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Mathematical Theory in non-linear Diffusion Processes: The Porous Medium Equation

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Abstract

This work is devoted to the study of the asymptotic behaviour of the 1 dimensional porous medium equation via a unifying argument rooted in Sturmian intersection theory. After considering a rescaling θ of solutions u to the porous medium equation with continuous compactly supported nonnegative initial data, we show uniform convergence of $\theta(\tau)$ to the fixed profile F_C of the Barenblatt solution with the same mass as u. We also derive a new a priori estimate that allows one to conclude disappearance of critical points of $\theta(\tau)$, the boundedness of $\partial_\xi\theta(\tau)$ and the uniform convergence of $\partial_\xi\theta(\tau)$ to F_C'. We then implement these equations numerically so as to show how solutions in higher dimensions exhibit a similar behaviour.

Keywords: Porous medium equation, Sturmian argument, numerical simulation, asymptotic behaviour, Barenblatt solutions.

1. Introduction

In this work we study many of the mechanisms inherent to non-linear diffusion processes such as the finite speed of propagation of solutions, the appearance of a free boundary on which solutions are not smooth and the existence of a family of self-similar solutions that determine the asymptotic behaviour of all solutions. This analysis will be carried out using the Sturm theory of intersections and based on one sole idea which is that of using a bi-parametric family of special solutions $\{W_{C,x_0}\}$ that intersect only once with the initial data to derive the behaviour of any solution u. This provides a unifying approach in the sense that many concepts and results are explained through one sole method and geometric idea.

To ground our work we will operate within the framework of the 1 dimensional Cauchy problem for the porous medium equation (PME). In the general d dimensional case this equation is given by

$$\begin{aligned} \partial_t u &= \Delta(u^m) \quad && \text{in } \mathbb{R}^d \times (0, +\infty) \\ u(0) &= u_0 && \text{in } \mathbb{R}^d \end{aligned} \tag{1}$$

where $m > 1$ and $u_0 \in C_c(\mathbb{R}^d)$ is a continuous compactly supported non-negative function.

The PME is perhaps the prototypical example of a nonlinear diffusion equation (a kind of parabolic equation) and describes a wealth of physical phenomena such as the evolution of the density of a gas in a porous medium, non-linear heat transfer, groundwater flow and even population dynamics. This wide range of applications explains the sustained interest in these equations and the reason they remain a fruitful avenue of study in the field of partial differential equations (PDE) and mathematical physics.

The study of parabolic equations, and in particular of the PME, presents a high level of complexity and needs a powerful mathematical machinery to operate. The necessary tools to accomplish this work only became available as recently as the 20th century. During this period great advances in the field of Functional Analysis, often motivated by the study of PDE, facilitated mathematicians the framework necessary to progress in their research into these equations. Much of the study of parabolic equations of this period is reflected in the classic work of O.Ladyzhenskaya [1] and the more recent work of Lieberman [2].

Though at a first glance the PME (1) resemble closely another parabolic PDE in the heat equation, their behaviour and study is very different. This is due to the fact that, as may be seen by developing equation (1)

$$\partial_t u = \Delta(u^m) = m(m-1)u^{m-2}|\nabla u|^2 + mu^{m-1}\Delta u,$$

The coefficient mu^{m-1} of the highest order derivative Δu becomes 0. This makes the PME *degenerate parabolic* (as opposed to *uniformly parabolic* see for example [1] page 11 for a definition) and explains many of the complications and phenomena that arise in the study of the equation. Some examples of these are:

(i) The propagation speed of a solution u to the PME will be finite. That is, if u_0 is of compact support, then the spatial support of $u(t)$ is compact for each fixed time t. Note that this does not occur for the heat equation, where the speed of propagation is infinite.

(ii) As a consequence of the previous point we deduce that the strong maximum principle can no longer hold. As given $u_0 \geq 0$ of compact support and any $t \in \mathbb{R}$ we will always have that $u(x,t) = 0$ on any point on the complementary of the support of $u(t)$, which is non-empty by the finite speed of propagation.

(iii) The degeneracy of u on the boundary of its positivity set

$$\mathcal{P}_u := \{(x,t) \in \mathbb{R}^d \times \mathbb{R} : u(x,t) > 0\}.$$

On this boundary $\Gamma = \partial \mathcal{P}_u$ the solution u is no longer smooth and may not verify (1). In fact in [3] it was shown that, even if $u_0 \in C^\infty(\mathbb{R})$ the derivative $\partial_x u$ is discontinuous. For this reason, if $\Gamma \neq \emptyset$, the PME (1) is understood in the distributional sense.

Despite all these complications the intensive study of the PME during these last two centuries has led to the main theory such as well posedness and the asymptotic behaviour of solutions to the PME to be well understood. We will explain some of these results in the following sections.

1.1. Main results

In our work we will carry out an asymptotic analysis of the solutions to the 1 dimensional PME (13) by using *Sturmian intersection theory* or *Sturm theory* for short. This theory is essentially geometric in nature and establishes that the number of intersections (also called sign changes) $J(t)$ of two solutions u, U of the PME (this concept will be defined precisely later on) is non-increasing. That is

$$J(t) \leq J(0) \quad \forall t \in \mathbb{R}^+ \tag{2}$$

The inception of this theory was the article [4] published in the year 1836 by the French mathematician Charles Sturm. Sturmian theory was later used in the 20th century to prove a multitude of results such as in asymptotic stability for various nonlinear parabolic equations [5], unique continuation theory [6] or even Poincaré Bendixson theory for parabolic equations [7]. In fact many results have been proven for the PME by using this theory. For a more complete historical review see the introduction of the book by Galaktionov [8] which is itself exclusively dedicated to Sturm theory.

Interestingly enough the only real principle of Sturm theory is that of (2). This simplicity combined with it's clear geometric nature and the breadth of results it allows one to prove make it a valuable tool in the study of 1 dimensional PDE. By counting the intersections of an arbitrary solution u to the PME with a sufficiently "complete" family of exactly known solutions (in our case the *Barenblatt* solutions) that initially intersect only once with the initial data u_0, we will obtain a clear picture of the asymptotic nature of u. This in large part will be due to the usage of these techniques to obtain a new a priori estimate on u. In our work we obtain the following result.

Theorem 1.1. *Let u be the solution to the* 1*D Cauchy problem in* (1)*, let U_{C_M} be the Barenblatt solution with mass $M = \int_{\mathbb{R}} u_0$ and set their respective pressures to be*

$$v = \frac{m}{m-1} u^{m-1}; \quad V_M = \frac{m}{m-1} U_{C_M}^{m-1}$$

Then the following hold

$$\|v(t) - V_M(t)\|_{L^\infty(\mathbb{R})} \lesssim t^{-\frac{m}{m+1}} \tag{3}$$

$$\|\partial_x v(t) - \partial_x V_M(t)\|_{L^\infty(\mathcal{P}_u)} \lesssim t^{-1} \tag{4}$$

Where the order of convergence given in (3)-(4) *is optimal. In particular we have the uniform convergence*

$$\lim_{\tau\to+\infty} \|u(t) - \partial_x U_{C_M}(t)\|_{L^\infty(\mathbb{R})} = 0 \tag{5}$$

$$\lim_{\tau\to+\infty} \|\partial_x u(t) - U_{C_M}(t)\|_{L^\infty(\mathcal{P}_u)} = 0 \tag{6}$$

Furthermore all the critical points of $u(t)$ *are located within an interval*

$$I(u_0) = [-r(u_0), r(u_0)] \tag{7}$$

where $r(u_0) \in \mathbb{R}$ *is a real number depending only on the initial data* u_0*.*

1.2. Some results already in the literature

The convergence of u to U_{C_M} was first proven in the 1 dimensional case by Kamin (Kamenomostskaya) in the year 1973. In [9] the author showed that

$$\lim_{\tau\to+\infty} t^{\frac{1}{m+1}} \|\partial_x u(t) - U_{C_M}(t)\|_{L^\infty(\mathcal{P}_u)} = 0.$$

This result, which is weaker than the one we give in (3), was later generalized in the year 1980 to the general $d-$dimensional by the same author in collaboration with Friedman in [10]. Further generalization was obtained in the year 1990 when Kamin in collaboration with Vazquez proved in [11] convergence in the d dimensional case where u has changing sign. In 1983 it was shown by Vázquez in [12] that these rates of convergence can be improved in the 1 dimensional case by considering convergence to

$$U_{C_M,-x_0}(x,t) := U_{C_M}(x - x_0, t)$$

where $x_0 = \int_{\mathbb{R}} x u_0(x)dx$ is the center of mass of u_0. If we denote the pressure of U_{C_M} by

$$V_{M,-x_0} := \frac{m}{m-1} U_{C_M,-x_0},$$

then it is proved in [12] that

$$\lim_{t\to\infty} t^{\frac{m}{m+1}} \|v(t) - V_{C_M,-x_0}(t)\|_{L^\infty(\mathbb{R})} = 0.$$

$$\lim_{t\to\infty} t\|v(t) - V_{C_M,-x_0}(t)\|_{L^\infty(\mathbb{R})} = 0.$$

This shows that, though the result we give in (3) and (4) is optimal, one may obtain an "infinitesimal order of convergence more" if one considers convergence to $V_{M,-x_0}$ instead of to V_M.

Concerning the result in equation (79), it was shown in [13] that in the 1 dimensional case and for continuous compactly supported initial data it holds that the set of (spatial) maxima of $u(t)$ become 1 point for $t \geq T$ and converges to the Euclidean center of mass of u_0. Additionally, it was shown that the pressure v becomes an eventually concave function with

$$\lim_{t\to\infty} \partial_x^2 v = -\frac{1}{t(m+1)}.$$

As for the d dimensional case, it was proved in [14] that for non-negative initial data which is bounded and of compact support the set of spatial critical points is contained for all t in the convex hull of u_0 and also becomes one point for large time.

For completeness we note that, though we shall not do so here, the PME may also be studied for $m < 1$. In this case (1) is called the fast diffusion equation. This equation was studied for example in [15]. Though some properties are similar to those of the PME many are different. For example, in the fast diffusion equation the free boundary disappears and the solutions are smooth. Furthermore for $0 < m < (d-2)/d$ the asymptotic behaviour is given by a family of self-similar solutions with non-constant mass.

We structure the rest of our work as follows. In Section 2 we state various preliminary results we will need throughout our work. In Section 3 we give a physical derivation of the PME. In Section 4 we introduce Sturm theory and its concepts which will be the tool used in our asymptotic analysis. In Section 5 we introduce various explicit solutions to the PME and most importantly among them the Barenblatt solutions. In Section 6 we define a useful rescaling of solutions to the PME and define the family of solutions $\{W_{C,x_0}\}$ that we will use in our Sturmian arguments. In Section 7 we establish a preliminary asymptotic analysis based on the maximum principle and which is then improved on in Section 8 where we prove our main results through Sturm theory. Finally, in Section 9 we show a possible numerical implementation that serves to visualize many of the previously shown concepts and theorems.

1.3. Notation

We will write $\mathbb{R}^+ := [0, +\infty)$ for the non-negative real numbers and
$\mathbb{N} + 1 = \{1, 2, 3, \dots\}$ for the positive integers .

Throughout our work we will often be working with functions of the form:

$$u : \mathbb{R} \times I \to \mathbb{R}; \quad \theta : \mathbb{R} \times I \to \mathbb{R}$$

with I a (possibly unbounded) interval in $\mathbb{R}^+$. Due to the physical interpretation of the equations we will denote the first variable by x (or ξ) and call the space variable. The second variable we will denote by t (or τ) and call the time variable. Furthermore, we shall frequently write $u(t)$ (or $\theta(\tau)$) to stand for the slightly more cumbersome $u(\cdot, t) : \mathbb{R} \to \mathbb{R}$ (respectively $\theta(\cdot, \tau)$).

We will denote the support of u at time t by

$$\operatorname{supp} u(t) := \overline{\{x \in \mathbb{R} : u(x,t) \neq 0\}}$$

Furthermore, if $d = 1$ we will denote respectively the left and right interface at time t by

$$h_-(t) = \inf\{x \in \mathbb{R} : u(x,t) > 0\}; \quad h_+(t) = \sup\{x \in \mathbb{R} : u(x,t) > 0\},$$

where $h_-(t), h_+(t) \in [-\infty, +\infty]$.

To simplify notation we will often write expressions such as the previous one as

$$h_{\pm}(t) \in [-\infty, +\infty].$$

Given a topological space X we write $C_c(X)$ for the space of continuously compactly supported functions $u : X \to \mathbb{R}$.

Finally, given two functions u, f, we will employ the notation $u \lesssim f$ to mean that there exists some constant C such that $u \leq Cf$. If the value of C depends on another parameter that is not fixed, say $N \in \mathbb{R}$, we shall make this explicit by writing $u \lesssim_N f$.

2. Preliminary theory: well posedness and maximum principle

We now state some results among which are existence and uniqueness of solutions. First of all, for completeness we observe that for initial data $u_0 \in L^1(\mathbb{R}^d)$ there exists a unique "strong solution" to (1). Where a strong solution is a distributional solution in the sense

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}^d} (u\partial_t \varphi - \nabla(u^m) \cdot \nabla\varphi) dxdt = 0 \quad \forall \varphi \in C_C^\infty(\mathbb{R}^d \times \mathbb{R}^+) \tag{8}$$

$$\lim_{t\to 0} \|u(t) - u_0\|_{L^1(\mathbb{R}^d)} = 0$$

A precise definition of strong solution along with a statement of this result may be found in [16] pages 195, 204 respectively. There it is also shown that u is continuous on $Q = \mathbb{R}^d \times (0, +\infty)$. Since the speed of propagation of the free boundary is finite we deduce that, if u_0 is compactly supported, then $u(t)$ will be continuous and compactly supported for any $t > 0$. Thus we conclude that, for the asymptotic purposes of this text, it is equivalent to consider $u_0 \in L^1(\mathbb{R}^d)$ and of compact support to $u_0 \in C_c(\mathbb{R}^d)$. This justifies that, for simplicity in our argument, we consider in the remainder of our work this second case $u_0 \in C_c(\mathbb{R}^d)$. If we also impose that the initial data be non-negative (as it will be in any physical application) the following result holds.

Proposition 2.1 (Well posedness). *Given $u_0 \in C_c(\mathbb{R}^d)$ there exists a unique strong solution $u \in C(\mathbb{R}^d \times \mathbb{R}^+)$ to* (13) *in the sense of* (8)*. Furthermore, u is non-negative, u is smooth non the domain of positivity*

$$\mathcal{P}_u = \{(x,t) \in \mathbb{R}^d \times (0, +\infty) : u(x,t) > 0\}.$$

and verifies that $u(0) = u_0$.

