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A mathematical model for malware spread on WSNs with population dynamics

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ABSTRACT

The aim of this work is to describe and analyze a new theoretical model to simulate the spread of malicious code on wireless sensor networks. Specifically, this is a SCIRS model such that population dynamics, and vaccination and reinfection processes are considered. The local and global stability of the equilibrium points are studied and the most important security countermeasures are explicitly shown by means of the analysis of the epidemic threshold.

1. Introduction

Cybersecurity is one of the basis pillars of the digital and social environment defined by the new emerging paradigms such as the Internet of Everything or the Industry 4.0. Among the main techniques used in cyber-attacks we must highlight malicious code (also known as malware). It is possibly the most important threat to cybersecurity and the economic and social costs caused by their malicious actions (damage to devices, theft or deletion of personal information, etc.) are extremely high. The use of wireless sensor networks (WSNs for short) is crucial in the development of these paradigms [1,2] and their security is a major issue [3].

The scientific and technological efforts to combat malware are focused in two ways: the detection and the simulation of malware spreading. The main research line deals with the design and analysis of protocols for detecting malware [4]; the other approach tries to combat malware through designing theoretical models to simulate its propagation [5]. Most of these models are usually deterministic and global [6] and the dynamics is usually modeled using a system of ordinary differential equations. Moreover, in these models the devices can be classified into several classes like susceptible devices (S), exposed devices (E), infectious devices (I), recovered devices (R), etc.

Although several models have appeared in the scientific literature simulating not only biological agents spreading but also malware propagation over Internet or computer networks (see, for example, [7–10]), very few have dealt with WSNs (see, for example, [11–13]). On the other hand population dynamics is an important feature (see, for example, [14–29]) and, to our knowledge, only one WSNs model has been proposed taken into account it [30].

The main goal of this work is to introduce a new theoretical model considering population dynamics and carrier compartment. Moreover a vaccination process and a reinfection process are taken into account to define its dynamics. This work can be considered as an improvement of the model presented in [31]. We will use the theory of the stability to

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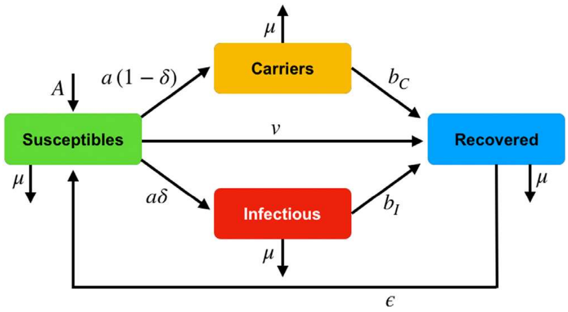

Fig. 1. Scheme that represents the dynamics of the mathematical model.

Table 1
Epidemiological coefficients of the model.

Symbol	Description
a	Transmission rate
δ	Fraction of susceptible devices targeted by malware
v	Temporal immunity rate
μ	Removal rate
b_C	Recovery rate from carrier compartment
b_I	Recovery rate from infectious compartment
A	Birth rate
ϵ	Loss of immunity

describe the behavior of the evolution of the compartments involved in the system and, also, the most efficient control measures will be obtained from the analysis of the expression of the basic reproductive number R_0.

The rest of the paper is organized as follows: the main characteristics of the model are presented in Section 2; in Section 3 the study of the local and global stability is detailed; in Section 4, the analysis of the most efficient security countermeasures is presented and, finally, the conclusions and future work are introduced in Section 5.

2. Description of the model to simulate the spread of malicious code

2.1. General dynamics and equations

The model presented in this work is global and deterministic such that the population of nodes is classified into the following compartments: susceptible $S(t)$, infectious $I(t)$, carrier $C(t)$, and recovered devices $R(t)$. Population dynamics is considered (new nodes can be added to the network and some nodes can be removed from it) and temporal immunity is assumed (see Fig. 1). Susceptible nodes are those that are free of malware; infectious nodes are those that have been reached by the malicious code and it can perform its malicious action (cause damage to the node and spread to other nodes); when malware is no able to cause some damage to the host node but the node serves as a transmission vector, the state of the node is that of carrier; finally, recovered are those (infectious or carrier) devices in which the malicious code has been removed by means of security countermeasures.

A susceptible node becomes infectious or carrier when the malicious code reaches it. The new infectious devices at time t are given by the expression $a\delta I(t)S(t)$ (incidence) such that a stands for the transmission coefficient, and δ represents the fraction of susceptible devices targeted by malware. Similarly, the number of new carrier devices at step of time t is given by $a(1-\delta)I(t)S(t)$. Thanks to preventive measures, a fraction of susceptible nodes can acquire temporal immunity at rate v: consequently the "vaccinated" devices at t are defined by $vS(t)$. Infectious and carrier devices can recover at rates b_C and b_I, respectively. Permanent immunity is not considered thus recovered devices become susceptible again at rate ϵ. Finally, as population dynamics is assumed then the nodes are removed at rate μ and new (susceptible) nodes appear at rate A. In Table 1 a brief description of these epidemiological parameters is done.

