ *for every* $i = 1, \ldots, n$.

Proof. Let us write $\hat{w}_i = w_i + \Delta w_i$.

Consider the profile $\hat{\theta} = (\hat{g}, \hat{\alpha})$ given by $\hat{g}_i = g_i^* + \Delta w_i$ and $\hat{\alpha} = \alpha^*$. Note that by definition $\hat{g}_i \in [0, \hat{w}_i]$ for every player i.

We will show that $\hat{\theta}$ is a Nash equilibrium for the game $\hat{\mathcal{G}}$ defined by the redistribution $\hat{w}$.

Let $\Gamma^* = \sum_{i=1}^n g_i^*$ and $\Gamma_{-i}^* = \sum_{j \neq i}^n g_j^*$.

The properties of f guarantee that θ^* is a Nash equilibrium of the original game if and only if for every player i the pair (Γ^*, α_i^*) solves the following individual problem:

$$\begin{aligned} \max_{(\Gamma, \alpha_i) \in \mathbf{R}_+ \times [0,1]} \quad & U_i(w_i - \Gamma + \Gamma_{-i}^*, F(\Gamma, \alpha_{-i}^*, \alpha_i), \alpha_{-i}^*, \alpha_i) \\ \text{s.t.} \quad & 0 \leq \Gamma - \Gamma_{-i}^* \leq w_i \end{aligned} \tag{1}$$

To show that $\hat{\theta}_i$ is a best strategy for player i given $\hat{\theta}_{-i}$, let $\hat{\Gamma}_{-i} = \sum_{j \neq i} \hat{g}_j$ and consider the problem:

$$\begin{aligned} \max_{(\Gamma, \alpha_i) \in \mathbf{R}_+ \times [0,1]} \quad & U_i(\hat{w}_i - \Gamma + \hat{\Gamma}_{-i}, F(\Gamma, \alpha_{-i}^*, \alpha_i), \alpha_{-i}^*, \alpha_i) \\ \text{s.t.} \quad & 0 \leq \Gamma - \hat{\Gamma}_{-i} \leq \hat{w}_i, \end{aligned} \tag{2}$$

Note that $\hat{\Gamma}_{-i} - \Gamma_{-i}^* = -\Delta w_i$, i.e., $\hat{w}_i + \hat{\Gamma}_{-i} = w_i + \Gamma_{-i}^*$.

It is immediate that players such that $\Delta w_i = 0$ have no incentive to deviate unilaterally from $\hat{\theta}$ since the corresponding individual problems above (1) and (2) are the same,

Consider now a player i such that $\Delta w_i < 0$. Then, player i's problem (2) is the same as (1) with a more demanding restriction that is held by the pair

(Γ^*, α_i^*) solving (1). This implies that $(\hat{\Gamma}, \alpha_i^*)$ solves the optimization problem (2) or equivalently $\hat{\theta}_i$ maximizes player i's payoff when the others choose $\hat{\theta}_{-i}$.

Finally, consider the case $\Delta w_i > 0$ and assume that there is a pair (Γ, α_i) solving problem (2) and differs from $(\hat{\Gamma}, \alpha_i^*)$. Then, for every λ sufficiently close to 1, $\lambda(\Gamma^*, \alpha_i^*) + (1-\lambda)(\Gamma, \alpha_i)$ is affordable[4] for agent i before the redistribution of endowments and by convexity this pair leads to a higher utility level than (Γ^*, α_i^*), which is a contradiction.

Q.E.D.

7 Some final remarks

In this paper, we have introduced an approach to the private provision of a collective good that incorporates a new variable capturing the levels of intensity of use of such a good, and the distribution of end users. As we have remarked, several models can be obtained as particular cases. In addition, we have provided several considerations regarding altruistic behavior, congestion problems, crowding externalities and neutrality results, among others, that differ from those already established in the related literature because of the effect of the new variable determining the profile of level of utilization of a public good.

Our modification of the standard public good provision game and the findings we present pretend to be a theoretical contribution. In spite of this, the modified strategy space may be of interest for empirical analysis or experimental studies that could be designed and used to test the predictions.

Finally, we point out that the extension of the standard set up we propose allows for further interpretations of the new variable we introduce and not only

[4]Note that $\lambda\Gamma^* + (1-\lambda)\Gamma \geq \Gamma^*_{-i}$ for λ close enough to 1, provided that the redistribution of endowments is among contributors

intensity of use of a collective good. For instance, by considering poverty alleviation as a public good, our proposal and arguments may pave the way for new insights regarding the theoretical and empirical literature on donations and, in particular, on charity private transfers to the needy or giving to reduce poverty. This may be part of future research.

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