The method of proof of these results is quite instructive. Essentially one considers a slightly modified version of (1) with initial data $u_{0,\epsilon}$ so that the new problem is uniformly parabolic. The well-posedness for such equations is well established and gives a smooth classical solution u_ϵ to the modified problem. If we have constructed our sequence well u_ϵ will converge to a solution u of the variational formulation (8). Smoothness of u in $\mathcal{P}_u$ follows by bounding u uniformly away from zero in a small neighbourhood of each point in $\mathcal{P}_u$ (using the method of barriers based on the maximum principle) and once again using theory of uniformly parabolic PDE. This proof also gives the following maximum principle

Proposition 2.2 (Maximum principle). *Let $u_0 \le v_0$ be two continuous compactly supported functions and let u, v be the solution to the Cauchy problem* (1) *with initial data u_0, v_0 respectively. Then*

$$u(x,t) \le v(x,t) \quad \forall (x,t) \in \mathbb{R}^d \times \mathbb{R}^+ \tag{9}$$

However note that, as was previously mentioned, the strong maximum principle no longer holds. Another result that we will often use is the conservation of mass of solutions to the PME.

Proposition 2.3 (Conservation of mass). *Let $u_0 \in C_c(\mathbb{R}^d)$ an let u be the unique strong solution to* (1)*. Then the function*

$$M(t) := \int_{\mathbb{R}^d} u(t)dx$$

is constant.

That $M(t)$ is in fact well defined may be seen as a consequence of the fact that, as previously explained, $u(t) \in C_c(\mathbb{R}^d)$. The conservation of mass may be justified essentially by the fundamental theorem of calculus and an integration by parts. See [16] page 20 for the details.

Another very useful fact is the invariance of solutions to the PME under a particular group of transformation, namely translations and some rescalings. The following lemma specifies this and will allow us to construct the family of solutions we will use for the development of our work.

Lemma 2.1 (Group of transformations)**.** *If u is a solution to the PME* (1) *with initial data u_0 then*

$$u(x + x_0, t + t_0); \quad Ku(Lx, Tt)$$

are also solutions to the PME with initial data respectively

$$u(x + x_0, t_0); \quad Ku_0(Lx).$$

Where $(x_0, t_0) \in \mathbb{R} \times \mathbb{R}^+$ is any and $(K, L, T) \in \mathbb{R}^2 \times \mathbb{R}^+$ are any verifying

$$T = K^{m-1}L^2.$$

Proof. In both cases this may be verified by a direct calculation, though for the first family of solutions $u(x + x_0, t + t_0)$ this is unnecessary as we see that the PME is homogeneous in time and space. □

In addition to the properties of u stated in this and the preceding section much more is known. In [17] it was shown that under these conditions we also have that u is not only continuous but Hölder continuous. If $d = 1$ and the support of u_0 is an interval it was shown in [18] that the free boundary consists of two curves $x = h_{n,i}(t)$ where $h_1 \leq h_2$, and there is a "waiting time" $t_i \geq 0$ such that

$$h_{n,i}(t_i) = h_{n,i}(0)$$

h_1(resp. h_2) is monotone decreasing (increasing) on$[t_1, +\infty)$ (resp.$[t_2, +\infty)$).

3. Physical derivation of the PME

So as to motivate the equations in (1) we begin by carrying out a physical deduction of them in the context of gas flow through a porous medium. To do so we will employ three equations which we now introduce.

1. Firstly, in continuum mechanics, a equation that often appears is that of conservation of mass; which in the case of a fluid of density $\rho(x,t)$ and velocity $v(x,t)$ in a none porous medium would take the following form.

   $$\partial_t \rho = \nabla \cdot (\rho v).$$

   However, in the case where the fluid is flowing through a porous medium it is necessary to introduce an additional term named the porosity. We denote this term by ϵ, consider it to be constant across time and space and take values in $(0,1)$.With this new term we have that the continuity equation now takes the slightly modified form

   $$\epsilon \partial_t \rho = -\nabla \cdot (\rho v) \tag{10}$$

   Note that, in particular, as ϵ approaches 0 we would have that the fluid in question would become incompressible.

2. Secondly we have *Darcy's law*

   $$\mu v = -k \nabla p \tag{11}$$

   Which is an empirical law expressing the relationship between the velocity v of the fluid, its viscosity $\mu \in \mathbb{R}^+$, its permeability $k \in \mathbb{R}^+$ and its pressure $p(x,t)$. Though Darcy's law is empirical, and thus not mathematically derivable from other basic rules of physics, its formulation is not difficult to interpret. Roughly speaking we have that:

   - The term $-\nabla p$ indicates that the fluid will flow in the direction opposite an increase in pressure. That is, the pressure "pushes the fluid away from it."

- The fact that v is inversely proportional to μ indicates that the "stickier" the fluid is, that is, the higher it's viscosity, the slower the fluid will flow through the medium.

- v being directly proportional to k indicates that, the more the fluid allows movement through it, that is, the higher it's permeability, the faster the fluid will flow.

- If the situation requires it the action of other forces can be incorporated on the right hand side. For example for a force of gravity pushing downward in the direction of the z-axis Darcy's equation would be

$$\mu v = -k\nabla(p + \rho g z)$$

3. Finally, we have the *state equation* for ideal gases

$$p = p_0\rho^{\gamma}. \tag{12}$$

Where $p_0, \gamma \in \mathbb{R}^+$ are constants and $\gamma > 1$ is called the polytropic-exponent and is determined empirically.

From these three equations one obtains from a straightforward algebraic manipulation that

$$\partial_t \rho = c\Delta(\rho^m),$$

where we define $m = \gamma + 1$ and where the constant

$$c := \frac{kp_0\gamma}{\mu\epsilon(\gamma+1)} > 0$$

can be factored out by a rescaling such as $t \leftrightarrow ct$ to obtain the porous medium equation

$$\partial_t \rho = \Delta(\rho^m)$$

In practice we will use the symbol u instead of ρ, as is customary in the mathematical literature. Thus the previous equation will be written as

$$\partial_t u = \Delta(u^m) \tag{13}$$

By the state equation and since $m = \gamma + 1$ we may write the pressure (which we now denote as v as is also common in the literature) as a constant multiple of u^{m-1}. To simplify some expression obtained later on we take this constant to be $m/(m-1)$ and define the (normalized) pressure to be

$$v := \frac{m}{m-1} u^{m-1}. \tag{14}$$

The equation satisfied by the pressure can be derived from (13) and is given by

$$\partial_t v = (m-1) v \Delta v + |\nabla v|^2 \tag{15}$$

4. Sturm theory

As was previously explained Sturm theory is an important tool when analyzing solutions to 1 dimensional PDE that verify the maximum principle. Despite the great variety of equations that it applies to its theory is relatively simple. In this section we define the various concepts of this theory and the single result that defines it.

Definition 4.1. *Let $w \in C(\mathbb{R})$ be a continuous function and let $A(w) \subset \mathbb{N} + 1$ be the set of positive integers $k \geq 1$ such that there exist $x_1 < ... < x_{k+1} \in \mathbb{R}$ verifying*

$$w(x_i) \cdot w(x_{i+1}) < 0 \quad i = 1, ..., k$$

Then the number of sign changes *of w is*

$$Z(w) := \begin{cases} \sup(A(w) \cup \{0\}) & \text{if } A(w) \text{ is bounded} \\ +\infty & \text{if } A(w) \text{ is unbounded} \end{cases}$$

Definition 4.2. *Let $u, v : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ be two continuous functions. Then the* number of intersections of u, v at time t *are*

$$J_{u,v}(t) := Z(u(t) - v(t))$$

We will also say that $u(t), v(t)$ intersect $J_{u,v}(t)$ times.

In the case where there can be no confusion we will often simplify our notation by writing $J(t)$. With these concepts defined we may now state the main theorem of Sturm theory.

Theorem 4.1 (Sturm theorem)**.** *Let u, v be the unique strong solutions to the PME* (1) *with initial $u_0, v_0 \in C_c(\mathbb{R})$ respectively. Then $J(t)$ is a non-increasing function. That is*

$$J(t) \leq J(0) \quad \forall t \in \mathbb{R}^+.$$

4.1. Comments on the Sturm theorem

The preceding theorem was initially formulated for uniformly parabolic equations on a bounded interval $D \subset \mathbb{R}$ and is stongly based on the maximum principle. In this case one has that the number of intersections of u, v at time t is smaller than the number of intersections of u, v on the parabolic boundary

$$\Gamma_t = (D \times \{0\}) \cup (\partial D \times [0, t)).$$

See for example [19] page 84 (note that we have in fact not defined what the number of intersections on the parabolic boundary are, as we will not use this concept in the remainder of our work. However, as with the previous definitions, this concept follows the intuitive idea of sign changes of $u - v$. In this case on Γ_t).

To generalize this result to degenerate parabolic equations on a bounded domain one applies the same trick as the one mentioned under the existence proposition (2.1). That is once considers approximating sequences u_ϵ, v_ϵ that are classical solutions to respective uniformly parabolic PDE that converge uniformly to u, v respectively.By taking limits one concludes that, even for degenerate parabolic equations, the number of intersections of u, v at time t are fewer than those on the parabolic boundary Γ_t.

Finally, if $u(t), v(t)$ are of compact support for all $t \in \mathbb{R}^+$ (as occurs by the finite propagation property for solutions to the PME with initial data of compact support), then, by taking D large enough, the number of intersections on Γ_t is simply the number of intersections of u_0, v_0. In this way one recover Theorem 4.1 (which we note is for the Cauchy problem and thus on an unbounded domain). For more details on the proof see page 11 of Galaktionov's book [8].

5. Special solutions to the PME

In this section we introduce various solutions to the PME whose expression is explicitly known. These solutions show various behaviours that are commonplace in the study of the PME. Importantly, in this section we introduce the family of Barenblatt solutions which will be of critical use in the remainder of our text.

5.1. Self-similar solutions: The Barenblatt solutions

We begin by deriving the explicit formula for a kind of non-negative, self similar, radially symmetric and compactly supported (in space) solution to the PME known as the Barenblatt solutions. We conduct our argument in the general d-dimensional case, as this supposes no extra difficulty, and will later on restrict ourselves to $d = 1$.

To derive the expression of these solutions we begin by postulating the existence of a self-similar solution of the form

$$u(x,t) = (1+t)^{-\alpha} f((1+t)^{-\beta} x) \tag{16}$$

By differentiating with respect to time and setting $\xi := (1+t)^{-\beta} x$ we obtain that

$$\partial_t u(x) = -(1+t)^{-\alpha-1}(\alpha f(\xi) + \beta \xi \cdot \nabla f(\xi)) \tag{17}$$

Another calculation shows that

$$\Delta u^m = (1+t)^{-\alpha m - 2\beta} \Delta f^m(\xi) \tag{18}$$

By equating (17) and (18) and using a separation of variables argument, we obtain that necessarily $f = 0$ or

$$-\alpha - 1 = -\alpha m - 2\beta \tag{19}$$

Furthermore, by the conservation of mass of Proposition 2.3 and by applying a change of variables we deduce that

$$\int_{\mathbb{R}^d} u(x,t)dx = (1+t)^{-\alpha+\beta d} \int f(x)dx = c \in \mathbb{R}^+$$

If we impose that $u \neq 0$ then, since f is non-negative, we deduce from the above and the expression of u in (16) that

$$\alpha = \beta d \tag{20}$$

By now solving the system given by (19) and (20) we obtain

$$\alpha = \frac{d}{d(m-1)+2}; \quad \beta = \frac{1}{d(m-1)+2}$$

Factoring out the expression in t in the PME we obtain the profile equation

$$\mathbf{A}(f) := \Delta f^m(\xi) + \beta \xi \cdot \nabla f(\xi) + \alpha f(\xi) = 0 \quad \forall \xi \in \mathbb{R}^d. \tag{21}$$

We now impose that f be radially symmetric and thus

$$f(\xi) = F \circ \mathrm{abs}(\xi); \quad \mathrm{abs}(\xi) := |\xi|,$$

for some function $F : \mathbb{R}^+ \to \mathbb{R}^+$. By now using the radial expression for the Laplacian and for the divergence

$$\nabla f(\xi) = F'(r)\frac{\xi}{r}; \quad \Delta f^m(\xi) = \frac{1}{r^{d-1}}((r^{d-1}F^{m\prime}(r))'(r) \quad \xi \neq 0, \quad r := |\xi|.$$

We obtain that the profile equation (21) becomes

$$\frac{1}{r^{d-1}}(r^{d-1}F^{m\prime}(r))' + \beta r F'(r) + \alpha F(r) = 0 \quad \forall r \in \mathbb{R}_+.$$

Which may be rewritten by an algebraic manipulation as

$$(r^{d-1}F^{m\prime}(r) + \beta r^d F(r))' = 0$$

Integrating the above equation gives

$$r^{d-1}F^{m\prime} + \beta r^d F(r) = c \in \mathbb{R}$$

If we also wish for $u(t)$ to have compact support for all t we must have $c = 0$. This gives

$$F^{m\prime}(r) + \beta r F(r) = 0 \tag{22}$$

This is an ODE in separate variables and we may solve it to obtain the implicit expression

$$F^{m-1}(r) = C - \frac{\beta(m-1)}{2m} r^2$$

Where C is an arbitrary constant. Note that F verifying the previous equation does not necessarily exist (or more precisely is not necessarily a real function due to the exponentiation of a negative number that occurs for $r >> 0$). However, we "solve" this issue by taking

$$F_C(r) := (C - kr^2)_+^{\frac{1}{m-1}}; \quad k = \frac{\beta(m-1)}{2m} \tag{23}$$

A direct computation shows that, if we set as in (16)

$$U_C(x,t) := (1+t)^{-\alpha} F_C((1+t)^{-\beta}|x|),$$

where α, β are as previously, then U_C is a solution to the PME on its domain of positivity and is thus a strong solution to (13) in the sense of (8).

Definition 5.1. *We will call a solution belonging to the family* $\{U_C\}_{C\in\mathbb{R}^+}$ *a* Barenblatt solution*. A function in the family* $\{F_C\}_{C\in\mathbb{R}^+}$ *will be called a* fixed profile *(as it is fixed and time and does not depend on* t*).*

We note that many authors take the family of Barenblatt solutions to be

$$\tilde{U}_C(x,t) := U_C(x, t-1)$$

In this case when t approaches 0 we have that $\tilde{U}_C$ takes as initial data

$$\tilde{U}_C(x,t) = t^{-\alpha} F_C(t^{-\beta}|x|) = t^{-\alpha}\left(C - k\frac{|x|^2}{t^{2\beta}}\right)_+^{\frac{1}{m-1}} \xrightarrow{t\to 0} M_C \delta_0(x),$$

where δ_0 is the Dirac distribution and M_C is the mass of U_C defined by

$$M_C := \int_{\mathbb{R}^d} U_C dx.$$

Since we wish to work only with continuous solutions so as to be able to apply Sturm theory this justifies the time translation we used in considering U_C instead of the more common $\tilde{U}_C$.