Consequently, the dynamics of the system is governed by a SODE whose equations are the following:

$$S'(t) = A + \epsilon R(t) - aI(t)S(t) - vS(t) - \mu S(t), \quad (1)$$

$$C'(t) = a(1-\delta)I(t)S(t) - b_C C(t) - \mu C(t), \quad (2)$$

$$I'(t) = a\delta I(t)S(t) - b_I I(t) - \mu I(t), \quad (3)$$

$$R'(t) = b_C C(t) + b_I I(t) + vS(t) - \epsilon R(t) - \mu R(t), \quad (4)$$

$$N'(t) = A - \mu N(t), \quad (5)$$

where $N(t)$ denotes the total number of devices at step of time t.

From Eq. (5) it is

$$N(t) = \frac{A}{\mu} + \left(N(0) - \frac{A}{\mu}\right) e^{-\mu t}, \tag{6}$$

and considering the limit system, we obtain:

$$\begin{aligned} S'(t) &= A + \epsilon(N - S(t) - I(t) - C(t)) \\ &\quad - aI(t)S(t) - vS(t) - \mu S(t), && (7) \\ C'(t) &= a(1-\delta)I(t)S(t) - b_C C(t) - \mu C(t), && (8) \\ I'(t) &= a\delta I(t)S(t) - b_I I(t) - \mu I(t), && (9) \end{aligned}$$

where $N = \lim_{t\to\infty} N(t) = A/\mu$. As a consequence, the feasible region obtained is

$$\Omega = \{(S, C, I) \in \mathbb{R}_3^+ \text{ such that } 0 \le S + C + I \le A/\mu\}, \tag{10}$$

such that its boundary is defined by the following four faces:

$$\begin{aligned} F_1 &= \{(S, C, I) \in \mathbb{R}_3^+ \text{ such that } S + C + I = A/\mu, 0 \le S, C, I \le A/\mu\}, && (11) \\ F_2 &= \{(S, C, I) \in \mathbb{R}_3^+ \text{ such that } S = 0, 0 \le C + I \le A/\mu\}, && (12) \\ F_3 &= \{(S, C, I) \in \mathbb{R}_3^+ \text{ such that } C = 0, 0 \le S + I \le A/\mu\}, && (13) \\ F_4 &= \{(S, C, I) \in \mathbb{R}_3^+ \text{ such that } I = 0, 0 \le S + C \le A/\mu\}, && (14) \end{aligned}$$

where $\vec{n}_1 = (1, 1, 1)$, $\vec{n}_2 = (-1, 0, 0)$, $\vec{n}_3 = (0, -1, 0)$, and $\vec{n}_4 = (0, 0, -1)$ are their outer normal vectors respectively. Furthermore:

$$\begin{aligned} \left(S'(t), C'(t), I'(t)\right)_{F_1} \bullet \vec{n}_1 &= A - \mu N - b_C C - b_I I - vS \le 0, && (15) \\ \left(S'(t), C'(t), I'(t)\right)_{F_2} \bullet \vec{n}_2 &= -A + (C + I - N)\epsilon \le 0, && (16) \\ \left(S'(t), C'(t), I'(t)\right)_{F_3} \bullet \vec{n}_3 &= -aIS(1-\delta) \le 0, && (17) \\ \left(S'(t), C'(t), I'(t)\right)_{F_4} \bullet \vec{n}_4 &= 0. && (18) \end{aligned}$$

Consequently, as Ω is closed then it is compact and invariant [32]. Thus, for all $t \ge 0$ the solutions in Ω of the SODE (7)–(9) exist and are unique [33].

2.2. Computation of the steady states and the basic reproductive number

The solutions of the non-linear system:

$$\begin{aligned} 0 &= A + \epsilon(N - S(t) - I(t) - C(t)) - aI(t)S(t) - vS(t) - \mu S(t), && (19) \\ 0 &= a(1-\delta)I(t)S(t) - b_C C(t) - \mu C(t), && (20) \\ 0 &= a\delta I(t)S(t) - b_I I(t) - \mu I(t). && (21) \end{aligned}$$

are the steady states of the system (1)–(4). This system has two solutions: the most simple is the disease-free steady state:

$$E_0 = (S_0, C_0, I_0) = \left(\frac{A + \epsilon N}{v + \epsilon + \mu}, 0, 0\right), \tag{22}$$

and the other is called endemic steady state $E^* = (S^*, C^*, I^*)$ such that:

$$\begin{aligned} S^* &= \frac{b_I + \mu}{a\delta}, && (23) \\ C^* &= \frac{(-1+\delta)(b_I + \mu)(-a\delta(A + N\epsilon) + b_I(v + \epsilon + \mu) + \mu(v + \epsilon + \mu))}{a\delta(\mu(\epsilon + \mu) + b_I(\epsilon - \delta\epsilon + \mu) + b_C(b_I + \delta\epsilon + \mu))}, && (24) \\ I* &= -\frac{(b_C + \mu)(-a\delta(A + N\epsilon) + b_I(v + \epsilon + \mu) + \mu(v + \epsilon + \mu))}{a(\mu(\epsilon + \mu) + b_I(\epsilon - \delta\epsilon + \mu) + b_C(b_I + \delta\epsilon + \mu))}. && (25) \end{aligned}$$

This second solution (the endemic steady state) exists if the following holds:

$$\frac{a\delta(A + N\epsilon)}{(b_I + \mu)(v + \epsilon + \mu)} > 1. \tag{26}$$

On the other hand, we can compute the expression of the basic reproductive number R_0 considering the next-generation matrix method [34]. In this sense, the next-generation matrix at E_0 is $G = F \cdot V^{-1}$ such that:

$$F = \begin{pmatrix} 0 & \frac{a(1-\delta)(A+N\epsilon)}{v+\epsilon+\mu} \\ 0 & \frac{a\delta(A+N\epsilon)}{v+\epsilon+\mu} \end{pmatrix}, \qquad V = \begin{pmatrix} b_C+\mu & 0 \\ 0 & b_I+\mu \end{pmatrix}. \tag{27}$$

Thus, R_0 is its spectral radius:

$$R_0 = \rho(G) = \frac{a\delta(A+N\epsilon)}{(b_I+\mu)(v+\epsilon+\mu)}. \tag{28}$$

As a consequence, the condition (26) can be reformulated as $R_0 > 1$.