In summary we have obtained a family of solutions to (1) given by

$$U_C(t,x) = (t+1)^{-\alpha} \left(C - k \frac{|x|^2}{(t+1)^{2\beta}} \right)_+^{\frac{1}{m-1}} \tag{24}$$

and that verify the following properties

- Regularity: U_C are Hölder continuous on $\mathbb{R}^d \times [0, +\infty)$ and smooth on their domain of positivity $\mathcal{P}_{U_C}$. However, $\partial_x U_C$ is discontinuous on $\partial \mathcal{P}_{U_C}$.
- Support: The support of $U_C(t)$ is increasing in t and is compactly supported for each t with

$$\operatorname{supp} U_C(t) = (1+t)^\beta \cdot \bar{B}(0, \sqrt{C/k}) \tag{25}$$

- Dependence on C: The mass of U_C is an increasing continuous function of C, so is the support of $U_C(t)$ for each fixed t.

Compare these properties to points i-iii of Section 1. Below we plot the functions F_C^{m-1} for different values of C (we don't plot F_C directly as it's derivative becomes infinite on the boundary of its support). As we can see two profiles $F_C, F_{C'}$ do not intersect in the intersection of their regions of positivity $I_{C,C'}$. This is due to the fact that both profiles are solutions to the first order ODE (22) in $I_{C,C'}$ and thus, by uniqueness of solutions to ODE, may not intersect. We also observe that $F_C \leq F_{C'}$ if $C \leq C'$. This may be directly deduced by observing that

$$F_C(0) = C^{\frac{1}{m-1}} < C'^{\frac{1}{m-1}} = F_{C'}(0)$$

and the non-intersection property previously discussed.

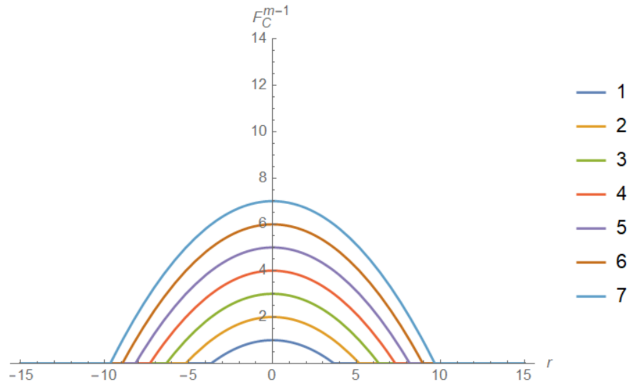

Figure 1: Plot of the profile of the pressure, F_C^{m-1} for $(d, m) = (1, 4)$

5.2. Other solutions

We now introduce a variety of other solutions to the PME so as to show other possible behaviours of solutions to these equations.

- Solutions with finite blow-up time: We may engineer such a solution by considering in a similar fashion to the heat equation a solution of the form

  $$u = T(t)F(x)$$

  A possible solution of this form is

  $$u(x,t) = \left(\frac{T_0|x|^2}{T_0 - t}\right)^{\frac{1}{m-1}}.$$

  Where the blowup time is

  $$T_0 := \frac{m-1}{2m(2 + d(m-1))},$$

  and u is defined on $\mathbb{R}^d \times [0, t_0)$. Note that the initial data for this solution is given by

  $$u_0(x) = |x|^{\frac{2}{m-1}}$$

  and thus grows to infinity when $|x| \to \infty$. The growth that this initial data presents is in fact the maximum growth that allows a solution to (1). This fact was proved in [20] page 54.

- Solutions with waiting time: As has previously been explained, solutions to the PME with initial data of compact support have a free boundary that propagates at finite speed and is non-decreasing. Consider the one dimensional case, then even if u_0 is not of compact support, we have that

$$-\infty < h_-(0) \implies -\infty < h_-(t); \quad h_+(0) < +\infty \implies h_+(t) < +\infty$$

and h_-, h_+ are respectively non-increasing and non-decreasing functions of t. Note that if we modify the previous solution to

$$\tilde{u}(x,t) := \begin{cases} u(x,t) & x \leq 0 \\ 0 & x > 0, \end{cases}$$

then we have that the right hand boundary $h_+(t)$ is in fact constant for all $t \in [0, T_0)$. This leads to the concept of *waiting time*, the time that it takes till the interface begins to move. In the above case we would have a finite waiting time of T_0. More generally the waiting time is always finite (possibly being zero) and may be precisely bounded. See [16] page 373 though this will arise as a natural byproduct of our own study.

- Travelling waves: In the 1 dimensional case we also have another special kind of solution of the from

$$u(x,t) = f(x - ct)$$

for some function $f : \mathbb{R} \to \mathbb{R}$. The above equation expresses that u has the form of a travelling wave of velocity c. To derive the expression for f we substitute the above equation in the PME and solve the resulting ODE which leads us to

$$u(x,t) = \left(\frac{m-1}{m} c(x_0 - x + ct)\right)_+^{\frac{1}{m-1}}.$$

- Focusing problem: This is the name given to the PME when the initial data is supported outside of a open ball B, which we may suppose for example to be centered at the origin. In [21] the following was shown:

1. For $d \geq 2$ and radially symmetric initial data the focusing problem has a unique radially symmetric self-similar solution u.
2. At some finite time T called the *focusing time* the fluid u fills the initially empty region B. At the focusing time T the solution presents a singularity at the origin where it has infinite velocity.
3. The solution has pressure given by

   $$v(x,t) = \frac{|x|^2}{t}\varphi(t^{-\alpha}|x|)$$

   Though φ is known to be the solution to an ODE there is no explicit formula for φ.

The previous solutions show the complexity and wide range behaviour of solutions to the PME. Despite all these factors our geometric tool will obtain surprisingly strong results.

6. Rescaled equation, fixed profile and a convergent family of solutions

Since every solution to the PME goes asymptotically to zero [11] instead of dealing explicitly with u it will prove simpler to work with an appropriately rescaled function θ which verifies its own parabolic PDE. This rescaling serves to fix the Barenblatt solutions and any results shown for θ may themselves be translated back into terms of u without difficulty be considering the inverse rescaling. Furthermore, rescaling the set of spatially translated Barenblatt solutions will give us a sufficiently complete family of solutions to this new PDE. This is the family of solutions we will use to be able to apply Sturm theory.

6.1. Rescaled solutions to the PME

Given any solution u to the PME we set the rescaled function

$$\theta(\xi,\tau;u) = (1+t)^{\alpha}u(x,t),$$

where $\xi := x(1+t)^{-\beta}$ and $\tau := log(1+t)$. Or equivalently

$$\theta(\xi, \tau; u) = e^{\alpha\tau} u(e^{\beta\tau}\xi, e^{\tau} - 1) \tag{26}$$

In the case where there is no doubt as to which solution u is being rescaled we write simply $\theta(\xi, \tau)$. As previously stated this rescaling fixes the Barenblatt solutions, in the sense that if we take u to be a Barenblatt solution U_C, we get that the rescaled function is constant in time with

$$\theta(\xi, \tau; U_C) = F_C(\xi).$$

In fact we would get the same result without rescaling the time variable to τ. The rescaling in time serves only to simplify the expression of the PDE verified by θ and which we introduce in the next subsection. Now, due to Lemma 2.1, we have that

$$U_{C,x_0} := U_C(x + x_0, t)$$

is also a solution to the PME for any $x_0 \in \mathbb{R}^d$. A simple calculation shows that the associated rescaled function to U_{C,x_0} is

$$W_{C,x_0}(\xi, \tau) := \theta(\xi, \tau; U_{C,x_0}) = F_C(\xi + e^{-\beta\tau} x_0). \tag{27}$$

For this reason we will call such functions *translated profiles.* Since our study will take place in dimension $d = 1$ the family

$$\{W_{C,x_0} : (C, x_0) \in \mathbb{R}^+ \times \mathbb{R}\}$$

is the one we will use in the remainder of our text. When $C = 0$ we have $W_{0,x_0} = F_0 = 0$ in other cases the following properties hold.

Property 6.1. *Given $(C, x_0) \in (0, +\infty) \times \mathbb{R}$ the following hold*

(p1) $W_{C,0} = F_C$

(p2) W_{C,x_0} *is continuous and smooth on its domain of positivity.*

(p3) *The support of W_{C,x_0} at time τ converges to that of F_C and is given by*

$$\operatorname{supp} W_{C,x_0}(\tau) = \left[-\sqrt{\frac{C}{k}} - e^{-\beta\tau} x_0, \sqrt{\frac{C}{k}} - e^{-\beta\tau} x_0 \right]$$

(p4) W_{C,x_0} *is increasing in* C *and strictly increasing on its domain of positivity. That is, given* $C < C'$

$$W_{C,x_0}(x,\tau) < W_{C',x_0}(x,\tau) \quad \forall(\xi,\tau) \in \mathcal{P}_{W_{C,x_0}}$$

(p5) W_{C,x_0} *converges uniformly to* F_C*, that is*

$$\lim_{\tau\to+\infty} \|W_{C,x_0}(\tau) - F_C\|_{L^\infty(\mathbb{R})} = 0$$

(p6) W_{C,x_0} *is uniformly continuous in* C, x_0*, and has mass equal to that of* F_C*. That is,*

$$\int_{\mathbb{R}} W_{C,x_0} dx = \int_{\mathbb{R}} F_C dx.$$

(p7) *Given* $M \in \mathbb{R}^+$ *there exists* $C_M \in \mathbb{R}^+$ *such that the mass of* F_{C_M} *is* M*.*

Proof. The proof of all these properties follow immediately from the expression for W_{C,x_0} in (27) along with the fact that, by definition of F_C (see equation (23)): F_C is continuous, smooth on its domain of positivity and has support

$$\operatorname{supp} F_C = \left[-\sqrt{\frac{C}{k}}, \sqrt{\frac{C}{k}}\right] \tag{28}$$

□

Remark 6.1. *The previous properties also hold in dimension* $\mathbb{R}^d$ *where it is only necessary to replace the closed interval in the support of* W_{C,x_0} *for the closed ball. The proof of this fact is identical to the* 1 *dimensional case.*

Below we plot some members of this family at different times so as to show visually the aforementioned properties.

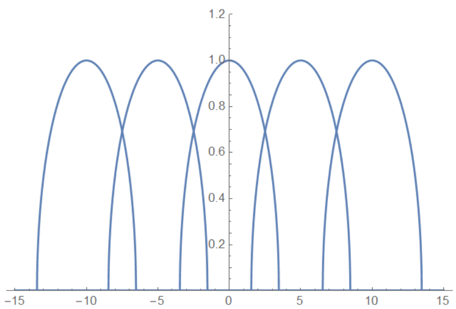

Figure 2: Family W_{C,x_0} at $\tau = 0$

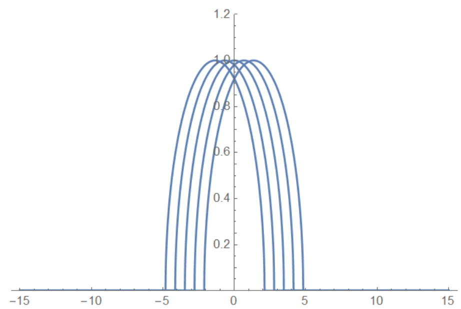

Figure 3: Family W_{C,x_0} at $\tau = 8$

Another very important property is that the family $\{W_{C,x_0}\}$ is "complete" in the sense that it has enough elements to describe the behaviour of any rescaled solution θ by means of counting intersections of θ with its elements. We will make this precise later in Lemma 8.2.

6.2. Rescaled PDE

We now wish to see what differential equation θ verifies when u is any solution to the PME. We first calculate that

$$\partial_\tau \theta(\xi,\tau) = \partial_\tau (e^{\alpha\tau} u(\xi e^{\beta\tau}, e^{\tau}-1)) = \alpha\theta(\xi,\tau) + e^{\alpha\tau}\left(\beta x \cdot \nabla u(x,t) + e^{\tau}\partial_t u(x,t)\right)$$

$$= (1+t)^{\alpha}((1+t)\partial_t u + \beta x \cdot \nabla u + \alpha u) = (1+t)^{\alpha}((1+t)(\Delta u^m) + \beta x \cdot \nabla u + \alpha u)$$

Where x, t are as in the beginning of the previous subsection and it was used that u solves the PME. We now calculate term by term $\mathbf{A}(\theta)$ as defined in the profile equation (21)

$$\mathbf{A}(\theta) = \Delta\theta^m + \beta\xi \cdot \nabla\theta + \alpha\theta = (1) + (2) + (3)$$

$$(1) = (1+t)^{m\alpha}\Delta\{u^m((1+t)^{\beta}\xi, t)\} = (1+t)^{\alpha+1}(\Delta u^m)(x,t)$$

$$(2) = \beta(1+t)^{\alpha}(1+t)^{\beta}\xi \cdot (\nabla u)(x,t) = (1+t)^{\alpha}\beta x \cdot (\nabla u)(x,t)$$

$$(3) = \alpha(1+t)^{\alpha} u(x,t).$$

Where we used that $m\alpha + 2\beta = \alpha + 1$. Thus we conclude that the rescaled equation verifies the PDE

$$\partial_\tau \theta = \mathbf{A}(\theta) \tag{29}$$

Note that, by construction, $\theta(\xi, 0; u) = u(\xi, 0)$ and thus

$$u \text{ solves (1) with initial data } u_0 \iff \theta \text{ solves (29) with initial data } u_0 \quad (30)$$

6.3. PDE of θ^{m-1}

It will also be useful to have an expression for the PDE verified by $\tilde{\theta} := \theta^{m-1}$. The expression may be found by using (29) and substituting θ for $\tilde{\theta}^{\frac{1}{m-1}}$ to obtain

$$\partial_\tau \tilde{\theta} = m\tilde{\theta}\Delta\tilde{\theta} + \frac{m}{m-1}\left|\nabla\tilde{\theta}\right|^2 + \beta\xi \cdot \nabla\tilde{\theta} + \alpha(m-1)\tilde{\theta} \quad (31)$$

Note that in particular the function preceding from the fixed profile of a Barenblatt solution

$$\tilde{\theta}(\xi, \tau) := (C - k|\xi|^2)_+,$$

is a solution of the above PDE.

7. Asymptotic analysis using the maximum principle

From here on out we set $d = 1$ and conduct our asymptotic analysis of u or equivalently θ. We begin by working directly with u and the maximum principle (2.2). However, to obtain more precise results it will soon become necessary to switch our study to that of the rescaling θ.

Proposition 7.1 (Rate of decay and increase in suport)**.** *Let u be the solution to the PME* (1) *with non-negative initial data $u_0 \in C_c(\mathbb{R})$. Then, if $u_0 \neq 0$,*

$$c_1 \leq \|(1+t)^\alpha u(t)\|_{L^\infty(\mathbb{R})} \leq c_2 \quad \forall t \in \mathbb{R}^+$$
$$-a_1(1+t)^\beta \leq h_-(t) < h_+(t) \leq a_2(1+t)^\beta$$

for some strictly positive constants c_1, c_2, a_1, a_2.

Proof. Since $u_0 \neq 0$ is continuous and nonnegative it is possible to consider two translated Barenblatt solutions U_{C_1,x_1}, U_{C_2,x_2} such that

$$U_{C_1,x_1}(x, 0) \leq u_0(x) \leq U_{C_2,x_2}(x, 0) \quad \forall x \in \mathbb{R},$$

and $C_1, C_2 > 0$. By now applying the maximum principle in Proposition 2.2 we deduce that also

$$U_{C_1,x_1}(x,t) \leq u(x,t) \leq U_{C_2,x_2}(x,t) \quad \forall (x,t) \in \mathbb{R} \times \mathbb{R}_+ \tag{32}$$

Furthermore, by the expression of a Barenblatt solution in (24)we have that

$$\|(1+t)^{\alpha} U_{C_i,x_i}(t)\|_{L^\infty(\mathbb{R})} = \|F_{C_i}\|_{L^\infty(\mathbb{R})} = c_i,$$

where c_i is the maximum of the respective profiles, $c_i = F_{C_i}(0)$. This in combination with (32) gives the first part of the proposition.

To derive the second part of the theorem we likewise use the maximum principle which tells us that

$$\operatorname{supp} U_{C_1,x_1}(t) \subset \operatorname{supp} u(t) \subset \operatorname{supp} U_{C_2,x_2}(t)$$

By the previously studied properties of the Barenblatt solution (its support was explicitly given in (25)) this concludes our proof. $\square$

This is as far as we go by using the base variables x, t. From now on we use the rescaling $\theta(\xi, \tau)$ introduced in subsection 5. As a immediate corollary of the previous result we have the following

Corollary 7.1. *Let θ be the solution to* (29) *with non-negativr initial data $\theta_0 \in C_c(\mathbb{R})$ and let $h_\pm(\tau)$ be the right hand and left hand interface of θ. Then if $\theta_0 \neq 0$*

$$c_1 \leq \|\theta(\xi,\tau)\|_{L^\infty(\mathbb{R})} \leq c_2 \quad \forall \tau \in \mathbb{R}^+$$
$$a_1 \leq h_-(\tau) < h_+(\tau) \leq a_2$$

for some strictly positive constants c_1, c_2, a_1, a_2.

Proof. This follows by construction of θ and by Lemma 7.1 in conjunction with the observation in (30). $\square$

8. Asymptotic analysis using Sturm theory

We now begin the main section of our work. Here we will derive precise properties describing the asymptotic behaviour of θ by using the Sturm theory of section 4.1. Note that, though we stated Sturm's theorem based on the number of intersections of two solutions u, v to the PME (1), this theorem can also be stated analogously for two solutions θ_1, θ_2 to the rescaled PDE (29). This can be seen in two ways

1. θ is obtained from u by a change of variables and thus it is equivalent to speak of the number of intersections of u, v as to speak of the number of intersections of θ_1, θ_2. Precisely speaking

$$J_{u,v}(t) = k \iff J_{\theta_1,\theta_2}(log(1+t)) = k$$

2. Sturm's theorem can be proven in an identical fashion for equation (29), as solutions to this equation verify the maximum principle.

In conclusion we have that

Proposition 8.1 ((Sturm's theorem))**.** *Let θ_1, θ_2 be two solutions to* (29) *with continous and compactly supported initial data. Then*

$$J_{\theta_1,\theta_2}(\tau) \leq J_{\theta_1,\theta_2}(0)$$

We will use this proposition in almost all of the remaining proofs by taking θ_1 to be an arbitrary solution of (29) and θ_2 to be a member of the family $\{W_{C,x_0}\}$.

We note that the Sturm Theorem on non-increase of the number of intersections between solutions, can be treated as a dissipativity-like property of parabolic evolution. Moreover, under special geometric configurations and properties of the initial data, the number of intersections $J(\tau)$ between two given solutions can be fixed, since $J(\tau)$ *does not increase* by Sturm's Theorem, and, on the other hand, *does not decrease* if it is the minimal admissible. In this framework, the fixed number of intersections can be used as a *oriented* intersection constraint, characterizing, both, the number and the character of the available intersections. We next prove that this geometrical property holds for solutions to (29) with the same mass and such that $J(0) = 1$. We first introduce the following notation: we say that $f_1 \prec f_2$ if the difference $f_1(\xi) - f_2(\xi)$ has a change of sign from $-$ to $+$. Then $f_1 \preceq f_2$ means that either $f_1 \prec f_2$ or $f_1 = f_2$.

Lemma 8.1 (*oriented* intersection constraint lemma)**.** *Let $\theta_1, \theta_2 \in C(\mathbb{R} \times \mathbb{R}^+)$ be two different solutions to* (29) *with the same mass and such that $\theta_1(0) \prec \theta_2(0)$ Then it holds for every $\tau > 0$ that*

$$\theta_1(\tau) \prec \theta_2(\tau)$$

Proof. To simplify the notation in what follows we introduce the difference function

$$w := \theta_1 - \theta_2.$$

We begin by proving that $J(\tau) = 1$. Suppose not, then we would have that, since J is non-increasing, $J(\tau_0) = 0$ at some time $\tau_0 \in \mathbb{R}^+$. From definition of J we then have that either

$$w(\xi, \tau_0) \leq 0 \quad \forall \xi \in \mathbb{R} \quad \text{or} \quad w(\xi, \tau_0) \geq 0 \quad \forall \xi \in \mathbb{R}.$$

That is, either θ_1 is below θ_2 at τ_0 or vice versa, leading, after integration, to a contradiction since θ_1 and θ_2 are solutions with the same mass.

In order to finish the proof, we consider the set of "good times"

$$A := \{\tau \in \mathbb{R}^+ : \theta_1(\tau) \prec \theta_2(\tau)\}.$$

Note that, since $J(\tau) = 1$ for all τ, we deduce from the definition of intersection number that the complement of A is

$$A^c = \{\tau \in \mathbb{R}^+ : \theta_2(\tau) \prec \theta_1(\tau)\}.$$

With this notation, we have by the hypothesis on the initial data that $0 \in A$ and to prove our lemma we must show that $A = \mathbb{R}^+$. By continuity of w we deduce that both A and A^c are closed. Since $\mathbb{R}^+$ is connected and $0 \in A \neq \emptyset$ we deduce that $A = \mathbb{R}^+$ which concludes our proof. $\square$

The preceding lemma will has the following use. Suppose that θ_1, θ_2 are two solutions to (29) with the same mass and such that the support of $\theta_2(0)$ is to the right of that of $\theta_1(0)$. Then if the support of $\theta_2(\tau)$ moves to the right as τ increases it will also push θ_1 along with it. As, by the lemma, at no point can the left hand interface of θ_1 be surpassed by the right hand interface of θ_2. Analogously if it is instead θ_1 which we know to move, this time to the right, then it must pull θ_2 along with it. In this way we can obtain the following theorem which improves on the result obtained for the interface of $\theta(\tau)$ in Corollary 7.1.

Proposition 8.2 (Convergence of suppport)**.** *Let θ be the solution to* (29) *with initial data θ_0 and take $C_M \in \mathbb{R}^+$ so that*

$$\int_{\mathbb{R}} F_{C_M} dx = \int_{\mathbb{R}} \theta_0 dx.$$

Then the left and right interface $h_-(\tau), h_+(\tau)$ of $\theta(\xi, \tau)$ verify that, for some $a, b \in \mathbb{R}$.

$$e^{-\beta\tau} a + R_M \leq h_+(\tau) \leq e^{-\beta\tau} b + R_M \tag{33}$$

$$e^{-\beta\tau} a - R_M \leq h_-(\tau) \leq e^{-\beta\tau} b - R_M \tag{34}$$

where $R_M = \sqrt{C_M/k}$.

Proof. Notice that C_M exists by property (p7). Additionally, by construction (see equation (28)) we have that

$$\operatorname{supp} F_{C_M} = [-R_M, R_M].$$

This said let us set $b := h_+(0) + R_M$ and consider the translated profile

$$W_{C_M,-b}(\xi, \tau) = F_{C_M}(\xi - e^{-\beta\tau} b).$$

As was established in property (p3) we have that

$$\operatorname{supp} W_{C_M,-b}(t) = \left[-R_M + e^{-\beta\tau} b, R_M + e^{-\beta\tau} b\right] \tag{35}$$

$$\operatorname{supp} W_{C_M,-b}(0) = [h_+(0), h_+(0) + 2R_M] \tag{36}$$

Thus, we deduce from (36) that the intersection number of θ and $W_{C_M,-b}$ at time zero is

$$J_{\theta,W_{C_M,-b}}(0) = 1,$$

and as a result (by conservation of mass) the conditions of Lemma 8.1 are verified. In consequence we have by (35) and (36) that necessarily

$$h_+(t) \leq R_M + e^{-\beta\tau} b; \quad h_-(\tau) \leq -R_M + e^{-\beta\tau} b \tag{37}$$

By now arguing with

$$W_{C_M,-a}(\xi,\tau) = F_{C_M}(\xi - e^{-\beta\tau} a)$$

where $a := h_-(0) - R_M$ we obtain in an identical fashion that

$$R_M + e^{-\beta\tau} a \leq h_+(t); \quad -R_M + e^{-\beta\tau} a \leq h_-(\tau)$$

$\square$

Corollary 8.1 (convergence of support)**.** *Let θ be the the solution to* (29) *with nonnegative initial data $\theta_0 \in C_c(\mathbb{R})$. Then if we denote by $h_-(\tau), h_+(\tau)$ the left and right interface of $\theta(\tau)$ respectively, we have that $h_\pm(\tau)$ converges to $\pm R_M$ with*

$$|h_-(\tau) + R_M| \lesssim e^{-\beta\tau}; \quad |h_+(\tau) - R_M| \lesssim e^{-\beta\tau}. \tag{38}$$

Furthermore, the rate of convergence in (38) *is sharp.*

Proof. The convergence in (38) follows directly from the previous lemma 8.2. The sharpness of this estimate is a consequence of the speed of convergence of the interface of

$$W_{C,x_0}(\xi,\tau) = F_C(\xi + e^{-\beta\tau} x_0)$$

to $\pm R_M$. $\square$

Our next theorem shows one of the principal results of this work. Namely that $\theta(\xi,\tau)$ converges uniformly to the fixed profile F_{C_M} with the same mass as θ. We also derive through the method of proof the optimal the rate of convergence.

Theorem 8.1 (Uniform convergence to the fixed profile)**.** *Let θ be the solution to* (29) *with initial data θ_0 and take $C_M \in \mathbb{R}^+$ so that*

$$\int_{\mathbb{R}} F_{C_M} dx = \int_{\mathbb{R}} \theta_0 dx.$$

Then we have the uniform convergence

$$\lim_{\tau \to +\infty} \|\theta(\tau) - F_{C_M}\|_{L^\infty(\mathbb{R})} = 0. \tag{39}$$

Moreover, the order of convergence is given by

$$\left\|\theta^{m-1}(\tau) - F_{C_M}^{m-1}\right\|_{L^\infty(\mathbb{R})} \lesssim e^{-\beta\tau} \tag{40}$$

Proof. Let $h_-(\tau), h_+(\tau)$ be the left and right interface of θ. Our idea is the following, we will consider for each large time $\tau' \in \mathbb{R}^+$ a translated profile with slightly larger mass

$$W_{C_M+\delta(\tau'),x_0}(\xi,\tau) = F_{C_M+\delta(\tau')}(\xi + e^{-\beta\tau}x_0).$$

Where we will choose $|x_0|$ large enough so that

$$\operatorname{supp} W_{C_M+\delta(\tau'),x_0}(0) \cap \operatorname{supp}\theta_0 = \emptyset, \tag{41}$$

and where we will choose $\delta(\tau')$ big enough, but as small as possible, so that at time τ'

$$-\sqrt{\frac{C_M+\delta(\tau')}{k}} - e^{-\beta\tau'}x_0 < h_-(\tau'); \quad h_+(\tau') < \sqrt{\frac{C_M+\delta(\tau')}{k}} - e^{-\beta\tau'}x_0 \tag{42}$$

This will be achieved by the convergence of support of Lemma 8.1 and reflects a support inclusion due to property (p3). By (41) we have that $J(0) = 1$ and thus we derive from (42) that

$$W_{C_M+\delta(\tau'),x_0}(\xi,\tau') \geq \theta(\xi,\tau') \quad \forall \xi \in \mathbb{R}. \tag{43}$$

This is so because otherwise, we would have that by the intermediate value theorem $J(\tau') = 2$. Which is impossible as J is non-increasing by Sturm's Theorem 8.1. This will give us an upper bound on $\theta(\tau')$. We can later obtain a lower bound in the same way by now considering a translated profile of lower mass $W_{C_M-\delta(\tau'),x_0}$ so that

$$h_-(\tau') < -\sqrt{\frac{C_M-\delta(\tau')}{k}} - e^{-\beta\tau'}x_0; \quad \sqrt{\frac{C_M-\delta(\tau')}{k}} - e^{-\beta\tau'}x_0 < h_+(\tau') \tag{44}$$

We now put our plan in motion. To simplify some expressions we set as in previous occasions $R_M = \sqrt{C_M/k}$ and take two constants

$$x_0 < \min\{-h_+(0) - R_M, 0\}; \quad \lambda > \max\{(x_0+a)^2, (x_0+b)^2, |x_0+a|, |x_0+b|\},$$

where a, b are as in Lemma 8.2. We also define the function $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\delta(\tau) := \max\{2e^{-\beta\tau}\lambda\sqrt{kC_M} + e^{-2\beta\tau}k\lambda, -kx_0(2R_M - e^{-\beta\tau}x_0)e^{-\beta\tau}\}$$

The first quantity in the maximum is taken so as to be able to verify (42) and the second is taken so that at time τ'

$$F_{C_M}^{m-1}(\xi) \le W_{C_M+\delta(\tau'),x_0}^{m-1}(\xi,\tau') \quad \forall \xi \in \mathbb{R}. \tag{45}$$