3. Local and global stability of the steady states

3.1. Local stability

Theorem 1. *The disease-free steady state, E_0, is locally asymptotically stable if $R_0 < 1$.*

Proof. The eigenvalues of the matrix

$$F - V = \begin{pmatrix} -b_C-\mu & \frac{a(1-\delta)(A+N\epsilon)}{v+\epsilon+\mu} \\ 0 & \frac{a\delta(A+N\epsilon)}{v+\epsilon+\mu} - b_I - \mu \end{pmatrix} \tag{29}$$

are

$$\lambda_1 = -b_C - \mu, \tag{30}$$

$$\lambda_2 = \frac{a\delta(A+N\epsilon)}{v+\epsilon+\mu} - b_I - \mu, \tag{31}$$

whose real parts are negative if $R_0 < 1$. Moreover, as

$$\frac{\partial}{\partial S}\left(-vS + \epsilon\,(N-S) - \mu S + A\right) = -v - \epsilon - \mu < 0 \tag{32}$$

then E_0 is locally asymptotically stable (see [35]). $\square$

Theorem 2. *The epidemic steady state, E^*, is locally asymptotically stable if $R_0 > 1$.*

Proof. The characteristic polynomial of the jacobian matrix of the SODE (7)–(9) in E^* is $P(\lambda) = p_0\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3$, where

$$p_0 = 1, \tag{33}$$

$$p_1 = b_C + R_0 v + \epsilon + (1-R_0)\mu + \frac{(-1+R_0)\epsilon(b_I v(-1+\delta) - v\mu + b_I\delta\mu + b_C(b_I+\mu-\delta(v+\mu)))}{\mu(\epsilon+\mu) + b_I(\epsilon-\delta\epsilon+\mu) + b_C(b_I+\delta\epsilon+\mu)}, \tag{34}$$

$$p_2 = \frac{(b_C+\mu)(v+\epsilon+\mu)}{\mu(\epsilon+\mu) + b_I(\epsilon-\delta\epsilon+\mu) + b_C(b_I+\delta\epsilon+\mu)} \cdot \left(b_I^2(-1+R_0) + b_C\delta\epsilon + R_0(b_C+\epsilon)\mu + (-1+2R_0)\mu^2 + b_I(b_C R_0 + R_0\epsilon - \delta\epsilon - 2\mu + 3R_0)\right), \tag{35}$$

$$p_3 = (-1+R_0)(b_C+\mu)(b_I+\mu)(v+\epsilon+\mu). \tag{36}$$

A simple calculus shows that $p_0 > 0, p_1 > 0, p_3 > 0$, and $p_1p_2 - p_3 > 0$, if $R_0 > 1$. Consequently, taking into account the Routh–Hurwitz criterion (see [36]), the local stability of the epidemic disease states follows when $R_0 > 1$. $\square$

3.2. Global stability

Theorem 3. *The disease-free steady state, E_0, is globally asymptotically stable in Ω if $R_0 \leq 1$.*

Proof. From Eq. (7) and considering $X(t)$ as an auxiliary variable, we have

$$S'(t) \leq A + \epsilon N - S(v+\epsilon+\mu), \tag{37}$$

$$X'(t) = A + \epsilon N - X(v+\epsilon+\mu). \tag{38}$$

By applying the Comparison Theorem (see [37]) we obtain that $X(t) > S(t), t > 0$. As a consequence, when t tends to infinity, the following inequality holds:

$$S \leq \frac{A+\epsilon N}{\upsilon+\epsilon+\mu}, \tag{39}$$

because of $\lim_{t\to\infty} X(t) = \frac{A+\epsilon N}{\upsilon+\epsilon+\mu}$. Now, from (39) and considering the Lyapunov function $V = I$, we obtain

$$\begin{aligned} V'(t) &= I(aS\delta - (b_I+\mu)) \leq I\left(a\left(\frac{A+N\epsilon}{\upsilon+\epsilon+\mu}\right)\delta - (b_I+\mu)\right) \\ &= I(b_I+\mu)(R_0-1). \end{aligned} \tag{40}$$

Note that if $R_0 \leq 1$ and $(S, C, I) \in \Omega$ then $\frac{dV}{dt} \leq 0$. In addition, $\frac{dV}{dt} = 0$ if and only if $R_0 = 1$ and $I = 0$ or $S = \frac{A+\epsilon N}{\upsilon+\epsilon+\mu}$. As a consequence, (S, I, C) tends to E_0 as $t \to \infty$, and consequently the maximum invariant set in $\{(S, C, I) \in \Omega : \frac{dV}{dt} = 0\}$ is E_0. Finally, taking into account $V = I$ and LaSalle invariance principle [38], the result follows. □

Theorem 4. *The endemic steady state, E^*, is globally asymptotically stable if $R_0 > 1$ when the following inequalities hold:*

$$b_I + aN - \upsilon - ac - b_C - \mu - \delta ac < 0, \tag{41}$$

$$-\mu - b_C + \epsilon + aN\delta < 0. \tag{42}$$

Proof. First of all, by applying [39, Theorem 4.3] the system (7)–(9) is uniformly persistent for $R_0 > 1$. As a consequence, E^* is the unique equilibrium point belonging to int(Ω) since there exists an absorbent compact in it which is simply connected [40].