Which will simplify some calculations. We will prove this now but first note that $\delta(\tau) = O(e^{-\beta\tau})$ is a decreasing function of τ and converges to 0 when $\tau \to \infty$. Thus, by construction of x_0 and the expression for the support of $W_{C_M+\delta(\tau'),x_0}$ in (p3), we have that for τ_0 sufficiently large and $\tau' \ge \tau_0$, (41) holds. Additionally, by the convergence of support shown in Corollary 8.1 we may also take τ_0 large enough so that

$$h_-(\tau') + e^{-\beta\tau'}x_0 < 0 < h_+(\tau') + e^{-\beta\tau'}x_0 \quad \forall \tau' \ge \tau_0 \tag{46}$$

We now show that (42) holds for $\tau' \ge \tau_0$. Using the lhs of (46) we deduce that the first inequality of (42) is equivalent to

$$\delta(\tau') > k\left(e^{-\beta\tau'}x_0 + h_-(\tau')\right)^2 - C_M$$

Now, by the bound on the free boundary (34) established in Proposition 8.2

and once more by (46), to verify the above it is sufficient for

$$\delta(\tau') \geq k\left(e^{-\beta\tau'}(x_0+a) - \sqrt{\frac{C_M}{k}}\right)^2 - C_M$$
$$= k\left(e^{-2\beta\tau'}(x_0+a)^2 - 2e^{-\beta\tau'}(x_0+a)\sqrt{\frac{C_M}{k}} + \frac{C_M}{k}\right) - C_M$$
$$= -2e^{-\beta\tau'}(x_0+a)\sqrt{kC_M} + e^{-2\beta\tau'}k(x_0+a)^2.$$

Where it was in in the first inequality that (34) and (46) were used. A completely analogous process shows that he second inequality in (42) is also verified if

$$\delta(\tau') > 2e^{-\beta\tau'}(x_0+b)\sqrt{kC_M} + e^{-2\beta\tau'}(x_0+b)^2.$$

By construction of λ both of the preceding equations hold. Thus, due to our discussion at the beginning of this theorem we deduce from Sturm's theorem that

$$W_{C_M+\delta(\tau'),x_0}(\xi,\tau') \geq \theta(\xi,\tau') \quad \forall(\xi,\tau') \in \mathbb{R}\times(\tau_0,+\infty) \tag{47}$$

It remains to see what kind of bound this gives on $\theta(\tau')$. The second term in the maximum that defines δ gives

$$\left|k\left((\xi+e^{-\beta\tau'}x_0)^2 - \xi^2\right)\right| = -ke^{-\beta\tau'}x_0\left|2\xi + e^{-\beta\tau'}x_0\right|$$
$$\leq -kx_0(2R_M - e^{-\beta\tau'}x_0)e^{-\beta\tau'} \leq \delta(\tau') \quad \forall\xi \in [-R_M, R_M] = \operatorname{supp} F_{C_M}. \tag{48}$$

Where it was used that, by construction, $x_0 < 0$. We deduce that, as a result, (45) holds. Another consequence of (48) is that

$$0 \leq W^{m-1}_{C_M+\delta(\tau'),x_0}(\xi,\tau') - F^{m-1}_{C_M}(\xi) = \left(\delta(\tau') - k((\xi+e^{-\beta\tau'}x_0)^2 - \xi^2)\right)_+$$
$$\leq 2\delta(\tau') \tag{49}$$

Thus we obtain from (47), (49) that

$$\theta^{m-1}(\xi,\tau') - F^{m-1}_{C_M}(\xi) \leq W^{m-1}_{C_M+\delta(\tau'),x_0}(\xi,\tau') - F^{m-1}_{C_M}(\xi) \leq 2\delta(\tau'). \tag{50}$$

As was previously explained, by reasoning with $W_{C_M-\delta(\tau'),x_0}$ we obtain in identical fashion a lower bound

$$-2\delta(\tau') \leq \theta^{m-1}(\xi,\tau') - F^{m-1}_{C_M}(\xi) \tag{51}$$

Since the bounds in (50),(51) are independent of $\xi \in \mathbb{R}$ and valid for all $\tau' \geq \tau_0$ we obtain that

$$\left\|\theta^{m-1}(\tau') - F_{C_M}^{m-1}\right\|_{L^\infty(\mathbb{R})} \leq 2\delta(\tau') \lesssim e^{-\beta\tau'} \quad \forall \tau' \in [\tau_0, +\infty)$$

$\square$

Remark 8.1. *The order of convergence*

$$\left\|\theta^{m-1}(\tau) - F_{C_M}^{m-1}\right\|_{L^\infty(\mathbb{R})} \lesssim e^{-\beta\tau}$$

obtained in the preceding theorem is sharp. To note this take $\theta = W_{C_M, x_0}$ for any $x_0 \neq 0$.

As a result of the previous theorem we also quickly obtain convergence of $\partial_\tau \theta$.

Remark 8.2. *By using the estimate obtained in* (48) *we see that if we reduce our study to an interval around the origin of the form $[Ne^{-\beta\tau}, Ne^{\beta\tau}]$ with $N > 0$ a constant then we obtain*

$$\begin{aligned}\left|k\left((\xi + e^{-\beta\tau}x_0)^2 - \xi^2\right)\right| &\leq -kx_0(2Ne^{-\beta\tau} - e^{-\beta\tau}x_0)e^{-\beta\tau} \\ &\lesssim_N e^{-2\beta\tau} \quad \forall \xi \in [-Ne^{-\beta\tau}, Ne^{-\beta\tau}]\end{aligned}$$

If we translate the preceding results back into terms of u and its pressure we obtain the following result

Theorem 8.2 (Uniform convergence of the pressure)**.** *Let u be the solution to* (1)*, let U_{C_M} be the Barenblatt solution with mass $M = \int_{\mathbb{R}} u_0 dx$ and set their respective pressures to be*

$$v = \frac{m}{m-1}u^{m-1}; \quad V_M = \frac{m}{m-1}U_{C_M}^{m-1}$$

Then the following hold

$$\|v(t) - V_M(t)\|_{L^\infty(\mathbb{R})} \lesssim t^{-\frac{m}{m+1}} \tag{52}$$

$$\|v(t) - V_M(t)\|_{L^\infty([-N,N])} \lesssim_N t^{-1} \quad \forall N > 0 \tag{53}$$

Where the order of convergence in (52),(53) *is optimal.*

Proof. Equation (52) follows from the fact that, by definition of the rescaling θ,

$$u^{m-1}(x,t) = (1+t)^{-\alpha(m-1)}\theta^{m-1}((1+t)^{\beta}x, log(1+t)).$$

Thus, by using the result (40) obtained in Theorem (8.1), we get

$$\begin{aligned}
\|v(t) - V_M(t)\|_{L^\infty(\mathbb{R})} &= (1+t)^{-\alpha(m-1)}\left\|\theta^{m-1}(log(1+t)) - F_{C_M}^{m-1}\right\|_{L^\infty(\mathbb{R})} \\
&\lesssim (1+t)^{-\alpha(m-1)}e^{-\beta log(1+t)} = (1+t)^{-\frac{m}{m+1}} \sim t^{-\frac{m}{m+1}}
\end{aligned}$$

Where it was used in the last inequality that $\alpha = \beta = 1/(m-1)$. The estimate (53) is obtained similarly. Note that

$$x \in [-N, N] \iff \xi = (1+t)^{-\beta}x \in [-Ne^{-\beta log(1+t)}, Ne^{-\beta log(1+t)}] \tag{54}$$

In consequence it holds that

$$\begin{aligned}
&\|v(t) - V_M(t)\|_{L^\infty([-N,N])} \\
&\quad = (1+t)^{-\alpha(m-1)}\left\|\theta^{m-1}(log(1+t)) - F_{C_M}^{m-1}\right\|_{L^\infty([-N(1+t)^{-\beta}, N(1+t)^{-\beta}])} \\
&\quad\quad \lesssim (1+t)^{-\alpha(m-1)-2\beta} \sim t^{-1}
\end{aligned}$$

Where in the first equality we used the definition of θ and (54) and in the inequality we used Observation 8.2. The optimality is a consequence of the convergence of the pressure of a translatedBarenblatt solution

$$V_{M,x_0} = \frac{m}{m-1}(1+t)^{-\alpha(m-1)}\left(C_M - k\frac{(x+x_0)^2}{(1+t)^{2\beta}}\right)_+$$

to the pressure of a Barenblatt solution centered at the origin

$$V_M = \frac{m}{m-1}(1+t)^{-\alpha(m-1)}\left(C_M - k\frac{x^2}{(1+t)^{2\beta}}\right)_+$$

This concludes the proof. $\square$

In the future it will be of great interest of us to, given a point $p_0 = (\xi_0, \tau_0)$, find a translated profile W_{C,x_0} that takes on p_0 the same value and same spatial derivative as θ while having

$$\operatorname{supp} W_{C,x_0} \cap \operatorname{supp} \theta_0 = \emptyset.$$

Our following lemma serves exactly this purpose and exemplifies the previously mentioned completeness of the family $\{W_{C,x_0}\}$.

Lemma 8.2 (Geometric density)**.** *There exist unique continuous functions*

$$C_* : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+; \quad \xi_* : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R},$$

such that for all $(\lambda, \mu) \in \mathbb{R} \times \mathbb{R}^+$*,*

$$F_{C_*(\lambda,\mu)}(\xi_*(\lambda,\mu)) = \lambda; \quad F'_{C_*(\lambda,\mu)}(\xi_*(\lambda,\mu)) = \mu. \tag{55}$$

Proof. The proof is obtained by simply solving the system of equations obtained from (55) for C_*, ξ_* to obtain that, necessarily,

$$C(\lambda,\mu) = \lambda^{m-1} + \frac{(m-1)^2\mu^2}{4k}\lambda^{2m-4}; \quad \xi_*(\lambda,\mu) = \frac{(1-m)\mu}{2k}\lambda^{m-2}. \tag{56}$$

Continuity follows from the above expression which concludes the proof. □

We now show that if a translated profile W_{C,x_0} and θ share the same value and spatial derivative on some point $p_0 = (\xi_0, \tau_0)$ then there support must initially intersect. From the previous lemma we will be able to precisely derive the expression of W in terms of C_*, ξ_*. Further on we will be able to use this to obtain a priori estimates on the terms involved in C_*, ξ_*. Which, as we shall see later in Theorem 8.3, contain a treasure trove of information on the behaviour of θ.

Lemma 8.3. *Let θ be the solution to*(29) *with non-negative initial data $\theta_0 \in C_c(\mathbb{R})$. For any translated fixed profile*

$$W_{C,x_0}(\xi,\tau) = F_C(\xi + e^{-\beta\tau}x_0).$$

only one of the following alternatives holds:

(i) *Initially their support does not intersect, that is*

$$\operatorname{supp}\theta_0 \cap \operatorname{supp} W_{C,x_0}(0) = \emptyset. \tag{57}$$

(ii) *There exists a "contact point"* $(\xi_0, \tau_0) \in \mathbb{R} \times \mathbb{R}^+$ *such that*

$$W_{C,x_0}(\xi_0,\tau_0) = \theta(\xi_0,\tau_0); \quad \partial_\xi W_{C,x_0}(\xi_0,\tau_0) = \partial_\xi\theta(\xi_0,\tau_0) \tag{58}$$

Proof. Suppose to the contrary that both (i) and (ii) hold. We will first consider the case in which ξ_0 is a local minimum or maximum of $W_{C,x_0}(\tau_0) - \theta(\tau_0)$, as both cases are solved in a similar way. Assume for instance that ξ_0 is a local minimum of $W_{C,x_0}(\tau_0) - \theta(\tau_0)$. Then we would have by the first equality in (58) that there exist $\xi_1 < \xi_0 < \xi_2$ such that

$$W_{C,x_0}(\xi, \tau_0) < \theta(\xi, \tau_0) \quad \forall \xi \in [\xi_1, \xi_2] \setminus \{\xi_0\} \tag{59}$$

Where we used the maximum principle to obtain the strict inequalities, as by basic theory of parabolic equations two different solutions can't be equal on a nontrivial interval within the support. We now take a rescaled and translated Barenblatt solution with slightly larger mass $W_{C+\epsilon,x_0}$. Since the support of the fixed profile depends continuously on its mass (see property p3) we deduce from (57) that we may take ϵ sufficiently small so that

$$\operatorname{supp} W_{C+\epsilon,x_0}(0) \cap \operatorname{supp} \theta_0 = \emptyset.$$

As a result, if we write $J(\tau)$ for the number of intersections of $W_{C+\epsilon,x_0}(\tau)$ with $\theta(\tau)$, we obtain $J(0) = 1$. Furthermore, since the fixed profile itself depends continuously on its mass, by (59) we may also take ϵ sufficiently small so that

$$W_{C+\epsilon,x_0}(\xi_1, \tau_0) < \theta(\xi_1, \tau_0); \quad W_{C+\epsilon,x_0}(\xi_2, \tau_0) < \theta(\xi_2, \tau_0) \tag{60}$$

Finally, as a byproduct of incrementing the mass of W_{C,x_0}, its value at every point in its domain of positivity increases (see property p7) and thus

$$W_{C+\epsilon,x_0}(\xi_0, \tau_0) > W_{C,x_0}(\xi_0, \tau_0) = \theta(\xi_0, \tau_0) \tag{61}$$

From equations (60), (61) we obtain on using the intermediate value theorem (IVT) two intersection points. This gives $J(\tau_0) = 2$, but this is impossible, as we had $J(0) = 1$. This shows that $\theta(\tau_0) - W(\tau_0)$ cannot have a minimum at ξ_0.

The case in which ξ_0 is a maximum of $W_{C,x_0}(\tau_0) - \theta(\tau_0)$ is dealt with in an identical fashion by now slightly decreasing its mass. Thus it only remains to solve the case where $W(\tau_0)$ and $\theta(\tau_0)$ "change sign" at ξ_0. That is, there are $\xi_1 < \xi_2$ so that one of the two following equations hold

$$\begin{aligned} W_{C,x_0}(\xi,\tau_0) &< \theta(\xi,\tau_0) \quad \forall \xi \in [\xi_1,\xi_0) \\ W_{C,x_0}(\xi,\tau_0) &> \theta(\xi,\tau_0) \quad \forall \xi \in (\xi_0,\xi_2] \end{aligned} \tag{62}$$

$$\begin{aligned} W_{C,x_0}(\xi,\tau_0) &> \theta(\xi,\tau_0) \quad \forall \xi \in [\xi_1,\xi_0) \\ W_{C,x_0}(\xi,\tau_0) &< \theta(\xi,\tau_0) \quad \forall \xi \in (\xi_0,\xi_2] \end{aligned} \tag{63}$$

Each case may be solved similarly. First we define the continuous functions $\tilde{C}, \tilde{x}_0 : \mathbb{R} \to \mathbb{R}$

$$\begin{aligned} \tilde{C}(\epsilon) &:= C_*(\theta(\xi_0,\tau_0), \partial_\xi \theta(\xi_0,\tau_0) + \epsilon) \\ \tilde{x}_0(\epsilon) &:= (\xi_*(\theta(\xi_0,\tau_0), \partial_\xi \theta(\xi_0,\tau_0) + \epsilon) - \xi_0) e^{\beta\tau_0} \end{aligned}$$

Where C_*, ξ_* are as in the previous lemma, Lemma 8.2. Our introduction of these functions is motivated by the desire to slightly modify the derivative of $W_{C,x_0}(\tau_0)$ at ξ_0 so as to surpass our quota of intersections, which is currently set to 1 as a direct result of (57). Note that in particular, by the hypothesis placed on W_{C,x_0} in (58), we have that

$$\tilde{C}(0) = C; \quad \tilde{x}_0(0) = x_0. \tag{64}$$

We now consider the "'translated profile with altered derivative"

$$W_\epsilon := W_{\tilde{C}(\epsilon),\tilde{x}_0(\epsilon)}.$$

By continuity of $\tilde{C}, \tilde{x}_0$, by (64), by continuity of the support of the fixed profile in the parameter C and by the initial hypothesis on the support (57) we deduce that if ϵ is sufficiently small in absolute value

$$\operatorname{supp} W_\epsilon(\xi,\tau) \cap \operatorname{supp} \theta_0 = \emptyset \tag{65}$$

Our setup is now complete and we are ready to conclude. Suppose first that equation (62) holds and fix $\epsilon_0 < 0$ sufficiently small so that (65) holds and also

$$W_{\epsilon_0}(\xi_1,\tau_0) < \theta(\xi_1,\tau_0); \quad W_{\epsilon_0}(\xi_2,\tau_0) > \theta(\xi_2,\tau_0) \tag{66}$$

Where this last equation is made possible for small ϵ_0 by (62) and (64) in combination with the uniform continuity of W_ϵ in ϵ. Furthermore, by construction of W_{ϵ_0} we have that

$$W_{\epsilon_0}(\xi_0,\tau_0)=\theta(\xi_0,\tau_0); \quad \partial_\xi W_{\epsilon_0}(\xi_0,\tau_0)=\partial_\xi\theta(\xi_0,\tau_0)+\epsilon<\partial_\xi\theta(\xi_0,\tau_0) \tag{67}$$

From equation (67) we deduce that there exist $\tilde{\xi}_1,\tilde{\xi}_2$ with

$$\xi_1<\tilde{\xi}_1<\xi_0<\tilde{\xi}_2<\xi_1 \tag{68}$$

$$W_{\epsilon_0}(\tilde{\xi}_1,\tau_0)>\theta(\tilde{\xi}_1,\tau_0); \quad W_{\epsilon_0}(\tilde{\xi}_2,\tau_0)<\theta(\tilde{\xi}_2,\tau_0) \tag{69}$$

By using equations (66) and (68),(69) we obtain two intersection points for $\tilde{W}(\tau_0),\theta(\tau_0)$ (in fact, though unnecessary, we may obtain at least three as ξ_0 will also be one) which is a contradiction with (65).

If on the other hand (63) holds instead of (62) we would conclude analogously by slightly increasing the derivative. That is by by now taking a small $\epsilon_0>0$. Since this remained the only remaining possibility we arrive at a contradiction which stemmed from supposing that (i) and (ii) held simultaneously. This concludes our proof. □

Our next theorem introduces an a priori estimate that gives very rich geometrical information as to the behaviour of θ. We first recall that the positivity set of θ is

$$\mathcal{P}_\theta:=\{(\xi,\tau)\in\mathbb{R}\times\mathbb{R}^+:\theta(\xi,\tau)>0\}$$

Theorem 8.3 (A priori estimate). *Let θ be the solution to* (29) *with non-negative initial data $\theta_0\in C_c(\mathbb{R})$. Then for all $(\xi,\tau)\in\mathcal{P}_\theta$*

$$\left|\frac{\partial_\xi(\theta^{m-1})(\xi,\tau)}{2k}+\xi\right|e^{\beta\tau}-\sqrt{\frac{\theta^{m-1}(\xi,\tau)}{k}+\left(\frac{\partial_\xi(\theta^{m-1})(\xi,\tau)}{2k}\right)^2} \leq \max\{-h_-(0),h_+(0)\}$$

Proof. The proof relies heavily on the result just proved, Lemma 8.3. Consider $(\xi_0,\tau_0)\in\mathcal{P}_\theta$, to apply the aforementioned lemma we must choose an appropriate translated profile W_{C,x_0} so that at the desired point we have

$$W_{C,x_0}(\xi_0,\tau_0)=\theta(\xi_0,\tau_0);\quad \partial_\xi W_{C,x_0}(\xi_0,\tau_0)=\partial_\xi\theta(\xi_0,\tau_0)$$

Such a construction has been done before and is possible by Lemma 8.2 as we may set

$$C:=C_*(\theta(\xi_0,\tau_0),\partial_\xi\theta(\xi_0,\tau_0));\quad x_0:=(\xi_*(\theta(\xi_0,\tau_0),\partial_\xi\theta(\xi_0,\tau_0))-\xi_0)e^{\beta\tau_0} \tag{70}$$

. Lemma 8.3 now tells us that initially the support of W_{C,x_0} and of θ must intersect. Thus, if we consider the left and right interface $h_\pm(0)$ of θ_0 we have that, since the support of $W_{C,x_0}(0)$ is

$$\operatorname{supp} W_{C,x_0}(0)=\left[-\sqrt{\frac{C}{k}}-x_0,\sqrt{\frac{C}{k}}-x_0\right],$$

then

$$\sqrt{\frac{C}{k}}-x_0>h_-(0);\quad -\sqrt{\frac{C}{k}}-x_0<h_+(0); \tag{71}$$

Suppose first that $x_0\geq 0$, then the first inequality in (71) shows that, on multiplying both sides by -1,

$$x_0-\sqrt{\frac{C}{k}}=|x_0|-\sqrt{\frac{C}{k}}<-h_-(0) \tag{72}$$

If on the other hand we have that $x_0\leq 0$ we have from the second inequality in (71) that

$$-x_0-\sqrt{\frac{C}{k}}=|x_0|-\sqrt{\frac{C}{k}}<h_+(0) \tag{73}$$

From the two preceding equations (72),(73) we obtain that the estimate

$$|x_0|-\sqrt{\frac{C}{k}}<\max\{-h_-(0),h_+(0)\} \tag{74}$$

always holds. Finally, from the expression of the ξ_*, C_* obtained in (56), the chain rule

$$\partial_\xi\theta^{m-1}=(m-1)\theta^{m-2}\partial_\xi\theta,$$

and the construction of C, x_0 in (70) we have

$$C=\theta^{m-1}(\xi_0,\tau)+\left(\frac{\partial_\xi(\theta^{m-1})(\xi_0,\tau)}{2\sqrt{k}}\right)^2;\quad x_0=\left(-\frac{\partial_\xi(\theta^{m-1})(\xi_0,\tau_0)}{2k}-\xi_0\right)e^{\beta\tau_0}$$

Substituting the values for C, x_0 in the estimate (74) completes our proof as $(\xi_0, \tau_0) \in \mathcal{P}_\theta$ were any. $\square$

From now on to slightly simplify the notation we shall write

$$H(0) := \max\{-h_-(0), h_+(0)\} \tag{75}$$

The preceding theorem will serve to show two strong results as to the asymptotic behaviour of θ. To show the first of these we need an auxiliary result bounding the spatial derivative $\partial_\xi \theta$.

Corollary 8.2. *Let θ be the solution to* (29) *with non-negative initial data $\theta_0 \in C_c(\mathbb{R})$. Then given $\tau_0 > 0$ it holds that*

$$\sup\{\left|\partial_\xi(\theta^{m-1})(\xi,\tau)\right| : (\xi,\tau) \in \mathcal{P}_\theta \cap ([\tau_0, +\infty) \times \mathbb{R})\} < +\infty$$

Proof. Consider $(\xi,\tau) \in \mathcal{P}_\theta \cap ([\tau_0, +\infty) \times \mathbb{R})$, by the uniform convergence of θ to its fixed profile proved in Theorem 8.1 and convergence of support in Corollary (8.1) we have that the supremums

$$L := \sup_{(\xi,\tau)\in\mathcal{P}_\theta} \{\theta^{m-1}(\xi,\tau)\}; \quad H := \sup\{|\xi| : (\xi,\tau) \in \mathcal{P}_\theta \text{ for some } \tau \in \mathbb{R}^+\}$$

both exist. By the a priori estimate just proven in Theorem 8.3 and the triangle inequality we have that

$$\left|\frac{\partial_\xi(\theta^{m-1})(\xi,\tau)}{2k}\right|(e^{\beta\tau} - 1) \leq H(0) + \sqrt{\frac{L^{m-1}}{k}} + He^{\beta\tau} \tag{76}$$

Since $\tau \geq \tau_0 > 0$ we may divide by $e^{\beta\tau} - 1$. Thus, if we write

$$\tilde{H} := \sup_{\tau \geq \tau_0} \frac{2kHe^{\beta\tau}}{e^{\beta\tau} - 1},$$

we obtain from (76) that

$$\left|\partial_\xi(\theta^{m-1})(\xi,\tau)\right| \leq 2k((e^{\beta\tau_0} - 1))^{-1}\left(H(0) + \sqrt{\frac{L^{m-1}}{k}}\right) + \tilde{H}$$

Since the right hand side of the above inequality is constant (u_0, τ_0 being fixed) we conclude our proof. $\square$

Remark 8.3. *Note that in the above proof it was necessary to restrict ourselves to the set $\mathcal{P}_\theta \cap ([\tau_0, +\infty) \times \mathbb{R})$. This may be justified a priori by the fact that*

1. *The derivative $\partial_\xi \theta$ is only defined on the domain of positivity $\mathcal{P}_\theta$.*
2. *Our only supposition on u_0 is that it be continuous and thus $\partial_\xi \theta$ may become unbounded as $\tau \to 0$.*

We now prove the first of the strong results obtained from the a priori estimate of Theorem 8.3. Namely that $\partial_\xi \theta$ converges to $\partial_\xi F_C$ uniformly on its domain of definition $\mathcal{P}_\theta$. The machinery needed for this result is all in place and as we shall see the proof is almost immediate.

Proposition 8.3 (Uniform convergence of the derivative)**.** *Let θ be the solution to* (29) *with non-negative initial data $\theta_0 \in C_c(\mathbb{R})$. Then we have uniform convergence of the following order*

$$\left\| \partial_\xi(\theta^{m-1})(\xi, \tau) - 2k\xi \right\|_{L^\infty(\mathcal{P}_\theta)} \lesssim e^{-\beta\tau}$$

Furthermore, the above order of convergence is sharp and if F_C is the fixed profile with the same mass as θ_0 then

$$\lim_{\tau\to\infty} \|\partial_\xi \theta(\xi, \tau) - F_C'(\xi)\|_{L^\infty(\mathcal{P}_\theta)} = 0$$

Proof. The first part of the theorem is a direct consequence of the eventual boundedness of $\partial_\xi(\theta^{m-1})$ (proven in Lemma 8.2) and of θ (due to the uniform convergence of Theorem (8.1)) in conjunction with the estimate of Theorem 8.3. The optimality of the estimate can be proved by considering a translated profile W_{C,x_0} as

$$\left\| \partial_\xi(W_{C,x_0}^{m-1})(\xi, \tau) - 2k\xi \right\|_{L^\infty(\mathcal{P}_\theta)} \sim e^{-\beta\tau}$$

where $C > 0$ and $x_0 \neq 0$.

The second part of the theorem follows from the first fragment of this same theorem and another application of the uniform convergence of Theorem (8.1). As, by the chain rule,

$$\partial_\xi \theta = \frac{\partial_\xi(\theta^{m-1})}{(m-1)\theta^{m-2}} \to \frac{-2k\xi}{(m-1)F_C^{m-2}} = F_C'.$$

Where the convergence is uniform by the previous results and where in the last equality we once again used the chain rule in conjunction with the equality $(F_C^{m-1})'(\xi) = -2k\xi$. $\square$

As a final result of our a priori estimate we obtain an interesting result on the convergence of maximums and minimums. Note that any fixed profile F_C has a unique critical point within its support at $\xi = 0$. In our past theorem we showed uniform convergence of $\theta(\tau)$ to F_M and of $\partial_\xi \theta(\tau)$ to F_C'. We now also show that the critical points (and thus the maxima and minima) of θ must all converge to that of F_C', that is the origin. The tools needed to prove this result have been developed far in advance and in fact, once arrived at this point, the proof is surprisingly simple.

Corollary 8.3 (Disappearance of maxima and minima)**.** *Let θ be the solution to* (29) *with initial data $\theta_0 \in C_c(\mathbb{R})$ and set*

$$R(u_0) := H(0) + \sqrt{\frac{\|\theta^{m-1}\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)}}{k}}$$

Then all the critical points of $\theta(\tau)$ are located within the interval

$$I(\tau) := e^{-\beta\tau} \cdot [-R(u_0), R(u_0)]$$

Proof. Consider $(\xi, \tau) \in \mathcal{P}_\theta$ such that ξ is a critical point of $\theta(\tau)$. Then by substituting $\partial_\xi \theta(\xi, \tau) = 0$ into our a priori estimate of Theorem 8.3 we obtain that

$$|\xi| \le e^{-\beta\tau} \left(H(0) + \sqrt{\frac{\theta^{m-1}(\xi,\tau)}{k}} \right) \le R(\theta_0).$$

This concludes the proof. $\square$

The above theorem gives us a precise knowledge of where the maxima and minima of $\theta(\tau)$ are located based only on information of the initial data. In fact note that, though the expression of $R(\theta_0)$ depends on $\left\|\theta^{m-1}\right\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)}$, this norm can be bounded without precise knowledge of θ and using only θ_0. It suffices to consider as has been done on previous occasions a fixed profile F_C with $F_C \geq u_0 = \theta_0$. As then by the maximum principle 2.2

$$F_C(0) \geq F_C(\xi) \geq \theta(\xi,\tau) \quad \forall \xi, \tau \in \mathbb{R} \times \mathbb{R}^+.$$

Finally we observe that, in the style of Theorem 8.2, the information obtained in the previous results can be translated in terms of u and its pressure without difficulty. This gives the following theorem

Theorem 8.4. *Let u be the solution to the PME* (1) *with non-negative initial data $u_0 \in C_c(\mathbb{R})$ and let U_{C_M} the Barenblatt solution with the same mass as u_0. Set v, V_M to be the pressure of u, U_{C_M} respectively. Then the following hold*

$$\|\partial_x v(t) - \partial_x V_M(t)\|_{L^\infty(\mathcal{P}_u)} \lesssim t^{-1} \tag{77}$$

Where the order of convergence given in (77) *is sharp. In particular we have the uniform convergence*

$$\lim_{\tau\to+\infty} \|\partial_x u(t) - U_{C_M}(t)\|_{L^\infty(\mathcal{P}_u)} = 0 \tag{78}$$

Furthermore, all the critical points of $u(t)$ are located within an interval

$$I(u_0) = [-r(u_0), r(u_0)] \tag{79}$$

9. Numerical simulation

In this section we perform a numerical simulation of the 2D porous medium equations. To do so we consider a linearized version of the porous medium equation obtained via a Newton method. We then use this linearized equation to obtain a variational formulation which is then solved numerically via the open-source program Freefem++. The numerically obtained solutions show how solutions in two dimensions have a similar behaviour to that which we proved for solutions in the one dimensional case.