The second additive compound matrix of Jacobian matrix is given by the following explicit expression:

$$J^{[2]} = \begin{pmatrix} -b_C - aI - \upsilon - \epsilon - 2\mu & aS(1-\delta) & aS+\epsilon \\ 0 & -b_I - aI - \upsilon + aS\delta - \epsilon - 2\mu & -\epsilon \\ -aI\delta & aI(1-\delta) & -b_C - b_I + aS\delta - 2\mu \end{pmatrix}. \tag{43}$$

Set A_f the directional derivative of the diagonal matrix $A = \operatorname{diag}\left(\frac{S}{I}, \frac{S}{I}, \frac{S}{I}\right)$ along (S, C, I), then:

$$A_f \cdot A^{-1} = \operatorname{diag}\left(\frac{S'(t)}{S} - \frac{I'(t)}{I}, \frac{S'(t)}{S} - \frac{I'(t)}{I}, \frac{S'(t)}{S} - \frac{I'(t)}{I}\right). \tag{44}$$

Set the matrix $B = A_f A^{-1} + AJ^{[2]}A^{-1} = (b_{ij})_{1\leq i,j\leq 3}$ where:

$$b_{11} = b_C + aI - \frac{I'}{I} + \frac{S' - S(\upsilon+\epsilon+2\mu)}{S}, \tag{45}$$

$$b_{12} = -aS(-1+\delta), \tag{46}$$

$$b_{13} = aS + \epsilon, \tag{47}$$

$$b_{21} = 0, \tag{48}$$

$$b_{22} = -b_I - aI - \frac{I'}{I} + aS\delta + \frac{S' - S(\upsilon+\epsilon+2\mu)}{S}, \tag{49}$$

$$b_{23} = -\epsilon, \tag{50}$$

$$b_{31} = -aI\delta, \tag{51}$$

$$b_{32} = -aI(-1+\delta), \tag{52}$$

$$b_{33} = -b_C - b_I - \frac{I'}{I} + \frac{S'}{S} + aS\delta - 2\mu. \tag{53}$$

Note that the Lozinskii measure of B is given by the following expression [41]:

$$\mu_L(B) = \inf\{c \in \mathbb{R} : \mathcal{D}_+\|z\| \leq c\|z\| \text{ such that } z' = Bz\}, \tag{54}$$

with $\mathcal{D}_+$ being the right-hand derivative [42,43]. In addition, one can estimate $\mathcal{D}_+\|z\|$ by means of two cases supposing that $\|z\| = \max\{\|z_2\| + \|z_3\|, \|z_1\|\}$ with $z = (z_1, z_2, z_3)$:

(1) If $\|z\| = \|z_2\| + \|z_3\|$, then $\mathcal{D}_+\|z\| \leq (-\mu - b_C + \epsilon + aI\delta)\,\|z\|$.
(2) If $\|z\| = \|z_1\|$, then

$$\mathcal{D}_+\|z\| \leq (b_I + aS - \upsilon - aI - b_C - \mu - \delta aS)\,\|z\|. \tag{55}$$

Now, considering (54)–(55) and assumptions (41)–(42), we have

$$\mu_L(B) \leq \frac{S'(t)}{S} - \theta, \quad \theta > 0. \tag{56}$$

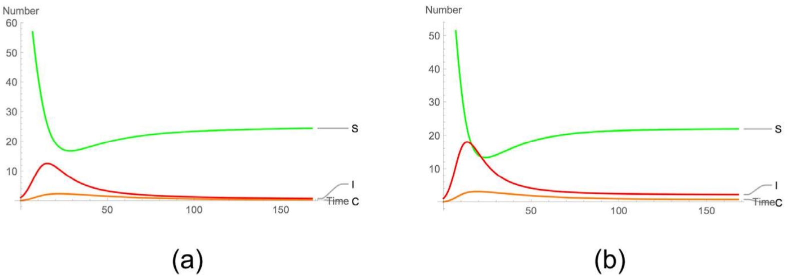

Fig. 2. Behavior of the compartments when E_0 is reached (a), and when E^* is obtained (b).

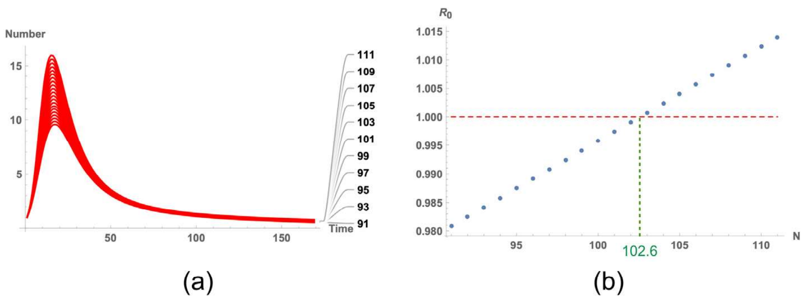

Fig. 3. (a) Evolution of $I(t)$ when N varies. (b) Evolution of R_0 when N varies.