9.1. Formulation of the problem

In this final section we implement a numerical simulation of the previous results. To do so we will use the opensource program Freefem++. This program is based on the finite element method and thus gives a numerical approximation to the solution h of a variational problem of the form

$$A(h,v) = L(v) \quad \forall v \in V \tag{80}$$

Where V is the Hilbert space of "test functions", $L : V \to \mathbb{R}$ is linear and continuous $A : V \times V \to \mathbb{R}$ is bilinear, continuous and α-coercive. Due to the nature of the finite element method we work in dimension $d = 2$. As was noted previously many of the asymptotic results that hold in the 1 dimensional case also carry over to the general dimensional case. In particular those of convergence of u to the Barenblatt solution of the same mass, the convergence of their respective derivatives and convergence of extremum points to the origin.

To be able to employ the finite element method we must work with a bounded domain Ω and consider the Dirichlet problem (DP)

$$\begin{aligned} \partial_t u &= \Delta\Phi(x,u) + f && \text{in } \Omega \times [0,T] \\ u(x,0) &= u_0(x) && \text{in } \Omega, \\ u(x,t) &= 0 && \text{in } \partial\Omega \times [0,T] \end{aligned} \tag{81}$$

We will take $\Omega = B(0,R) \subset \mathbb{R}^2$ where R is large enough to contain the support of $u_0 \in C_c(\mathbb{R}^2)$. By the homogeneous boundary condition and the finite speed of propagation of solutions to the PME, if we take T small enough so that the support of $u(T)$ is within Ω, then

$$u \text{ solves the DP (81)} \iff u \text{ solves the PME (1) on } \mathbb{R}^2 \times [0,T]$$

Thus, if we take the radius R of Ω to be large enough, we may also take T large and hope to be able to verify the asymptotic results shown for the Cauchy problem (13) by using a numerical solution to the DP (81).

To obtain a variational formulation as in (80) we begin by partitioning the time interval $[0, T]$ as

$$[0,T] = \bigcup_{n=1}^{N} [t_{n-1}, t_n]; \quad t_n := n\Delta t, \quad n = 0, ..., N; \quad \Delta t := \frac{T}{N},$$

where $N >> 0$ is taken to be large. We now use an implicit Euler method to obtain an approximation u_n to $u(t_n)$ where u_n solves

$$\frac{u_n(x) - u_{n-1}(x)}{\Delta t} = \Delta u_n^m(x) \tag{82}$$

Since the PME (1) is non-linear so is the problem obtained in (82). Thus obtaining the operators L, A in (80) is not immediately possible and we must first conduct a linearization of this equation. To do so we will employ a Newton Method. More information on the Newton Method for PDE may be found for example in chapter 8 of [22]. In the ordinary scalar case the Newton method is based on the Taylor expansion of a function $F : \mathbb{R} \to \mathbb{R}$. In our case F will have its domain in the Banach space V instead of $\mathbb{R}$. We set

$$F(u_n) := \frac{u_n - u_{n-1}}{\Delta t} - \Delta({u_n}^m). \tag{83}$$

and wish to solve

$$F(u_n) = 0 \quad n = 1, \ldots, N. \tag{84}$$

As in the ordinary case the Newton method for PDE one must first make a guess as to the solution of (84). Since we are taking $N >> 0$ and u is continuous we initially guess

$$u_{n,0} := u_{n-1}.$$

Then one must search for $h \in V$ such that $u_{n,0} + h$ solves (84). This is in turn equivalent to

$$0 = F(u_{n,0} + h) = F(u_{n,0}) + dF(u_{n,0})(h) + o(h^2).$$

This leads to an initial approximation h_0 to h as the solution to

$$dF(u_{n,0})(h_{n,0}) = -F(u_{n,0}), \tag{85}$$

Iterating this some nomber of times $I \in \mathbb{N}$ we obtain

$$dF(u_{n,i})(h_{n,i}) = -F(u_{n,i}); \quad u_{n,i+1} = u_{n,i} + h_{n,i}, \quad i = 0, ..., I. \tag{86}$$

And where we set as our final approximation to u_n solving (84)

$$u_n = u_{n,I+1}.$$

Thus by solving iteratively equations (86) we obtain an approximation to u_n. It only remains to see what form these equations take. By using the linearity of the Laplacian Δ and the definition of F in (83) we see that $dF(u_{n,i})$ is the linear function defined by

$$dF(u_{n,i})(h_{n,i}) = \frac{h_{n,i}}{\Delta t} - \Delta(m u_{n,i}^{m-1} h_{n,i}) \qquad \forall h_{n,i} \in V. \tag{87}$$

Thus, substituting in (86) the expressions in (83) and (87), we obtain that the equations to solve are for each $n \in \{1, ..., N\}$

$$\frac{h_{n,i}}{\Delta t} - \Delta(m u_n^{m-1} h_{n,i}) + \frac{u_{n,i} - u_{n-1}}{\Delta t} - \Delta(u_{n,i}^m) = 0; \quad i = 0, ..., I \tag{88}$$

Observe that, excepting $h_{n,i}$, all the expressions in (88) are known having been obtained in previous steps. In consequence (88) are linear equations for $h_{n,i}$ and we may now obtain a variational formulation for them of the form (80). This formulation is obtained from (88) by using Green's theorem and is

$$\int_\Omega \left(\frac{h_{n,i} v}{\Delta t} + \nabla(m u_{n,i}^{m-1} h_{n,i}) \cdot \nabla v \right) dx = -\int_\Omega \left(\frac{u_{n,i} - u_{n-1}}{\Delta t} v + \nabla(u_{n,i}^m) \cdot \nabla v \right) dx$$
$$h_{n,i}\big|_{\partial\Omega} = -u_{n,i}\big|_{\partial\Omega} \tag{89}$$

On solving (89) for each i, n we obtain, as was explained previously, the value of u_n from the iteration

$$u_{n,0} = u_{n-1} \quad u_{n,i+1} = u_{n,i} + h_{n,i}; \quad u_n = u_{n,I+1} \tag{90}$$

9.2. Implementation and results

In this section we discuss the results obtained from numerically solving the equations (89)-(90) of the previous subsection. In the following simulation we take as initial data

$$u_0 = \left(4 - e^{(x-2)^2+(y-2)^2}\right)_+ + 2 \cdot \left(4 - e^{(x+1)^2+(y+2)^2}\right)_+ .$$

It is clear that $u_0 \in C_c(\mathbb{R})$ has two peaks, one at $(2,2)$ and another at $(-1,-2)$. So that the support of u_0 is within $\Omega = B(0,R)$ we take $R = 20$. Furthermore to generate the mesh for the finite element method we take 50 nodes on $\partial\Omega$. As for the time step we set $\Delta t = 0.01$.

To obtain the Barenblatt solution to which u converges we must calculate C_M where M is the mass of u_0. The formula for C_M in the general d dimensional case being

$$C_M = \left(\frac{2k^{\frac{d}{2}}M}{dw_d\mathrm{B}\left(\frac{d}{2}, \frac{m}{m-1}\right)}\right)^{\frac{2(m-1)\alpha}{d}}$$

Where

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt,$$

is the beta function and

$$w_d = \frac{2\pi^{d/2}}{d\Gamma(d/2)}$$

is the volume (measure) of a d dimensional sphere of radius 1 (see for example [16] page 447). As a final step, so as to avoid the degeneracy of the equation at the free boundary we approximate u_0 by

$$\tilde{u}_0 := \max\{u_0, 1/N\}$$

where N is taken to be very large. We show below the rescaled solution θ at various instants in time. As can be seen in the images below the maxima of $\theta(t)$ converge to the origin. Likewise, as can be seen from the iso-values (it will be necessary to zoom in) $\theta(t)$ converges to W_{C_M}

Below we also include a screenshot of the distance in the uniform norm. For completeness we also include the numerical estimate of the exponent γ_i which appears in the convergence:

$$\|v(\tau) - V_M\|_{L^\infty(\mathbb{R})} \sim e^{-\gamma_1\tau}; \quad \|\nabla v(\xi,\tau) - \nabla V_M\|_{L^\infty(\mathcal{P}_\theta)} \sim e^{-\gamma_2\tau} \tag{91}$$

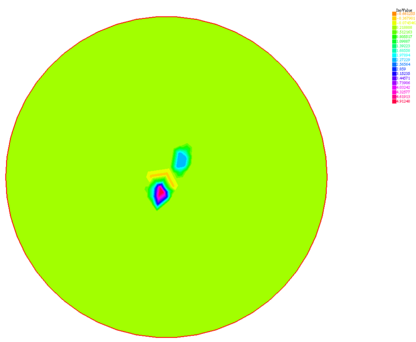

Figure 4: Plot of θ_0

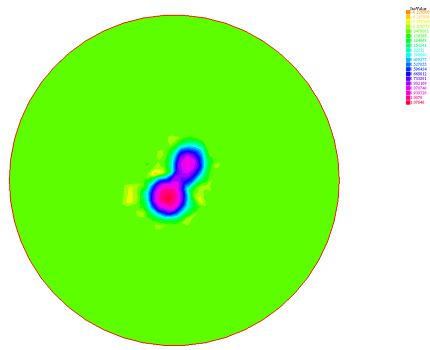

Figure 5: Plot of θ at $\tau = 2$

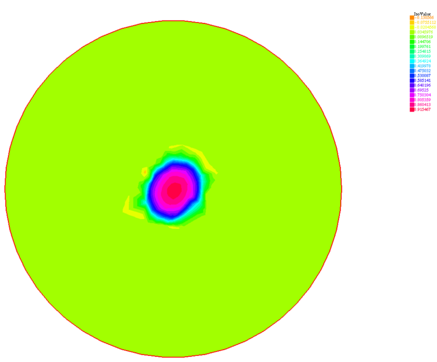

Figure 6: Plot of θ at $\tau = 6$

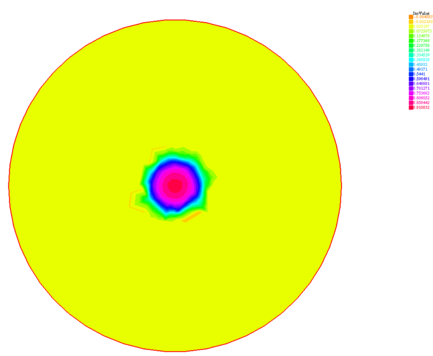

Figure 7: Plot of W_{C_M}

Where $v := \theta^{m-1}, V_M := F_{C_M}^{m-1}$. The minimum value for these exponents was obtained in Theorem 8.1 to be β. Thus, with the value taken in our simulation of $m = 3, d = 2$ we have

$$\gamma_i \geq \beta = \frac{1}{d(m-1)+2} = \frac{1}{6} \quad i = 1, 2. \tag{92}$$

In the figure 8 below we observe that the numerical estimate obtained for γ_1 and γ_2 fluctuate widely. However, on average, the bounds in (92) hold. Despite this, it is possible that for other initial data we will not recover (92), as the main goal of the simulation was to obtain a visual representation of the theoretical results in the text and a study of the error or convergence of this method was not carried out.

The mass is 26.4553, C = 0.789686, m= 3 alpha=0.333333, beta=0.166667

Proximity of solutions

Tau (time)	Linfty(v-V_M)	ROC	Linfty(gradient(v-V_M))	ROC
0.0953102	4.00009	----------	3.72513	----------
1.80829	0.423219	1.31127	0.327998	1.41849
2.40695	0.374646	0.203637	0.342449	-0.0720173
2.77882	0.353892	0.153248	0.298274	0.37139
3.04927	0.350714	0.0333514	0.247894	0.68407
3.26194	0.35002	0.00931868	0.247763	0.00247787
3.43721	0.344915	0.0838225	0.241495	0.1462
3.58629	0.327377	0.35005	0.239269	0.0621255
3.71601	0.310663	0.403976	0.251083	-0.371539
3.83081	0.303493	0.203387	0.259064	-0.272569
3.93378	0.3002	0.105948	0.258044	0.0382941
4.02714	0.296219	0.143002	0.251705	0.26646
4.11251	0.290217	0.239788	0.243452	0.390486
4.19117	0.282493	0.34296	0.2366	0.362934
4.26409	0.273706	0.433345	0.231731	0.285153
4.33205	0.264363	0.511053	0.228032	0.23677
4.39568	0.254834	0.576869	0.224365	0.254752
4.45551	0.24551	0.62308	0.220026	0.326481
4.51196	0.236849	0.636194	0.215095	0.401522
4.56539	0.229307	0.605698	0.209859	0.461164
4.61611	0.223202	0.532039	0.216357	-0.601186
4.66438	0.218632	0.42849	0.219608	-0.308913
4.71043	0.215471	0.316246	0.21995	-0.0338539

Figure 8: The uniform distance and rates of convergence of the simulation

Lastly, the finite element method is in fact better adapted to the calculation of L^p norms for $p < +\infty$. Thus we also include the rate of convergence obtained by measuring the distances in (91) in the L^2 norms. As can be seen below in Figure 9 the convergence rates obtained numerically for the L^2 distance also vary widely. This said they remain mostly positive and the distance is seen to converge to 0.

```
The mass is 24.2523, C = 0.745214,  m= 3 alpha=0.333333,  beta=0.166667
-----------------------------------------------------------------
Proximity of solutions
-----------------------------------------------------------------
```

Tau (time)	L2norm(v-V_M)	ROC	L2norm(gradient(v-V_M))	ROC
0.0953102	3.72897	----------	9.26463	----------
1.80829	2.44744	0.245823	6.56097	0.201442
2.40695	2.16062	0.208208	5.7409	0.223037
2.77882	1.93608	0.295081	5.00783	0.367363
3.04927	1.80181	0.265754	4.6329	0.287739
3.26194	1.713	0.237681	4.45312	0.186109
3.43721	1.64747	0.222511	4.28064	0.225383
3.58629	1.59193	0.230028	4.14137	0.221853
3.71601	1.54194	0.245994	4.02023	0.228856
3.83081	1.49537	0.267141	3.94029	0.174967
3.93378	1.45463	0.268239	3.90599	0.0848929
4.02714	1.4212	0.249014	3.89608	0.0272138
4.11251	1.3938	0.228025	3.88404	0.0362438
4.19117	1.37131	0.206887	3.8814	0.00867369
4.26409	1.35037	0.211011	3.87532	0.0214873
4.33205	1.33039	0.219355	3.86167	0.051917
4.39568	1.31246	0.213141	3.84854	0.0535057
4.45551	1.29594	0.211742	3.83207	0.0717035
4.51196	1.27946	0.226714	3.81118	0.0968383
4.56539	1.26403	0.227113	3.78895	0.109483
4.61611	1.2501	0.218499	3.77226	0.0870426
4.66438	1.23865	0.190635	3.75666	0.085865

Figure 9: The L^2 distance and rates of convergence of the simulation

10. Conclusion and future work

The approach to the study of some of the fundamental properties concerning the behavior of the solutions of the Porous Medium Equation has demonstrated the strength of Sturm theory as a key tool in the analysis of parabolic PDE. We note that in most, if not all our arguments, we relied on the following facts

1. The existence of a family radially symmetric self-similar solution $\{F_C\}$.
2. The existence of a sufficiently complete family of solutions $\{W_{C,x_0}\}$ lying with the stable variety of $\{F_C\}$ and such that $\{W_{C,x_0}\}$ lies within the stable variety of $\{F_C\}$.
3. The maximum princile and Sturm theory

Thus, a future avenue of research is to investigate how the results we proved in our work can be generalized to equations verifying the preceding three properties. An example of such equation would for example be the $p-$Laplacian equation.