As a consequence we can fiend a real number $T > 0$ in such a way $I(t) < e^{(\theta t/2)}$ when $t > T$. Consequently, $\frac{1}{t}\log S(t) < \frac{\theta}{2}$ for each solution of SODE (7)–(9) in $\mathrm{int}(\Omega)$. For $t \gg 0$, it is

$$\overline{q}_2 = \limsup_{t\to\infty} \sup_{(S(0),C(0),I(0))\in \mathrm{int}(\Omega)} \frac{1}{t}\int_0^t \mu(B)dt < -\frac{1}{2}\theta < 0, \tag{57}$$

and applying the geometrical approach [44], the statement is proved. □

3.3. Numerical and illustrative simulations

In this section we will introduce some illustrative simulations where the different behaviors of the system are shown. Assume that the population is initially formed by $N = 101$ devices such that there is only one infectious device and the rest are susceptible at $t = 0$: $S(0) = 100, I(0) = 1, C(0) = R(0) = 0$. In addition, suppose that $A = 2$, $\mu = 0.04$, $\upsilon = 0.05$, $\epsilon = 0.004$, $b_C = 0.005$, $b_I = 0.1$ and $\delta = 0.91$. In our case $0 \le t \le 168$ where t is measured in hours (the first week after the outbreak is simulated).

If the transmission coefficient is given by $a = 0.006$ then $R_0 \approx 0.99 \le 1$ and consequently E_0 is locally and globally asymptotically stable (see Fig. 2(a)). This equilibrium point is given by the expression $E_0 = (S_0, C_0, I_0) \approx (25.64, 0, 0)$.

Now, if $a = 0.007$ and the rest of coefficients are the same than in the previous example, then $R_0 \approx 1.164 > 1$. As a consequence the malware outbreak becomes epidemic (see Fig. 2(b)). Note that the endemic steady state is $E^* = (S^*, C^*, I^*) \approx (21.98, 0.6539, 2.125)$.

Now assume that the total number of devices, N, varies. Suppose that the rest of values of epidemiological coefficients, with $a = 0.006$, remains constant as stated in the first paragraph of this subsection. In this case the impact of the malware outbreak (that is, the maximum number of infected devices) grows as N grows as can shown in Fig. 3(a) where $91 \le N \le 111$. Furthermore, in Fig. 3(b) the evolution of the basic reproductive number is shown. As is introduced in Sections 3.1 and 3.2 this is a threshold coefficient that determines the local and global stability of steady states. This parameter grows linearly as N increases and when $N \approx 102.6$ then $R_0 = 1$. Note that the system exhibits a similar behavior when a, δ and A are studied since the basic reproductive number depends linearly on these parameters.

On the other hand suppose that the temporal immunity rate is varied such that $\upsilon = 0.01, 0.02, \ldots, 0.1$. If each phase diagram for susceptible, carrier and infected devices is computed and all trajectories start at the same initial point $(S(0), C(0), I(0)) = (100, 0, 1)$, it is observed that as the vaccination coefficient increases, these trajectories tend to disease-free steady states with low epidemiological impact (see Fig. 4). A similar behavior is obtained when the non-constant coefficient is the recovery rate from infected b_I.

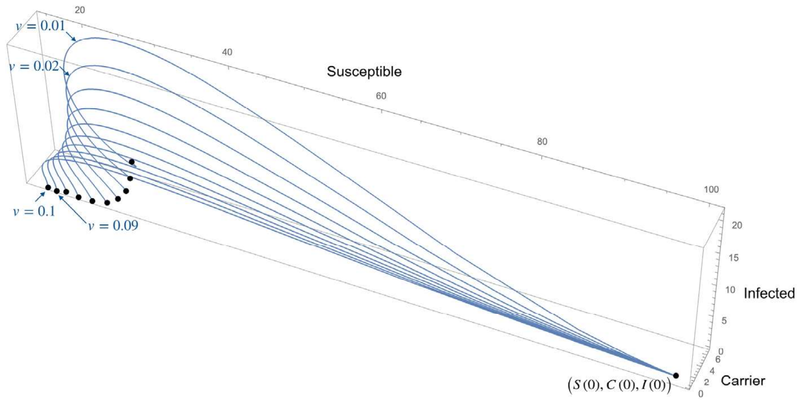

Fig. 4. Phase diagram of $S(t)$, $C(t)$ and $I(t)$ with ten different values of the temporal immunity rate.

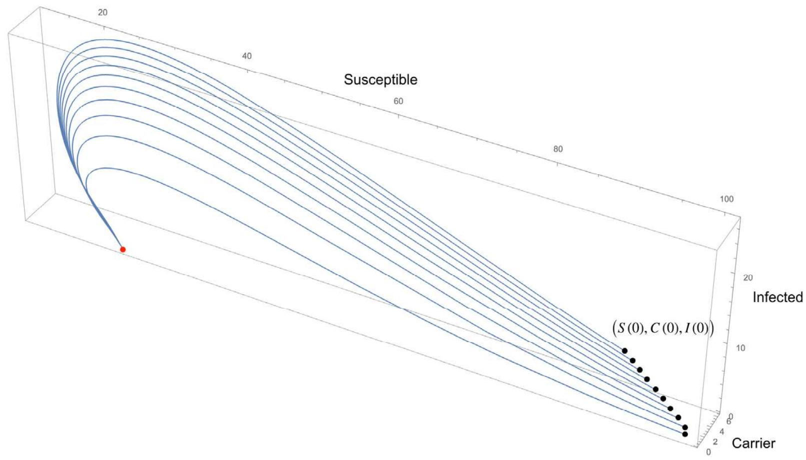

Fig. 5. Phase diagram of $S(t)$, $C(t)$ and $I(t)$ with ten different initial points $(S(0), C(0), I(0))$.

Finally, if ten simulations are computed from different initial points (it is supposed that $I(0) = 1, 2, \ldots, 10$ with $N = 101$, $C(0) = 0$, and $S(0) = N - I(0)$), then the phase diagrams are computed (see Fig. 5). As is shown in this figure all trajectories starting from different initial points finally converge to the same equilibrium point.

4. Determination of efficient security countermeasures

It is well known that R_0 is an important epidemiological threshold. Moreover, it is crucial in the design of efficient security countermeasures. In this sense if $R_0 < 1$ the malware outbreak does not start an epidemic. Consequently, it is extremely important to reduce the value of R_0 below 1, and this must be the main goal of the security countermeasures.

4.1. Analysis considering only one parameter

Taking into account the expression of R_0 (28) and assuming $0 < a, \mu, \epsilon, b_I, b_C, v, \delta, \leq 1$ and $A, N \geq 1$, we obtain the following:

$$\frac{\partial R_0}{\partial a} = \frac{\delta(A + N\epsilon)}{(b_I + \mu)(v + \epsilon + \mu)} > 0, \tag{58}$$

$$\frac{\partial R_0}{\partial \epsilon} = \frac{a\delta(-A + N(v + \mu))}{(b_I + \mu)(v + \epsilon + \mu)^2}, \tag{59}$$

$$\frac{\partial R_0}{\partial b_I} = -\frac{a\delta(A+N\epsilon)}{(b_I+\mu)^2(v+\epsilon+\mu)} < 0, \tag{60}$$

$$\frac{\partial R_0}{\partial b_C} = 0, \tag{61}$$

$$\frac{\partial R_0}{\partial v} = -\frac{a\delta(A+N\epsilon)}{(b_I+\mu)(v+\epsilon+\mu)^2} < 0, \tag{62}$$

$$\frac{\partial R_0}{\partial N} = \frac{a\delta\epsilon}{(b_I+\mu)(v+\epsilon+\mu)} > 0, \tag{63}$$

$$\frac{\partial R_0}{\partial \delta} = \frac{a(A+N\epsilon)}{(b_I+\mu)(v+\epsilon+\mu)} > 0, \tag{64}$$

$$\frac{\partial R_0}{\partial A} = \frac{a\delta}{(b_I+\mu)(v+\epsilon+\mu)} > 0, \tag{65}$$

$$\frac{\partial R_0}{\partial \mu} = -\frac{a\delta(A+N\epsilon)}{(b_I+\mu)(v+\epsilon+\mu)^2} - \frac{a\delta(A+N\epsilon)}{(b_I+\mu)^2(v+\epsilon+\mu)} < 0, \tag{66}$$

It is easy to check that R_0 decreases when N, A, a or δ decreases assuming that the remaining coefficients are constant. Furthermore, R_0 decreases when b_I, v and μ increases (the remaining epidemiological coefficients are considered constant). If $A < N(v+\mu)$ then R_0 decreases as ϵ decreases and if $A > N(v+\mu)$ then R_0 decreases as ϵ increases. Furthermore, R_0 does not change if b_C changes. Consequently the following security countermeasures are efficient:

- To decrease the transmission rate of infective devices by strengthening the awareness and knowledge about security of the users.
- To increase the vaccination and recovery rates taking into account efficient security software.

The epidemiological coefficients v, b_I, a, δ and ϵ are involved to a greater or lesser extent in these measures. Note that the rest of the security countermeasures derived from the above lead to the control of the number of the sensors initially deployed, N, and those that are subsequently added A or removed μ; this could not be realistic due to the characteristics of WSNs since it is not always possible to control both the number of sensors that are correctly deployed, or the number of sensors removed from the WSN due to, for example, energy consumption and end of battery life. Finally, if we change recovery rate of carrier devices, R_0 does not change.

Finally note that considering the algebraic structure of Eq. (28) the coefficients with greatest effect on the basic reproductive number is $A > 1$ since it appears in a summation where the other two (positive) coefficients, a and δ, are less than 1. Moreover, also μ has a great influence on R_0 when it decreases — note that it appears in the denominator of the explicit expression of the basic reproductive number as $\mu^2 + (v+\epsilon+b_I) + b_I(v+\epsilon)$-. On the other hand, the least effect is due to ϵ.

4.2. Analysis considering two epidemiological parameters

Now we analyze R_0 to determine security countermeasures by considering it as a function of two variables. In what follows the basic procedure to obtain such security actions is described.