Additionally we note that, though the orders of convergence obtained in our work are sharp, it is possible to obtain higher rates of convergence for radially symmetric solutions. This was accomplished by Vázquez in [12]. This suggests that it may be possible to improve the rate of converge we obtained by considering convergence to a fixed profile centered at the center of mass of the initial data and opens another possible line of research into the asymptotic behaviour of the solutions to the PME.

11. Appendix: Code for the numerical simulation

Below we include the code used for the numerical simulation. The code used for the obtention of the L^2 distance is practically identical to that of the L^∞ distance. Additionally both codes are included in program form in the submission through Studium

11.1. Code for uniform error

```
//Numerical verification that solution to PME converges to Barenblatt solution with same mas
//Note that the simulation will hold only for times where the domain of the solution is insi
verbosity=0;
//PARAMETERS OF THE PROBLEM
real m=3;  //parameter of equation
real dim=2;//the dimension
real alpha=dim/(dim*(m - 1) + 2);
real beta= alpha/dim;
real k= alpha*(m - 1)/(2*m*dim);
real gamma= dim/(2*(m-1)*alpha);
real betafunc = (m-1)/m; //value of B(d/2,m/m-1)=B(1,m/m-1)
```

```
//PARAMETERS OF THE DOMAIN
real R=20; //radius of domain, WARNING once the support of u leaves the domain the solution
int M=50; // número of nodes on the boundary

//PARAMETERS OF THE NUMERICAL METHOD
real n=320000;// The min value of u_0 is 1/n
int imax=4; //the number of iterations of the newton method is imax+1
real t0=0; //The initial time where the solution is considered
real t=t0; //time set to initial time t0
real dt=0.1; //Time step
int stepstoshow=50;// Steps till error is showed
int steps=50; //Number of time steps where solution is displayed
real N=(steps-1)*stepstoshow+1; //number of time steps where u_n is calculated
int counter=-1;//for printing
int scounter=-1;//counter to store error


//DISTANCE VECTORS
real[int] distLinfty(steps); //stores L^infty distance
real[int] distLinftyder(steps); //stores L^infty distance
func real eoc(real e, real eo, real tnew, real told)//Function that gives eoc

if (eo > 0.0)
 return -log(e/eo)/(tnew-told);
else
return 0.0;
;
```

```
//DOMAIN AND INITIAL DATA
border d(t=0,2*pi) x = R*cos(t); y = R*sin(t);label=1;; //definimos la frontera del círculo
func f=0; // fuente
//Define initial data for solution here.
//Initial data must be supported in disk of radius R so as to calculate its mass

real lambda = 4;//some parameters of the initial data
real mu = 2;
mesh disk = buildmesh(d(M));  // Mallado de disco de radio R
plot(disk, wait=1);  //dibujar malla
func uu0= fmax(lambda-exp((x-2)^2+(y-2)^2),1/n)+2*fmax(lambda-exp((x+1)^2+(y+2)^2),1/n);  //
//func uu0= fmax(lambda-mu*(x^2+y^6),0);   //SIMULATION 2

//FUNCTION SPACE AND VARIATIONAL PROBLEM
fespace Vh(disk,P2); Vh u=uu0,uold=uu0,h,w,WC,theta,difference,v,FC,vx,vy,FCx,FCy,diffder; /
macro dot(a,b) (dx(a)*dx(b)+dy(a)*dy(b))// Macro for ∇ u·∇ w
problem pme(h,w)=int2d(disk)(h*w)+int2d(disk)(dt*m*(m-1)*u^(m-2)*h*dot(u,w))+int2d(disk)(dt*
+int2d(disk)(u*w)-int2d(disk)(uold*w)+int2d(disk)(dt*m*u^(m-1)*dot(u,w))-int2d(disk)(dt*f*w)

//PARAMETERS FOR BARENBLATT SOLUTION
real mass = int2d(disk)(uu0); //mass M of u0
real C = (mass*2*k^(dim/2)/(dim*pi*betafunc))^(1/gamma); //C_M
//Prints value of parameters and name of distances
cout<<"The mass is "<< mass<<", C = "<<C<< ",  m= "<<m<<" alpha="<<alpha<<",  beta="<<beta<<
cout << "-------------------------------------------------------------------" << endl;
cout << "Proximity of solutions" << endl;
cout << "-------------------------------------------------------------------" << endl;
cout << "Tau (time)(v-V_M)~~~(gradient(v-V_M))~¬" << endl;

//PLOT OF FUNCTIONS AT T_0
```

```
WC = fmax(C-k*(x^2+y^2),0)^(1/(m-1));//Rescaled Barenblatt solution to which theta(t) conver
plot(WC,wait=1,value=true,fill=1);// Plots W_C
plot(u,wait=0,value=true,fill=1);//Plots initial data



//NUMERICAL SOLUTION
for (int j=1;j<N+1;j+=1) // Time iteration j in 1,2,...N
t+=dt; //time t in t0+dt,t0+2dt,....t0+Ndt
for (int i=0; i<imax+1;i++) // Newton method iteration, i in 0,...,imax
pme; // solves to find h_i
u=u+h;//u_j,i+1=u_j,i+h_i at final step u_j,I+1 is calculated and u==u_j is set to= u_j,I+1

uold=u; //old u is now uold==u_j, as in the next time step we find u_j+1

counter=(counter+1)%stepstoshow;//If counter=j-1 is a multiple of stepstoshow it shows plot
if(counter==0)
scounter+=1;// scounter will start at 0 and go till (N-1)/stepstoshow=steps
real tau=log(1+t);// time in tau
real tauold=log(1+t-stepstoshow*dt);// tau of prevousl calculated error
theta = (1+t)^alpha*u((1+t)^(beta)*x,(1+t)^(beta)*y);//rescaled solution ==theta(t_j)
v = theta^(m-1);//pressure of theta=v
vx = dx(v);//derivatives of v
vy = dx(v);
FC = fmax(C-k*(x^2+y^2),0);//fixed profile and its derivatives
FCx= dx(FC);
FCy= dy(FC);
difference= abs(v-FC);//difference of theta and fixed profile
diffder=vx-FCx;//difference of derivatives
distLinfty[scounter] =difference[].linfty;
distLinftyder[scounter] =diffder[].linfty;
real error0 =difference(0,0);
```

```
if(scounter==0)
cout << tau<<"         " << distLinfty[scounter]<<"                  "<< "-----------" <<"

cout << endl;

else

cout << tau<<"           " << distLinfty[scounter]<<"                    "<< eoc(distLinfty[

cout << endl;
plot(theta,wait=1,value=true,fill=1);// dibuja la solWCión reescalada para ser eventualmente

elsecontinue;
```

11.2. Code for L^2 error

In[1]:=

```
//Numerical verification that solution to PME converges to Barenblatt solution with same mas
//You must manually change the value of betafunc=B(d/2,m/(m-1)) if you change m from 2 to so
//Note that the simulation will hold only for times where the domain of the solution is insi
verbosity=0;


//PARAMETERS OF THE PROBLEM
real m=3;  //parameter of equation
real dim=2;//the dimension
real alpha=dim/(dim*(m - 1) + 2);
real beta= alpha/dim;
```

```
real k= alpha*(m - 1)/(2*m*dim);
real gamma= dim/(2*(m-1)*alpha);
real betafunc = (m-1)/m; //value of B(d/2,m/m-1)=B(1,m/m-1)


//PARAMETERS OF THE DOMAIN
real R=20; //radius of domain, WARNING once the support of u leaves the domain the solution
int M=50; // número of nodes on the boundary

//PARAMETERS OF THE NUMERICAL METHOD
real n=320000;// The min value of u_0 is 1/n
int imax=4; //the number of iterations of the newton method is imax+1
real t0=0; //The initial time where the solution is considered
real t=t0; //time set to initial time t0
real dt=0.1; //Time step
int stepstoshow=50;// Steps till error is showed
int steps=1000; //Number of time steps where solution is displayed
real N=(steps-1)*stepstoshow+1; //number of time steps where u_n is calculated
int counter=-1;//for printing
int scounter=-1;//counter to store error


//DISTANCE VECTORS
real[int] distL2(steps);//stores L^2 distance
real[int] distH1(steps); //stores H^1_0 distance
func real eoc(real e, real eo, real tnew, real told)//Function that gives eoc

if (eo > 0.0)
 return -log(e/eo)/(tnew-told);
else
return 0.0;
```

```
;

//DOMAIN AND INITIAL DATA
border d(t=0,2*pi) x = R*cos(t); y = R*sin(t);label=1;; //definimos la frontera del círculo
func f=0; // fuente
//Define initial data for solution here.
//Initial data must be supported in disk of radius R so as to calculate its mass

real lambda = 4;//some parameters of the initial data
real mu = 2;
mesh disk = buildmesh(d(M));  // Mallado de disco de radio R
plot(disk, wait=1);  //dibujar malla
func uu0= fmax(lambda-exp((x-2)^2+(y-2)^2),1/n)+2*fmax(lambda-exp((x+1)^2+(y+2)^2),1/n);  //
//func uu0= fmax(lambda-mu*(x^2+y^6),0);   //SIMULATION 2

//FUNCTION SPACE AND VARIATIONAL PROBLEM
fespace Vh(disk,P2); Vh u=uu0,uold=uu0,h,w,WC,theta,difference,v,FC,vx,vy,FCx,FCy,diffder; /
macro dot(a,b) (dx(a)*dx(b)+dy(a)*dy(b))// Macro for ∇ u·∇ w
problem pme(h,w)=int2d(disk)(h*w)+int2d(disk)(dt*m*(m-1)*u^(m-2)*h*dot(u,w))+int2d(disk)(dt*
+int2d(disk)(u*w)-int2d(disk)(uold*w)+int2d(disk)(dt*m*u^(m-1)*dot(u,w))-int2d(disk)(dt*f*w)

//PARAMETERS FOR BARENBLATT SOLUTION
real mass = int2d(disk)(uu0); //mass M of u0
real C = (mass*2*k^(dim/2)/(dim*pi*betafunc))^(1/gamma); //C_M
//Prints value of parameters and name of distances
cout<<"The mass is "<< mass<<", C = "<<C<< ",  m= "<<m<<" alpha="<<alpha<<",  beta="<<beta<<
cout << "------------------------------------------------------------------" << endl;
cout << "Proximity of solutions" << endl;
cout << "------------------------------------------------------------------" << endl;
cout << "Tau (time)2norm(v-V_M)~~~2norm(gradient(v-V_M))~~" << endl;
```

```
//PLOT OF FUNCTIONS AT T_0

WC = fmax(C-k*(x^2+y^2),0)^(1/(m-1));//Rescaled Barenblatt solution to which theta(t) conver
plot(WC,wait=1,value=true,fill=1);// Plots W_C
plot(u,wait=0,value=true,fill=1);//Plots initial data



//NUMERICAL SOLUTION
for (int j=1;j<N+1;j+=1) // Time iteration j in 1,2,...N
t+=dt; //time t in t0+dt,t0+2dt,....t0+Ndt
for (int i=0; i<imax+1;i++) // Newton method iteration, i in 0,...,imax
pme; // solves to find h_i
u=u+h;//u_j,i+1=u_j,i+h_i at final step u_j,I+1 is calculated and u==u_j is set to= u_j,I+1

uold=u; //old u is now uold==u_j, as in the next time step we find u_j+1

counter=(counter+1)%stepstoshow;//If counter=j-1 is a multiple of stepstoshow it shows plot
if(counter==0)
scounter+=1;// scounter will start at 0 and go till (N-1)/stepstoshow=steps
real tau=log(1+t);// time in tau
real tauold=log(1+t-stepstoshow*dt);// tau of prevousl calculated error
theta = (1+t)^alpha*u((1+t)^(beta)*x,(1+t)^(beta)*y);//rescaled solution ==theta(t_j)
v = theta^(m-1);//pressure of theta=v
vx = dx(v);//derivatives of v
vy = dx(v);
FC = fmax(C-k*(x^2+y^2),0);//fixed profile and its derivatives
FCx= dx(FC);
FCy= dy(FC);
difference= abs(v-FC);//difference of theta and fixed profile
diffder=vx-FCx;//difference of derivatives
```

```
distL2[scounter]=sqrt(int2d(disk)(difference^2));//error L^2 de theta(xi,t)
distH1[scounter] = int2d(disk)((vx-FCx)^2+(vy-FCy)^2);
real error0 =difference(0,0);
if(scounter==0)
cout << tau<<"         " << distL2[scounter]<<"                  "<< "-----------" <<"

cout << endl;

else

cout << tau<<"           " << distL2[scounter]<<"                   "<< eoc(distL2[scounter
cout << endl;
plot(theta,wait=0,value=true,fill=1);// dibuja la solWCión reescalada para ser eventualmente

elsecontinue;
```

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