Assume that the coefficients considered as variables are x and y, thus $R_0 = R_0(x, y)$. Let $p_0 = (x_0, y_0)$ be the starting point located in the endemic region defined by $R_0 > 1$ of the xy-plane. Set $\bar{p} = (\bar{x}, \bar{y})$ the point belonging to the curve $R_0 = 1$ such that $d(p_0, \{R_0 = 1\}) = d(p_0, \bar{p})$. As a consequence, the best way to get the malware-free region ($R_0 < 1$) is defined by the segment between the points p_0 and $\bar{p}$. As the segment $p_0\bar{p}$ is defined by:

$$x = \lambda x_0 + (1-\lambda)\bar{x} \tag{67}$$

$$y = \lambda y_0 + (1-\lambda)\bar{y} \tag{68}$$

with $0 \le \lambda \le 1$, the control measure obtained consists of modifying the coefficients x and y through the segment $p_0\bar{p}$ such that $\lambda \to 0$. That is, the best strategy is to make (x_0, y_0) tend to $(\bar{x}, \bar{y})$ through the last mentioned segment.

For the sake of simplicity we will suppose that one of these variables is the vaccination coefficient v, while the other variable varies between the rest of epidemiological coefficients a, δ, ϵ, b_I and population parameters N, A, and μ.

Case I: $R_0 = R_0(v, a)$. Set $p_0 = (v_0, a_0)$ the initial point (initial state of epidemiological coefficients of the system) and let $R_0(v, a) = 1$ be the threshold curve between both regions of the va-plane (endemic and disease-free regions) whose explicit equation is

$$a = \frac{b_I+\mu}{\delta(A+N\epsilon)}\left(v+(\epsilon+\mu)\right). \tag{69}$$

The desired point $\bar{p} = (\bar{v}, \bar{a})$ is obtained studying the minimum distance from p_0 to $R_0(v, a) = 1$. A calculus, using simple analytical tools, shows that $\bar{p} = (\bar{v}, \bar{a})$ is defined as follows:

$$\bar{v} = \frac{\bar{a}\delta(A+N\epsilon) - (b_I+\mu)(\epsilon+\mu)}{b_I+\mu}, \tag{70}$$

$$\bar{a} = \frac{(b_I+\mu)(a_0(b_I+\mu)+\delta(A+N\epsilon)(v_0+\epsilon+\mu))}{\delta^2(A+N\epsilon)^2+(b_I+\mu)^2}. \tag{71}$$

Case II: $R_0 = R_0(v,\delta)$. If $p_0 = (v_0,\delta_0)$ is in the endemic region, the point belonging to $R_0 = 1$ that is the nearest to p_0 is $\bar{p} = \left(\bar{v},\bar{\delta}\right)$ given by:

$$\bar{v} = \frac{a\bar{\delta}(A+N\epsilon)-(b_I+\mu)(\epsilon+\mu)}{b_I+\mu}, \tag{72}$$

$$\bar{\delta} = \frac{(b_I+\mu)(\delta_0(b_I+\mu)+a(A+N\epsilon)(v_0+\epsilon+\mu))}{a^2(A+N\epsilon)+(b_I+\mu)^2}. \tag{73}$$

Case III: $R_0 = R_0(v,\epsilon)$. Similarly, when the variables are defined by the coefficients v and ϵ the nearest point to $p_0 = (v_0,\epsilon_0)$ on the straight line $R_0 = 1$ is $\bar{p} = (\bar{v},\bar{\epsilon})$ such that:

$$\bar{v} = \frac{aA\delta - b_I\epsilon - \epsilon + aN\delta\epsilon - b_I\mu - \epsilon\mu - \mu^2}{b_I+\mu}, \tag{74}$$

$$\begin{aligned}\bar{\epsilon} = &\frac{-a^2AN\delta^2 - b_I^2(v_0-\epsilon_0+\mu) - \mu^2(v_0+\epsilon_0+\mu) + a\delta\mu(A+N(v_0+\mu))}{2(b_I^2-2aN\delta+a^2N^2\delta^2-2aN\delta\mu+2\mu^2+b_I(-2aN\delta+4\mu))}\\ &+\frac{b_I(-2\mu(v_0-e_0+\mu)+a\delta(A+N(v_0+\mu)))}{2(b_I^2-2aN\delta+a^2N^2\delta^2-2aN\delta\mu+2\mu^2+b_I(-2aN\delta+4\mu))}.\end{aligned} \tag{75}$$

Case IV: $R_0 = R_0(v,b_I)$. If $R_0 = R_0(v,b_I)$ and $p_0 = (v_0,b_{I0})$ belongs to the endemic region, $\bar{p} = \left(\bar{v},\bar{b}_I\right)$ is the nearest point to p_0 belonging the straight line $R_0 = 1$ whose coordinates are the following:

$$\bar{v} = \frac{a\delta(A+N\epsilon)-(\bar{b}_I+\mu)(\epsilon+\mu)}{\bar{b}_I+\mu}, \tag{76}$$

and $\bar{b}_I$ is a real and positive solution of the following equation:

$$2\bar{b}_I^4 + (-2b_{I0}+6)\,\bar{b}_I^3 + \left(-6b_{I0}\mu+6\mu^2\right)\bar{b}_I^2 + \alpha_1\bar{b}_I + \alpha_0 = 0, \tag{77}$$

where

$$\alpha_0 = -2a^2\delta^2(A+N\epsilon)^2 - 2b_{I0}\mu^3 + 2a\delta(A+N\epsilon)\mu(v_0+\epsilon+\mu), \tag{78}$$

$$\alpha_1 = -6b_{I0}\mu^2 + 2\mu^3 + 2a\delta(A+N\epsilon)(v_0+\epsilon+\mu). \tag{79}$$

Case V: $R_0 = R_0(v,N)$. When $p_0 = (v_0,N_0)$ is in the endemic region, the point $\bar{p} = \left(\bar{v},\bar{N}\right)$ is given by:

$$\bar{v} = \frac{a\delta(A+N\epsilon)-(b_I+\mu)(\epsilon+\mu)}{b_I+\mu}, \tag{80}$$

$$\bar{N} = \frac{N_0\left(b_I^2a^2A\delta^2\epsilon+2b_I\mu+\mu^2\right)+a\delta\epsilon(v_0+\epsilon+\mu)\,(b_I+\mu)}{a^2\delta^2\epsilon^2+(b_I+\mu)^2}. \tag{81}$$

Case VI: $R_0 = R_0(v,A)$. If $p_0 = (v_0,A_0)$ then $\bar{p} = \left(\bar{v},\bar{A}\right)$ is defined as follows:

$$\bar{v} = \frac{a\delta(\bar{A}+N\epsilon)-(b_I+\mu)(\epsilon+\mu)}{b_I+\mu}, \tag{82}$$

$$\bar{A} = \frac{A_0(b_I+\mu)^2 + a\delta(-aN\delta\epsilon + b_I(v_0+\epsilon+\mu)+\mu(v_0+\epsilon+\mu))}{a^2\delta^2+(b_I+\mu)^2}. \tag{83}$$

Case VII: $R_0 = R_0(v,\mu)$. Finally, if $R_0 = R_0(v,\mu)$ and $p_0 = (v_0,\mu_0)$ is in the endemic zone, it is easy to check that the nearest point to p_0 over $R_0 = 1$ is given by $\bar{p} = (\bar{v},\bar{\mu})$ such that:

$$\bar{v} = \frac{a\delta(A+N\epsilon)-(b_I+\bar{\mu})(\epsilon+\bar{\mu})}{b_I+\bar{\mu}}, \tag{84}$$

and $\bar{\mu}$ is a real and positive solution of

$$4\bar{\mu}^4 + (12b_I+2v_0+2\epsilon-2\mu_0)\,\bar{\mu}^3 + \alpha_2\bar{\mu}^2 + \alpha_1\bar{\mu} + \alpha_0 = 0, \tag{85}$$

where

$$\begin{aligned}\alpha_0 = &\ 2b_I^3v_0 + 2b_I^3\epsilon - 2ab_I^2\delta(A+N\epsilon) + 2ab_Iv_0\delta(A+N\epsilon) + 2ab_I\delta\epsilon(A+N\epsilon)\\ &-2a^2\delta^2(A+N\epsilon)^2 - 2b_I^3\mu_0\end{aligned} \tag{86}$$

$$\begin{aligned}\alpha_1 = &\ 4b_I^3 + 6b_I^2v_0 + 6b_I^2\epsilon - 2ab_I\delta(A+N\epsilon) + 2av_0\delta(A+N\epsilon)\\ &+2a\delta\epsilon(A+N\epsilon) - 6b_I^2\mu_0\end{aligned} \tag{87}$$

$$\alpha_2 = 12b_I^2 + 6b_I v_0 + 6b_I \epsilon - 6b_I \mu_0 \quad (88)$$

In order to decide which is the most effective security countermeasure when two parameters can be varied, one has to determine the pair of coefficients that minimizes the value of the distance $d(p, \bar{p})$.

5. Conclusions

A novel model to simulate the propagation of a specimen of malware on a wireless sensor network is presented and analyzed. It is a compartmental model considering susceptible, infectious, carrier and recovered devices. Moreover, population dynamics is considered (new sensor devices can be deployed in the sensor environment and also sensors are removed at every step of time due to the battery powers run out. Moreover, vaccination and reinfected processes are taken into account.) The infection process depends on the contact between susceptible and infectious -not carrier- devices.

The basic reproductive number is computed and it does not depends on the carrier recovery rate. This threshold influences on both the local and global stabilities of the equilibrium points. An analysis of R_0 allows us to obtain some control measures when all epidemiological coefficients are fixed with the exception of one or two. The most important and realistic security countermeasures when one can change only one parameter are the following:

- Strength the awareness and knowledge about security of the WSN users.
- Employment of efficient detection models and recovery tools.

When two epidemiological coefficients can be modified, one has to determine in each case (considering the numeric specific values assigned in this case) the pair of parameters that minimizes certain distance.

The great majority of theoretical models to simulate malware that has been proposed during the last years deal with generic computer networks and very few are devoted to the study of malicious code spread over wireless sensor networks. In fact, to our knowledge, only one WSN malware epidemic model has been proposed considering population dynamics (see [30]). In this model susceptible, infectious, quarantined and recovered sensors are considered and vaccination process is not taken into account; as a consequence the basic reproductive number associated to this model does not depend on such coefficients or the total number of devices. On the contrary, with the purpose to design a more realistic model, in our proposed model we do not take into account the quarantined class -since it is difficult to consider such compartment in an WSN environment- but we include the carrier sensors (those sensors that are not targeted by malware but are effectively deployed in the sensor area). Moreover, in our model a vaccination process is also considered which reflects the awareness of WSN users.

Future work aimed at designing a networked model based on that proposed in this work where different contact topologies could be considered (scale-free networks, small-world networks, etc.)

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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